Topological Models

Step 1:

The logic of open sets

Logika	Topologija
$\top$	$\mathcal{X}$
$\bot$	$\emptyset$
$\varphi \wedge \psi$	$[\![\varphi]\!] \cap [\![\psi]\!]$
$\varphi \vee \psi$	$[\![\varphi]\!] \cup [\![\psi]\!]$
$\varphi \Rightarrow \psi$	$\mathrm{Int}\left([\![\psi]\!] \cup [\![\varphi]\!]^c\right)$
$\forall a : \underline{A} \,.\, \varphi(a)$	$\mathrm{Int}\left(\bigcap_{a \in A} [\![\varphi(a)]\!]\right)$
$\exists a : \underline{A} \,.\, \varphi(a)$	$\bigcup_{a \in A} [\![\varphi(a)]\!]$

Theorem: $\mathcal{X}$ validates LEM iff $\mathcal{X}$ is discrete.

Proof.

If $\mathcal{X}$ is discrete then $\mathcal{O}\mathcal{X}$ is Boolean.

If $\mathcal{X} \models$ LEM then $\forall U \overset{op}{\subseteq} \mathcal{X}.\ U \vee \neg U$

$$\neg U = U \Rightarrow \bot = \mathrm{Int}(\emptyset \cup U^c) = \mathrm{Int}(U^c).$$

Then if $\mathcal{X} = U \cup \mathrm{Int}(U^c)$ we must have

$$U^c = \mathrm{Int}(U^c),$$

so U is closed.

$\square$

Step 2: Heyting valued sets

Let $\mathcal{F}$ be a sheaf.

Define $F := \sum_{U \overset{op}{\subseteq} X} \mathcal{F}(U)$ and $\|a\| = pr_1 a$.

Let $F' \subseteq F$. Then F' generates $\mathcal{F}$ when

(a) $\mathcal{F}$ is the least subsheaf containing F

(b) $\forall f \in \mathcal{F}(U).\ \exists f_i \in F',\ U_i \overset{op}{\subseteq} U,\ f_i \restriction U_j = f_j \restriction U_i$

$$f = f_1 \restriction U_1 \cup \dots \cup f_n \restriction U_n$$

Definition: An $\mathcal{L}$-set is a set A with $[\![- = -]\!] : A \times A \longrightarrow \mathcal{L}$ s.t.

1. $[\![a = b]\!] \leq [\![b = a]\!]$

2. $[\![a = b]\!] \wedge [\![b = c]\!] \leq [\![a = c]\!]$

Definition. An $\mathcal{L}$-morphism is a map

$$[\![- = f(-)]\!] : B \times A \longrightarrow \mathcal{L} \quad \text{s.t.}$$

1. $[\![b = f(a)]\!] \leq \|b\| \wedge \|a\|$

2. $[\![b' = b]\!] \wedge [\![b = f(a)]\!] \wedge [\![a = a']\!] \leq [\![b' = f(a')]\!]$

3. $[\![b = f(a)]\!] \wedge [\![b' = f(a)]\!] \leq [\![b = b']\!]$

4. $\|a\| \leq \bigvee_{b \in B} [\![b = f(a)]\!]$

Step 3:

Internal language

Logika	Topologija
$\top$	$\mathcal{X}$
$\bot$	$\emptyset$
$\varphi \land \psi$	$[\![\varphi]\!] \cap [\![\psi]\!]$
$\varphi \lor \psi$	$[\![\varphi]\!] \cup [\![\psi]\!]$
$\varphi \Rightarrow \psi$	$\mathrm{Int}([\![\psi]\!] \cup [\![\varphi]\!]^c)$
$\forall a{:}A \,.\, \varphi(a)$	$\mathrm{Int}\left(\bigcap_{a \in A} [\![\lVert a \rVert \Rightarrow \varphi(a)]\!]\right)$
$\exists a{:}A \,.\, \varphi(a)$	$\bigcup_{a \in A} [\![\lVert a \rVert \land \varphi(a)]\!]$
$a = b$	$[\![a = b]\!]$

Definition: An $\mathcal{L}$-set is a set A with

$[\![- = -]\!] : A \times A \longrightarrow \mathcal{L}$ s.t.

1. $\vdash a = b \Rightarrow b = a$

2. $\vdash a = b \wedge b = c \Rightarrow a = c$

Definition. An $\mathcal{L}$-morphism is a map

$$[\![- = f(-)]\!] : B \times A \longrightarrow \mathcal{L} \quad \text{s.t.}$$

1. $\vdash b = f(a) \Rightarrow \|b\| \wedge \|a\|$

2. $\vdash b' = b \wedge b = f(a) \wedge a = a' \Rightarrow b' = f(a')$

3. $\vdash b = f(a) \wedge b' = f(a) \Rightarrow b = b'$

4. $\vdash \forall a : A.\ \exists b : B.\ b = f(a)$

Facts:

- $[\![- =_A -]\!]$ is id_A.

- $c = g \circ f(a)$ is $\exists b : B.\ c = g(b) \wedge b = f(a)$.

- $R(f(a))$ is $\exists b : B.\ b = f(a) \wedge R(b)$.

  ↪ $f(a) = g(a)$ is $\exists b : B.\ b = f(a) \wedge b = g(a)$

Ex. $\mathbb{1} := \{*\}$, $\|*\| = \top$.

Ex. $A/_U := A$, $[\![a = a']\!] = [\![a = a']\!] \wedge U$.

Ex. $\Omega = \mathcal{L}$, $[\![p = q]\!] = p \Leftrightarrow q$.

Ex. $\underline{A} = A$; $[\![a = a']\!] = \bigvee\{\top \mid a = a'\}$.

Ex. $B^A = A \nrightarrow B$, $[\![f = g]\!] = [\![\forall a{:}A.\ f(a) = g(a)]\!]$.

Ex. $A/_\sim = A$, $[\![a = a']\!] = a \sim a'$.

Lemma (funext). $f = g$ iff $\vdash f =_{B^A} g$.

Theorem. Monomorphisms are precisely injections.

$$\vdash f(x) = f(y) \Rightarrow x = y.$$

Proof. Let $f : B \rightarrowtail C$.

($\Leftarrow$) f inj. $g, h : A \rightarrowtail B$, $f \circ g = f \circ h$, $a : A$.

Then let $x :\equiv g(a)$, $y = h(a)$.

Then $f(x) = f(g(a)) = f(h(a)) = f(y) \rightsquigarrow x = y$,

($\Rightarrow$) Let $x, y : B$, $f(x) = f(y)$.

Def. $\hat{x}, \hat{y} : \mathbb{1} \rightarrowtail B$, $x = \hat{x}(*)$, $y = \hat{y}(*)$.

Then $f \circ \hat{x} = f \circ \hat{y} \rightsquigarrow \hat{x} = \hat{y} \rightsquigarrow x = y$. $\square$

Theorem. Epimorphisms are precisely surjections.

$\vdash \forall b : B\ \exists a : A.\ b = f(a).$

Proof. Let $f : A \hookrightarrow B$.

$(\Leftarrow)$ f sur. $g, h : B \hookrightarrow C$, $g \circ f = h \circ f$, $b : B$

Then let $a : A$ such that $b = f(a)$

Then $g(b) = g(f(a)) = h(f(a)) = h(b)$.

$(\Rightarrow)$ let $b : B$.

Let $i_1, i_2 : B \hookrightarrow \mathrm{coker}(f)$

Then $i_1 \circ f = i_2 \circ f \rightsquigarrow i_1 = i_2 \rightsquigarrow \exists a : A.\ b = f(a).$ $\square$

Corrolary. Isomorphisms are bijections.