Math 42 Assignment 6

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Instructions:

- On a separate sheet of paper typed with latex, please complete the following questions from your textbook. Then, scan and upload your work to Homework Ass 03 in our Gradescope Math 42 course.

  Textbook: A Course in Mathematical Modeling: Mooney and Swift
  — Exercises 2, 3 (Page: 136-137)
  — Project : 3.7, 3.8 (Page 143-146)

  Type your solutions in a latex file.

Solution

1. Exercise 2: Showing that a model is valid can be difficult. Often it is much easier to show that a model is invalid. Consider the following model that has been proposed [4] to describe the growth of redwoods (Sequoia sempervirens). Redwoods frequently live 1,000 years or more. The model used classifies redwoods into three age stages : 0 to 200 years (young), 200 to 800 years (mature), and older than 800 (old). A current census in a particular stand finds that currently there are 1,696 young redwoods, 485 mature redwoods, and 82 old redwoods. The transition between stages over a 50 year period is given by the matrix T $\begin{pmatrix} 12 & 26 & 6 \\ 0.30 & 0.92 & 0 \\ 0 & 0.18 & 0.67 \end{pmatrix}$

   Trace this stand of trees through five time steps(five 50 year periods for a total of 250 years) and explain how we know that this model cannot possibly be valid.

   Answer: Let L = $\begin{bmatrix} 12 & 26 & 6 \\ 0.30 & 0.92 & 0 \\ 0 & 0.18 & 0.67 \end{bmatrix}$. Then we have eigenvalues of this matrix are $\begin{pmatrix} 12.67 \\ 0.33 \\ 0.59 \end{pmatrix}$. The dominant eigenvalue is 12.67 corresponding eigenvector is $\begin{pmatrix} 0.9997 \\ 0.0255 \\ 0.0004 \end{pmatrix}$. Let's normalize the eigenvector, then we have $\begin{pmatrix} 0.9747 \\ 0.0249 \\ 0.0004 \end{pmatrix}$.

So, the stable population distribution is young = 97.47%, mature = 2.49%, and old = 0.04%. A current census in a particular stand finds that currently there are 1696 young redwoods, 485 mature redwoods, and 82 old redwoods ($\Rightarrow I = \begin{pmatrix} 1696 \\ 485 \\ 82 \end{pmatrix}$).

For tracing population for 5 step times, $P1 = L * I = \begin{pmatrix} 33454 \\ 955 \\ 142 \end{pmatrix}$, $P2 = L * P1 = \begin{pmatrix} 426131 \\ 10915 \\ 267 \end{pmatrix}$, $P3 = L * P2 = \begin{pmatrix} 5410965 \\ 138181 \\ 2144 \end{pmatrix}$, $P4 = L * P3 = \begin{pmatrix} 68537152 \\ 1750416 \\ 26308 \end{pmatrix}$, and $P5 = L * P4 = \begin{pmatrix} 868114506 \\ 22171528 \\ 332701 \end{pmatrix}$.

Thus, the normalized population vector is also N = $\begin{pmatrix} 0.9747 \\ 0.0249 \\ 0.0004 \end{pmatrix}$. The Leslie matrix is stablizing in 5 iterations. However, the population distribution obviously looks unrealistic. So, we can conclude discarding this model.

2. Exercise 3: Find the trace and determinant of each matrix. Determine whether each matrix is singular (has no inverse).

   - Matrix $\begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix}$: The trace of matrix is 1 + 2 = 3. The determinant of matrix is (1)(2) - (0)(3) = 2. Since determinant of matrix is not 0, this matrix is nonsingular matrix.
   - Matrix $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 1 & 2 & 1 \end{pmatrix}$: The trace of matrix is 1 + 3 + 1 = 5. The determinant of matrix is (1)det$\begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}$ - (2)det$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ + (3)det$\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$ = 1 - 2 + 3 = 2. Since determinant of matrix is not 0, this matrix is nonsingular matrix.

- Matrix $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 5 & 4 \end{pmatrix}$: The trace of matrix is $1 + 3 + 4 = 8$. The determinant of matrix is (1)det$\begin{pmatrix} 3 & 1 \\ 5 & 4 \end{pmatrix}$ - (2)det$\begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}$ + (3)det$\begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix}$ = 7 - 10 + 3 = 0. Since determinant of matrix is 0, this matrix is singular matrix.

3. Project 3.7 (pg. 143- 156):

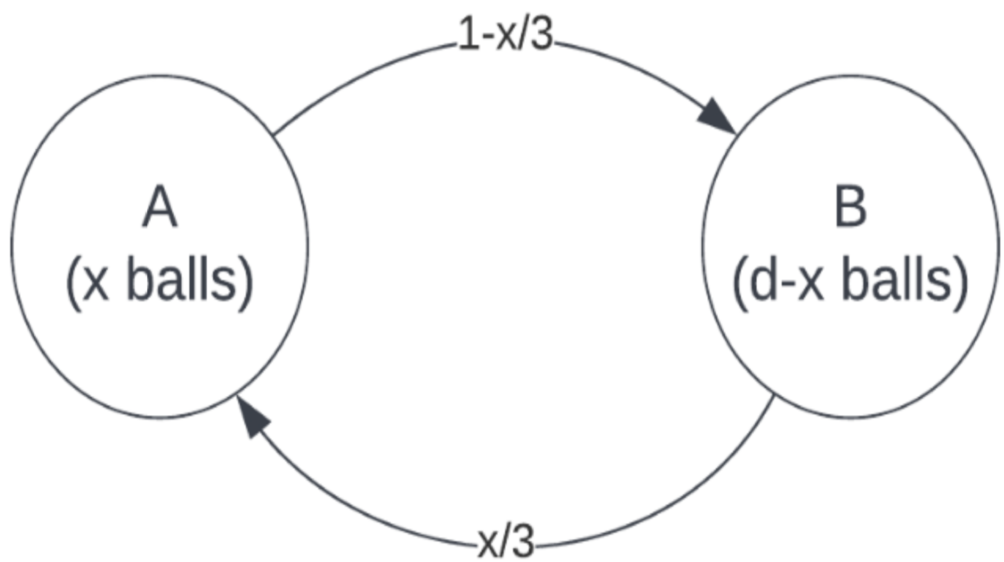
   Answer 1)

   Answer 2)

   Answer 3)

4. Project 3.8:

   Answer 1)



This ehrenfest state diagram with d = 3

Answer 2)
A matrix is called regular if two properties satisfies. First one is M is

stochastic, and second one is M^n for $n > 1$ has only non-zero positive entries.
Since all columns to E and O add up to 1, we can state that both matrices are stochastic.
However, for any $n > 1$, $E^n = E$ and $O^n = O$:
So,

$$E^n = \begin{pmatrix} \frac{1}{4} & 0 & \frac{1}{4} & 0 \\ 0 & \frac{3}{4} & 0 & \frac{3}{4} \\ \frac{3}{4} & 0 & \frac{3}{4} & 0 \\ 0 & \frac{1}{4} & 0 & \frac{1}{4} \end{pmatrix} \quad \text{and} \quad O^n = \begin{pmatrix} 0 & \frac{1}{4} & 0 & \frac{1}{4} \\ \frac{3}{4} & 0 & \frac{3}{4} & 0 \\ 0 & \frac{3}{4} & 0 & \frac{3}{4} \\ \frac{1}{4} & 0 & \frac{1}{4} & 0 \end{pmatrix}.$$

Therefore, E and O are not regular matrices. Hence, the Ehrenfest chain is not regular. The Matlab code:

```
n = 3;

E = [1/4 0 1/4 0;
     0 3/4 0 3/4;
     3/4 0 3/4 0;
     0 1/4 0 1/4;];

O = [0 1/4 0 1/4;
     3/4 0 3/4 0;
     0 3/4 0 3/4;
     1/4 0 1/4 0;];

E_power = E^n;
O_power = O^n;

% disp(E)
disp(E_power)
disp(O_power)
```

Answer 3)
Let $X_0 = (1,0,0,0)^T, X_1 = (0,1,0,0)^T$, and $X_0 = (0,0,1,0)^T$
Then we have,

$$\overline{X} = \tfrac{1}{2}(EX_0 + OX_0) = \begin{pmatrix} 0.1250 \\ 0.3750 \\ 0.3750 \\ 0.1250 \end{pmatrix} = (\tfrac{1}{8}, \tfrac{3}{8}, \tfrac{3}{8}, \tfrac{1}{8})^T$$

$$\overline{X} = \tfrac{1}{2}(EX_1 + OX_1) = \begin{pmatrix} 0.1250 \\ 0.3750 \\ 0.3750 \\ 0.1250 \end{pmatrix} = (\tfrac{1}{8}, \tfrac{3}{8}, \tfrac{3}{8}, \tfrac{1}{8})^T$$

$$\overline{X} = \tfrac{1}{2}(EX_2 + OX_2) = \begin{pmatrix} 0.1250 \\ 0.3750 \\ 0.3750 \\ 0.1250 \end{pmatrix} = (\tfrac{1}{8}, \tfrac{3}{8}, \tfrac{3}{8}, \tfrac{1}{8})^T$$

The Matlab code:

```
E = [1/4 0 1/4 0;
     0 3/4 0 3/4;
     3/4 0 3/4 0;
     0 1/4 0 1/4;];

O = [0 1/4 0 1/4;
     3/4 0 3/4 0;
     0 3/4 0 3/4;
     1/4 0 1/4 0;];

X_0 = [1; 0; 0; 0;];
X_1 = [0; 1; 0; 0;];
X_2 = [0; 0; 1; 0;];

X_bar_0 = 0.5*(E*X_0 + O*X_0);
X_bar_1 = 0.5*(E*X_1 + O*X_1);
X_bar_2 = 0.5*(E*X_2 + O*X_2);


disp(X_bar_0)
disp(X_bar_1)
disp(X_bar_2)
```