Generalized linear models, binary data

Regression models

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Key ideas

- Frequently we care about outcomes that have two values
  - Alive/dead
  - Win/loss
  - Success/Failure
  - etc
- Called binary, Bernoulli or 0/1 outcomes
- Collection of exchangeable binary outcomes for the same covariate data are called binomial outcomes.

Example Baltimore Ravens win/loss

Ravens Data

```
download.file("https://dl.dropboxusercontent.com/u/7710864/data/ravensData.rda"
              , destfile="./data/ravensData.rda",method="curl")
load("./data/ravensData.rda")
head(ravensData)
```

```
  ravenWinNum ravenWin ravenScore opponentScore
1           1        W         24             9
2           1        W         38            35
3           1        W         28            13
4           1        W         34            31
5           1        W         44            13
6           0        L         23            24
```

Linear regression

try to predict whether they win from score

$$RW_i = b_0 + b_1 RS_i + e_i$$

RW_i - 1 if a Ravens win, 0 if not

RS_i - Number of points Ravens scored

b_0 - probability of a Ravens win if they score 0 points

b_1 - increase in probability of a Ravens win for each additional point

e_i - residual variation due

Um Voraussagen zu machen, ist so ein Modell ev ganz gut (Machine learning)
Aber es ist schwer zu interpretieren. Dieser Kurs geht v.a.
um statistische Interpretation.

Linear regression in R

```
lmRavens <- lm(ravensData$ravenWinNum ~ ravensData$ravenScore)
summary(lmRavens)$coef
```

```
                      Estimate Std. Error t value Pr(>|t|)
(Intercept)             0.2850   0.256643   1.111  0.28135
ravensData$ravenScore   0.0159   0.009059   1.755  0.09625
```

Odds

Binary Outcome 0/1

$$\mathrm{RW_i}$$

1 oder 0: ob sie gewonnen haben oder nicht

Probability (0,1)

$$\Pr(\mathrm{RW_i}|\mathrm{RS_i}, b_0, b_1)$$

Probability of winning, given the score and the parameters.

Odds $(0, \infty)$

$$\frac{\Pr(\mathrm{RW_i}|\mathrm{RS_i}, b_0, b_1)}{1 - \Pr(\mathrm{RW_i}|\mathrm{RS_i}, b_0, b_1)}$$

Log odds $(-\infty, \infty)$

$$\log\left(\frac{\Pr(\mathrm{RW_i}|\mathrm{RS_i}, b_0, b_1)}{1 - \Pr(\mathrm{RW_i}|\mathrm{RS_i}, b_0, b_1)}\right)$$

Linear vs. logistic regression

Linear

$$RW_i = b_0 + b_1 RS_i + e_i$$

or

$$E[RW_i | RS_i, b_0, b_1] = b_0 + b_1 RS_i$$

Logistic

$$\Pr(RW_i | RS_i, b_0, b_1) = \frac{\exp(b_0 + b_1 RS_i)}{1 + \exp(b_0 + b_1 RS_i)}$$

or

wenn man den Log nimmt:

$$\log\left(\frac{\Pr(RW_i | RS_i, b_0, b_1)}{1 - \Pr(RW_i | RS_i, b_0, b_1)}\right) = b_0 + b_1 RS_i$$

log(mu_i) = eta_i

Interpreting Logistic Regression

$$\log\left(\frac{\Pr(RW_i|RS_i, b_0, b_1)}{1 - \Pr(RW_i|RS_i, b_0, b_1)}\right) = b_0 + b_1 RS_i$$ RS: RavenScore

b_0 - Log odds of a Ravens win if they score zero points

b_1 - Log odds ratio of win probability for each point scored (compared to zero points)

$\exp(b_1)$ - Odds ratio of win probability for each point scored (compared to zero points) mit allen Kovariaten fixiert (nur hat es in diesem Bsp keine).

beta1 = log(odds(score is x+1)) - log(odds(score is x))

-> subtraction of 2 logs: daher

$$\text{beta1} = \log \frac{\text{odds(score is x+1)}}{\text{odds(score is x)}}$$

daher:

$$\text{e\^{}beta1} = \frac{\text{odds(score is x+1)}}{\text{odds(score is x)}}$$

Odds

- Imagine that you are playing a game where you flip a coin with success probability p.
- If it comes up heads, you win X. If it comes up tails, you lose Y.
- What should we set X and Y for the game to be fair? fair = fuer beide Spieler gleich

$$E[\text{earnings}] = Xp - Y(1-p) = 0$$

-> ich gewinne X$ mit Prob = p
und verliere Y$ mit Prob = 1-p

- Implies

$$\frac{Y}{X} = \frac{p}{1-p}$$ <- rechte Seite sind die odds

wenn man X = 1 setzt:

- The odds can be said as "How much should you be willing to pay for a p probability of winning a dollar?"
  - (If $p > 0.5$ you have to pay more if you lose than you get if you win.)
  - (If $p < 0.5$ you have to pay less if you lose than you get if you win.)

Visualizing fitting logistic regression curves

```
x <- seq(-10, 10, length = 1000)
manipulate(
    plot(x, exp(beta0 + beta1 * x) / (1 + exp(beta0 + beta1 * x)),
         type = "l", lwd = 3, frame = FALSE),
    beta1 = slider(-2, 2, step = .1, initial = 2),
    beta0 = slider(-2, 2, step = .1, initial = 0)
    )
```

Ravens logistic regression

family="binomial" -> i.e. logistic regr.

```
logRegRavens <- glm(ravensData$ravenWinNum ~ ravensData$ravenScore,family="binomial")
summary(logRegRavens)
```

(oder selbe Syntax wie fuer lm, mit data=…)

```
Call:
glm(formula = ravensData$ravenWinNum ~ ravensData$ravenScore,
    family = "binomial")

Deviance Residuals:
   Min      1Q  Median      3Q     Max
-1.758  -1.100   0.530   0.806   1.495

Coefficients:
                      Estimate Std. Error z value Pr(>|z|)
(Intercept)            -1.6800     1.5541   -1.08     0.28
ravensData$ravenScore   0.1066     0.0667    1.60     0.11

(Dispersion parameter for binomial family taken to be 1)

    Null deviance: 24.435  on 19  degrees of freedom
Residual deviance: 20.895  on 18  degrees of freedom
AIC: 24.89
```

beta0 (Intercept)
beta1 ravensData$ravenScore

Estimate:
beta0 = logodds of Ravens winning when they score nothing
To get the odds: do e^beta0
beta1 = increase in logodds for every point that they score.
e^beta1: relative increase= odds ratio for 1 unit increase in score!

Uebrigens:
Die Parameter e^x (aka exp(x)) sind meist sehr klein.
In der Naehe von 0 ist exp(x) ~= 1+x

exp(0.1066) = 1.1125 ~= 1.1066

Interpretation:
estimated odds of winning for the Ravens increases by 11% per point scored!

Fuer Intercept: odds of Ravens winning when they score 0 points.
But: since they aren't any such games, that's just an extrapolation.

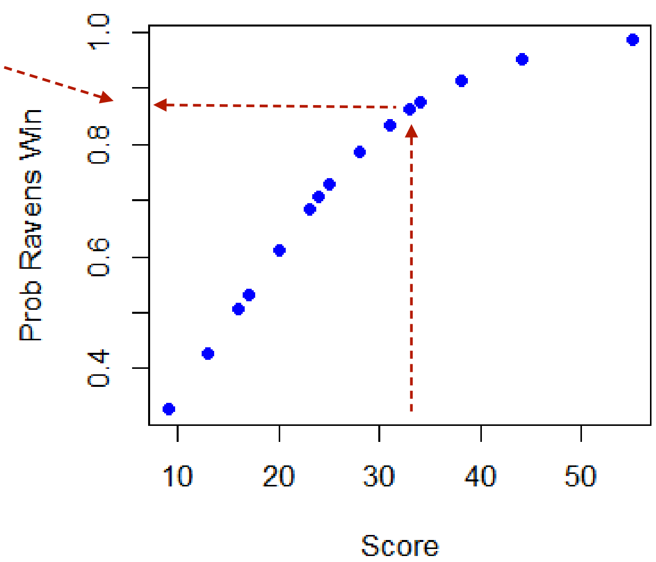
To get the probability (instead of the odds):

$$p = \frac{\exp(\ \text{beta0}\hat{}\ +\ \text{beta1}\hat{}\ *\ x)}{1\ +\ \exp(\ \text{beta0}\hat{}\ +\ \text{beta1}\hat{}\ *\ x)}$$

Ravens fitted values

```
plot(ravensData$ravenScore,logRegRavens$fitted,pch=19,col="blue",xlab="Score",ylab="Prob Ravens Win")
```

estimated prob of winning for a given score



Diese S-Kurve ist:

$$p = \frac{\exp(\hat{\beta}_0 + \hat{\beta}_1 * x)}{1 + \exp(\hat{\beta}_0 + \hat{\beta}_1 * x)}$$

Odds ratios and confidence intervals

```
exp(logRegRavens$coeff)
```

```
        (Intercept) ravensData$ravenScore
             0.1864                 1.1125  also ~11% increase of odds of winning for 1 point scored
        = exp(-1.6800)          = exp(0.1066)
```

```
exp(confint(logRegRavens))
```

```
                          2.5 % 97.5 %
(Intercept)           0.005675  3.106
ravensData$ravenScore 0.996230  1.303
```

ANOVA for logistic regression

```
anova(logRegRavens,test="Chisq")
```

```
Analysis of Deviance Table

Model: binomial, link: logit

Response: ravensData$ravenWinNum

Terms added sequentially (first to last)


                      Df Deviance Resid. Df Resid. Dev Pr(>Chi)
NULL                                     19       24.4
ravensData$ravenScore  1     3.54        18       20.9     0.06 .
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

If you fit multiple models, it adds them sequentially.
“Ideally with nested models” (?)

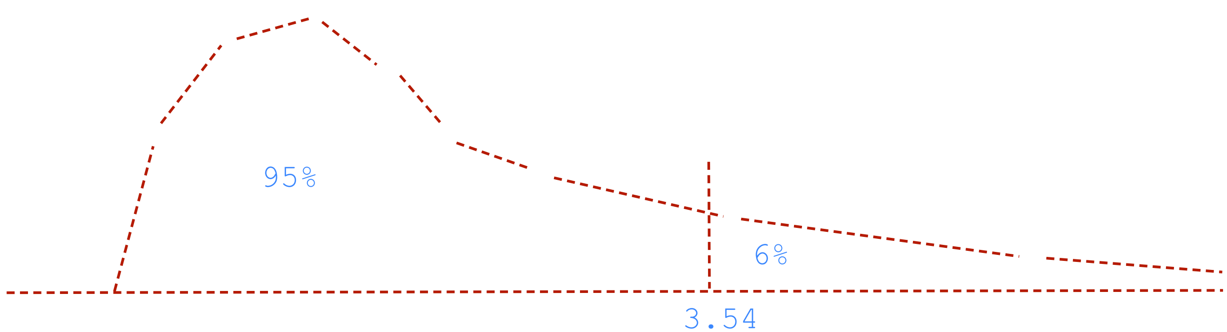
Resid.Dev von erstem minus zweitem Modell

NULL = just the intercept

minus

Differenz in Anzahl Parameter der beiden Modelle (19 - 18 = 1)

Der Chisquare-Test testet die Wahrscheinlichkeit, eine solche Deviance (3.54) mit einem solchen Degree of Freedom (1) zu erhalten: p = 0.06



Interpreting Odds Ratios

- Not probabilities
- Odds ratio of 1 = no difference in odds
- Log odds ratio of 0 = no difference in odds `Koeffizienten sind dann 0`
- Odds ratio < 0.5 or > 2 commonly a "moderate effect"
- Relative risk $\frac{\Pr(RW_i|RS_i=10)}{\Pr(RW_i|RS_i=0)}$ often easier to interpret, harder to estimate
- For small probabilities RR ≈ OR but **they are not the same**!

```
RR gibt Probleme, wenn man
Probability nahe bei 0 oder
1 hat (weil
-infinity < log(prob) <= 0
```

[Wikipedia on Odds Ratio]

```
Use of odds in retrospective studies.
```

Further resources

- Wikipedia on Logistic Regression
- Logistic regression and glms in R
- Brian Caffo's lecture notes on: Simpson's paradox, Case-control studies
- Open Intro Chapter on Logistic Regression `the classic text book on Log. Regr.`