ON THE SECOND BOUNDARY VALUE PROBLEM FOR A CLASS OF FULLY NONLINEAR FLOWS I

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ABSTRACT. In this paper, a class of fully nonlinear flows with nonlinear Neumann type boundary condition is considered. This problem was solved partly by the first author under the assumption that the flow is the parabolic type special Lagrangian equation in \mathbb{R}^{2n} . We show that the convexity is preserved for solutions of the fully nonlinear parabolic equations and prove the long time existence and convergence of the flow. In particular, we can prescribe the second boundary value problems for a family of special Lagrangian graphs in Euclidean and pseudo-Euclidean space.

1. Introduction

In this paper we will study the long time existence and convergence of convex solutions solving

(1.1)
$$\frac{\partial u}{\partial t} = F(D^2 u), \quad \text{in} \quad \Omega_T = \Omega \times (0, T),$$

associated with the second boundary value problem

$$(1.2) Du(\Omega) = \tilde{\Omega}, \quad t > 0,$$

and the initial condition

$$(1.3) u = u_0, t = 0,$$

for given F, u_0 , Ω and $\tilde{\Omega}$. Specifically, Ω , $\tilde{\Omega}$ are uniformly convex bounded domains with smooth boundary in \mathbb{R}^n . F is a $C^{2+\alpha_0}$ function for some $0<\alpha_0<1$ defined on the cone Γ_+ of positive definite symmetric matrices, which is monotonically increasing and

(1.4)
$$\begin{cases} F[A] := F(\lambda_1, \lambda_2, \cdots, \lambda_n) \\ F(\cdots, \lambda_i, \cdots, \lambda_j, \cdots) = F(\cdots, \lambda_j, \cdots, \lambda_i, \cdots), & \text{for } 1 \le i < j \le n, \end{cases}$$

with

$$\lambda_1 < \lambda_2 < \dots < \lambda_n$$

being the eigenvalues of the $n \times n$ symmetric matrix A.

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Our motivation for studying equations (1.1)-(1.3) comes from providing a parabolic approach to prescribe the boundary value problems for a family of special Lagrangian graphs. In Euclidean space \mathbb{R}^{2n} , Brendle-Warren's theorem [1] says that there exists a diffeomorphism f: $\Omega \to \tilde{\Omega}$ such that

$$\Sigma = \{(x, f(x)) | x \in \Omega\}$$

is a special Lagrangian submanifold. It's well known that Σ is special Lagrangian if and only if the Lagrangian angle is a constant, a proof of which can be found in Harvey and Lawson's work (see Proposition 2.17 in [2]). To find a special Lagrangian graph Σ with f being a diffeomorphism : $\Omega \to \tilde{\Omega}$ is equivalent to the following problem (f = Du):

(1.5)
$$\begin{cases} \Sigma_{i=1}^n \arctan \lambda_i = c, & x \in \Omega, \\ Du(\Omega) = \tilde{\Omega}. \end{cases}$$

Using the continuity method of solving fully nonlinear elliptic equations with second boundary condition, S. Brendle and M. Warren [1] obtained the existence and uniqueness of the solution to (1.5). In [3], the first author studied the parabolic type special Lagrangian equation with second boundary condition:

(1.6)
$$\begin{cases} \frac{\partial u}{\partial t} = \sum_{i=1}^{n} \arctan \lambda_{i}, & t > 0, \quad x \in \Omega, \\ Du(\Omega) = \tilde{\Omega}, & t > 0, \\ u = u_{0}, & t = 0, \quad x \in \Omega. \end{cases}$$

The first author proved long time existence and convergence of the flow and then showed the existence result of special Lagrangian submanifold with the same boundary condition in $\mathbb{R}^n \times \mathbb{R}^n$. In this paper, we will see that there exists relative interesting geometric flow with second boundary condition, where the stable solutions correspond to a family of special Lagrangian graphs in pseudo-Euclidean space. To obtain the long time existence and convergence of the flow, it is often important to know whether the convexity of solutions of the evolution equations involved remain unchanged in time, seeing the work of B. Andrews [4], P. L.Lions and M. Musiela [5].

Let δ_0 be the standard Euclidean metric and g_0 be the pseudo-Euclidean metric where

$$g_0 = \frac{1}{2} \sum_{i} (dx_i \bigotimes dy_i + dy_i \bigotimes dx_i).$$

By taking linear combinations of the metrics δ_0 and g_0 , M. Warren constructed a family of metrics on $\mathbb{R}^n \times \mathbb{R}^n$:

$$(1.7) g_{\tau} = \cos \tau g_0 + \sin \tau \delta_0.$$

In calibrated geometry, we adapt the details of the extremal Lagrangian surfaces in $(\mathbb{R}^n \times \mathbb{R}^n, g_{\tau})$ which were firstly obtained by M. Warren [6] and arise as solutions to a family of special Lagrangian equations:

(1.8)
$$(\tau = 0) \qquad \sum_{i} \ln \lambda_i = c,$$

(1.9)
$$(0 < \tau < \frac{\pi}{4}) \qquad \sum_{i} \ln(\frac{\lambda_i + a - b}{\lambda_i + a + b}) = c,$$

(1.10)
$$(\tau = \frac{\pi}{4}) \qquad \sum_{i} \frac{1}{1 + \lambda_i} = c,$$

(1.11)
$$(\frac{\pi}{4} < \tau < \frac{\pi}{2}) \qquad \sum_{i} \arctan(\frac{\lambda_i + a - b}{\lambda_i + a + b}) = c,$$

(1.12)
$$(\tau = \frac{\pi}{2}) \qquad \sum_{i} \arctan \lambda_i = c,$$

where $a = \cot \tau$ and $b = \sqrt{|\cot^2 \tau - 1|}$.

Here we consider the following fully nonlinear elliptic equations with second boundary condition:

(1.13)
$$\begin{cases} F_{\tau}(D^2u) = c, & x \in \Omega, \\ Du(\Omega) = \tilde{\Omega}, \end{cases}$$

where

$$F_{\tau}(A) = \begin{cases} \sum_{i} \ln \lambda_{i}, & \tau = 0, \\ \sum_{i} \ln(\frac{\lambda_{i} + a - b}{\lambda_{i} + a + b}) & 0 < \tau < \frac{\pi}{4}, \\ \sum_{i} \frac{1}{1 + \lambda_{i}}, & \tau = \frac{\pi}{4}, \\ \sum_{i} \arctan(\frac{\lambda_{i} + a - b}{\lambda_{i} + a + b}), & \frac{\pi}{4} < \tau < \frac{\pi}{2}, \\ \sum_{i} \arctan \lambda_{i}, & \tau = \frac{\pi}{2}, \end{cases}$$

and Du are the special Lagrangian diffeomorphisms from Ω to $\tilde{\Omega}$ in Euclidean and pseudo-Euclidean space. In dimension 2, P. Delanoë [7] obtained a unique smooth solution for the second boundary value problem of the Monge-Ampère equation with respect to $\tau=0$ in (1.13) if both domains are uniformly convex. Later the generalization of P. Delanoë's theorem to higher dimensions was given by L. Caffarelli [8] and J. Urbas [9]. Using the parabolic methods, O.C. Schnürer and K. Smoczyk [10] also obtained the existence of solutions to (1.13) for $\tau=0$. As far as $\tau=\frac{\pi}{2}$ is concerned, S. Brendle and M. Warren [1] proved the existence and uniqueness of the solution by the elliptic methods and the first author [3] obtained the existence of solution by the parabolic methods.

The aim of the paper is to study the existence of solutions of the equations (1.13) for $\tau = \frac{\pi}{4}$, i.e.,

(1.14)
$$\begin{cases} \sum_{i} \frac{1}{1+\lambda_{i}} = c, & x \in \Omega, \\ Du(\Omega) = \tilde{\Omega}. \end{cases}$$

We will see that $-F_{\frac{\pi}{4}}$ can be viewed as the map F defined by (1.4).

For any $\mu_1 > 0, \mu_2 > 0$, we define

$$\Gamma^+_{|\mu_1,\mu_2|} = \{(\lambda_1,\lambda_2,\cdots,\lambda_n)|0 \le \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n, \lambda_1 \le \mu_1, \lambda_n \ge \mu_2\}.$$

We assume that there exist positive constants λ , Λ depending only on μ_1, μ_2 such that for any $(\lambda_1, \lambda_2, \dots, \lambda_n) \in \Gamma^+_{|\mu_1, \mu_2|}$:

(1.15)
$$\Lambda \ge \sum_{i=1}^{n} \frac{\partial F}{\partial \lambda_i} \ge \lambda,$$

(1.16)
$$\Lambda \ge \sum_{i=1}^{n} \frac{\partial F}{\partial \lambda_i} \lambda_i^2 \ge \lambda.$$

In addition,

(1.17)
$$F(A) \text{ and } F^*(A) \triangleq -F(A^{-1}) \text{ are concave on } \Gamma_+.$$

Moreover, we assume that there exist two functions f_1 , f_2 which are monotonically increasing in $(0, +\infty)$ satisfying

(1.18)
$$f_1(\lambda_1) \leq F(\lambda_1, \lambda_2, \dots, \lambda_n) \leq f_2(\lambda_n) \quad (\forall \ 0 \leq \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n),$$
 and for any $\Phi, \Psi \in \mathcal{L}$,

(1.19)
$$\begin{cases} f_1(t) \le \Phi \Rightarrow \exists t_1 > 0, \ t \le t_1, \\ f_2(t) \ge \Psi \Rightarrow \exists t_2 > 0, \ t \ge t_2, \end{cases}$$

where

$$\pounds = \{\Upsilon | \exists (\lambda_1, \lambda_2, \cdots, \lambda_n), 0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n, \Upsilon = F(\lambda_1, \lambda_2, \cdots, \lambda_n) \}.$$

We can't expect that F satisfies (1.15) and (1.16) for the universal constants λ and Λ on the cone Γ_+ . The reason is in the following: for any $\varepsilon > 0$, by taking

$$\lambda_1 = \lambda_2 = \dots = \lambda_n = \varepsilon,$$

we obtain

$$\varepsilon^2 \Lambda \ge \varepsilon^2 \sum_{i=1}^n \frac{\partial F}{\partial \lambda_i} = \sum_{i=1}^n \frac{\partial F}{\partial \lambda_i} \lambda_i^2 \ge \lambda.$$

In view of the above fact, we introduce the domain $\Gamma_{]\mu_1,\mu_2[}^+$ such that the two conditions are compatible. As same as the statement in [1], the range of c should be limited for the solvability of the equation (1.13) and the condition (1.18) reflect the issue in some way. On the other hand, in Section 3, we will prove that there exist universal constants μ_1 and μ_2 such that $(\lambda_1, \lambda_2, \dots, \lambda_n)$ are always in $\Gamma_{]\mu_1,\mu_2[}^+$ under

the flow. So F satisfies the structure conditions (1.15) and (1.16) for the constants λ and Λ under the flow.

By [3] and Corollary 5.4 of the present paper, the examples of functions satisfying (1.15)- (1.19) are those corresponding to

$$\begin{cases} F(\lambda_1, \lambda_2, \cdots, \lambda_n) := F_{\frac{\pi}{2}}(\lambda_1, \lambda_2, \cdots, \lambda_n) = \sum_i \arctan \lambda_i, \\ f_1(t) = n \arctan t, & f_2(t) = n \arctan t, \end{cases}$$

and

$$\begin{cases} F(\lambda_1, \lambda_2, \cdots, \lambda_n) := -F_{\frac{\pi}{4}}(\lambda_1, \lambda_2, \cdots, \lambda_n) = -\sum_i \frac{1}{1 + \lambda_i}, \\ f_1(t) = -\frac{n}{1 + t}, \quad f_2(t) = -\frac{n}{1 + t}. \end{cases}$$

The main results of this paper can be summarized as follows

Theorem 1.1. Assume that Ω , $\tilde{\Omega}$ are bounded, uniformly convex domains with smooth boundary in \mathbb{R}^n , $0 < \alpha_0 < 1$ and the map F satisfies (1.4), (1.15), (1.16), (1.17), (1.18), (1.19). Then for any given initial function $u_0 \in C^{2+\alpha_0}(\bar{\Omega})$ which is uniformly convex and satisfies $Du_0(\Omega) = \tilde{\Omega}$, the strictly convex solution of (1.1)-(1.3) exists for all $t \geq 0$ and $u(\cdot,t)$ converges to a function $u^{\infty}(x,t) = u^{\infty}(x) + C_{\infty} \cdot t$ in $C^{1+\zeta}(\bar{\Omega}) \cap C^{4+\alpha}(\bar{D})$ as $t \to \infty$ for any $D \subset\subset \Omega$, $\zeta < 1, 0 < \alpha < \alpha_0$, i.e.,

$$\lim_{t \to +\infty} \|u(\cdot,t) - u^{\infty}(\cdot,t)\|_{C^{1+\zeta}(\bar{\Omega})} = 0, \qquad \lim_{t \to +\infty} \|u(\cdot,t) - u^{\infty}(\cdot,t)\|_{C^{4+\alpha}(\bar{D})} = 0.$$
and $u^{\infty}(x) \in C^{1+1}(\bar{\Omega}) \cap C^{4+\alpha}(\Omega)$ in a colution of

And $u^{\infty}(x) \in C^{1+1}(\bar{\Omega}) \cap C^{4+\alpha}(\Omega)$ is a solution of

(1.20)
$$\begin{cases} F(D^2u) = C_{\infty}, & x \in \Omega, \\ Du(\Omega) = \tilde{\Omega}. \end{cases}$$

The constant C_{∞} depends only on Ω , $\tilde{\Omega}$ and F. The solution to (1.20) is unique up to additions of constants.

Corollary 1.2. Let Ω , $\tilde{\Omega}$ and F satisfy the conditions in the above theorem. If Fis C^{∞} , then there exist $u^{\infty}(x) \in C^{\infty}(\bar{\Omega})$ and the constant C_{∞} which satisfy (1.20).

Remark 1.3. Theorem 3.3 of B.Andrews in [4] proved that the convexity for solutions of fully nonlinear parabolic equations under conditions (1.4) and (1.17) can be preserved if the solutions on the boundary are strictly convex. Here we do not make any assumption of u to be convex on the boundary.

Remark 1.4. The first author [11] used the elliptic methods to obtain the similar results of (1.20), but the structure conditions of F in [11] are more complicated.

As a consequence of Theorem 1.1, we will prove the existence and uniqueness of classical solution of (1.14).

Theorem 1.5. Let Ω , Ω be bounded, uniformly convex domains with smooth boundary in \mathbb{R}^n . Then there exist $u \in C^{\infty}(\bar{\Omega})$ and the constant c such that u is a solution of the second boundary value problem (1.14). The solution to (1.14) is unique up to additions of constants.

Definition 1.6. We say that $\Sigma = \{(x, f(x)) | x \in \Omega\}$ is a special Lagrangian graph in $(\mathbb{R}^n \times \mathbb{R}^n, g_{\tau})$ if

$$f = Du$$

and u satisfies

$$F_{\tau}(D^2u(x)) = c, \quad x \in \Omega.$$

By the above result, we can extend Brendle-Warren's theorem [1] to the case in $(\mathbb{R}^n \times \mathbb{R}^n, g_{\frac{\pi}{4}})$:

Corollary 1.7. Let Ω , $\tilde{\Omega}$ be bounded, uniformly convex domains with smooth boundary in \mathbb{R}^n . Then there exists a diffeomorphism $f: \Omega \to \tilde{\Omega}$ such that

$$\Sigma = \{(x, f(x)) | x \in \Omega\}$$

is a special Lagrangian graph in $(\mathbb{R}^n \times \mathbb{R}^n, g_{\frac{\pi}{4}})$.

The monotone increasing of F implies the ellipticity of the equation (1.20). As the statement in [1], the range of c should be limited for the solvability of the equation (1.20), so the conditions (1.18)-(1.19) are just natural in some degree. Condition (1.17) is essentially the ones used in [4] to preserve the convexity for the solutions of the fully nonlinear parabolic equations. And conditions (1.15), (1.16) are essential part to carry out the upcoming C^2 priori estimates for the solutions on the boundary.

The rest of this paper is organized as follows. In Section 2, we establish the short time existence result to the flow (1.1)-(1.3) by using the inverse function theorem. In Section 3, we collect necessary preliminaries which will be used in the proof of Theorem 1.1. The techniques used in this section are reflective of those in [3], [9], and [10] to the second boundary value problems for fully nonlinear differential equations. We use barrier arguments to obtain the C^2 upper bound estimates on the boundary. Here the structure conditions (1.15), (1.16) play an important role to construct suitable auxiliary functions as barriers. The conditions are also used to establish the C^2 lower bounds for the solutions on the boundary. In Section 4, we give the proof of Theorem 1.1 and Corollary 1.2. In Section 5, we replace F by $-F_{\frac{\pi}{4}}$ in (1.1) and solve the corresponding problems. Under the flow (1.1)-(1.3), we show that $-F_{\frac{\pi}{4}}$ satisfies the conditions (1.15)-(1.19), and then we give the proof of Theorem 1.5 and Corollary 1.7. In Section 6, we show that the applications of Theorem 1.1 include a number of previously established results of various authors as consequences.

2. The short time existence of the parabolic flow

Throughout the following, Einstein's convention of summation over repeated indices will be adopted. Denote

$$u_i = \frac{\partial u}{\partial x_i}, u_{ij} = \frac{\partial^2 u}{\partial x_i \partial x_j}, u_{ijk} = \frac{\partial^3 u}{\partial x_i \partial x_j \partial x_k}, \cdots$$

and

$$[u^{ij}] = [u_{ij}]^{-1}, \quad F^{ij}(D^2u) = \frac{\partial F}{\partial u_{ij}}, \quad \Omega_T = \Omega \times (0, T).$$

We first recall the relevant Sobolev spaces (cf. Chapter 1 in [12]). For every multiindex $\beta = (\beta_1, \beta_2, \dots, \beta_n)(\beta_i \ge 0 \text{ for } i = 1, 2, \dots, n)$ with length $|\beta| = \sum_{i=1}^n \beta_i$ and $j \ge 0$, we set

$$D^{\beta}u = \frac{\partial^{|\beta|}u}{\partial x_1^{\beta_1}\partial x_2^{\beta_2}\cdots\partial x_n^{\beta_n}}, \qquad D^{\beta}D_t^ju = \frac{\partial^{|\beta|+j}u}{\partial x_1^{\beta_1}\partial x_2^{\beta_2}\cdots\partial x_n^{\beta_n}\partial t^j}.$$

We remind the definition of the usual functional spaces $(k \ge 0)$:

$$C^k(\Omega) = \{u : \Omega \to \mathbb{R} | \forall \beta, |\beta| \le k, D^{\beta}u \text{ is continuous in } \Omega\},\$$

$$C^k(\bar{\Omega}) = \{ u \in C^k(\Omega) | \forall \beta, |\beta| \le k, D^\beta u \text{ can be extended by continuity to } \partial \Omega \};$$

$$C^{k,\frac{k}{2}}(\Omega_T) = \{u : \Omega_T \to \mathbb{R} | \forall \beta, j \ge 0, |\beta| + 2j \le k, D^{\beta}D_t^j u \text{ is continuous in } \Omega_T \},$$

 $C^{k,\frac{k}{2}}(\bar{\Omega}_T) = \{u \in C^{k,\frac{k}{2}}(\Omega_T) | \forall \beta, j \geq 0, |\beta| + 2j \leq k, D^{\beta}D_t^j u \text{ can be extended by continuity to } \partial\Omega_T \}.$

Moreover $C^k(\bar{\Omega}), C^{k,\frac{k}{2}}(\bar{\Omega}_T)$ are Banach spaces equipped with the norm respectively:

$$||u||_{C^k(\bar{\Omega})} = \sum_{i=0}^k \sup_{|\beta|=i} \sup_{\bar{\Omega}} |D^{\beta}u|,$$

$$||u||_{C^{k,\frac{k}{2}}(\bar{\Omega}_T)} = \sum_{j>0,|\beta|+2j \le k} \sup_{\bar{\Omega}_T} |D^{\beta}D_t^j u|.$$

We now remind the definition of Hölder spaces. Let $\alpha \in [0,1]$. We define the α -Hölder coefficient of u in Ω :

$$[u]_{\alpha,\Omega} = \sup_{x \neq u, x, u \in \Omega} \frac{|u(x) - u(y)|}{|x - y|^{\alpha}}.$$

If $[u]_{\alpha,\Omega} < +\infty$, then we call u Hölder continuous with exponent α in Ω . If there are no ambiguity about the domains Ω , we denote $[u]_{\alpha,\Omega}$ by $[u]_{\alpha}$. Similarly, the $(\alpha, \frac{\alpha}{2})$ -Hölder coefficient of u in Ω_T can be defined by

$$[u]_{\alpha,\frac{\alpha}{2},\Omega_T} = \sup_{(x,t)\neq (y,\tau),(x,t),(y,\tau)\in\Omega_T} \frac{|u(x,t)-u(y,\tau)|}{\max\{|x-y|^\alpha,|t-\tau|^\frac{\alpha}{2}\}},$$

and u is Hölder continuous with exponent $(\alpha, \frac{\alpha}{2})$ in Ω_T if $[u]_{\alpha, \frac{\alpha}{2}, \Omega_T} < +\infty$. Meanwhile, we denote $[u]_{\alpha, \frac{\alpha}{2}, \Omega_T}$ by $[u]_{\alpha, \frac{\alpha}{2}}$. We denote $C^{k+\alpha}(\bar{\Omega})$ as the set of functions belonging to $C^k(\bar{\Omega})$ whose k-order partial derivatives are Hölder continuous with exponent α in Ω and $C^{k+\alpha}(\bar{\Omega})$ is a Banach space equipped with the following norm:

$$||u||_{C^{k+\alpha}(\bar{\Omega})} = ||u||_{C^k(\bar{\Omega})} + [u]_{k+\alpha}$$

where

$$[u]_{k+\alpha} = \sum_{|\beta|=k} [D^{\beta}u]_{\alpha}.$$

Likewise, we denote $C^{k+\alpha,\frac{k+\alpha}{2}}(\bar{\Omega}_T)$ as the set of functions belonging to $C^{k,\frac{k}{2}}(\bar{\Omega}_T)$ whose $(k,\frac{k}{2})$ -order partial derivatives are Hölder continuous with exponent $(\alpha,\frac{\alpha}{2})$ in Ω_T and $C^{k+\alpha,\frac{k+\alpha}{2}}(\bar{\Omega}_T)$ is a Banach space equipped with the following norm:

$$\|u\|_{C^{k+\alpha,\frac{k+\alpha}{2}}(\bar{\Omega}_T)} = \|u\|_{C^{k,\frac{k}{2}}(\bar{\Omega}_T)} + [u]_{k+\alpha,\frac{k+\alpha}{2}},$$

where

$$[u]_{k+\alpha,\frac{k+\alpha}{2}} = \sum_{|\beta|+2j=k} [D^{\beta}D_t^j u]_{\alpha,\frac{\alpha}{2}}.$$

By the methods on the second boundary value problems for equations of Monge-Ampère type [9], the parabolic boundary condition in (1.2) can be reformulated as

$$h(Du) = 0, \qquad x \in \partial\Omega, \quad t > 0,$$

where h is a smooth function on $\tilde{\Omega}$:

$$\tilde{\Omega} = \{ p \in \mathbb{R}^n | h(p) > 0 \}, \qquad |Dh|_{\partial \tilde{\Omega}} = 1.$$

The so called boundary defining function is strictly concave, i.e., $\exists \theta > 0$,

$$\frac{\partial^2 h}{\partial y_i \partial y_j} \xi_i \xi_j \le -\theta |\xi|^2, \quad \text{for} \quad \forall p = (y_1, y_2, \cdots, y_n) \in \tilde{\Omega}, \quad \xi = (\xi_1, \xi_2, \cdots, \xi_n) \in \mathbb{R}^n.$$

We also give the boundary defining function according to Ω (cf.[1]):

$$\Omega = \{ p \in \mathbb{R}^n | \tilde{h}(p) > 0 \}, \qquad |D\tilde{h}|_{\partial\Omega} = 1,$$

$$\exists \tilde{\theta} > 0, \quad \frac{\partial^2 \tilde{h}}{\partial y_i \partial y_j} \xi_i \xi_j \le -\tilde{\theta} |\xi|^2, \quad \text{for} \quad \forall p = (y_1, y_2, \cdots, y_n) \in \Omega, \quad \xi = (\xi_1, \xi_2, \cdots, \xi_n) \in \mathbb{R}^n.$$

Thus the parabolic flow (1.1)-(1.3) is equivalent to the evolution problem:

(2.1)
$$\begin{cases} \frac{\partial u}{\partial t} = F(D^2 u), & t > 0, \quad x \in \Omega, \\ h(Du) = 0, & t > 0, \quad x \in \partial \Omega, \\ u = u_0, & t = 0, \quad x \in \Omega. \end{cases}$$

To establish the short time existence of classical solutions of (2.1), we use the inverse function theorem in Fréchet spaces and the theory of linear parabolic equations for oblique boundary condition. The method is along the idea of proving the short time existence of convex solutions on the second boundary value problem for Lagrangian mean curvature flow [3]. We include the details for the convenience of the readers.

Lemma 2.1. (I. Ekeland, see Theorem 2 in [13].) Let X and Y be Banach spaces. Suppose

$$happa: X \to Y$$

is continuous and Gâteaux-differentiable, with $\hbar[0] = 0$. Assume that the derivative $D\hbar[x]$ has a right inverse T[x], uniformly bounded in a neighbourhood of 0 in X:

$$\forall y \in Y, \quad D\hbar[x]\Upsilon[x]y = y;$$

$$||x|| \le R \Longrightarrow ||T[x]|| \le m.$$

For every $y \in Y$ if

$$\parallel y \parallel < \frac{R}{m},$$

then there is some $x \in X$ such that :

and

$$\hbar[x] = y.$$

As an application of I. Ekeland's theorem, we obtain the following inverse function theorem which will be used to prove the short time existence result for equation (2.1).

Lemma 2.2. Let X and Y be Banach spaces. Suppose

$$J: X \to Y$$

is continuous and Gâteaux-differentiable, with $J(v_0) = w_0$. Assume that the derivative DJ[v] has a right inverse L[v], uniformly bounded in a neighbourhood of v_0 :

$$\begin{aligned} \forall y \in Y, \quad DJ[v]L[v]y &= y; \\ \|v - v_0\| &\leq R \Longrightarrow \|L[v]\| \leq m. \end{aligned}$$

For every $w \in Y$ if

$$\parallel w - w_0 \parallel < \frac{R}{m},$$

then there is some $v \in X$ such that:

$$||v - v_0|| < R$$

and

$$J(v) = w$$
.

Proof. Denote $v=x+v_0$ and $\hbar[x]\triangleq J[x+v_0]-w_0$, then $\hbar[0]=0$. Since the derivative DJ[v] has a right inverse L[v], we deduce that $D\hbar[x]=DJ[v]$ has a right inverse L[v]. Set $T[x]=L[x+v_0]$. Following Lemma 2.1, for every $y\triangleq w-w_0\in Y$, if

$$\parallel y \parallel < \frac{R}{m},$$

then there is some $x \in X$ such that:

and

$$\hbar[x] = y = w - w_0.$$

So we have:

$$||v - v_0|| < R,$$

and

$$J(v) = w$$
.

Lemma 2.3. (See Theorem 8.8 and 8.9 in [14].) Assume that $f \in C^{\alpha_0, \frac{\alpha_0}{2}}(\bar{\Omega}_T)$ for some $0 < \alpha_0 < 1$, T > 0, and G(x, p), $G_p(x, p)$ are in $C^{1+\alpha_0}(\Xi)$ for any compact subset Ξ of $\partial\Omega \times \mathbb{R}^n$ such that $\inf_{\partial\Omega}\langle G_p, \nu \rangle > 0$ where ν is the inner normal vector of $\partial\Omega$. Let $u_0 \in C^{2+\alpha_0}(\bar{\Omega})$ be strictly convex and satisfy $G(x, Du_0) = 0$. Then there exists T' > 0 ($T' \leq T$) such that we can find a unique solution which is strictly convex in x variable in the class $C^{2+\alpha_0}(\bar{\Omega}_T)$ to the following equations

$$\begin{cases} \frac{\partial u}{\partial t} - a^{ij}(x, t)u_{ij} = f(x, t), & T' > t > 0, \quad x \in \Omega, \\ G(x, Du) = 0, & T' > t > 0, \quad x \in \partial\Omega, \\ u = u_0, & t = 0, \quad x \in \Omega, \end{cases}$$

where $a^{ij}(x,t)(1 \leq i,j \leq n) \in C^{\alpha_0,\frac{\alpha_0}{2}}(\bar{\Omega}_T)$ and $[a^{ij}(x,t)] \geq a_0I$ for some positive constant a_0 .

By the property of $C^{2+\alpha_0,\frac{2+\alpha_0}{2}}(\bar{\Omega}_{T'})$ and $u(x,t)|_{t=0}=u_0(x)$, we obtain

(2.2)
$$\lim_{t \to 0} \| u(\cdot, t) - u_0(\cdot) \|_{C^{2+\alpha_0}(\bar{\Omega})} = 0.$$

For any $\alpha < \alpha_0$, we have

$$\begin{split} &\frac{|(D^2u(x,t)-D^2u_0(x))-(D^2u(y,\tau)-D^2u_0(y))|}{\max\{|x-y|^{\alpha},|t-\tau|^{\frac{\alpha}{2}}\}} \\ &\leq \frac{|(D^2u(x,t)-D^2u_0(x))-(D^2u(y,t)-D^2u_0(y))|}{|x-y|^{\alpha}} \\ &+|t-\tau|^{\frac{\alpha_0-\alpha}{2}}\frac{|(D^2u(y,t)-D^2u_0(y))-(D^2u(y,\tau)-D^2u_0(y))|}{|t-\tau|^{\frac{\alpha_0}{2}}}. \end{split}$$

Then we get

(2.3)
$$\|D^{2}u - D^{2}u_{0}\|_{C^{\alpha,\frac{\alpha}{2}}(\bar{\Omega}_{T'})} \leq \max_{0 \leq t \leq T'} \|D^{2}u(\cdot,t) - D^{2}u_{0}(\cdot)\|_{C^{\alpha}(\bar{\Omega})} + T'^{\frac{\alpha-\alpha_{0}}{2}} \|D^{2}u - D^{2}u_{0}\|_{C^{\alpha_{0},\frac{\alpha_{0}}{2}}(\bar{\Omega}_{T'})}$$

Combining (2.2) with (2.3), we obtain

(2.4)
$$\lim_{T' \to 0} \| D^2 u - D^2 u_0 \|_{C^{\alpha, \frac{\alpha}{2}}(\bar{\Omega}_{T'})} = 0$$

which will be used later.

According to the proof in [9], we can verify the oblique boundary condition.

Lemma_2.4. (See J. Urbas [9].)

 $u \in C^2(\bar{\Omega})$ with $D^2u > 0 \Longrightarrow \inf_{\partial\Omega} h_{p_k}(Du)\nu_k > 0$ where $\nu = (\nu_1, \nu_2, \cdots, \nu_n)$ is the unit inward normal vector of $\partial\Omega$, i.e., h(Du) = 0 is strictly oblique.

We are now in a position to prove the short time existence of solutions of (2.1) which is equivalent to the problem (1.1)-(1.3).

Proposition 2.5. According to the conditions in Theorem 1.1, there exist some T''>0 and $u\in C^{2+\alpha,\frac{2+\alpha}{2}}(\bar{\Omega}_{T''})$ which depend only on Ω , $\tilde{\Omega}$, u_0 , such that u is a solution of (2.1) and is strictly convex in x variable.

Proof. Denote the Banach spaces

$$X = C^{2+\alpha,1+\frac{\alpha}{2}}(\bar{\Omega}_T), \quad Y = C^{\alpha,\frac{\alpha}{2}}(\bar{\Omega}_T) \times C^{1+\alpha,\frac{1+\alpha}{2}}(\partial\Omega \times (0,T)) \times C^{2+\alpha}(\bar{\Omega}),$$

where

$$\|\cdot\|_{Y} = \|\cdot\|_{C^{\alpha,\frac{\alpha}{2}}(\bar{\Omega}_{T})} + \|\cdot\|_{C^{1+\alpha,\frac{1+\alpha}{2}}(\partial\Omega\times(0,T))} + \|\cdot\|_{C^{2+\alpha}(\bar{\Omega})}.$$

Define a map

$$J: X \to Y$$

by

$$J(u) = \begin{cases} \frac{\partial u}{\partial t} - F(D^2 u), & (x, t) \in \Omega_T, \\ h(Du), & (x, t) \in \partial\Omega \times (0, T), \\ u, & (x, t) \in \Omega \times \{t = 0\}. \end{cases}$$

The strategy is now to use the inverse function theorem to obtain the short time existence result. The computation of the $G\hat{a}$ teaux derivative shows that:

$$\forall u, v \in X, \quad DJ[u](v) \triangleq \frac{d}{d\tau}J(u+\tau v)|_{\tau=0} = \begin{cases} \frac{\partial v}{\partial t} - F^{ij}(D^2u)v_{ij}, & (x,t) \in \Omega_T, \\ h_{p_i}(Du)v_i, & (x,t) \in \partial\Omega \times (0,T), \\ v, & (x,t) \in \Omega \times \{t=0\}. \end{cases}$$

Using Lemma 2.3 and Lemma 2.4, there exists $T_1 > 0$ such that we can find

$$\hat{u} \in C^{2+\alpha_0,1+\frac{\alpha_0}{2}}(\bar{\Omega}_{T_1}) \subset X$$

to be strictly convex in x variable, which satisfies the following equations:

(2.5)
$$\begin{cases} \frac{\partial \hat{u}}{\partial t} - \triangle \hat{u} = F(D^2 u_0) - \triangle u_0, & T_1 > t > 0, \quad x \in \Omega, \\ h(D\hat{u}) = 0, & T_1 > t > 0, \quad x \in \partial \Omega. \\ \hat{u} = u_0, & t = 0, \quad x \in \Omega. \end{cases}$$

We see that $\exists R > 0$, such that u is strictly convex in x variable if

$$||u - \hat{u}||_{C^{2+\alpha, \frac{2+\alpha}{2}}(\bar{\Omega}_{T_1})} < R,$$

For each $Z \triangleq (f, g, w) \in Y$ and using Lemma 2.3 again, we know that there exists a unique $v \in X(T = T_1)$ satisfying DJ[u](v) = (f, g, w), i.e.

$$\begin{cases} \frac{\partial v}{\partial t} - F^{ij}(D^2 u)v_{ij} = f, & T_1 > t > 0, \quad x \in \Omega, \\ h_{p_i}(D u)v_i = g, & T_1 > t > 0, \quad x \in \partial\Omega, \\ v = w, & t = 0, \quad x \in \Omega. \end{cases}$$

Using Schauder estimates for linear parabolic equation to oblique boundary condition (cf. Theorem 8.8 and 8.9 in [14]), we obtain

$$\|v\|_{C^{2+\alpha,\frac{2+\alpha}{2}}(\bar{\Omega}_{T_1})} \le m(\|f\|_{C^{\alpha,\frac{\alpha}{2}}(\bar{\Omega}_{T_1})} + \|g\|_{C^{1+\alpha,\frac{1+\alpha}{2}}(\partial\Omega\times(0,T_1))} + \|w\|_{C^{2+\alpha}(\bar{\Omega})}),$$

for some positive constant m. Using the definition of the Banach spaces X and Y with $T = T_1$, we can rewrite the above Schauder estimates as

$$\parallel v \parallel_X \leq m \parallel Z \parallel_Y$$
.

If $||Z||_Y \leq 1$, then we have

$$||v||_X \leq m$$
.

It means that the derivative DJ[u](v) = Z has a right inverse v = L[u](Z) and

$$||L[u]|| \triangleq \sup_{||Z||_Y \le 1} ||L[u](Z)||_X \le m.$$

If we set

$$\hat{f} = \frac{\partial \hat{u}}{\partial t} - F(D^2 \hat{u}), \quad w_0 = (\hat{f}, 0, u_0), \quad w = (0, 0, u_0),$$

then we can show that

$$\| \hat{f} - 0 \|_{C^{\alpha, \frac{\alpha}{2}}(\bar{\Omega}_{T_{1}})} = \| \triangle \hat{u} - \triangle u_{0} + F(D^{2}u_{0}) - F(D^{2}\hat{u}) \|_{C^{\alpha, \frac{\alpha}{2}}(\bar{\Omega}_{T_{1}})}$$

$$\leq \| \triangle \hat{u} - \triangle u_{0} \|_{C^{\alpha, \frac{\alpha}{2}}(\bar{\Omega}_{T_{1}})} + \| F(D^{2}u_{0}) - F(D^{2}\hat{u}) \|_{C^{\alpha, \frac{\alpha}{2}}(\bar{\Omega}_{T_{1}})}$$

$$\leq C \| D^{2}\hat{u} - D^{2}u_{0} \|_{C^{\alpha, \frac{\alpha}{2}}(\bar{\Omega}_{T_{1}})},$$

where C is a constant depending only on the known data. Using (2.4), we conclude: $\exists T'' > 0 \ (T'' \leq T_1)$ to be small enough such that

$$\|\hat{f} - 0\|_{C^{\alpha,\frac{\alpha}{2}}(\bar{\Omega}_{T''})} \le C \|D^2\hat{u} - D^2u_0\|_{C^{\alpha,\frac{\alpha}{2}}(\bar{\Omega}_{T''})} < \frac{R}{m}.$$

Thus we obtain

$$\| w - w_0 \|_Y = \| 0 - \hat{f} \|_{C^{\alpha, \frac{\alpha}{2}}(\bar{\Omega}_{T''})} < \frac{R}{m}.$$

By Lemma 2.2, we give the desired results.

Remark 2.6. By the strong maximum principle, the strictly convex solution to (2.1) is unique.

3. Preliminary results

In this section, the C^2 a priori bound is accomplished by making the second derivative estimates on the boundary for the solutions of fully nonlinear parabolic equations. This treatment is similar to the problems presented in [3], [9] and [10], but requires some modification to accommodate the more general situation. Specifically, the structure conditions (1.15) and (1.16) are needed in order to derive differential inequalities from barriers which can be used.

For convenience of the computation, we set

$$\beta^k \triangleq \frac{\partial h(Du)}{\partial u_k} = h_{p_k}(Du)$$

and $\langle \cdot, \cdot \rangle$ be the inner product in \mathbb{R}^n . By Proposition 2.5 and the regularity theory of parabolic equations, we may assume that u is a strictly convex solution of (1.1)-(1.3) in the class $C^{2+\alpha,1+\frac{\alpha}{2}}(\bar{\Omega}_T) \cap C^{4+\alpha,2+\frac{\alpha}{2}}(\Omega_T)$ for some T>0.

Lemma 3.1 (\dot{u} -estimates).

As long as the convex solution to (1.1)-(1.3) exists, the following estimates hold, i.e.,

$$\Theta_0 \triangleq \min_{\bar{\Omega}} F(D^2 u_0) \le \dot{u} \triangleq \frac{\partial u}{\partial t} \le \Theta_1 \triangleq \max_{\bar{\Omega}} F(D^2 u_0).$$

Proof. We use the methods known from Lemma 2.1 in [10].

From (1.1), a direct computation shows that

$$\frac{\partial(\dot{u})}{\partial t} - F^{ij}\partial_{ij}(\dot{u}) = 0.$$

Using the maximum principle, we see that

$$\min_{\bar{\Omega}_T}(\dot{u}) = \min_{\partial \bar{\Omega}_T}(\dot{u}).$$

Without loss of generality, we assume that $\dot{u} \neq constant$. If $\exists x_0 \in \partial \Omega, t_0 > 0$, such that $\dot{u}(x_0, t_0) = \min_{\bar{\Omega}_T}(\dot{u})$. On one hand, since $\langle \beta, \nu \rangle > 0$, by the Hopf Lemma (cf.[15]) for parabolic equations, there must hold in the following

$$\dot{u}_{\beta}(x_0, t_0) \neq 0.$$

On the other hand, we differentiate the boundary condition and then obtain

$$\dot{u}_{\beta} = h_{p_k}(Du) \frac{\partial \dot{u}}{\partial x_k} = \frac{\partial h(Du)}{\partial t} = 0.$$

It is a contradiction. So we deduce that

$$\dot{u} \ge \min_{\bar{\Omega}_T}(\dot{u}) = \min_{\partial \bar{\Omega}_T|_{t=0}}(\dot{u}) = \min_{\bar{\Omega}} F(D^2 u_0).$$

For the same reason, we have

$$\frac{\partial u}{\partial t} \le \Theta_1 \triangleq \max_{\bar{\Omega}} F(D^2 u_0).$$

Putting these facts together, the assertion follows.

Lemma 3.2. Let (x,t) be an arbitrary point of Ω_T , and $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ be the eigenvalues of D^2u at (x,t). As long as the convex solution to (1.1)-(1.3) exists, then $\exists \mu_1 > 0, \mu_2 > 0$ depending only on $F(D^2u_0)$, such that

$$\lambda_1 \le \mu_1, \ \lambda_n \ge \mu_2.$$

Proof. Using Lemma 3.1 and condition (1.18), we obtain

$$f_1(\lambda_1) \le \Theta_1, \ f_2(\lambda_n) \ge \Theta_0.$$

By the condition (1.19), we get the desired results.

By Lemma 3.2, the points $(\lambda_1, \lambda_2, \dots, \lambda_n)$ are always in $\Gamma^+_{]\mu_1,\mu_2[}$ under the flow. So we obtain the next lemma.

Lemma 3.3. Let (x,t) be an arbitrary point of Ω_T , and $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ be the eigenvalues of D^2u at (x,t). As long as the convex solution to (1.1)-(1.3) exists, then $\exists \lambda > 0, \Lambda > 0$ depending only on $F(D^2u_0)$, such that F satisfies the structure conditions (1.15), (1.16).

In the following, we always assume that λ and Λ are universal constants depending on the known data. In order to establish the C^2 estimates, first we make use of the method to do the strict obliqueness estimates, a parabolic version of a result of J.Urbas [9] which was given in [10]. Returning to Lemma 2.4, we get a uniform positive lower bound of the quantity $\inf_{\partial\Omega} h_{p_k}(Du)\nu_k$ which does not depend on t under the structure conditions of F.

Lemma 3.4. As long as the strictly convex solution to (1.1)-(1.3) exists, the strict obliqueness estimates can be obtained by

$$\langle \beta, \nu \rangle \ge \frac{1}{C_1} > 0,$$

where the constant C_1 is independent of t.

Proof. Let $(x_0, t_0) \in \partial \Omega \times [0, T]$ such that

$$\langle \beta, \nu \rangle (x_0, t_0) = h_{p_k}(Du)\nu_k = \min_{\partial \Omega \times [0, T]} \langle \beta, \nu \rangle.$$

By the computation in [9], we know

(3.2)
$$\langle \beta, \nu \rangle = \sqrt{u^{ij} \nu_i \nu_j h_{p_k} h_{p_l} u_{kl}}.$$

Further on, we may assume that $t_0 > 0$ and $\nu(x_0) = (0, 0, \dots, 1) \triangleq e_n$. As in the proof of Lemma 8.1 in [10], by the convexity of Ω and its smoothness, we extend ν smoothly to a tubular neighborhood of $\partial\Omega$ such that in matrix sense

$$(3.3) D_k \nu_l \equiv \nu_{kl} \le -\frac{1}{C} \delta_{kl}$$

for some positive constant C. Let

$$v = \langle \beta, \nu \rangle + h(Du).$$

By the above assumptions and the boundary condition, we obtain

$$v(x_0,t_0) = \min_{\partial \Omega \times [0,T]} v = \min_{\partial \Omega \times [0,T]} \langle \beta, \nu \rangle.$$

In (x_0, t_0) , we have

$$(3.4) 0 = v_r = h_{p_n p_k} u_{kr} + h_{p_k} \nu_{kr} + h_{p_k} u_{kr}, 1 \le r \le n - 1,$$

We assume that the following key estimate holds which will be proved later,

$$(3.5) v_n(x_0, t_0) \ge -C,$$

where C is a constant depending only on Ω , u_0 , h, ε_0 , h and we will use the conditions of (1.4), (1.17), (1.15), (1.16). It's not hard to check that (3.5) can be rewritten as

$$(3.6) h_{n_n n_k} u_{kn} + h_{n_k} \nu_{kn} + h_{n_k} u_{kn} \ge -C.$$

Multiplying (3.6) with h_{p_n} and (3.4) with h_{p_r} respectively, and summing up together, we obtain

$$(3.7) h_{p_k} h_{p_l} u_{kl} \ge -C h_{p_n} - h_{p_k} h_{p_l} \nu_{kl} - h_{p_k} h_{p_n p_l} u_{kl}.$$

By the concavity of h, we can easily check that

$$-h_{p_n p_n} \ge 0$$
, $h_{p_k} u_{kr} = \frac{\partial h(Du)}{\partial x_r} = 0$, $h_{p_k} u_{kn} = \frac{\partial h(Du)}{\partial x_n} \ge 0$.

Substituting those into (3.7) and using (3.3), we have

$$h_{p_k} h_{p_l} u_{kl} \ge -C h_{p_n} + \frac{1}{C} |Dh|^2 = -C h_{p_n} + \frac{1}{C}.$$

For the last term of the above inequality, we distinguish two cases in (x_0, t_0) . Case (i).

$$-Ch_{p_n} + \frac{1}{C} \le 0.$$

Then

$$h_{p_k}(Du)\nu_k = h_{p_n} \ge \frac{1}{C^2}.$$

It shows that there is a uniform positive lower bound for the quantity $\min_{\partial\Omega\times[0,T]}h_{p_k}(Du)\nu_k$. Case (ii).

$$-Ch_{p_n} + \frac{1}{C} > 0.$$

Then we obtain a positive lower bound of $h_{p_k}h_{p_l}u_{kl}$. Introduce the Legendre transformation of u:

$$y_i = \frac{\partial u}{\partial x_i}, \ i = 1, 2, \dots, n, \ u^*(y_1, \dots, y_n, t) := \sum_{i=1}^n x_i \frac{\partial u}{\partial x_i} - u(x, t).$$

In terms of $y_1, \dots, y_n, u^*(y_1, \dots, y_n)$, we can easily check that

$$\frac{\partial^2 u^*}{\partial y_i \partial y_j} = \left[\frac{\partial^2 u}{\partial x_i \partial x_j}\right]^{-1}.$$

Then u^* satisfies

(3.8)
$$\begin{cases} \frac{\partial u^*}{\partial t} - F^*(D^2 u^*) = 0, & T > t > 0, \quad x \in \tilde{\Omega}, \\ \tilde{h}(Du^*) = 0, & T > t > 0, \quad x \in \partial \tilde{\Omega}, \\ u^* = u_0^*, & t = 0, \quad x \in \tilde{\Omega}, \end{cases}$$

where \tilde{h} is a smooth and strictly concave function on $\bar{\Omega}$:

$$\Omega = \{ p \in \mathbb{R}^n | \tilde{h}(p) > 0 \}, \qquad |D\tilde{h}|_{\partial \tilde{\Omega}} = 1,$$

and u_0^* is the Legendre transformation of u_0 . Here F^* is defined by (1.17). Set

$$F^*(\lambda_1, \lambda_2, \dots, \lambda_n) = -F(\mu_1, \mu_2, \dots, \mu_n), \qquad \mu_i = \lambda_i^{-1} (i = 1, 2, \dots, n).$$

A simple calculation shows that

$$\sum_{i=1}^{n} \frac{\partial F^*}{\partial \lambda_i} = \sum_{i=1}^{n} \mu_i^2 \frac{\partial F}{\partial \mu_i}, \qquad \sum_{i=1}^{n} \lambda_i^2 \frac{\partial F^*}{\partial \lambda_i} = \sum_{i=1}^{n} \frac{\partial F}{\partial \mu_i}.$$

Then the structure conditions of F imply that F^* also satisfies (1.15) and (1.16). We also define

$$\tilde{v} = \tilde{\beta}^k \tilde{\nu}_k + \tilde{h}(Du^*) = \langle \tilde{\beta}, \tilde{\nu} \rangle + \tilde{h}(Du^*),$$

where

$$\tilde{\beta}^k \triangleq \frac{\partial \tilde{h}(Du^*)}{\partial u_k^*} = \tilde{h}_{p_k}(Du^*),$$

and $\tilde{\nu} = (\tilde{\nu}_1, \tilde{\nu}_2, \dots, \tilde{\nu}_n)$ is the inner unit normal vector of $\partial \tilde{\Omega}$. Using the same methods, under the assumption of

$$\tilde{v}_n(y_0, t_0) \ge -C,$$

we obtain the positive lower bounds of $\tilde{h}_{p_k}\tilde{h}_{p_l}u_{kl}^*$ or

$$h_{p_k}(Du)\nu_k = \tilde{h}_{p_k}(Du^*)\tilde{\nu}_k(y_0) = \tilde{h}_{p_n} \ge \frac{1}{C^2}.$$

We notice that

$$\tilde{h}_{p_k}\tilde{h}_{p_l}u_{kl}^* = \nu_i\nu_j u^{ij}.$$

Then the claim follows from (3.2) by the positive lower bounds of $h_{p_k}h_{p_l}u_{kl}$ and $\tilde{h}_{p_k}\tilde{h}_{p_l}u_{kl}^*$.

It remains to prove the key estimate (3.5). We generalize the proof of Lemma 8.1 in [10] for the goal.

Define the linearized operator by

$$L = F^{ij}\partial_{ij} - \partial_t.$$

Since $D^2 \tilde{h} \leq -\tilde{\theta} I$, we obtain

$$(3.9) L\tilde{h} \le -\tilde{\theta} \sum F^{ii}.$$

On the other hand,

(3.10)
$$Lv = h_{p_k p_l p_m} \nu_k F^{ij} u_{li} u_{mj} + 2h_{p_k p_l} F^{ij} \nu_{kj} u_{li} + h_{p_k p_l} F^{ij} u_{lj} u_{ki} + h_{p_k p_l} \nu_k L u_l + h_{p_k} L \nu_k.$$

By estimating the first term in the diagonal basis, we have

$$|h_{p_k p_l p_m} \nu_k F^{ij} u_{li} u_{mj}| \le C \sum_{i=1}^n \frac{\partial F}{\partial \lambda_i} \lambda_i^2 \le C,$$

where we use the assumption of (1.16) and C is a constant depending only on h, Ω , λ , Λ . For the second term, by Cauchy inequality, we obtain

$$|2h_{p_k p_l} F^{ij} \nu_{kj} u_{li}| \leq C \sum_{i=1}^n \frac{\partial F}{\partial \lambda_i} \lambda_i = C \sum_{i=1}^n \sqrt{\frac{\partial F}{\partial \lambda_i}} \sqrt{\frac{\partial F}{\partial \lambda_i}} \lambda_i$$

$$\leq C (\sum_{i=1}^n \frac{\partial F}{\partial \lambda_i}) (\sum_{i=1}^n \frac{\partial F}{\partial \lambda_i} \lambda_i^2)$$

$$\leq C.$$

By the same reason, we get

$$|h_{p_k p_l} F^{ij} u_{lj} u_{ki}| \leq C.$$

After simple calculation, we give

$$Lu_l=0.$$

Obviously we have

$$|h_{p_k}L\nu_k| \le C \sum F^{ii}.$$

So there exists a positive constant C such that

$$(3.11) Lv \le C \sum F^{ii}.$$

Here we use the condition (1.15) and C depends only on h, Ω , λ , Λ .

Denote a neighborhood of x_0 :

$$\Omega_{\delta} \triangleq \Omega \cap B_{\delta}(x_0),$$

where δ is a positive constant such that ν is well defined in Ω_{δ} . We consider

$$\Phi \triangleq v(x,t) - v(x_0,t_0) + C_0 \tilde{h}(x) + A|x - x_0|^2,$$

where C_0 and A are positive constants to be determined. On $\partial\Omega \times [0,T)$, it is clear that $\Phi \geq 0$. Since v is bounded, we can select A large enough such that

$$(v(x,t) - v(x_0,t_0) + C_0\tilde{h}(x) + A|x - x_0|^2)|_{(\Omega \cap \partial B_{\delta}(x_0)) \times [0,T]}$$

$$\geq v(x,t) - v(x_0,t_0) + A\delta^2$$

$$> 0.$$

Using the strict concavity of \tilde{h} , we have

$$\triangle (C_0 \tilde{h}(x) + A|x - x_0|^2) \le C(-C_0 \tilde{\theta} + 2A) \sum_i F^{ii}.$$

Then by choosing the constant $C_0 \gg A$, we can show that

$$\triangle(v(x,0) - v(x_0, t_0) + C_0\tilde{h}(x) + A|x - x_0|^2) \le 0.$$

We calculate by using the maximum principle to get

$$(v(x,0) - v(x_0,t_0) + C_0\tilde{h}(x) + A|x - x_0|^2)|_{\Omega_{\delta}}$$

$$\geq \min_{(\partial\Omega \cap B_{\delta}(x_0)) \cup (\Omega \cap \partial B_{\delta}(x_0)} (v(x,0) - v(x_0,t_0) + C_0\tilde{h}(x) + A|x - x_0|^2)$$

$$> 0.$$

Combining (3.9) with (3.11) and letting C_0 be large enough, we obtain

$$L\Phi \le (-C_0\tilde{\theta} + C + 2A)\sum F^{ii} \le 0.$$

From the above arguments, we verify that Φ satisfies

(3.12)
$$\begin{cases} L\Phi \leq 0, & (x,t) \in \Omega_{\delta} \times [0,T], \\ \Phi \geq 0, & (x,t) \in (\partial \Omega_{\delta} \times [0,T] \cup (\Omega_{\delta} \times \{t=0\}. \end{cases}$$

Using the maximum principle, we deduce that

$$\Phi \ge 0, \qquad (x,t) \in \Omega_{\delta} \times [0,T].$$

Combining the above inequality with $\Phi(x_0, t_0) = 0$, we obtain $\langle \nabla \Phi, e_n \rangle|_{(x_0, t_0)} \geq 0$ which gives the desired key estimate (3.5). Thus we complete the proof of the lemma.

By making use of (3.11), we can state the following result which is similar to Proposition 2.6 in [1].

Lemma 3.5. Fix a smooth function $H: \Omega \times \tilde{\Omega} \to R$ and define $\varphi(x,t) = H(x,Du(x,t))$. Then there holds

$$|L\varphi| \le C \sum F^{ii}, \quad (x,t) \in \Omega_T,$$

where C is a positive constant depending on h, H, Ω , λ , Λ

We now proceed to carry out the C^2 estimates. The procedure is similar to a priori estimates on the second boundary value problem for Lagrangian mean curvature flow [3]. The strategy is to bound the interior second derivative first.

Lemma 3.6. For each $t \in [0,T]$, the following estimates hold:

(3.13)
$$\sup_{\Omega} |D^2 u| \leq \max_{\partial \Omega \times [0,T]} |D^2 u| + \max_{\bar{\Omega}} |D^2 u_0|.$$

Proof. Given any unit vector ξ , according to the concavity of F, we know that $u_{\xi\xi}$ satisfies

$$\partial_t u_{\xi\xi} - F^{ij} \partial_{ij} u_{\xi\xi} = \frac{\partial^2 F}{\partial u_{ij} \partial u_{kl}} u_{ij\xi} u_{kl\xi} \le 0.$$

Combining with the convexity of u and using the maximum principle, we obtain

$$0 \le |u_{\xi\xi}| = u_{\xi\xi}(x,t) \le \max_{\partial\Omega_T} u_{\xi\xi}$$
$$\le \max_{\partial\Omega \times [0,T]} |D^2 u| + \max_{\bar{\Omega}} |D^2 u_0|.$$

Therefore we obtain the desired estimates (3.13).

By taking tangential differentiation on the boundary condition h(Du) = 0, we have

$$(3.14) u_{\beta\tau} = h_{n_k}(Du)u_{k\tau} = 0,$$

where τ denotes a tangential vector. The second order derivative estimates on the boundary are controlled by $u_{\beta\tau}, u_{\beta\beta}, u_{\tau\tau}$. In the following, we give the arguments

as in [9] which can be found in [3]. For $x \in \partial\Omega$, any unit vector ξ can be written in terms of a tangential component $\tau(\xi)$ and a component in the direction β by

$$\xi = \tau(\xi) + \frac{\langle \nu, \xi \rangle}{\langle \beta, \nu \rangle} \beta,$$

where

$$\tau(\xi) = \xi - \langle \nu, \xi \rangle \nu - \frac{\langle \nu, \xi \rangle}{\langle \beta, \nu \rangle} \beta^{T}$$

and

$$\beta^T = \beta - \langle \beta, \nu \rangle \nu.$$

After a simple computation, we get

$$|\tau(\xi)|^{2} = 1 - \left(1 - \frac{|\beta^{T}|^{2}}{\langle \beta, \nu \rangle^{2}}\right) \langle \nu, \xi \rangle^{2} - 2\langle \nu, \xi \rangle \frac{\langle \beta^{T}, \xi \rangle}{\langle \beta, \nu \rangle}$$

$$\leq 1 + C\langle \nu, \xi \rangle^{2} - 2\langle \nu, \xi \rangle \frac{\langle \beta^{T}, \xi \rangle}{\langle \beta, \nu \rangle}$$

$$\leq C,$$
(3.15)

where we use the strict obliqueness estimates (3.1). Let $\tau \triangleq \frac{\tau(\xi)}{|\tau(\xi)|}$. Then by (3.14) and (3.1), we obtain

$$(3.16) u_{\xi\xi} = |\tau(\xi)|^2 u_{\tau\tau} + 2|\tau(\xi)| \frac{\langle \nu, \xi \rangle}{\langle \beta, \nu \rangle} u_{\beta\tau} + \frac{\langle \nu, \xi \rangle^2}{\langle \beta, \nu \rangle^2} u_{\beta\beta}$$
$$= |\tau(\xi)|^2 u_{\tau\tau} + \frac{\langle \nu, \xi \rangle^2}{\langle \beta, \nu \rangle^2} u_{\beta\beta}$$
$$\leq C(u_{\tau\tau} + u_{\beta\beta}).$$

Along with specifying the boundary conditions, we can carry out the double derivative estimates in the direction β .

Lemma 3.7. For any $t \in [0,T]$, we have the estimates

$$\max_{\beta \Omega} u_{\beta \beta} \le C_2,$$

where $C_2 > 0$ depends only on u_0 , h, \tilde{h} , Ω , λ , Λ .

Proof. We use the barrier functions for any $x_0 \in \partial \Omega$ and thus consider

$$\Psi \triangleq \pm h(Du) + C_0\tilde{h} + A|x - x_0|^2.$$

As in the proof of (3.12), we can find constants C_0 and A, such that

$$\begin{cases} L\Psi \leq 0, & (x,t) \in \Omega_{\delta} \times [0,T], \\ \Psi \geq 0, & (x,t) \in (\partial \Omega_{\delta} \times [0,T] \cup (\Omega_{\delta} \times \{t=0\}. \end{cases}$$

By the maximum principle, we get

$$\Psi \ge 0, \qquad (x,t) \in \Omega_{\delta} \times [0,T].$$

Combining the above inequality with $\Psi(x_0, t_0) = 0$ and using Lemma 3.4, we obtain $\Psi_{\beta}(x_0, t_0) \geq 0$. Furthermore we see from $\beta = (\frac{\partial h}{\partial p_1}, \frac{\partial h}{\partial p_2}, \cdots, \frac{\partial h}{\partial p_n})$ that

$$\frac{\partial h}{\partial \beta} = \langle Dh(Du), \beta \rangle = \Sigma_{k,l} \frac{\partial h}{\partial p_k} u_{kl} \beta^l = \Sigma_{k,l} \beta^k u_{kl} \beta^l = u_{\beta\beta}.$$

Then we show that

$$|u_{\beta\beta}| = |\frac{\partial h}{\partial \beta}| \le C_2.$$

We are now in a position to obtain the bound of double tangential derivative on the boundary.

Lemma 3.8. There exists a constant $C_3 > 0$ depending only on u_0 , h, \tilde{h} , Ω , λ , Λ , such that

$$\max_{\partial \Omega \times [0,T]} \max_{|\tau|=1, \langle \tau, \nu \rangle = 0} u_{\tau\tau} \le C_3.$$

Proof. Assume that $x_0 \in \partial\Omega$, $t_0 \in [0,T]$ and $\nu(x_0) = e_n$ is the inner unit normal of $\partial\Omega$ at x_0 . We can also choose the coordinates such that

$$\max_{\partial\Omega\times[0,T]}\max_{|\tau|=1,\langle\tau,\nu\rangle=0}u_{\tau\tau}=u_{11}(x_0,t_0).$$

For any $x \in \partial\Omega$, combining (3.15) with (3.16), we have

$$u_{\xi\xi} = |\tau(\xi)|^2 u_{\tau\tau} + \frac{\langle \nu, \xi \rangle^2}{\langle \beta, \nu \rangle^2} u_{\beta\beta}$$

$$\leq (1 + C\langle \nu, \xi \rangle^2 - 2\langle \nu, \xi \rangle \frac{\langle \beta^T, \xi \rangle}{\langle \beta, \nu \rangle}) u_{\tau\tau} + \frac{\langle \nu, \xi \rangle^2}{\langle \beta, \nu \rangle^2} u_{\beta\beta}$$

$$\leq (1 + C\langle \nu, \xi \rangle^2 - 2\langle \nu, \xi \rangle \frac{\langle \beta^T, \xi \rangle}{\langle \beta, \nu \rangle}) u_{11}(x_0, t_0) + \frac{\langle \nu, \xi \rangle^2}{\langle \beta, \nu \rangle^2} u_{\beta\beta}.$$

Without loss of generality, we assume that $u_{11}(x_0, t_0) \ge 1$. Then by Lemma 3.4 and Lemma 3.7, we get

$$\frac{u_{\xi\xi}}{u_{11}(x_0, t_0)} + 2\langle \nu, \xi \rangle \frac{\langle \beta^T, \xi \rangle}{\langle \beta, \nu \rangle} \le 1 + C\langle \nu, \xi \rangle^2.$$

Let $\xi = e_1$, then we have

$$\frac{u_{11}}{u_{11}(x_0, t_0)} + 2\langle \nu, e_1 \rangle \frac{\langle \beta^T, e_1 \rangle}{\langle \beta, \nu \rangle} \le 1 + C\langle \nu, e_1 \rangle^2.$$

We see that the function

$$w \triangleq A|x - x_0|^2 - \frac{u_{11}}{u_{11}(x_0, t_0)} - 2\langle \nu, e_1 \rangle \frac{\langle \beta^T, e_1 \rangle}{\langle \beta, \nu \rangle} + C\langle \nu, e_1 \rangle^2 + 1$$

satisfies

$$w|_{\partial\Omega\times[0,T]}\geq 0, \quad w(x_0,t_0)=0.$$

As before, by (3.13), we can select the constant A such that

$$w|_{(\partial B_{\delta}(x_0)\cap\Omega)\times[0,T]}\geq 0.$$

Consider

$$-2\langle \nu, e_1 \rangle \frac{\langle \beta^T, e_1 \rangle}{\langle \beta, \nu \rangle} + C\langle \nu, e_1 \rangle^2 + 1$$

as a known function depending on x and Du. Then by Lemma 3.5, we obtain

$$|L(-2\langle \nu, e_1 \rangle \frac{\langle \beta^T, e_1 \rangle}{\langle \beta, \nu \rangle} + C\langle \nu, e_1 \rangle^2 + 1)| \le C \sum F^{ii}.$$

Combining the above inequality with the proof of Lemma 3.6, we have

$$Lw \le C \sum F^{ii}$$
.

As in the proof of Lemma 3.7, we consider the function

$$\Upsilon \triangleq w + C_0 \tilde{h}$$
.

Using the standard barrier argument, we show that

$$\Upsilon_{\beta}(x_0, t_0) \geq 0.$$

A direct computation deduce that

$$(3.17) u_{11\beta} \le Cu_{11}(x_0, t_0).$$

On the other hand, differentiating h(Du) twice in the direction e_1 at (x_0, t_0) , we have

$$h_{p_k}u_{k11} + h_{p_kp_l}u_{k1}u_{l1} = 0.$$

The concavity of h yields

$$h_{p_k}u_{k11} = -h_{p_kp_l}u_{k1}u_{l1} \ge \tilde{C}u_{11}(x_0, t_0)^2.$$

Combining the above inequality with $h_{p_k}u_{k11}=u_{11\beta}$, and using (3.17), we obtain

$$\tilde{C}u_{11}(x_0, t_0)^2 \le Cu_{11}(x_0, t_0).$$

Then we get the upper bound estiamtes of $u_{11}(x_0, t_0)$ and the conclusion follows. \square

Using Lemma 3.7, 3.8, and (3.16), we obtain the C^2 a priori bound for the solution on the boundary:

Lemma 3.9. There exists a constant $C_4 > 0$ depending on h, \tilde{h} , u_0 , λ , Λ , and Ω , such that

$$\sup_{\partial \Omega_T} |D^2 u| \le C_4.$$

Combining Lemma 3.6 with Lemma 3.9, we obtain the following result:

Lemma 3.10. There exists a constant $C_5 > 0$ depending on h, \tilde{h} , u_0 , Ω , λ , Λ such that

$$\sup_{\bar{\Omega}_T, |\xi|=1} D_{ij} u \xi_i \xi_j \le C_5.$$

By the Legendre transformation of u, using (3.8) and repeating the proof of the above lemmas, we get the following statement:

Lemma 3.11. There exists a constant $C_6 > 0$ depending on h, Ω , \tilde{h} , $\tilde{\Omega}$, u_0 , λ , Λ such that

(3.18)
$$\frac{1}{C_6} \le \inf_{\bar{\Omega}_T, |\xi|=1} D_{ij} u \xi_i \xi_j \le \sup_{\bar{\Omega}_T, |\xi|=1} D_{ij} u \xi_i \xi_j \le C_6.$$

Remark 3.12. The differential inequality (3.11) plays a central role in the C^2 estimates for the solution on the boundary. The conditions (1.15) and (1.16) imposed on F is used to prove (3.11). If we assume (3.11), (1.4) and (1.17), we can also prove the above lemma instead of (1.15) and (1.16).

4. Longtime existence and convergence

Proof of Theorem 1.1:

We divide the proof of the Theorem into two parts.

Part 1: The long time existence.

By Lemma 3.1 and Lemma 3.10, we know global $C^{2,1}$ estimates for the solutions of the flow (2.1). Using Theorem 14.22 in Lieberman [14], we show that the solutions of the oblique derivative problem (2.1) have global $C^{2+\alpha,1+\frac{\alpha}{2}}$ estimates.

Now let u_0 be a $C^{2+\alpha_0}$ strictly convex function as in the conditions of Theorem 1.1. We assume that T is the maximal time such that the solution to the flow (2.1) exists. Suppose that $T < +\infty$. Combining Proposition 2.5 with Lemma 3.11 and using Theorem 14.23 in [14], there exists $u \in C^{2+\alpha,1+\frac{\alpha}{2}}(\bar{\Omega}_T)$ which satisfies (2.1) and

$$||u||_{C^{2+\alpha,1+\frac{\alpha}{2}}(\bar{\Omega}_T)} < +\infty.$$

Then we can extend the flow (2.1) beyond the maximal time T. So that we deduce that $T = +\infty$. Then there exists the solution u(x, t) for all times t > 0 to (1.1)-(1.3).

Part 2: The convergence.

Using the boundary condition, we have

$$(4.1) |Du| \le C_7,$$

where C_7 be a constant depending on Ω and $\tilde{\Omega}$. By intermediate Schauder estimates for parabolic equations (cf. Lemma 14.6 and Proposition 4.25 in [14]), for any $D \subset\subset \Omega$, we have

$$[D^{2}u]_{\alpha,\frac{\alpha}{2},D_{T}} \leq C \sup |D^{2}u| \leq C_{8},$$

$$\sup_{t\geq 1} \|D^{3}u(\cdot,t)\|_{C(\bar{D})} + \sup_{t\geq 1} \|D^{4}u(\cdot,t)\|_{C(\bar{D})}$$

$$+ \sup_{x_{i}\in D,t_{i}\geq 1} \frac{|D^{4}u(x_{1},t_{1}) - D^{4}u(x_{2},t_{2})}{\max\{|x_{1} - x_{2}|^{\alpha}, |t_{1} - t_{2}|^{\frac{\alpha}{2}}\}}$$

$$\leq C_{9},$$

where C_8 , C_9 are constants depending on the known data and dist $(\partial\Omega,\partial D)$.

To finish the proof of Theorem 1.1, we use a trick that we learned from J. Kitagawa[16] and O.C. Schnürer[17] which proving that some curvature flows with second boundary condition converge to translating solutions. Now fixing some positive t_0 and writing

$$v(x,t) = u(x,t) - u(x,t+t_0),$$

then we have

$$\dot{v} = F(D^{2}u(x,t) - F(D^{2}u(x,t+t_{0}))$$

$$= \int_{0}^{1} \frac{d}{ds} F(sD^{2}u(x,t) + (1-s)D^{2}u(x,t+t_{0}))ds$$

$$= \left(\int_{0}^{1} \nabla_{r_{ij}} F(sD^{2}u(x,t) + (1-s)D^{2}u(x,t+t_{0}))ds\right) v_{ij}$$

$$= a^{ij}v_{ij},$$
(4.3)

where we denote

$$a^{ij} = \int_0^1 \nabla_{r_{ij}} F(sD^2 u(x,t) + (1-s)D^2 u(x,t+t_0)) ds.$$

Since the smallest eigenvalue of u_{ij} has a strictly positive lower bound and upper bound uninformly in t and x by (3.18) and $u \in C^{2+\alpha,1+\frac{\alpha}{2}}(\bar{\Omega}_T)$, we see that

$$||u(\cdot,t) - u(\cdot,t+t_0)||_{C^2(\bar{\Omega})} \le C \cdot t_0^{\frac{\alpha}{2}},$$

where C is independent of t by Theorem 14.22 in Lieberman [14].

By taking t_0 sufficiently small, we can ensure that the convex combination $sD^2u(\cdot,t)+(1-s)D^2u(\cdot,t+t_0)$ is close to $u(\cdot,t)$ in $C^2(\bar{\Omega})$ norm. By the uniform positive lower bound and upper bound on the eigenvalues of u_{ij} in Lemma 3.11, there exist positive constants $\tilde{\lambda}$, $\tilde{\Lambda}$ such that

$$\tilde{\lambda}I \leq [a^{ij}] \leq \tilde{\Lambda}I$$

and hence the equation (4.3) is uniformly parabolic.

Additionally, we see that for $x \in \partial \Omega$, v satisfies

(4.4)
$$0 = h(Du(x,t)) - h(Du(x,t+t_0))$$
$$= (\int_0^1 h_{p_k}(x,sDu(x,t) + (1-s)Du(x,t+t_0))ds)v_k$$
$$=: \alpha^k v_k.$$

By Lemma 3.4, we have $h_{p_k}(x, Du(x,t))\nu_k \geq C > 0$ for some C uniform in t and x. Therefore as in the above, by choosing t_0 small enough, we can deduce that $\alpha^k \nu_k \geq \frac{C}{2} > 0$ and we see that v satisfies a linear, uniformly oblique boundary condition.

By the proof of section 6.2 in [17], the uniformly parabolic equation (4.3) with the uniformly oblique boundary condition (4.4) ensures that we can obtain a translating solution of the same regularity as $u: u^{\infty}(x,t) = u^{\infty}(x) + C_{\infty} \cdot t$ for some constant C_{∞} which satisfies equation (1.1) and $u^{\infty}(x,t)$ satisfies

(4.5)
$$\lim_{t \to +\infty} ||u(\cdot,t) - u^{\infty}(\cdot,t)||_{C(\bar{\Omega})} = 0.$$

Using (4.5) and the interpolation inequalities of the following form: (cf. [17])

$$(4.6) \qquad ||D(u(\cdot,t) - u^{\infty}(\cdot,t))||_{C(\bar{\Omega})}^{2} \\
\leq c(\Omega)||u(\cdot,t) - u^{\infty}(\cdot,t)||_{C(\bar{\Omega})}(||D^{2}(u(\cdot,t) - u^{\infty}(\cdot,t))||_{C(\bar{\Omega})}) \\
+ ||D(u(\cdot,t) - u^{\infty}(\cdot,t))||_{C(\bar{\Omega})}) \\
\leq c(\Omega)||u(\cdot,t) - u^{\infty}(\cdot,t)||_{C(\bar{\Omega})}(2n^{2}C_{6} + 2C_{7}),$$

we obtain

(4.7)
$$\lim_{t \to +\infty} \|u(\cdot, t) - u^{\infty}(\cdot, t)\|_{C^{1}(\overline{\Omega})} = 0.$$

By the interpolation inequalities and (4.2), we get

$$||D^{2}(u(\cdot,t)-u^{\infty}(\cdot,t))||_{C(\bar{D})}^{2}$$

$$\leq c(D)||D(u(\cdot,t)-u^{\infty}(\cdot,t))||_{C(\bar{D})}(||D^{3}(u(\cdot,t)-u^{\infty}(\cdot,t))||_{C(\bar{D})})$$

$$+||D^{2}(u(\cdot,t)-u^{\infty}(\cdot,t))||_{C(\bar{D})})$$

$$\leq c(D)||D(u(\cdot,t)-u^{\infty}(\cdot,t))||_{C(\bar{D})}(2C_{9}+2n^{2}C_{6}).$$

Using (4.7), we also obtain

(4.8)
$$\lim_{t \to +\infty} \|u(\cdot, t) - u^{\infty}(\cdot, t)\|_{C^{2}(\bar{D})} = 0.$$

Repeating the above procedure and using (4.2), we have

(4.9)
$$\lim_{t \to +\infty} \|u(\cdot, t) - u^{\infty}(\cdot, t)\|_{C^{4+\alpha}(\bar{D})} = 0.$$

By Lemma 3.11 and (4.8), we get

$$\lim_{t \to +\infty} \|u(\cdot, t) - u^{\infty}(\cdot, t)\|_{C^{1+\zeta}(\bar{\Omega})} = 0.$$

Therefore by using the equation (1.1) and letting $t \to \infty$, we obtain

$$C_{\infty} = \frac{\partial u^{\infty}(x,t)}{\partial t} = F(D^2 u^{\infty}(x,t)) = F(D^2 u^{\infty}(x)),$$

$$0 = \lim_{t \to \infty} h(Du(x,t)) = \lim_{t \to \infty} h(Du^{\infty}(x,t)) = h(Du^{\infty}(x)).$$

Then the proof of Theorem 1.1 is completed.

Proof of Corollary 1.2: As the arguments in Section 2, we know that (1.20) is equivalent to

(4.10)
$$\begin{cases} F(D^2u) = C_{\infty}, \ x \in \Omega, \\ h(Du) = 0, \ x \in \partial \Omega. \end{cases}$$

By Evans-Krylov theorem and interior Schauder estimates, we have

$$u \in C^{\infty}(\Omega)$$
.

For any $l \in \{1, 2, \dots, n\}$, set $w = u_l$. Then w satisfies

$$\begin{cases} F^{ij}w_{ij} = 0, \ x \in \Omega, \\ \beta^k w_k = 0, \ x \in \partial\Omega, \end{cases}$$

where

$$\langle \beta, \nu \rangle \ge \frac{1}{C_1} > 0$$

by Lemma 3.4 and $\nu=(\nu_1,\nu_2,\cdots,\nu_n)$ is the unit inward normal vector of $\partial\Omega$. Using Theorem 6.30 in Gilbarg-Trudinger [18], we have $w\in C^{2+\alpha}(\bar{\Omega})$. Then

$$u \in C^{3+\alpha}(\bar{\Omega}).$$

By the smoothness of F and $\partial\Omega$ and using bootstrap argument, we obtain

$$u \in C^{\infty}(\bar{\Omega}).$$

5. Proof of theorem 1.5 and corollary 1.7

Let us consider the case $\tau = \frac{\pi}{4}$. Then we write

(5.1)
$$F(\lambda_1, \lambda_2, \cdots, \lambda_n) := -F_{\frac{\pi}{4}}(\lambda_1, \lambda_2, \cdots, \lambda_n) = -\sum_i \frac{1}{1 + \lambda_i}.$$

The proof of Theorem 1.5 is based on the following conclusions.

Lemma 5.1.

F and F^* are concave on the cone Γ_+ .

Proof. We calculate directly to obtain:

$$\sum \frac{\partial^2 F}{\partial \lambda_i \lambda_j} \xi_i \xi_j = -2 \sum \frac{\xi_i^2}{(1 + \lambda_i)^3} \le 0,$$
$$\sum \frac{\partial^2 F^*}{\partial \lambda_i \lambda_j} \xi_i \xi_j = -2 \sum \frac{\xi_i^2}{(1 + \lambda_i)^3} \le 0.$$

Thus, the above inequalities imply the desired result.

By Lemma 3.1, we obtain the following statements.

Lemma 5.2 (\dot{u} -estimates).

As long as the convex solution to (1.1)-(1.3) exists, the following estimates hold, i.e.,

$$\Theta_0 \triangleq \min_{\bar{\Omega}} F_{\frac{\pi}{4}}(D^2 u_0) \le -\dot{u} \triangleq -\frac{\partial u}{\partial t} \le \Theta_1 \triangleq \max_{\bar{\Omega}} F_{\frac{\pi}{4}}(D^2 u_0).$$

Since $u_0 \in C^{2+\alpha}(\bar{\Omega})$ is strictly convex, it is clear that $0 < \Theta_0 \le \Theta_1 < n$.

Lemma 5.3. Let (x,t) be an arbitrary point of Ω_T , and $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ be the eigenvalues of D^2u at (x,t). As long as the convex solution to (1.1)-(1.3) exists, then

(5.2)
$$0 \le \lambda_1 \le \frac{n}{\Theta_0} - 1, \quad \frac{n}{\Theta_1} - 1 \le \lambda_n.$$

Proof. Following the definition of $F_{\frac{\pi}{4}}(D^2u)$ and Lemma 5.2:

$$\Theta_0 \le -\dot{u} = \sum_i \frac{1}{1 + \lambda_i} \le \frac{n}{1 + \lambda_1}.$$

Combining with the convexity of u, we obtain

$$1 \le 1 + \lambda_1 \le \frac{n}{\Theta_0}.$$

Using the same methods, we get

$$\frac{n}{1+\lambda_n} \le -\dot{u} = \sum_i \frac{1}{1+\lambda_i} \le \Theta_1.$$

Then we have

$$\frac{n}{\Theta_1} \le 1 + \lambda_n.$$

From the above arguments, we prove the lemma.

Now we show the operator $-F_{\frac{\pi}{4}}$ satisfying (1.15) and (1.16) which play an important role in the barrier arguments.

Corollary 5.4. As long as the convex solution to (1.1)-(1.3) exists, then for any $(x,t) \in \Omega_T$ we have

(5.3)
$$\frac{\Theta_0^2}{n^2} \le \sum_{i=1}^n \frac{\partial F}{\partial \lambda_i} \le n,$$

(5.4)
$$\frac{(n-\Theta_1)^2}{n^2} \le \sum_{i=1}^n \frac{\partial F}{\partial \lambda_i} \lambda_i^2 \le n.$$

Proof. We observe

$$\sum_{i=1}^{n} \frac{\partial F}{\partial \lambda_i} = \sum_{i=1}^{n} \frac{1}{(1+\lambda_i)^2}.$$

By Lemma 5.3 and the convexity of u, we have

$$\frac{\Theta_0^2}{n^2} \le \frac{1}{(1+\lambda_1)^2} \le \sum_{i=1}^n \frac{1}{(1+\lambda_i)^2} \le n.$$

We further obtain

$$\frac{(n - \Theta_1)^2}{n^2} \le \frac{\lambda_n^2}{(1 + \lambda_n)^2} \le \sum_{i=1}^n \frac{\lambda_i^2}{(1 + \lambda_i)^2} \le n.$$

Corollary 5.4 is established.

Proof of Theorem 1.5:

It follows from Lemma 5.1, Corollary 5.4 and Theorem 1.1 . \Box

Proof of Corollary 1.7:

Let u be a convex solution of (1.14) and we define f = Du. Then f is a diffeomorphism from Ω to $\tilde{\Omega}$. It follows from [6] that

$$\Sigma = \{(x, f(x)) | x \in \Omega\}$$

is a special Lagrangian graph in $(\mathbb{R}^n \times \mathbb{R}^n, g_{\frac{\pi}{4}})$. The required properties are thus proved.

6. Applications to special Lagrangian diffeomorphism problems Let

$$F(\lambda_1, \lambda_2, \cdots, \lambda_n) := F_{\frac{\pi}{2}}(\lambda_1, \lambda_2, \cdots, \lambda_n) = \sum_i \arctan \lambda_i.$$

A direct computation shows that F and F^* are concave on the cone Γ_+ . Using Corollary 3.3 in [3] we see that F satisfies (1.15) and (1.16). By Theorem 1.1, we reprove the existence result on the second boundary value problem for special Lagrangian equations which belongs to S.Brendle and M.Warren.

Theorem 6.1. (see Theorem 1.1 in [1]) Let Ω , $\tilde{\Omega}$ be bounded, uniformly convex domains with smooth boundary in \mathbb{R}^n . Then there exist $u \in C^{\infty}(\bar{\Omega})$ and the constant c which satisfy

(6.1)
$$\begin{cases} \sum_{i} \arctan \lambda_{i} = c, & x \in \Omega, \\ Du(\Omega) = \tilde{\Omega}, \end{cases}$$

where the solution is unique up to additions of constants.

By [1] and [6], we present the similar special Lagrangian diffeomorphism problem in the following. The two convex domains Ω , $\tilde{\Omega} \subset \mathbb{R}^n$ with smooth boundary are fixed. Given a diffeomorphism $f: \Omega \longrightarrow \tilde{\Omega}$, the graph $\Sigma = \{(x, f(x)) : x \in \Omega\} \subset \mathbb{R}_n^{2n}$ is considered.

Question: How to find a diffeomorphism $f: \Omega \longrightarrow \tilde{\Omega}$ such that Σ is a special Lagrangian graph in \mathbb{R}^{2n}_n . By [19], the graph Σ is special Lagrangian if and only if there is a convex potential function $u: \Omega \to \mathbb{R}$ such that f = Du, and

$$\ln \det D^2 u = c,$$

where c is some constant. Hence, we present the problem as

(6.2)
$$\begin{cases} \ln \det D^2 u = c, & x \in \Omega, \\ Du(\Omega) = \tilde{\Omega}. \end{cases}$$

In 1997, J.Urbas proved the existence and uniqueness of global smooth convex solutions to (6.2), i.e.,

Theorem 6.2. (see Theorem 1.1 in [9]) Let Ω , $\tilde{\Omega}$ be bounded, uniformly convex domains with smooth boundary in \mathbb{R}^n . Then there exist $u \in C^{\infty}(\bar{\Omega})$ and the constant c which satisfy (6.2) where the solution is unique up to additions of constants.

Later, O.C. Schnürer and K. Smoczyk also obtained the above result by parabolic methods [10]. In fact, we can consider Lagrangian mean curvature flow in pseudo-Euclidean space \mathbb{R}_n^{2n} (cf.[20]) with second boundary condition:

(6.3)
$$\begin{cases} \frac{\partial u}{\partial t} = \ln \det D^2 u, & t > 0, \quad x \in \Omega, \\ Du(\Omega) = \tilde{\Omega}, & t > 0, \\ u = u_0, & t = 0, \quad x \in \Omega. \end{cases}$$

If we define

$$F(\lambda_1, \lambda_2, \cdots, \lambda_n) := F_{\tau}(\lambda_1, \lambda_2, \cdots, \lambda_n)|_{\tau=0} = \sum_{i} \ln \lambda_i,$$

then a direct computation shows that F and F^* are concave on the cone Γ_+ . Here we need modify v slightly which occur in Lemma 3.1, i.e.,

$$v = \langle \beta, \nu \rangle + Ah(Du)$$

which was introduced by J.Urbas [9]. Using the computation of Lemma 8.1 in [10], we have

$$Lv \le C \sum F^{ii},$$

where $L \triangleq F^{ij}\partial_{ij} - \partial_t$ as before. By Remark 3.12, we also obtain Theorem 6.2 as a corollary of Theorem 1.1.

In summary, by Theorem 1.1, we obtain the existence and uniqueness of global smooth convex solutions to (1.13), i.e.,

$$\begin{cases} F_{\tau}(D^2 u) = c, & x \in \Omega, \\ Du(\Omega) = \tilde{\Omega}, \end{cases}$$

for $\tau = 0, \frac{\pi}{4}, \frac{\pi}{2}$. Finally, we give another version of the above results as following

Theorem 6.3. Let Ω , $\tilde{\Omega}$ be bounded, uniformly convex domains with smooth boundary in \mathbb{R}^n . Then there exist smooth diffeomorphisms $f_{\tau} \colon \Omega \to \tilde{\Omega}$ such that

$$\Sigma = \{(x, f_{\tau}(x)) | x \in \Omega\}$$

are special Lagrangian graphs in $(\mathbb{R}^n \times \mathbb{R}^n, g_{\tau})$ for $\tau = 0, \frac{\pi}{4}, \frac{\pi}{2}$.

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