# Category Theory: First Homework

Deadline: 18. 11. 2022

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# Preface:

Throughout this homework I use covariant composition (so (f;g)(x) = (fg)(x) = g(f(x))) on morphisms. Composition (and use) of functors and natural transformations will stay as usual. I also denote the equivalence class of a partial map representative as [m, f].

I will also write  $\ell : (m, f) \sqsubseteq (m', f')$  (and  $\equiv$ ) when I talk about the (unique) map which makes the (in)equality hold.

## Problem 1

Show that partial maps in  $\mathcal{C}$  form a category. That is, define a category  $\mathcal{C}_{par}$  with  $|\mathcal{C}_{par}| = |\mathcal{C}|$  and  $\mathcal{C}_{par}(X,Y) =$  the collection of partial maps from X to Y.

**Solution** We define  $\mathcal{C}_{par}$  via  $|\mathcal{C}_{par}| := |\mathcal{C}|$  and for  $X, Y \in |\mathcal{C}_{par}|$  we define

$$\mathcal{C}_{\mathrm{par}}(X,Y) \coloneqq \left\{ [f,m] \subseteq \mathcal{C}(X_0,Y) \times \mathcal{C}(X_0,X) \mid X_0 \in |\mathcal{C}|, m \text{ monic} \right\}.$$

It remains to define the composition and identity, and show that the identity and associativity laws hold for the above.

Define  $1_X \in \mathcal{C}_{par}$  as  $[1_X, 1_X]$  and the composite of partial maps [m, f] and [n, g] as follows: taking the pullback of the cospan  $X_0 \stackrel{f}{\to} Y \stackrel{n}{\leftarrow} Y_0$  gives us the following diagram

$$\overline{X_0} \xrightarrow{\overline{f}} Y_0 \xrightarrow{g} Z$$

$$\downarrow^{\overline{n}} \qquad \downarrow^n$$

$$X_0 \xrightarrow{f} Y$$

$$\downarrow^m$$

$$X$$

We define [m, f];  $[n, g] := [\overline{n}m, \overline{f}g]$ . The map  $\overline{n}m$  is monic, as  $\overline{n}$  is monic, since it's the pullback of a monic map. It remains to show that composition is well-defined according to  $\equiv$ .

Let  $(m_1, f_1)$  and  $(m_2, f_2)$  be two partial map representatives of the same map. Then the two compositions (via pullbacks) with [n, g] are  $\left[\overline{n}_i m_i, \overline{f_i} g\right]$ . Because  $\varphi: (m_1, f_1) \equiv (m_2, f_2)$ , we can form the following diagrams:

The outer squares on the right two diagrams commute, thus we get the (unique) morphisms  $\psi$  and  $\psi'$  from the pullback square. This gives us the identities  $\psi \overline{n}_2 = \overline{n}_1 \varphi$ ,  $\psi \overline{f}_2 = \overline{f}_1$ ,  $\psi' \overline{n}_1 = \overline{n}_2 \varphi^{-1}$ , and  $\psi' \overline{f}_1 = \overline{f}_2$ , which are exactly the required identities to conclude both  $(\overline{n}_1 m_1, \overline{f}_1 g) \sqsubseteq (\overline{n}_2 m_2, \overline{f}_2 g)$  and  $(\overline{n}_2 m_2, \overline{f}_2 g) \sqsubseteq (\overline{n}_1 m_1, \overline{f}_1 g)$ . We can similarly prove that composition is also well-defined on the other argument.

Define the identity morphism on  $X \in |\mathcal{C}_{par}|$  as  $(1_X, 1_X)$ . Pullback along an identity give us the same morphism, so [m, f];  $[1_X, 1_X] = [1_X; m, f; 1_X] = [m, f]$  and  $[1_X, 1_X]$ ;  $[m, f] = [m; 1_X, 1_X; f] = [m, f]$ .

It remains to show associativity. Consider the pullback diagrams

$$P \xrightarrow{p} Z_0 \xrightarrow{h} W \qquad P' \xrightarrow{\overline{f}} P'_{YZ} \xrightarrow{\overline{g}} Z_0 \xrightarrow{h} W$$

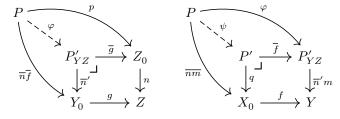
$$\downarrow^{\overline{n}} \qquad \downarrow^{n} \qquad \downarrow^{n}$$

$$\downarrow^{\overline{m}} \qquad \downarrow^{m} \qquad \text{and} \qquad \downarrow^{q} \qquad Y_0 \xrightarrow{g} Z$$

$$\downarrow^{\overline{m}} \qquad \downarrow^{m} \qquad X_0 \xrightarrow{f} Y \qquad \downarrow^{i}$$

$$X \qquad X$$

We wish to show that  $(\overline{nm}i, ph) \equiv (qi, \overline{fgh})$ . Consider now the following sub-diagrams of the above:



The outer squares are present in the original diagrams, so they commute. Then by pullback laws the maps  $\varphi$  and  $\psi$  exist and  $\psi$  is a witness for  $(\overline{nm}i, ph) \sqsubseteq (qi, \overline{fg}h)$  (since  $\psi qi = \overline{nm}i$  and  $\psi \overline{fg}h = \varphi \overline{g}h = ph$ ).

Define an identity-on-objects faithful functor  $I: \mathcal{C} \to \mathcal{C}_{par}$ .

**Solution** Define the functor as follows:

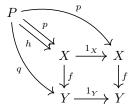
$$\begin{split} I:\mathcal{C} &\to \mathcal{C}_{\mathrm{par}} \\ X &\mapsto X \\ (f:X \to Y) &\mapsto [1_X,f]:I(X) \to I(Y). \end{split}$$

We need to verify it respects identities and compositions. Let  $X, Y, Z \in |\mathcal{C}|, f \in \mathcal{C}(X, Y),$  and  $g \in \mathcal{C}(Y, Z).$ 

$$\begin{split} I(1_X) &= [1_X, 1_X] = 1_X \in \mathcal{C}_{\mathrm{par}}, \\ I(f;g) &= [1_X, f;g] = [1_X, f] \, ; [1_Y, g] = I(f) ; I(g). \end{split}$$

Thus, I is indeed a functor. Let now f and g both be from  $\mathcal{C}(X,Y)$ , such that I(f)=I(g). Then we have  $\varphi:(1_X,f)\equiv(1_X,g)$ . From the equation  $\varphi 1_X=1_X$  it follows that  $\varphi=1_X$ . Then it follows from  $\varphi f=g$  that f=g and I is faithful.

We used the fact that a pullback along an identity is again the same morphism. Consider the following diagram, with the inner and outer squares commuting.



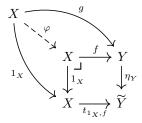
Then the required map from P to X as required for a pullback exists (and is p). It is also unique, since if there is another map h such that the triangles commute, then p = h follows from the upper triangle.

Suppose that every object X of  $\mathcal{C}$  has a partial map classifier. Show that the operation  $X \mapsto \widetilde{X}$  extends to a faithful functor  $(-): \mathcal{C} \to \mathcal{C}$  with respect to which the maps  $(\eta_X)_{X \in |\mathcal{C}|}$  form the components of a natural transformation  $\eta: 1_{\mathcal{C}} \Rightarrow (-)$ .

#### Solution

**Lemma:** The maps  $\eta_X$  are monic.

*Proof of lemma.* Let  $f, g: X \to Y$  be such that  $f\eta_Y = g\eta_Y$ . Then consider the following diagram:



The inner square is a pullback, since it arises from the partial map classifier of  $(1_X, f)$ , and the outer square commutes by assumption  $(1_X; t_{1_X, f} = f; \eta_Y = g; \eta_Y)$ . Then by pullback laws there exists a unique  $\varphi : X \to X$  such that  $\varphi; 1_X = 1_X$  and  $\varphi; f = g$ . From the first equation we get  $\varphi = 1_X$  which means that f = g follows and thus,  $\eta_Y$  is a mono.

*Proof.* Let  $f: X \to Y$ . Define then  $\tilde{f}$  to be the map  $t_{\eta_X,f}$  arising from the partial map classifier of the partial

map representative  $(\eta_X, f)$  (as shown above,  $\eta_X$  is monic). This makes the diagram  $\begin{array}{c} X \stackrel{f}{\longrightarrow} Y \\ \downarrow^{\eta_X} & \downarrow^{\eta_Y} \text{ commute,} \\ \widetilde{X} \stackrel{\widetilde{f}}{\longrightarrow} \widetilde{Y} \end{array}$ 

which will satisfy the naturality condition later.

Next we have to show  $\widetilde{(-)}$  is a functor. We have already defined the functions  $F_0: |\mathcal{C}| \to |\mathcal{C}|$  and  $F_{1,X,Y}: \mathcal{C}(X,Y) \to \mathcal{C}\big(\widetilde{X},\widetilde{Y}\big)$  appropriately, so we only need to verify that  $\widetilde{1_X} = 1_{\widetilde{X}}$  and  $\widetilde{f;g} = \widetilde{f};\widetilde{g}$ . First consider the following diagrams:

$$X \xrightarrow{1_X} X \qquad X \xrightarrow{1_X} X$$

$$\downarrow^{\eta_X} \qquad \downarrow^{\eta_X} \qquad \downarrow^{\eta_X} \qquad \downarrow^{\eta_X}$$

$$\widetilde{X} \xrightarrow{\widetilde{1_X}} \widetilde{X} \qquad \widetilde{X} \xrightarrow{1_{\widetilde{X}}} \widetilde{X}$$

They are both pullbacks, so by uniqueness of  $t_{\eta_X,1_X}$  they are equal, so  $\widetilde{1_X}=1_{\widetilde{X}}$ . Next consider the following diagrams:

$$\begin{array}{cccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & & X & \xrightarrow{f;g} & Z \\ \downarrow^{\eta_X} & & \downarrow^{\eta_Y} & & \downarrow^{\eta_Z} & & \downarrow^{\eta_X} & & \downarrow^{\eta_Z} \\ \widetilde{X} & \xrightarrow{\widetilde{f}} & \widetilde{Y} & \xrightarrow{\widetilde{g}} & \widetilde{Z} & & \widetilde{X} & \xrightarrow{\widetilde{f};g} & \widetilde{Z} \end{array}$$

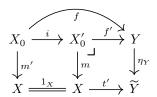
On the left diagram the inner two squares are pullbacks, thus by the pullback lemma so is the outer square. But then, as the right square is also a pullback diagram, we get  $\widetilde{f;g} = \widetilde{f}; \widetilde{g}$  by uniqueness via the partial map classifier.

Together then (-) is a functor and  $\eta: 1_{\mathcal{C}} \Rightarrow (-)$  is a natural transformation.

Suppose that Y has a partial map classifier. Show that, for any two representatives (m, f) and (m', f') of partial maps from X to Y, it holds that  $(m, f) \equiv (m', f')$  if and only if  $t_{m, f} = t_{m', f'}$ .

#### Solution

*Proof.* Consider the diagram



Then if  $i:(m,f)\equiv (m',f')$  the above diagram commutes. We wish to show the left square is a pullback. Take  $X\stackrel{p}{\leftarrow} Z\stackrel{J}{\rightarrow} X'_0$ , such that p=qm. Then, since i is an iso  $qi^{-1}$  is a morphism from Z to  $X_0$ . Because Y has a partial map classifier that morphism is also unique (as it maps into a pullback), so the left square is also a pullback. Then, by the pullback lemma the outer square is also a pullback. From there, we conclude that t=t', since t is the unique morphism, that makes that square a pullback.

In the other direction, from the assumption that t = t' we see that the outer and right squares are pullbacks, and  $i: X_0 \to X'_0$  exists because  $X'_0$  is a pullback and the diagram (without i) commutes. This i also makes the whole diagram commute. Then again by the pullback lemma we conclude that the left square is a pullback. Then i is an iso and  $i: (m, f) \equiv (m', f')$ .

Show that an object Y has a partial map classifier if and only if the functor  $F := \mathcal{C}_{par}(-,Y) \circ I^{op} : \mathcal{C}^{op} \to \mathbf{Set}$  is representable. (Here  $I^{op} : \mathcal{C}^{op} \to \mathcal{C}^{op}_{par}$  is the opposite-category version of  $I : \mathcal{C} \to \mathcal{C}_{par}$ ).

#### Solution

Let us first break down both definitions:

- $\bullet \ \ \exists \ \widetilde{Y} \in |\mathcal{C}|, \eta_Y \colon Y \to \widetilde{Y} \ni \colon \ \forall \ [m,f] \in \mathcal{C}_{\mathrm{par}}(X,Y). \ \exists ! \ t_{m,f} \in \mathcal{C}\big(X,\widetilde{Y}\big) \ni \colon \ \text{the square is a pullback}.$
- $\exists Z \in |\mathcal{C}|, \alpha : \mathcal{C}_{\mathrm{par}}(-,Y) \circ I^{\mathrm{op}} \Rightarrow \mathcal{C}(-,Z) \ni: \alpha \text{ is a natural isomorphism.}$ 
  - For all  $X \in |\mathcal{C}| \alpha_X$  is an iso
  - For all  $f \in \mathcal{C}(X, X')$  we have

$$\begin{array}{ccc} X & & \mathcal{C}_{\mathrm{par}}(X,Y) & \xrightarrow{\alpha_X} \mathcal{C}(X,Z) \\ \downarrow^f & & f^{\uparrow} & \uparrow^{f^*} \\ X' & & \mathcal{C}_{\mathrm{par}}(X',Y) & \xrightarrow{\alpha_{X'}} \mathcal{C}(X',Z) \end{array}$$

where  $Ff = [m, g] \mapsto [1_X, f]; [m, g] = \left[\overline{m}, \overline{f}; g\right].$ 

The equivalence will obviously take  $\widetilde{Y} = Z$ .

 $(\Rightarrow)$ . Define  $\alpha_X[m,f]:=t_{m,f}$ , the unique map arising from the partial map classifier  $\widetilde{Y}$ . It is well-defined by problem 4  $([m,f]=[m',f']\Rightarrow t_{m,f}=t_{m',f'})$ .

Take  $[m,g] \in \mathcal{C}_{\mathrm{par}}(X',Y)$  and  $f \in \mathcal{C}(X,X')$ . Then we have  $(Ff;\alpha_X)[m,g] = t_{\overline{m},\overline{f};g}$  and  $(\alpha_{X'};f^*)[m,g] = f;t_{m,g}$ . As before by pullback lemma on the diagram

$$X_{0} \xrightarrow{\overline{f}} X'_{0} \xrightarrow{g} Y$$

$$\downarrow^{\overline{m}} \qquad \downarrow^{m} \qquad \downarrow^{\eta_{Y}}$$

$$X \xrightarrow{f} X' \xrightarrow{t_{m,g}} \widetilde{Y}$$

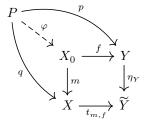
and uniqueness of  $t_{\overline{m},\overline{f};g}$  we see that  $f;t_{m,g}=t_{\overline{m},\overline{f};g}$ . This makes the naturality square commute, so  $\alpha$  is a natural transformation.

Define  $\beta: \mathcal{C}\left(-,\widetilde{Y}\right) \Rightarrow F$  by  $\beta_X h := [p,q]$  where  $X \stackrel{p}{\leftarrow} P \stackrel{q}{\rightarrow} Y$  is the pullback of the cospan  $X \stackrel{h}{\rightarrow} \widetilde{Y} \stackrel{\eta_Y}{\leftarrow} Y$ . Then  $(\alpha\beta)_X h = t_{p,q} = h$  by uniqueness of  $t_{p,q}$  and  $(\beta\alpha)_{X'} [m,g] = [m,g]$ . Thus,  $\alpha$  is a natural isomorphism.  $\square$ 

 $(\Leftarrow)$ . Let now  $\alpha: \mathcal{C}_{\mathrm{par}}(-,Y) \circ I^{\mathrm{op}} \Rightarrow \mathcal{C}(-,Z)$  be a natural isomorphism and  $X \overset{m}{\leftarrow} X_0 \overset{f}{\rightarrow} Y$  be a partial map representative from X to Y.

Define  $\eta_Y := \alpha_Y[1_Y, 1_Y]$  and  $t_{m,f} := \alpha_X[m, f]$ .

Keep the following diagram in mind (note that we do not assert the commutativity or existence of any of the maps indicated by convention):



First we have to prove the inner square commutes. Consider the following diagrams that arise from the naturality of  $\alpha$ :

We use commutativity of the diagrams on  $1_Y$  and [m,f] respectively to get  $f;\eta_Y=\alpha_{X_0}\left[1_{X_0},f\right]$  and  $m;t_{m,f}=1$  $\alpha_{X_0}\left[1_{X_0},f\right].$  It then follows that the inner square commutes.

Next we wish to show that it is a pullback. Consider  $X \stackrel{p}{\leftarrow} P \stackrel{q}{\rightarrow} Y$  arbitrary, such that  $p; \eta_Y = q; t_{m,f}$ . By similar two naturality diagrams for p and q as above we get  $p; \eta_Y = \alpha_P(Fp(1_Y)) = \alpha_P[1_P, p]$  and  $q; t_{m,f} = \alpha_P(Fp(1_Y)) = \alpha_P[1_P, p]$ .

 $\alpha_P[\overline{m},\overline{q}f], \text{ which are equal by assumption}.$  (Note: the maps  $\overline{m}$  and  $\overline{q}$  arise from the pullback diagram  $\bigvee_{P \longrightarrow q}^{\overline{q}} X_0$   $\downarrow_{\overline{m}}^{Q} \bigvee_{P \longrightarrow q}^{q} X$ 

Since  $\alpha_P$  is an iso it then follows that  $[1_P, p] = [\overline{m}, \overline{q}f]$ . Writing out the definition of equivalence we see that  $\overline{m}$ is an iso and  $\overline{m}p = \overline{q}f$ . We can then define  $\varphi := \overline{m}^{-1}\overline{q}$ .

The pullback square above gives us the equation  $\overline{q}m = \overline{m}q$ . Putting it all together we get  $\varphi f = \overline{m}^{-1}\overline{q}f = p$  and  $\varphi m = \overline{m}^{-1} \overline{q} m = \overline{m}^{-1} \overline{m} q = q$ , making the square a pullback.

 $\varphi m = \overline{m}^{-1} \overline{q} m = \overline{m}^{-1} \overline{m} q = q, \text{ making the square a parameter } X_0 \xrightarrow{f} Y$  It remains to show uniqueness of  $t_{m,f}$ . Let t' be another map for which  $\bigvee_{m}^{J} \bigvee_{\gamma} \eta_{\gamma} \text{ is a pullback. Then,} X \xrightarrow{t'} \widetilde{Y}$ 

because  $\alpha_X$  is iso, we have  $t' = \alpha_X[n,g]$  for some  $[n,g] \in \mathcal{C}_{par}(X,\widetilde{Y})$ . Then by the first part of this proof we

know that  $\bigvee_{n}^{P} \xrightarrow{g} Y$   $\downarrow_{\eta_{Y}}$  is also a pullback. Because pullbacks are unique up to isomorphism, we have an iso

 $X_0 \to P$  witnessing  $(m, f) \equiv (n, g)$ , so  $t_{m, f} = \alpha_X[m, f] = \alpha_X[n, g] = t'$ .