Math 42 Assignment 7

Mingyeong Kim
UID: 905 974 280

December 8, 2022

Instructions:

- On a separate sheet of paper typed with latex, please complete the following questions from your textbook. Then, scan and upload your work to Homework Ass 03 in our Gradescope Math 42 course.

  Textbook: A Course in Mathematical Modeling: Mooney and Swift
  — Exercises 4, 8 (Page: 307 - 308)
  — Project : 2 (Page 311 - 312)

  Type your solutions in a latex file.

Solution

1. Exercise 4: Show that $\frac{dx}{dt} = ax + by$ and $\frac{dy}{dt} = cx + dy$ can be rewritten as $\frac{d^2x}{dt^2} - \beta\frac{dx}{dt} + \gamma x = 0$.

   Answer: Let $\frac{dx}{dt} = ax + by$ and $\frac{dy}{dt} = cx + dy$
   By taking differential for $\frac{dx}{dt} = ax + by$, we then have $\frac{d^2x}{dt^2} = a\frac{dx}{dt} + b\frac{dy}{dt}$
   $\frac{d^2x}{dt^2} = a\frac{dx}{dt} + b(cx + dy)$
   $\frac{d^2x}{dt^2} = a\frac{dx}{dt} + bcx + bdy$
   $\frac{d^2x}{dt^2} = a\frac{dx}{dt} + bcx + \frac{bd}{b}(\frac{dx}{dt} - ax)$
   $\frac{d^2x}{dt^2} + (-a - d)\frac{dx}{dt} + (ad - bc)x = 0$
   $\frac{d^2x}{dt^2} - \beta\frac{dx}{dt} + \gamma x = 0$, where $\beta = a + d$ and $\gamma = ad - bc$

2. Exercise 8: For a final project, a student of ours was attempting to build a model for an ecosystem with a number of interacting variables changing over time. He set the equations as a system of first-order homogenous linear equations with constant coefficients. Without consulting us, he was

arbitrarily assigning values to the coefficients trying to find a set of coefficients that would result in a stable ecosystem.
Try to help him. Set up a software model that solves (numerically or symbolically) a system of three first-order linear homogenous differential equations with constant coefficients. Arbitrarily (and without mathematical analysis) assign coefficients and initial conditions to try to get a system with solution curves that level out and are not all zero. Keep trying. It can be done. It can be done, but probably not by arbitrarily picking numbers. Try again, this time carefully choosing numbers and, perhaps, referring to some of the discussions on these types of systems in the test.
If this student wanted to stable model, what advice would you give him? Also, feel free to advise about randomly choosing parameters which are supposed to have some meaning in an ecosystem.

Answer:

3. Project 2 (page: 311 - 312):

   Answer a)

   Answer b)
   The fixed point is $(\frac{ax+nr}{mn-ab}, \frac{ms-br}{mn-ab}$. Since all the constants are positive, then the numerator will be positive. This means that the sign of the denominator will determine what quadrant the point will be in. Since the denominator for both coordinates are the same, then the point can either have both positive coordinates or both negative coordinates.
   This corresponds to the first and third quadrant respectively.
   We have the two differential equations with their respective null clines.
   $\frac{dx}{dt} = ky - ax + g$
   $\frac{dy}{dt} = bx - ny + s$
   $y = \frac{mx-r}{a}$
   $y = \frac{bx+s}{n}$

   Focusing on $\frac{dx}{dt}$, if we let $y > \frac{mx-r}{a}$, then $\frac{dx}{dt}$ will be positive and vice versa. For $\frac{dy}{dt}$, if we let $y > \frac{bx+s}{n}$, then $\frac{dy}{dt}$ will be negative and vice versa. Using phase plane analysis, we can see that if mn - ab ¿0 (meaning that $\frac{dx}{dt}$ has a steeper slope), then we have a stable point.
   If $mn - ab < 0$ (meaning that $\frac{dy}{dt}$ has a steeper slope), then we heave an unstable point, using the same idea as above.

   Answer c)
   If we let r,s be negative, and we focus on the case of mn - ab ¡ 0, then we

have our fixed point in the first quadrant with $\frac{dy}{dt} = 0$ having a steeper slope that than of $\frac{dx}{dt}$.
Using phase plane analysis, we can see that we have a saddle point meaning that the critical point is unstable.
On one side, if the initial values (or initial level of expenditure) for x and y are both greater than the equilibrium point, then x and y will tend to infinity. If x and y are both smaller than the equilibrium point, then x and y will go towards 0.
So, we focus on when mn - ab ¿ 0. This is the case when $\frac{dx}{dt} = 0$ has a steeper slope than that of $\frac{dy}{dt}$.
in this case, we have an unstable point using phase plane analysis, but when x or y are greater than the equilibrium point, then x and y tend towards the origin, meaning that the arms race results in total disarmament.

Answer d)