Math 42 Assignment 4

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Instructions:

- On a separate sheet of paper typed with latex, please complete the following questions from your textbook. Then, scan and upload your work to Homework Ass 03 in our Gradescope Math 42 course.

  - Exercise 1.8 (TextBook: A Course in Mathematical Modeling: Mooney and Swift)
  — Exercises 2, 3 (Page: 38)
  — Project : 1.4, 1.5 (Pages: 42-43)

  Type your solutions in a latex file.

Solution

1. Exercise 1.8.2: Find a closed-form solution for the affine recurrence relation $x(n) = Rx(n-1) + a$

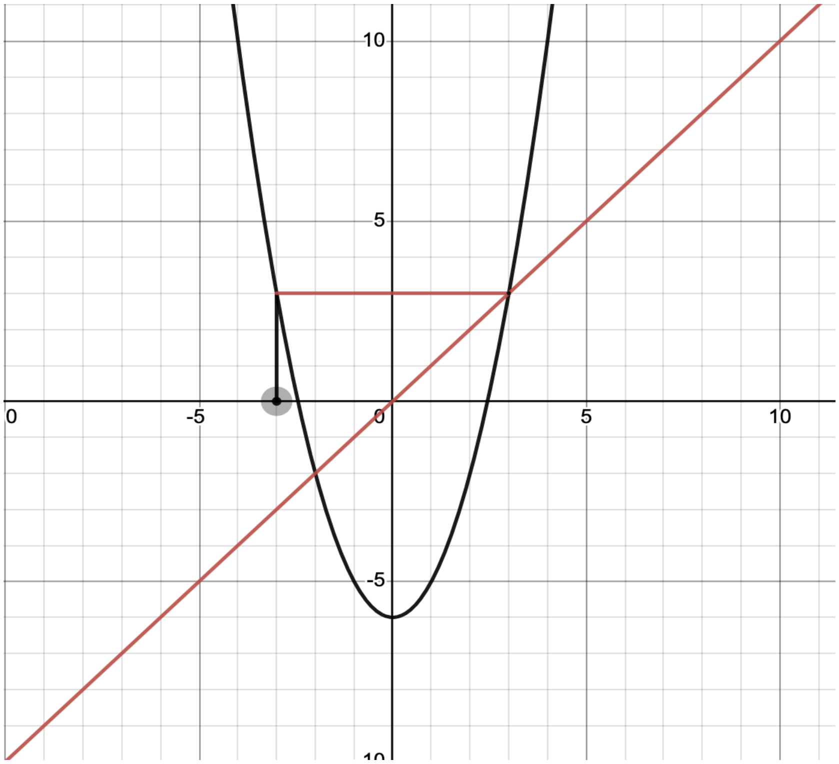
   Answer: Given recurrence relation is $x(n) = Rx(n-1) + a$
   Let begin with assumption that x(0) is initial value when n = 1
   So, when n = 1, then $x(1) = R(1-1) + a = Rx(0) + a$
   when n = 2, then $x(2) = R(2-1) + a = Rx(1) + a = R[Rn(0) + a] + a = R^2n(0) + Ra + a$
   when n = 3, then $x(3) = R(3-1) + a = Rx(2) + a = R[R^2n(0) + Ra + a] + a = R^3n(0) + R^2a + Ra + a$
   So, if we keep going up to n(th) term.
   $x(n) = R^n x(0) + R^{n-1}a + R^{n-2}a + ... + R^2a + Ra + a$
   $x(n) = R^n x(0) + a[1 + R + R^2 + ... + R^{n-1}]$
   Here, $1 + R + R^2 + ... + R^{n-1}$ is a geometric progression. Then sum y0 n terms is $S_n = (\frac{1-R^n}{1-R})$
   Thus, $x(n) = x(0)R^n + a\frac{(1-R^n)}{1-R}$

2. Exercise 1.8.3: Find fixed points for the following recursion relations, and test for stability. Draw a cobweb diagram for parts c and d.

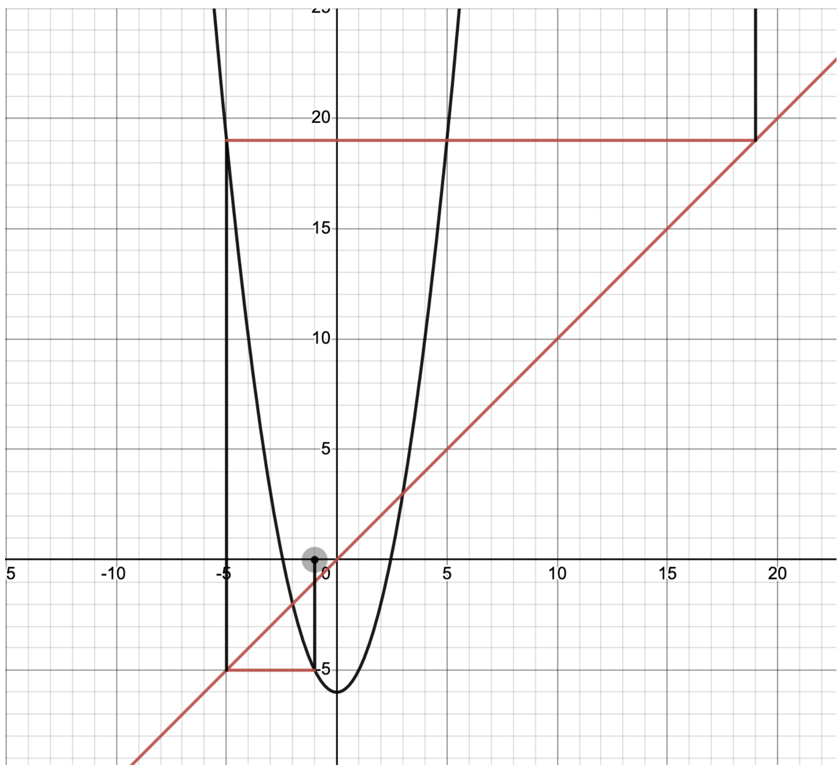
   Part a) $x(n) = \frac{x(n-1)}{1+x(n-1)}$
   Answer: Let $\bar{x}$ be the fixed point, like $\bar{x} = x(n) = x(n-1)$
   Then we have $\bar{x} = \frac{\bar{x}}{1+\bar{x}}$
   $\Rightarrow \bar{x} + \bar{x}^2 = \bar{x}$
   $\Rightarrow \bar{x}^2 = 0$ so, $\bar{x} = 0$ which is fixed point
   For f(x) that we have is $f(x) = \frac{x}{1+x}$ and $f'(x) = \frac{1}{(x+1)^2}$
   When $\bar{x} = 0$, $|f'(0)| = |\frac{1}{0+1}^2| = 1 = 1$ which means the test for stability is inconclusive.

   Part b) $x(n) = x(n-1)e^{rx(n-1)}$ where r is a constant
   Answer: Let $\bar{x}$ be the fixed point, like $\bar{x} = x(n) = x(n-1)$
   Then we have $\bar{x} = \bar{x}e^{r\bar{x}}$
   $\Rightarrow \bar{x}(1 - e^{r\bar{x}}) = 0$, so $\bar{x} = 0$ which is fixed point.
   For f(x) that we have is $f(x) = xe^{rx}$ and $f'(x) = (rx+1)e^{rx}$
   When $\bar{x} = 0$, $|f'(0)| = |(r0+1)e^{r0}| = 1 = 1$ which means the test for stability is inconclusive.

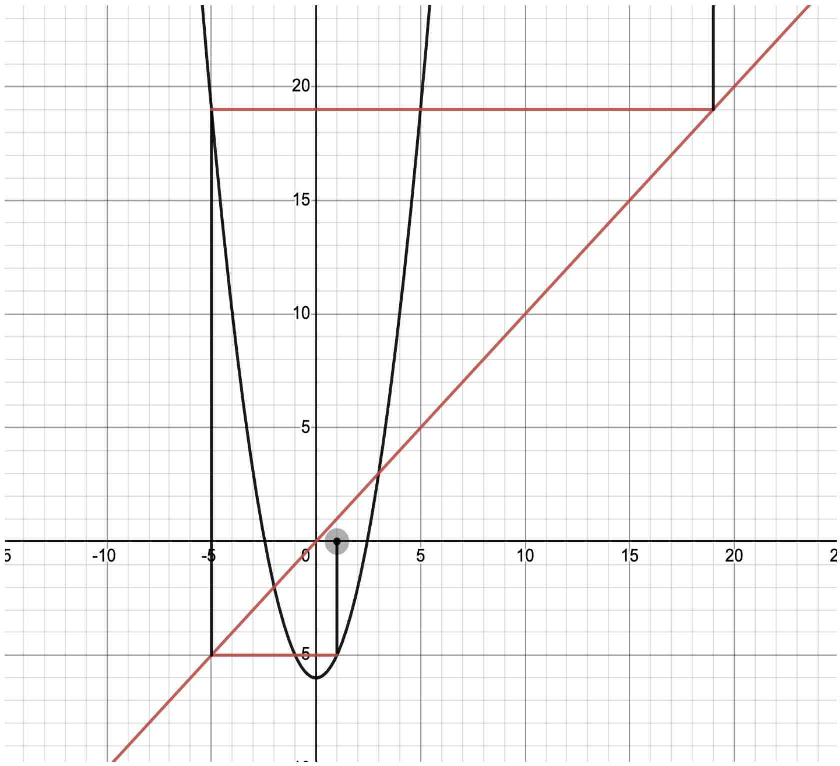
   Part c) $x(n) = x(n-1)^2 - 6$
   Answer: Let $\bar{x}$ be the fixed point, like $\bar{x} = x(n) = x(n-1)$
   Then we have $\bar{x} = \bar{x}^2 - 6$
   $\Rightarrow \bar{x}^2 - \bar{x} - 6 = 0$
   $\Rightarrow (\bar{x} - 3)(\bar{x} + 2) = 0$ so, $\bar{x} = 3$ and $\bar{x} = -2$
   For f(x) that we have is $f(x) = x^2 - 6$ and $f'(x) = 2x$ which are fixed points.
   When $\bar{x} = 3$, $|f'(3)| = |2 * 3| = 6 > 1$ which means unstable fixed point.
   When $\bar{x} = -2$, $|f'(-2)| = |2 * (-2)| = 4 > 1$ which means unstable fixed point too.



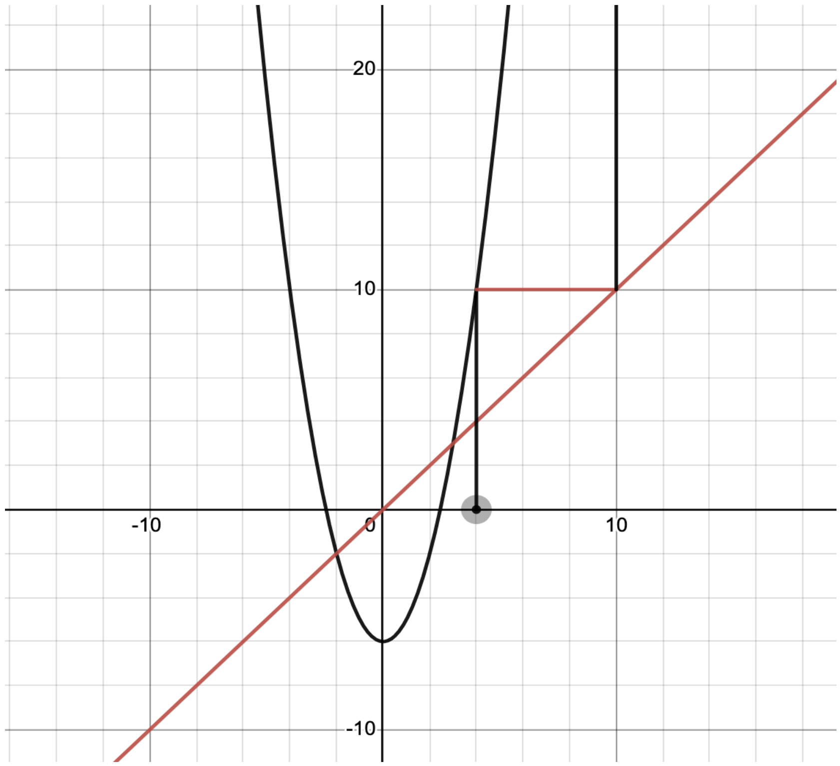
$f(x) = x^2 - 6$ cobweb diagram with $x_0 = -3$



$f(x) = x^2 - 6$ cobweb diagram with $x_0 = -1$

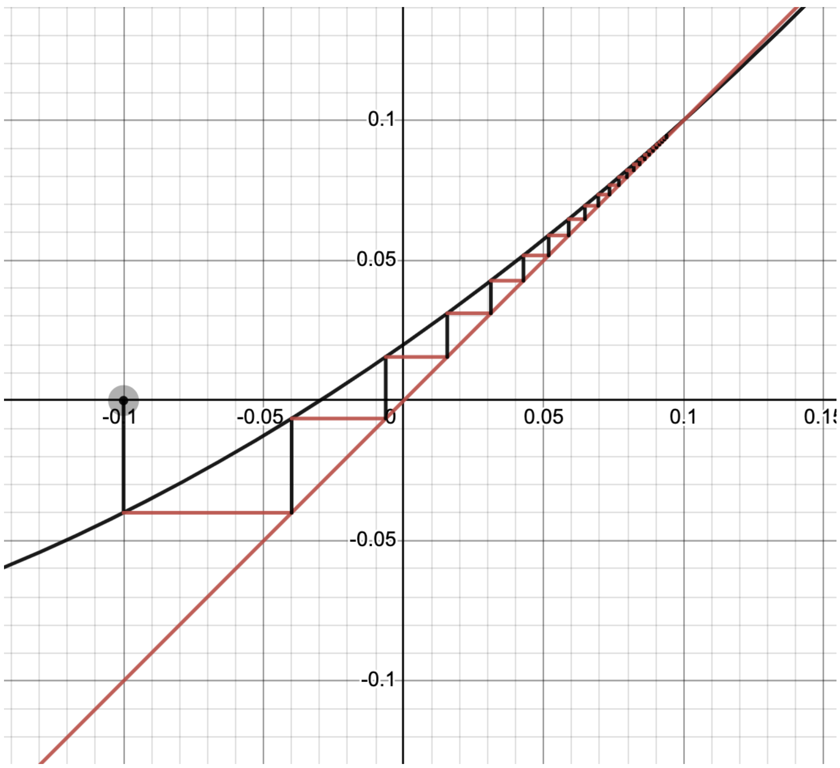


$f(x) = x^2 - 6$ cobweb diagram with $x_0 = 1$

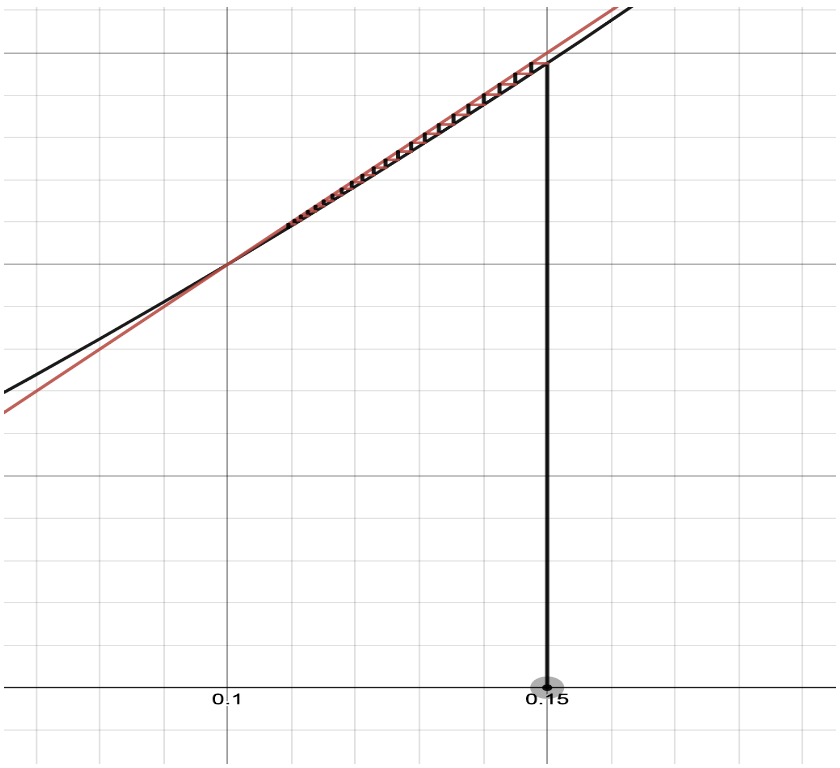


$f(x) = x^2 - 6$ cobweb diagram with $x_0 = 4$

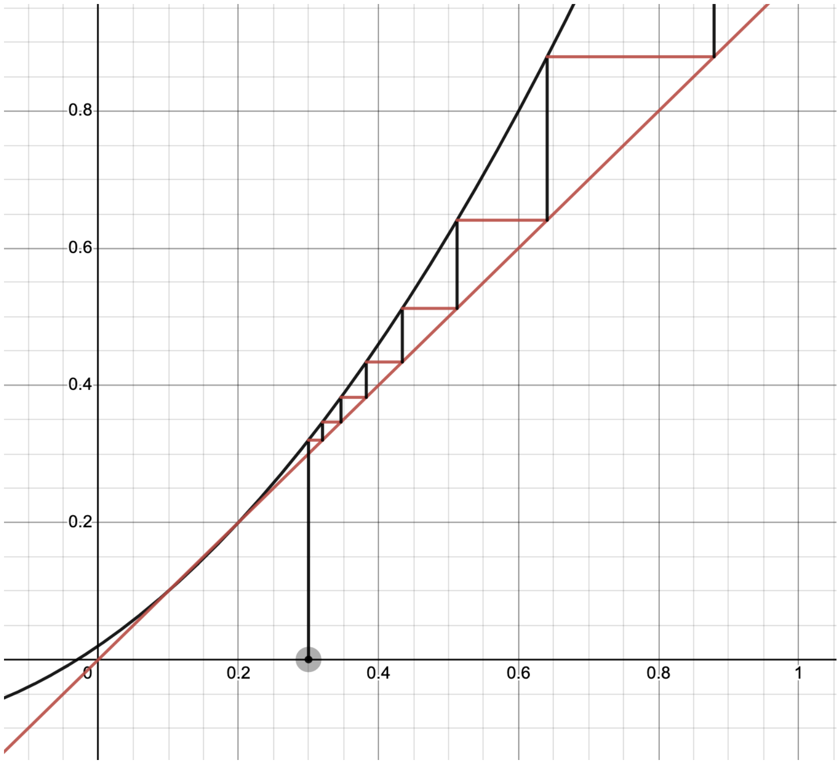
Part d) $x(n) = x(n-1)^2 + 0.7x(n-1) + 0.02$
Answer: Let $\bar{x}$ be the fixed point, like $\bar{x} = x(n) = x(n-1)$
Then we have $\bar{x} = \bar{x}^2 + 0.7\bar{x} + 0.02$
$\Rightarrow \bar{x}^2 + 0.7\bar{x} - \bar{x} + 0.02 = 0$
$\Rightarrow \bar{x}^2 - 0.3\bar{x} + 0.02 = 0$
$\Rightarrow (\bar{x} - 0.1)(\bar{x} - 0.2) = 0$ so, $\bar{x} = 0.2$ and $\bar{x} = 0.1$ which are fixed points.
For f(x) that we have is $f(x) = x^2 + 0.7x - 0.02$ and $f'(x) = 2x + 0.7$
When $\bar{x} = 0.2$, $|f'(0.2)| = |2(0.2) + 0.7| = 1.1 > 1$ which means unstable fixed point.
When $\bar{x} = 0.1$, $|f'(0.1)| = |2(0.1) + 0.7| = 0.9 < 1$ which means stable fixed point.



$f(x) = x^2 + 0.7 + 0.02$ cobweb diagram with $x_0 = -0.1$



$f(x) = x^2 + 0.7 + 0.02$ cobweb diagram with $x_0 = -0.15$



$f(x) = x^2 + 0.7 + 0.02$ cobweb diagram with $x_0 = 0.3$

3. Project 1.4: Compound Interest

Answer 1) To set up the recurrence relation between x(t+1) and x(t) is given by $x(t+1) = x(t)(12.05/12)$; x(0) = initial deposit which is 1000. To track the amount of money in the account over two year,
$x(1) = x(0)(12.05/12) = 1000(1.00417) = 1004.17$
$x(2) = x(1)(12.05/12) = (1004.17)(1.00417) = 1008.35$
$x(3) = x(2)(12.05/12) = (1008.35)(1.00417) = 1012.55$
$x(4) = x(3)(12.05/12) = (1012.55)(1.00417) = 1016.77$
$x(5) = x(4)(12.05/12) = (1016.77)(1.00417) = 1021.01$
$x(6) = x(5)(12.05/12) = (1021.01)(1.00417) = 1025.26$
$x(7) = x(6)(12.05/12) = (1025.26)(1.00417) = 1029.54$
$x(8) = x(7)(12.05/12) = (1029.54)(1.00417) = 1033.83$
$x(9) = x(8)(12.05/12) = (1033.83)(1.00417) = 1038.14$
$x(10) = x(9)(12.05/12) = (1038.14)(1.00417) = 1042.47$
$x(11) = x(10)(12.05/12) = (1008.35)(1.00417) = 1046.82$
$x(12) = x(11)(12.05/12) = (1046.82)(1.00417) = 1051.19$

$x(13) = x(12)(12.05/12) = (1051.19)(1.00417) = 1055.57$

$x(14) = x(13)(12.05/12) = (1055.57)(1.00417) = 1059.97$
$x(15) = x(14)(12.05/12) = (1059.97)(1.00417) = 1064.39$
$x(16) = x(15)(12.05/12) = (1064.39)(1.00417) = 1068.83$
$x(17) = x(16)(12.05/12) = (1068.83)(1.00417) = 1073.27$
$x(18) = x(17)(12.05/12) = (1073.27)(1.00417) = 1077.76$
$x(19) = x(18)(12.05/12) = (1077.76)(1.00417) = 1082.26$
$x(20) = x(19)(12.05/12) = (1082.26)(1.00417) = 1086.77$
$x(21) = x(20)(12.05/12) = (1086.77)(1.00417) = 1091.30$
$x(22) = x(21)(12.05/12) = (1091.30)(1.00417) = 1095.98$
$x(23) = x(22)(12.05/12) = (1095.98)(1.00417) = 1100.42$
$x(24) = x(23)(12.05/12) = (1100.42)(1.00417) = 1105.01$
Answer 2) $x(i+1) = 1000(1 + \frac{0.05}{12})^i$

Answer 3) The amount initial is x(0) = P. When i is 1, we then have $x(1) = P + \frac{r}{100n} * P$. When i is 2, $x(2) = x(1) + \frac{r}{100n} * P$. Thus, we can conclude the recurrence relation as $x(i+1) = x(i)(1 + \frac{r}{100n})$

Answer 4) Let the initial value is $x(0) = P$
When t = 1, $x(1) = x(0) + \frac{r}{100n}x(0) = P + \frac{r}{100n}P = P(1 + \frac{r}{100n})$
When t = 2, $x(2) = x(1) + \frac{r}{100n}x(1) = P(1 + \frac{r}{100n}) + \frac{r}{100n}P(1 + \frac{r}{100n}) = P(1 + \frac{r}{100n})(1 + \frac{r}{100n}) = P(1 + \frac{r}{100n})^2$
When t = 3, $x(3) = x(2) + \frac{r}{100n}x(2) = P(1 + \frac{r}{100n})^2 + \frac{r}{100n}P(1 + \frac{r}{100n}) = P(1 + \frac{r}{100n})^2(1 + \frac{r}{100n}) = P(1 + \frac{r}{100n})^3$
...
we can conclude that the closed-form solution for the recurrence relation is $x(t) = P(1 + \frac{r}{100n})^t$

Answer 5) Let initial money x(0) is 1000.
$x(1) = x(0)(12.05/12) = 1004.17$
$x(2) = x(1)(12.051/12) = 1008.43$
$x(3) = x(2)(12.052/12) = 1012.80$
$x(4) = x(3)(12.053/12) = 1017.28$
$x(5) = x(4)(12.052/12) = 1021.69$
$x(6) = x(5)(12.052/12) = 1026.11$
$x(7) = x(6)(12.053/12) = 1030.64$
$x(8) = x(7)(12.053/12) = 1035.20$
$x(9) = x(8)(12.053/12) = 1039.77$
$x(10) = x(9)(12.052/12) = 1044.27$
$x(11) = x(10)(12.052/12) = 1048.80$
$x(12) = x(11)(12.051/12) = 1053.26$

$x(13) = x(12)(12.051/12) = 1057.73$
$x(14) = x(13)(12.049/12) = 1062.05$
$x(15) = x(14)(12.049/12) = 1066.39$
$x(16) = x(15)(12.048/12) = 1070.66$

$x(17) = x(16)(12.048/12) = 1074.94$
$x(18) = x(17)(12.048/12) = 1079.24$
$x(19) = x(18)(12.047/12) = 1083.46$
$x(20) = x(19)(12.048/12) = 1087.80$
$x(21) = x(20)(12.047/12) = 1092.06$
$x(22) = x(21)(12.046/12) = 1096.25$
$x(23) = x(22)(12.045/12) = 1100.36$
$x(24) = x(23)(12.045/12) = 1104.39$

4. Project 1.5: Inflation

Answer 1) By using the compound interest rate equation which is $A = P(1 + \frac{r}{n})^t$, where P = initial money(principal), r = interest rate in decimal, t = time, and n = number of times interest is compounded per unit 't'.
When t = 1, $A = 100(1 + \frac{0.03}{1})^{1*1} = 103$
When t = 5, $A = 100(1 + \frac{0.03}{1})^{1*5} = 115.93$
When t = 10, $A = 100(1 + \frac{0.03}{1})^{1*10} = 134.39$
When t = 30, $A = 100(1 + \frac{0.03}{1})^{1*30} = 242.73$

Answer 2) By using the compound interest rate equation which is $A = P(1 + \frac{r}{n})^{nt}$, where P = initial money(principal), r = interest rate in decimal, t = time, and n = number of times interest is
When t = -10, $A = 100(1 + \frac{0.03}{1})^{1*-10} = 74.41$

Answer 3)
If an item cost \$100 thirty years ago(1992), let set thirty years ago as t = 0.
when t = 1 which is 29 later (1993), the item cost is \$102.99
when t = 5 which is 25 later (1997), the item cost is \$114.40
when t = 10 which is 20 later (2002), the item cost is \$128.23
when t = 30 which is 30 later (2022), the item cost is \$211.55

So, we can calculate the average inflation rate over those time period: 2.33%
When t = 30, then $x(30) = A = 100(1 + \frac{2.33}{100})^{1*30} = \$199.57$

The inflation rate 30 years ago (1992) is 3.03%
When t = 30, then $x(30) = A = 100(1 + \frac{3.03}{100})^{1*30} = \$244.856$

We could check that the average inflation rate underestimates the price after 30 years than the actual model. Also, using the inflation rate 30 years ago overestimate the price more than the actual price. Since the actual price is in between two inflation rates, it will be good to take the average to find a better model.