Topological Models

Step 1:
The logic of open sets

Logika	Topologija
$\top$	$\mathcal{X}$
$\bot$	$\emptyset$
$\varphi \wedge \psi$	$\llbracket \varphi \rrbracket \cap \llbracket \psi \rrbracket$
$\varphi \vee \psi$	$\llbracket \varphi \rrbracket \cup \llbracket \psi \rrbracket$
$\varphi \Rightarrow \psi$	$\mathrm{Int}\left(\llbracket \psi \rrbracket \cup \llbracket \varphi \rrbracket^c \right)$
$\forall a : \underline{A} \,.\, \varphi(a)$	$\mathrm{Int}\left(\bigcap_{a \in A} \llbracket \varphi(a) \rrbracket \right)$
$\exists a : \underline{A} \,.\, \varphi(a)$	$\bigcup_{a \in A} \llbracket \varphi(a) \rrbracket$

Theorem: $\mathcal{X}$ validates LEM iff $\mathcal{X}$ is discrete.

Proof. LEM says

"every $\underbrace{\text{truth value}}_{\text{open set}}$ is $\underbrace{\text{complemented}}_{\text{closed}}$."

Points are closed. By assumption they are open. $\square$

Step 2:

Heyting valued sets

Let $\mathcal{F}$ be a sheaf.

Define $F := \sum_{U \in \mathcal{O}_X} \mathcal{F}(U)$ and $\|a\| = pr_1\, a$

Let $F' \subseteq F$. Then F' generates $\mathcal{F}$ when

(a) $\mathcal{F}$ is the least subsheaf containing F'

(b) $\forall f \in \mathcal{F}(U).\ \exists f_i \in F',\ U_i \overset{op}{\subseteq} U,\ f_i \restriction U_j = f_j \restriction U_i$

$$f = f_1 \restriction U_1 \cup \dots \cup f_n \restriction U_n$$

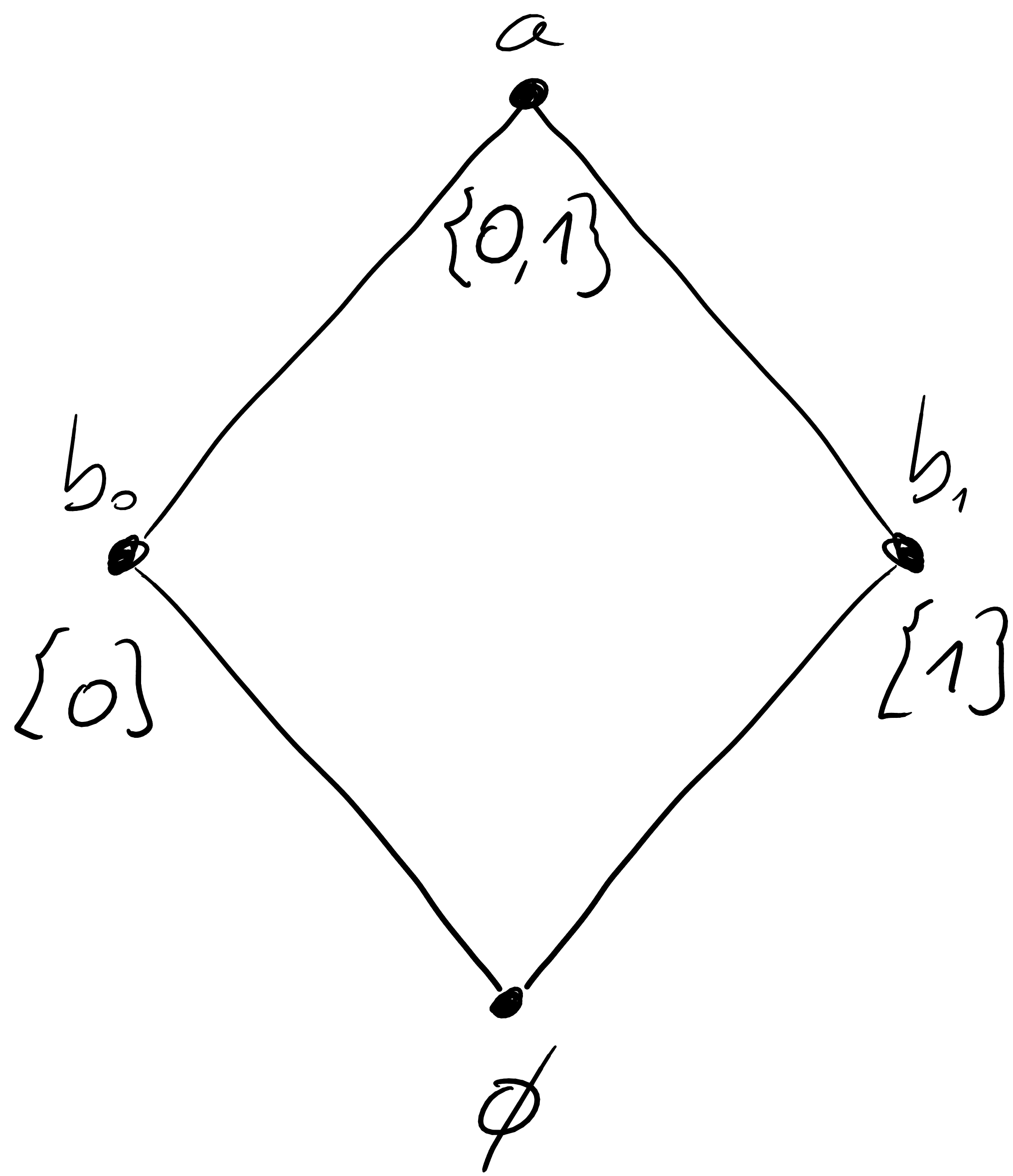
Definition: An $\mathcal{L}$-set is a set A with $[\![- = -]\!] : A \times A \longrightarrow \mathcal{L}$ s.t.

1. $[\![a = b]\!] \leq [\![b = a]\!]$

2. $[\![a = b]\!] \wedge [\![b = c]\!] \leq [\![a = c]\!]$

Define $\|a\| := [\![a = a]\!]$.

$$A = \{a\}, \; B = \{b_0, b_1\}$$



Definition. An $\mathcal{L}$-morphism is a map

$$[\![- = f(-)]\!] : B \times A \longrightarrow \mathcal{L} \quad \text{s.t.}$$

1. $[\![b = f(a)]\!] \leq \|b\| \wedge \|a\|$

2. $[\![b' = b]\!] \wedge [\![b = f(a)]\!] \wedge [\![a = a']\!] \leq [\![b' = f(a')]\!]$

3. $[\![b = f(a)]\!] \wedge [\![b' = f(a)]\!] \leq [\![b = b']\!]$

4. $\|a\| \leq \bigvee_{b \in B} [\![b = f(a)]\!]$

Step 3: Internal language

Logika	Topologija
$\top$	$\mathcal{X}$
$\bot$	$\emptyset$
$\varphi \wedge \psi$	$[\![\varphi]\!] \cap [\![\psi]\!]$
$\varphi \vee \psi$	$[\![\varphi]\!] \cup [\![\psi]\!]$
$\varphi \Rightarrow \psi$	$\mathrm{Int}([\![\psi]\!] \cup [\![\varphi]\!]^c)$
$\forall a : A \,.\, \varphi(a)$	$\mathrm{Int}\left(\bigcap_{a \in A} [\![\, \lVert a \rVert \Rightarrow \varphi(a)]\!]\right)$
$\exists a : A \,.\, \varphi(a)$	$\bigcup_{a \in A} [\![\, \lVert a \rVert \wedge \varphi(a)]\!]$
$a = b$	$[\![a = b]\!]$

Logika	Topologija
$\forall a{:}A\ .\ \varphi(a)$	$\mathrm{Int}\left(\bigcap_{a\in A} [\![\lVert a \rVert \Rightarrow \varphi(a)]\!]\right)$
$\exists a{:}A\ .\ \varphi(a)$	$\bigcup_{a\in A} [\![\lVert a \rVert \wedge \varphi(a)]\!]$
$a = b$	$[\![a = b]\!]$
$R(a_1, \dots, a_n)$	$R(a_1, \dots, a_n)$
$b = f(a)$	$[\![b = f(a)]\!]$

Definition: An $\mathcal{L}$-set is a set A with $[\![- = -]\!] : A \times A \longrightarrow \mathcal{L}$ s.t.

1. $\vdash a = b \Rightarrow b = a$

2. $\vdash a = b \wedge b = c \Rightarrow a = c$

Definition. An $\mathcal{L}$-morphism is a map

$$[\![- = f(-)]\!] : B \times A \longrightarrow \mathcal{L} \quad \text{s.t.}$$

1. $\vdash b = f(a) \Rightarrow \|b\| \wedge \|a\|$

2. $\vdash b' = b \wedge b = f(a) \wedge a = a' \Rightarrow b' = f(a')$

3. $\vdash b = f(a) \wedge b' = f(a) \Rightarrow b = b'$

4. $\vdash \forall a : A.\ \exists b : B.\ b = f(a)$

Step 4:

Properties of $\mathcal{L}$-sets and morphisms

Facts:

- $[\![- =_A -]\!]$ is id_A.

- $c = g \circ f(a)$ is $\exists b : B.\ c = g(b) \wedge b = f(a)$.

- $R(f(a))$ is $\exists b : B.\ b = f(a) \wedge R(b)$.

  $\hookrightarrow$ $f(a) = g(a)$ is $\exists b : B.\ b = f(a) \wedge b = g(a)$

Ex. $\mathbb{1} := \{*\}$, $\|*\| := \top$, $\mathbb{O} := \emptyset$

Ex. $\underline{A} := A$, $[\![a = a']\!] := \bigvee\{\top \mid a = a'\}$.

Ex. $\Omega := \mathcal{L}$, $[\![p = q]\!] := p \Leftrightarrow q$.

Ex. $A \times B := \{(a,b) \in A \times B \mid \|a\| = \|b\|\}$

Ex. $B^A := A \multimap B$, $[\![f = g]\!] := [\![\forall a{:}A.\ f(a) = g(a)]\!]$

Ex. $A/_\sim := A$, $[\![a = a']\!] := a \sim a'$.

Lemma (funext). $f = g$ iff $\vdash f =_{B^A} g$.

Proof. Assume $\vdash f = g$.

Let $a : A$, $b : B$ s.t. $b = f(a)$.

By ass., $\exists b' : B.\ b' = f(a) \wedge b' = g(a)$

Then $b = f(a) = b'$, so $b = b' = g(a)$. $\square$

Theorem. $f : A \to B$ is mono iff

$$\forall x, y : A.\ f(x) = f(y) \Rightarrow x = y.$$

Proof. ($\Leftarrow$) Let $g, h : C \to A$, $f \circ g = f \circ h$.

Then $f(g(c)) = f(h(c))$, so $g = h$.

($\Rightarrow$) Let $x, y : B$ s.t. $f(x) = f(y)$.

Define $\hat{x}, \hat{y} : \mathbb{1} \to B$, $\hat{x}(*) := x$, $\hat{y}(*) := y$

Then $f \circ \hat{x} = f \circ \hat{y}$, so $\hat{x} = \hat{y}$. $\square$

Theorem $f : A \to B$ is epi iff

$$\forall b : B\ \exists a : A.\ b = f(a).$$

Proof. ($\Leftarrow$) let $g, h : B \to C$, $g \circ f = h \circ f$.

Let $b : B$ Then $\exists a : A$ $b = f(a)$

Then $g(b) = g(f(a)) = h(f(a)) = h(b)$.

($\Rightarrow$) Let $b : B$ and $i_1, i_2 : B \to \operatorname{coker} f$.

Then $i_1 \circ f = i_2 \circ f \rightsquigarrow i_1 = i_2 \rightsquigarrow \exists a : A.\ b = f(a)$. $\square$

Karakterizacija podobjektov

1. monomorphisms $S \rightarrowtail A$
2. sections of $\mathcal{P}(A) = \Omega^A$
3. predicates on A

$f : A \rightarrowtail \Omega$:

1. $\vdash U = f(a) \Rightarrow \|U\| \wedge \|a\|$
2. $\vdash U' = U \wedge U = f(a) \wedge a = a' \Rightarrow U' = f(a')$
3. $\vdash U = f(a) \wedge U' = f(a) \Rightarrow U = U'$
4. $\vdash \forall a : A.\ \exists U : \Omega.\ U = f(a)$

Define $\hat{f}(a) := [\![\top = f(a)]\!]$

1. $\hat{f}(a) \Rightarrow \|a\|$
2. $\hat{f}(a) \wedge a = a' \Rightarrow \hat{f}(a')$

Image and preimage constructions

Let $f : A \rightarrowtail B$, $S \subseteq A$, $T \subseteq B$.

Define $f_*(S) := \{y : B \mid \exists a \in S.\ y = f(a)\}$

$$f^*(T) := \{x : A \mid \exists b \in T.\ b = f(x)\}$$

If $S = \{a\}$ and $T = \{b\}$

- $f(a)(y) = [\![y = f(a)]\!]$
- $f^{-1}(b)(x) = [\![b = f(x)]\!]$

(Sub)singletons

Consider $f : \mathbb{1} \hookrightarrow A$.

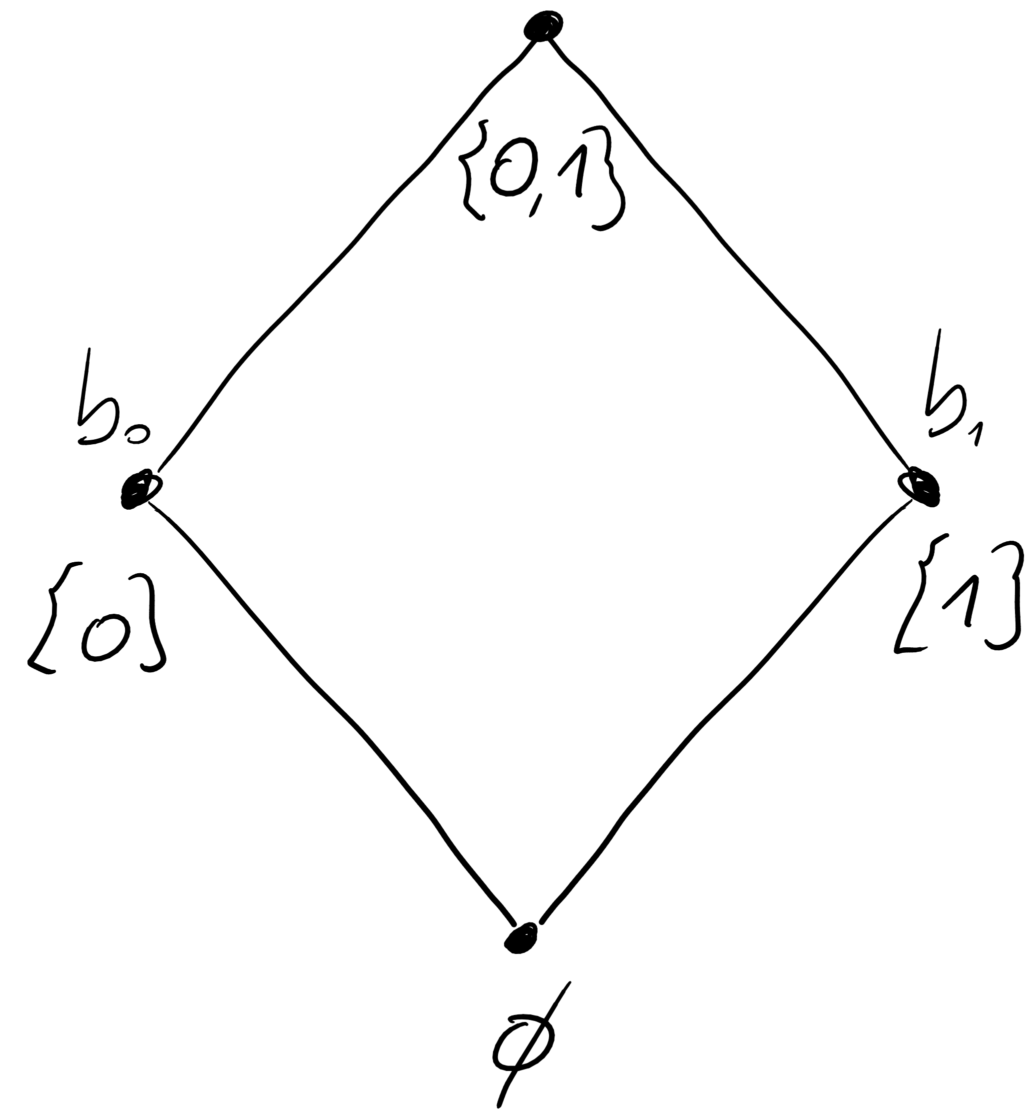
1. $\vdash a \in f(*) \Rightarrow \|a\| \wedge \|*\|$
2. $\vdash a' = a \wedge a \in f(*) \Rightarrow a' \in f(*)$
3. $\vdash a \in f(*) \wedge a' \in f(*) \Rightarrow a = a'$
4. $\vdash \|*\| \Rightarrow \exists a : A .\ a \in f(*)$

Definition: $S \subseteq A$ is a (sub)singleton when

$$\vdash x \in S \wedge y \in S \Rightarrow x = y.$$

$A = \{a\}, \; B = \{b_0, b_1\}$

- $A = \mathbb{1}_\top \rightsquigarrow f(*) = \{b_0 + b_1\}$
- $A = \mathbb{1}_{\{0\}} \rightsquigarrow f(*) = \{b_0\}$
- $A = \mathbb{1}_{\{1\}} \rightsquigarrow f(*) = \{b_1\}$
- $A = \mathbb{1}_\perp \rightsquigarrow f(*) = \{\emptyset\}$



Definition. $\sigma(A) := \{\sigma \subseteq A \mid \sigma \text{ singleton}\}$ is an $\mathcal{L}$-set

with $\sigma = \tau \iff \exists a : A.\ a \in \sigma \wedge a \in \tau$.

Lemma. 1. $\vdash \sigma = \{a\} \iff a \in \sigma$,

2. $\vdash \{a\} = \{b\} \iff a = b$,

3. $\vdash a = b \iff \exists \sigma : \sigma(A).\ a \in \sigma \wedge b \in \sigma$.

Definition. A is complete when the map

$\{-\} : A \longrightarrow \sigma(A)$ is an isomorphism.

Restrictions: Let $a \in A$, $U \subseteq \|a\|$.

Define $x \in \sigma \iff x \in \{a\} \wedge U$

By completeness $\sigma = \{a'\}$.

$a = a' \iff \{a\} = \{a'\} \iff \exists x : A.\ x \in \{a\} \wedge x \in \{a'\}$

$\iff \exists x : A.\ x \in \{a\} \wedge U \iff \|a\| \wedge U \iff U.$

Gluing: Let $\{a_i\}_{i\in I}$ be a compatible family

Define $x \in \sigma \iff \exists i : I.\ x \in \{a_i\}$

If $x \in \{a_i\}$ and $y \in \{a_j\}$ then $x = a_i = a_j = y$.

So $\exists! a : A.\ \sigma = \{a\}$.

$$\begin{aligned}
x \in a|_{U_i} &\iff x \in a \wedge U_i \\
&\iff \exists j : I.\ x \in \{a_j\} \wedge U_i \\
&\iff x \in \{a_j\} \wedge U_i \\
&\iff x \in \{a_i\}
\end{aligned}$$

Theorem. If B is complete then every $f : A \rightarrowtail B$ comes from an $\hat{f} : A \to B$.

Proof. For $a : A$ $f(a)$ is a singleton so $\hat{f}(a) :\equiv \{f(a)\}^{-1}$. $\square$

Ex. $\mathbb{1} := \{*\}$, $\|*\| := \top$, $\mathbb{0} := \emptyset$

Ex. $\underline{A} := A$, $[\![a = a']\!] := \bigvee\{\top \mid a = a'\}$.

Ex. $\Omega := \mathcal{L}$, $[\![p = q]\!] := p \Leftrightarrow q$.

Ex. $A \times B := \{(a,b) \in A \times B \mid \|a\| = \|b\|\}$

Ex. $B^A := A \looparrowright B$, $[\![f = g]\!] := [\![\forall a : A.\ f(a) = g(a)]\!]$

Ex. $A/_{\sim} := A$, $[\![a = a']\!] := a \sim a'$.

Theorem. $\sigma(A)$ is complete.

Proof. Let $\Sigma : \sigma(A)$.

Define $a \in \tau \iff \{a\} \in \Sigma$.

Take $a : A$ s.t. $a \in \tau$.

Then $\tau = \{a\}$ and $\{a\} \in \Sigma$ so $\tau \in \Sigma$, or $\Sigma = \{\tau\}$. $\square$

Theorem. $A \cong \sigma(A)$.

Proof. Define $\eta(\sigma, a) := a \in \sigma$.

3. $a \in \sigma \wedge a \in \tau \Rightarrow \exists a : A.\ a \in \sigma \wedge a \in \tau$,

4. $a \in \{a\} \Rightarrow \exists \sigma : \sigma(A).\ a \in \sigma$,

inj. $a \in \sigma \wedge b \in \sigma \Rightarrow a = b$,

surj. $\forall \sigma : \sigma(A).\ \sigma = \sigma$, so $\exists a : A.\ a \in \sigma$. $\square$

Natural numbers

$\sigma(\underline{\mathbb{N}})$:

$\sigma : \mathbb{N} \longrightarrow \mathcal{L}$

1. $\vdash n \in \sigma \wedge m \in \sigma \Rightarrow n = m$,
2. $\vdash n = m \wedge m \in \sigma \Rightarrow n \in \sigma$.

For $x \in [\![\exists n : \underline{\mathbb{N}} . n \in \sigma]\!]$, $x \in \bigcup_{n \in \mathbb{N}} \sigma(n)$

Thm. $f = \hat{f}$ $\qquad \hat{f}(u,a) = [\![u =_A [\![\top = f(a)]\!]]\!]$

Pf. Let $a : A$. $\quad f(a) = \hat{f}(a)$ is $f(a) = [\![\top = f(a)]\!]$

or $f(a) = (\top \Leftrightarrow f(a))$

Thm. $\hat{f} = \hat{g} \Rightarrow f = g$.

Pf. Let $a : A$. Show $f(a) = g(a)$

By ass. $[\![\top = f(a)]\!] = \hat{f}(a) = \hat{g}(a) = [\![\top = g(a)]\!]$.

Then $f(a) = g(a)$.