Category Theory: First Homework

Deadline: 18. 11. 2022

Luna Strah, 27222025

Preface:

Throughout this homework I use covariant composition (so $(f;g)(x) = (fg)(x) = g(f(x))$) on morphisms. Composition (and use) of functors and natural transformations will stay as usual. I also denote the equivalence class of a partial map representative as $[m, f]$.
I will also write $\ell : (m, f) \sqsubseteq (m', f')$ (and $\equiv$) when I talk about the (unique) map which makes the (in)equality hold.

Problem 1

Show that partial maps in $\mathcal{C}$ form a category. That is, define a category $\mathcal{C}_{\mathrm{par}}$ with $|\mathcal{C}_{\mathrm{par}}| = |\mathcal{C}|$ and $\mathcal{C}_{\mathrm{par}}(X, Y) =$ the collection of partial maps from X to Y.

Solution We define $\mathcal{C}_{\mathrm{par}}$ via $|\mathcal{C}_{\mathrm{par}}| := |\mathcal{C}|$ and for $X, Y \in |\mathcal{C}_{\mathrm{par}}|$ we define

$$\mathcal{C}_{\mathrm{par}}(X, Y) := \{[f, m] \subseteq \mathcal{C}(X_0, Y) \times \mathcal{C}(X_0, X) \mid X_0 \in |\mathcal{C}|, m \text{ monic}\}.$$

It remains to define the composition and identity, and show that the identity and associativity laws hold for the above.
Define $1_X \in \mathcal{C}_{\mathrm{par}}$ as $[1_X, 1_X]$ and the composite of partial maps $[m, f]$ and $[n, g]$ as follows: taking the pullback of the cospan $X_0 \xrightarrow{f} Y \xleftarrow{n} Y_0$ gives us the following diagram

$$\begin{tikzcd}
\overline{X_0} \arrow[r, "\overline{f}"] \arrow[d, "\overline{n}"] \arrow[dr, phantom, "\lrcorner", very near start] & Y_0 \arrow[r, "g"] \arrow[d, "n"] & Z \\
X_0 \arrow[r, "f"] \arrow[d, "m"] & Y & \\
X & &
\end{tikzcd}$$

We define $[m, f]\,;[n, g] := \left[\overline{n}m, \overline{f}g\right]$. The map $\overline{n}m$ is monic, as $\overline{n}$ is monic, since it's the pullback of a monic map. It remains to show that composition is well-defined according to $\equiv$.
Let (m_1, f_1) and (m_2, f_2) be two partial map representatives of the same map. Then the two compositions (via pullbacks) with $[n, g]$ are $\left[\overline{n}_i m_i, \overline{f}_i g\right]$. Because $\varphi : (m_1, f_1) \equiv (m_2, f_2)$, we can form the following diagrams:

$$\begin{tikzcd}
P_i \arrow[r, "\overline{f_i}"] \arrow[d, "\overline{n}_i"] \arrow[dr, phantom, "\lrcorner", very near start] & Y_0 \arrow[r, "g"] \arrow[d, "m"] & Z \\
X_i \arrow[r, "f_i"] \arrow[d, "m_i"] & Y & \\
X & &
\end{tikzcd}
\qquad
\begin{tikzcd}
P_1 \arrow[dr, dashed, "\psi"] \arrow[drr, bend left, "\overline{f_1}"] \arrow[ddr, bend right, "\overline{n}_1\varphi"'] & & \\
& P_2 \arrow[r, "\overline{f_2}"] \arrow[d, "\overline{n}_2"] \arrow[dr, phantom, "\lrcorner", very near start] & Y_0 \\
& X_2 &
\end{tikzcd}
\qquad
\begin{tikzcd}
P_2 \arrow[dr, dashed, "\psi'"] \arrow[drr, bend left, "\overline{f_2}"] \arrow[ddr, bend right, "\overline{n}_2\varphi^{-1}"'] & & \\
& P_1 \arrow[r, "\overline{f_1}"] \arrow[d, "\overline{n}_1"] \arrow[dr, phantom, "\lrcorner", very near start] & Y_0 \\
& X_1 &
\end{tikzcd}$$

The outer squares on the right two diagrams commute, thus we get the (unique) morphisms ψ and ψ' from the pullback square. This gives us the identities $\psi\overline{n}_2 = \overline{n}_1\varphi$, $\psi\overline{f_2} = \overline{f_1}$, $\psi'\overline{n}_1 = \overline{n}_2\varphi^{-1}$, and $\psi'\overline{f_1} = \overline{f_2}$, which are exactly the required identities to conclude both $\left(\overline{n}_1 m_1, \overline{f_1}g\right) \sqsubseteq \left(\overline{n}_2 m_2, \overline{f_2}g\right)$ and $\left(\overline{n}_2 m_2, \overline{f_2}g\right) \sqsubseteq \left(\overline{n}_1 m_1, \overline{f_1}g\right)$. We can similarly prove that composition is also well-defined on the other argument.
Define the identity morphism on $X \in |\mathcal{C}_{\mathrm{par}}|$ as $(1_X, 1_X)$. Pullback along an identity give us the same morphism, so $[m, f]\,;[1_X, 1_X] = [1_X; m, f; 1_X] = [m, f]$ and $[1_X, 1_X]\,;[m, f] = [m; 1_X, 1_X; f] = [m, f]$.

Problem 1 continued on next page...

It remains to show associativity. Consider the pullback diagrams

$$
\begin{tikzcd}
P \arrow[rr, "p"] \arrow[d, "\overline{n}"] \arrow[dr, phantom, very near start, "\lrcorner"] & & Z_0 \arrow[r, "h"] \arrow[d, "n"] & W \\
P_{XY} \arrow[r, "\overline{f}"] \arrow[d, "\overline{m}"] \arrow[dr, phantom, very near start, "\lrcorner"] & Y_0 \arrow[r, "g"] \arrow[d, "m"] & Z & \\
X_0 \arrow[r, "f"] \arrow[d, "i"] & Y & & \\
X & & &
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
P' \arrow[r, "\overline{f}"] \arrow[dd, "q"] \arrow[dr, phantom, very near start, "\lrcorner"] & P'_{YZ} \arrow[r, "\overline{g}"] \arrow[d, "\overline{n}'"] \arrow[dr, phantom, very near start, "\lrcorner"] & Z_0 \arrow[r, "h"] \arrow[d, "n"] & W \\
& Y_0 \arrow[r, "g"] \arrow[d, "m"] & Z & \\
X_0 \arrow[r, "f"] \arrow[d, "i"] & Y & & \\
X & & &
\end{tikzcd}
$$

We wish to show that $(\overline{n}\overline{m}i, ph) \equiv \left(qi, \overline{f}\overline{g}h\right)$. Consider now the following sub-diagrams of the above:

$$
\begin{tikzcd}
P \arrow[dr, dashed, "\varphi"] \arrow[drr, bend left, "p"] \arrow[ddr, bend right, "\overline{n}\overline{f}"'] & & \\
& P'_{YZ} \arrow[r, "\overline{g}"] \arrow[d, "\overline{n}'"] \arrow[dr, phantom, very near start, "\lrcorner"] & Z_0 \arrow[d, "n"] \\
& Y_0 \arrow[r, "g"] & Z
\end{tikzcd}
\qquad
\begin{tikzcd}
P \arrow[dr, dashed, "\psi"] \arrow[drr, bend left, "\varphi"] \arrow[ddr, bend right, "\overline{nm}"'] & & \\
& P' \arrow[r, "\overline{f}"] \arrow[d, "q"] \arrow[dr, phantom, very near start, "\lrcorner"] & P'_{YZ} \arrow[d, "\overline{n}'m"] \\
& X_0 \arrow[r, "f"] & Y
\end{tikzcd}
$$

The outer squares are present in the original diagrams, so they commute. Then by pullback laws the maps φ and ψ exist and ψ is a witness for $(\overline{n}\overline{m}i, ph) \sqsubseteq \left(qi, \overline{f}\overline{g}h\right)$ (since $\psi qi = \overline{n}\overline{m}i$ and $\psi\overline{f}\overline{g}h = \varphi\overline{g}h = ph$).

Problem 2

Define an identity-on-objects faithful functor $I : \mathcal{C} \to \mathcal{C}_{\text{par}}$.

Solution Define the functor as follows:

$$\begin{gathered} I : \mathcal{C} \to \mathcal{C}_{\text{par}} \\ X \mapsto X \\ (f : X \to Y) \mapsto [1_X, f] : I(X) \to I(Y). \end{gathered}$$

We need to verify it respects identities and compositions. Let $X, Y, Z \in |\mathcal{C}|$, $f \in \mathcal{C}(X, Y)$, and $g \in \mathcal{C}(Y, Z)$.

$$\begin{aligned} &I(1_X) = [1_X, 1_X] = 1_X \in \mathcal{C}_{\text{par}}, \\ &I(f; g) = [1_X, f; g] = [1_X, f] ; [1_Y, g] = I(f); I(g). \end{aligned}$$

Thus, I is indeed a functor. Let now f and g both be from $\mathcal{C}(X, Y)$, such that $I(f) = I(g)$. Then we have $\varphi : (1_X, f) \equiv (1_X, g)$. From the equation $\varphi 1_X = 1_X$ it follows that $\varphi = 1_X$. Then it follows from $\varphi f = g$ that $f = g$ and I is faithful.

We used the fact that a pullback along an identity is again the same morphism. Consider the following diagram, with the inner and outer squares commuting.

$$\begin{tikzcd}
P \arrow[dr, shift left, "p"] \arrow[dr, shift right, "h"'] \arrow[drr, bend left, "p"] \arrow[ddr, bend right, "q"'] & & \\
& X \arrow[r, "1_X"] \arrow[d, "f"] & X \arrow[d, "f"] \\
& Y \arrow[r, "1_Y"] & Y
\end{tikzcd}$$

Then the required map from P to X as required for a pullback exists (and is p). It is also unique, since if there is another map h such that the triangles commute, then $p = h$ follows from the upper triangle.

Problem 3

Suppose that every object X of $\mathcal{C}$ has a partial map classifier. Show that the operation $X \mapsto \widetilde{X}$ extends to a faithful functor $\widetilde{(-)} : \mathcal{C} \to \mathcal{C}$ with respect to which the maps $(\eta_X)_{X \in |\mathcal{C}|}$ form the components of a natural transformation $\eta : 1_{\mathcal{C}} \Rightarrow \widetilde{(-)}$.

Solution
Lemma: The maps η_X are monic.

Proof of lemma. Let $f, g : X \to Y$ be such that $f\eta_Y = g\eta_Y$. Then consider the following diagram:

$$\begin{tikzcd}
X \arrow[dr, dashed, "\varphi"] \arrow[drr, bend left, "g"] \arrow[ddr, bend right, "1_X"'] & & \\
& X \arrow[r, "f"] \arrow[d, "1_X"] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "\eta_Y"] \\
& X \arrow[r, "t_{1_X, f}"'] & \widetilde{Y}
\end{tikzcd}$$

The inner square is a pullback, since it arises from the partial map classifier of $(1_X, f)$, and the outer square commutes by assumption ($1_X; t_{1_X, f} = f; \eta_Y = g; \eta_Y$). Then by pullback laws there exists a unique $\varphi : X \to X$ such that $\varphi; 1_X = 1_X$ and $\varphi; f = g$. From the first equation we get $\varphi = 1_X$ which means that $f = g$ follows and thus, η_Y is a mono. $\square$

Proof. Let $f : X \to Y$. Define then $\tilde{f}$ to be the map $t_{\eta_X, f}$ arising from the partial map classifier of the partial map representative (η_X, f) (as shown above, η_X is monic). This makes the diagram $\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\eta_X"] & Y \arrow[d, "\eta_Y"] \\ \widetilde{X} \arrow[r, "\tilde{f}"] & \widetilde{Y} \end{tikzcd}$ commute, which will satisfy the naturality condition later.
Next we have to show $\widetilde{(-)}$ is a functor. We have already defined the functions $F_0 : |\mathcal{C}| \to |\mathcal{C}|$ and $F_{1,X,Y} : \mathcal{C}(X, Y) \to \mathcal{C}\big(\widetilde{X}, \widetilde{Y}\big)$ appropriately, so we only need to verify that $\widetilde{1_X} = 1_{\widetilde{X}}$ and $\widetilde{f; g} = \tilde{f}; \tilde{g}$.
First consider the following diagrams:

$$\begin{tikzcd}
X \arrow[r, "1_X"] \arrow[d, "\eta_X"] & X \arrow[d, "\eta_X"] \\
\widetilde{X} \arrow[r, "\widetilde{1_X}"] & \widetilde{X}
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "1_X"] \arrow[d, "\eta_X"] & X \arrow[d, "\eta_X"] \\
\widetilde{X} \arrow[r, "1_{\widetilde{X}}"] & \widetilde{X}
\end{tikzcd}$$

They are both pullbacks, so by uniqueness of $t_{\eta_X, 1_X}$ they are equal, so $\widetilde{1_X} = 1_{\widetilde{X}}$.
Next consider the following diagrams:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\eta_X"] & Y \arrow[r, "g"] \arrow[d, "\eta_Y"] & Z \arrow[d, "\eta_Z"] \\
\widetilde{X} \arrow[r, "\tilde{f}"] & \widetilde{Y} \arrow[r, "\tilde{g}"] & \widetilde{Z}
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "f;g"] \arrow[d, "\eta_X"] & Z \arrow[d, "\eta_Z"] \\
\widetilde{X} \arrow[r, "\widetilde{f;g}"] & \widetilde{Z}
\end{tikzcd}$$

On the left diagram the inner two squares are pullbacks, thus by the pullback lemma so is the outer square. But then, as the right square is also a pullback diagram, we get $\widetilde{f; g} = \tilde{f}; \tilde{g}$ by uniqueness via the partial map classifier.
Together then $\widetilde{(-)}$ is a functor and $\eta : 1_{\mathcal{C}} \Rightarrow \widetilde{(-)}$ is a natural transformation. $\square$

Problem 4

Suppose that Y has a partial map classifier. Show that, for any two representatives (m, f) and (m', f') of partial maps from X to Y, it holds that $(m, f) \equiv (m', f')$ if and only if $t_{m,f} = t_{m',f'}$.

Solution

Proof. Consider the diagram

$$\begin{tikzcd}
X_0 \arrow[r, "i"] \arrow[d, "m'"] \arrow[rr, bend left, "f"] & X_0' \arrow[r, "f'"] \arrow[d, "m"] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "\eta_Y"] \\
X \arrow[r, equal, "1_X"] & X \arrow[r, "t'"] & \widetilde{Y}
\end{tikzcd}$$

Then if $i : (m, f) \equiv (m', f')$ the above diagram commutes. We wish to show the left square is a pullback. Take $X \xleftarrow{p} Z \xrightarrow{q} X_0'$, such that $p = qm$. Then, since i is an iso qi^{-1} is a morphism from Z to X_0. Because Y has a partial map classifier that morphism is also unique (as it maps into a pullback), so the left square is also a pullback. Then, by the pullback lemma the outer square is also a pullback. From there, we conclude that $t = t'$, since t is the unique morphism, that makes that square a pullback.
In the other direction, from the assumption that $t = t'$ we see that the outer and right squares are pullbacks, and $i : X_0 \to X_0'$ exists because X_0' is a pullback and the diagram (without i) commutes. This i also makes the whole diagram commute. Then again by the pullback lemma we conclude that the left square is a pullback. Then i is an iso and $i : (m, f) \equiv (m', f')$. $\square$

Problem 5

Show that an object Y has a partial map classifier if and only if the functor $F \coloneqq \mathcal{C}_{\text{par}}(-,Y)\circ I^{\text{op}} : \mathcal{C}^{\text{op}} \to \mathbf{Set}$ is representable. (Here $I^{\text{op}} : \mathcal{C}^{\text{op}} \to \mathcal{C}^{\text{op}}_{\text{par}}$ is the opposite-category version of $I : \mathcal{C} \to \mathcal{C}_{\text{par}}$).

Solution
Let us first break down both definitions:

- $\exists\, \widetilde{Y} \in |\mathcal{C}|, \eta_Y : Y \to \widetilde{Y} \ni: \ \forall\, [m,f] \in \mathcal{C}_{\text{par}}(X,Y).\ \exists!\, t_{m,f} \in \mathcal{C}\big(X, \widetilde{Y}\big) \ni:$ the square is a pullback.
- $\exists\, Z \in |\mathcal{C}|, \alpha : \mathcal{C}_{\text{par}}(-,Y) \circ I^{\text{op}} \Rightarrow \mathcal{C}(-,Z) \ni:$ α is a natural isomorphism.
  - For all $X \in |\mathcal{C}|$ α_X is an iso
  - For all $f \in \mathcal{C}(X,X')$ we have

$$\begin{tikzcd} X \arrow[d, "f"] \\ X' \end{tikzcd} \qquad \begin{tikzcd} \mathcal{C}_{\text{par}}(X,Y) \arrow[r, "\alpha_X"] & \mathcal{C}(X,Z) \\ \mathcal{C}_{\text{par}}(X',Y) \arrow[r, "\alpha_{X'}"] \arrow[u, "Ff"] & \mathcal{C}(X',Z) \arrow[u, "f^*"'] \end{tikzcd}$$

where $Ff = [m,g] \mapsto [1_X, f]\,;[m,g] = \big[\overline{m}, \overline{f}; g\big]$.

The equivalence will obviously take $\widetilde{Y} = Z$.

$(\Rightarrow)$. Define $\alpha_X\,[m,f] \coloneqq t_{m,f}$, the unique map arising from the partial map classifier $\widetilde{Y}$. It is well-defined by problem 4 ($[m,f] = [m',f'] \Rightarrow t_{m,f} = t_{m',f'}$).
Take $[m,g] \in \mathcal{C}_{\text{par}}(X',Y)$ and $f \in \mathcal{C}(X,X')$. Then we have $(Ff;\alpha_X)\,[m,g] = t_{\overline{m},\overline{f};g}$ and $(\alpha_{X'};f^*)\,[m,g] = f;t_{m,g}$. As before by pullback lemma on the diagram

$$\begin{tikzcd} X_0 \arrow[r, "\overline{f}"] \arrow[d, "\overline{m}"'] \arrow[dr, phantom, very near start, "\lrcorner"] & X_0' \arrow[r, "g"] \arrow[d, "m"] \arrow[dr, phantom, very near start, "\lrcorner"] & Y \arrow[d, "\eta_Y"] \\ X \arrow[r, "f"] & X' \arrow[r, "t_{m,g}"] & \widetilde{Y} \end{tikzcd}$$

and uniqueness of $t_{\overline{m},\overline{f};g}$ we see that $f;t_{m,g} = t_{\overline{m},\overline{f};g}$. This makes the naturality square commute, so α is a natural transformation.
Define $\beta : \mathcal{C}\big(-,\widetilde{Y}\big) \Rightarrow F$ by $\beta_X h \coloneqq [p,q]$ where $X \xleftarrow{p} P \xrightarrow{q} Y$ is the pullback of the cospan $X \xrightarrow{h} \widetilde{Y} \xleftarrow{\eta_Y} Y$.
Then $(\alpha\beta)_X\, h = t_{p,q} = h$ by uniqueness of $t_{p,q}$ and $(\beta\alpha)_{X'}\,[m,g] = [m,g]$. Thus, α is a natural isomorphism. $\square$

$(\Leftarrow)$. Let now $\alpha : \mathcal{C}_{\text{par}}(-,Y) \circ I^{\text{op}} \Rightarrow \mathcal{C}(-,Z)$ be a natural isomorphism and $X \xleftarrow{m} X_0 \xrightarrow{f} Y$ be a partial map representative from X to Y.
Define $\eta_Y \coloneqq \alpha_Y[1_Y, 1_Y]$ and $t_{m,f} \coloneqq \alpha_X\,[m,f]$.
Keep the following diagram in mind (note that we do not assert the commutativity or existence of any of the maps indicated by convention):

$$\begin{tikzcd} P \arrow[dr, dashed, "\varphi"] \arrow[drr, bend left, "p"] \arrow[ddr, bend right, "q"'] & & \\ & X_0 \arrow[r, "f"] \arrow[d, "m"] & Y \arrow[d, "\eta_Y"] \\ & X \arrow[r, "t_{m,f}"'] & \widetilde{Y} \end{tikzcd}$$

First we have to prove the inner square commutes. Consider the following diagrams that arise from the naturality of α:

$$\begin{tikzcd} X_0 \arrow[d, "f"] & \mathcal{C}_{\mathrm{par}}(X_0, Y) \arrow[r, "\alpha_{X_0}"] & \mathcal{C}(X_0, Z) \\ Y & \mathcal{C}_{\mathrm{par}}(Y, Y) \arrow[u, "Ff"] \arrow[r, "\alpha_Y"] & \mathcal{C}(Y, Z) \arrow[u, "f^*"'] \end{tikzcd} \qquad \begin{tikzcd} X_0 \arrow[d, "m"] & \mathcal{C}_{\mathrm{par}}(X_0, Y) \arrow[r, "\alpha_{X_0}"] & \mathcal{C}(X_0, Z) \\ X & \mathcal{C}_{\mathrm{par}}(X, Y) \arrow[u, "Fm"] \arrow[r, "\alpha_X"] & \mathcal{C}(X, Z) \arrow[u, "m^*"'] \end{tikzcd}$$

We use commutativity of the diagrams on 1_Y and $[m, f]$ respectively to get $f; \eta_Y = \alpha_{X_0}\left[1_{X_0}, f\right]$ and $m; t_{m,f} = \alpha_{X_0}\left[1_{X_0}, f\right]$. It then follows that the inner square commutes.
Next we wish to show that it is a pullback. Consider $X \overset{p}{\leftarrow} P \overset{q}{\rightarrow} Y$ arbitrary, such that $p; \eta_Y = q; t_{m,f}$.
By similar two naturality diagrams for p and q as above we get $p; \eta_Y = \alpha_P(Fp(1_Y)) = \alpha_P[1_P, p]$ and $q; t_{m,f} = \alpha_P[\overline{m}, \overline{q}f]$, which are equal by assumption.

(Note: the maps $\overline{m}$ and $\overline{q}$ arise from the pullback diagram $\begin{tikzcd} \overline{P} \arrow[r, "\overline{q}"] \arrow[d, "\overline{m}"] \arrow[dr, phantom, "\lrcorner", very near start] & X_0 \arrow[d, "m"] \\ P \arrow[r, "q"] & X \end{tikzcd}$)

Since α_P is an iso it then follows that $[1_P, p] = [\overline{m}, \overline{q}f]$. Writing out the definition of equivalence we see that $\overline{m}$ is an iso and $\overline{m}p = \overline{q}f$. We can then define $\varphi \coloneqq \overline{m}^{-1}\overline{q}$.
The pullback square above gives us the equation $\overline{q}m = \overline{m}q$. Putting it all together we get $\varphi f = \overline{m}^{-1}\overline{q}f = p$ and $\varphi m = \overline{m}^{-1}\overline{q}m = \overline{m}^{-1}\overline{m}q = q$, making the square a pullback.

It remains to show uniqueness of $t_{m,f}$. Let t' be another map for which $\begin{tikzcd} X_0 \arrow[r, "f"] \arrow[d, "m"] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "\eta_Y"] \\ X \arrow[r, "t'"] & \widetilde{Y} \end{tikzcd}$ is a pullback. Then, because α_X is iso, we have $t' = \alpha_X[n, g]$ for some $[n, g] \in \mathcal{C}_{\mathrm{par}}\left(X, \widetilde{Y}\right)$. Then by the first part of this proof we know that $\begin{tikzcd} P \arrow[r, "g"] \arrow[d, "n"] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "\eta_Y"] \\ X \arrow[r, "t'"] & \widetilde{Y} \end{tikzcd}$ is also a pullback. Because pullbacks are unique up to isomorphism, we have an iso $X_0 \to P$ witnessing $(m, f) \equiv (n, g)$, so $t_{m,f} = \alpha_X[m, f] = \alpha_X[n, g] = t'$. □