Category Theory: First Homework

Deadline: 18. 11. 2022

Luna Strah, 27222025

Preface:

Throughout this homework I use contravariant composition (so (f;g)(x) = (fg)(x) = g(f(x))) on morphisms. Composition (and use) of functors and natural transformations will stay as usual. I also denote the equivalence class of a partial map representative as [m, f].

I will also write $\ell : (m, f) \equiv (m', f')$ (and \sqsubseteq) when I talk about the (unique) map which makes the (in)equality hold.

Problem 1

Show that partial maps in \mathcal{C} form a category. That is, define a category \mathcal{C}_{par} with $|\mathcal{C}_{par}| = |\mathcal{C}|$ and $\mathcal{C}_{par}(X,Y) =$ the collection of partial maps from X to Y.

Solution We define \mathcal{C}_{par} via $|\mathcal{C}_{par}| := |\mathcal{C}|$ and for $X, Y \in |\mathcal{C}_{par}|$ we define

$$\mathcal{C}_{\mathrm{par}}(X,Y) \coloneqq \left\{ [f,m] \subseteq \mathcal{C}(X_0,Y) \times \mathcal{C}(X_0,X) \mid X_0 \in |\mathcal{C}|, m \text{ monic} \right\}.$$

It remains to define the composition and identity, and show that the identity and associativity laws hold for the above.

Define $1_X \in \mathcal{C}_{par}$ as $[1_X, 1_X]$ and the composite of partial maps [m, f] and [n, g] as follows: taking the pullback of the cospan $X_0 \stackrel{f}{\to} Y \stackrel{n}{\leftarrow} Y_0$ gives us the following diagram

$$\overline{X_0} \xrightarrow{\overline{f}} Y_0 \xrightarrow{g} Z$$

$$\downarrow^{\overline{n}} \downarrow^n$$

$$X_0 \xrightarrow{f} Y$$

$$\downarrow^m$$

$$X$$

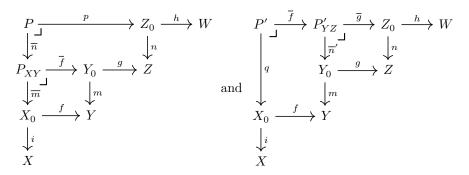
We define [m, f]; $[n, g] := [\overline{n}m, \overline{f}g]$. The map $\overline{n}m$ is monic, as \overline{n} is monic, since it's the pullback of a monic map. It remains to show that composition is well-defined according to \equiv .

Let (m_1, f_1) and (m_2, f_2) be two partial map representatives of the same map. Then the two compositions (via pullbacks) with [n, g] are $\left[\overline{n}_i m_i, \overline{f}_i g\right]$. Because $\varphi: (m_1, f_1) \equiv (m_2, f_2)$, we can form the following diagrams:

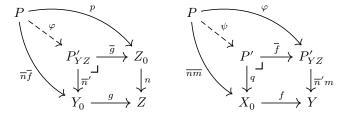
The outer squares on the right two diagrams commute, thus we get the (unique) morphisms ψ and ψ' from the pullback square. This gives us the identities $\psi \overline{n}_2 = \overline{n}_1 \varphi$, $\psi \overline{f}_2 = \overline{f}_1$, $\psi' \overline{n}_1 = \overline{n}_2 \varphi^{-1}$, and $\psi' \overline{f}_1 = \overline{f}_2$, which are exactly the required identities to conclude both $(\overline{n}_1 m_1, \overline{f}_1 g) \sqsubseteq (\overline{n}_2 m_2, \overline{f}_2 g)$ and $(\overline{n}_2 m_2, \overline{f}_2 g) \sqsubseteq (\overline{n}_1 m_1, \overline{f}_1 g)$. We can similarly prove that composition is also well-defined on the other argument.

Define the identity morphism on $X \in |\mathcal{C}_{par}|$ as $(1_X, 1_X)$. Pullback along an identity give us the same morphism, so [m, f]; $[1_X, 1_X] = [1_X; m, f; 1_X] = [m, f]$ and $[1_X, 1_X]$; $[m, f] = [m; 1_X, 1_X; f] = [m, f]$.

It remains to show associativity. Consider the pullback diagrams



We wish to show that $(\overline{nm}i, ph) \equiv (qi, \overline{fgh})$. Consider now the following sub-diagrams of the above:



The outer squares are present in the original diagrams, so they commute. Then by pullback laws the maps φ and ψ exist and ψ is a witness for $(\overline{nm}i, ph) \sqsubseteq (qi, \overline{fg}h)$ (since $\psi qi = \overline{nm}i$ and $\psi \overline{fg}h = \varphi \overline{g}h = ph$).

Define an identity-on-objects faithful functor $I: \mathcal{C} \to \mathcal{C}_{par}$.

Solution Define the functor as follows:

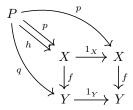
$$\begin{split} I:\mathcal{C} &\to \mathcal{C}_{\mathrm{par}} \\ X &\mapsto X \\ (f:X \to Y) &\mapsto [1_X,f]:I(X) \to I(Y). \end{split}$$

We need to verify it respects identities and compositions. Let $X, Y, Z \in |\mathcal{C}|, f \in \mathcal{C}(X, Y),$ and $g \in \mathcal{C}(Y, Z).$

$$\begin{split} I(1_X) &= [1_X, 1_X] = 1_X \in \mathcal{C}_{\mathrm{par}}, \\ I(f;g) &= [1_X, f;g] = [1_X, f]; [1_Y, g] = I(f); I(g). \end{split}$$

Thus, I is indeed a functor. Let now f and g both be from $\mathcal{C}(X,Y)$, such that I(f)=I(g). Then we have $\varphi:(1_X,f)\equiv(1_X,g)$. From the equality $\varphi\,1_X=1_X$ it follows that $\varphi=1_X$. Then it follows from $\varphi f=g$ that f=g so I is faithful.

We used the fact that a pullback along an identity is again the same morphism. Consider the following diagram, with the inner and outer squares commuting.



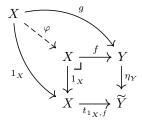
Then the required map from P to X as required for a pullback exists (and is p). It is also unique, since if there is another map h such that the triangles commute, then p = h follows from the upper triangle.

Suppose that every object X of \mathcal{C} has a partial map classifier. Show that the operation $X \mapsto \widetilde{X}$ extends to a faithful functor $(-): \mathcal{C} \to \mathcal{C}$ with respect to which the maps $(\eta_X)_{X \in |\mathcal{C}|}$ form the components of a natural transformation $\eta: 1_{\mathcal{C}} \Rightarrow (-)$.

Solution

Lemma: The maps η_X are monic.

Proof of lemma. Let $f, g: X \to Y$ be such that $f\eta_Y = g\eta_Y$. Then consider the following diagram:



The inner square is a pullback, since it arises from the partial map classifier of $(1_X, f)$, and the outer square commutes by assumption $(1_X; t_{1_X, f} = f; \eta_Y = g; \eta_Y)$. Then by pullback laws there exists a unique $\varphi : X \to X$ such that $\varphi; 1_X = 1_X$ and $\varphi; f = g$. From the first equality we get $\varphi = 1_X$ which means that f = g follows and thus, η_Y is a mono.

Proof. Let $f: X \to Y$. Define then \tilde{f} to be the map $t_{\eta_X,f}$ arising from the partial map classifier of the partial

map representative (η_X, f) (as shown above, η_X is monic). This makes the diagram $\begin{array}{c} X \stackrel{f}{\longrightarrow} Y \\ \downarrow^{\eta_X} & \downarrow^{\eta_Y} \text{ commute,} \\ \widetilde{X} \stackrel{\widetilde{f}}{\longrightarrow} \widetilde{Y} \end{array}$

which will satisfy the naturality condition later. Next we have to show (-) is a functor. We have already defined the functions $F_0: |\mathcal{C}| \to |\mathcal{C}|$ and $F_{1,X,Y}: \mathcal{C}(X,Y) \to \mathcal{C}(\widetilde{X},\widetilde{Y})$ appropriately, so we only need to verify that $\widetilde{1_X} = 1_{\widetilde{X}}$ and $\widetilde{f}; g = \widetilde{f}; \widetilde{g}$. First consider the following diagrams:

$$\begin{array}{ccc} X \xrightarrow{1_X} X & X \xrightarrow{1_X} X \\ \downarrow^{\eta_X} & \downarrow^{\eta_X} & \downarrow^{\eta_X} & \downarrow^{\eta_X} \\ \widetilde{X} \xrightarrow{\widetilde{1_X}} \widetilde{X} & \widetilde{X} & \widetilde{X} \xrightarrow{1_{\widetilde{X}}} \widetilde{X} \end{array}$$

They are both pullbacks, so by uniqueness of $t_{\eta_X,1_X}$ they are equal, so $\widetilde{1_X}=1_{\widetilde{X}}$. Next consider the following diagrams:

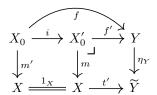
$$\begin{array}{cccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & & X & \xrightarrow{f;g} & Z \\ \downarrow^{\eta_X} & & \downarrow^{\eta_Y} & & \downarrow^{\eta_Z} & & \downarrow^{\eta_X} & & \downarrow^{\eta_Z} \\ \widetilde{X} & \xrightarrow{\widetilde{f}} & \widetilde{Y} & \xrightarrow{\widetilde{g}} & \widetilde{Z} & & \widetilde{X} & \xrightarrow{\widetilde{f};g} & \widetilde{Z} \end{array}$$

On the left diagram the inner two squares are pullbacks, thus by the pullback lemma so is the outer square. But then, as the right square is also a pullback diagram, we get $\widetilde{f}; g = \widetilde{f}; \widetilde{g}$ by uniqueness via the partial map classifier. Together then (-) is a functor and $\eta: 1_{\mathcal{C}} \Rightarrow (-)$ is a natural transformation.

Suppose that Y has a partial map classifier. Show that, for any two representatives (m, f) and (m', f') of partial maps from X to Y, it holds that $(m, f) \equiv (m', f')$ if and only if $t_{m,f} = t_{m',f'}$.

Solution

Proof. Consider the diagram



Then if $i:(m,f)\equiv (m',f')$ the above diagram commutes. We wish to show the left square is a pullback. Take $X\stackrel{p}{\leftarrow} Z\stackrel{q}{\rightarrow} X'_0$, such that p=qm. Then, since i is an iso, qi^{-1} is a morphism from Z to X_0 . Because Y has a partial map classifier that morphism is also unique (as it maps into a pullback), so the left square is also a pullback. Then, by the pullback lemma, the outer square is also a pullback. From there, we conclude that t=t', since t is the unique morphism that makes that square a pullback.

In the other direction, from the assumption that t = t' we see that the outer and right squares are pullbacks, and $i: X_0 \to X_0'$ exists because X_0' is a pullback and the diagram (without i) commutes. This i also makes the whole diagram commute. By the pullback lemma we again conclude that the left square is a pullback. Then i is an iso and $i: (m, f) \equiv (m', f')$.

Show that an object Y has a partial map classifier if and only if the functor $F := \mathcal{C}_{par}(-,Y) \circ I^{op} : \mathcal{C}^{op} \to \mathbf{Set}$ is representable. (Here $I^{op} : \mathcal{C}^{op} \to \mathcal{C}^{op}_{par}$ is the opposite-category version of $I : \mathcal{C} \to \mathcal{C}_{par}$).

Solution

Let us first break down both definitions:

- $\exists \ \widetilde{Y} \in |\mathcal{C}|, \eta_Y \colon Y \to \widetilde{Y} \ni \colon \ \forall \ [m,f] \in \mathcal{C}_{\mathrm{par}}(X,Y). \ \exists ! \ t_{m,f} \in \mathcal{C}\big(X,\widetilde{Y}\big) \ni \colon$ the square is a pullback.
- $\exists Z \in |\mathcal{C}|, \alpha : \mathcal{C}_{\mathrm{par}}(-,Y) \circ I^{\mathrm{op}} \Rightarrow \mathcal{C}(-,Z) \ni: \alpha \text{ is a natural isomorphism.}$
 - For all $X \in |\mathcal{C}|$ α_X is an iso
 - For all $f \in \mathcal{C}(X, X')$ we have

$$\begin{array}{ccc} X & & \mathcal{C}_{\mathrm{par}}(X,Y) & \xrightarrow{\alpha_X} \mathcal{C}(X,Z) \\ \downarrow^f & & f^{\uparrow} & \uparrow^{f^*} \\ X' & & \mathcal{C}_{\mathrm{par}}(X',Y) & \xrightarrow{\alpha_{X'}} \mathcal{C}(X',Z) \end{array}$$

where $Ff = [m, g] \mapsto [1_X, f]; [m, g] = \left[\overline{m}, \overline{f}; g\right].$

The equivalence will obviously take $\widetilde{Y} = Z$.

 (\Rightarrow) . Define $\alpha_X[m,f]:=t_{m,f}$, the unique map arising from the partial map classifier \widetilde{Y} . It is well-defined by problem 4 $([m,f]=[m',f']\Rightarrow t_{m,f}=t_{m',f'})$.

Take arbitrary $[m, g] \in \mathcal{C}_{par}(X', Y)$ and $f \in \mathcal{C}(X, X')$. Then we have $(Ff; \alpha_X)[m, g] = t_{\overline{m}, \overline{f}; g}$ and $(\alpha_{X'}; f^*)[m, g] = f; t_{m,g}$. As before by pullback lemma on the diagram

$$X_{0} \xrightarrow{\overline{f}} X'_{0} \xrightarrow{g} Y$$

$$\downarrow^{\overline{m}} \qquad \downarrow^{\eta_{Y}}$$

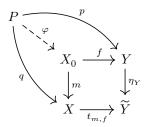
$$X \xrightarrow{f} X' \xrightarrow{t_{m,g}} \widetilde{Y}$$

and uniqueness of $t_{\overline{m},\overline{f};g}$ we see that $f;t_{m,g}=t_{\overline{m},\overline{f};g}$. This makes the naturality square commute, so α is a natural transformation.

Define $\beta: \mathcal{C}\left(-,\widetilde{Y}\right) \Rightarrow F$ by $\beta_X h := [p,q]$ where $X \stackrel{p}{\leftarrow} P \stackrel{q}{\rightarrow} Y$ is the pullback of the cospan $X \stackrel{h}{\rightarrow} \widetilde{Y} \stackrel{\eta_Y}{\leftarrow} Y$. Then $(\alpha\beta)_X h = t_{p,q} = h$ by uniqueness of $t_{p,q}$ and $(\beta\alpha)_{X'} [m,g] = [m,g]$. Thus, α is a natural isomorphism.

 (\Leftarrow) . Let now $\alpha: \mathcal{C}_{\mathrm{par}}(-,Y) \circ I^{\mathrm{op}} \Rightarrow \mathcal{C}(-,Z)$ be a natural isomorphism and $X \stackrel{m}{\leftarrow} X_0 \stackrel{f}{\rightarrow} Y$ be a partial map representative from X to Y. Define $\eta_Y := \alpha_Y[1_Y, 1_Y]$ and $t_{m,f} := \alpha_X[m,f]$.

For the rest of the proof keep the following diagram in mind (note that we do not assert the commutativity or existence of any of the maps indicated by convention):



First we have to prove the inner square commutes. Consider the following diagrams that arise from the naturality of α :

We use commutativity of the diagrams on 1_Y and [m,f] respectively to get $f; \eta_Y = \alpha_{X_0} \left[1_{X_0}, f \right]$ and $m; t_{m,f} = \alpha_{X_0} \left[1_{X_0}, f \right]$. It then follows that the inner square commutes.

Next we wish to show that it is a pullback. Consider $X \stackrel{p}{\leftarrow} P \stackrel{q}{\rightarrow} Y$ arbitrary, such that $p; \eta_Y = q; t_{m,f}$. By similar two naturality diagrams for p and q as above we get $p; \eta_Y = \alpha_P(Fp(1_Y)) = \alpha_P[1_P, p]$ and $q; t_{m,f} = \alpha_P[\overline{m}, \overline{q}f]$, which are equal by assumption.

(Note: the maps \overline{m} and \overline{q} arise from the pullback diagram $\begin{array}{c} \overline{P} \xrightarrow{\overline{q}} X_0 \\ \downarrow^{\overline{m}} & \downarrow^{m}) \\ P \xrightarrow{q} X \end{array}$

Since α_P is an iso it then follows that $[1_P, p] = [\overline{m}, \overline{q}f]$. Writing out the definition of equivalence we see that \overline{m} is an iso and $\overline{m}p = \overline{q}f$. We can then define $\varphi := \overline{m}^{-1}\overline{q}$.

The pullback square above gives us the equality $\overline{q}m = \overline{m}q$. Putting it all together we get the equalities $\varphi f = \overline{m}^{-1}\overline{q}f = p$ and $\varphi m = \overline{m}^{-1}\overline{q}m = \overline{m}^{-1}\overline{m}q = q$, making the square a pullback.

It remains to show uniqueness of $t_{m,f}$. Let t' be another map for which $X_0 \xrightarrow{f} Y$ Y is a pullback. Then, $X \xrightarrow{t'} \widetilde{Y}$

because α_X is iso, we have $t' = \alpha_X[n,g]$ for some $[n,g] \in \mathcal{C}_{\mathrm{par}}\big(X,\widetilde{Y}\big)$. Then by the first part of this proof we

know that $V \xrightarrow{g} Y$ $V \xrightarrow{\eta_Y} V$ is also a pullback. Because pullbacks are unique up to isomorphism, we have an iso $V \xrightarrow{t'} V$

 $X_0 \to P$ witnessing $(m, f) \equiv (n, g)$, so $t_{m, f} = \alpha_X[m, f] = \alpha_X[n, g] = t'$.