

GLOBAL GROUP LAWS AND EQUIVARIANT BORDISM RINGS

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ABSTRACT. For every abelian compact Lie group A , we prove that the homotopical A -equivariant complex bordism ring, introduced by tom Dieck (1970), is isomorphic to the A -equivariant Lazard ring, introduced by Cole–Greenlees–Kriz (2000). This settles a conjecture of Greenlees. We also show an analog for homotopical real bordism rings over elementary abelian 2-groups. Our results generalize classical theorems of Quillen (1969) on the connection between non-equivariant bordism rings and formal group laws, and extend the case $A = C_2$ due to Hanke–Wiemeler (2018).

We work in the framework of global homotopy theory, which is essential for our proof. In addition to the statements for a fixed group A , we also prove a global algebraic universal property that characterizes the collection of all equivariant complex bordism rings simultaneously. We show that they form the universal contravariant functor from abelian compact Lie groups to commutative rings that is equipped with a coordinate; the coordinate is given by the universal Euler class at the circle group. More generally, the ring of n -fold cooperations of equivariant complex bordism is shown to be universal among functors equipped with a strict n -tuple of coordinates.

MSC: 57R85, 55N22, 55P91, 14L05

1. INTRODUCTION

In [Qui69], Quillen proved that the complex bordism ring MU_* is isomorphic to the Lazard ring, and that the formal group law associated to the complex orientation of MU is the universal one. This relationship between the algebra of formal group laws and the structure of the stable homotopy category has proved to be one of the main organizational principles for understanding the latter. For example, Quillen’s theorem has been central for the construction of many new cohomology theories, the classification of thick subcategories of finite spectra [HS98], computations of homotopy groups of spheres via the Adams–Novikov spectral sequence [Nov67] and the solution of the Kervaire invariant 1 problem [HHR16].

In view of the importance of Quillen’s theorem, much work has been put into obtaining a similar understanding of equivariant bordism rings and their characterization in terms of formal group law data (see, e.g., [tD70, BH72, L f73, Com96, Kri99, CGK00, Gre01, Str01, Sin01, Han05, Str11, AK15, HW18, Uri18]); motivated for example by geometric questions about almost complex G -manifolds, the construction of equivariant higher chromatic K - and E -theory spectra or equivariant elliptic cohomology [Lur], computations of MU_*BG via the localization theorems of Greenlees–May [GM97], or for studying the global structure of the category of finite G -spectra [BS17, BHN⁺19, BGH20].

In the case where $G = A$ is an abelian compact Lie group, the notion of an A -equivariant formal group law was introduced in [CGK00] and further studied in [Gre01, Str11]. As in the non-equivariant case there exists a universal A -equivariant formal group law defined over an A -equivariant Lazard ring L_A ; and a map

$$L_A \rightarrow \pi_*^A(MU_A)$$

classifying the Euler class of the tensor product of A -equivariant complex line bundles. Here, MU_A is tom Dieck’s homotopical A -equivariant complex bordism spectrum [tD70]. In [Gre00, Gre01], Greenlees showed that when A is finite abelian, the map $L_A \rightarrow \pi_*^A(MU_A)$ is surjective with Euler-torsion and infinitely Euler-divisible kernel, and conjectured that it is an isomorphism for all abelian compact Lie groups A . Despite this strong evidence, the only non-trivial group for which the conjecture has been proved to date is the cyclic group of order 2, by work of Hanke–Wiemeler [HW18] who made use of an explicit presentation of $\pi_*^{C_2}(MU_{C_2})$ due to Strickland [Str01]. The main result of this paper is a proof of the conjecture in full generality:

Theorem A. *The map $L_A \rightarrow \pi_*^A(MU_A)$ is an isomorphism for every abelian compact Lie group A .*

We obtain a similar result for homotopical real bordism MO_A introduced in [BH72]. Here the analog of the family of abelian compact Lie groups is that of elementary abelian 2-groups, since those are the groups

whose irreducible real representations are all 1-dimensional (cf. [Remark 1.2](#) below). Let $L_A^{2-\text{tor}}$ be the ring carrying the universal A -equivariant 2-torsion formal group law, i.e., the universal A -equivariant formal group law with trivial 2-series. The universal real orientation of MO_A induces an A -equivariant 2-torsion formal group law over $\pi_*^A(MO_A)$, giving rise to a map $L_A^{2-\text{tor}} \rightarrow \pi_*^A(MO_A)$.

Theorem B. *The map $L_A^{2-\text{tor}} \rightarrow \pi_*^A(MO_A)$ is an isomorphism for every elementary abelian 2-group A .*

Again, the underlying non-equivariant theorem that MO_* carries the universal 2-torsion formal group law is due to Quillen [[Qui69](#)].

1.1. Global formulation. Our main tool in proving Theorems [A](#) and [B](#) is the global structure of equivariant bordism, i.e., the fact that the collections of the various MU_A and MO_A assemble to global equivariant spectra **MU** and **MO** in the sense of Schwede [[Sch18](#)]. In addition to its essential role in the proofs of Theorems [A](#) and [B](#), the global framework also gives rise to a universal property of the collection of equivariant bordism rings that is simpler to state than the one for a fixed group A .

To describe this universal property, we recall from [[Sch18](#), Section 4.2] that the existence of a global complex bordism theory **MU** implies that the collection of homotopy groups $\{\pi_*^A(MU_A)\}_A$ abelian has restriction maps along all group homomorphisms; it assembles to a contravariant functor from the category of abelian compact Lie groups to commutative rings. We call this structure an *Ab-algebra*, and write $\pi_*(\mathbf{MU})$ for the graded *Ab-algebra* given by the coefficients of the various MU_A . Moreover, given an *Ab-algebra* X , we call an element $e \in X(\mathbb{T})$ at the circle group \mathbb{T} a *coordinate* if for every torus A and split surjective character $V: A \rightarrow \mathbb{T}$ the sequence

$$0 \rightarrow X(A) \xrightarrow{V^*(e)\cdot} X(A) \xrightarrow{\text{res}_{\ker(V)}^A} X(\ker(V)) \rightarrow 0$$

is exact. Here, the first map denotes multiplication with the pullback of e along V , and the second map is the restriction map to the kernel of V .

A complex orientation of a global ring spectrum E gives rise to an Euler class $e_\tau \in \pi_{-2}^{\mathbb{T}}(E)$ for the tautological 1-dimensional complex \mathbb{T} -representation τ . The cofiber sequence of based \mathbb{T} -spaces

$$\mathbb{T}_+ \rightarrow S^0 \rightarrow S^\tau$$

(together with its pullbacks along split surjections $V: A \rightarrow \mathbb{T}$) implies that e_τ defines a coordinate on $\pi_*(E)$, see [Section 5.4](#) for details. In particular, this applies to the universal global complex oriented theory **MU**, and hence $\pi_*(\mathbf{MU})$ carries a canonical coordinate e_τ .

Theorem C. *The pair $(\pi_*(\mathbf{MU}), e_\tau)$ is the universal *Ab-algebra* with a coordinate, i.e., for any other *Ab-algebra* X with coordinate $e \in X(\mathbb{T})$ there exists a unique map of *Ab-algebras* $\pi_*(\mathbf{MU}) \rightarrow X$ sending e_τ to e .*

We propose a pair consisting of an *Ab-algebra* X and a coordinate $e \in X(\mathbb{T})$ as a definition for a *global group law*, where we leave out the word ‘formal’ since the rings involved are not necessarily complete. Hence with this definition, $(\pi_*(\mathbf{MU}), e_\tau)$ is the universal global group law. To be precise, $(\pi_*(\mathbf{MU}), e_\tau)$ is more naturally the universal *graded* global group law, while the 0-th homotopy ring $\pi_0(\mathbf{MUP})$ of periodic global complex bordism with e_τ shifted to degree 0 is the universal ungraded global group law, see [Sections 2.6](#) and [4.2.1](#) for a discussion of gradings. Every global group law (X, e) has an associated A -equivariant formal group law X_A^\wedge over $X(A)$ for all abelian compact Lie groups A , and these are suitably compatible under group homomorphisms ([Section 5.5](#)). Global group laws can be interpreted as an uncompleted version of ordinary formal group laws and are related to 1-dimensional linear algebraic groups and (not necessarily connected) p -divisible groups. We refer to [Section 5.3](#) for a list of examples.

In fact, the coordinate of $\pi_*(\mathbf{MU})$ has even stronger regularity properties than those required of a global group law: The sequence

$$0 \rightarrow \pi_*^A(MU_A) \xrightarrow{V^*(e)\cdot} \pi_*^A(MU_A) \xrightarrow{\text{res}_{\ker(V)}^A} \pi_*^{\ker(V)}(MU_{\ker(V)}) \rightarrow 0$$

is exact for every abelian compact Lie group A and surjective character $V: A \rightarrow \mathbb{T}$, not necessarily split. In terms of global group laws, we say that $\pi_*(\mathbf{MU})$ is *regular*, see [Sections 5.2](#) and [5.4](#). These exact sequences played a central role in Sinha’s work [[Sin01](#)], where he gave a description of $\pi_*^{\mathbb{T}}(MU_{\mathbb{T}})$ in terms of chosen

sections of the restriction maps $\pi_*^{\mathbb{T}}(MU_{\mathbb{T}}) \rightarrow \pi_*^{C_n}(MU_{C_n})$. [Theorem C](#) shows that when interpreted in the global equivariant setting, these exact sequences characterize the homotopy groups of \mathbf{MU} .

We also obtain a similar statement for $\pi_*(\mathbf{MO})$, replacing the family of abelian compact Lie groups by the family of elementary abelian 2-groups el_2 , and the circle \mathbb{T} by the cyclic group C_2 : Let X be an el_2 -algebra, i.e., a contravariant functor from el_2 to commutative rings. A *coordinate* on X is an element $e \in X(C_2)$ such that for every surjective character $V : A \rightarrow C_2$ the sequence

$$0 \rightarrow X(A) \xrightarrow{V^*(e)\cdot} X(A) \xrightarrow{\text{res}_{\ker(\alpha)}^A} X(\ker(\alpha)) \rightarrow 0$$

is exact. (Note that for elementary abelian 2-groups there is no difference between surjective and split surjective.) Again, the real orientation of a real oriented global ring spectrum E (such as \mathbf{MO}) gives rise to a universal Euler class $e_{\sigma} \in \pi_{-1}^{C_2}(E)$ for the real sign-representation σ of C_2 . The C_2 -cofiber sequence

$$(C_2)_+ \rightarrow S^0 \rightarrow S^{\sigma}$$

implies that it defines a coordinate on $\pi_*(E)$. We then have:

Theorem D. *The pair $(\pi_*(\mathbf{MO}), e_{\sigma})$ is universal among el_2 -algebras equipped with a coordinate.*

In light of this, we view a pair of an el_2 -algebra and a coordinate as a global version of a 2-torsion formal group law, which is again not necessarily complete. Every global 2-torsion group law (X, e) has an associated family of compatible A -equivariant 2-torsion formal group laws X_A^{\wedge} , see [Section 4.4](#).

Finally, we also give a description of the n -fold cooperations $\pi_*(\mathbf{MU}^{\wedge n})$, which are relevant for setting up an Adams-Novikov spectral sequence in this framework. We say that an n -tuple of coordinates $(e^{(1)}, \dots, e^{(n)})$ of an Ab -algebra X is *strict* if each $e^{(i)}$ is of the form $\lambda_i e^{(1)}$ for a unit $\lambda_i \in X(\mathbb{T})$ which restricts to 1 at the trivial group. The inclusion maps $i_1, \dots, i_n : \mathbf{MU} \rightarrow \mathbf{MU}^{\wedge n}$ give rise to a strict n -tuple $(e_{\tau}^{(1)}, \dots, e_{\tau}^{(n)})$ of coordinates for $\pi_*(\mathbf{MU})$. We show the following:

Theorem E. *The tuple $(\pi_*(\mathbf{MU}^{\wedge n}); e_{\tau}^{(1)}, \dots, e_{\tau}^{(n)})$ is universal among Ab -algebras equipped with a strict n -tuple of coordinates.*

There is also a similar interpretation of $\pi_*^A(MU_A^{\wedge n})$ at every fixed group A in terms of strict isomorphisms of A -equivariant formal group laws, see [Section 5.8](#).

Remark 1.1. Due to the failure of generic equivariant transversality, the homotopical bordism ring $\pi_*^A(MU_A)$ does not agree with the geometrically defined complex bordism ring Ω_*^A for any non-trivial A . However, there is a Thom-Pontrjagin map $\Omega_*^A \rightarrow \pi_*^A(MU_A)$ which was shown to be injective (in the abelian case considered here) by tom Dieck [[tD70](#)], Löffler [[Löff73](#)] and Comezana [[Com96](#)]. The ring $\pi_*^A(MU_A)$ can also be described as a stabilization of geometric A -equivariant bordism, see [[BH72](#)].

When A is a torus, Hanke [[Han05](#)] proved a more precise relationship between Ω_*^A and $\pi_*^A(MU_A)$ in terms of a pullback square involving certain classes $Y_{V,d}$. It is straightforward to express these classes in terms of formal group law data, see [Remark 4.39](#) for the real analog. Hence, combined with Hanke's result, the results of this paper also give an algebraic description of the geometric bordism rings Ω_*^A for tori, though this is not entirely satisfactory since $\pi_*^A(MU_A)$ has a universal property for maps out of rather than into it.

The same discussion applies to the geometric real bordism ring for elementary abelian 2-groups, for which the analog of Hanke's result has been obtained by Firsching [[Fir13](#)].

1.2. Outline of the proof. We first quickly recall Greenlees' method in his proof of the surjectivity of $L_A \rightarrow \pi_*^A(MU_A)$ for finite A : He proceeds by induction on the order of A via isotropy separation, using that both L_A and $\pi_*^A(MU_A)$ simplify after inverting or completing at certain collections of Euler classes; the precise description involves local homology. The reason his method does not lead to a full proof of the isomorphism is that it is a priori unclear that L_A can be recovered from the various localizations and completions considered. In particular, elements that are infinitely Euler-divisible and Euler-torsion are taken to 0 under all these constructions. For topological reasons, one knows that such elements do not exist in $\pi_*^A(MU_A)$. This non-existence is difficult to see directly for L_A , for which no simple presentation is known. Note that this stands in contrast to the non-equivariant Lazard ring L , which is a polynomial algebra over \mathbb{Z} by Lazard's theorem [[Laz55](#)].

Similar to Greenlees' work, our strategy is to reduce Theorems A and B to the already known statements that the maps $L_A \rightarrow \pi_*^A(MU_A)$ and $L_A^{2-\text{tor}} \rightarrow \pi_*^A(MO_A)$ are isomorphisms after inverting all non-trivial Euler classes. In order to be able to make this reduction, we derive the regularity of Euler classes in Lazard rings (at tori in the complex case, and elementary abelian 2-groups in the real case) essentially formally via global homotopy theory. The key definition is that of a global group law given above, which is inspired by the cofiber sequence $\mathbb{T}_+ \rightarrow S^0 \rightarrow S^\tau$ from topology. It is based on the observation that all the A -equivariant formal group laws associated to a complex oriented global spectrum E are encoded in the single Euler class $e \in \pi_{-2}^{\mathbb{T}}(E)$ and the global functoriality of $\pi_*^A E$, using only that the sequences

$$0 \rightarrow \pi_*^A(E) \xrightarrow{V^*(e) \cdot} \pi_*^A(E) \xrightarrow{\text{res}_{\ker(V)}^A} \pi_*^{\ker(V)}(E) \rightarrow 0$$

are exact, for all split characters $V \in A^*$. This construction carries over to algebra, showing that every global group law X has an associated A -equivariant formal group law X_A^\wedge over $X(A)$, defined as a certain completion of the ring $X(A \times \mathbb{T})$. The A -equivariant formal group laws that arise this way are special, since they inherit some of the structure of X . In particular, all Euler classes e_V for split characters V of A are regular elements. Since controlling the torsion of Euler classes in Lazard rings is our main concern, it is hence desirable to show that the collection $\{L_A\}_{A \text{ abelian}}$ assembles to such a global group law. Instead of trying to prove this directly, which would merely be a reformulation of the question we started with, we turn the problem around: We consider the universal global group law \tilde{L} and show that its evaluation on A is isomorphic to L_A , or rather that the A -equivariant formal group law \tilde{L}_A^\wedge is the universal one. This is done by showing that the functor $(-)_A^\wedge$ has a right adjoint, which implies that it must send universal objects to universal objects, as desired. Hence, the collection of the Lazard rings L_A even describes the universal global group law. In particular all split Euler classes are regular elements.

In the real bordism case, all non-trivial characters are split, so this argument allows us to prove regularity for all non-trivial Euler classes in $L_A^{2-\text{tor}}$, and Theorem B follows in a straightforward manner. In the complex case we need additional arguments to extend the regularity of split Euler classes to the regularity of all non-trivial Euler classes at tori. This is achieved by studying the effect of the p -th power map of \mathbb{T}^r on a suitable localization of the Lazard ring $L_{\mathbb{T}^r}$, see Section 5.6.

Remark 1.2. We say a few words on why we restrict to abelian rather than all compact Lie groups. The main reason is the splitting principle. It allows to reduce the theory of complex orientations to the behaviour of line bundles and their tensor product, giving the link to 1-dimensional formal group laws. For non-abelian G there are irreducible G -representations of dimension larger than 1, so that this reduction is no longer possible. We are unaware even of a definition of a G -equivariant formal group law for non-abelian G , though some steps are taken in [Gre01, Part 3]. It seems likely that the global point of view can also shed light on this.

Another reason to restrict to abelian groups is that we rely on the theorems of [Löf73, Com96] that $\pi_*^A(MU_A)$ is concentrated in even degrees. Whether this is also the case for non-abelian G is an open problem. See [Uri18] for a survey on the analog of this question for the geometric bordism rings Ω_*^G .

Organization: We first discuss the simpler case of real bordism and the proof of Theorems B and D in Section 4, after we have recalled the necessary background on equivariant formal group laws and global homotopy theory in Section 2 and Section 3. Finally, Section 5 discusses complex bordism and the proof of Theorems A, C and E.

1.3. Acknowledgements. I thank Bernhard Hanke, Lennart Meier and Stefan Schwede for helpful comments on a preliminary version of this paper, and two anonymous referees for their careful reading and the many useful suggestions. This research was supported by the Hausdorff Center for Mathematics at the University of Bonn (DFG GZ 2047/1, project ID 390685813) and by the Danish National Research Foundation through the Centre for Symmetry and Deformation at the University of Copenhagen (DNRF92).

2. EQUIVARIANT FORMAL GROUP LAWS

In this first section we recall the notion of an A -equivariant formal group law as defined in [CGK00] and list some of its properties.

2.1. Definition and basic properties. Let A be an abelian compact Lie group. We denote by $A^* = \text{Hom}(A, \mathbb{T})$ its group of characters, with trivial character ϵ . We often identify a character with its associated 1-dimensional complex representation. Given a commutative ring k , we view its ring of functions k^{A^*} as a topological ring via the product topology. In addition to its ring structure, k^{A^*} carries the ‘dual group ring’ coproduct

$$\Delta: k^{A^*} \rightarrow k^{A^* \times A^*} \cong k^{A^*} \hat{\otimes} k^{A^*}$$

which sends a function $f: A^* \rightarrow k$ to the function $\Delta(f): A^* \times A^* \rightarrow k$ mapping (V, W) to $f(VW) \in k$. Here and throughout the paper, we write VW for the product of V and W in the group A^* (i.e., the tensor product of V and W when thought of as complex representations). Cole–Greenlees–Kriz [CGK00, Definition 11.1] define the following:

Definition 2.1. An A -equivariant formal group law F is a quintuple $(k, R, \Delta, \theta, y(\epsilon))$ of a commutative ring k , a commutative complete topological k -algebra R , a continuous comultiplication $\Delta: R \rightarrow R \hat{\otimes} R$, a map of topological k -algebras $\theta: R \rightarrow k^{A^*}$ and an orientation $y(\epsilon) \in R$, such that the following hold:

- (1) The comultiplication is a map of k -algebras, cocommutative, coassociative and counital for the augmentation $\theta(\epsilon): R \rightarrow k$.
- (2) The map θ is compatible with the coproducts, and the topology on R is generated by finite products of the kernels of the component functions $\theta(V): R \rightarrow k$ for $V \in A^*$.
- (3) The element $y(\epsilon)$ is regular and generates the kernel of $\theta(\epsilon)$.

Definition/Notation 2.2. Given an A -equivariant formal group law, we obtain an A^* -action l on R via the formula

$$l_V: R \xrightarrow{\Delta} R \hat{\otimes} R \xrightarrow{\theta(V) \hat{\otimes} \text{id}_R} R,$$

for $V \in A^*$.

By the compatibility of Δ and θ , we have $\theta(V) = \theta(\epsilon) \circ l_V$, hence the A^* -action together with the augmentation $\theta(\epsilon)$ determine the map θ . The condition that θ is compatible with the comultiplication is then equivalent to demanding that Δ is $(A^* \times A^*)$ -equivariant, where $A^* \times A^*$ acts through the tensor factors on $R \hat{\otimes} R$ and via the multiplication map $A^* \times A^* \rightarrow A^*$ on R .

Definition/Notation 2.3. For $V \in A^*$, we write $y(V)$ for the element $l_V(y(\epsilon))$, which by the above discussion generates the ideal $I_{V^{-1}} = \ker(\theta(V^{-1}))$. Finally we define *Euler classes* $e_V \in k$ via

$$e_V = \theta(\epsilon)(y(V)) = \theta(V)(y(\epsilon)).$$

Note that $e_\epsilon = 0$, since $y(\epsilon)$ lies in the kernel of $\theta(\epsilon)$ by definition. Furthermore, we have $\theta(V)(y(W)) = e_{VW}$ for a pair $V, W \in A^*$.

Remark 2.4. We use a slightly different convention than [CGK00] for the definition of l_V , $y(V)$ and e_V . What we call l_V would be $l_{V^{-1}}$ in [CGK00], and consequently our $y(V)$ and e_V correspond to their $y(V^{-1})$ and $e_{V^{-1}}$. We make this choice to simplify notation later on in the comparison with global group laws.

Remark 2.5. As is shown in [CGK00, Appendix B], every A -equivariant formal group law allows a coinverse, i.e., a continuous self-map $\chi: R \rightarrow R$ of k -algebras such that the composite

$$R \xrightarrow{\Delta} R \hat{\otimes} R \xrightarrow{\text{id}_R \hat{\otimes} \chi} R \hat{\otimes} R \rightarrow R$$

equals the composite $R \xrightarrow{\theta(\epsilon)} k \rightarrow R$. The coinverse turns R into a complete topological Hopf algebra over k .

2.2. Flags and topological bases. Given $V \in A^*$, every element $x \in R$ can be written as

$$x = \theta(V^{-1})(x) + \tilde{x} \cdot y(V)$$

for a unique $\tilde{x} \in R$, since $y(V)$ is regular and generates the kernel of $\theta(V^{-1})$. It follows that for any sequence $V_1, \dots, V_n \in A^*$, the quotient $R/(I_{V_1^{-1}} \cdots I_{V_n^{-1}})$ has a k -basis given by

$$1, y(V_1), y(V_2)y(V_1), \dots, y(V_{n-1}) \cdots y(V_1).$$

Given a *complete flag* F , i.e., a sequence $V_1, V_2, \dots, V_n, \dots$ of characters such that every character occurs infinitely often, an element $x \in R$ can be written uniquely as

$$x = \sum_{i=0}^{\infty} a_i^F \cdot y(V_i) \cdots y(V_1)$$

with $a_i^F \in k$. Unless A is trivial, the choice of flag is not unique. The translation between the coefficients (a_i^F) and $(a_i^{F'})$ for different flags F and F' depends on the multiplicative structure of R , which is in general not simply given by a power series ring, see [Gre01, Section 3] for a discussion. Writing x in the above form, we see that

$$\theta(V)(x) = \sum_{i=0}^{\infty} a_i^F \cdot e_{V V_i} \cdots e_{V V_1} \in k,$$

where the sum is finite since the product $e_{V V_i} \cdots e_{V V_1}$ becomes trivial once there is a $j \leq i$ with $V_j = V^{-1}$. Hence, for every flag F , the map $\theta(V)$ is a polynomial in the coefficients a_i^F and the Euler classes.

For later use we note the following, where $F = (k, R, \Delta, \theta, y(\epsilon))$ is an A -equivariant formal group law.

Lemma 2.6. *An element $x \in R$ is invertible if and only if all augmentations $\theta(V)(x) \in k$ are invertible.*

Proof. Since the augmentations are ring maps, it is clear that if x is invertible, then so is $\theta(V)(x)$ for any $V \in A^*$. Conversely, assume all the $\theta(V)(x)$ are invertible. We need to see that x is a unit in $R/(I_{V_n} \cdots I_{V_1})$ for all $n \in \mathbb{N}$ and characters $V_1, \dots, V_n \in A^*$. To show this, we consider the kernel K of the projection $R/(I_{V_n} \cdots I_{V_1}) \rightarrow R/(I_{V_{n-1}} \cdots I_{V_1})$. By the preceding discussion, K is a free module of rank 1 over k , generated by the equivalence class of the element $y(V_{n-1}^{-1}) \cdots y(V_1^{-1})$. By induction, we need to show that x acts isomorphically on K . As we saw above, we can write $x = \theta(V_n)(x) + x' \cdot y(V_n^{-1})$ for some $x' \in R$. Computing in $R/(I_{V_n} \cdots I_{V_1})$, we obtain

$$\begin{aligned} x \cdot [y(V_{n-1}^{-1}) \cdots y(V_1^{-1})] &= (\theta(V_n)(x) + x' \cdot y(V_n^{-1})) \cdot [y(V_{n-1}^{-1}) \cdots y(V_1^{-1})] \\ &= \theta(V_n)(x) \cdot [y(V_{n-1}^{-1}) \cdots y(V_1^{-1})] + [y(V_n^{-1}) y(V_{n-1}^{-1}) \cdots y(V_1^{-1})] \\ &= \theta(V_n)(x) \cdot [y(V_{n-1}^{-1}) \cdots y(V_1^{-1})], \end{aligned}$$

since $[y(V_n^{-1}) y(V_{n-1}^{-1}) \cdots y(V_1^{-1})] = 0$ in $R/(I_{V_n} \cdots I_{V_1})$. Hence, x acts on K via $\theta(V_n)(x)$, which is a unit in k by assumption. This finishes the proof. \square

2.3. The category of A -equivariant formal group laws. We write FGL_A for the category of A -equivariant formal group laws, where a morphism from $F = (k, R, \Delta, \theta, y(\epsilon))$ to $F' = (k', R', \Delta', \theta', y(\epsilon'))$ is a pair of maps $f_1: k \rightarrow k'$ and $f_2: R \rightarrow R'$ preserving all the structure at hand. In fact, such a morphism is already determined by f_1 , since f_2 needs to send $y(V)$ to $y(V)'$ and the $y(V)$ generate R as a k -algebra topologically.

Said differently, given a map $f: k \rightarrow k'$ of commutative rings and an A -equivariant formal group law F over k , there is a push-forward A -equivariant formal group law f_*F over k' whose complete k' -algebra f_*R is given by $\lim_{V_1, \dots, V_n} (R/(I_{V_n} \cdots I_{V_1}) \otimes_k k')$. The induced map $R \rightarrow f_*R$ then extends f to a morphism of A -equivariant formal group laws. Up to canonical isomorphism fixing the coefficient ring, every morphism of A -equivariant formal group laws arises in this way.

In [CGK00, Sections 13 and 14] it is shown that the category FGL_A has an initial object (which will also follow from the adjunction in Section 5.5), whose ground ring the authors call the *A -equivariant Lazard ring* and denote by L_A . By the discussion above, this means that every A -equivariant formal group law over a commutative ring k arises as the push-forward of a unique map $L_A \rightarrow k$. Therefore, the category of A -equivariant formal group laws is in fact equivalent to the category of commutative rings under L_A .

2.4. The global Lazard ring. It will be crucial for us to consider the functoriality of A -equivariant formal group laws as A varies. For this let $\alpha: B \rightarrow A$ be a homomorphism of abelian compact Lie groups, and $F = (k, R, \Delta, \theta, y(\epsilon))$ a B -equivariant formal group law. Then we obtain an A -equivariant formal group law $\alpha_*F = (k, R_\alpha, \Delta_\alpha, \theta_\alpha, y_\alpha(\epsilon))$ over the same ground ring k , with

- R_α the completion of R at all finite products of elements $y(V)$ with V in the image of $\alpha^*: A^* \rightarrow B^*$,
- $\Delta_\alpha: R_\alpha \rightarrow R_\alpha \hat{\otimes} R_\alpha$ the completion of the comultiplication $\Delta: R \rightarrow R \hat{\otimes} R$,

- θ_α the map induced by the composite $R \xrightarrow{\theta} k^{B^*} \xrightarrow{(\alpha^*)^*} k^{A^*}$ on completion, and
- $y_\alpha(\epsilon) \in R_\alpha$ the image of $y(\epsilon)$ under the completion map $R \rightarrow R_\alpha$.

This defines a functor $\alpha_*: FGL_B \rightarrow FGL_A$, which induces a map $\alpha^*: L_A \rightarrow L_B$ between the representing rings. It is a straightforward check that we obtain a functor

$$\mathbf{L}: (\text{abelian compact Lie groups})^{op} \rightarrow \text{commutative rings},$$

which we call the *global Lazard ring*.

If $i: B \hookrightarrow A$ is the inclusion of a subgroup, then the induced map $i^*: A^* \rightarrow B^*$ is surjective. This implies that the pushforward $i_*F = (k, R_i, \Delta_i, \theta_i, y_i(\epsilon))$ of a B -equivariant formal group law $F = (k, R, \Delta, \theta, y(\epsilon))$ has $R_i = R$, $\Delta_i = \Delta$ and $y_i(\epsilon) = y(\epsilon)$. The only difference is that θ is replaced by the composite $R \rightarrow k^{B^*} \rightarrow k^{A^*}$, or equivalently, that the B^* -action on R is replaced by the A^* -action obtained via $i^*: A^* \rightarrow B^*$. This implies that FGL_B is equivalent to the full subcategory of FGL_A on those A -equivariant formal group laws where the A^* -action factors through i^* . This property can be characterized in terms of Euler classes as follows:

Lemma 2.7. *Let $(k, R, \Delta, \theta, y(\epsilon))$ be an A -equivariant formal group law, and $i: B \hookrightarrow A$ the inclusion of a subgroup. Then the action of A^* on R factors through i^* if and only if $e_V = 0$ for every $V \in \ker(i^*) \subseteq A^*$.*

Proof. If the action l factors through i^* , then any $V \in \ker(i^*)$ acts trivially. Hence $y(V) = y(\epsilon)$ and $e_V = e_\epsilon = 0$. If for the other direction we assume that $e_V = 0$ for all $V \in \ker(i^*)$, then for any such V the map $\theta(V): R \rightarrow k$ has $y(\epsilon)$ in the kernel. Since $\theta(V)$ is a k -algebra map and $R/y(\epsilon) \cong k$, this implies that $\theta(V) = \theta(\epsilon)$, and hence by definition $l_V = l_\epsilon = \text{id}_R$. This shows that the action factors through i^* , as desired. \square

Corollary 2.8. *Let $i: B \hookrightarrow A$ be a subgroup inclusion. Then the map $i^*: L_A \rightarrow L_B$ is surjective with kernel generated by the Euler classes e_V for $V \in \ker(i^*) \subseteq A^*$.*

In fact the proof of Lemma 2.7 shows that it always suffices to quotient out by finitely many Euler classes: When V_1, \dots, V_n are a set of generators of $\ker(i^*) \subseteq A^*$, then e_{V_1}, \dots, e_{V_n} generate the kernel of $i^*: L_A \rightarrow L_B$.

As every abelian compact Lie group embeds into a torus, Corollary 2.8 implies that the theory of equivariant formal group laws is controlled by its behaviour over tori. For example, there is a universal Euler class $e_\tau \in L_{\mathbb{T}}$ for the tautological \mathbb{T} -character $\tau \in \mathbb{T}^*$. The Euler class $e_V \in L_A$ of any other character $V \in A^*$ is obtained by pulling back e_τ along $V: A \rightarrow \mathbb{T}$. We will later simply write e instead of e_τ .

2.5. The global 2-torsion Lazard ring. There is also a 2-torsion version of this theory which is relevant for the description of equivariant real bordism. For this we restrict to elementary abelian 2-groups A , or in other words to finite dimensional \mathbb{F}_2 -vector spaces. Note that in this case the dual group A^* agrees with the dual \mathbb{F}_2 -vector space.

We say that an A -equivariant formal group law $F = (k, R, \Delta, \theta, y(\epsilon))$ is *2-torsion* if the composite

$$[2]_F: R \xrightarrow{\Delta} R \hat{\otimes} R \xrightarrow{m} R$$

sends the orientation class $y(\epsilon)$ (or phrased orientation-independently, the ideal I_ϵ) to 0. In other words, F is 2-torsion if the coinverse $\chi: R \rightarrow R$ (see Remark 2.5) equals the identity map of R . Since $[2]_F(y(\epsilon))$ can be written as $2 \cdot y(\epsilon) + x \cdot y(\epsilon)^2$, the ground ring k of every 2-torsion A -equivariant formal group law is an \mathbb{F}_2 -algebra.

The subcategory of 2-torsion A -equivariant formal group laws, which we denote by $FGL_A^{2\text{-tor}}$, also has an initial object defined over a 2-torsion Lazard ring $L_A^{2\text{-tor}}$. For example, $L_A^{2\text{-tor}}$ can be obtained from L_A by quotienting out the coefficients of $[2]_F(y(\epsilon))$ with respect to some choice of complete flag. Moreover, given a homomorphism of elementary abelian 2-groups $\alpha: B \rightarrow A$, the functor $\alpha_*: FGL_B \rightarrow FGL_A$ restricts to a functor $\alpha_*: FGL_B^{2\text{-tor}} \rightarrow FGL_A^{2\text{-tor}}$. We obtain a global 2-torsion Lazard ring

$$\mathbf{L}^{2\text{-tor}}: (\text{el}_2)^{op} = (\text{elementary abelian 2-groups})^{op} \rightarrow \text{commutative } \mathbb{F}_2\text{-algebras}.$$

Again, there is a universal 2-torsion Euler class, given by $e_\sigma \in L_{C_2}^{2\text{-tor}}$ associated to the sign character σ of C_2 , and the analog of Corollary 2.8 holds.

2.6. Gradings. It is sometimes convenient to view \mathbf{L} and $\mathbf{L}^{2-\text{tor}}$ as functors to \mathbb{Z} -graded rings. The relevant gradings are uniquely determined by the requirement that the Euler class $e_\tau \in L_\mathbb{T}$ is of degree -2 , and the Euler class $e_\sigma \in L_{C_2}^{2-\text{tor}}$ is of degree -1 . The existence and uniqueness of these gradings follows either by inspection of the construction in [CGK00, Section 14], or from the arguments in Sections 4.2 and 5.1. The grading on L_{C_2} is also discussed in [HW18], where it plays a central role in their proof of the isomorphism $L_{C_2} \xrightarrow{\cong} \pi_*^{C_2}(MU_{C_2})$.

Topologically, the graded version corresponds to considering $\pi_*(\mathbf{MU})$, while the ungraded version corresponds to $\pi_0(\mathbf{MUP})$, the 0-th homotopy ring of periodic complex bordism, see Section 5.4, and similarly for \mathbf{MO} and \mathbf{MOP} .

2.7. Inverting Euler classes. The goal of this subsection is to understand the structure of equivariant Lazard rings after inverting certain collections of Euler classes.

Proposition 2.9. *Let*

$$1 \rightarrow B \xrightarrow{i} A \xrightarrow{p} A/B \rightarrow 1$$

be a short exact sequence of abelian compact Lie groups. Let $V(B) \subset A^$ be the subset of those characters whose restriction to B^* is non-trivial, or in other words the complement of the image of the inclusion $p^*: (A/B)^* \hookrightarrow A^*$. Then the composite*

$$L_{A/B} \xrightarrow{p^*} L_A \rightarrow L_A[e_V^{-1} \mid V \in V(B)]$$

is flat.

Proof. The arguments are due to Greenlees ([Gre00, Section 11], [Gre01]), but not explicitly stated in this form and generality, so we recall the proof. The ring $L_A[e_V^{-1} \mid V \in V(B)]$ classifies A -equivariant formal group laws for which all Euler classes e_V with $V \in V(B)$ are invertible. We claim that the datum of such an A -equivariant formal group law is equivalent to the datum of an $(A/B)^*$ -equivariant formal group law \tilde{F} and an $(A/B)^*$ -equivariant function

$$z: V(B) \rightarrow \tilde{R}^\times \subset \tilde{R}.$$

Here, $(A/B)^*$ acts on $V(B) \subset A^*$ via left translation along p^* . The identification sends F to its pushforward p_*F together with the function z which takes $V \in V(B)$ to the image of $y(V^{-1})$ under the completion map $R \rightarrow R_p$ (note that R_p is given by the completion of R at finite products of the ideals I_V with $V \in \text{im}(p^*)$). The image of $z(V)$ under $\theta(V')$, for $V' \in (A/B)^*$, equals the Euler class $e_{V^{-1}V'} \in k$. Hence, as V does not lie in $(A/B)^*$, neither does $V^{-1}V'$ and the Euler class $e_{V^{-1}V'}$ is a unit by assumption. Therefore, $z(V)$ is a unit in R by Lemma 2.6, as required.

Assuming the claim, the proposition follows: To give an $(A/B)^*$ -equivariant function $z: V(B) \rightarrow \tilde{R}$ is equivalent to giving one element $z(W) \in \tilde{R}$ for a chosen representative W of every non-trivial $(A/B)^*$ -coset in $V(B)$. The elements $z(W)$ in turn are uniquely determined by their coefficients $a_i^F(z(W)) \in k$ for a chosen flag F . Finally, the requirement that these are units is, again by Lemma 2.6, equivalent to the requirement that the images under $\theta(V'): \tilde{R} \rightarrow k$ are units for all $V' \in (A/B)^*$. As we saw in Section 2.2, the functions $\theta(V')$ are polynomials in the coefficients a_i^F and the Euler classes. In summary, the representing ring of pairs $(\tilde{F} \in FGL_{A/B}, z: V(B) \rightarrow \tilde{R}^\times)$ is obtained from a polynomial ring over $L_{A/B}$ by inverting a collection of elements, and hence flat over $L_{A/B}$.

To prove the claim, we explain how to construct an A -equivariant formal group law $(k, R, \Delta, \theta, y(\epsilon))$ out of data \tilde{F} and z as above. We set

$$R = \text{map}_{(A/B)^*}(A^*, \tilde{R}),$$

the coinduction of \tilde{R} to a ring with A^* -action, with the product topology. The map θ is given by the composite

$$R = \text{map}_{(A/B)^*}(A^*, \tilde{R}) \xrightarrow{\text{map}_{(A/B)^*}(A^*, \tilde{\theta})} \text{map}_{(A/B)^*}(A^*, k^{(A/B)^*}) \cong k^{A^*}.$$

The comultiplication

$$\Delta: R = \text{map}_{(A/B)^*}(A^*, \tilde{R}) \rightarrow \text{map}_{(A/B)^* \times (A/B)^*}(A^* \times A^*, \tilde{R} \hat{\otimes} \tilde{R}) \cong R \hat{\otimes} R$$

is the unique $(A^* \times A^*)$ -equivariant map extending the $(A/B)^* \times (A/B)^*$ -equivariant map

$$R = \text{map}_{(A/B)^*}(A^*, \tilde{R}) \xrightarrow{ev(\epsilon)} \tilde{R} \xrightarrow{\tilde{\Delta}} \tilde{R} \hat{\otimes} \tilde{R}.$$

Finally, to define the coordinate $y(\epsilon)$ we note that R splits up as

$$R \cong \tilde{R} \times \text{map}_{(A/B)^*}(V(B), \tilde{R}),$$

where the first component of the isomorphism evaluates a function at ϵ , and the second restricts to $V(B)$. We then define the first component of $y(\epsilon)$ to be $\tilde{y}(\epsilon)$, and the second component to be the given map z . By assumption, z is a unit in $\text{map}_{(A/B)^*}(V(B), \tilde{R})$ and hence $\theta(\epsilon)$ defines an isomorphism $R/y(\epsilon) \cong \tilde{R}/\tilde{y}(\epsilon) \cong k$, as desired. The Euler classes e_V for $V \in V(B)$ are given by $\tilde{\theta}(\epsilon)(z(V^{-1}))$ and hence units, since $z(V^{-1})$ is a unit.

It remains to show that the two assignments are inverse to one another. It is immediate that the pushforward to an (A/B) -equivariant formal group law of the construction just described gives back \tilde{F} , since the completion amounts to projecting to the factor \tilde{R} of R . To see that it also gives back the same function z , note that by equivariance the V -th component of $y(\epsilon) \in \text{map}_{(A/B)^*}(A^*, \tilde{R})$ equals the ϵ -component of $y(V^{-1})$, or in other words the image of $y(V^{-1})$ in the completion, as desired. For the other direction, let $F = (k, R, \Delta, \theta, y(\epsilon))$ be an A -equivariant formal group law for which all Euler classes e_V with $V \in V(B)$ are units, and let p_*F be the pushforward to an (A/B) -equivariant formal group law. The completion map $R \rightarrow R_p$ is $(A/B)^*$ -equivariant (if we let $(A/B)^*$ act on R via $p^*: (A/B)^* \rightarrow A^*$), hence we obtain an induced A^* -equivariant map

$$f: R \rightarrow \text{map}_{(A/B)^*}(A^*, R_p).$$

It is straightforward to check that f is compatible with the comultiplication and augmentation, and by the same argument as above it preserves the coordinates. Since both sides are A -equivariant formal group laws and f is a map of k -algebras, it follows that f is an isomorphism (see the discussion in [Section 2.3](#)). This finishes the proof. \square

Remark 2.10. An alternative proof that R splits up into a product when the Euler classes e_V for $V(B)$ are invertible is given by the Chinese Remainder theorem, see [\[Gre00, Theorem 11.2\]](#).

In the special case where $A = B$ and we are hence inverting all non-trivial Euler classes, both the choice of coset representatives and the choice of flag in the proof of [Proposition 2.9](#) go away. This leads to the following description in terms of the ordinary Lazard ring L , again originally due to Greenlees ([\[Gre01, Theorem 6.3\]](#)). Given $V \in A^* - \{\epsilon\}$, we define elements $\gamma_i^V \in L_A$ via the property that

$$y(\epsilon) = e_V + \sum_{i=1}^n \gamma_i^V \cdot y(V^{-1})^i \in R/y(V^{-1})^{n+1} \cong L_A\{1, y(V^{-1}), y(V^{-1})^2, \dots, y(V^{-1})^n\}$$

for all $n \in \mathbb{N}$. In other words, γ_i^V is the coefficient of $y(V^{-1})^i$ in the power series expansion of $y(\epsilon)$ with respect to $y(V^{-1})$. Equivalently, γ_i^V can be defined as the coefficient of $y(\epsilon)^i$ in the power series expansion of $y(V)$ with respect to $y(\epsilon)$, which, in the notation of the proof of [Proposition 2.9](#), equals the i th coefficient of the power series $z(V^{-1}) \in R_p = k[[y_p(\epsilon)]]$. Then we have:

Proposition 2.11. *The canonical map*

$$L[e_V^{\pm 1}, \gamma_i^V \mid V \in A^* - \{\epsilon\}, i \in \mathbb{N}_{>0}] \rightarrow L_A[e_V^{-1} \mid V \in A^* - \{\epsilon\}]$$

is an isomorphism for every abelian compact Lie group A .

There is also a version for the 2-torsion Lazard ring. Again, given an elementary abelian 2-group A and $V \in A^*$ we write $\gamma_i^V \in L_A^{2\text{-tor}}$ for the coefficient of $y(V)^i$ in the power series expansion of $y(\epsilon)$ with respect to $y(V)$. The proof of the following is then entirely analogous to the one above:

Proposition 2.12. *The canonical map*

$$L^{2\text{-tor}}[e_V^{\pm 1}, \gamma_i^V \mid V \in A^* - \{\epsilon\}, i \in \mathbb{N}_{>0}] \rightarrow L_A^{2\text{-tor}}[e_V^{-1} \mid V \in A^* - \{\epsilon\}]$$

is an isomorphism for every elementary abelian 2-group A .

3. RECOLLECTIONS ON GLOBAL SPECTRA

In this section we recall various features of global spectra. We work in the framework of [Sch18], though our arguments rely only on formal properties of the global stable homotopy category. For our purposes it will be enough to know that (1) every global spectrum gives rise to a global cohomology theory in the sense below, and (2) that there exist global spectra **MU**, **MUP**, **MO** and **MOP** whose associated A -equivariant spectra are equivalent to the ones introduced by tom Dieck [tD70] and Bröcker-Hook [BH72].

3.1. Global cohomology theories. Throughout this paper we take *global spectrum* to mean an object of the global stable homotopy category with respect to the global family of abelian compact Lie groups, in the sense of [Sch18, Section 4.3]. Every global spectrum E has an underlying genuine A -spectrum E_A for every abelian compact Lie group A ; hence it defines a cohomology theory $E_A^*(-)$ on based A -spaces. Since the spectra E_A are genuine, these cohomology theories are graded over the real representation ring $RO(A)$, though there are some subtleties in making this precise. For us it will be enough to consider gradings of the form $E_A^{*+V}(-)$ for $*$ an integer and V an A -representation. Then there are natural isomorphisms

$$E_A^{*+V}(X \wedge S^V) \cong E_A^*(X)$$

for any based A -space X .

The fact that the E_A come from one global spectrum E induces extra functoriality between these cohomology theories (see [DHL⁺, Construction 3.3.6] for a construction in the more general setting of proper actions of arbitrary Lie groups): Any group homomorphism $\alpha: B \rightarrow A$ induces a natural restriction map

$$\alpha^*: E_A^{*+V}(X) \rightarrow E_B^{*+\alpha^*(V)}(\alpha^*(X)),$$

where $\alpha^*(X)$ denotes X with B -action pulled back along α . The restriction maps are contravariantly functorial, i.e., we have $(\alpha \circ \beta)^* = \beta^* \circ \alpha^*$ if $\beta: C \rightarrow B$ is another group homomorphism.

Example 3.1. In particular, setting $X = S^0$ and $V = 0$, the assignment

$$\pi_* E: A \mapsto \pi_*^A(E_A) = E_A^{-*}(S^0)$$

defines a contravariant functor from abelian compact Lie groups to graded abelian groups, see [Sch18, Section 4.2] for a discussion. In addition to the restriction maps there are also transfer maps for subgroup inclusions, but these do not play a role in this paper.

3.2. Inductions. The restriction maps satisfy an additional property, called the *induction isomorphism*: Given a group homomorphism $\alpha: B \rightarrow A$ and a based B -space X , we let $A_+ \wedge_\alpha X$ denote the induction of X to a based A -space along α . The resulting functor $A_+ \wedge_\alpha (-)$ is left adjoint to the restriction functor α^* , with unit

$$\eta_X: X \rightarrow \alpha^*(A_+ \wedge_\alpha X)$$

sending x to the class $[1, x]$. For a global spectrum E , we obtain an induction map

$$\text{ind}_\alpha: E_A^*(A_+ \wedge_\alpha X) \xrightarrow{\alpha^*} E_B^*(\alpha^*(A_+ \wedge_\alpha X)) \xrightarrow{(\eta_X)^*} E_B^*(X).$$

These satisfy the following (see [DHL⁺, Proposition 3.3.8] for a proof):

Induction isomorphism: If X is a based B -CW complex on which the kernel of α acts freely (in the based sense), then the induction map ind_α is an isomorphism $E_A^*(A_+ \wedge_\alpha X) \cong E_B^*(X)$.

The induction isomorphism can be viewed as a combination of the following two special cases:

- (1) If $i: B \hookrightarrow A$ is the inclusion of a subgroup, there is an induction isomorphism

$$E_A^*(A_+ \wedge_B X) \cong E_B^*(X)$$

for any based B -CW complex X . This is a formal consequence of the induction/restriction adjunction relating A -spectra and B -spectra, together with the fact that given a global spectrum E , its underlying B -spectrum E_B is equivalent to the restriction of its underlying A -spectrum E_A to B .

- (2) If $1 \rightarrow B \rightarrow A \rightarrow A/B \rightarrow 1$ is a short exact sequence of abelian compact Lie groups, and X is a based A -CW complex on which B acts freely, then there is an induction isomorphism $E_{A/B}^*(X/B) \cong E_A^*(X)$. This special case is closely related to the Adams isomorphism.

We further record the following:

Lemma 3.2. *Let $i: B \hookrightarrow A$ be a subgroup inclusion, X a based A -CW complex and consider the counit $\epsilon_X: A_+ \wedge_B \text{res}_B^A X \rightarrow X$. Then the composite*

$$E_A^*(X) \xrightarrow{\epsilon_X^*} E_A^*(A_+ \wedge_B \text{res}_B^A X) \xrightarrow[\cong]{\text{ind}_i} E_B^*(\text{res}_B^A X)$$

equals the restriction map.

Proof. We consider the diagram

$$\begin{array}{ccccc} E_A^*(X) & \xrightarrow{\epsilon_X^*} & E_A^*(A_+ \wedge_B \text{res}_B^A X) & & \\ \text{res}_B^A \downarrow & & \downarrow \text{res}_B^A & \searrow \text{ind}_i & \\ E_B^*(\text{res}_B^A X) & \xrightarrow{(\text{res}_B^A \epsilon_X)^*} & E_B^*(\text{res}_B^A(A_+ \wedge_B \text{res}_B^A X)) & \xrightarrow{\eta_{\text{res}_B^A(X)}^*} & E_B^*(\text{res}_B^A X) \end{array}$$

where the square commutes by naturality and the triangle commutes by definition. By the triangle equality, the composite of the two lower horizontal map equals the identity of $E_B^*(\text{res}_B^A X)$, which shows the claim. \square

Remark 3.3. Let A, C be two abelian compact Lie groups, X a based A -CW complex and $p: A \times C \rightarrow A$ the projection. Then $C = \ker(p)$ acts freely on $(A \times C)_+ \wedge_A X$ and hence there is an induction isomorphism

$$E_{A \times C}^*((A \times C)_+ \wedge_A X) \xleftarrow[\cong]{\text{ind}_p} E_A^*((A \times C)_+ \wedge_A X)/C \cong E_A^*(X).$$

We omit the straightforward proof that this isomorphism agrees with the induction isomorphism associated to the inclusion $A \hookrightarrow A \times C$.

3.3. Homotopy orbits. The induction isomorphisms give rise to the following construction regarding homotopy orbits: Given two abelian compact Lie groups A and C , let $E_A C$ be a universal C -space in the category of A -spaces. In other words, $E_A C$ is an $(A \times C)$ -CW complex such that given a subgroup $H \subseteq A \times C$, the H -fixed points $(E_A C)^H$ are contractible if H is the graph subgroup $\Gamma(\alpha)$ associated to a homomorphism $\alpha: B \rightarrow C$ for a subgroup B of A , and empty otherwise. Given a based $(A \times C)$ -space X , the quotient A -space $E_A C_+ \wedge_C X$ is called the (C, A) -homotopy orbit space of X and denoted $X_{h_A C}$. Now given a global spectrum E , we obtain a natural map

$$h_{A,C}(X): E_{A \times C}^*(X) \rightarrow E_A^*(X_{h_A C})$$

as the composite

$$E_{A \times C}^*(X) \xrightarrow{(E_A C \rightarrow *)^*} E_{A \times C}^*((E_A C)_+ \wedge X) \xleftarrow[\cong]{} E_A^*((E_A C)_+ \wedge_C X),$$

where the second map is the inverse of the induction isomorphism. These maps have the following property: Let $\alpha: A \rightarrow C$ be a homomorphism. Choosing a point $*_\alpha$ in the contractible space $(E_A C)^{\Gamma(\alpha)}$ yields an A -equivariant map $(\text{id}_A, \alpha)^*(X) \rightarrow X_{h_A C}$ sending x to the class $[\ast_\alpha, x]$.

Lemma 3.4. *The composite*

$$E_{A \times C}^*(X) \xrightarrow{h_{A,C}(X)} E_A^*(X_{h_A C}) \rightarrow E_A^*((\text{id}_A, \alpha)^*(X))$$

equals the restriction map $(\text{id}_A, \alpha)^$.*

Proof. We consider the graph subgroup $\Gamma(\alpha) \subseteq A \times C$. Our chosen point $*_\alpha$ induces a $\Gamma(\alpha)$ -equivariant map $[\ast_\alpha, -]: X \rightarrow E_A C_+ \wedge X$, which we can induce up to an $(A \times C)$ -equivariant map

$$(A \times C)_+ \wedge_{\Gamma(\alpha)} X \rightarrow E_A C_+ \wedge X.$$

Now, since the intersection of C with any conjugate of $\Gamma(\alpha)$ is empty, the C -action on $(A \times C)_+ \wedge_{\Gamma(\alpha)} X$ is free. The quotient A -space $((A \times C)_+ \wedge_{\Gamma(\alpha)} X)/C$ identifies with $(\text{id}_A, \alpha)^*(X)$ via the homeomorphism

sending x to the class $[1, x]$. Hence, by naturality of the induction isomorphism we obtain a commutative diagram

$$\begin{array}{ccccc}
 E_{A \times C}^*(X) & \xrightarrow{(E_A C \rightarrow *)^*} & E_{A \times C}^*(E_A C_+ \wedge X) & \xleftarrow[\cong]{\text{ind}} & E_A^*(E_A C_+ \wedge_C X) \\
 & \searrow (\epsilon_X)^* & \downarrow [*_{\alpha}, -]^* & & \downarrow [*_{\alpha}, -]^* \\
 & & E_{A \times C}^*((A \times C)_+ \wedge_{\Gamma(\alpha)} X) & \xleftarrow[\cong]{\text{ind}} & E_A^*((\text{id}_A, \alpha)^* X).
 \end{array}$$

The composite of the upper two horizontal maps equals $h_{A,C}(X)$ by definition. Hence it remains to see that the lower composite equals the restriction map $(\text{id}_A, \alpha)^*$. But identifying A with $\Gamma(\alpha)$ by sending a to $(a, \alpha(a))$, this follows from [Lemma 3.2](#) and [Remark 3.3](#). \square

3.4. Products. The functor that assigns to a global spectrum E the genuine A -spectrum E_A is strong symmetric monoidal. Hence, if E is a commutative global homotopy ring spectrum (i.e., a commutative monoid in the homotopy category of global spectra with the smash product [[Sch18](#), Corollary 4.3.26]), then each E_A is a commutative homotopy A -ring spectrum. Hence there are product maps of the form

$$E_A^{n+V}(X) \otimes E_A^{m+W}(Y) \rightarrow E_A^{n+m+V+W}(X \wedge Y)$$

and these are compatible with the restriction maps α^* . In particular, if E is a commutative global homotopy ring spectrum, then $\pi_* E$ is a contravariant functor from abelian compact Lie groups to graded commutative rings.

In the case $X = Y$, pulling back along the diagonal $X \rightarrow X \wedge X$ induces an internal multiplication on $E_A^{*+*}(X)$. Given X , we say that an element $t \in E_A^{n+V}(X \wedge S^W)$ is a unit if there exists an element $s \in E_A^{-n+W}(X \wedge S^V)$ such that $s \cdot t \in E_A^{W+V}(X \wedge S^{V+W})$ corresponds to 1 under the isomorphism

$$E_A^{W+V}(X \wedge S^{V+W}) \cong E_A^{W+V}(X \wedge S^{W+V}) \cong E_A^0(S^0) = \pi_0^A(E).$$

In terms of the $RO(A)$ -grading, the class t defines an element in $E_A^{n+V-W}(X)$, and being a unit with respect to the $RO(A)$ -graded multiplication amounts to the condition above.

4. REAL BORDISM

We now begin our study of equivariant bordism spectra and global group laws. We start with real bordism to demonstrate our general method while avoiding some of the complications that arise in the complex case.

4.1. Global 2-torsion group laws. We work over the family of elementary abelian 2-groups and write ‘ el_2 -algebra’ to mean a functor

$$(\text{elementary abelian 2-groups})^{op} \rightarrow \text{commutative rings}.$$

In this setup, we make the following definition:

Definition 4.1. A *global 2-torsion group law* is an el_2 -algebra X together with a class $e \in X(C_2)$, called the *coordinate*, such that for every elementary abelian 2-group A and non-trivial $V \in A^*$ the sequence

$$0 \rightarrow X(A) \xrightarrow{e_V} X(A) \xrightarrow{\text{res}_{\ker(V)}^A} X(\ker(V)) \rightarrow 0 \quad (4.2)$$

is exact. Here, e_V denotes the pullback $V^*e \in X(A)$.

A morphism of 2-torsion global group laws is a map of el_2 -algebras $X \rightarrow Y$ sending the coordinate of X to that of Y . We write $GL_{gl}^{2\text{-tor}}$ for the category of 2-torsion global group laws.

We will usually leave the choice of coordinate implicit and denote the pair (X, e) simply by X . We call the element e_V the *Euler class* of V . Note that the surjectivity in (4.2) holds for any el_2 -algebra, since every inclusion of elementary abelian 2-groups has a section. So the condition on e is that any e_V with $V \neq \epsilon$ is a regular element, and that divisibility by e_V is detected by restricting to $\ker(V)$ (in particular that $\text{res}_{\ker(V)}^A(e_V) = 0$, which follows from the universal case $\text{res}_1^{C_2}(e) = 0$).

Remark 4.3. At this point the notion of a 2-torsion global group law might seem a little mysterious. The definition is inspired by equivariant stable homotopy theory, where long exact sequences relating Euler classes and restrictions to subgroups are a standard tool. When combined with global structures and real orientations, these long exact sequences split up into short exact sequences of the form (4.2), as we will see shortly in Section 4.3.

In Section 4.4 we will show that a global 2-torsion group law X naturally encodes a 2-torsion A -equivariant formal group over $X(A)$, for every elementary abelian 2-group A . For now we list some elementary properties:

Lemma 4.4. (1) *Let X be a global 2-torsion group law and A an elementary abelian 2-group. If $V_1, \dots, V_n \in A^*$ are linearly independent, then the sequence $(e_{V_1}, \dots, e_{V_n}) \in X(A)$ is regular and generates the kernel of the restriction map*

$$\text{res}_{\cap \ker(V_i)}^A : X(A) \rightarrow X\left(\bigcap_{i=1}^n (\ker(V_i))\right).$$

In particular, given a basis V_1, \dots, V_r of A^ , the kernel of the restriction map to the trivial group is generated by the regular sequence $(e_{V_1}, \dots, e_{V_r})$.*

(2) *If X is a global 2-torsion group law, then $2 = 0$ in $X(1)$ and therefore $2 = 0$ in $X(A)$ for any $A \in \text{el}_2$. Hence, X takes image in \mathbb{F}_2 -algebras.*

Proof. (1): The proof is via induction on n . We assume the statement for $n-1$ and write K_{n-1} for the intersection $\cap_{i=1}^{n-1} (\ker(V_i))$. Then the quotient $X(A)/(e_{V_1}, \dots, e_{V_{n-1}})$ identifies with $X(K_{n-1})$ via the restriction map. Since the V_i are linearly independent, the restriction of V_n to a character of K_{n-1} is non-trivial. Hence, by the axioms of a global 2-torsion group law, the element $\text{res}_{K_{n-1}}^A e_{V_n}$ is a regular element in $X(K_{n-1})$ generating the kernel of the restriction map to $X(K)$, where K is the kernel of the restriction of V_n to K_{n-1} . Since $K = K_{n-1} \cap \ker(V_n)$, the claim follows.

(2): We let $e_{1,0}, e_{0,1}, e_{1,1} \in X(C_2 \times C_2)$ denote the Euler classes associated to the projection to the first component, projection to the second component, and the multiplication map, respectively. Then $e_{1,1} - e_{1,0}$ lies in the kernel of the restriction to the first component which is generated by $e_{0,1}$. Hence there exists a unique $x \in X(C_2 \times C_2)$ such that $e_{1,1} = e_{1,0} + x \cdot e_{0,1}$. Similarly, let $x_{0,1} \in X(C_2)$ be the restriction of x along the inclusion of C_2 in the second component. Then $x - \text{pr}_2^*(x_{0,1})$ lies in the kernel of the restriction to the second component and there exists an $x' \in X(C_2 \times C_2)$ such that $x = \text{pr}_2^*(x_{0,1}) + x' \cdot e_{1,0}$. Put together we have $e_{1,1} = e_{1,0} + \text{pr}_2^*(x_{0,1}) \cdot e_{0,1} + x' \cdot e_{1,0} \cdot e_{0,1}$. Now restricting to the second component, we find that $e = x_{0,1} \cdot e$, and hence by regularity $x_{0,1}$ must be 1. So in summary we see that

$$e_{1,1} = e_{1,0} + e_{0,1} + x' \cdot e_{1,0} \cdot e_{0,1}.$$

Restricting along the diagonal $\text{diag}: C_2 \rightarrow C_2 \times C_2$ this yields

$$0 = 2e + \text{diag}^*(x') \cdot e^2.$$

Regularity of e then implies that $0 = 2 + \text{diag}^*(x') \cdot e$, which restricts to $0 = 2$ at the trivial group, as claimed. \square

Remark 4.5. Let X be a global 2-torsion group law. Then for every element $x \in X(C_2)$ there exist unique elements $x_0 \in X(1)$ and $x' \in X(C_2)$ such that $x = p^*(x_0) + x' \cdot e$, where $p: C_2 \rightarrow 1$ is the projection. Indeed, the uniqueness follows from the fact that restricting to the trivial group shows that x_0 must be $\text{res}_1^{C_2}(x)$ and so regularity implies that x' is also unique. Conversely, the element $x - p^*(\text{res}_1^{C_2}(x))$ lies in the kernel of $\text{res}_1^{C_2}$, hence it is uniquely divisible by e , proving existence. Iterating this construction shows that for given $n \in \mathbb{N}$, every $x \in X(C_2)$ can be uniquely written as $x = \sum_{i=0}^{n-1} p^*(x_i) \cdot e^i + \tilde{x} \cdot e^n$ for elements $x_i \in X(1)$ and $\tilde{x} \in X(C_2)$. In particular, the completion of $X(C_2)$ at the augmentation ideal is isomorphic to the power series ring $X(1)[[e]]$.

Similarly, every $x \in X(C_2 \times C_2)$ can be written uniquely as

$$x = p^*(x_0) + \text{pr}_1^*(x_{1,0}) \cdot e_{1,0} + \text{pr}_2^*(x_{0,1}) \cdot e_{0,1} + x' \cdot e_{1,0} \cdot e_{0,1}$$

with $x_0 \in X(1)$, $x_{1,0}, x_{0,1} \in X(C_2)$ and $x' \in X(C_2 \times C_2)$. The special case $x = e_{1,1}$ was shown in the proof of Part (2) of Lemma 4.4. Applying the polynomial expansions from the first paragraph of the remark to the

elements $x_{1,0}, x_{0,1} \in X(C_2)$, this implies that modulo the ideal $(e_{1,0}^m, e_{0,1}^n)$, every element $x \in X(C_2 \times C_2)$ can be written uniquely as

$$x = \sum_{0 \leq i < m, 0 \leq j < n} p^*(x_{i,j}) \cdot e_{1,0}^i \cdot e_{0,1}^j$$

with coefficients $x_{i,j} \in X(1)$. There are analogous decompositions for $x \in X(B)$ for all $B \in \text{el}_2$. More generally, elements $x \in X(A \times B)$ can be expressed as polynomials with coefficients in $X(A)$ modulo certain products of Euler classes, for all pairs $A, B \in \text{el}_2$. These constructions are studied systematically in [Section 4.4](#) and give the link from global group laws to A -equivariant formal group laws.

The following are examples of global 2-torsion group laws:

Example 4.6 (Additive). Let \mathbb{G}_a be the el_2 -algebra defined via

$$\mathbb{G}_a(A) = \mathbb{F}_2[e_V, V \in A^*]/(e_V + e_W + e_{VW}),$$

with restriction maps $\alpha^*: \mathbb{G}_a(A) \rightarrow \mathbb{G}_a(B)$ sending e_V to $e_{\alpha^*(V)}$. Then the non-trivial element $\sigma \in (C_2)^*$ defines a coordinate $e = e_\sigma \in \mathbb{G}_a(C_2)$. This turns \mathbb{G}_a into a global 2-torsion group law, which we call the *additive 2-torsion formal group law*. Indeed, given a non-trivial character $V \in A^*$, we can extend it to a basis V, V_2, \dots, V_n of A^* . Then the ring $\mathbb{G}_a(A)$ identifies with the polynomial ring $\mathbb{F}_2[e_V, e_{V_2}, \dots, e_{V_n}]$ and restriction to $\ker(V)$ amounts to the map

$$\mathbb{F}_2[e_V, e_{V_2}, \dots, e_{V_n}] \rightarrow \mathbb{F}_2[e_{V_2}, \dots, e_{V_n}]$$

sending e_V to 0. So the sequence $0 \rightarrow \mathbb{G}_a(A) \xrightarrow{e_V} \mathbb{G}_a(A) \xrightarrow{\text{res}_{\ker(V)}^A} \mathbb{G}_a(\ker(V)) \rightarrow 0$ is short exact as required.

A more compact way to write down \mathbb{G}_a is via the global presentation

$$\mathbb{G}_a = \mathbb{F}_2[e]/(e_{1,0} + e_{0,1} + e_{1,1}),$$

where e is a free variable in $\mathbb{G}_a(C_2)$ and $e_{1,0}, e_{0,1}, e_{1,1} \in \mathbb{G}_a(C_2 \times C_2)$ are as in the proof of [Lemma 4.4](#).

Example 4.7 (Ordinary 2-torsion formal group laws). A global 2-torsion group law is said to be *complete* if each $X(A)$ is complete with respect to the kernel of the restriction map $X(A) \rightarrow X(1)$. When X is complete, $X(C_2)$ is given by the power series ring $X(1)[[e]]$, see [Remark 4.5](#). Moreover, given $A, B \in \text{el}_2$, the map

$$X(A) \hat{\otimes}_{X(1)} X(B) \rightarrow X(A \times B)$$

is an isomorphism, where the tensor product is completed with respect to the induced topologies on $X(A)$ and $X(B)$ (see [Lemma 4.24](#) for a proof). Choosing a basis V_1, \dots, V_n of A^* , this shows by induction that $X(A)$ is a power series ring over $X(1)$ on the variables e_{V_1}, \dots, e_{V_n} .

The category of complete global 2-torsion group laws is equivalent to the category of non-equivariant 2-torsion formal group laws. The equivalence sends a complete global 2-torsion group law X to the tuple $(X(1), X(C_2), e)$ with comultiplication

$$X(C_2) \xrightarrow{m^*} X(C_2 \times C_2) \xleftarrow{\cong} X(C_2) \hat{\otimes}_{X(1)} X(C_2).$$

In other words, the non-equivariant 2-torsion formal group law associated to X is given by the Euler class $e_{1,1} \in X(C_2 \times C_2) \cong X(1)[[e_{1,0}, e_{0,1}]]$.

Example 4.8 (More general complete Hopf algebras). More examples of the previous sort can be constructed by starting with a triple (k, R, y) where R is a complete topological Hopf algebra over k (with the topology not necessarily induced by the augmentation $R \rightarrow k$) and $y \in R$ a regular generator of the augmentation ideal. In particular, every A -equivariant 2-torsion formal group law for some $A \in \text{el}_2$ gives rise to a global 2-torsion group law. More details are given in [Section 4.5](#).

Many more examples follow from the adjunction defined in the following section, which shows that any el_2 -algebra X with a chosen element $e \in X(C_2)$ can be turned into a global 2-torsion group law in a universal way. For example, there is an initial global 2-torsion group law (which we will later see is isomorphic to $\mathbf{L}^{2\text{-tor}}$, of which we don't yet know that it satisfies the regularity properties of a global 2-torsion group law), a free global 2-torsion group law on a class $x \in X(A)$, or a global 2-torsion group law classifying the choice of two coordinates (see [Section 5.8](#)).

4.2. The universal global 2-torsion group law. In this section we study some formal properties of global 2-torsion group laws. Since we make use of similar constructions throughout the paper, we recall the following standard categorical lemma, see for example [AR94, Theorem 1.39]:

Lemma 4.9. *Let \mathcal{C} be a bicomplete category and $M = (f_i: c_i \rightarrow d_i)_{i \in I}$ a set of morphisms in \mathcal{C} . Let $\mathcal{D} \subseteq \mathcal{C}$ be the subcategory of M -local objects, i.e., those objects d for which $\text{Hom}_{\mathcal{C}}(f_i, d)$ is a bijection for all $i \in I$. Assume further that the sources c_i are small in the sense that $\text{Hom}_{\mathcal{C}}(c_i, -)$ commutes with filtered colimits. Then the inclusion $\mathcal{D} \rightarrow \mathcal{C}$ has a left adjoint l and \mathcal{D} has all colimits and limits.*

In our situation, we take \mathcal{C} to be the following category of $(\mathbb{F}_2)_e$ -algebras:

Definition 4.10. An $(\mathbb{F}_2)_e$ -algebra is an el_2 -algebra

$$X: (\text{elementary abelian 2-groups})^{op} \rightarrow \text{commutative rings},$$

equipped with an element $e \in X(C_2)$ which restricts to 0 at the trivial group.

In other words, an $(\mathbb{F}_2)_e$ -algebra is an el_2 -algebra X together with a map

$$(\mathbb{F}_2)_e = \mathbb{F}_2[e]/(\text{res}_1^{C_2}(e)) \rightarrow X,$$

where $\mathbb{F}_2[e]$ is the free el_2 -algebra over \mathbb{F}_2 on a class e in level C_2 . In particular, the category of $(\mathbb{F}_2)_e$ -algebras is bicomplete, with colimits and limits computed levelwise. For the set of morphisms M in the lemma we take the collection

$$f_{A,V}: (\mathbb{F}_2)_e[x]/(\text{res}_{\ker(V)}^A(x)) \rightarrow (\mathbb{F}_2)_e[y],$$

indexed over all pairs of an elementary abelian 2-group A (or rather a set of representatives of isomorphism classes of such) and a non-trivial element $V \in A^*$. Here, x and y are free variables in level A , and $f_{A,V}$ sends x to $e_V \cdot y$. Note that every global 2-torsion group law is in particular an $(\mathbb{F}_2)_e$ -algebra, since the short exact sequence

$$0 \rightarrow X(C_2) \xrightarrow{e} X(C_2) \xrightarrow{\text{res}_1^{C_2}} X(1) \rightarrow 0$$

(associated to the tautological C_2 -character $\sigma = \text{id}_{C_2}: C_2 \rightarrow C_2$) implies that e restricts to 0 at the trivial group. Moreover, the following holds:

Lemma 4.11. *An $(\mathbb{F}_2)_e$ -algebra X is a global 2-torsion group law if and only if it is local with respect to M .*

Proof. An $(\mathbb{F}_2)_e$ -algebra X is local with respect to a fixed $f_{A,V}$ if and only if every element $x \in X(A)$ which restricts to 0 at $X(\ker(V))$ is uniquely divisible by e_V . Since any $(\mathbb{F}_2)_e$ -algebra satisfies $\text{res}_{\ker(V)}^A(e_V) = 0$, this is precisely the definition of a global 2-torsion group law. \square

Corollary 4.12. *The category of global 2-torsion group laws is a reflective subcategory of the category of $(\mathbb{F}_2)_e$ -algebras and closed under limits. In particular, it has all limits and colimits. We write $\tilde{\mathbf{L}}^{2\text{-tor}}$ for the initial object.*

One can make the construction of the left adjoint $l: ((\mathbb{F}_2)_e\text{-algebras}) \rightarrow GL_{gl}^{2\text{-tor}}$ slightly more explicit than just an existence statement: Given an $(\mathbb{F}_2)_e$ -algebra X , we form a new $(\mathbb{F}_2)_e$ -algebra X_1 by

- (1) quotienting out by all e_V -torsion in $X(A)$, and
- (2) adjoining an element $x' \in X(A)$ and the relation $x' \cdot e_V = x$ for every element x in the kernel of the restriction $\text{res}_{\ker(V)}^A$,

both for all pairs $(A \in \text{el}_2, V \in A^*)$ with V non-trivial. Iterating this process leads to a sequence

$$X \rightarrow X_1 \rightarrow X_2 \rightarrow \dots \rightarrow X_n \rightarrow \dots \quad (4.13)$$

The colimit of this sequence is a model for $l(X)$. Note that this shows that if each $X(A)$ is a countable ring, then so is each $l(X)(A)$. In particular the universal A -torsion formal group law $\tilde{\mathbf{L}}^{2\text{-tor}}$ is countable in each degree, since this is the case for $(\mathbb{F}_2)_e$.

4.2.1. *Gradings.* We say that an $(\mathbb{F}_2)_e$ -algebra (or global 2-torsion group law) is graded if it is equipped with a grading where e is of degree -1 .

Lemma 4.14. *If an $(\mathbb{F}_2)_e$ -algebra X is graded, there is a unique grading on $l(X)$ such that the unit $X \rightarrow l(X)$ respects the gradings. Via these gradings, the adjunction lifts to an adjunction between graded $(\mathbb{F}_2)_e$ -algebras and graded global 2-torsion group laws.*

Proof. This follows from the iterated construction above. If an $(\mathbb{F}_2)_e$ -algebra X is graded, it suffices to check the conditions of a global 2-torsion group law on the homogeneous elements. This means that we can alternatively form the sequence (4.13) above by performing both operations only to the homogeneous elements of $X(A)$. It then follows that each X_i has a unique grading extending the one of X_{i-1} , giving the newly adjoined elements x' satisfying $x' \cdot e_V = x$ the degree one larger than that of x . Consequently, the colimit inherits a unique compatible grading. The adjunction on graded objects is then a formal consequence. \square

Corollary 4.15. *There exists a grading on $\tilde{L}^{2-\text{tor}}$, uniquely determined by the property that e is homogeneous of degree -1 . With this grading, $\tilde{L}^{2-\text{tor}}$ is initial among all graded global 2-torsion group laws.*

4.3. **Global real oriented ring spectra.** In this section we turn to topology and show that the homotopy groups of a global real oriented ring spectrum equipped with the universal Euler class form a graded global 2-torsion group law. For the study of real bordism we restrict the global category to the global family of elementary abelian 2-groups, in the sense of [Sch18, Section 4.3]. We call the resulting spectra *el₂-global spectra*.

Let E be a commutative el₂-global homotopy ring spectrum, i.e., a commutative monoid in the el₂-global homotopy category with respect to the smash product.

Definition 4.16. A *real orientation* of E is an element $t \in E_{C_2}^1(S^\sigma) = \pi_{\sigma-1}^{C_2}(E)$ which is a unit with respect to the $RO(C_2)$ -graded multiplication discussed in Section 3.4 and restricts to $1 \in E^1(S^1)$ at the trivial group.

Every real orientation t gives rise to a unit Thom class $t_V = V^*(t) \in \pi_{V-1}^A(E)$ for any character $V: A \rightarrow C_2$ of an elementary abelian 2-group A . Furthermore, pulling back t along the inclusion of fixed points $S^0 \rightarrow S^\sigma$ yields the universal Euler class $e \in E_{C_2}^1(S^0) = \pi_{-1}^{C_2}(E)$.

Proposition 4.17. *If E is a real oriented el₂-global ring spectrum, then the pair $(\pi_* E, e)$ is a graded 2-torsion global group law.*

Proof. The sphere for the sign-representation σ sits inside a C_2 -cofiber sequence

$$(C_2)_+ \rightarrow S^0 \rightarrow S^\sigma.$$

More generally, given a non-trivial character $V: A \rightarrow C_2$, we can pull back the C_2 -actions via V , yielding a cofiber sequence of based A -spaces

$$(A/\ker(V))_+ \rightarrow S^0 \rightarrow S^V.$$

Applying the A -equivariant cohomology theory E_A to this cofiber sequence gives a long exact sequence

$$\dots \rightarrow E_A^*(S^V) \rightarrow E_A^*(S^0) \rightarrow E_A^*((A/\ker(V))_+) \rightarrow \dots$$

The induction isomorphism from Section 3.2 implies that $E_A^*((A/\ker(V))_+)$ identifies with $E_{\ker(V)}^*(S^0)$, and the resulting map $E_A^*(S^0) \rightarrow E_{\ker(V)}^*(S^0)$ equals the restriction $\text{res}_{\ker(V)}^A$ (Lemma 3.2). Since E is a global theory and $\ker(V) \hookrightarrow A$ has a section, this restriction map also has a section and it follows that the long exact sequence collapses into a collection of (split) short exact sequences. Moreover, the real orientation of E yields an identification $E_A^*(S^V) \cong E_A^{*-1}(S^0)$, and the map $E_A^{*-1}(S^0) \rightarrow E_A^*(S^0)$ becomes multiplication with the Euler class (since it is by definition the composition of the inclusion $S^0 \hookrightarrow S^V$ and the Thom class). Putting everything together and replacing the cohomological notation $E_A^*(S^0)$ with the homological $\pi_*^A(E)$, we see that the sequence

$$0 \rightarrow \pi_{*+1}^A(E) \xrightarrow{e_V \cdot} \pi_*^A(E) \xrightarrow{\text{res}_{\ker(V)}^A} \pi_*^{\ker(V)}(E) \rightarrow 0$$

is exact, as desired. \square

Remark 4.18. For $A = C_2$, $V = \sigma$ the sign-representation and $E = \mathbf{MO}$ the real equivariant bordism spectrum (see below), the above argument first appeared in [Sin02, Theorem 2.2]. Generally, the Gysin sequence relating the restriction map $\text{res}_1^{C_2}$ to the Euler class $S^0 \rightarrow S^\sigma$ is a standard tool in equivariant stable homotopy theory, going back at least to [Bre67].

We are particularly interested in the case where E is the universal global real oriented theory \mathbf{MO} , or its periodic version $\mathbf{MOP} = \sum_{i \in \mathbb{Z}} (\mathbf{MO} \wedge S^i)$. These are discussed in detail in [Sch18, Section 6.1]. In particular, the real orientation of \mathbf{MOP} is introduced in [Sch18, Construction 6.1.15]. Note that there is an isomorphism $\pi_0(\mathbf{MOP}) \cong \bigoplus_{n \in \mathbb{Z}} \pi_n(\mathbf{MO})$. Hence, $\pi_0(\mathbf{MOP})$ carries an ungraded global 2-torsion group law, which we will show to be the universal one. In this sense, the difference between \mathbf{MOP} and \mathbf{MO} corresponds roughly to the difference between ungraded and graded group laws on the algebraic side.

4.3.1. Relation to real oriented A -ring spectra. Some words are in order on how this story relates to real orientations at a fixed group A . For this we recall the notion of a real orientation of an A -ring spectrum. Let $\mathcal{U}_A^{\mathbb{R}}$ denote a complete real A -universe, i.e., a sum of countable infinitely many copies of each character $V \in A^*$. Let $\mathbb{R}P(\mathcal{U}_A^{\mathbb{R}})$ be its real projective space. The tautological line bundle γ over $\mathbb{R}P(\mathcal{U}_A^{\mathbb{R}})$ is universal among all line bundles in A -spaces. The Thom space of γ is again A -homeomorphic to $\mathbb{R}P(\mathcal{U}_A^{\mathbb{R}})$. Now recall (see [CGK00, Definition 4.1] or [Gre01, Definition 9.5] for the complex case) that a real orientation of a commutative A -homotopy-ring spectrum Y is a class $t^{(A)} \in Y^1(\mathbb{R}P(\mathcal{U}_A^{\mathbb{R}}))$ whose restriction to $Y^1(\mathbb{R}P(\epsilon \oplus V))$ for a character $V \in A^*$ is given by

$$\begin{cases} 1 & \text{if } V = \epsilon \\ \text{unit} & \text{otherwise.} \end{cases}$$

Here, the condition that the restriction of $t^{(A)}$ to $Y^1(\mathbb{R}P(\epsilon \oplus V))$ is a unit is again meant in the $RO(A)$ -graded sense, identifying $\mathbb{R}P(\epsilon \oplus V)$ with S^V by sending $v \in V$ to $[1 : v]$ and ∞ to $[0 : 1]$. Our definition differs slightly from the one given in [CGK00] in that they allow the restriction to the trivial sphere S^1 to be an arbitrary unit, not necessarily the element 1.

The restriction $y(\epsilon)$ of $t^{(A)}$ to $Y^1(\mathbb{R}P(\mathcal{U}_A^{\mathbb{R}})_+)$ via the 0-section is called the *universal Euler class* or *coordinate* for Y . Given such a coordinate, the quintuple

$$F(Y) = (Y^*(S^0), Y^*(\mathbb{R}P(\mathcal{U}_A^{\mathbb{R}})_+), \Delta, \theta, y(\epsilon))$$

is a graded A -equivariant formal group law, where

$$\Delta: Y^*(\mathbb{R}P(\mathcal{U}_A^{\mathbb{R}})_+) \rightarrow Y^*((\mathbb{R}P(\mathcal{U}_A^{\mathbb{R}}) \times \mathbb{R}P(\mathcal{U}_A^{\mathbb{R}}))_+) \cong Y^*(\mathbb{R}P(\mathcal{U}_A^{\mathbb{R}})_+) \hat{\otimes}_{Y^*} Y^*(\mathbb{R}P(\mathcal{U}_A^{\mathbb{R}})_+)$$

is induced from the multiplication map $\mathbb{R}P(\mathcal{U}_A^{\mathbb{R}}) \times \mathbb{R}P(\mathcal{U}_A^{\mathbb{R}}) \rightarrow \mathbb{R}P(\mathcal{U}_A^{\mathbb{R}})$ classifying the tensor product of A -equivariant real line bundles, and

$$\theta(V): Y^*(\mathbb{R}P(\mathcal{U}_A^{\mathbb{R}})_+) \rightarrow Y^*(S^0)$$

is induced from the map $S^0 \rightarrow \mathbb{R}P(\mathcal{U}_A^{\mathbb{R}})_+$ classifying V thought of as an A -equivariant line bundle over a point. The proof that $F(Y)$ is indeed an A -equivariant formal group law is analogous to the complex version proved in [CGK00]. Since the tensor product of any A -equivariant real line bundle with itself is trivial, it follows that $F(Y)$ is 2-torsion.

The link from the global definition to this one is via the following observation: The unit sphere $S(\mathcal{U}_A^{\mathbb{R}})$ is a model for the universal C_2 -space $E_A C_2$ in A -spaces, where C_2 acts by multiplication with -1 . Moreover, applying the unreduced (C_2, A) -homotopy orbit construction (see Section 3.3) to the C_2 -line bundle $\sigma \rightarrow *$ (with $A \times C_2$ acting through the projection to C_2) via this model yields the tautological A -line bundle γ_A . It follows that $(S^\sigma)_{h_A C_2}$ is equivalent to the Thom space $\mathbb{R}P(\mathcal{U}_A^{\mathbb{R}})$ of γ_A . Hence, given a real orientation t of an el_2 -global ring spectrum E , we can apply the homotopy orbit construction $h_{A, C_2}(S^\sigma)$ of Section 3.3 to obtain an element $t^{(A)} = h_{A, C_2}(S^\sigma)(t) \in E_A^1(\mathbb{R}P(\mathcal{U}_A^{\mathbb{R}}))$.

Lemma 4.19. *The element $t^{(A)}$ is a real orientation of E_A .*

Proof. We have to show that the restriction of $t^{(A)}$ to $E_A^1(\mathbb{R}P(\epsilon \oplus V))$ equals 1 when $V = \epsilon$ and an $RO(A)$ -graded unit otherwise. Under the A -homeomorphisms $\mathbb{R}P(\epsilon \oplus V) \cong S^V$ and $\mathbb{R}P(\mathcal{U}_A^{\mathbb{R}}) \cong S(\mathcal{U}_A^{\mathbb{R}})_+ \wedge_{C_2} (S^\sigma)$, the inclusion $\mathbb{R}P(\epsilon \oplus V) \rightarrow \mathbb{R}P(\mathcal{U}_A^{\mathbb{R}})$ corresponds to the map $S^V = V^*(S^\sigma) \rightarrow S(\mathcal{U}_A^{\mathbb{R}})_+ \wedge_{C_2} (S^\sigma)$ associated

to a $\Gamma(V)$ -fixed point $*_V = v \in S(\mathcal{U}_A^{\mathbb{R}})$ spanning a character isomorphic to V . Hence, using the notation of [Section 3.3](#), the restriction of $t^{(A)}$ is given by the image of t under the composition

$$E_{C_2}^1(S^\sigma) \xrightarrow{\text{pr}_{C_2}^*} E_{A \times C_2}^1(\text{pr}_{C_2}^* S^\sigma) \xrightarrow{h_{A,C_2}(\text{pr}_{C_2}^* S^\sigma)} E_A^1((S^\sigma)_{h_{A,C_2}}) \xrightarrow{[*_V, -]^*} E_A^1(S^V).$$

By [Lemma 3.4](#), this composition equals restriction along $V: A \rightarrow C_2$. The restriction map V^* is compatible with the $RO(A)$ -graded multiplications, hence it takes t to a unit in general, and to 1 if $V = \epsilon$ since that is required of a global orientation. This finishes the proof. \square

Hence, for every real oriented el_2 -global ring spectrum E we have a global 2-torsion group law $(\pi_*(E), e)$, as well as an A -equivariant 2-torsion formal group law $F(E_A)$ for every elementary abelian 2-group A . For the rest of this subsection we explain that each $F(E_A)$ can be constructed out of $(\pi_*(E), e)$ purely algebraically.

We start by applying the argument from the proof of [Lemma 4.19](#) above to the base space instead of the total space of the universal line bundle. This shows that there is a commutative diagram

$$\begin{array}{ccc} \pi_*^{C_2}(E) & \xrightarrow{\text{pr}_{C_2}^*} \pi_*^{A \times C_2}(E) & \xrightarrow{h_{A,C_2}} E_A^{-*}(\mathbb{R}P(\mathcal{U}_A^{\mathbb{R}})_+) \\ & \searrow V^* & \downarrow (\text{id}_A, V)^* \\ & & \pi_*^A(E) \xrightarrow{\cong} E_A^{-*} \end{array} \quad (4.20)$$

Furthermore, naturality implies that the comultiplication map on $E_A^{-*}(\mathbb{R}P(\mathcal{U}_A^{\mathbb{R}})_+)$ fits into the diagram

$$\begin{array}{ccccc} \pi_*^{A \times C_2 \times C_2}(E) & \xrightarrow{h_{A,C_2 \times C_2}} & E_A^{-*}(\mathbb{R}P(\mathcal{U}_A^{\mathbb{R}}) \times \mathbb{R}P(\mathcal{U}_A^{\mathbb{R}})_+) & \xrightarrow{\cong} & E_A^{-*}(\mathbb{R}P(\mathcal{U}_A^{\mathbb{R}})_+) \hat{\otimes} E_A^{-*}(\mathbb{R}P(\mathcal{U}_A^{\mathbb{R}})_+) \\ (\text{id}_A \times m)^* \uparrow & & \uparrow m^* & \nearrow \Delta & \\ \pi_*^{A \times C_2}(E) & \xrightarrow{h_{A,C_2}} & E_A^{-*}(\mathbb{R}P(\mathcal{U}_A^{\mathbb{R}})) & & \\ \text{pr}_A^* \uparrow & & \uparrow & & \\ \pi_*^A(E) & \xrightarrow{\cong} & E_A^{-*} & & \end{array} \quad (4.21)$$

Now let $I_V \subseteq \pi_*^{A \times C_2}(E)$ denote the kernel of the augmentation $(\text{id}_A, V)^*: \pi_*^{A \times C_2}(E) \rightarrow \pi_*^A(E)$, and denote by $(\pi_*^{A \times C_2}(E))_A^\wedge$ the completion at finite products of the I_V with $V \in A^*$. More generally, we write $(\pi_*^{A \times B}(E))_A^\wedge$ for the completion at finite products of the kernels of the maps $(\text{id}_A, \alpha)^*: \pi_*^{A \times B}(E) \rightarrow \pi_*^A(E)$ with $\alpha: A \rightarrow B$. Then we have the following:

Proposition 4.22. *Let E be a real oriented el_2 -global ring spectrum. Then the map*

$$h_{A,C_2}: \pi_*^{A \times C_2}(E) \rightarrow E_A^{-*}(\mathbb{R}P(\mathcal{U}_A^{\mathbb{R}})_+)$$

sends an Euler class of the form $e_{(V,\text{id})}$ to $y(V)$ and induces an isomorphism

$$(\pi_*^{A \times C_2}(E))_A^\wedge \xrightarrow{\cong} E_A^{-*}(\mathbb{R}P(\mathcal{U}_A^{\mathbb{R}})_+).$$

More generally, the homotopy orbit maps induce isomorphisms

$$(\pi_*^{A \times (C_2)^{\times n}}(E))_A^\wedge \xrightarrow{\cong} E_A^{-*}(\mathbb{R}P(\mathcal{U}_A^{\mathbb{R}})_+^{\times n}) \xrightarrow{\cong} E_A^{-*}(\mathbb{R}P(\mathcal{U}_A^{\mathbb{R}})_+)^{\hat{\otimes} n}.$$

Proof. The 0-section $\mathbb{R}P(\mathcal{U}_A^{\mathbb{R}})_+ \rightarrow \mathbb{R}P(\mathcal{U}_A^{\mathbb{R}})$ is obtained by applying the (C_2, A) -homotopy orbit construction to the C_2 -map $S^0 \rightarrow S^\sigma$. Hence, the universal Euler class $y(\epsilon)$ is the image of $\text{pr}_{C_2}^*(e) = e_{(\epsilon, \text{id})}$. For the other characters V recall that the A^* -action l_V on $E_A^{-*}(\mathbb{R}P(\mathcal{U}_A^{\mathbb{R}})_+)$ is given by the composite

$$E_A^{-*}(\mathbb{R}P(\mathcal{U}_A^{\mathbb{R}})_+) \xrightarrow{\Delta} E^{-*}(\mathbb{R}P(\mathcal{U}_A^{\mathbb{R}})_+) \hat{\otimes} E^{-*}(\mathbb{R}P(\mathcal{U}_A^{\mathbb{R}})_+) \xrightarrow{\theta(V) \hat{\otimes} \text{id}} E_A^{-*}(\mathbb{R}P(\mathcal{U}_A^{\mathbb{R}})_+).$$

The commutative diagrams (4.20) and (4.21) hence imply that the homotopy orbit map h_{A,C_2} is A^* -equivariant for the A^* -action on $\pi_*^{A \times C_2}(E)$ defined via the composite

$$\pi_*^{A \times C_2}(E) \xrightarrow{(\text{id}_A \times m)^*} \pi_*^{A \times C_2 \times C_2}(E) \xrightarrow{((\text{id}_A, V) \times \text{id}_{C_2})^*} \pi_*^{A \times C_2}(E),$$

or in other words by $l_V = \begin{pmatrix} \text{id}_A & 0 \\ V & \text{id}_{C_2} \end{pmatrix}^* : \pi_*^{A \times C_2}(E) \rightarrow \pi_*^{A \times C_2}(E)$. It follows that $e_{(V, \text{id})} = l_V(e_{(\epsilon, \text{id})})$ is sent to $y(V) = l_V(y(\epsilon))$.

Knowing this, the two statements about completion follow from a more general statement (Lemma 4.24) about completions of the form $X(A \times B)_A^\wedge$ for arbitrary global 2-torsion group laws X , which we will prove in the next section. Indeed, Lemma 4.24 together with the first part of the proposition guarantees that the map $\pi_*^{A \times C_2}(E) \rightarrow E_A^{-*}(\mathbb{R}P(\mathcal{U}_A^\mathbb{R})_+)$ is an isomorphism modulo $I_{V_1} \cdots I_{V_n}$ (it sends the basis $1, e_{(V_1, \text{id})}, e_{(V_2, \text{id})}e_{(V_1, \text{id})}, \dots, e_{(V_{n-1}, \text{id})} \cdots e_{(V_1, \text{id})}$ to the basis $1, y(V_1), y(V_2)y(V_1), \dots, y(V_{n-1}) \cdots y(V_1)$). Since we know that the target $E_A^{-*}(\mathbb{R}P(\mathcal{U}_A^\mathbb{R})_+^{\times n})$ is complete, the map must be a completion. Part (2) of Lemma 4.24 then gives the statement about higher $A \times C_2^{\times n}$. \square

Remark 4.23. Alternatively, the proposition has the following more topological proof: We can write

$$E_A^{-*}(\mathbb{R}P(\mathcal{U}_A^\mathbb{R})_+) \cong \lim_{\text{fin. dim. } W \subset \mathcal{U}_A^\mathbb{R}} E_A^{-*}(\mathbb{R}P(W)_+) \cong \lim_{V_1, \dots, V_i \in A^*} E_A^{-*}(\mathbb{R}P(V_1 \oplus \dots \oplus V_i)_+).$$

Let $W = V_1 \oplus \dots \oplus V_i$. The projective A -space $\mathbb{R}P(W)$ is the C_2 -orbit space of the unit sphere $S(W \otimes \sigma)$ of the $(A \times C_2)$ -representation $W \otimes \sigma$. Since the C_2 -action on the unit sphere is free, the induction isomorphism (Section 3.2) yields an identification

$$E_A^{-*}(\mathbb{R}P(W)_+) \cong E_{A \times C_2}^{-*}(S(W \otimes \sigma)_+).$$

There is an $(A \times C_2)$ -cofiber sequence

$$S(W \otimes \sigma)_+ \rightarrow S^0 \rightarrow S^{W \otimes \sigma}$$

which induces a long exact sequence

$$\dots \rightarrow E_{A \times C_2}^{-*}(S^{W \otimes \sigma}) \rightarrow E_{A \times C_2}^{-*} \rightarrow E_{A \times C_2}^{-*}(S(W \otimes \sigma)_+) \rightarrow E_{A \times C_2}^{-*+1}(S^{W \otimes \sigma}) \rightarrow \dots$$

The orientation gives an isomorphism $E_{A \times C_2}^{-*}(S^{W \otimes \sigma}) \cong E_{A \times C_2}^{-*- \dim W}$, under which the map $E_{A \times C_2}^{-*}(S^{W \otimes \sigma}) \rightarrow E_{A \times C_2}^{-*}$ becomes multiplication with the Euler class $e_{W \otimes \sigma}$. By the decomposition into characters $W \otimes \sigma \cong (V_1 \otimes \sigma) \oplus \dots \oplus (V_i \otimes \sigma)$, the Euler class $e_{W \otimes \sigma}$ factors as the product $e_{W \otimes \sigma} = e_{V_1 \otimes \sigma} \cdots e_{V_i \otimes \sigma}$, or in our previous notation $e_{W \otimes \sigma} = e_{(V_1, \text{id})} \cdots e_{(V_i, \text{id})}$. Since all these Euler classes are regular elements, the long exact sequence above decomposes into short exact sequences, yielding an isomorphism

$$E_A^{-*}(\mathbb{R}P(W)_+) \cong E_{A \times C_2}^{-*}(S(W \otimes \sigma)_+) \cong E_{A \times C_2}^{-*}/e_{W \otimes \sigma} \cong \pi_*^{A \times C_2}(E)/e_{(V_1, \text{id})} \cdots e_{(V_i, \text{id})},$$

which upon passing to the limit over all tuples (V_1, \dots, V_i) yields the desired statement. An iteration of this argument can be used for $A \times C_2^{\times n}$.

Together with the diagrams (4.20) and (4.21), Proposition 4.22 shows that the full structure of the graded A -equivariant formal group $F(E_A)$ can be obtained from the global homotopy groups $\pi_*(E)$ by passing to a suitable completion. The observation that $F(Y)$ can often be described as the completion of a simpler structure if Y is part of a family of equivariant spectra first appeared in [CGK00, Section 7], predating a rigorous definition of a global spectrum.

Note moreover that given $V \in A^*$, the Euler class $e_V \in \pi_*^A(E)$ defined by pulling back the coordinate $e \in \pi_*^{C_2}(E)$ along V agrees with the Euler class $e_V \in E_A^*(S^0) = \pi_*^A(E)$ associated to $F(E_A)$. Hence, the Euler classes in $\pi_*(E)$ play two different roles: Those in $\pi_*^A(E)$ correspond to the Euler classes of $F(E_A)$, while the ones of the form $e_{(V, \text{id})} \in \pi_*^{A \times C_2}(E)$ are taken to the coordinates $y(V)$ of $F(E_A)$ under the completion map. As we will see in the next section, this interplay of Euler classes and coordinates exists more generally for global 2-torsion group laws and their associated A -equivariant formal group laws. It ultimately allows us to derive the regularity of universal Euler classes e_V from the regularity of coordinates $y(V)$, which is required by the axioms of an A -equivariant formal group law.

4.4. From global group laws to A -equivariant formal group laws. We now mimic these topological constructions algebraically to associate an A -equivariant 2-torsion formal group law X_A^\wedge to an arbitrary global 2-torsion group law X .

Let X be a global 2-torsion group law, and A an elementary abelian 2-group. Given another $B \in \text{el}_2$ and homomorphism $\alpha: A \rightarrow B$, we write I_α for the kernel of $(\text{id}, \alpha)^*: X(A \times B) \rightarrow X(A)$. We then set $X(A \times B)_A^\wedge$ to be the completion of $X(A \times B)$ at finite products of the ideals I_α .

Lemma 4.24. *Let X be a global 2-torsion group law and $A \in \text{el}_2$.*

- (1) *Given n characters $V_1, \dots, V_n \in A^*$, the quotient $X(A \times C_2)/(I_{V_1} \cdot \dots \cdot I_{V_n})$ is free of rank n over $X(A)$, and a basis is given by*

$$1, e_{(V_1, \text{id})}, e_{(V_2, \text{id})} e_{(V_1, \text{id})}, \dots, e_{(V_{n-1}, \text{id})} \cdots e_{(V_1, \text{id})}.$$

- (2) *Given $B, B' \in \text{el}_2$, the map*

$$X(A \times B) \otimes_{X(A)} X(A \times B') \xrightarrow{(\text{pr}_{A \times B}^*, \text{pr}_{A \times B'}^*)} X(A \times B \times B')$$

induces an isomorphism

$$X(A \times B)_A^\wedge \hat{\otimes}_{X(A)} X(A \times B')_A^\wedge \xrightarrow{\cong} X(A \times B \times B')_A^\wedge.$$

Proof. For Part (1), note that given $V \in A^*$ there is a short exact sequence of groups

$$1 \rightarrow A \xrightarrow{(\text{id}, V)} A \times C_2 \xrightarrow{(V, \text{id})} C_2 \rightarrow 1$$

that is split by the projection $\text{pr}_A: A \times C_2 \rightarrow A$. Therefore,

$$0 \rightarrow X(A \times C_2) \xrightarrow{e_{(V, \text{id})}^*} X(A \times C_2) \xrightarrow{(\text{id}, V)^*} X(A) \rightarrow 0$$

is also short exact by the axioms of a global 2-torsion group law, and split by pr_A^* . It follows that every element $x \in X(A \times C_2)$ can be written uniquely as $x = \text{pr}_A^*(x_0) + x_1 \cdot e_{(V, \text{id})}$ with $x_0 \in X(A)$. Given that I_V is generated by $e_{(V, \text{id})}$, Part (1) follows by an easy inductive argument.

Part (2): By choosing a basis we can assume that $B' = (C_2)^{\times n}$. Via induction on n we can further reduce to the case $n = 1$, hence we are considering the map

$$X(A \times B) \otimes_{X(A)} X(A \times C_2) \xrightarrow{(\text{pr}_{A \times B}^*, \text{pr}_{A \times C_2}^*)} X(A \times B \times C_2).$$

Now take $(\alpha, V): A \rightarrow B \times C_2$ and let $x \in I_{(\alpha, V)} \subseteq X(A \times B \times C_2)$. We consider the map $i_V: A \times B \rightarrow A \times B \times C_2$ whose first two components are the projections to A and B , and whose last component equals $A \times B \rightarrow A \xrightarrow{V} C_2$. Then the kernel of $i_V^*: X(A \times B \times C_2) \rightarrow X(A \times B)$ is generated by the Euler class $e_{(V, \epsilon, \text{id})}$ associated to the map $(V, \epsilon, \text{id}): A \times B \times C_2 \rightarrow C_2$. Now the element $x_0 = (i_V)^*(x) \in X(A \times B)$ lies in $I_\alpha \subseteq X(A \times B)$, and $x - \text{pr}_{A \times B}^*(x_0)$ is an element of the kernel of $(i_V)^*$, hence it is divisible by $e_{(V, \epsilon, \text{id})}$. So we see that $I_{(\alpha, V)}$ is generated by $\text{pr}_{A \times B}^*(I_\alpha)$ and $e_{(V, \epsilon, \text{id})}$. Therefore we can form the completion in two steps: Completing $X(A \times B \times C_2)$ at the Euler classes $e_{(V, \epsilon, \text{id})}$ for $V: A \rightarrow C_2$ yields $X(A \times B) \hat{\otimes}_{X(A)} X(A \times C_2)_A^\wedge$, and further completing at the ideal generated by $\text{pr}_{A \times B}^*(I_\alpha)$ gives $X(A \times B)_A^\wedge \hat{\otimes}_{X(A)} X(A \times C_2)_A^\wedge$, as desired. \square

We obtain an A -equivariant 2-torsion formal group law X_A^\wedge as the tuple

$$(X(A), X(A \times C_2)_A^\wedge, \Delta, \theta, y(\epsilon)),$$

where

- Δ is the induced map on completion by the composite

$$X(A \times C_2) \xrightarrow{(\text{id}, m)^*} X(A \times C_2 \times C_2) \rightarrow X(A \times C_2 \times C_2)_A^\wedge \cong X(A \times C_2)_A^\wedge \hat{\otimes}_{X(A)} X(A \times C_2)_A^\wedge.$$

- θ is the induced map on completion by the map $X(A \times C_2) \rightarrow X(A)^{A^*}$ whose V -th component is the map $(\text{id}, V)^*: X(A \times C_2) \rightarrow X(A)$.
- $y(\epsilon)$ is the image of $e_{(\epsilon, \text{id})}$ under the completion map $X(A \times C_2) \rightarrow X(A \times C_2)_A^\wedge$.

Checking that this is indeed an A -equivariant 2-torsion formal group law is straightforward: Coassociativity, cocommutativity and counitality of Δ follow directly from the corresponding properties of $m: C_2 \times C_2 \rightarrow C_2$, and compatibility of Δ and θ follows from the commutative diagram

$$\begin{array}{ccc} X(A \times C_2) & \xrightarrow{(\text{id}, VW)^*} & X(A) \\ (\text{id}_A, m)^* \downarrow & \nearrow (\text{id}_A, V, W)^* & \\ X(A \times C_2 \times C_2) & & \end{array}$$

The topology on $X(A \times C_2)_A^\wedge$ is by definition described by the kernels of θ , and regularity of $y(\epsilon)$ is an immediate consequence of Part (1) of [Lemma 4.24](#), since the coefficients of $y(\epsilon) \cdot x$ with respect to a flag F starting with ϵ are given by the shift of the coefficients of x with respect to the flag F' obtained by removing ϵ at the beginning of F .

Finally, the discussion in [Section 4.3.1](#) shows that for an el_2 -global real oriented ring spectrum E with associated global 2-torsion group law $\pi_* E$, the A -equivariant formal group law $(\pi_* E)_A^\wedge$ is isomorphic to the one associated to the induced real orientation of E_A .

4.5. From A -equivariant 2-torsion formal group laws to global 2-torsion group laws. Having seen how to assign an A -equivariant formal group law to a global group law, we now study how to go the other way, as indicated in [Example 4.8](#). Throughout this section we fix an A -equivariant 2-torsion formal group law $F = (k, R, \Delta, \theta, y(\epsilon))$, to which we want to associate a global 2-torsion group law F_{gl} . The idea is simple: We want to set $F_{gl}(1) = k$, $F_{gl}(C_2) = R$, $F_{gl}(C_2 \times C_2) = R \hat{\otimes} R$ etc., and use the comultiplication Δ to define the functoriality in the groups.

To make this precise, we use the following coordinate-free description. Let \mathcal{C}_k denote the category of complete commutative linear topological k -algebras, i.e., commutative topological k -algebras S which have a basis $\{U_i\}_{i \in I}$ of open neighborhoods of 0 consisting of ideals U_i such that the induced map

$$S \rightarrow \lim_{i \in I} S/U_i$$

is an isomorphism (it can be shown that this condition is independent of the chosen basis of ideals). Given two objects $S, S' \in \mathcal{C}_k$, we denote by $S \hat{\otimes} S'$ their completed tensor product over k . Choosing ideal neighborhood bases $\{U_i\}_{i \in I}$ and $\{V_j\}_{j \in J}$ for S and S' respectively, $S \hat{\otimes} S'$ can be computed as

$$S \hat{\otimes} S' = \lim_{i \in I, j \in J} (S/U_i \otimes S'/V_j)$$

equipped with the limit topology (with each $S/U_i \otimes S'/V_j$ discrete). The completed tensor product $-\hat{\otimes}-$ is the coproduct in \mathcal{C}_k , making the complete Hopf-algebra R a commutative cogroup object. Hence, given an object $S \in \mathcal{C}_k$, we obtain a natural abelian group structure on the set $R(S)$ of morphisms $R \rightarrow S$ via the formula

$$f + g = (R \xrightarrow{\Delta} R \hat{\otimes} R \xrightarrow{f \hat{\otimes} g} S \hat{\otimes} S \xrightarrow{m} S),$$

with neutral element

$$R \xrightarrow{\theta(\epsilon)} k \rightarrow S.$$

Since comultiplication by 2 is trivial on R , the abelian groups $R(S)$ are in fact \mathbb{F}_2 -vector spaces.

Now let B be an elementary abelian 2-group (or in other words a finite dimensional \mathbb{F}_2 -vector space). Then we have the following:

Lemma 4.25. *The functor*

$$\text{Hom}(B^*, R(-)): \mathcal{C}_k \rightarrow \text{Vect}_{\mathbb{F}_2}$$

is representable.

Proof. Choosing a basis of B , we find that there is a natural isomorphism

$$\text{Hom}(B^*, R(S)) \cong R(S)^{\times n}$$

for some n , which implies that $R^{\hat{\otimes} n}$ is a representing object. □

We let $F_{gl}(B)$ denote a choice of representing object. Since the assignment $\text{Hom}(B^*, R(S))$ is also covariantly functorial in B , it follows that $F_{gl}(-)$ extends canonically to a contravariant functor from elementary abelian 2-groups to commutative k -algebras, in particular to an el_2 -algebra. Note that for any $B, B' \in \text{el}_2$ the canonical map

$$F_{gl}(B) \hat{\otimes} F_{gl}(B') \rightarrow F_{gl}(B \times B')$$

is an isomorphism, since $\text{Hom}((B \times B')^*, R(-))$ is naturally isomorphic to $\text{Hom}(B^*, R(-)) \times \text{Hom}((B')^*, R(-))$.

Lemma 4.26. *The element $y(\epsilon) \in R \cong F_{gl}(C_2)$ is a coordinate for F_{gl} .*

Proof. Let $V \in B^*$ be a non-trivial character. Extending V to a basis V_2, \dots, V_n of B^* , we see that $V^*(y(\epsilon))$ identifies with $y(\epsilon) \otimes 1_{R^{\hat{\otimes}(n-1)}}$ under the induced isomorphism $F_{gl}(B) \cong R^{\hat{\otimes}n}$, and restriction to $\ker(V)$ equals the map

$$\theta(\epsilon) \hat{\otimes} \text{id}_{R^{\hat{\otimes}(n-1)}} : R^{\hat{\otimes}n} \rightarrow R^{\hat{\otimes}(n-1)}.$$

By assumption, the sequence $0 \rightarrow R \xrightarrow{y(\epsilon)} R \xrightarrow{\theta(\epsilon)} k \rightarrow 0$ is split exact, hence it remains so after tensoring with $R^{\hat{\otimes}(n-1)}$, which finishes the proof. \square

Remark 4.27. Note that this construction works more generally for any 2-torsion commutative cocommutative complete linear topological Hopf-algebra R over k together with a regular element generating the augmentation ideal.

Remark 4.28. Like $(-)_A^\wedge$, the functor $(-)_{gl}$ also has a topological analog: The functor that assigns to a global spectrum an A -equivariant spectrum has a lax symmetric monoidal right adjoint r_A (see [Sch18, Theorem 4.5.24]). A real orientation of an A -ring spectrum Y induces a global real orientation on $r_A Y$, and one can show that there is an isomorphism

$$\pi_*(r_A Y) \cong F(Y)_{gl},$$

where $F(Y)$ is the A -equivariant formal group law associated to the real orientation of Y .

4.5.1. Extended functoriality. There is one piece of structure of F that we have not made use of yet: The augmentation map θ (or rather the components $\theta(V)$ with $V \neq \epsilon$). This additional structure can be interpreted as additional functoriality of F_{gl} , in the following way: Sending $V \in A^*$ to the map $R \xrightarrow{\theta(V)} k$ defines a group homomorphism $A^* \rightarrow R(k)$. Since k is initial, this means that $R(S)$ is in fact naturally an \mathbb{F}_2 -vector space under A^* , and we can rewrite the defining property of $F_{gl}(B)$ as

$$\text{map}(F_{gl}(B), S) \cong \text{Hom}(B^*, R(S)) \cong \text{Hom}_{A^*/}(A^* \times B^*, R(S)),$$

the set of group homomorphisms from $A^* \times B^*$ to $R(S)$ under A^* . The upshot is that there is an induced map $F_{gl}(B) \rightarrow F_{gl}(C)$ for any map $A \times C \rightarrow A \times B$ that lies over the projections to A , or in other words for any map $C \times A \rightarrow B$, not only for maps $C \rightarrow B$. Using this extended functoriality, the full structure of F can be reconstructed from F_{gl} : k is given by $F_{gl}(1)$, R by $F_{gl}(C_2)$, the comultiplication identifies with $(\text{id}_A \times m)^* : F_{gl}(C_2) \rightarrow F_{gl}(C_2 \times C_2)$, the maps $\theta(V)$ correspond to $(\text{id}_A, V)^* : F_{gl}(C_2) \rightarrow F_{gl}(1)$ and the class $y(\epsilon)$ is just the coordinate of F_{gl} .

Note that for a global 2-torsion group law X with associated A -equivariant formal group law X_A^\wedge , Lemma 4.24 shows that there is a natural isomorphism

$$(X_A^\wedge)_{gl}(B) \cong X(A \times B)_A^\wedge.$$

Moreover, the additional functoriality of $(X_A^\wedge)_{gl}$ in maps over A agrees with the apparent one on $X(A \times -)$ induced on completion.

4.6. The adjunction. We now put the two constructions of the previous sections together and show:

Proposition 4.29. *The functors $(-)^\wedge_A$ and $(-)_{gl}$ form an adjunction*

$$(-)_A^\wedge : GL_{gl}^{2\text{-tor}} \rightleftarrows FGL_A^{2\text{-tor}} : (-)_{gl}.$$

The unit $X \rightarrow (X_A^\wedge)_{gl}$ is given in level B by the composite

$$X(B) \xrightarrow{\text{pr}_B^*} X(A \times B) \rightarrow X(A \times B)_A^\wedge \cong (X_A^\wedge)_{gl}(B),$$

and the counit $(F_{gl})_A^\wedge \rightarrow F$ is the pair of maps $F_{gl}(A) \rightarrow F_{gl}(1) \cong k$ and $F_{gl}(A \times C_2)_A^\wedge \rightarrow F_{gl}(C_2) \cong R$ induced by the diagonals $\text{diag} : A \rightarrow A \times A$ and $\text{diag} \times \text{id}_{C_2} : A \times C_2 \rightarrow A \times A \times C_2$ respectively, using the extended functoriality of F_{gl} .

Proof. The adjunction is most easily described as a composite of two adjunctions of the following form:

$$\begin{array}{ccccc}
 & & (-)^\wedge_A & & \\
 & \swarrow & & \searrow & \\
 GL_{gl}^{2-\text{tor}} & \xrightarrow{(-)_A} & GL_{gl/A}^{2-\text{tor}} & \xrightarrow{(-)^\wedge} & FGL_A^{2-\text{tor}} \\
 & \nwarrow & & \nearrow & \\
 & & (-)_{gl} & &
 \end{array}$$

$u: GL_{gl/A}^{2-\text{tor}} \rightarrow GL_{gl}^{2-\text{tor}}$ $(-)_{gl/A}: FGL_A^{2-\text{tor}} \rightarrow GL_{gl/A}^{2-\text{tor}}$

Here, the intermediate category $GL_{gl/A}^{2-\text{tor}}$, whose objects one might call ‘ A -global 2-torsion group laws’, is the category of global 2-torsion group laws Y with extended functoriality as described at the end of the previous section. This means that there are restriction maps $Y(B) \rightarrow Y(C)$ for any map $A \times C \rightarrow A \times B$ over A . The categories $GL_{gl/A}^{2-\text{tor}}$ and $GL_{gl}^{2-\text{tor}}$ are related by an adjunction, where the right adjoint $u: GL_{gl/A}^{2-\text{tor}} \rightarrow GL_{gl}^{2-\text{tor}}$ forgets the extra functoriality, and the left adjoint $(-)^\wedge_A: GL_{gl}^{2-\text{tor}} \rightarrow GL_{gl/A}^{2-\text{tor}}$ is defined via $X_A(B) = X(A \times B)$. The unit $X \rightarrow u(X_A)$ is then induced by the projections $A \times B \rightarrow A$, while the counit $u(Y)_A \rightarrow Y$ is restriction along the maps $\text{diag} \times B: A \times B \rightarrow A \times A \times B$ via the extended functoriality of Y .

As we just saw, the functor $(-)_{gl}$ naturally factors through u , yielding $(-)_{gl/A}: FGL_A^{2-\text{tor}} \rightarrow GL_{gl/A}^{2-\text{tor}}$. The remaining functor $(-)^\wedge: GL_{gl/A}^{2-\text{tor}} \rightarrow FGL_A^{2-\text{tor}}$ is the evident generalization of our definition of $(-)^\wedge_A$ to an arbitrary object of $GL_{gl/A}^{2-\text{tor}}$. It sends $Y \in GL_{gl/A}^{2-\text{tor}}$ to the quintuple

$$(Y(1), Y(C_2)^\wedge, \Delta, \theta, y(\epsilon)),$$

where the completion on $Y(C_2)$ is formed with respect to all kernels of augmentation ideals $(\text{id}_A, V)^*: Y(C_2) \rightarrow Y(1)$ (with $(\text{id}_A, V): A \rightarrow A \times C_2$, using the extended functoriality of Y), and Δ is induced by $(\text{id}_A \times m)^*$ on completions. To see that this is indeed left adjoint to $(-)_{gl/A}$, we note that [Lemma 4.24](#) also applies in this more general setting, showing that there is a natural isomorphism

$$(Y^\wedge)_{gl/A}(B) \cong Y(B)_A^\wedge$$

for any $B \in \text{el}_2$. We can then define the unit of the adjunction $Y \rightarrow (Y^\wedge)_{gl/A}$ to be the completion map. For the other direction, note that $F_{gl/A}(B)$ is already complete for any B . In particular there are natural isomorphisms $F_{gl/A}(1) \cong k$ and $F_{gl/A}(C_2)^\wedge \cong R$ and we find that $F_{gl/A}^\wedge$ is naturally isomorphic to F itself. We let the counit be this natural isomorphism. The composite unit and counit then work out as stated, finishing the proof of [Proposition 4.29](#). \square

Remark 4.30. Note that this shows that $(-)_{gl/A}$ identifies the category of A -equivariant 2-torsion formal group laws with the full subcategory of A -global 2-torsion group laws consisting of the complete objects. When A is the trivial group, this embeds ordinary 2-torsion formal group laws into global 2-torsion group laws, as described in [Example 4.7](#).

Now consider the universal global 2-torsion group law $\tilde{\mathbf{L}}^{2-\text{tor}}$, the initial object of $GL_{gl}^{2-\text{tor}}$. Since any left adjoint preserves initial objects, it follows that the associated A -equivariant 2-torsion formal group law $(\tilde{\mathbf{L}}^{2-\text{tor}})_A^\wedge$ is also universal. In particular, $\tilde{\mathbf{L}}^{2-\text{tor}}(A)$ is canonically isomorphic to the equivariant Lazard ring $L_A^{2-\text{tor}}$, and for any global 2-torsion group law X the map

$$\tilde{\mathbf{L}}^{2-\text{tor}}(A) \rightarrow X(A),$$

given by evaluating the unique map $\tilde{\mathbf{L}}^{2-\text{tor}} \rightarrow X$ on A , classifies the A -equivariant formal group law X_A^\wedge . In fact, we have:

Proposition 4.31. *The adjunctions of [Proposition 4.29](#) induce an isomorphism $\mathbf{L}^{2-\text{tor}} \cong \tilde{\mathbf{L}}^{2-\text{tor}}$ of el_2 -algebras with coordinate $e \in \tilde{\mathbf{L}}^{2-\text{tor}}$ corresponding to the universal Euler class $e_\sigma \in \mathbf{L}^{2-\text{tor}}$. In particular, $\mathbf{L}^{2-\text{tor}}$ equipped with the universal Euler class e_σ is a global 2-torsion group law and hence for every elementary abelian 2-group A all Euler classes of non-trivial characters are regular elements in $L_A^{2-\text{tor}}$.*

Proof. The relation between the coordinate and the universal Euler class is clear by construction: In the C_2 -equivariant formal group law $X_{C_2}^\wedge$ associated to any $X \in GL_{gl}^{2-\text{tor}}$, the Euler class $e_\sigma \in X(C_2)$ is given as the pullback of $e_{(\epsilon, \text{id})} \in X(C_2 \times C_2)$ along the diagonal $C_2 \rightarrow C_2 \times C_2$, which gives back the coordinate

$e \in X(C_2)$. To show that the isomorphisms $(\tilde{\mathbf{L}}^{2-\text{tor}})(A) \cong L_A^{2-\text{tor}}$ are compatible with restriction maps, let $\alpha: B \rightarrow A$ be a homomorphism. For every $X \in GL_{gl}^{2-\text{tor}}$ we obtain a map of A -equivariant formal group laws $(X)_A^\wedge \rightarrow \alpha_*(X)_B^\wedge$ (where α_* denotes the push-forward along α as in [Section 2.4](#)) via

$$X(A) \xrightarrow{\alpha^*} X(B)$$

and

$$X(A \times C_2)_A^\wedge \xrightarrow{(\alpha \times \text{id}_{C_2})^*} \alpha_* X(B \times C_2)_B^\wedge.$$

For $X = \tilde{\mathbf{L}}^{2-\text{tor}}$, this must give the unique map of A -equivariant formal group laws $(\tilde{\mathbf{L}}^{2-\text{tor}})_A^\wedge \rightarrow \alpha_*(\tilde{\mathbf{L}}^{2-\text{tor}})_B^\wedge$. In particular, the underlying map $L_A^{2-\text{tor}} \cong \tilde{\mathbf{L}}^{2-\text{tor}}(A) \rightarrow \tilde{\mathbf{L}}^{2-\text{tor}}(B) \cong L_B^{2-\text{tor}}$ classifies the push-forward along α of the universal B -equivariant 2-torsion formal group law. This equals by definition the functoriality in $\mathbf{L}^{2-\text{tor}}$, so we find that the levelwise isomorphisms between $\mathbf{L}^{2-\text{tor}}$ and $\tilde{\mathbf{L}}^{2-\text{tor}}$ are indeed compatible with the structure maps. \square

Remark 4.32. We stress that the regularity of Euler classes in Lazard rings is a priori very unclear. The Euler classes of arbitrary A -equivariant formal group laws are typically not regular, so this is a special feature of the universal ones. Historically, getting control of the Euler torsion in Lazard rings has been the main obstacle to proving Theorems [A](#) and [B](#), see [Section 1.2](#) in the introduction. It is a major benefit of working globally that the regularity follows essentially formally.

Corollary 4.33. *All 2-torsion Lazard rings $L_A^{2-\text{tor}}$ are integral domains.*

Proof. The non-equivariant 2-torsion Lazard ring $L^{2-\text{tor}}$ is a polynomial algebra over \mathbb{F}_2 , hence an integral domain. By [Proposition 2.11](#), the ring $L_A^{2-\text{tor}}[e_V^{-1} \mid V \in A^* - \{\epsilon\}]$ is obtained from $L^{2-\text{tor}}$ by adjoining polynomial and Laurent ring generators, in particular it is again a domain. But since we just saw that the Euler classes are regular elements of $L_A^{2-\text{tor}}$, this means that $L_A^{2-\text{tor}}$ injects into this localization and hence must be a domain itself. \square

4.7. Geometric fixed points and proof of the main theorem. Before we turn to the proof of Theorems [B](#) and [D](#), we first generalize the construction of inverting Euler classes to arbitrary global 2-torsion group laws.

Definition 4.34 (Geometric fixed points). Let X be a global 2-torsion group law, and A an elementary abelian 2-group. Then we define

$$\Phi^A(X) = X(A)[e_V^{-1} \mid V \in A^* - \{\epsilon\}]$$

and call it the A -geometric fixed points of X .

Since any other coordinate only differs from e by a unit in $X(C_2)$, the definition of $\Phi^A(X)$ is independent of the choice of coordinate. The collection of the $\Phi^A(X)$ for varying A does not assemble to an el_2 -algebra, since the pullback along an arbitrary homomorphism $\alpha: B \rightarrow A$ can send a non-trivial $V \in A^*$ to the trivial class in B^* . This does not happen if α is surjective, so the $\Phi^A(X)$ still form a contravariant functor from the category of elementary abelian 2-groups and surjective group homomorphisms to commutative rings.

The name ‘geometric fixed points’ is motivated from topology: Given a genuine A -spectrum Y , the A -geometric fixed points $\Phi^A(Y)$ are the non-equivariant spectrum defined as the genuine fixed points $(Y \wedge \tilde{E}P_A)^A$, where $\tilde{E}P_A$ is a based A -CW complex uniquely characterized by the fact that its A -fixed points are equivalent to S^0 and the fixed points at all other subgroups are contractible. If E is a global spectrum, $\Phi^A(E)$ is defined as $\Phi^A(E_A)$.

Lemma 4.35. *For a real oriented global ring spectrum E the geometric fixed point maps $\pi_*^A(E) \rightarrow \pi_*(\Phi^A(E))$ induce a natural isomorphism*

$$\Phi^A(\pi_*(E)) \cong \pi_*(\Phi^A(E)) = \Phi_*^A(E).$$

Proof. This follows from the fact that the sphere

$$\text{hocolim}_{V \subseteq \mathcal{U}_A; V^A=0} S^V$$

is a model for $\tilde{E}P_A$, where the colimit is taken along all finite dimensional subrepresentations $V \subseteq \mathcal{U}_A$ with trivial fixed points. Hence there is an equivalence

$$(E_A \wedge \tilde{E}P_A)^A \simeq \operatorname{hocolim}_{V \subseteq \mathcal{U}_A; V^A=0} (E_A \wedge S^V)^A \simeq (E_A \wedge S^{\dim(V)})^A,$$

where the last equivalence comes from the real orientation of E_A . Decomposing each V into a direct sum of irreducible representations shows that $\Phi_*^A(E_A)$ is obtained from $\pi_*((E_A)^A) = \pi_*^A(E_A)$ via a filtered colimit of iterated multiplications by Euler classes, each appearing infinitely often, which proves the claim. \square

Now we come to the proof of Theorems B and D. By Proposition 4.17 applied to the real orientation of \mathbf{MO} , the coefficients $\pi_*(\mathbf{MO})$ form a global 2-torsion group law. Hence there exists a unique map

$$\mathbf{L}^{2-\text{tor}} \rightarrow \pi_*(\mathbf{MO})$$

of global 2-torsion group laws, and we want to show that this map is an isomorphism. With the structural understanding of $\mathbf{L}^{2-\text{tor}}$ and $\pi_*(\mathbf{MO})$ we gathered in the previous sections, the only input from topology we require is a comparison of their geometric fixed points.

Proposition 4.36. *The map $\mathbf{L}^{2-\text{tor}} \rightarrow \pi_*(\mathbf{MO})$ induces an isomorphism on geometric fixed points Φ^A for every elementary abelian 2-group A .*

Proof. By Proposition 2.12 we understand the geometric fixed points of the Lazard ring $L_A^{2-\text{tor}}$. The computation of the geometric fixed points of MO_A goes back to tom Dieck (see [tD70] for the complex analog) and is also described in [Fir13, Proposition 3.2]. From these descriptions it follows that $\Phi^A(\mathbf{L}^{2-\text{tor}})$ and $\Phi_*^A(\mathbf{MO})$ are abstractly isomorphic, assuming Quillen's non-equivariant isomorphism $L^{2-\text{tor}} \cong MO_*$ [Qui69]. The proof that the map $\Phi^A(\mathbf{L}^{2-\text{tor}}) \rightarrow \Phi_*^A(\mathbf{MO})$ realizes such an isomorphism can then be given analogously to the one in the complex case due to Greenlees (see [Gre01, Proposition 13.2] and [Gre00, Section 11]).

For the convenience of the reader, we sketch the argument. Let Y be a real oriented A -ring spectrum. The geometric fixed point functor Φ^A defines a bijection

$$[\mathbb{R}P(\mathcal{U}_A^{\mathbb{R}})_+, Y \wedge \tilde{E}P_A]^A \cong [\mathbb{R}P(\mathcal{U}_A^{\mathbb{R}})_+, \Phi^A(Y)]_* \cong \prod_{A^*} [\mathbb{R}P_+^{\infty}, \Phi^A(Y)]_*, \quad (4.37)$$

using that $\mathbb{R}P(\mathcal{U}_A^{\mathbb{R}})^A$ decomposes into a disjoint union of copies of $\mathbb{R}P^{\infty}$ indexed over all $V \in A^*$ (see [Gre01, Section 11.A], for example). The ring $[\mathbb{R}P_+^{\infty}, \Phi^A(Y)]_* = \Phi^A(Y)^*(\mathbb{R}P_+^{\infty})$ is the associated non-equivariant formal group law over $\Phi_*^A(Y)$, and the splitting (4.37) corresponds to the splitting of Euler-invertible formal group laws described in the proof of Proposition 2.9.

Now we consider the case $Y = MO_A$. There is an equivalence

$$\Phi^A(MO_A) \simeq S[\tilde{A}^*] \wedge MO \wedge \left(\prod_{V \in \tilde{A}^*} BO \right)_+, \quad (4.38)$$

(see [Fir13, Proposition 3.2], or [Sin01, Theorem 4.9] for the complex analog) where \tilde{A}^* denotes the set of non-trivial characters and $S[\tilde{A}^*]$ is the homotopy ring spectrum $\bigvee_{W \in \mathbb{Z}[\tilde{A}^*]} S^{|W|}$. In the latter, $\mathbb{Z}[\tilde{A}^*]$ denotes the free abelian group on \tilde{A}^* , and $|W|$ denotes the sum of the coefficients of W . The Euler class $e_V \in \Phi_*^A(MO_A)$ corresponds to the inclusion $S^{-1} \rightarrow S[\tilde{A}^*]$ associated to $-V$.

Now we recall from non-equivariant homotopy theory that there is an isomorphism $MO_*(BO_+) = MO_*[b_1, b_2, \dots]$, where b_i is the image of the generator in $MO_i(\mathbb{R}P_+^{\infty})$ dual to the i -th power of the standard coordinate x of MO . Recall moreover that in terms of the pushforward of $x \in MO^1(\mathbb{R}P_+^{\infty})$ under $MO \rightarrow MO \wedge BO_+$, the map $\mathbb{R}P_+^{\infty} \rightarrow BO_+ \rightarrow MO \wedge BO_+$ is given by the power series $1 + b_1 x + b_2 x^2 + b_3 x^3 + \dots$. For $V \in \tilde{A}^*$, we write $b_i^V \in$ for the image of b_i under the inclusion $MO \wedge BO_+ \xrightarrow{MO \wedge i_V} \Phi^A(MO_A)$ associated to V . Then we find that there is an isomorphism

$$\Phi_*^A(MO_A) = MO_*[b_i^V, e_V^{\pm 1} \mid V \in \tilde{A}^*, i \in \mathbb{N}_{>0}].$$

In Proposition 2.9 we saw that $\Phi^A(\mathbf{L}^{2-\text{tor}})$ is isomorphic to $L^{2-\text{tor}}[\gamma_i^V, e_V^{\pm 1} \mid V \in \tilde{A}^*, i \in \mathbb{N}_{>0}]$, where the γ_i^V are determined by the fact that under the Euler-invertible splitting, the V th component of the coordinate $y(\epsilon)$ equals $e_V + \gamma_1^V x + \gamma_2^V x^2 + \dots$, where x is the coordinate of the associated non-equivariant formal group law.

Hence, we need to consider the orientation class in the A -equivariant formal group law over $\Phi_*^A(MO_A)$. As described above, this A -equivariant formal group law is given by

$$\prod_{A^*} [\mathbb{R}P_+^\infty, \Phi^A(MO_A)]_* \cong \prod_{A^*} [\mathbb{R}P_+^\infty, S[\tilde{A}^*] \wedge MO \wedge (\prod_{V \in \tilde{A}^*} BO)_+]_*.$$

By taking fixed points, the orientation $(\mathbb{R}P(\mathcal{U}_A^\mathbb{R})_+) \wedge S^{-1} \rightarrow MO_A \wedge \tilde{E}P_A$ then corresponds to a map

$$(\bigsqcup_{A^*} \mathbb{R}P^\infty)_+ \wedge S^{-1} \rightarrow S[\tilde{A}^*] \wedge MO \wedge (\prod_{V \in \tilde{A}^*} BO)_+.$$

Inspection of the equivalence (4.38) shows that this map is given by the wedge of the maps

$$\lambda_V: \mathbb{R}P_+^\infty \wedge S^{-1} \rightarrow BO_+ \wedge S^{-1} \xrightarrow{i_V \wedge e_V} \Phi^A(MO_A)$$

for non-trivial characters V , and the usual orientation

$$t: \mathbb{R}P_+^\infty \wedge S^{-1} \rightarrow MO \rightarrow \Phi^A(MO_A)$$

for the trivial character.

By what we recalled above, the map λ_V can be expressed in terms of the coordinate x as

$$\lambda_V = e_V(1 + b_1^V x + b_2^V x^2 + \dots) = e_V + e_V b_1^V x + e_V b_2^V x^2 + \dots$$

Hence, γ_i^V is sent to the element $e_V b_i^V$, a unit multiple of b_i^V . Since we know that the map $L^{2-\text{tor}} \rightarrow MO_*$ is an isomorphism by [Qui69], it follows that so is $\Phi^A(\mathbf{L}^{2-\text{tor}}) \rightarrow \Phi_*^A(MO_A)$, as desired. \square

Remark 4.39. In terms of the classes described in the proof, the classes $Y_{d,V} \in \Phi_*^A(MO_A)$ that feature in the pullback square of [Fir13] relating geometric and homotopical bordism are defined via

$$Y_{d,V} = e_V^{-1} b_{d-1}^V,$$

for $d \geq 2$ (see [Fir13, Definition 3.5]), or in other words the image of the element $e_V^{-2} \gamma_{d-1}^V \in \Phi^A(\mathbf{L}^{2-\text{tor}})$. This is the real analog of Hanke's definition in the complex version [Han05].

Theorems B and D are then a direct consequence of the following ‘Whitehead theorem for global 2-torsion group laws’.

Proposition 4.40. *Let $f: X \rightarrow Y$ be a morphism of global 2-torsion group laws such that $\Phi^A(f)$ is an isomorphism for every elementary abelian 2-group A . Then f is an isomorphism.*

Proof. We fix $A \in \text{el}_2$ and show that the map $f(A): X(A) \rightarrow Y(A)$ is an isomorphism. Since all non-trivial Euler classes are regular in X and Y , it follows that $X(A)$ and $Y(A)$ embed into $\Phi^A(X)$ and $\Phi^A(Y)$ respectively. By assumption, $\Phi^A(f)$ is an isomorphism, thus $f(A)$ is injective. Furthermore, surjectivity of $\Phi^A(f)$ implies that for every $y \in Y(A)$ there exist non-trivial $V_1, \dots, V_n \in A^*$ and $x \in X(A)$ such that $f(x) = e_{V_1} \dots e_{V_n} \cdot y$. If $n = 0$ and hence $f(x) = y$, we are done. If $n \geq 1$, note that

$$f(\text{res}_{\ker(V_n)}^A(x)) = \text{res}_{\ker(V_n)}^A(f(x)) = 0,$$

since $f(x)$ is a multiple of e_{V_n} . But $f(\ker(V_n))$ is also injective, thus $\text{res}_{\ker(V_n)}^A(x) = 0$. As X is a global 2-torsion group law, this implies that x can be written as $x = e_{V_n} \cdot x'$. It follows that

$$e_{V_n} \cdot f(x') = f(e_{V_n} \cdot x') = f(x) = e_{V_1} \dots e_{V_n} \cdot y,$$

which means that

$$f(x') = e_{V_1} \dots e_{V_{n-1}} \cdot y$$

by regularity of e_{V_n} . Repeating this process for V_{n-1}, \dots, V_1 produces the desired preimage of y , which finishes the proof of the proposition and Theorems B and D. \square

Remark 4.41. One can show that more generally, $\pi_*(\mathbf{MO}^{\wedge n})$ is universal among el_2 -algebras equipped with a so-called strict n -tuple of coordinates. We carry this out in the complex case in Section 5.8.

5. COMPLEX BORDISM

We now move on to global complex bordism \mathbf{MU} and its relation to equivariant formal group laws via Theorems A, C and E. The general strategy is similar to the one in Section 4, but it is complicated by the facts that (1) not every abelian compact Lie group is a torus and especially (2) not every non-trivial character $V: \mathbb{T}^r \rightarrow \mathbb{T}$ is split. Proceeding analogously to the 2-torsion case in Section 4 will show that the Euler classes $e_V \in L_{\mathbb{T}^r}$ for split characters $V: \mathbb{T}^r \rightarrow \mathbb{T}$ are regular elements. This is not sufficient to reduce to geometric fixed points since these are formed by inverting the Euler classes for all non-trivial characters. The step from split surjective characters to all non-trivial ones requires an additional argument, which is given in Section 5.6 and relies on studying the effect of the p -th power map $\mathbb{T}^r \xrightarrow{(-)^p} \mathbb{T}^r$ on Lazard rings.

In the following we leave out details where they are analogous to those in Section 4 and focus on the differences.

5.1. Global group laws. We begin with the definition of a global group law. For many of the constructions it is technically convenient to first restrict to the family of tori and later extend formally to all abelian compact Lie groups, in the way explained in Section 5.1.1 below. See the discussion in Section 5.4 for the reasons behind this.

Definition 5.1. A *global group law* is a functor

$$X: (\text{tori})^{op} \rightarrow \text{commutative rings}$$

together with a class $e \in X(\mathbb{T})$, called a *coordinate*, such that for every torus A and every split surjective character $V: A \rightarrow \mathbb{T}$ the sequence

$$0 \rightarrow X(A) \xrightarrow{e_V} X(A) \xrightarrow{\text{res}_{\ker(V)}^A} X(\ker(V)) \rightarrow 0 \quad (5.2)$$

is exact, where we write e_V for the pullback V^*e . A morphism of global group laws $(X, e) \rightarrow (Y, e')$ is a natural transformation $X \rightarrow Y$ sending e to e' . We denote the category of global group laws by GL_{gl} .

Remark 5.3. Since the opposite of the category of tori is equivalent to the category of finitely generated free abelian groups via Pontryagin duality, the more algebraically minded reader might prefer to phrase global group laws as functors

$$\text{finitely generated free abelian groups} \rightarrow \text{commutative rings}.$$

This point of view suggests a relationship between global group laws and abelian cogroup objects in commutative rings, i.e., commutative cocommutative Hopf algebras. We say more about this in Example 5.12 and Section 5.5.

Again, we usually simply write X for the tuple (X, e) . We also make use of the weaker notion of a \mathbb{Z}_e -algebra, which is just a contravariant functor X from tori to commutative rings equipped with an element $e \in X(\mathbb{T})$ which restricts to 0 at the trivial group. Every \mathbb{Z}_e -algebra can be formally turned into a global group law:

Proposition 5.4. *The category of global group laws is a reflective subcategory of the category of \mathbb{Z}_e -algebras which is closed under limits. In particular, it has all limits and colimits.*

Proof. The surjectivity of the restriction maps in (5.2) is implied by the existence of a section of V , and hence the global group laws are exactly the \mathbb{Z}_e -algebras that are local with respect to the morphisms

$$f_{A,V}: \mathbb{Z}_e[x]/(\text{res}_{\ker(V)}^A(x)) \rightarrow \mathbb{Z}_e[y]$$

sending x to $y \cdot e_V$. Here, (A, V) ranges through all pairs of a torus A and a split character $V \in A^*$, and x and y are formal variables at the group A . The proposition then follows from Lemma 4.9. \square

In particular there exists an initial global group law $\tilde{\mathbf{L}}$ (which we will later prove to be isomorphic to the global Lazard ring \mathbf{L} defined in Section 2.4).

Definition 5.5. A *graded* global group law is a contravariant functor from tori to graded commutative rings equipped with a coordinate e of homogeneous degree -2 such that the sequences (5.2) are exact.

We obtain the following, cf. [Section 4.2.1](#).

Corollary 5.6. *There exists a unique grading on $\tilde{\mathbf{L}}$, which is concentrated in even degrees. With this grading, $\tilde{\mathbf{L}}$ is initial among all graded formal group laws.*

5.1.1. *From tori to all abelian compact Lie groups.* It is useful to extend global group laws to functors defined on all abelian compact Lie groups (i.e., to an *Ab*-algebra) via left Kan extension. This process has a more explicit description in terms of quotienting by Euler classes, which we now describe. We first need a preparatory lemma.

Lemma 5.7. *Let T_1, T_2 be tori and $f, g : T_1 \rightarrow T_2$ two group homomorphisms which agree on a closed subgroup B of T_1 . Then for every global group law X , the maps f^* and $g^* : X(T_2) \rightarrow X(T_1)$ agree modulo the ideal generated by all Euler classes e_V for those $V \in T_1^*$ which vanish on B .*

Proof. Let $g' = f^{-1} \cdot g$ be the pointwise product of f^{-1} and g . Then g' vanishes on B , and g factors as

$$T_1 \xrightarrow{(f, g')} T_2 \times T_2 \xrightarrow{m} T_2.$$

Given $x \in X(T_2)$, the difference $m^*(x) - \text{pr}_1^*(x) \in X(T_2 \times T_2)$ lies in the kernel of the restriction to the first T_2 -factor. Hence we can write $m^*(x)$ as

$$m^*(x) = \text{pr}_1^*(x) + x',$$

with x' in the ideal generated by all e_V for characters $V \in \text{im}(\text{pr}_2^* : T_2^* \rightarrow (T_2 \times T_2)^*)$ (cf. [Lemma 4.4](#) for the real analog). So we have $g^*(x) = f^*(x) + (f, g')^*(x')$, and $(f, g')^*(x')$ lies in the ideal generated by Euler classes e_V with $V \in \text{im}((g')^* : T_2^* \rightarrow T_1^*)$. Since any V in $\text{im}((g')^*)$ becomes trivial when restricted to B (because g' vanishes on B), this yields the desired statement. \square

We can now extend a global group law X as follows: For any abelian compact Lie group A we choose an embedding $i_A : A \hookrightarrow T_A$ into a torus. For simplicity we choose i_A to be the identity if A already is a torus. Then we set

$$X(A) = X(T_A) / (e_V \mid V \in T_A^*, i_A^* V = \epsilon).$$

Note that it is enough to quotient out by e_V for V ranging through a set of generators of $\ker(i_A^*)$. For a group homomorphism $\alpha : B \rightarrow A$, we choose a dotted arrow in the diagram

$$\begin{array}{ccc} T_B & \xrightarrow{\tilde{\alpha}} & T_A \\ i_B \uparrow & & \uparrow i_A \\ B & \xrightarrow{\alpha} & A \end{array}$$

which exists since T_A is an injective object in the category of abelian compact Lie groups. We then set $\alpha^* : X(A) \rightarrow X(B)$ to be the unique map making the diagram

$$\begin{array}{ccc} X(T_B) & \xleftarrow{\tilde{\alpha}^*} & X(T_A) \\ \downarrow & & \downarrow \\ X(B) & \xleftarrow{\alpha^*} & X(A) \end{array}$$

commute. This is well-defined: If $V \in (T_A)^*$ restricts to the trivial element in A^* , then, by commutativity of the square, $\tilde{\alpha}^*(V)$ restricts to the trivial element in B^* , and hence the relevant Euler classes are indeed sent to 0 in $X(B)$. Furthermore, the definition is independent of the choice of $\tilde{\alpha}$ by [Lemma 5.7](#). It then follows easily that this construction preserves identities and composition, that its restriction to the full subcategory of tori agrees with the original X , and that $i_A^* : X(T_A) \rightarrow X(A)$ is given by the projection map. Finally, this extension is a left Kan extension by the following two observations:

- (1) By definition, every class of $X(A)$ is the image of a class in $X(T)$ for some torus T and group homomorphism $A \rightarrow T$. In fact, the chosen embedding i_A does the job.

- (2) Let X' be any extension of X to a functor defined on all abelian compact Lie groups. Then, if $V \in T_A^*$ restricts to ϵ in A^* , its Euler class e_V must restrict to 0 in $X'(A)$ since the composite

$$A \rightarrow T_A \xrightarrow{V} \mathbb{T}$$

factors through the trivial group, where the unique Euler class is trivial.

Since left Kan extensions are unique up to canonical isomorphism, this also implies that the description of $X(A)$ is independent of the chosen embedding i_A . From now on we will not distinguish between a global group law defined only on tori and its extension to all abelian compact Lie groups obtained in this way.

5.2. Regular global group laws. Given a global group law, all Euler classes e_V for split characters V are regular elements by definition, while no condition is required for the other Euler classes.

Lemma 5.8. *Given a global group law X and number $k \in \mathbb{N}$, the following two conditions are equivalent:*

- (1) *For every torus T , natural number $l \leq k$ and linearly independent l -tuple (V_1, \dots, V_l) of characters of T , the sequence*

$$(e_{V_1}, \dots, e_{V_l})$$

is regular in $X(T)$.

- (2) *For every abelian compact Lie group A with $\pi_0(A)$ a product of at most $(k-1)$ cyclic groups, and every surjective character $V: A \rightarrow \mathbb{T}$, the sequence*

$$0 \rightarrow X(A) \xrightarrow{e_V} X(A) \xrightarrow{\text{res}_{\ker(V)}^A} X(\ker(V)) \rightarrow 0$$

is exact.

Definition 5.9. If the two equivalent conditions are satisfied, we say that X is a k -regular global group law. If X is k -regular for all k , we say that X is *regular*.

Proof of Lemma 5.8. (1) \Rightarrow (2): Let (A, V) be as in (2) above. We choose an embedding $i_A: A \hookrightarrow T_A$ and a basis V_1, \dots, V_l of $\ker(i_A^*)$. We can arrange that $l \leq k-1$ since $\pi_0(A)$ is a product of at most $(k-1)$ cyclic groups by assumption. Furthermore, we lift V to a character \tilde{V} of T_A . Note that $X(A)$ identifies with $X(T_A)/(e_{V_1}, \dots, e_{V_l})$, and $X(\ker(V))$ identifies with $X(T_A)/(e_{V_1}, \dots, e_{V_l}, e_{\tilde{V}})$. It follows that the map $X(A) \rightarrow X(\ker(V))$ is surjective with kernel generated by e_V . Moreover, the surjectivity of V implies that $(V_1, \dots, V_l, \tilde{V})$ is a linearly independent tuple, and hence $e_V = i_A^*(e_{\tilde{V}})$ is a regular element of

$$X(T_A)/(e_{V_1}, \dots, e_{V_l}) \cong X(A),$$

so the statement follows.

- (2) \Rightarrow (1): By induction on $l \leq k$ it suffices to show that e_{V_l} is a regular element in

$$X(T)/(e_{V_1}, \dots, e_{V_{l-1}}),$$

which identifies with $X(A)$ for $A = \ker(V_1) \cap \dots \cap \ker(V_{l-1})$. Since the sequence is regular, the restriction of V_l to A is a surjective character, so by assumption its Euler class must be regular, which finishes the proof. \square

Hence, if X is a regular global group law, then the sequence

$$0 \rightarrow X(A) \xrightarrow{e_V} X(A) \xrightarrow{\text{res}_{\ker(V)}^A} X(\ker(V)) \rightarrow 0$$

is exact for every abelian compact Lie group A and surjective character V . We will later see that the initial global group law $\tilde{\mathbf{L}}$ is regular, but this is unclear at this point.

5.3. Examples. We list some examples of global group laws.

Example 5.10 (Additive global group law). Let k be a commutative ring. Then the *additive global group law* $(\mathbb{G}_a)_k$ over k is defined as

$$(\mathbb{G}_a)_k(A) = k[e_V, V \in A^*] / (e_{VW} - e_V - e_W \mid V, W \in A^*),$$

with coordinate $e = e_\tau \in (\mathbb{G}_a)_k(\mathbb{T})$ for the tautological \mathbb{T} -character τ . Choosing a basis W_1, \dots, W_r of $(\mathbb{T}^r)^*$ gives an isomorphism

$$(\mathbb{G}_a)_k(\mathbb{T}^r) \cong k[e_{W_1}, \dots, e_{W_r}].$$

It follows analogously to [Example 4.6](#) that $(\mathbb{G}_a)_k$ is indeed a global group law. Its regularity depends on the ground ring k : Given an l -tuple of linearly independent characters $V_1, \dots, V_l \in (\mathbb{T}^r)^*$, we can choose a basis W_1, \dots, W_r of $(\mathbb{T}^r)^*$ such that each V_i is a non-trivial multiple of W_i modulo the subgroup generated by W_1, \dots, W_{i-1} (and hence e_{V_i} is the same multiple of e_{W_i} modulo $(e_{W_1}, \dots, e_{W_{i-1}})$). We then have an isomorphism $(\mathbb{G}_a)_k(\mathbb{T}^r) \cong k[e_{W_1}, \dots, e_{W_r}]$. It follows that if $k = \mathbb{Q}$, the sequence e_{V_1}, \dots, e_{V_l} is regular in $(\mathbb{G}_a)_k(\mathbb{T}^r)$, since by induction each e_{V_i} is a unit multiple of e_{W_i} modulo $(e_{V_1}, \dots, e_{V_{i-1}}) = (e_{W_1}, \dots, e_{W_{i-1}})$. So $(\mathbb{G}_a)_\mathbb{Q}$ is a regular global group law. If $k = \mathbb{F}_p$, then $e_{V_i} = 0$ if p divides V_i , so $(\mathbb{G}_a)_{\mathbb{F}_p}$ is not even 1-regular. When $k = \mathbb{Z}$, then each single Euler class e_V with $V \in (\mathbb{T}^r)^* - \{\epsilon\}$ is regular, but for example any sequence (e_{V_1}, e_{V_2}) with some prime p dividing both V_1 and V_2 is not regular as

$$e_{V_2} \cdot e_{V_1/p} = p \cdot e_{V_2/p} \cdot e_{V_1/p} = e_{V_2/p} \cdot e_{V_1}$$

is trivial modulo e_{V_1} , while $e_{V_1/p}$ is not. Therefore, $(\mathbb{G}_a)_\mathbb{Z}$ is 1-regular but not 2-regular.

Example 5.11 (Multiplicative global group law). The *multiplicative global group law* over a commutative ring k is defined via

$$(\mathbb{G}_m)_k(A) = kA^* \cong RU(A) \otimes_\mathbb{Z} k,$$

the group ring of A^* over k , or in other words the complex representation ring $RU(A)$ tensored with k . The coordinate e is the difference $[\tau] - 1 \in RU(\mathbb{T}) \otimes k$. A basis W_1, \dots, W_r of $(\mathbb{T}^r)^*$ gives an isomorphism to the Laurent polynomial ring

$$(\mathbb{G}_m)_k(\mathbb{T}^r) \cong k[[W_1]^{\pm 1}, \dots, [W_r]^{\pm 1}].$$

Given a linearly independent tuple (V_1, \dots, V_l) in $(\mathbb{T}^r)^*$, we can again choose the basis in such a way that each V_i is a non-trivial multiple $n_i W_i$ of W_i modulo the subgroup generated by W_1, \dots, W_{i-1} . Then the quotient $(\mathbb{G}_m)_k(\mathbb{T}^r) / (e_{V_1}, \dots, e_{V_{i-1}})$ takes the form

$$(k[[W_1]^{\pm 1}, \dots, [W_{i-1}]^{\pm 1}] / (e_{V_1}, \dots, e_{V_{i-1}})) [[W_i]^{\pm 1}, \dots, [W_r]^{\pm 1}],$$

with e_{V_i} being of the form $x + [W_i]^{n_i}$ for some $x \in k[[W_1]^{\pm 1}, \dots, [W_{i-1}]^{\pm 1}] / (e_{V_1}, \dots, e_{V_{i-1}})$. Since any element of this form is regular, we conclude that $(\mathbb{G}_m)_k$ is a regular global group law for all ground rings k .

The multiplicative global group law over $k = \mathbb{Z}$ is realized as π_0 of the periodic global K -theory spectrum \mathbf{KU} , see [\[Joa04\]](#) and [\[Sch18, Section 6.4\]](#).

Example 5.12 (Hopf algebras/1-dimensional linear algebraic groups). The category of global group laws for which all maps

$$(\mathrm{pr}_A^* \otimes \mathrm{pr}_B^*): X(A) \otimes_{X(1)} X(B) \rightarrow X(A \times B) \tag{5.13}$$

are isomorphisms is equivalent to the category of triples $(k, R, y(\epsilon))$ of a commutative ring k , a cocommutative commutative Hopf algebra R over k and a regular element $e \in R$ generating the augmentation ideal. The correspondence sends a global group law X with the above property to the triple $(X(1), X(\mathbb{T}), e)$, where the comultiplication on $X(\mathbb{T})$ is given by

$$X(\mathbb{T}) \xrightarrow{m^*} X(\mathbb{T} \times \mathbb{T}) \xleftarrow{\cong} X(\mathbb{T}) \otimes_{X(1)} X(\mathbb{T}).$$

The previous two examples are special cases of this construction, for the Hopf algebras given by the polynomial ring $k[x]$ (the additive group) and the Laurent ring $k[x^{\pm 1}]$ (the multiplicative group). A further specific example is the circle $S^1 = \mathbb{R}[x, y] / (x^2 + y^2 - 1)$ over \mathbb{R} , with standard comultiplication

$$\Delta: \mathbb{R}[x, y] / (x^2 + y^2 - 1) \rightarrow \mathbb{R}[x_1, y_1] / (x_1^2 + y_1^2 - 1) \otimes_\mathbb{R} \mathbb{R}[x_2, y_2] / (x_2^2 + y_2^2 - 1)$$

sending $[x]$ to $[x_1 \otimes y_1 - x_2 \otimes y_2]$ and $[y]$ to $[x_1 \otimes y_2 + x_2 \otimes y_1]$. A coordinate is given by $[x] - 1$.

Example 5.14 (Complete global group laws). We say that a global group law is *complete* if each $X(\mathbb{T}^r)$ is complete with respect to the augmentation ideal $\ker(X(\mathbb{T}^r) \rightarrow X(1))$. Then $X(\mathbb{T})$ is isomorphic to the power series ring over $X(1)$ on the coordinate e , and each $X(\mathbb{T}^r)$ becomes a power series ring on r generators. The subcategory of complete global group laws is equivalent to the category of ordinary formal group laws, see [Section 5.5](#). Topologically, the complete global group laws correspond to the global Borel theories of ordinary complex oriented ring spectra (see [\[Sch18, Example 4.5.19\]](#)).

When X is complete as above, then $X(C_n)$ is isomorphic to $X(1)[[e]]/([n]_F(e))$, where $[n]_F(e)$ denotes the n -series of the associated formal group law F . This ring classifies the n -torsion in F . If F is a divisible formal group law over a complete local ring $X(1)$ whose residue field is of characteristic p ('divisible' meaning that the multiplication-by- p map $X(1)[[e]] \xrightarrow{[p]_F} X(1)[[e]]$ is an isogeny), then the sequence

$$X(1) \leftarrow X(C_p) \leftarrow X(C_{p^2}) \leftarrow \dots \leftarrow X(C_{p^n}) \leftarrow \dots$$

is the p -divisible group associated to F , see [\[Tat67\]](#).

Example 5.15 (Complete Hopf algebras). More generally, complete topological Hopf algebras with a regular generator of the augmentation ideal give rise to global group laws, see [Remark 5.31](#). In particular, every A -equivariant formal group law F over any abelian compact Lie group A defines a global group law F_{gl} ([Section 5.5](#)).

Example 5.16 (Base change). If X is a global group law, k is a commutative ring and $X(1) \rightarrow k$ is a ring map, then one obtains a new global group law X_k by tensoring each degree $X(A)$ with k over $X(1)$. To see that this construction preserves the relevant short exact sequences, note that these are split and hence preserved under any additive functor.

Base change can be combined with the previous examples to obtain new global group laws. In particular, we can start with a complete global group law F_{gl} associated to an ordinary formal group law F and base change to another ground ring k . Then $(F_{gl})_k$ is in general *not* complete. From the viewpoint of the divisible case described at the end of [Example 5.14](#), the base change on global group laws computes the base change of the associated p -divisible group, which in particular preserves the height. This shows that global group laws can encode p -divisible groups that are not connected, unlike ordinary formal group laws.

Example 5.17 (Universal constructions). We already saw that every \mathbb{Z}_e -algebra can be universally turned into a global group law. For example this gives rise to the initial global group law $\tilde{\mathbf{L}}$, or a universal global group law equipped with a choice of n coordinates (see [Section 5.8](#)), or one which is universal with the property that $e_{V^{-1}} = -e_V$ for all V (i.e., the universal 'odd global group law', cf. [\[BM14\]](#)).

5.4. Global complex oriented ring spectra. We now come to the relation between global group laws and complex orientated global ring spectra. Here, again, 'global' is meant with respect to the family of abelian compact Lie groups. Let E be a commutative global homotopy ring spectrum, i.e., a commutative monoid with respect to the smash product on the global homotopy category [\[Sch18, Corollary 4.3.26\]](#). We from now on simply write 'global ring spectrum' for this notion. Let τ denote the tautological \mathbb{T} -character.

Definition 5.18. A *complex orientation* of E is an $RO(\mathbb{T})$ -graded unit $t \in \pi_{2-\tau}^{\mathbb{T}} E = E_{\mathbb{T}}^2(S^{\tau})$ which restricts to 1 at the trivial group.

Given a complex orientation t of E , we obtain the universal Euler class $e \in E_{\mathbb{T}}^2(S^0) = \pi_{-2}^{\mathbb{T}}(E)$ by pulling back t along the inclusion $S^0 \hookrightarrow S^{\tau}$.

Proposition 5.19. *Let E be a complex oriented global ring spectrum. Then the pair $(\pi_*(E), e)$ is a graded global group law. In fact, the sequence*

$$0 \rightarrow \pi_*^A(E) \xrightarrow{e_V} \pi_*^A(E) \xrightarrow{\text{res}_{\ker(V)}^A} \pi_*^{\ker(V)}(E) \rightarrow 0 \quad (5.20)$$

is exact for every abelian compact Lie group A (not necessarily a torus) and split character $V: A \rightarrow \mathbb{T}$.

Proof. We consider the cofiber sequence

$$\mathbb{T}_+ \rightarrow S^0 \rightarrow S^{\tau} \quad (5.21)$$

of based \mathbb{T} -spaces. Pulling back along a surjective character $V: A \rightarrow \mathbb{T}$ yields a cofiber sequence

$$A/\ker(V)_+ \rightarrow S^0 \rightarrow S^V$$

of based A -spaces, which is taken to a long exact sequence

$$\dots \rightarrow E_A^*(S^V) \rightarrow E_A^*(S^0) \rightarrow E_A^*(A/\ker(V)_+) \rightarrow \dots$$

Via the isomorphism $E_A^*(A/\ker(V)_+) \cong E_{\ker(V)}^*(S^0)$, the second map agrees with restriction to $\ker(V)$. If V is split, the fact that E is a global theory implies that this restriction has a section and the long exact sequence decomposes into short exact sequences. Finally, the complex orientation gives an isomorphism $E_A^*(S^V) \cong E_A^{*-2}$, under which the first map becomes multiplication with the Euler class e_V , so the statement follows. \square

Remark 5.22. The line of argument in the above proof can be found in [Sin01, Theorem 1.2], in the universal case $E = \mathbf{MU}$. One difference is that Sinha makes use of the fact that $\pi_*(\mathbf{MU})$ is concentrated in even degrees to deduce the surjectivity of restriction maps $\pi_*^A(MU_A) \rightarrow \pi_*^B(MU_B)$ for all inclusions $B \hookrightarrow A$ of abelian groups. This produces short exact sequences of the above form also for non-split surjective characters $V \in A^*$. For split characters $V \in A^*$ the surjectivity of $\pi_*^A(MU_A) \rightarrow \pi_*^{\ker(V)}(MU_{\ker(V)})$ follows directly from the global structure and hence generalizes to all complex oriented global ring spectra E , while the short exact sequences for non-split characters require additional assumptions on E , see Lemma 5.23 below.

Warning: When E is a global complex oriented ring spectrum, it is in general not true that $\pi_*(E)$ is left Kan extended from tori. An example is given by the Borel theory $bH\mathbb{F}_p$ for ordinary cohomology with \mathbb{F}_p coefficients (cf. [Sch18, Example 4.5.19]), since the restriction map on group cohomology $H^*(B\mathbb{T}, \mathbb{F}_p) \rightarrow H^*(BC_p, \mathbb{F}_p)$ is not surjective.

Therefore, the proposition should be read as ‘the restriction of $\pi_*(E)$ to tori is a global group law’, while the generalized exact sequences (5.20) refer to the actual values of $\pi_*^A(E)$ and $\pi_*^{\ker(V)}(E)$, not the left Kan extension of its restriction to tori. To make this distinction clear, we write $\pi_*^{\text{toral}} E$ for the left Kan extension.

In light of this, one might wonder why we have not defined a global group law to be a contravariant functor from all abelian compact Lie groups to commutative rings for which the analogs of (5.20) are exact. The reason to restrict to tori instead is of technical nature. With this more general definition one runs into problems when defining the adjunctions of Section 5.5, for example it is unclear how to compute the completion $X(C_2)^\wedge$ at the augmentation ideal when the restriction map $X(\mathbb{T}) \rightarrow X(C_2)$ is not surjective.

Given these issues, it is useful to know when $\pi_*(E)$ does agree with the left Kan extension from tori:

Lemma 5.23. *Let E be a complex oriented global ring spectrum.*

- (1) *The Ab-algebra $\pi_*(E)$ is left induced from tori if and only if $\pi_*^{\text{toral}}(E)$ is regular as a global group law.*
- (2) *A sufficient condition for the equivalent conditions of (1) to hold is that $\pi_* E$ be concentrated in even degrees.*

Proof. Let A be an abelian compact Lie group and $V: A \rightarrow \mathbb{T}$ a character that is surjective but not necessarily split. Again we can pullback the cofiber sequence (5.21) and obtain a long exact sequence

$$\dots \rightarrow \pi_*^A(E) \xrightarrow{e_V} \pi_*^A(E) \xrightarrow{\text{res}_{\ker(V)}^A} \pi_*^{\ker(V)}(E) \rightarrow \dots \quad (5.24)$$

Now, since V does not split, it is not generally true that $\text{res}_{\ker(V)}^A$ is surjective, so the long exact sequence does not split up into short exact ones. In fact, we see that the restriction map is surjective if and only if e_V is a regular element. Note that if $\pi_*(E)$ is concentrated in even degrees, then the boundary maps are necessarily trivial, which proves Part (2).

If $\pi_*(E)$ is left induced from tori (i.e., $\pi_*^{\text{toral}}(E) = \pi_*(E)$), then all restriction maps are surjective. Hence, all Euler classes e_V for surjective characters are regular and $\pi_*^{\text{toral}}(E)$ is regular, as desired. We now conversely assume that $\pi_*^{\text{toral}}(E)$ is a regular global group law. We show by induction on the rank of $\pi_0(A)$ that the natural map $\pi_*^{\text{toral}}(E)(A) \rightarrow \pi_*^A(E)$ is an isomorphism for every abelian compact Lie group A . For this we choose an embedding $i_A: A \hookrightarrow T_A$ into a torus and a basis V_1, \dots, V_k for the kernel of $i_A^*: T_A^* \rightarrow A^*$. We can assume that k equals the rank of $\pi_0(A)$. Let $B = \ker(V_1) \cap \dots \cap \ker(V_{k-1})$. Then by the induction hypothesis the natural map $\pi_*^{\text{toral}}(E)(B) \rightarrow \pi_*^B(E)$ is an isomorphism. Furthermore, the restriction of V_k to B^* is a surjective character and hence $e_{\text{res}_{B^*}^{T_A}(V_k)} \in \pi_*^{\text{toral}}(E)(B) \cong \pi_*^B(E)$ is a regular element. The long

exact sequence (5.24) associated to $\text{res}_B^{T_A}(V_k)$ together with the formula for the left Kan extension from Section 5.1.1 then implies that we have an isomorphism

$$\pi_*^{\text{toral}}(E)(A) \xrightarrow{\cong} \pi_*^{\text{toral}}(E)(B)/(e_{\text{res}_B^{T_A}(V_k)}) \xrightarrow{\cong} \pi_*^B(E)/(e_{\text{res}_B^{T_A}(V_k)}) \xrightarrow{\cong} \pi_*^A(E),$$

which finishes the proof. \square

Hence, the relationship between topology and algebra is tightest if one restricts to regular global group laws. For general complex oriented theories one either needs to somehow implement the long exact sequences (5.24) in the definition of a global group law, making the theory very cumbersome to work with, or to disregard the classes that are not restrictions of toral ones. Since the global Lazard ring is left Kan extended from tori, all the equivariant formal group law data lies in the restrictions of toral classes. So the latter simplification is quite harmless.

We will particularly be interested in global complex bordism \mathbf{MU} and its periodic version $\mathbf{MUP} = \bigvee_{i \in \mathbb{Z}} \mathbf{MU} \wedge S^{2i}$, both described in [Sch18, Example 6.1.53].

Corollary 5.25. *The global group laws of \mathbf{MU} and \mathbf{MUP} are regular.*

Proof. The coefficients of MU_A are concentrated in even degrees for every abelian compact Lie group A , which was announced by Löffler in [Löff73]. A proof by Comezaña can be found in [Com96, XXVIII Theorem 5.3], Hence the coefficients of MUP_A are also concentrated in even degree. Therefore, Part (2) of Lemma 5.23 gives the statement. \square

5.4.1. *Relation to complex oriented A -ring spectra.* We discuss the relation to complex orientations for a fixed abelian compact Lie group A , in analogy to Section 4.3.1 in the real case. Let Y be a commutative A -homotopy ring spectrum (short: A -ring spectrum). We write $\mathbb{C}P(\mathcal{U}_A^{\mathbb{C}})$ for a complete complex A -universe, i.e., a countably infinite dimensional complex A -representation into which any finite dimensional A -representation embeds. Given a character $V \in A^*$, there is an A -homeomorphism $\mathbb{C}P(\epsilon \oplus V) \cong S^V$ sending v to $[1 : v]$.

Definition 5.26. A complex orientation of Y is an element $t^{(A)} \in Y^2(\mathbb{C}P(\mathcal{U}_A^{\mathbb{C}}))$ which restricts to 1 in $Y^*(\mathbb{C}P(\epsilon \oplus \epsilon))$ and to an $RO(A)$ -graded unit in $Y^*(\mathbb{C}P(\epsilon \oplus V))$ for all other characters $V \in A^*$.

Again, we differ slightly from the definition in [CGK00] in that they do not require $t^{(A)}$ to restrict to 1 at the trivial group but merely to a unit. Pulling back $t^{(A)}$ along the 0-section of the universal A -equivariant line bundle gives the universal Euler class or coordinate $y(\epsilon) \in Y^2(\mathbb{C}P(\mathcal{U}_A^{\mathbb{C}})_+)$. Cole–Greenlees–Kriz [CGK00] then show that the quintuple

$$(Y^*, Y^*(\mathbb{C}P(\mathcal{U}_A^{\mathbb{C}})_+), \Delta, \theta, y(\epsilon))$$

is a graded A -equivariant formal group law, where Δ is induced by the multiplication map $m: (\mathbb{C}P(\mathcal{U}_A^{\mathbb{C}}) \times \mathbb{C}P(\mathcal{U}_A^{\mathbb{C}}))_+ \rightarrow \mathbb{C}P(\mathcal{U}_A^{\mathbb{C}})_+$ classifying tensor products of line bundles via

$$Y^*(\mathbb{C}P(\mathcal{U}_A^{\mathbb{C}})_+) \xrightarrow{m^*} Y^*((\mathbb{C}P(\mathcal{U}_A^{\mathbb{C}}) \times \mathbb{C}P(\mathcal{U}_A^{\mathbb{C}}))_+) \cong Y^*(\mathbb{C}P(\mathcal{U}_A^{\mathbb{C}})_+) \hat{\otimes}_{Y^*} Y^*(\mathbb{C}P(\mathcal{U}_A^{\mathbb{C}})_+).$$

The map $\theta(V): Y^*(\mathbb{C}P(\mathcal{U}_A^{\mathbb{C}})_+) \rightarrow Y^*$ is induced by the map $S^0 \rightarrow \mathbb{C}P(\mathcal{U}_A^{\mathbb{C}})_+$ classifying V .

When E is a complex oriented global ring spectrum, we again make use of the homotopy orbit map

$$E_{\mathbb{T}}^2(S^{\tau}) \xrightarrow{\text{pr}_{\mathbb{C}_2}^*} E_{A \times \mathbb{T}}^2(S^{\tau}) \xrightarrow{h_{A, \mathbb{T}}(S^{\tau})} E_A^2((S^{\tau})_{h_{\mathbb{T}}A}) \cong E_A^2(\mathbb{C}P(\mathcal{U}_A^{\mathbb{C}}))$$

to push the global orientation t forward to an orientation $t^{(A)}$ for E_A . This uses that, similarly to the real case, the universal A -equivariant line bundle can be constructed by applying (\mathbb{T}, A) -homotopy orbits to the \mathbb{T} -bundle $\tau \rightarrow *$, and that its Thom space is again \mathbb{T} -homeomorphic to $\mathbb{C}P(\mathcal{U}_A^{\mathbb{C}})$. That $t^{(A)}$ indeed defines an orientation for E_A follows from the fact that the composite

$$E_{\mathbb{T}}^2(S^{\tau}) \rightarrow E_A^2((S^{\tau})_{h_{\mathbb{T}}A}) \rightarrow E_A^2(S^V)$$

equals restriction along $V: A \rightarrow \mathbb{T}$, as a consequence of Lemma 3.4.

Furthermore, the homotopy orbit maps give rise to commutative diagrams

$$\begin{array}{ccccc} \pi_*^{\mathbb{T}}(E) & \xrightarrow{\text{pr}_{\mathbb{T}}^*} & \pi_*^{A \times \mathbb{T}}(E) & \longrightarrow & E_A^{-*}(\mathbb{CP}(\mathcal{U}_A^{\mathbb{C}})_+) \\ & \searrow V^* & \downarrow (\text{id}_A, V)^* & & \downarrow \theta(V) \\ & & \pi_*^A(E) & \xrightarrow{\cong} & E_A^{-*} \end{array}$$

and

$$\begin{array}{ccccc} \pi_*^{A \times \mathbb{T} \times \mathbb{T}}(E) & \longrightarrow & E^{-*}(\mathbb{CP}(\mathcal{U}_A^{\mathbb{C}}) \times \mathbb{CP}(\mathcal{U}_A^{\mathbb{C}})_+) & \xrightarrow{\cong} & E^{-*}(\mathbb{CP}(\mathcal{U}_A^{\mathbb{C}})_+) \hat{\otimes} E^{-*}(\mathbb{CP}(\mathcal{U}_A^{\mathbb{C}})_+) \\ \uparrow (\text{id}_A \times m)^* & & \uparrow m^* & \nearrow \Delta & \\ \pi_*^{A \times \mathbb{T}}(E) & \longrightarrow & E_A^{-*}(\mathbb{CP}(\mathcal{U}_A^{\mathbb{C}})_+) & & \\ \uparrow \text{pr}_A^* & & \uparrow & & \\ \pi_*^A(E) & \xrightarrow{\cong} & E_A^{-*} & & \end{array}$$

We also have:

Proposition 5.27. *The homotopy orbit maps*

$$h_{A, \mathbb{T}^r}(S^0): \pi_*^{A \times \mathbb{T}^r}(E) \rightarrow E_A^{-*}(\mathbb{CP}(\mathcal{U}_A^{\mathbb{C}})^{\times r}) \cong E_A^{-*}(\mathbb{CP}(\mathcal{U}_A^{\mathbb{C}})_+)^{\hat{\otimes} r}$$

are completion with respect to finite products of the augmentation ideals

$$I_\alpha = \ker((\text{id}_A, \alpha)^*: \pi_*^{A \times \mathbb{T}^r}(E) \rightarrow \pi_*^A(E))$$

for homomorphisms $\alpha: A \rightarrow \mathbb{T}^r$.

Proof. The proof is analogous to that of [Proposition 4.22](#). □

Moreover, $h_{A, \mathbb{T}}(S^0)$ sends the Euler class $e_{(\epsilon, \text{id})} \in \pi_{-2}^{A \times \mathbb{T}}(E)$ to $y(\epsilon) \in E_{\mathbb{T}}^2(\mathbb{CP}(\mathcal{U}_A^{\mathbb{C}})_+)$. Hence, the A -equivariant formal group law associated to E_A is encoded in the functoriality of the coefficients $\pi_*(E)$ and the universal Euler class.

5.5. The adjunction between global group laws and A -equivariant formal group laws. We now describe the process of constructing an A -equivariant formal group law out of an arbitrary global group law not necessarily induced by a complex oriented global ring spectrum. For the reasons described earlier, we first restrict to the case where A is a torus. The constructions are very similar to those in [Sections 4.4, 4.5 and 4.6](#), so we will be brief.

Let X be a global group law and A, B tori. We denote by $X(A \times B)_A^\wedge$ the completion at finite products of ideals I_α for maps $\alpha: A \rightarrow B$ (as in [Proposition 5.27](#)). Then we have:

Lemma 5.28. (1) *Given another torus B' , the projection maps induce an isomorphism*

$$X(A \times B)_A^\wedge \hat{\otimes}_{X(A)} X(A \times B')_A^\wedge \xrightarrow{\cong} X(A \times B \times B')_A^\wedge.$$

(2) *The quintuple*

$$(X(A), X(A \times \mathbb{T})_A^\wedge, \Delta, \theta, y(\epsilon))$$

is an A -equivariant formal group law, where Δ is given by

$$(\text{id}_A \times m)^*: X(A \times \mathbb{T})_A^\wedge \rightarrow X(A \times \mathbb{T} \times \mathbb{T})_A^\wedge \cong X(A \times \mathbb{T})_A^\wedge \hat{\otimes}_{X(A)} X(A \times \mathbb{T})_A^\wedge,$$

$\theta(V)$ is induced by $(\text{id}_A, V)^: X(A \times \mathbb{T}) \rightarrow X(A)$ and $y(\epsilon)$ is the image of $e_{(\epsilon, \text{id})} \in X(A \times \mathbb{T})$ under the completion map. We denote this A -equivariant formal group law by X_A^\wedge .*

Proof. See [Section 4.4](#) for the proof in the 2-torsion case. □

Here, we make use of A being a torus, because otherwise the element $e_{(\epsilon, \text{id})} \in X(A \times \mathbb{T})$ would in general not be regular. In [Section 5.5.1](#) we describe a way to get around this problem and associate an A -equivariant formal group law X_A^\wedge to X also when A is not a torus.

To define the functor $(-)_\text{gl}$ in the other direction we do not need to restrict to tori. Let A be an arbitrary abelian compact Lie group, and let $F = (k, R, \Delta, \theta, y(\epsilon))$ be an A -equivariant formal group law. We recall from [Section 4.5](#) that \mathcal{C}_k denotes the category of complete commutative linear topological k -algebras. Given $S \in \mathcal{C}_k$, let $R(S)$ denote the set of morphisms $R \rightarrow S$ in \mathcal{C}_k equipped with the abelian group structure defined via

$$f + g = (R \xrightarrow{\Delta} R \hat{\otimes} R \xrightarrow{f \hat{\otimes} g} S \hat{\otimes} S \xrightarrow{m} S).$$

The existence of inverses for this addition follows from R having a coinverse, see [Remark 2.5](#). Furthermore, the augmentation θ induces a group homomorphism $A^* \rightarrow R(k)$, implying that the functor $R(-)$ takes values in abelian groups under A^* . We have:

Lemma 5.29. *The functor*

$$\text{Hom}_{A^*/}(A^* \times B^*, R(-))$$

is representable for every torus B .

Proof. Choosing a basis V_1, \dots, V_n for B^* , we find that $R^{\hat{\otimes} n}$ is a representing object. □

We choose one such representing object $F_\text{gl}(B)$ for every B . Then the assignment $B \mapsto F_\text{gl}(B)$ has a canonical extended functoriality, as in [Section 4.5.1](#). This means that every map $A \times B \rightarrow A \times B'$ which lies over the projections to A induces a restriction map $F_\text{gl}(B') \rightarrow F_\text{gl}(B)$. If we only remember the restriction maps along $A \times B \xrightarrow{\text{id}_A \times f} A \times B'$ for homomorphisms $f: B \rightarrow B'$, we obtain a functor

$$F_\text{gl}: (\text{tori})^{\text{op}} \rightarrow \text{commutative rings}.$$

Furthermore we have:

Lemma 5.30. *The class $y(\epsilon) \in R \cong F_\text{gl}(\mathbb{T})$ is a coordinate for F_gl .*

Proof. See the proof of [Lemma 4.26](#). □

Remark 5.31. The construction $(-)_\text{gl}$ works more generally for any cocommutative commutative complete linear topological algebra over k which is equipped with a regular generator of its augmentation ideal and which allows an antipode. In this case one needs to set A to be the trivial group in the above.

Proposition 5.32. *Let A be a torus. Then:*

(1) *There is an adjunction*

$$(-)_A^\wedge: GL_\text{gl} \rightleftarrows FGL_A: (-)_\text{gl}.$$

The unit $X \rightarrow (X_A^\wedge)_\text{gl}$ is given in degree B by the composite

$$X(B) \xrightarrow{\text{pr}_B^*} X(A \times B) \rightarrow X(A \times B)_A^\wedge \cong (X_A^\wedge)_\text{gl}(B).$$

The counit $(F_\text{gl})_A^\wedge \rightarrow F$ is the pair of maps $F_\text{gl}(A) \rightarrow F_\text{gl}(1) \cong k$ and $F_\text{gl}(A \times \mathbb{T})_A^\wedge \rightarrow F_\text{gl}(\mathbb{T}) \cong R$ induced by the diagonals $\text{diag}: A \rightarrow A \times A$ and $\text{diag} \times \text{id}_\mathbb{T}: A \times \mathbb{T} \rightarrow A \times A \times \mathbb{T}$ via the extended functoriality of F_gl .

(2) *Given a complex oriented global ring spectrum E , there is a natural isomorphism*

$$(\pi_*(E))_A^\wedge \cong F(E_A),$$

where $F(E_A)$ is the A -equivariant formal group law associated to the induced complex orientation of E_A .

Proof. As in the proof of [Proposition 4.29](#), the adjunction factors naturally through the category $GL_{\text{gl}/A}$ of global group laws with extended functoriality. The functor from $GL_{\text{gl}/A}$ forgets the extended functoriality, with left adjoint sending X to $X(A \times -)$. For the second adjunction, the functor $(-)_\text{gl}/A$ embeds A -equivariant formal group laws inside $GL_{\text{gl}/A}$ as the complete objects, and the adjunction unit is the completion map. We refer to the proof of [Proposition 4.29](#) for more details.

Part (2) follows from [Proposition 5.27](#) and the diagrams preceding it. □

Corollary 5.33. *There is an isomorphism of Ab-algebras $\mathbf{L} \cong \tilde{\mathbf{L}}$ sending the universal Euler class $e = e_\tau \in L_\mathbb{T}$ to the universal coordinate $e \in \tilde{\mathbf{L}}(\mathbb{T})$.*

Hence, the pair (\mathbf{L}, e) is the universal global group law.

Proof. Since any left adjoint preserves initial objects, the A -equivariant formal group law $\tilde{\mathbf{L}}_A^\wedge$ must be the universal one for any torus A . In particular, $\tilde{\mathbf{L}}(A)$ is canonically isomorphic to L_A . Moreover, given any global group law X , the Euler class $e_\tau \in X(\mathbb{T})$ of the associated \mathbb{T} -equivariant formal group law is given as the pullback of $e_{(\epsilon, \text{id})} \in X(\mathbb{T} \times \mathbb{T})$ along the diagonal $\mathbb{T} \rightarrow \mathbb{T} \times \mathbb{T}$, which gives back the orientation class $e \in X(\mathbb{T})$. The same arguments as in the 2-torsion case (see Section 4.6) then apply to show that these degreewise isomorphisms assemble to a natural isomorphism of functors $(\text{tori})^{\text{op}} \rightarrow (\text{commutative rings})$. Now the universal global group law $\tilde{\mathbf{L}}$ is by definition left Kan extended from tori , and Corollary 2.8 shows that the same is true for \mathbf{L} , hence we obtain a canonical isomorphism $\mathbf{L} \cong \tilde{\mathbf{L}}$ of Ab-algebras, as desired. \square

In particular we have the following regularity of Euler classes in Lazard rings, which we will later strengthen to all Euler classes for surjective characters in Corollary 5.50:

Corollary 5.34. *If A is a torus and $V: A \rightarrow \mathbb{T}$ is split, then the Euler class $e_V \in L_A$ is regular.*

5.5.1. *Adjunction for non-tori.* Corollary 5.33 can be used to define an associated A -equivariant formal group law X_A^\wedge for general abelian compact Lie groups A , in the following way. Evaluating the unique map $\mathbf{L} \rightarrow X$ of global group laws at A yields a map $L_A \rightarrow X(A)$. We define X_A^\wedge to be the A -equivariant formal group law classified by this map. Unravelling the definitions, X_A^\wedge can be computed as follows: Choose an embedding $i_A: A \hookrightarrow T_A$ into a torus, and consider the pushforward of the T_A -equivariant formal group law $X_{T_A}^\wedge$ along $(i_A)^*: X(T_A) \rightarrow X(A)$. This defines a T_A -equivariant formal group law over $X(A)$ for which an Euler class e_V vanishes if V restricts to ϵ in A^* (simply because i_A^* sends these classes to ϵ , see Section 5.1.1). By Lemma 2.7, this is the same datum as an A -equivariant formal group law; one checks that it agrees with X_A^\wedge .

Note that the functor $(-)_\text{gl}: FGL_A \rightarrow GL_\text{gl}$ factors as $FGL_A \xrightarrow{(i_A)^*} FGL_{T_A} \xrightarrow{(-)_\text{gl}} GL_\text{gl}$. Since $(i_A)_*$ is right adjoint to the functor quotienting out all Euler classes e_V for V in the kernel of i_A^* , it follows formally that there is an adjunction

$$(-)_A^\wedge: GL_\text{gl} \rightleftarrows FGL_A: (-)_\text{gl}.$$

for general A .

If E is a complex oriented global ring spectrum, then it is in general not quite true that $\pi_*^{\text{toral}}(E)_A^\wedge$ agrees with the A -equivariant formal group law $F(E_A)$ associated to E_A , since $\pi_*^A(E_A)$ does not necessarily agree with $\pi_*^{\text{toral}}(E)(A)$. However, there is a natural map $\pi_*^{\text{toral}}(E)(A) \rightarrow \pi_*^A(E_A)$ and one checks that $F(E_A)$ is the pushforward of $\pi_*^{\text{toral}}(E)_A^\wedge$ along this map. If $\pi_*^{\text{toral}}(E)_A^\wedge$ is regular, these problems disappear by Lemma 5.23 and $\pi_*^{\text{toral}}(E)_A^\wedge$ is canonically isomorphic to $F(E_A)$.

5.6. The global Lazard ring is 2-regular. Our strategy for proving Theorems A and C is again to reduce to geometric fixed points. For this we need to extend our understanding of the regularity of Euler classes in the global Lazard ring beyond split characters. As $\pi_*(\mathbf{MU})$ is concentrated in even degrees, we know that it is a regular global group law (Corollary 5.25). Hence every $e_V \in \pi_*^A(\mathbf{MU})$ with $V \in A^*$ surjective is regular, for any abelian compact Lie group A . We want to prove that \mathbf{L} is isomorphic to $\pi_*(\mathbf{MU})$, so the same must be true for \mathbf{L} . The goal of this subsection is to first show something weaker:

Proposition 5.35. *The global Lazard ring \mathbf{L} is 2-regular in the sense of Section 5.2: For every torus A and every pair of linearly independent characters $V_1, V_2 \in A^*$ the sequence (e_{V_1}, e_{V_2}) is regular in L_A , i.e., e_{V_1} is a regular element of L_A and e_{V_2} is a regular element of $L_A/(e_{V_1}) \cong L_{\ker(V_1)}$.*

This turns out to be enough to prove Theorems A and C. Knowing that $\pi_*(\mathbf{MU})$ is regular, we can then deduce the regularity of \mathbf{L} from its isomorphism to $\pi_*(\mathbf{MU})$. I know of no purely algebraic proof that \mathbf{L} is fully regular.

We now turn to the proof of Proposition 5.35. Let A be a torus. By now we only know that the Euler classes for split characters are regular in L_A . If we want to apply our previous method for showing Euler class regularity once more, we run into the following problem: It is in general not the case that the global group law F_gl associated to an A -equivariant formal group law F is 2-regular. For example, the Euler class

$e_p \in F_{gl}(\mathbb{T})$ for the p -th power map on the circle is given by the p -series $[p]_F(y(\epsilon))$. Taking A to be the trivial group and F the additive formal group law over \mathbb{F}_p , this p -series is trivial and hence far from being a regular element. This means that the adjunction of [Proposition 5.32](#) does not restrict to an adjunction between 2-regular global group laws and all A -equivariant formal group laws. Instead one needs to restrict to a subclass of A -equivariant formal group laws satisfying extra conditions. This is problematic because we do not know a priori that the universal A -equivariant formal group law satisfies these conditions. We overcome this problem in two steps:

- Step 1: Localizing at a prime p gives us control over all sequences (e_{V_1}, e_{V_2}) for linearly independent characters $V_1, V_2 \in A^*$ which remain linearly independent in the quotient A^*/pA^* .
- Step 2: The Euler classes for the remaining linearly independent pairs of characters are then studied via the effect of the p -th power map on L_A and $L_{\ker(V_1)}$, using [Proposition 2.9](#) on the interplay of localized Lazard rings and short exact sequences of abelian compact Lie groups.

We start with **Step 1**: Let p be a prime number.

Definition 5.36. We say that a global group law X is $(p, 2)$ -regular if

- (1) $X(1)$ is a $\mathbb{Z}_{(p)}$ -algebra (and hence so is $X(A)$ for any A), and
- (2) for all tori A and characters $V_1, V_2 \in A^*$ which are linearly independent modulo p , i.e., their images in the \mathbb{F}_p -vector space A^*/pA^* are linearly independent, the Euler classes (e_{V_1}, e_{V_2}) form a regular sequence in $X(A)$.

Lemma 5.37. Let k be a $\mathbb{Z}_{(p)}$ -algebra and B an abelian compact Lie group. Then for every B -equivariant formal group law F with ground ring k , the global group law F_{gl} is $(p, 2)$ -regular.

Proof. Let A be a torus and $V_1, V_2 \in A^*$ be linearly independent modulo p . We first make a few preliminary reductions on V_1 and V_2 which apply to any global group law X . To start, we can choose coordinates on $A \cong \mathbb{T}^r$ and hence $A^* \cong \mathbb{Z}^r$ so that V_1 and V_2 are given by the vectors $(n_1, 0, \dots, 0)$ and $(a, n_2, 0, \dots, 0)$ respectively, where n_1 and n_2 are natural numbers coprime to p . We now argue that for showing that (e_{V_1}, e_{V_2}) form a regular sequence in $X(A)$ for all such pairs V_1, V_2 , we can reduce to the case where $a = 0$ and hence the coefficient vector for V_2 is of the form $(0, n_2, 0, \dots, 0)$ with n_2 coprime to p . For this we consider the character $V_2^{n_1}$, the n_1 -fold power of V_2 , whose coefficient vector is $(a \cdot n_1, n_2 \cdot n_1, 0, \dots, 0)$. Since the Euler class e_{V_2} divides the Euler class $e_{V_2^{n_1}}$ (this is a consequence of the universal Euler class $e_\tau = e \in X(\mathbb{T})$ dividing the Euler class e_{τ^n} , which follows from the fact that e_{τ^n} is sent to 0 under restriction to the trivial group), it suffices to show that $e_{V_2^{n_1}}$ is a regular element in the quotient $X(A)/(e_{V_1})$. But modulo e_{V_1} , the Euler class $e_{V_2^{n_1}}$ agrees with the Euler class associated to the vector $(0, n_2 \cdot n_1, 0, \dots, 0)$, since the character with coefficient vector $(a \cdot n_1, 0, \dots, 0)$ vanishes in the quotient $A^*/(V_1)$ (cf. [Section 5.1.1](#)). This proves the claim.

We now specialize to the case F_{gl} for a B -equivariant formal group law B . By the above, we have to show that for every pair of natural numbers n_1, n_2 coprime to p and $r \geq 2$, the Euler classes e_{V_1}, e_{V_2} associated to the vectors $V_1 = (n_1, 0, \dots, 0)$ and $V_2 = (0, n_2, 0, \dots, 0)$ form a regular sequence in $F(\mathbb{T}^r)$. We have an isomorphism

$$F_{gl}(A) \cong R \hat{\otimes} R \hat{\otimes} R^{\hat{\otimes} r-2},$$

with Euler classes e_{V_1} and e_{V_2} given by $[n_1]_F(y(\epsilon)) \otimes 1 \otimes 1$ and $1 \otimes [n_2]_F(y(\epsilon)) \otimes 1$. Modulo $y(\epsilon)^2$, the n_i -series $[n_i]_F(y(\epsilon))$ equals $n_i y(\epsilon)$. By assumption, k is a $\mathbb{Z}_{(p)}$ -algebra, and hence n_1 and n_2 are units in k . It follows that both $[n_i]_F(y(\epsilon))$ are regular elements in R (note that in order to be a regular element in R , it is sufficient that the leading term with respect to any choice of flag is a regular element in k), which directly implies that (e_{V_1}, e_{V_2}) is a regular sequence in $F_{gl}(A)$. \square

Remark 5.38. The proof shows that the p -regularity of global group laws associated to p -local B -equivariant formal group laws is not specific to the number two. More generally, let A be a torus and B an abelian compact Lie group. Further, let F be a B -equivariant formal group law defined over a p -local ring k and $V_1, \dots, V_n \in A^*$ an n -tuple of characters which is linearly independent modulo p . Then $(e_{V_1}, \dots, e_{V_n})$ is a regular sequence in $F_{gl}(A)$.

The lemma implies that the adjunction between global group laws and B -equivariant formal group laws restricts to an adjunction

$$(-)_B^\wedge : GL_{gl}^{(p,2)\text{-reg}} \rightleftarrows FGL_B^{(p)} : (-)_{gl},$$

where $FGL_B^{(p)} \subseteq FGL_B$ denotes the category of B -equivariant formal group laws over p -local rings.

Once again we want to apply [Lemma 4.9](#) (this time to the category of $(\mathbb{Z}_e)_{(p)}$ -algebras) to see that there exists a universal $(p, 2)$ -regular global group law. For this we need to see that $(p, 2)$ -regular global group laws are the local objects for some set of morphisms. Note that we can rephrase the conditions for a $(\mathbb{Z}_e)_{(p)}$ -algebra X to be a $(p, 2)$ -regular global group law in the following way, where A ranges through all tori:

- (1) If $V \in A^*$ is a split character, then

$$0 \rightarrow X(A) \xrightarrow{e_V} X(A) \xrightarrow{\text{res}_{\ker(V)}^A} X(\ker(V)) \rightarrow 0$$

is exact.

- (2) Let $V_1, V_2 \in A^*$ be characters which are linearly independent modulo p . If two elements $x, y \in X(A)$ satisfy

$$x \cdot e_{V_1} = y \cdot e_{V_2},$$

then y is uniquely divisible by e_{V_1} .

To see the equivalence, note that in the second item, divisibility of y by e_{V_1} guarantees that e_{V_2} is a regular element in $X(A)/(e_{V_1})$, and the uniqueness implies that e_{V_1} is a regular element in $X(A)$ (set $x = y = 0$). We have already seen that properties of type (1) are equivalent to being local with respect to the morphisms $(f_{A,V})_{(p)}$, see [Proposition 5.4](#). For the second type we can take

$$g_{A,V_1,V_2} : (\mathbb{Z}_e)_{(p)}[x, y] / (x \cdot e_{V_1} - y \cdot e_{V_2}) \rightarrow (\mathbb{Z}_e)_{(p)}[x, y, y'] / (x \cdot e_{V_1} - y \cdot e_{V_2}, e_{V_1} \cdot y' - y).$$

Hence $(p, 2)$ -regular global group laws are the local objects with respect to this set of morphisms, and we obtain a universal $(p, 2)$ -regular global group law $\mathbf{L}^{(p,2)}$.

Remark 5.39. The existence of a universal object is less clear for (p, n) -regularity with $n > 2$ (defined in the obvious way). This is the reason why in this section we have to restrict to showing 2-regularity of \mathbf{L} , and later use the comparison with $\pi_*(\mathbf{MU})$ to deduce the full regularity.

It follows that for any torus A , the left adjoint functor $(-)_A^\wedge$ must take $\mathbf{L}^{(p,2)}$ to the universal A -equivariant formal group law over p -local rings. By the same arguments as in [Sections 4.6 and 5.5](#) we deduce that $\mathbf{L}^{(p,2)}$ is canonically isomorphic to $\mathbf{L}_{(p)}$, the p -localization of the global Lazard ring. In particular we have shown that $\mathbf{L}_{(p)}$ is $(p, 2)$ -regular, finishing the proof of Step 1. We note that this in particular implies the following:

Corollary 5.40. *For every torus A and character $V \in A^*$ which is not divisible by p , the Euler class $e_V \in (L_A)_{(p)}$ is a regular element.*

Proof. We consider the larger torus $\mathbb{T} \times A$ and the pair of characters (V_1, q^*V) , where V_1 is the projection to the first component and $q : \mathbb{T} \times A \rightarrow A$ is the projection. Then (V_1, q^*V) are linearly independent modulo p , and hence the reduction of e_{q^*V} to the quotient $(L_{\mathbb{T} \times A})_{(p)} / (e_{V_1})$ is a regular element. Under the isomorphism $(L_{\mathbb{T} \times A})_{(p)} / (e_{V_1}) \cong L_A$, this reduction agrees with e_V , proving the claim. \square

We now come to **Step 2**: Let A be a torus. Continuing to work p -locally, it remains to consider Euler classes of pairs of characters $V_1, V_2 \in A^*$ which are linearly independent, but not necessarily linearly independent modulo p . We first discuss the case where V_1 is not divisible by p but V_2 is arbitrary. Hence we know by [Corollary 5.40](#) that $e_{V_1} \in (L_A)_{(p)}$ is regular, and our goal is to show that $\text{res}_{\ker(V_1)}^A(e_{V_2})$ is regular in $(L_{\ker(V_1)})_{(p)}$. For this we choose a homomorphism $\beta_p : \ker(V_1) \rightarrow \ker(V_1)$ which induces the identity on $\pi_0(\ker(V_1))$ and equals the p -th power map on the identity component. We have the following lemmata:

Lemma 5.41. *Let $V \in \ker(V_1)^*$ be a character not in the image of $\beta_p^* : \ker(V_1)^* \rightarrow \ker(V_1)^*$. Then the associated Euler class $e_V \in (L_{\ker(V_1)})_{(p)}$ is a regular element.*

Proof. A character $V \in \ker(V_1)^*$ lies in the image of β_p^* if and only if its restriction to the identity component $(\ker(V_1))_1^*$ is divisible by p . Hence, if V is not in the image, it is (1) itself not divisible by p and (2) a surjective character (i.e., of infinite order in $\ker(V_1)^*$), since any torsion character is trivial on the identity component and hence in the image of β_p^* . Let W be a lift of V to a character for A . Then combining (1) and (2) with the assumption that V_1 is not divisible by p shows that the pair (V_1, W) is linearly independent modulo p . So Step 1 implies that $e_V \in (L_{\ker(V_1)})_{(p)}$ is regular, as desired. \square

Lemma 5.42. *Let $W \in \ker(V_1)^*$ be a character and assume that $e_W \in (L_{\ker(V_1)})_{(p)}$ is regular. Then $e_{\beta_p^*(W)} \in (L_{\ker(V_1)})_{(p)}$ is also regular.*

Proof. Since β_p is surjective, we obtain a short exact sequence

$$1 \rightarrow B \rightarrow \ker(V_1) \xrightarrow{\beta_p} \ker(V_1) \rightarrow 1.$$

Applying [Proposition 2.9](#), we see that the composition

$$(L_{\ker(V_1)})_{(p)} \xrightarrow{(\beta_p)^*} (L_{\ker(V_1)})_{(p)} \rightarrow (L_{\ker(V_1)})_{(p)}[e_V^{-1} \mid V \notin \text{im}(\beta_p^*)]$$

is a flat map. This means that if e_W is regular in $(L_{\ker(V_1)})_{(p)}$, then its image under $(\beta_p)^*$, i.e., $e_{\beta_p^*(W)}$, is a regular element in $(L_{\ker(V_1)})_{(p)}[e_V^{-1} \mid V \notin \text{im}(\beta_p^*)]$. By [Lemma 5.41](#), each $e_V \in (L_{\ker(V_1)})_{(p)}$ for $V \notin \text{im}(\beta_p^*)$ is a regular element. Hence the map

$$(L_{\ker(V_1)})_{(p)} \rightarrow (L_{\ker(V_1)})_{(p)}[e_V^{-1} \mid V \notin \text{im}(\beta_p^*)]$$

is injective and therefore $e_{\beta_p^*(W)}$ must also be regular in $(L_{\ker(V_1)})_{(p)}$, proving the lemma. \square

Now, since $\text{res}_{\ker(V_1)}^A(V_2) \in \ker(V_1)^*$ is surjective, it can be written as $\text{res}_{\ker(V_1)}^A(V_2) = (\beta_p^*)^m(W)$ for some $m \in \mathbb{N}$ and surjective character $W \in \ker(V_1)^*$ whose restriction to the component of the identity is not divisible by p , i.e., which is not in the image of β_p^* . By [Lemma 5.41](#), the associated Euler class e_W of W is then regular in $(L_{\ker(V_1)})_{(p)}$. Iterated application of [Lemma 5.42](#) shows that $e_{\text{res}_{\ker(V_1)}^A(V_2)}$ is also regular in $(L_{\ker(V_1)})_{(p)}$. This finishes the proof of the case where V_1 is not divisible by p and V_2 is arbitrary. Again, this in particular implies the following:

Corollary 5.43. *Let A be a torus and $V \in A^*$ a non-trivial character. Then $e_V \in (L_A)_{(p)}$ is regular.*

Proof. The proof is analogous to that of [Corollary 5.40](#): Let $V_1 \in (\mathbb{T} \times A)^*$ be the projection to the first component and $q: \mathbb{T} \times A \rightarrow A$ the projection to the second component. Then the pair of characters $(V_1, q^*V) \in (\mathbb{T} \times A)^*$ is linearly independent with V_1 not divisible by p . Hence by what we just saw the Euler class $\text{res}_A^{\mathbb{T} \times A}(e_{q^*V}) = e_V \in (L_A)_{(p)}$ is regular. \square

Knowing this, we see that for tori A and non-trivial characters $V_1, V_2 \in A^*$, the pair (e_{V_1}, e_{V_2}) is a regular sequence if and only if the pair (e_{V_2}, e_{V_1}) is a regular sequence. Indeed, if $x \cdot e_{V_1} = y \cdot e_{V_2}$ and we assume that (e_{V_1}, e_{V_2}) is regular, the element y must be divisible by e_{V_1} , say $y = y' \cdot e_{V_1}$. This yields that $x \cdot e_{V_1} = y' \cdot e_{V_2} \cdot e_{V_1}$, which by regularity of e_{V_1} implies that x is divisible by e_{V_2} . Since we already know that e_{V_2} is regular, this proves that the sequence (e_{V_2}, e_{V_1}) is regular.

Hence, we also know that (e_{V_1}, e_{V_2}) is a regular sequence in $(L_A)_{(p)}$ if (V_1, V_2) are linearly independent and V_2 is not divisible by p , with no further condition on V_1 . But now we can choose $\beta_p: \ker(V_1) \rightarrow \ker(V_1)$ as above and repeat the exact same argument to deduce the case where V_2 is also arbitrary. This finishes the proof that $\mathbf{L}_{(p)}$ is 2-regular for all p . Since regularity can be checked locally, [Proposition 5.35](#) follows.

Corollary 5.44. *If A is a torus, the Lazard ring L_A is an integral domain.*

Proof. By [Proposition 2.11](#) we know that for any abelian compact Lie group A the Lazard ring L_A becomes an integral domain after inverting all Euler classes of non-trivial characters, since the non-equivariant Lazard ring L is a polynomial algebra over the integers [\[Laz55\]](#). If A is a torus we just saw that these Euler classes are regular elements, so it follows that L_A itself is a domain. \square

5.7. Geometric fixed points and proof of the main theorem. Let X be a global group law. As in [Section 4.7](#), we define the geometric fixed points $\Phi^A(X)$ at any abelian compact Lie group A by

$$\Phi^A(X) = X(A)[e_V^{-1} \mid V \in A^* - \{\epsilon\}].$$

If $X = \pi_*(E)$ is a regular global group law associated to a global complex oriented ring spectrum, then there is a natural isomorphism

$$\Phi^A(\pi_*(E)) \cong \pi_*(\Phi^A(E)) = \Phi_*^A(E).$$

For general complex oriented E , such an isomorphism only holds at tori.

We now move towards the proof of [Theorems A and C](#) and consider the unique map of global group laws

$$\alpha: \mathbf{L} \rightarrow \pi_*(\mathbf{MU}).$$

Again, the only input from topology that we need is the following:

Proposition 5.45 (Greenlees). *The morphism $\alpha: \mathbf{L} \rightarrow \pi_*(\mathbf{MU})$ of global group laws induces an isomorphism on geometric fixed points Φ^A for every abelian compact Lie group A .*

Proof. The computation of $\Phi^A(MU_A)$ is due to tom Dieck [[tD70](#)] and also appears in [[Sin01](#), Theorem 4.9]. In [Proposition 2.11](#) we saw a formula for $\Phi^A(\mathbf{L})$ (originally also due to Greenlees), showing that the rings are abstractly isomorphic. The proof that φ_A is indeed an isomorphism is given in [[Gre00](#), Section 11] for finite A , but applies without change to infinite A (it is also stated in this generality in [[Gre01](#), Section 13.B.]). We refer to [Proposition 4.36](#) for a proof of the real analog, which straightforwardly translates to the complex case. \square

For real bordism there was a simple Whitehead theorem ([Proposition 4.40](#)) that allowed us to conclude directly from the geometric fixed point statement that the map $\mathbf{L}^{2-\text{tor}} \rightarrow \pi_*(\mathbf{MO})$ is an isomorphism. In the complex case we have to work a little harder. We first show the following structural property of $L_{\mathbb{T}}$, where we denote by $e_n \in L_{\mathbb{T}}$ the Euler class associated to the n -th power map on \mathbb{T} :

Proposition 5.46. *There exists a family of elements $\{\psi_n\}_{n \in \mathbb{N}_{>0}}$ of $\mathbf{L}(\mathbb{T})$ uniquely characterized by the equations*

$$e_n = \prod_{m|n} \psi_m \tag{5.47}$$

for all $n \in \mathbb{N}_{>0}$. Moreover, each ψ_n is a prime element and generates the kernel of the composite

$$\varphi_{C_n}^{\mathbb{T}}: \mathbf{L}(\mathbb{T}) \xrightarrow{\text{res}_{C_n}^{\mathbb{T}}} \mathbf{L}(C_n) \rightarrow \Phi^{C_n}(\mathbf{L}).$$

Example 5.48 (Cyclotomic polynomials). Once the proposition is proved, it defines elements $\psi_n \in X(\mathbb{T})$ for any global group law X . To understand the nature of these elements, it is helpful to consider the example of the multiplicative global group law over \mathbb{Z} ([Example 5.11](#)). Its value at the circle group is the Laurent polynomial ring $\mathbb{Z}[t^{\pm 1}]$ with universal Euler class $t - 1$. The element ψ_n is then the n -th cyclotomic polynomial, and equation (5.47) becomes the standard decomposition of $t^n - 1$ into a product of cyclotomic polynomials. To see why the ψ_n are as claimed in this case, note that the multiplicative global group law is regular and hence the $\psi_n \in \mathbb{Z}[t^{\pm 1}]$ are uniquely characterized by the equations (5.47). Since $\psi_1 = e_1 = t - 1$, the claim follows.

Proof of Proposition 5.46. We prove the proposition by induction on n . By definition, ψ_1 is given by e_1 . We now fix an $n > 1$ and assume we have proved the existence and uniqueness of ψ_m for all m properly dividing n , and that these classes satisfy the properties stated in the proposition. Recall from [Corollary 5.44](#) that $\mathbf{L}(\mathbb{T})$ is a domain, so there can be at most one ψ_n satisfying (5.47).

It remains to prove existence. Note that e_n maps to 0 in $\mathbf{L}(C_m)$ for all m dividing n . Hence e_n is uniquely divisible by e_m for all these m . In particular, each individual such ψ_m divides e_n . We need to show that their product also divides e_n , as then we can (and have to) define ψ_n as the unique element satisfying equation (5.47). Since the ψ_m are prime elements, this will follow once we know that no two ψ_m and $\psi_{m'}$ with $m \neq m'$ agree up to a unit. To see that this is indeed not the case it suffices to find a single example of a global group law for which (the push-forwards of) ψ_m and $\psi_{m'}$ do not differ by a unit. The multiplicative global group law described in [Example 5.48](#) does the job, since it is clear that no two cyclotomic polynomials differ by a unit

in $\mathbb{Z}[t^{\pm 1}]$ as the roots of ψ_m in \mathbb{C} are given by the primitive m -th roots of unity. This proves the existence and uniqueness of ψ_n . The properties of the multiplicative global group law also imply that ψ_n does not divide any of the ψ_m with $m < n$ and vice versa, since this is not the case for cyclotomic polynomials.

It remains to see that ψ_n generates the kernel of the map $\varphi_{C_n}^{\mathbb{T}}$. First note that the product $\psi_n \cdot e_1 \cdots e_{n-1}$ is divisible by e_n , since each ψ_m divides e_m . Each e_i with $0 < i < n$ is by definition sent to a unit under $\varphi_{C_n}^{\mathbb{T}}$, while e_n is sent to 0. It follows that ψ_n lies in the kernel of $\varphi_{C_n}^{\mathbb{T}}$. For the other direction, assume that $x \in \mathbf{L}(\mathbb{T})$ is sent to 0 under $\varphi_{C_n}^{\mathbb{T}}$. This means that its restriction to $\mathbf{L}(C_n)$ is Euler class torsion. In other words there exist numbers m_1, \dots, m_{n-1} such that $x \cdot e_1^{m_1} \cdots e_{n-1}^{m_{n-1}}$ is divisible by e_n . This uses that the kernel of the restriction map is generated by e_n and that e_1, \dots, e_{n-1} restrict to all the non-trivial Euler classes of C_n . Since ψ_n divides e_n , we can hence write

$$x \cdot e_1^{m_1} \cdots e_{n-1}^{m_{n-1}} = x' \cdot \psi_n$$

for some $x' \in \mathbf{L}(\mathbb{T})$. Now each e_i is a product of the regular prime elements ψ_j with $j < n$. As we argued above, none of the ψ_j divide ψ_n , so it follows that the whole product $e_1^{m_1} \cdots e_{n-1}^{m_{n-1}}$ must divide x' . We can cancel by $e_1^{m_1} \cdots e_{n-1}^{m_{n-1}}$ on both sides, showing that x is divisible by ψ_n , as desired. Finally, that ψ_n is again prime follows from the fact that $\varphi_{C_n}^{\mathbb{T}}$ defines an embedding of $\mathbf{L}(\mathbb{T})/(\psi_n)$ into $\Phi^{C_n}(\mathbf{L})$, which is an integral domain by [Proposition 2.9](#). This finishes the proof. \square

In fact, more is true:

Lemma 5.49. *Let A be a torus, $V: A \rightarrow \mathbb{T}$ be a split character and $n \in \mathbb{N}_{>0}$. Then $V^*\psi_n \in \mathbf{L}(A)$ is a prime element and generates the kernel of the composition*

$$\varphi_{\ker(V^n)}^A: \mathbf{L}(A) \xrightarrow{\text{res}_{\ker(V^n)}^A} \mathbf{L}(\ker(V^n)) \rightarrow \Phi^{\ker(V^n)} \mathbf{L}.$$

Proof. Since the global Lazard ring \mathbf{L} is 2-regular ([Proposition 5.35](#)), all Euler classes e_W for surjective characters $W \in (\ker(V^n))^*$ are regular elements of $\mathbf{L}(\ker(V^n))$. Hence, in order to determine the kernel of $\mathbf{L}(\ker(V^n)) \rightarrow \Phi^{\ker(V^n)} \mathbf{L}$, it suffices to consider the Euler-torsion for the non-surjective, non-trivial characters of $\ker(V^n)$. These are given precisely by the restrictions of the characters $V, V^2, \dots, V^{n-1} \in A^*$ to $\ker(V^n)^*$. It then follows that an element $x \in \mathbf{L}(A)$ lies in the kernel of $\varphi_{\ker(V^n)}^A$ if and only if $x \cdot e_{V^n}^{m_1} \cdots e_{V^{n-1}}^{m_{n-1}}$ is divisible by e_{V^n} for some numbers m_1, \dots, m_{n-1} . Furthermore, no $V^*\psi_m$ with $m < n$ can divide $V^*\psi_n$ as this would imply that also ψ_m divides ψ_n by pulling back along a section $\mathbb{T} \rightarrow A$ of V . Knowing this, the proof proceeds as the one above for [Proposition 5.46](#). \square

[Lemma 5.49](#) further shows that for any global group law X and torus A , the geometric fixed points $\Phi^A X$ can also be obtained from $X(A)$ by inverting $V^*\psi_n$ for all split characters $V: A \rightarrow \mathbb{T}$ and $n \in \mathbb{N}_{>0}$. This is because any non-trivial character W can be written as V^n for some split character V , and e_{V^n} decomposes as

$$e_{V^n} = V^*e_n = \prod_{m|n} V^*\psi_m.$$

Proof of Theorems A and C. We wish to show that the canonical map

$$\alpha: \mathbf{L} \rightarrow \pi_*(\mathbf{MU})$$

is an isomorphism, and we have already seen in [Proposition 5.45](#) that it induces an isomorphism on geometric fixed points for all abelian compact Lie groups. Since both \mathbf{L} and $\pi_*(\mathbf{MU})$ are left induced from tori, it is sufficient to show that α induces an isomorphism on tori. Let A be a torus. All Euler classes $e_V \in \mathbf{L}(A)$ with $V \neq \epsilon \in A^*$ are regular elements, so it follows that the maps $\mathbf{L}(A) \rightarrow \Phi^A(\mathbf{L})$ are injective. Since $\Phi^A(\alpha)$ is an isomorphism, this implies that $\alpha(A)$ is injective. It remains to prove that $\alpha(A)$ is also surjective. Again using that $\Phi^A(\alpha)$ is an isomorphism, it is sufficient to show that if $V^*\psi_n \cdot y \in \pi_*^A(\mathbf{MU})$ is in the image for some split character V and $n \in \mathbb{N}_{>0}$, then so is y . For this, let $x \in \mathbf{L}(A)$ be a preimage of $V^*\psi_n \cdot y$. Since $\varphi_{\ker(V^n)}^A(V^*\psi_n) = 0 \in \Phi_*^A(\mathbf{MU})$, it follows that $\varphi_{\ker(V^n)}^A(x)$ lies in the kernel of $\Phi^{\ker(V^n)}(\alpha)$. But $\Phi^{\ker(V^n)}(\alpha)$ is an isomorphism, so x lies in the kernel of $\varphi_{\ker(V^n)}^A$. Hence [Lemma 5.49](#) implies that x is divisible by $V^*\psi_n$, yielding an equation of the form

$$V^*\psi_n \cdot y = \alpha(x) = \alpha(V^*\psi_n \cdot x') = V^*\psi_n \cdot \alpha(x') \in \pi_*^A(\mathbf{MU})$$

for some $x' \in \mathbf{L}(A)$. Since $\pi_*^A(\mathbf{MU})$ is an integral domain and $V^*\psi_n$ is non-trivial (both statements follow from the embedding $\pi_*^A(\mathbf{MU}) \hookrightarrow \Phi_*^A(\mathbf{MU})$), we can conclude that $y = \alpha(x')$. This finishes the proof of Theorems A and C. \square

Corollary 5.50. *The global Lazard ring \mathbf{L} is a regular global group law. Moreover, for every abelian compact Lie group A the ring L_A is free as a module over the Lazard ring \mathbf{L} .*

Proof. We know that $\pi_*(\mathbf{MU})$ is regular, since it is concentrated in even degrees. Moreover, Comezaña [Com96] showed that $\pi_*^A(\mathbf{MU}_A)$ is free as a module over $\pi_*(\mathbf{MU})$. Hence, by Theorems A and C, the analogous statements hold for \mathbf{L} . \square

Remark 5.51. It is useful to abstract what properties of \mathbf{L} and $\pi_*(\mathbf{MU})$ we used to conclude that the map $\mathbf{L} \rightarrow \pi_*(\mathbf{MU})$ is an isomorphism from only knowing that it is an isomorphism on geometric fixed points. Let $f: X \rightarrow Y$ be a morphism of global group laws such that

- (1) X is 2-regular, all geometric fixed points $\Phi^A(X)$ are integral domains, and the classes $\psi_n, \psi_m \in X(\mathbb{T})$ for $n \neq m$ are not multiples of one another.
- (2) Y is 1-regular.

Note that the condition on the ψ_i follows if X maps to any global group law with this property, for example the multiplicative one. Then if $\Phi^A f$ is an isomorphism for all A (in fact it suffices to consider A with $\pi_0(A)$ cyclic), the map f is already an isomorphism. Indeed, the conditions on X guarantee that the classes ψ_n satisfy the analogs of Proposition 5.46 and Lemma 5.49, and the 1-regularity of Y implies that $Y(A)$ is a domain for all tori A , so the same proof applies.

5.8. Cooperations. Several results in chromatic homotopy theory rely not only on understanding the coefficients MU_* but also the cooperations MU_*MU and higher analogs $\pi_*(MU^{\wedge n})$. For example, these play an essential role in the Adams-Novikov spectral sequence. The goal of this final section is to give a description of the cooperations for global equivariant bordism.

Let X be an Ab -algebra equipped with two coordinates $e^{(1)}$ and $e^{(2)}$. Then $e^{(2)} = \lambda e^{(1)}$ for a unit $\lambda \in X(\mathbb{T})$. We say that the tuple $(e^{(1)}, e^{(2)})$ is *strict* if the unit λ restricts to 1 at $X(1)$. More generally we say that an n -tuple $(e^{(1)}, \dots, e^{(n)})$ of coordinates is strict if every pair $(e^{(i)}, e^{(j)})$ is strict.

Now assume given n complex oriented global ring spectra E_1, \dots, E_n . We obtain n orientations t_1, \dots, t_n in $(E_1 \wedge \dots \wedge E_n)_{\mathbb{T}}^2(S^\tau)$, differing from one another by units in $\pi_0^\mathbb{T}(E_1 \wedge \dots \wedge E_n)$. Since each of the orientation classes restrict to 1 at the trivial group, the same must be true for the units relating them. So it follows that the n -tuple of coordinates $(e^{(1)}, \dots, e^{(n)})$ associated to these orientations is strict.

Our goal is to show the following:

Theorem 5.52 (Theorem E in the introduction). *The tuple $(\pi_*(\mathbf{MU}^{\wedge n}), e^{(1)}, \dots, e^{(n)})$ is universal among Ab -algebras equipped with a strict n -tuple of coordinates.*

We can obtain a universal Ab -algebra with n coordinates $\mathbf{L}^{(n)}$ by applying the left adjoint from Proposition 5.4 to the \mathbb{Z}_e -algebra

$$\mathbb{Z}_e[\lambda_2^{\pm 1}, \dots, \lambda_n^{\pm 1}] / (\text{res}_1^\mathbb{T}(\lambda_i) - 1),$$

where λ_i are free unit variables at the circle group \mathbb{T} . The associated strict n -tuple is then $(e, \lambda_2 e, \lambda_3 e, \dots, \lambda_n e)$. We hence obtain a map

$$\alpha^{(n)}: \mathbf{L}^{(n)} \rightarrow \pi_*(\mathbf{MU}^{\wedge n}).$$

To show that $\alpha^{(n)}$ is an isomorphism we again need to describe the values $\mathbf{L}^{(n)}(A)$ in terms of A -equivariant formal group laws. By the adjunction of Proposition 5.32, the ring $\mathbf{L}^{(n)}(A)$ is initial among A -equivariant formal group laws equipped with a strict n -tuple of coordinates. This shows that $\mathbf{L}^{(n)}(A)$ is flat over $\mathbf{L}(A) = L_A$ (where we view $\mathbf{L}^{(n)}$ as an algebra over \mathbf{L} via the first coordinate), since once the A -equivariant formal group law and the first coordinate are specified, it remains to give the units $\lambda_2, \dots, \lambda_n \in R^\times$ which augment to 1. Choosing a flag F starting with the trivial character, this amounts to giving the coefficients $a_j^F(\lambda_i)$ for $j \geq 1$ (as $a_0^F(\lambda_i)$ is required to be 1), such that the associated Euler classes, which can be expressed as polynomials in the coefficients $a_j^F(\lambda_i)$, are units (see Lemma 2.6 and the paragraph preceding it). Hence, the ring classifying such data is a localization of a polynomial ring over L_A . This also shows that $\mathbf{L}^{(n)}(\mathbb{T}^r)$

is again an integral domain for all r , and that $\Phi^A(\mathbf{L}^{(n)})$ is an integral domain for all abelian compact Lie groups A .

We can express the universal property of $\mathbf{L}^{(n)}(A)$ a little differently: It is initial among rings k equipped with an n -tuple of A -equivariant formal group laws (F_1, \dots, F_n) together with strict isomorphisms $F_i \xrightarrow{\cong} F_{i+1}$ for $i = 1, \dots, n-1$. By the latter we mean isomorphisms $R_i \rightarrow R_{i+1}$ over k that are compatible with the maps Δ and θ and send the coordinate $y_i(\epsilon)$ to $\lambda y_{i+1}(\epsilon)$ for a unit $\lambda \in R_{i+1}^\times$ which augments to $1 \in k$.

Note that every strict isomorphism $\varphi: F \xrightarrow{\cong} F'$ gives rise to a strict isomorphism $\tilde{\varphi}: \tilde{F} \xrightarrow{\cong} \tilde{F}'$ between the associated non-equivariant formal group laws by completion.

Lemma 5.53. *Let A be an abelian compact Lie group, and F and F' be A -equivariant formal group laws over k , all of whose Euler classes e_V with $V \neq \epsilon$ are invertible. Then the assignment $\varphi \mapsto \tilde{\varphi}$ is a bijection from the set of strict isomorphisms $F \xrightarrow{\cong} F'$ to the set of strict isomorphisms $\tilde{F} \xrightarrow{\cong} \tilde{F}'$.*

Proof. Let R and R' be the complete Hopf algebras of F and F' , respectively, and \tilde{R} and \tilde{R}' their completions at the augmentation ideal. Then we saw in the proof of [Proposition 2.9](#) that R is canonically isomorphic to $\text{map}(A^*, \tilde{R})$ and likewise for R' , with A^* -action permuting the factors. Hence any isomorphism $F \cong F'$ is determined by the composite $R \rightarrow R' \rightarrow \tilde{R}'$ under the A^* -action. This composite factors as $R \rightarrow \tilde{R}$ followed by the induced isomorphism on completions $\tilde{R} \xrightarrow{\cong} \tilde{R}'$. Hence it is determined by the latter. Since φ is strict if and only if $\tilde{\varphi}$ is strict (the units λ and $\tilde{\lambda}$ have the same augmentation), the claim follows. \square

Corollary 5.54. *The ring $\Phi^A(\mathbf{L}^{(n)})$ is universal among rings equipped with an n -tuple of Euler-invertible A -equivariant formal groups (F_1, \dots, F_n) plus strict isomorphisms $\tilde{F}_i \xrightarrow{\cong} \tilde{F}_{i+1}$ between their associated non-equivariant formal group laws.*

Using this, we show:

Proposition 5.55. *The map $\alpha^{(n)}: \mathbf{L}^{(n)} \rightarrow \pi_*(\mathbf{MU}^{\wedge n})$ induces an isomorphism on all geometric fixed points.*

Proof. Geometric fixed points are strong symmetric monoidal, so the maps $i_1, \dots, i_n: \mathbf{MU} \rightarrow \mathbf{MU}^{\wedge n}$ induce an equivalence

$$\Phi^A(\mathbf{MU}^{\wedge n}) \cong (\Phi^A(\mathbf{MU}))^{\wedge n}.$$

The spectrum $\Phi^A(\mathbf{MU})$ is a wedge of shifted copies of MU (see [[Sin01](#), Theorem 4.9]), hence in particular it is Landweber exact. Now recall that given two Landweber exact theories E and E' , the ring $(E \wedge E')_*$ is universal among rings R equipped with two maps $E_* \rightarrow R$ and $E'_* \rightarrow R$ and a strict isomorphism between the pushforwards of the given formal group laws over E and E' . This follows from the case $(MU \wedge MU)_*$ by base-changing one copy of MU to E_* and the other to E'_* . Hence, [Proposition 5.45](#) shows that the coefficients of $(\Phi^A(\mathbf{MU}))^{\wedge n}$ are universal among rings equipped with an n -tuple of Euler-invertible A -equivariant formal group laws, together with strict isomorphisms between their associated non-equivariant formal group laws. We just saw that this is exactly the universal property of $\Phi^A(\mathbf{L}^{(n)})$, so the claim follows. \square

We now apply the same proof as in the previous section to conclude that $\mathbf{L}^{(n)} \rightarrow \pi_*(\mathbf{MU}^{\wedge n})$ is an isomorphism, following [Remark 5.51](#). We argued above that the first coordinate map $\mathbf{L}(A) \rightarrow \mathbf{L}^{(n)}(A)$ is flat for all A , which proves that $\mathbf{L}^{(n)}$ is also a regular global group law. Furthermore, we saw that the geometric fixed points $\Phi^A(\mathbf{L}^{(n)})$ are integral domains for all A . Since \mathbf{L} is a retract of $\mathbf{L}^{(n)}$, the different ψ_i cannot be multiples of one another. Finally, $\pi_*(\mathbf{MU}^{\wedge n})$ is again concentrated in even degrees as a consequence of the results of [[CGK02](#)], hence it is regular. So we see that all the conditions of [Remark 5.51](#) are satisfied, which concludes the proof of [Theorem 5.52](#).

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