Topological Models

Topological Models

Today I will be giving an introduction to topological models in the sense of categories of Heyting valued sets. Some weeks/months ago prof. Simpson presented a tutorial on sheaf semantics, in much greater generality than what I will present today, and there you can just say “topological models are categories of sheaves over a topological space” and be done. However in this case the objects involved are quite complicated. Conceptually, sheaves are simple, but any particular sheaf is going to be unwieldy. Take for example the sheaf of natural numbers. It is the sheaf of locally constant maps from open subsets to the naturals. But in the world of Heyting valued sets, you can just say the natural numbers object is the set $\mathbb{N}$.

Topological Models

Topological Models

So my primary sources for this is Michael Fourman and Dana Scott's paper from the 70s, Sheaves and Logic, which introduced the concept and the 3rd volume of Francis Borceux's Handbook of Categorical Algebra, which has quite a few useful lemmas. A big difference to those though is that I develop as much of the theory as possible in the internal language, which I believe works quite well.

But to start small, we wish to construct a topos, whose internal logic has as its truth values the open sets of our topological space.

Step 1:
The logic of
open sets

Step 1:
The logic of
open sets

So first we can construct a propositional logic with infinitary conjunctions and disjuncitons, and that is basically just the structure of the topology as a complete Heyting algebra.

Logika	Topologija
$\top$	X
$\bot$	$\emptyset$
$\varphi \wedge \psi$	$[\![\varphi]\!] \cap [\![\psi]\!]$
$\varphi \vee \psi$	$[\![\varphi]\!] \cup [\![\psi]\!]$

2025-06-18

Topological models

Logika	Topologija
$\top$	X
$\bot$	$\emptyset$
$\varphi \wedge \psi$	$[\![\varphi]\!] \cap [\![\psi]\!]$
$\varphi \vee \psi$	$[\![\varphi]\!] \cup [\![\psi]\!]$

So true and false are the top and bottom elements respectivelly, so the whole space and the empty set, and conjunction and disjunction are intersection and union.

Implication is more novel, we don't see that in the intro to topology course. The correct definition of $U \Rightarrow V$ would be something like "the union of opens W such that if $W \subseteq U$ then $W \subseteq V$", but it turns out it has a much nicer characterisation as the interior of $V \cup U^c$. Now that's similar to saying that implication is defined as $V \vee \neg U$, but the key difference is when we take that interior.

Then for the infinitary ones, we just take infinitary intersections and unions, though for the conjunciton we need to take the interior.

Logika	Topologija
$\top$	$\mathcal{X}$
$\bot$	$\emptyset$
$\varphi \wedge \psi$	$[\![\varphi]\!] \cap [\![\psi]\!]$
$\varphi \vee \psi$	$[\![\varphi]\!] \cup [\![\psi]\!]$
$\varphi \Rightarrow \psi$	$\mathrm{Int}([\![\psi]\!] \cup [\![\varphi]\!]^c)$

2025-06-18

Topological models

Logika	Topologija
$\top$	$\mathcal{X}$
$\bot$	$\emptyset$
$\varphi \wedge \psi$	$[\![\varphi]\!] \cap [\![\psi]\!]$
$\varphi \vee \psi$	$[\![\varphi]\!] \cup [\![\psi]\!]$
$\varphi \Rightarrow \psi$	$\mathrm{Int}([\![\psi]\!] \cup [\![\varphi]\!]^c)$

So true and false are the top and bottom elements respectivelly, so the whole space and the empty set, and conjunction and disjunction are intersection and union.

Implication is more novel, we don't see that in the intro to topology course. The correct definition of $U \Rightarrow V$ would be something like "the union of opens W such that if $W \subseteq U$ then $W \subseteq V$", but it turns out it has a much nicer characterisation as the interior of $V \cup U^c$. Now that's similar to saying that implication is defined as $V \vee \neg U$, but the key difference is when we take that interior.

Then for the infinitary ones, we just take infinitary intersections and unions, though for the conjunciton we need to take the interior.

2025-06-18

Topological models

Logika	Topologija
$\top$	$\mathcal{X}$
$\bot$	$\emptyset$
$\varphi \wedge \psi$	$[\![\varphi]\!] \cap [\![\psi]\!]$
$\varphi \vee \psi$	$[\![\varphi]\!] \cup [\![\psi]\!]$
$\varphi \Rightarrow \psi$	$\mathrm{Int}([\![\psi]\!] \cup [\![\varphi]\!]^c)$
$\forall a:\underline{A} \,.\, \varphi(a)$	$\mathrm{Int}(\bigcap_{a \in A} [\![\varphi(a)]\!])$
$\exists a:\underline{A} \,.\, \varphi(a)$	$\bigcup_{a \in A} [\![\varphi(a)]\!]$

So true and false are the top and bottom elements respectivelly, so the whole space and the empty set, and conjunction and disjunction are intersection and union.

Implication is more novel, we don't see that in the intro to topology course. The correct definition of $U \Rightarrow V$ would be something like "the union of opens W such that if $W \subseteq U$ then $W \subseteq V$", but it turns out it has a much nicer characterisation as the interior of $V \cup U^c$. Now that's similar to saying that implication is defined as $V \vee \neg U$, but the key difference is when we take that interior.

Then for the infinitary ones, we just take infinitary intersections and unions, though for the conjunciton we need to take the interior.

Theorem: $\mathcal{X}$ validates LEM iff $\mathcal{X}$ is discrete.

2025-06-18

Topological models

Theorem: $\mathcal{X}$ validates LEM iff $\mathcal{X}$ is discrete.

Now, to demonstrate how expressive this logic is, we can actually prove the following.

Theorem: $\mathcal{X}$ validates LEM iff $\mathcal{X}$ is discrete.

Proof. LEM says "every $\underbrace{\text{truth value}}_{\text{open set}}$ is $\underbrace{\text{complemented}}_{\text{closed}}$."

Points are closed. By assumption they are open. $\square$

Theorem: $\mathcal{X}$ validates LEM iff $\mathcal{X}$ is discrete.
Proof. LEM says "every $\underbrace{\text{truth value}}_{\text{open set}}$ is $\underbrace{\text{complemented}}_{\text{closed}}$."
Points are closed. By assumption they are open. $\square$

Now, to demonstrate how expressive this logic is, we can actually prove the following.

Step 2:
Heyting valued
sets

Step 2:
Heyting valued
sets

As I mentioned earlier the idea goes back to Peter Fourman and Dana Scott's work in the 70s, but they didn't explicitly write what the motivation for this construction is, at least not one that immediately stands out to me. To be precise, they say this is "the obvious extension of Heyting algebras to predicate logic".

This is also why I also rely heavily on Borceux's textbook, which does explicitly provide a motivation. In algebra, it is often useful to give an algebraic structure via its presentation, aka a set of generators and some relations they need to satisfy. We wish to apply the same idea to sheaves, and it turns out it works quite well. We know sheaves are closed under restrictions and gluing, and those are the only forms of generation that sheaves support. Furthermore restrictions distribute over gluing, so this gives a somewhat natural characterisation of generation.

Let $\mathcal{F}$ be a sheaf.

Define $F := \sum_{U \in \mathcal{O}X} \mathcal{F}(U)$ and $\|a\| = \mathrm{pr}_1\, a$

2025-06-18

Topological models

Let $\mathcal{F}$ be a sheaf.
Define $F := \sum_{U \in \mathcal{O}X} \mathcal{F}(U)$ and $\|a\| = \mathrm{pr}_1\, a$

So in particular, how to define that a set generates a sheaf, we first say it's a subset of it's set of elements, and it generates $\mathcal{F}$ when $\mathcal{F}$ is the least subsheaf of $\mathcal{F}$ that contains S.

Let $\mathcal{F}$ be a sheaf.

Define $F := \sum_{U \in \mathcal{O}_X} \mathcal{F}(U)$ and $\|a\| = pr_1\, a$

Let $F' \subseteq F$. Then F' generates $\mathcal{F}$ when

(a) $\mathcal{F}$ is the least subsheaf containing F'

Let $\mathcal{F}$ be a sheaf.
Define $F := \sum_{U \in \mathcal{O}_X} \mathcal{F}(U)$ and $\|a\| = pr_1\, a$
Let $F' \subseteq F$. Then F' generates $\mathcal{F}$ when
(a) $\mathcal{F}$ is the least subsheaf containing F'

Equivalently, this is a set of generators as discussed above. Key to note here is, that restriction is a bona fide operation, so we don't need any equivalence relation. So if we wish to consider generating sets in absence of the sheaf we are generating, we also need to define it. And it turns out a set of elements and an appropriate notion of equivalence relation is exactly what we need to recover a whole sheaf.

Let $\mathcal{F}$ be a sheaf.

Define $F := \sum_{U \in \mathcal{O}X} \mathcal{F}(U)$ and $\|a\| = pr_1\, a$

Let $F' \subseteq F$. Then F' generates $\mathcal{F}$ when

(a) $\mathcal{F}$ is the least subsheaf containing F'

(b) $\forall f \in \mathcal{F}(U).\ \exists f_i \in F',\ U_i \overset{op}{\subseteq} U,\ f_i{\restriction}_{U_j} = f_j{\restriction}_{U_i}$

$$f = f_1{\restriction}_{U_1} \cup \dots \cup f_n{\restriction}_{U_n}$$

Let $\mathcal{F}$ be a sheaf.
Define $F := \sum_{U \in \mathcal{O}X} \mathcal{F}(U)$ and $\|a\| = pr_1\, a$
Let $F' \subseteq F$. Then F' generates $\mathcal{F}$ when
(a) $\mathcal{F}$ is the least subsheaf containing F'
(b) $\forall f \in \mathcal{F}(U).\ \exists f_i \in F',\ U_i \overset{op}{\subseteq} U,\ f_i{\restriction}_{U_j} = f_j{\restriction}_{U_i}$
$f = f_1{\restriction}_{U_1} \cup \dots \cup f_n{\restriction}_{U_n}$

Definition: An $\mathcal{L}$-set is a set A with $[\![- = -]\!] : A \times A \longrightarrow \mathcal{L}$ s.t.

1. $[\![a = b]\!] \leq [\![b = a]\!]$

2. $[\![a = b]\!] \wedge [\![b = c]\!] \leq [\![a = c]\!]$

Define $\|a\| := [\![a = a]\!]$.

2025-06-18

Topological models

Definition: An $\mathcal{L}$-set is a set A with $[\![- = -]\!] : A \times A \longrightarrow \mathcal{L}$ st

1. $[\![a = b]\!] \leq [\![b = a]\!]$

2. $[\![a = b]\!] \wedge [\![b = c]\!] \leq [\![a = c]\!]$

Define $\|a\| := [\![a = a]\!]$.

Let $\mathcal{L}$ be the topology on X. Here $\mathcal{L}$ stands for locale, but I don't say this in my thesis. So an $\mathcal{L}$-set is just a set, with a (partial) equivalence relation in the internal language, which we call the $\mathcal{L}$-equality. But because we don't have an internal language yet, we have to say it's a map from $A \times A$ to $\mathcal{L}$, which satisfies the symmetry-like condition and the transitivity-like condition. We ommit reflexivity in a sense, since our elements are not necessarily total, e.g. $\frac{1}{x}$ is not defined at 0. However we will have that reflexivity holds up to where the element is defined, and in fact, we define the extent of definition by the diagonal of the $\mathcal{L}$-equality.

I'm not quite set on the name, alternatively this is called a "Heyting valued set", but that's somewhat of a mouthful in English and a lot of a mouthful in Slovene, so for now I'm sticking with the short $\mathcal{L}$-set.

Definition: An $\mathcal{L}$-set is a set A with
$[\![- = -]\!] : A \times A \longrightarrow \mathcal{L}$ s.t.

1. $[\![a = b]\!] \leq [\![b = a]\!]$

2. $[\![a = b]\!] \wedge [\![b = c]\!] \leq [\![a = c]\!]$

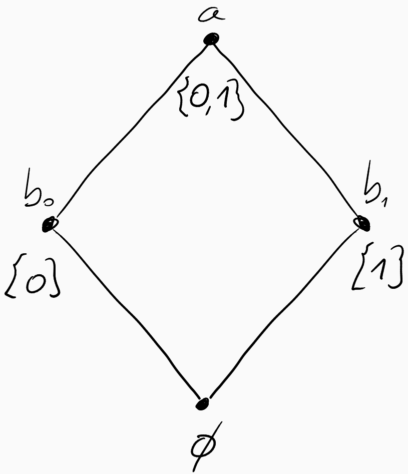
Define $\|a\| := [\![a = a]\!]$.

Definition: An $\mathcal{L}$-set is a set A with
$[\![- = -]\!] : A \times A \longrightarrow \mathcal{L}$ st
1. $[\![a = b]\!] \leq [\![b = a]\!]$
2. $[\![a = b]\!] \wedge [\![b = c]\!] \leq [\![a = c]\!]$
Define $\|a\| := [\![a = a]\!]$.

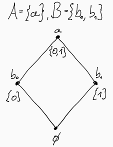
The next building block in our story would be morphisms, and we might be tempted to just define them as structure preserving maps, so maps $f : A \to B$ such that $a = a' \Rightarrow f(a) = f(a')$ and $\|f(a)\| = \|a\|$. Why exactly these maps? Well, because they will work. But why am I writing this on the whiteboard? Because they don't work yet.

Consider this example of a map we would be able to define internally.

$A = \{a\}, B = \{b_0, b_1\}$



Topological models



Let A be the $\mathcal{L}$-set with one global element a and B the $\mathcal{L}$-set with two elements b_0 and b_1, defined at 0 and 1 respectivelly. Then there should actually be only one map from A to B. This is because b_0 and b_1 should actually glue to a single element b at the top, and then both A and B are one-element sets, so we only have one map.

But we can define it sort of internally. At 0 the map maps a to b_0 and at 1 it maps a to b_1. So in essence, we can define a relation between A and B, with the meaning of "a is mapped to b by f". And it will satisfy the internal version of functionality.

Definition. An $\mathcal{L}$-morphism is a map
$$[\![- = f(-)]\!] : B \times A \longrightarrow \mathcal{L} \quad \text{s.t.}$$

1. $[\![b = f(a)]\!] \leq \|b\| \wedge \|a\|$

2. $[\![b' = b]\!] \wedge [\![b = f(a)]\!] \wedge [\![a = a']\!] \leq [\![b' = f(a')]\!]$

3. $[\![b = f(a)]\!] \wedge [\![b' = f(a)]\!] \leq [\![b = b']\!]$

4. $\|a\| \leq \bigvee_{b \in B} [\![b = f(a)]\!]$

Definition. An $\mathcal{L}$-morphism is a map $[\![- = f(-)]\!] : B \times A \longrightarrow \mathcal{L}$ s.t.
1. $[\![b = f(a)]\!] \leq \|b\| \wedge \|a\|$
2. $[\![b' = b]\!] \wedge [\![b = f(a)]\!] \wedge [\![a = a']\!] \leq [\![b' = f(a')]\!]$
3. $[\![b = f(a)]\!] \wedge [\![b' = f(a)]\!] \leq [\![b = b']\!]$
4. $\|a\| \leq \bigvee_{b \in B} [\![b = f(a)]\!]$

So an $\mathcal{L}$-morphism will be a functional relation in the internal language, and again that means we have a map from $B \times A$ to $\mathcal{L}$ that satisfies the appropriate axioms, so that's 3 for single-valuedness and 4 for totality. And the other two are just saying that this relation sufficiently agrees with the $\mathcal{L}$-equality. And in fact, this is exactly what it means for a map like that to be a relation. Those are called strictness and extensionality, and will be essentialy tautologies in the internal language. But lets actually define the internal language first, so I stop gesturing to the air.

Step 3:
Internal language

Step 3:
Internal language

And considering I already cheated a bit and gave myself infinitary conjunctions and disjunctions there's not much to add. The basic connectives are the same, and the quantifiers are essentially the same, except we account for the extents of elements. We can also interpret relations and equality as their $\mathcal{L}$-set counterparts.

Logika	Topologija
$\forall a : A \,.\, \varphi(a)$	$\mathrm{Int}\left(\bigcap_{a \in A} [\![\lVert a \rVert \Rightarrow \varphi(a)]\!]\right)$
$\exists a : A \,.\, \varphi(a)$	$\bigcup_{a \in A} [\![\lVert a \rVert \wedge \varphi(a)]\!]$
$a = b$	$[\![a = b]\!]$
$R(a_1, \ldots, a_n)$	$R(a_1, \ldots, a_n)$
$b = f(a)$	$[\![b = f(a)]\!]$

2025-06-18

Topological models

Logika	Topologija
$\forall a : A \,.\, \varphi(a)$	$\mathrm{Int}\left(\bigcap_{a \in A} [\![\lVert a \rVert \Rightarrow \varphi(a)]\!]\right)$
$\exists a : A \,.\, \varphi(a)$	$\bigcup_{a \in A} [\![\lVert a \rVert \wedge \varphi(a)]\!]$
$a = b$	$[\![a = b]\!]$
$R(a_1, \ldots, a_n)$	$R(a_1, \ldots, a_n)$
$b = f(a)$	$[\![b = f(a)]\!]$

So what changes is that for universal quantification we add that $\varphi(a)$ must at least where a is defined, and for existential quantification a exists at most where a exists.

We do this because sometimes we don't use all bound variables, e.g. in $\exists x : A.\ \top$, if A only has one element defined at the open point of the Sierpinski space. Similarly, for universal quantification this allows us to say for example that $\forall x : A.\ x = x$ holds.

There is also a term language, but it's not that interesting, elements of $\mathcal{L}$-sets are interpreted as themselves, and internal operations are just external maps (that are strict and extensional).

It's important to note though, that we can't actually form the term $f(a)$ for an $\mathcal{L}$-morphism. We need an actual (extensional) mapping for that, though in certain contexts we will be able to use $f(a)$ as a bona fide element of the codomain.

Definition: An $\mathcal{L}$-set is a set A with

$[\![- = -]\!] : A \times A \longrightarrow \mathcal{L}$ s.t.

1. $\vdash a = b \Rightarrow b = a$

2. $\vdash a = b \wedge b = c \Rightarrow a = c$

2025-06-18

Topological models

Definition: An $\mathcal{L}$-set is a set A with
$[\![- = -]\!] : A \times A \longrightarrow \mathcal{L}$ st
1. $\vdash a = b \Rightarrow b = a$
2. $\vdash a = b \wedge b = c \Rightarrow a = c$

So rephrasing the conditions in the definitions in terms of the internal language we get the following, which is saying exactly "a set with an internal equivalence relation".

Definition. An $\mathcal{L}$-morphism is a map

$[\![- = f(-)]\!] : B \times A \longrightarrow \mathcal{L}$ s.t.

1. $\vdash b = f(a) \Rightarrow \|b\| \wedge \|a\|$
2. $\vdash b' = b \wedge b = f(a) \wedge a = a' \Rightarrow b' = f(a')$
3. $\vdash b = f(a) \wedge b' = f(a) \Rightarrow b = b'$
4. $\vdash \forall a:A.\ \exists b:B.\ b = f(a)$

Definition. An $\mathcal{L}$-morphism is a map
$[\![- = f(-)]\!] : B \times A \longrightarrow \mathcal{L}$ s.t.
1. $\vdash b = f(a) \Rightarrow \|b\| \wedge \|a\|$
2. $\vdash b' = b \wedge b = f(a) \wedge a = a' \Rightarrow b' = f(a')$
3. $\vdash b = f(a) \wedge b' = f(a) \Rightarrow b = b'$
4. $\vdash \forall a:A.\ \exists b:B.\ b = f(a)$

And similarly here we get “functional relation” with 3 and 4, and the first two are essentially tautologies. For the first, to have formed “$b = f(a)$” we needed to have b and a exist. But then the consequent is just true, so the first condition doesn’t say anything. And in the second one, if we have equalities we can substitute with them, so it’s a tautology again.

Step 4:

Properties of $\mathcal{L}$-sets and morphisms

Step 4:
Properties of $\mathcal{L}$-sets and morphisms

Now we're ready to explore the landscape of topological models in this context, from the internal view.

Facts:

- $[\![- =_A -]\!]$ is id_A.
- $c = g \circ f(a)$ is $\exists b : B.\ c = g(b) \wedge b = f(a)$.

Facts:

- $[\![- =_A -]\!]$ is id_A.
- $c = g \circ f(a)$ is $\exists b : B.\ c = g(b) \wedge b = f(a)$.

Obviously we need $\mathcal{L}$-sets to form a category, so we need an identity morphism and composition. But since morphisms are internal relations, we can just take whatever the identity and composition are for relations. In our case, the identity is just the $\mathcal{L}$-equality, and the composition is just what it is.

And as promised, we can evaluate formulas at $f(a)$, since propositionally there exists a b that is equal to $f(a)$.

Facts:

- $[\![- =_A -]\!]$ is id_A.
- $c = g \circ f(a)$ is $\exists b : B.\ c = g(b) \wedge b = f(a)$.
- $R(f(a))$ is $\exists b : B.\ b = f(a) \wedge R(b)$.
  - ↪ $f(a) = g(a)$ is $\exists b : B.\ b = f(a) \wedge b = g(a)$

Facts:
- $[\![- =_A -]\!]$ is id_A.
- $c = g \circ f(a)$ is $\exists b : B.\ c = g(b) \wedge b = f(a)$.
- $R(f(a))$ is $\exists b : B.\ b = f(a) \wedge R(b)$.
  - ↪ $f(a) = g(a)$ is $\exists b : B.\ b = f(a) \wedge b = g(a)$

Obviously we need $\mathcal{L}$-sets to form a category, so we need an identity morphism and composition. But since morphisms are internal relations, we can just take whatever the identity and composition are for relations. In our case, the identity is just the $\mathcal{L}$-equality, and the composition is just what it is.

And as promised, we can evaluate formulas at $f(a)$, since propositionally there exists a b that is equal to $f(a)$.

2025-06-18

Topological models

I could have also done this earlier, but we can construct the usual suspects.

Ex. $\mathbb{1} := \{*\}$, $\|*\| := \top$, $\mathbb{0} := \emptyset$

Ex. $\underline{A} := A$, $[\![a = a']\!] := \bigvee\{\top \mid a = a'\}$.

Ex. $\Omega := \mathcal{L}$, $[\![p = q]\!] := p \Leftrightarrow q$.

Ex. $\mathbb{1} := \{*\}$, $\|*\| := \top$, $\mathbb{0} := \emptyset$
Ex. $\underline{A} := A$, $[\![a = a']\!] := \bigvee\{\top \mid a = a'\}$
Ex. $\Omega := \mathcal{L}$, $[\![p = q]\!] := p \Leftrightarrow q$

2025-06-18

Topological models

Ex. $\mathbb{1} := \{*\}$, $\|*\| := \top$, $\mathbb{0} := \emptyset$

Ex. $\underline{A} := A$, $[\![a = a']\!] := \bigvee\{\top \mid a = a'\}$.

Ex. $\Omega := \mathcal{L}$, $[\![p = q]\!] := p \Leftrightarrow q$.

Ex. $A \times B := \{(a,b) \in A \times B \mid \|a\| = \|b\|\}$

Ex. $B^A := A \looparrowright B$, $[\![f = g]\!] := [\![\forall a{:}A.\, f(a) = g(a)]\!]$

Ex. $A/{\sim} := A$, $[\![a = a']\!] := a \sim a'$.

I could have also done this earlier, but we can construct the usual suspects.

Lemma (funext). $f = g$ iff $\vdash f =_{B^A} g$.

Lemma (funext). $f = g$ iff $\vdash f =_{B^A} g$.

Lemma (funext). $f = g$ iff $\vdash f =_{B^A} g$.

Proof. Assume $\vdash f = g$.

Let $a : A$, $b : B$ s.t. $b = f(a)$.

By ass., $\exists b' : B.\ b' = f(a) \wedge b' = g(a)$

Then $b = f(a) = b'$, so $b = b' = g(a)$. $\square$

Lemma (funext). $f = g$ iff $\vdash f =_{B^A} g$.
Proof. Assume $\vdash f = g$.
Let $a : A$, $b : B$ s.t. $b = f(a)$.
By ass., $\exists b' : B.\ b' = f(a) \wedge b' = g(a)$
Then $b = f(a) = b'$, so $b = b' = g(a)$. $\square$

Theorem. $f : A \rightarrowtail B$ is mono iff
$$\vdash \forall x, y : A.\ f(x) = f(y) \Rightarrow x = y.$$

2025-06-18

Topological models

Theorem. $f : A \rightarrowtail B$ is mono iff
$\vdash \forall x, y : A.\ f(x) = f(y) \Rightarrow x = y.$

Theorem. $f : A \rightarrowtail B$ is mono iff
$+\forall x,y : A.\ f(x) = f(y) \Rightarrow x = y.$

Proof. ($\Leftarrow$) Let $g,h : C \rightarrowtail A$, $f \circ g = f \circ h$.
Then $f(g(c)) = f(h(c))$, so $g = h$.

Theorem. $f : A \rightarrowtail B$ is mono iff
$+\forall x,y : A.\ f(x) = f(y) \Rightarrow x = y.$
Proof. ($\Leftarrow$) Let $g,h : C \rightarrowtail A$, $f \circ g = f \circ h$.
Then $f(g(c)) = f(h(c))$, so $g = h$.

Theorem. $f : A \hookrightarrow B$ is mono iff

$\forall x,y : A.\ f(x) = f(y) \Rightarrow x = y$.

Proof. $(\Leftarrow)$ Let $g,h : C \hookrightarrow A$, $f \circ g = f \circ h$.

Then $f(g(c)) = f(h(c))$, so $g = h$.

$(\Rightarrow)$ Let $x,y : B$ s.t. $f(x) = f(y)$.

Define $\hat{x}, \hat{y} : \mathbb{1} \hookrightarrow B$, $\hat{x}(*) := x$, $\hat{y}(*) := y$

Then $f \circ \hat{x} = f \circ \hat{y}$, so $\hat{x} = \hat{y}$. $\square$

Theorem. $f : A \hookrightarrow B$ is mono iff
$\forall x,y : A.\ f(x) = f(y) \Rightarrow x = y$.
Proof. $(\Leftarrow)$ Let $g,h : C \hookrightarrow A$, $f \circ g = f \circ h$.
Then $f(g(c)) = f(h(c))$, so $g = h$.
$(\Rightarrow)$ Let $x,y : B$ s.t. $f(x) = f(y)$.
Define $\hat{x}, \hat{y} : \mathbb{1} \hookrightarrow B$, $\hat{x}(*) := x$, $\hat{y}(*) := y$
Then $f \circ \hat{x} = f \circ \hat{y}$, so $\hat{x} = \hat{y}$. $\square$

Theorem $f : A \hookrightarrow B$ is epi iff

$\vdash \forall b : B \; \exists a : A . \; b = f(a).$

Theorem $f : A \hookrightarrow B$ is epi iff

$\vdash \forall b : B \; \exists a : A . \; b = f(a).$

Theorem $f : A \looparrowright B$ is epi iff

$\forall b : B\ \exists a : A.\ b = f(a)$.

Proof. $(\Leftarrow)$ let $g, h : B \looparrowright C$, $g \circ f = h \circ f$.

Let $b : B$ Then $\exists a : A$ $b = f(a)$

Then $g(b) = g(f(a)) = h(f(a)) = h(b)$.

Theorem $f : A \looparrowright B$ is epi iff
$\forall b : B\ \exists a : A.\ b = f(a)$.
Proof. $(\Leftarrow)$ let $g, h : B \looparrowright C$, $g \circ f = h \circ f$.
Let $b : B$ Then $\exists a : A$ $b = f(a)$
Then $g(b) = g(f(a)) = h(f(a)) = h(b)$

Theorem $f : A \looparrowright B$ is epi iff

$\vdash \forall b : B\ \exists a : A.\ b = f(a).$

Proof. $(\Leftarrow)$ Let $g, h : B \looparrowright C$, $g \circ f = h \circ f$.

Let $b : B$ Then $\exists a : A\ b = f(a)$

Then $g(b) = g(f(a)) = h(f(a)) = h(b)$.

$(\Rightarrow)$ Let $b : B$ and $i_1, i_2 : B \looparrowright \mathrm{coker} f$.

Then $i_1 \circ f = i_2 \circ f \rightsquigarrow i_1 = i_2 \rightsquigarrow \exists a : A.\ b = f(a).$ $\square$

Theorem $f : A \looparrowright B$ is epi iff $\vdash \forall b : B\ \exists a : A.\ b = f(a).$ Proof. $(\Leftarrow)$ Let $g, h : B \looparrowright C$, $g \circ f = h \circ f$. Let $b : B$ Then $\exists a : A\ b = f(a)$ Then $g(b) = g(f(a)) = h(f(a)) = h(b)$. $(\Rightarrow)$ Let $b : B$ and $i_1, i_2 : B \looparrowright \mathrm{coker} f$. Then $i_1 \circ f = i_2 \circ f \rightsquigarrow i_1 = i_2 \rightsquigarrow \exists a : A.\ b = f(a).$ $\square$

$$
\begin{aligned}
[\![c = g(b)]\!] &= [\![c = g(b)]\!] \wedge [\![b = b]\!] \\
&= \bigvee_{a \in A} [\![c = g(b)]\!] \wedge [\![b = f(a)]\!] \\
&\leq \bigvee_{a \in A, b' \in B} [\![c = g(b')]\!] \wedge [\![b' = f(a)]\!] \wedge [\![b = f(a)]\!] \\
&= \bigvee_{a \in A} [\![c = g \circ f(a)]\!] \wedge [\![b = f(a)]\!] \\
&= \bigvee_{a \in A} [\![c = h \circ f(a)]\!] \wedge [\![b = f(a)]\!] \\
&= \bigvee_{a \in A, b' \in B} [\![c = h(b')]\!] \wedge [\![b' = f(a)]\!] \wedge [\![b = f(a)]\!] \\
&\leq \bigvee_{b' \in B} [\![c = h(b')]\!] \wedge [\![b' = b]\!] \\
&= [\![c = h \circ 1_B(b)]\!] \\
&= [\![c = h(b)]\!]
\end{aligned}
$$

2025-06-18

Topological models

$$
\begin{aligned}
[\![c = g(b)]\!] &= [\![c = g(b)]\!] \wedge [\![b = b]\!] \\
&= \bigvee_{a \in A} [\![c = g(b)]\!] \wedge [\![b = f(a)]\!] \\
&\leq \bigvee_{a \in A, b' \in B} [\![c = g(b')]\!] \wedge [\![b' = f(a)]\!] \wedge [\![b = f(a)]\!] \\
&= \bigvee_{a \in A} [\![c = g \circ f(a)]\!] \wedge [\![b = f(a)]\!] \\
&= \bigvee_{a \in A} [\![c = h \circ f(a)]\!] \wedge [\![b = f(a)]\!] \\
&= \bigvee_{a \in A, b' \in B} [\![c = h(b')]\!] \wedge [\![b' = f(a)]\!] \wedge [\![b = f(a)]\!] \\
&\leq \bigvee_{b' \in B} [\![c = h(b')]\!] \wedge [\![b' = b]\!] \\
&= [\![c = h \circ 1_B(b)]\!] \\
&= [\![c = h(b)]\!]
\end{aligned}
$$

Karakterizacija podobjektov

1. monomorphisms $S \hookrightarrow A$
2. sections of $\mathcal{P}(A) = \Omega^A$
3. predicates on A

Karakterizacija podobjektov

1. monomorphisms $S \hookrightarrow A$
2. sections of $\mathcal{P}(A) = \Omega^A$
3. predicates on A

$f : A \hookrightarrow \Omega :$

1. $\vdash U = f(a) \Rightarrow \|U\| \wedge \|a\|$
2. $\vdash U' = U \wedge U = f(a) \wedge a = a' \Rightarrow U' = f(a')$
3. $\vdash U = f(a) \wedge U' = f(a) \Rightarrow U = U'$
4. $\vdash \forall a : A .\ \exists U : \Omega .\ U = f(a)$

2025-06-18

Topological models

$f : A \hookrightarrow \Omega :$

1. $\vdash U = f(a) \Rightarrow \|U\| \wedge \|a\|$
2. $\vdash U' = U \wedge U = f(a) \wedge a = a' \Rightarrow U' = f(a')$
3. $\vdash U = f(a) \wedge U' = f(a) \Rightarrow U = U'$
4. $\vdash \forall a : A .\ \exists U : \Omega .\ U = f(a)$

If $U = f(a)$ then $f(a)$ is U, so when you compare it with $\top$ you should get U back.

$f : A \hookrightarrow \Omega$:

1. $\vdash U = f(a) \Rightarrow \|U\| \wedge \|a\|$
2. $\vdash U' = U \wedge U = f(a) \wedge a = a' \Rightarrow U' = f(a')$
3. $\vdash U = f(a) \wedge U' = f(a) \Rightarrow U = U'$
4. $\vdash \forall a : A.\ \exists U : \Omega.\ U = f(a)$

Define $\hat{f}(a) := [\![\top = f(a)]\!]$

1. $\hat{f}(a) \Rightarrow \|a\|$
2. $\hat{f}(a) \wedge a = a' \Rightarrow \hat{f}(a')$

Topological models

2025-06-18

$f : A \hookrightarrow \Omega$:
1. $\vdash U = f(a) \Rightarrow \|U\| \wedge \|a\|$
2. $\vdash U' = U \wedge U = f(a) \wedge a = a' \Rightarrow U' = f(a')$
3. $\vdash U = f(a) \wedge U' = f(a) \Rightarrow U = U'$
4. $\vdash \forall a : A.\ \exists U : \Omega.\ U = f(a)$
Define $\hat{f}(a) := [\![\top = f(a)]\!]$
1. $\hat{f}(a) \Rightarrow \|a\|$
2. $\hat{f}(a) \wedge a = a' \Rightarrow \hat{f}(a')$

If $U = f(a)$ then $f(a)$ is U, so when you compare it with $\top$ you should get U back.

Image and preimage constructions

Let $f : A \hookrightarrow B$, $S \subseteq A$, $T \subseteq B$.

Define $f_*(S) := \{y : B \mid \exists a \in S.\ y = f(a)\}$

$f^*(T) := \{x : A \mid \exists b \in T.\ b = f(x)\}$

Image and preimage constructions
Let $f : A \hookrightarrow B$, $S \subseteq A$, $T \subseteq B$.
Define $f_*(S) := \{y : B \mid \exists a \in S.\ y = f(a)\}$
$f^*(T) := \{x : A \mid \exists b \in T.\ b = f(x)\}$

Image and preimage constructions

Let $f : A \mathrel{\circ\!\!\to} B$, $S \subseteq A$, $T \subseteq B$.

Define $f_*(S) := \{y : B \mid \exists a \in S.\ y = f(a)\}$

$f^*(T) := \{x : A \mid \exists b \in T.\ b = f(x)\}$

If $S = \{a\}$ and $T = \{b\}$

- $f(a)(y) = [\![y = f(a)]\!]$
- $f^{-1}(b)(x) = [\![b = f(x)]\!]$

Image and preimage constructions
Let $f : A \mathrel{\circ\!\!\to} B$, $S \subseteq A$, $T \subseteq B$.
Define $f_*(S) := \{y : B \mid \exists a \in S.\ y = f(a)\}$
$f^*(T) := \{x : A \mid \exists b \in T.\ b = f(x)\}$
If $S = \{a\}$ and $T = \{b\}$
- $f(a)(y) = [\![y = f(a)]\!]$
- $f^{-1}(b)(x) = [\![b = f(x)]\!]$

But then the third propery of $\mathcal{L}$-morphisms says $y \in f(a) \wedge y' \in f(a) \Rightarrow y = y'$.

(Sub)singletons

Consider $f : \mathbb{1} \hookrightarrow A$.

1. $\vdash a \in f(*) \Rightarrow \|a\| \wedge \|*\|$
2. $\vdash a' = a \wedge a \in f(*) \Rightarrow a' \in f(*)$
3. $\vdash a \in f(*) \wedge a' \in f(*) \Rightarrow a = a'$
4. $\vdash \|*\| \Rightarrow \exists a : A.\ a \in f(*)$

(Sub)singletons

Consider $f : \mathbb{1} \hookrightarrow A$.

1. $\vdash a \in f(*) \Rightarrow \|a\| \wedge \|*\|$
2. $\vdash a' = a \wedge a \in f(*) \Rightarrow a' \in f(*)$
3. $\vdash a \in f(*) \wedge a' \in f(*) \Rightarrow a = a'$
4. $\vdash \|*\| \Rightarrow \exists a : A.\ a \in f(*)$

(Sub)singletons

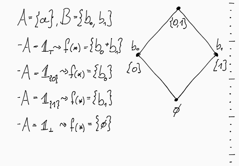
Consider $f : \mathbb{1} \looparrowright A$.

1. $\vdash a \in f(*) \Rightarrow \|a\| \wedge \|*\|$
2. $\vdash a' = a \wedge a \in f(*) \Rightarrow a' \in f(*)$
3. $\vdash a \in f(*) \wedge a' \in f(*) \Rightarrow a = a'$
4. $\vdash \|*\| \Rightarrow \exists a : A.\ a \in f(*)$

Definition: $S \subseteq A$ is a (sub)singleton when

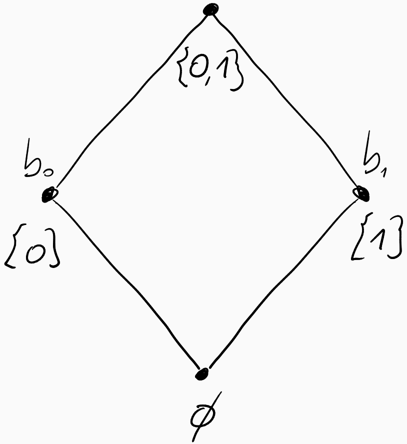
$$\vdash x \in S \wedge y \in S \Rightarrow x = y.$$

(Sub)singletons
Consider $f : \mathbb{1} \looparrowright A$.
1. $\vdash a \in f(*) \Rightarrow \|a\| \wedge \|*\|$
2. $\vdash a' = a \wedge a \in f(*) \Rightarrow a' \in f(*)$
3. $\vdash a \in f(*) \wedge a' \in f(*) \Rightarrow a = a'$
4. $\vdash \|*\| \Rightarrow \exists a : A.\ a \in f(*)$
Definition: $S \subseteq A$ is a (sub)singleton when $\vdash x \in S \wedge y \in S \Rightarrow x = y$.



$A = \{\alpha\}, \; B = \{b_0, b_1\}$

- $A = \mathbb{1}_{\top} \rightsquigarrow f(*) = \{b_0 + b_1\}$
- $A = \mathbb{1}_{\{0\}} \rightsquigarrow f(*) = \{b_0\}$
- $A = \mathbb{1}_{\{1\}} \rightsquigarrow f(*) = \{b_1\}$
- $A = \mathbb{1}_{\bot} \rightsquigarrow f(*) = \{\emptyset\}$



Definition. $\sigma(A) := \{\sigma \leq A \mid \sigma \text{ singleton}\}$ is an $\mathcal{L}$-set with $\sigma = \tau \iff \exists a : A.\ a \in \sigma \wedge a \in \tau$.

Definition. $\sigma(A) := \{\sigma \leq A \mid \sigma \text{ singleton}\}$ is an $\mathcal{L}$-set with $\sigma = \tau \iff \exists a : A.\ a \in \sigma \wedge a \in \tau$.

Definition. $\sigma(A) :\equiv \{\sigma \le A \mid \sigma \text{ singleton}\}$ is an $\mathcal{L}$-set
with $\sigma = \tau \Leftrightarrow \exists a : A.\ a \in \sigma \wedge a \in \tau$.

Lemma.
1. $\vdash \sigma = \{a\} \Leftrightarrow a \in \sigma$,
2. $\vdash \{a\} = \{b\} \Leftrightarrow a = b$,
3. $\vdash a = b \Leftrightarrow \exists \sigma : \sigma(A).\ a \in \sigma \wedge b \in \sigma$.

Definition. $\sigma(A) :\equiv \{\sigma \le A \mid \sigma \text{ singleton}\}$ is an $\mathcal{L}$-set
with $\sigma = \tau \Leftrightarrow \exists a : A.\ a \in \sigma \wedge a \in \tau$.
Lemma. 1. $\vdash \sigma = \{a\} \Leftrightarrow a \in \sigma$,
2. $\vdash \{a\} = \{b\} \Leftrightarrow a = b$,
3. $\vdash a = b \Leftrightarrow \exists \sigma : \sigma(A).\ a \in \sigma \wedge b \in \sigma$.

Definition. $\sigma(A) := \{\sigma \leq A \mid \sigma \text{ singleton}\}$ is an $\mathcal{L}$-set
with $\sigma = \tau \iff \exists a : A.\ a \in \sigma \wedge a \in \tau$.

Lemma. 1. $\vdash \sigma = \{a\} \iff a \in \sigma$,
2. $\vdash \{a\} = \{b\} \iff a = b$,
3. $\vdash a = b \iff \exists \sigma : \sigma(A).\ a \in \sigma \wedge b \in \sigma$.

Definition. A is complete when the map
$\{-\} : A \longrightarrow \sigma(A)$ is an isomorphism.

Definition. $\sigma(A) := \{\sigma \leq A \mid \sigma \text{ singleton}\}$ is an $\mathcal{L}$-set with $\sigma = \tau \iff \exists a : A.\ a \in \sigma \wedge a \in \tau$.
Lemma. 1. $\vdash \sigma = \{a\} \iff a \in \sigma$, 2. $\vdash \{a\} = \{b\} \iff a = b$, 3. $\vdash a = b \iff \exists \sigma : \sigma(A).\ a \in \sigma \wedge b \in \sigma$.
Definition. A is complete when the map $\{-\} : A \longrightarrow \sigma(A)$ is an isomorphism.

Restrictions: Let $a \in A$, $\mathcal{U} \subseteq \|a\|$.

Restrictions: Let $a \in A$, $\mathcal{U} \subseteq \|a\|$.

Restrictions: Let $a \in A$, $\mathcal{U} \subseteq \|a\|$.

Define $x \in \alpha \iff x \in \{a\} \wedge \mathcal{U}$

By completeness $\alpha = \{a'\}$.

$$a = a' \iff \{a\} = \{a'\} \iff \exists x : A.\ x \in \{a\} \wedge x \in \{a'\}$$

$$\iff \exists x : A.\ x \in \{a\} \wedge \mathcal{U} \iff \|a\| \wedge \mathcal{U} \iff \mathcal{U}.$$

Restrictions: Let $a \in A$, $\mathcal{U} \subseteq \|a\|$.
Define $x \in \alpha \iff x \in \{a\} \wedge \mathcal{U}$
By completeness $\alpha = \{a'\}$.
$a = a' \iff \{a\} = \{a'\} \iff \exists x : A.\ x \in \{a\} \wedge x \in \{a'\}$
$\iff \exists x : A.\ x \in \{a\} \wedge \mathcal{U} \iff \|a\| \wedge \mathcal{U} \iff \mathcal{U}.$

Gluing: Let $\{a_i\}_{i\in I}$ be a compatible family

2025-06-18

Topological models

Gluing: Let $\{a_i\}_{i\in I}$ be a compatible family

Gluing: Let $\{a_i\}_{i \in I}$ be a compatible family

Define $x \in \sigma \iff \exists i : I.\ x \in \{a_i\}$

If $x \in \{a_i\}$ and $y \in \{a_j\}$ then $x = a_i = a_j = y$.

So $\exists! a : A.\ \sigma = \{a\}$.

$$\begin{aligned} x \in a|_{U_i} &\iff x \in a \wedge U_i \\ &\iff \exists j : I.\ x \in \{a_j\} \wedge U_i \\ &\iff x \in \{a_j\} \wedge U_i \\ &\iff x \in \{a_i\} \end{aligned}$$

Gluing: Let $\{a_i\}_{i \in I}$ be a compatible family
Define $x \in \sigma \iff \exists i : I.\ x \in \{a_i\}$
If $x \in \{a_i\}$ and $y \in \{a_j\}$ then $x = a_i = a_j = y$
So $\exists! a : A.\ \sigma = \{a\}$.
$x \in a|_{U_i} \iff x \in a \wedge U_i \iff \exists j : I.\ x \in \{a_j\} \wedge U_i \iff x \in \{a_j\} \wedge U_i \iff x \in \{a_i\}$

Theorem. If B is complete then every $f : A \looparrowright B$ comes from an $\hat{f} : A \to B$.

Proof. For $a : A$ $f(a)$ is a singleton so $\hat{f}(a) := \{f(a)\}^{-1}$. $\square$

Theorem. If B is complete then every $f : A \looparrowright B$ comes from an $\hat{f} : A \to B$.
Proof. For $a : A$ $f(a)$ is a singleton so $\hat{f}(a) := \{f(a)\}^{-1}$. $\square$

Ex. $\mathbb{1} := \{*\}$, $\|*\| := \top$, $\mathbb{0} := \emptyset$

Ex. $\underline{A} := A$, $[\![a = a']\!] := \bigvee\{\top \mid a = a'\}$.

Ex. $\Omega := \mathcal{L}$, $[\![p = q]\!] := p \Leftrightarrow q$.

Ex. $A \times B := \{(a,b) \in A \times B \mid \|a\| = \|b\|\}$

Ex. $B^A := A \looparrowright B$, $[\![f = g]\!] := [\![\forall a : A.\ f(a) = g(a)]\!]$

Ex. $A/_{\sim} := A$, $[\![a = a']\!] := a \sim a'$.

2025-06-18

Topological models

Ex. $\mathbb{1} := \{*\}$, $\|*\| := \top$, $\mathbb{0} := \emptyset$

Ex. $\underline{A} := A$, $[\![a = a']\!] := \bigvee\{\top \mid a = a'\}$

Ex. $\Omega := \mathcal{L}$, $[\![p = q]\!] := p \Leftrightarrow q$

Ex. $A \times B := \{(a,b) \in A \times B \mid \|a\| = \|b\|\}$

Ex. $B^A := A \looparrowright B$, $[\![f = g]\!] := [\![\forall a : A.\ f(a) = g(a)]\!]$

Ex. $A/_{\sim} := A$, $[\![a = a']\!] := a \sim a'$.

Theorem. $\sigma(A)$ is complete.

Proof. Let $\Sigma : \sigma(A)$.

Define $a \in \tau \iff \{a\} \in \Sigma$.

Take $a : A$ s.t. $a \in \tau$.

Then $\tau = \{a\}$ and $\{a\} \in \Sigma$ so $\tau \in \Sigma$, or $\Sigma = \{\tau\}$. $\square$

Theorem. $\sigma(A)$ is complete.
Proof. Let $\Sigma : \sigma(A)$.
Define $a \in \tau \iff \{a\} \in \Sigma$.
Take $a : A$ s.t. $a \in \tau$.
Then $\tau = \{a\}$ and $\{a\} \in \Sigma$ so $\tau \in \Sigma$, or $\Sigma = \{\tau\}$. $\square$

Topological models

Theorem. $A \cong \sigma(A)$.

Proof. Define $\eta(\sigma, a) := a \in \sigma$.

3. $a \in \sigma \wedge a \in \tau \Rightarrow \exists a : A.\ a \in \sigma \wedge a \in \tau$,
4. $a \in \{a\} \Rightarrow \exists \sigma : \sigma(A).\ a \in \sigma$,

inj. $a \in \sigma \wedge b \in \sigma \Rightarrow a = b$,

surj. $\forall \sigma : \sigma(A).\ \sigma = \sigma$, so $\exists a : A.\ a \in \sigma$. $\square$

Theorem. $A \cong \sigma(A)$.
Proof. Define $\eta(\sigma, a) := a \in \sigma$.
3. $a \in \sigma \wedge a \in \tau \Rightarrow \exists a : A.\ a \in \sigma \wedge a \in \tau$,
4. $a \in \{a\} \Rightarrow \exists \sigma : \sigma(A).\ a \in \sigma$,
inj. $a \in \sigma \wedge b \in \sigma \Rightarrow a = b$,
surj. $\forall \sigma : \sigma(A).\ \sigma = \sigma$, so $\exists a : A.\ a \in \sigma$. $\square$

Natural numbers

$\sigma(\mathbb{N})$:

$\sigma : \mathbb{N} \longrightarrow \mathcal{L}$

1. $\vdash n \in \sigma \wedge m \in \sigma \Rightarrow n = m$,
2. $\vdash n = m \wedge m \in \sigma \Rightarrow n \in \sigma$.

For $x \in [\![\exists n : \mathbb{N}.\, n \in \sigma]\!]$, $x \in \bigcup_{n \in \mathbb{N}} \sigma(n)$

Natural numbers
$\sigma(\mathbb{N})$:
$\sigma : \mathbb{N} \longrightarrow \mathcal{L}$
1. $\vdash n \in \sigma \wedge m \in \sigma \Rightarrow n = m$,
2. $\vdash n = m \wedge m \in \sigma \Rightarrow n \in \sigma$.
For $x \in [\![\exists n : \mathbb{N}.\, n \in \sigma]\!]$, $x \in \bigcup_{n \in \mathbb{N}} \sigma(n)$

Thm. $f = \hat{f}$ $\qquad \hat{f}(U,a) = [\![U =_A [\![\top = f(a)]\!]]\!]$

Pf. Let $a : A$. $\quad f(a) = \hat{f}(a)$ is $f(a) = [\![\top = f(a)]\!]$

or $f(a) = (\top \Leftrightarrow f(a))$

Thm. $\hat{f} = \hat{g} \Rightarrow f = g$.

Pf. Let $a : A$. Show $f(a) = g(a)$

By ass. $[\![\top = f(a)]\!] = \hat{f}(a) = \hat{g}(a) = [\![\top = g(a)]\!]$.

Then $f(a) = g(a)$.

Objekti

Objekti

Objekte v topoloških modelih se da konstruirat na veliko načinov, lahko so snopi, étale prostori, ali pa Heytingovo vrednotene množice. Jaz v delu uporabljam slednjo od teh, je pa bolj praktično rečt snopi. So pa te konstrukcije v vsakem primeru precej komplicirane, tako da se mi zdi da nima smisla, da katerokoli točno razpišem, tako da mi boste morali malo verjeti na besedo.

Sicer je pa naša zgodba itak, da se stvari spreminjajo vzdolž topološkega prostora, tako da bi tudi želeli da se elementi spreminjajo vzdolž prostora. Tako da kar rečemo, da na vsaki točki prostora definiramo vrednost elementa, to je pa ubistvu kar funkcija iz prostora nekam (še ne vemo točno kam). Edino kar moramo paziti je, da je ta funkcija dovolj lepa (beri zvezna). In to dejansko večinoma dela, je pa kar dosti dela to dejansko preveriti, tko da ja, mi morte verjet :)

Objekti

A množica, T topološki prostor

Objekti

A množica, T topološki prostor

Če malo fiksiramo oznake, naj bo ...

Najprej vložimo prostor T, ker je še najbolj očitno kako se to naredi. Lahko bi vzeli kar zvezne funkcije iz X v T. To bi delalo, ampak spomnimo se, da so naše resničnostne vrednosti odprte podmnožice X. In obstoj elementa ima resničnostno vrednost, tako da je smiselno, da dovoljujemo tako imenovane delne elemente, torej elemente, ki niso definirani na celem X. To pa pomeni, da je množica...

Realna števila so potem kar realne funkcije, in recimo če je $X = \mathbb{R}$ je identiteta neko realno število (reče se mu generični element).

Za splošne množice pa vzamemo kar isto stvar. Ampak zdaj je vprašanje, kakšne funkcije vzamemo. Izkaže se da kar zvezne, kjer A opremimo z diskretno topologijo.

Objekti

A množica, T topološki prostor

$$T_X := \{f : U \to T \mid U \in \mathcal{O}(X)\}$$

Objekti

A množica, T topološki prostor

$T_X := \{f : U \to T \mid U \in \mathcal{O}(X)\}$

Če malo fiksiramo oznake, naj bo ...

Najprej vložimo prostor T, ker je še najbolj očitno kako se to naredi. Lahko bi vzeli kar zvezne funkcije iz X v T. To bi delalo, ampak spomnimo se, da so naše resničnostne vrednosti odprte podmnožice X. In obstoj elementa ima resničnostno vrednost, tako da je smiselno, da dovoljujemo tako imenovane delne elemente, torej elemente, ki niso definirani na celem X. To pa pomeni, da je množica...

Realna števila so potem kar realne funkcije, in recimo če je $X = \mathbb{R}$ je identiteta neko realno število (reče se mu generični element).

Za splošne množice pa vzamemo kar isto stvar. Ampak zdaj je vprašanje, kakšne funkcije vzamemo. Izkaže se da kar zvezne, kjer A opremimo z diskretno topologijo.

Objekti

A množica, T topološki prostor

$$T_X := \{f : U \to T \mid U \in \mathcal{O}(X)\}$$

$$\mathbb{R}_X := \{f : U \to \mathbb{R} \mid U \in \mathcal{O}(X)\} = \bigcup_{U \in \mathcal{O}(X)} \mathcal{C}(U)$$

Nad realnimi števili je torej $\mathrm{id} : \mathbb{R} \to \mathbb{R}$ realno število.

Objekti
A množica, T topološki prostor
$T_X := \{f : U \to T \mid U \in \mathcal{O}(X)\}$
$\mathbb{R}_X := \{f : U \to \mathbb{R} \mid U \in \mathcal{O}(X)\} = \bigcup_{U \in \mathcal{O}(X)} \mathcal{C}(U)$
Nad realnimi števili je torej $\mathrm{id} : \mathbb{R} \to \mathbb{R}$ realno število.

Če malo fiksiramo oznake, naj bo …
Najprej vložimo prostor T, ker je še najbolj očitno kako se to naredi. Lahko bi vzeli kar zvezne funkcije iz X v T. To bi delalo, ampak spomnimo se, da so naše resničnostne vrednosti odprte podmnožice X. In obstoj elementa ima resničnostno vrednost, tako da je smiselno, da dovoljujemo tako imenovane delne elemente, torej elemente, ki niso definirani na celem X. To pa pomeni, da je množica…
Realna števila so potem kar realne funkcije, in recimo če je $X = \mathbb{R}$ je identiteta neko realno število (reče se mu generični element).

Za splošne množice pa vzamemo kar isto stvar. Ampak zdaj je vprašanje, kakšne funkcije vzamemo. Izkaže se da kar zvezne, kjer A opremimo z diskretno topologijo.

Objekti

A množica, T topološki prostor

$$T_X := \{f : U \to T \mid U \in \mathcal{O}(X)\}$$

$$\mathbb{R}_X := \{f : U \to \mathbb{R} \mid U \in \mathcal{O}(X)\} = \bigcup_{U \in \mathcal{O}(X)} \mathcal{C}(U)$$

Nad realnimi števili je torej $\mathrm{id} : \mathbb{R} \to \mathbb{R}$ realno število.

$$\underline{A} := \{f : U \to A \mid U \in \mathcal{O}(X)\}$$

$$\underline{\mathbb{N}} := \{f : U \to \mathbb{N} \mid U \in \mathcal{O}(X)\}$$

Objekti

A množica, T topološki prostor

$$T_X := \{f : U \to T \mid U \in \mathcal{O}(X)\}$$

$$\mathbb{R}_X := \{f : U \to \mathbb{R} \mid U \in \mathcal{O}(X)\} = \bigcup_{U \in \mathcal{O}(X)} \mathcal{C}(U)$$

Nad realnimi števili je torej $\mathrm{id} : \mathbb{R} \to \mathbb{R}$ realno število.

$$\underline{A} := \{f : U \to A \mid U \in \mathcal{O}(X)\}$$

$$\underline{\mathbb{N}} := \{f : U \to \mathbb{N} \mid U \in \mathcal{O}(X)\}$$

Če malo fiksiramo oznake, naj bo …
Najprej vložimo prostor T, ker je še najbolj očitno kako se to naredi. Lahko bi vzeli kar zvezne funkcije iz X v T. To bi delalo, ampak spomnimo se, da so naše resničnostne vrednosti odprte podmnožice X. In obstoj elementa ima resničnostno vrednost, tako da je smiselno, da dovoljujemo tako imenovane delne elemente, torej elemente, ki niso definirani na celem X. To pa pomeni, da je množica…
Realna števila so potem kar realne funkcije, in recimo če je $X = \mathbb{R}$ je identiteta neko realno število (reče se mu generični element).

Za splošne množice pa vzamemo kar isto stvar. Ampak zdaj je vprašanje, kakšne funkcije vzamemo. Izkaže se da kar zvezne, kjer A opremimo z diskretno topologijo.