Choice axioms and real numbers

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I plan to take you on the journey I took to discovering a cute minor result.

There are various ways to define the real numbers, in particular we're interested in the Dedekind and Cauchy reals.

Classically, these give the same objects, in particular LEM or CC are enough.

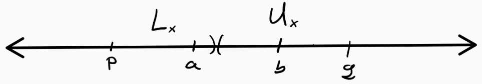
Countable choice implies the Dedekind and Cauchy reals are the same

Proof.

Take Dedekind cut (L_x, U_x). We construct a sequence of rational interval approximations (p_n, q_n) as follows: Take arbitrary $p_0 \in L_x$ and $q_0 \in U_x$. Then at step $n+1$ define

$$a = \frac{2p_n + q_n}{3} \qquad b = \frac{p_n + 2q_n}{3}$$

Now define $(p_{n+1}, q_{n+1}) := \begin{cases} (a, q_n) & ; a \in L_x \\ (p_n, b) & ; b \in U_x \end{cases}.$ □



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The proof is simple once you see it. We represent Dedekind reals by two-sided Dedekind cuts and the Cauchy reals by sequences of rational interval approximations.

Take arbitrary points in each of the cuts, then inductively at step n look at the thirds and choose an interval that contains the real x.

So, this uses countable choice, and in particular, CC for binary sets. Call it $\mathrm{AC}(\mathbb{N}, 2)$.

Now Andrej Bauer has been asking people for more than 10 years whether we can recover some sort of choice principle by assuming that Dedekind reals are Cauchy. I answer this in the negative.

The reals, really

For sheaf toposes over topological spaces $\mathbb{R}_d = \mathcal{C}(-, \mathbb{R})$.

Further, if the space is locally connected, $\mathbb{R}_c = \underline{\mathbb{R}}$.

So the objects $\mathbb{R}_d$ and $\mathbb{R}_c$ are the same when every continuous function from a subset of X to the reals is locally constant.

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Now in sheaf topoi over spaces we know what each of the constructions of reals look like, externally as sheaves.

The Dedekind reals are simply the functions from open subsets to the reals, while the Cauchy reals are the locally constant such functions (for locally connected spaces).

So, in the topos $\mathbb{R}_d = \mathbb{R}_c$ when all local functions to reals are locally constant.

$\mathrm{AC}(\mathbb{N},2)$ in $\mathrm{Sh}(X)$

Call a P-space a topological space in which any countable intersection of open sets is open.

Then if a space is T_1 and in the sheaf topos over it $\mathrm{AC}(\mathbb{N},2)$ holds it must be a P-space.

Thus, we want a space such that

(a) It is T_1
(b) It is not a P-space
(c) Continuous functions are locally constant
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Now lets look at $\mathrm{AC}(\mathbb{N},2)$. A P-space is a topological space where ctbl intersections of opens are open.
And I've shown a T_1 space satisfying $\mathrm{Sh}(X) \Vdash \mathrm{AC}(\mathbb{N},2)$ is a P-space.
So now, we know that if a space is not a P-space it is either weird (not T_1) or $\mathrm{AC}(\mathbb{N},2)$ does not hold. The second thing is what we want, so we look for a T_1 space that is not a P-space, but also satisfies the requirements for reals.

And luckily, there exists the pi-base, a database of most interesting topological with their properties. And we can search on it.

pi-base to the rescue!

π-Base Explore Spaces Properties Theorems Advanced Contribute

Filter by Text

e.g. plank

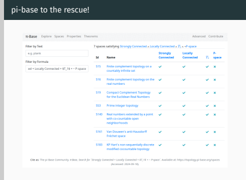
Filter by Formula

:ed + Locally Connected + T_1 + ~ P-space

7 spaces satisfying Strongly Connected ∧ Locally Connected ∧ T_1 ∧ ¬P-space

Id	Name	Strongly Connected	Locally Connected	T_1	P-space
S15	Finite complement topology on a countably infinite set	✓	✓	✓	✗
S16	Finite complement topology on the real numbers	✓	✓	✓	✗
S19	Compact Complement Topology for the Euclidean Real Numbers	✓	✓	✓	✗
S53	Prime integer topology	✓	✓	✓	✗
S140	Real numbers extended by a point with co-countable open neighborhoods	✓	✓	✓	✗
S161	Van Douwen's anti-Hausdorff Fréchet space	✓	✓	✓	✗
S183	KP Hart's non-sequentially discrete modified cocountable topology	✓	✓	✓	✗

Cite as: The pi-Base Community. *π-Base, Search for `Strongly Connected + Locally Connected + $T_1$ + ~ P-space`*. Available at: https://topology.pi-base.org/spaces (Accessed: 2024-09-18).



So we find the desired topos of sheaves over the cofinite topology on the naturals.

But, we still need to manually verify it actually works, since the topological properies in the pi-base do not exactly mirror the interesting properties arising from topos theory.

And in fact, there was a lot of fussing with translation before I could type something into the pi-base, even though I knew the exact properties I want. So wouldn't it be nice to have a similar site but whose objects are toposes? Actually, this inspired me to start working on this, so if anybody is interested please do ask me about it.