Math 42 Assignment 3

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Instructions:

- On a separate sheet of paper typed with latex, please complete the following questions from your textbook. Then, scan and upload your work to Homework Ass 03 in our Gradescope Math 42 course.

  - Exercise 5.6 (TextBook: A Course in Mathematical Modeling: Mooney and Swift)
  — Exercises 1, 2, 3, 4, 5 (Page: 309)
  - Project : 5.1 (Pages: 309-310)

  Type your solutions in a latex file.

Solution

1. Exercise 5.6.1: Use the separation of variables technique to solve. $\frac{dx}{dt} = k$.

   Answer: $\frac{dx}{dt} = k \Rightarrow dx = k(dt) \Rightarrow \int dx = \int k(dt) \Rightarrow x(t) = kt + c_1$ where, c_1 is constant.

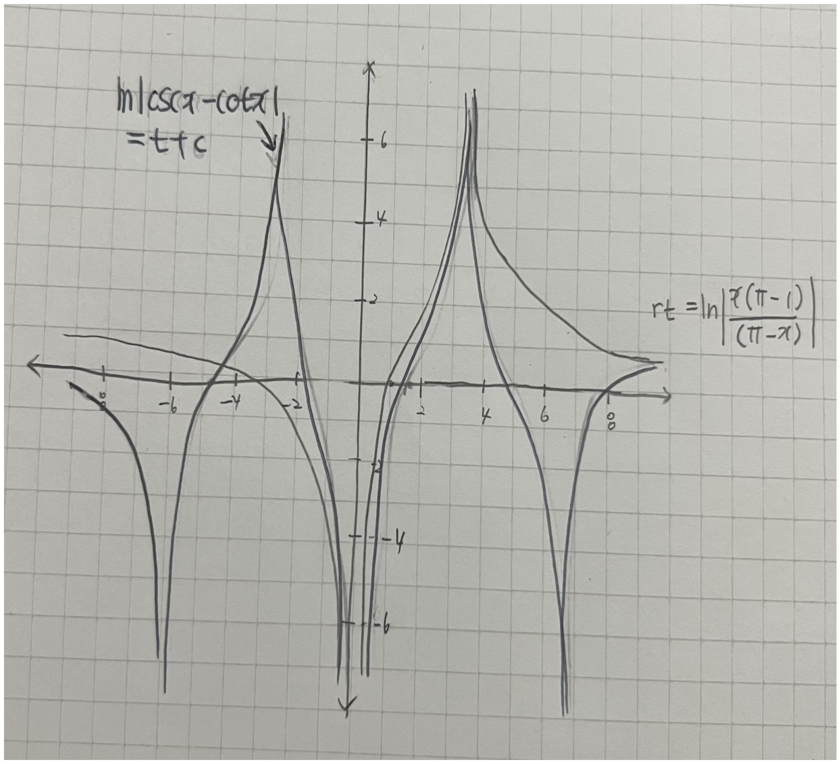
2. Exercise 5.6.2: Use the separation of variables technique to solve the logistic differential equation $\frac{dx}{dt} = rx(1 - \frac{x}{K})$.

   Answer: $\frac{dx}{dt} = rx(1 - \frac{x}{K})$
   $\Rightarrow \frac{dx}{x(K-x)} = \frac{r}{K}dt$
   $\Rightarrow \frac{K-x+x}{x(K-x)}dx = rdt$
   $\Rightarrow [\frac{1}{x} + \frac{1}{K-x}]dx = rdt$
   $\Rightarrow [\frac{1}{x} - \frac{1}{x-K}]dx = rdt$
   $\Rightarrow \int[\frac{1}{x} - \frac{1}{x-K}]dx = r\int dt$
   $\Rightarrow ln(x) - ln(x-K) = rt + ln(a)$
   $\Rightarrow ln(\frac{x}{x-K}) = rt + ln(a)$

$\Rightarrow ln(\frac{x}{a(x-K)}) = rt$
$\Rightarrow \frac{x}{a(x-K)} = e^{rt}$
$\Rightarrow x(t) = ax(t)e^{rt} - aKe^{rt}$
$\Rightarrow (ae^{rt} - 1)x(t) = aKe^{rt}$
$\Rightarrow x(t) = aKe^{rt}/(ae^{rt} - 1)$

3. Exercise 5.6.3: Use separation of variables to solve $\frac{dx}{dt} = sin(x)$ and using a graphing program, compare this solution with the solution of the logistic equation $\frac{dx}{dt} = rx(1 - \frac{x}{\pi})$ for several different values r and $x(0) = 1$.

   Answer: $\frac{dx}{dt} = sin(x) \rightarrow cosec(x)dx = dt \rightarrow \int cosec(x)dx = \int dt \rightarrow ln|cosec(x) - cot(x)| = t + c_1$, where c_1 is constant.
   To find the solution of the logistic equation, $\frac{dx}{dt} = rx(1 - \frac{x}{\pi})$
   $\Rightarrow \frac{dx}{x(1-(x/\pi)} = (r)dt$
   $\Rightarrow \frac{x+(1-(x/\pi)\pi}{\pi x(1-x/\pi)}dx = (r)dt$
   $\Rightarrow \frac{1}{\pi(1-x/\pi)}dx + \frac{1}{x}dx = (r)dt$
   $\Rightarrow \int \frac{1}{\pi(1-x/\pi)}dx + \int \frac{1}{x}dx = \int(r)dt$
   For substitution, $\pi(1 - \frac{x}{\pi}) = z \rightarrow \pi - x = z \rightarrow dx = -dz$
   $\Rightarrow \int -\frac{1}{z}dz + \int \frac{1}{x}dx = \int rdt$
   $\Rightarrow -ln|z| + ln|x| = rt + c$, where c = constant
   $\Rightarrow ln|\frac{1}{z}| + ln|x| = rt + c$
   $\Rightarrow ln|\frac{x}{z}| = rt + c$
   $\Rightarrow ln|\frac{x}{\pi(1-x/\pi)}| = rt + c$
   Since we have x(0) = 1, then we have $c = ln|\frac{1}{\pi-1}|$
   Thus, $ln|\frac{x}{\pi-x}| = rt + ln|\frac{1}{\pi-1}|$
   $\Rightarrow ln|\frac{x}{\pi-x}| - ln|\frac{1}{\pi-1}| = rt$
   $\Rightarrow ln|\frac{x(\pi-1)}{\pi-x}| = rt$
   To compare the obtained solutions $ln|csc(x) - cot(x)| = t + c_1$ and $rt = ln|\frac{x(\pi-1)}{(\pi-x)}|$ by graphing, we can get like



4. Exercise 5.6.4: Show that $\frac{dx}{dt} = ax + by$ and $\frac{dy}{dt} = cx + dy$ can be rewritten as $\frac{d^2x}{dt^2} - \beta\frac{dx}{dt} + \gamma x = 0$.

   Answer: Let $\frac{dx}{dt} = ax + by$ and $\frac{dy}{dt} = cx + dy$.
   By differentiating $\frac{dx}{dt} = ax + by$ about "t", we then have $\frac{d^2x}{dt^2} = a\frac{dx}{dt} + b\frac{dy}{dt}$
   $\Rightarrow \frac{d^2x}{dt^2} = a\frac{dx}{dt} + b(cx + dy)$
   $\Rightarrow \frac{d^2x}{dt^2} = a\frac{dx}{dt} + bcx + bdy$
   $\Rightarrow \frac{d^2x}{dt^2} = a\frac{dx}{dt} + bcx + \frac{bd}{b}(\frac{dx}{dt} - ax)$
   $\Rightarrow \frac{d^2x}{dt^2} + (-a - \frac{bd}{b})\frac{dx}{dt} + (ad - bc)x = 0$
   $\Rightarrow \frac{d^2x}{dt^2}(-a - d)\frac{dx}{dt} + (ad - bc)x = 0$
   $\Rightarrow \frac{d^2x}{dt^2} - \beta\frac{dx}{dt} + \gamma x = 0$, where $\beta = a + b$ and $\gamma = ad - bc$
   Thus, we conclude $\frac{dx}{dt} = ax + by$ and $\frac{dy}{dt} = cx + dy$ can be rewritten as $\frac{d^2x}{dt^2} - \beta\frac{dx}{dt} + \gamma x = 0$.

5. Exercise 5.6.5: Find values for a, b, c, and d so that the fixed point (the origin) of $\frac{dx}{dt} = ax + by$ and $\frac{dy}{dt} = cx + dy$ is a

   Let $\frac{dx}{dt} = ax + by = f_1(x, y)$ and $\frac{dy}{dt} = cx + dy = f_2(x, y)$.
   The equilibrium point is (x,y) = (0,0). So, the Jacobian matrix J at (0,0)

is J(0,0) $= \begin{bmatrix} f_{1x} & f_{1y} \\ f_{2x} & f_{2y} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. We then have that characteristic equation is $\lambda^2 - (a+d)\lambda + (ad - bc) = 0$.

a) source node: J(0,0) $= \begin{bmatrix} 4 & 2 \\ 0 & 2 \end{bmatrix} \Rightarrow \lambda = 2, 4$ So, the characteristic roots are unequal and positive. Thus, the fixed point of system has source node.
b) sink node: J(0,0) $= \begin{bmatrix} -4 & 2 \\ 0 & -2 \end{bmatrix} \Rightarrow \lambda = -2, -4$ So, the characteristic roots are unequal and negative. Thus, the fixed point of system has sink node.
c) spiral source: J(0,0) $= \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix} \Rightarrow \lambda = 3 \pm i$ Thus, the fixed point of system has source spiral.
d) spiral sink: J(0,0) $= \begin{bmatrix} -3 & -1 \\ 1 & -3 \end{bmatrix} \Rightarrow \lambda = -3 \pm i$ Thus, the fixed point of system has sink spiral.
e) saddle point: J(0,0) $= \begin{bmatrix} 3 & -1 \\ 0 & -2 \end{bmatrix} \Rightarrow \lambda = 3, -2$ Thus, the fixed point of system has saddle point.

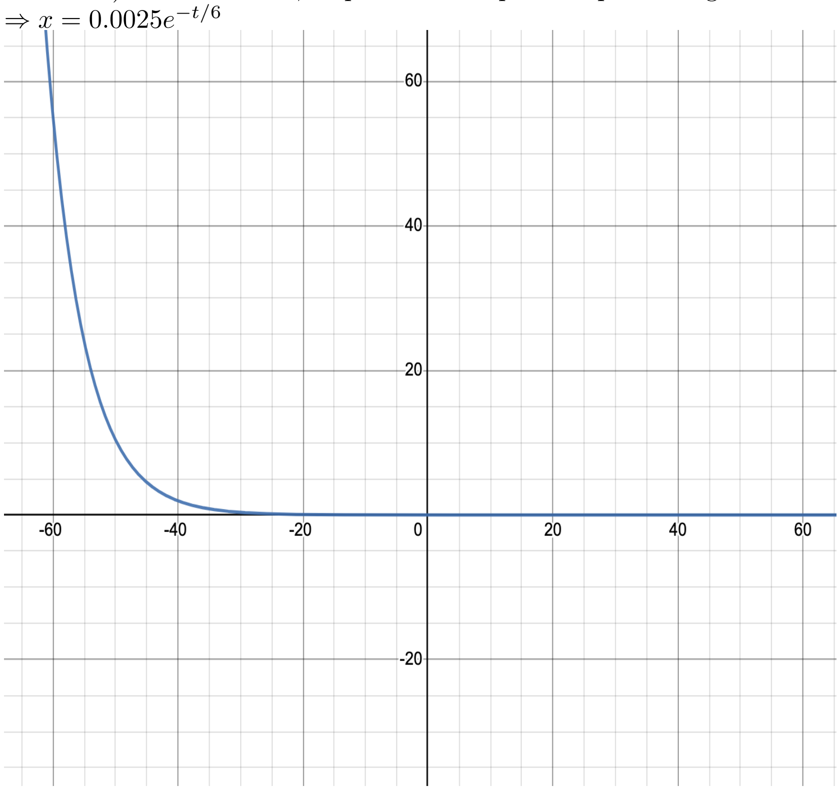
6. Project 5.1: The Michaelis-Menton Equation I
The given differential equation is $\frac{dx}{dt} = \frac{-kx}{A+x}$

Answer 1) When $A >> x(t) \Rightarrow x(t) + A \approx A$. The differential equation can be rewritten as $\frac{dx}{dt} = \frac{-kx}{A}$.
The given values are A = 6 and x(0) = 0.0025. We then have $\frac{dx}{dt} = \frac{-kx}{6}$
$\Rightarrow \frac{dx}{x} = \frac{-k}{6} dt$
$\Rightarrow \int \frac{dx}{x} = \frac{-k}{6} \int dt$, where k is constant.
$\Rightarrow ln(x) = \frac{-k}{6} t + c$
When $t = 0$ and $x = 0.0025$, then we have $ln(0.0025) = c$
$\Rightarrow ln(x) = \frac{-kt}{6} + ln(0.0025)$
$\Rightarrow ln(x) - ln(0.0025) = \frac{-k}{6} t$
$\Rightarrow ln(\frac{x}{0.0025}) = \frac{-k}{6} t$
$\Rightarrow \frac{x}{0.0025} = e^{-k/6*t}$
$\Rightarrow x = 0.0025e^{-k/6*t}$
Thus, $x = x(0)e^{-k/6*t}$, where $x(0) = 0.0025$

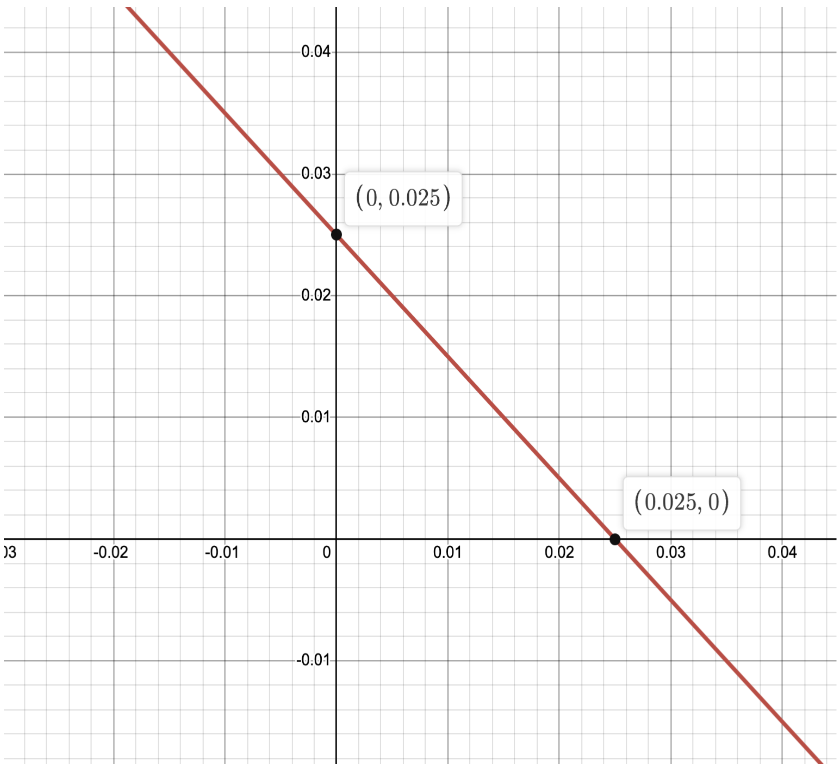
Answer 2) When $x(t) >> A \Rightarrow x(t) + A \approx x(t)$. The differential equation can be rewritten as $\frac{dx}{dt} = \frac{-kx}{x}$
$\Rightarrow \frac{dx}{dt} = -k$
$\Rightarrow dx = -kdt$
$\rightarrow \int dx = -k \int dt$, where k is constant
$\Rightarrow x = -kt + c$
When t = 0 and x = x(0) = 0.025, we then have x(0) = c

$\Rightarrow x = -kt + x(0)$
$\Rightarrow x = x(0) - kt$
As x(0) = 0.025 then $x = 0.025 - kt$

Answer 3) Assume k = 1, so put k = 1 in part 1 equation I got
$\Rightarrow x = 0.0025e^{-t/6}$



By putting k = 1 in part 2 equation I got
$\Rightarrow x = 0.025 - t$



Answer 4) By putting value of k = 1, A = 0.025 in the differential equation, we get $\frac{dx}{dt} = -\frac{kx}{A+x}$

$\Rightarrow \frac{dx}{dt} = -\frac{x}{0.025+x}$

$\Rightarrow -\frac{0.025+x}{x} dx = dt$

$\Rightarrow (\frac{0.025}{x} + 1)dx = -dt$

$\Rightarrow \int (\frac{0.025}{x} + 1)dx = -\int dt$

$\Rightarrow \int \frac{0.025}{x} dx + \int dx = -\int dt$

$\Rightarrow 0.025 \int \frac{dx}{x} + x = -t + c$

$\Rightarrow 0.025ln(x) + x = -t + c$

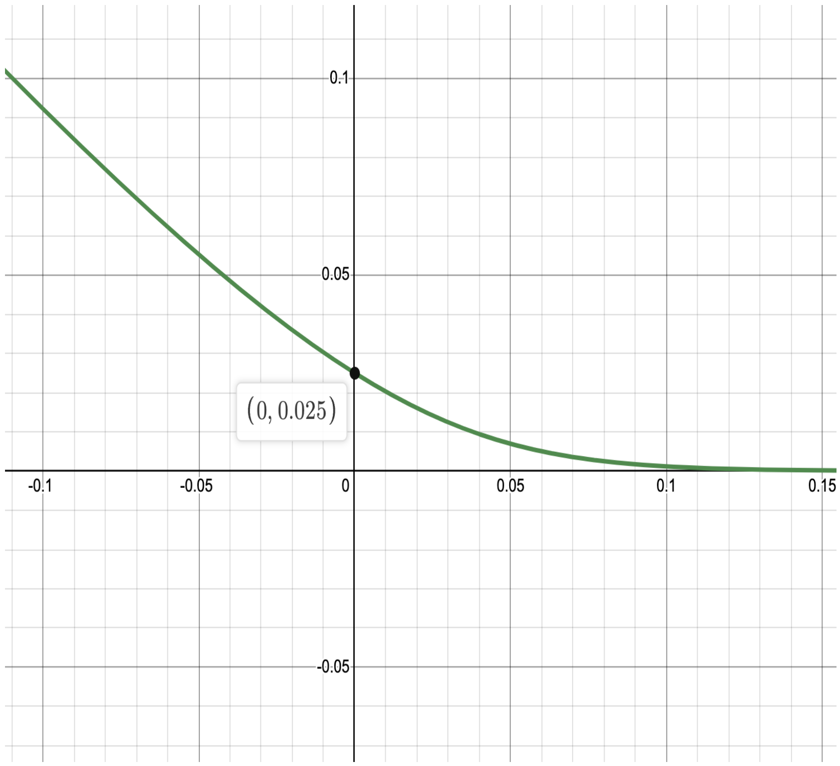
When t = 0 and x = x(0), then we have $0.025ln(x(0)) + x(0) = c$

We have $0.025ln(x) + x = -t + (0.025ln(x(0)) + x(0))$

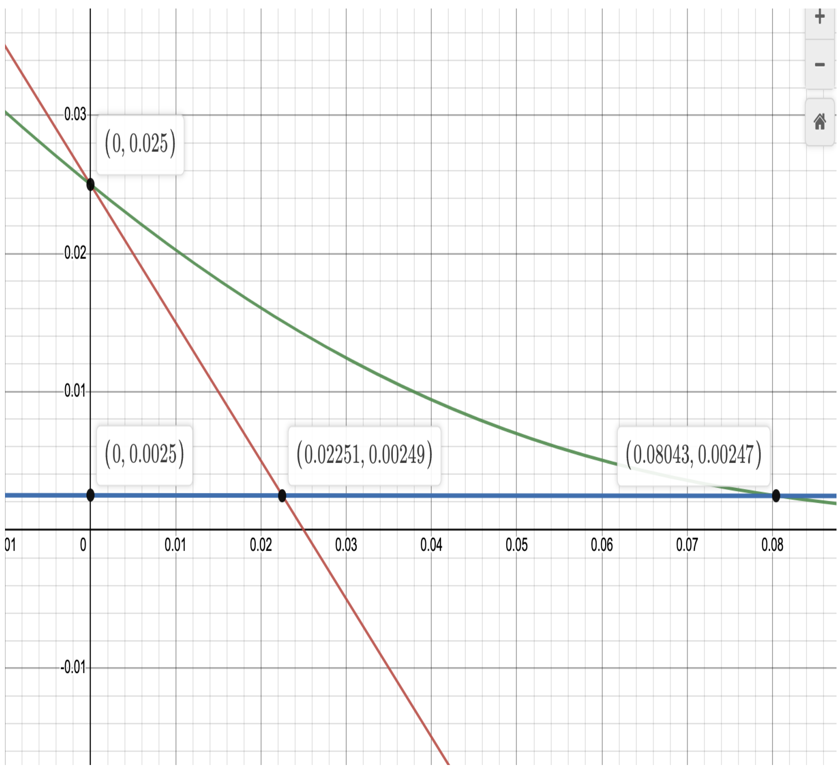
$\Rightarrow 0.025(ln(x) - ln(x(0))) + (x - x(0)) = -t$

$\Rightarrow 0.025ln(\frac{x}{x(0)} + (x - x(0)) = -t$

$\Rightarrow 0.025ln(\frac{x}{0.025}) + (x - 0.025) + t = 0$



To compare all three curves from part 3 and part 4,


0.03
0.02
0.01
-0.01
01
0
0.01
0.02
0.03
0.04
0.05
0.06
0.07
0.08
(0, 0.025)
(0, 0.0025)
(0.02251, 0.00249)
(0.08043, 0.00247)