Residuals and residual variation

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Residuals

die beta0 und beta1 sind die "richtigen" Parameter fuer das Model.

Die haben wir aber nicht. Wir haben nur die estimated, also beta^0, beta^1.

- Model $Y_i = \beta_0 + \beta_1 X_i + \epsilon_i$ where $\epsilon_i \overset{\text{iid}}{\sim} N(0, \sigma^2)$.
- Observed outcome i is Y_i at predictor value X_i
- Predicted outcome i is $\hat{Y}_i$ at predictor valuve X_i is

$$\hat{Y}_i = \hat{\beta}_0 + \hat{\beta}_1 X_i$$

der Hut (^) heisst einfach 'estimated', also estimated interceptor, estimated slope.

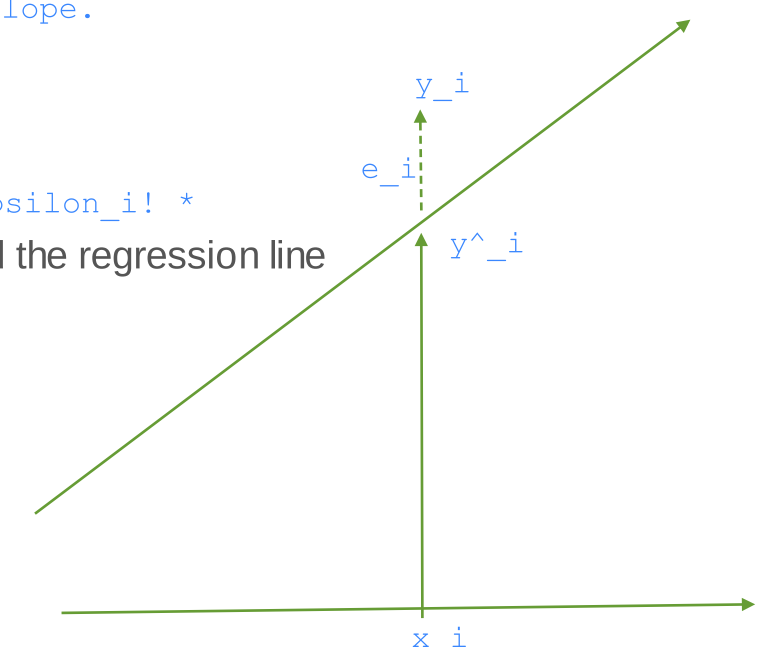
- Residual, the between the observed and predicted outcome

$$e_i = Y_i - \hat{Y}_i$$

Achtung: e_i ist /nicht/ epsilon_i! *

  - The vertical distance between the observed data point and the regression line

- Least squares minimizes $\sum_{i=1}^n e_i^2$
- The e_i can be thought of as estimates of the ϵ_i.



* epsilon_i ist der "richtige" Fehler,
e_i die beobachtete Distanz zwischen
unserer Vorhersage Y^_i und dem richtigen Y_i.

Properties of the residuals

- $E[e_i] = 0$.

  dh die e_is koennen nicht unabhaengig voneinander sein, da sie sich zu 0 aufsummieren. Wenn man n-1 eins kennt, kann man das nte ausrechnen.

- If an intercept is included, $\sum_{i=1}^{n} e_i = 0$

- If a regressor variable, X_i, is included in the model $\sum_{i=1}^{n} e_i X_i = 0$.

  Falls X=1, sei diese 3. Aussage equivalent zur zweiten, sagt er.
  X, X_i, x, x_i ??

- Residuals are useful for investigating poor model fit.
- Positive residuals are above the line, negative residuals are below.
- Residuals can be thought of as the outcome (Y) with the linear association of the predictor (X) removed. = "Y adjusted for X"
- One differentiates residual variation (variation after removing the predictor) from systematic variation (variation explained by the regression model).
- Residual plots highlight poor model fit.

Code

```
data(diamond)
y <- diamond$price; x <- diamond$carat; n <- length(y)
fit <- lm(y ~ x)
e <- resid(fit)
yhat <- predict(fit)
max(abs(e -(y - yhat)))
```

```
[1] 9.486e-13
```

residuals = predicted - observed

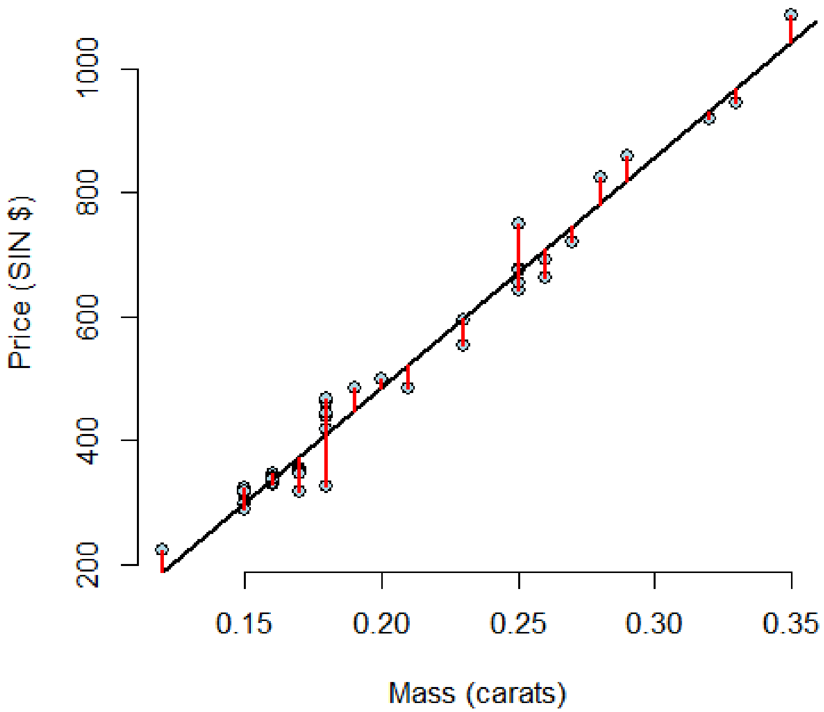
beta^0 beta^1

```
max(abs(e - (y - coef(fit)[1] - coef(fit)[2] * x)))
```

dasselbe:

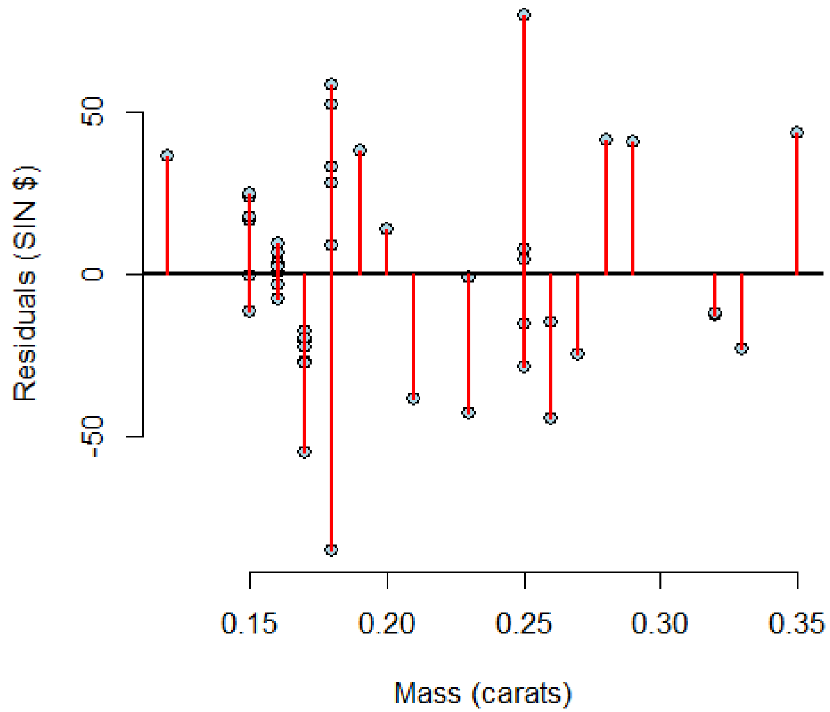
```
[1] 9.486e-13
```

Residuals are the signed length of the red lines



Residuals versus X

Wichtig: die Residuals sollten keinen Zusammenhang haben mit dem Predictor. Keine Muster.

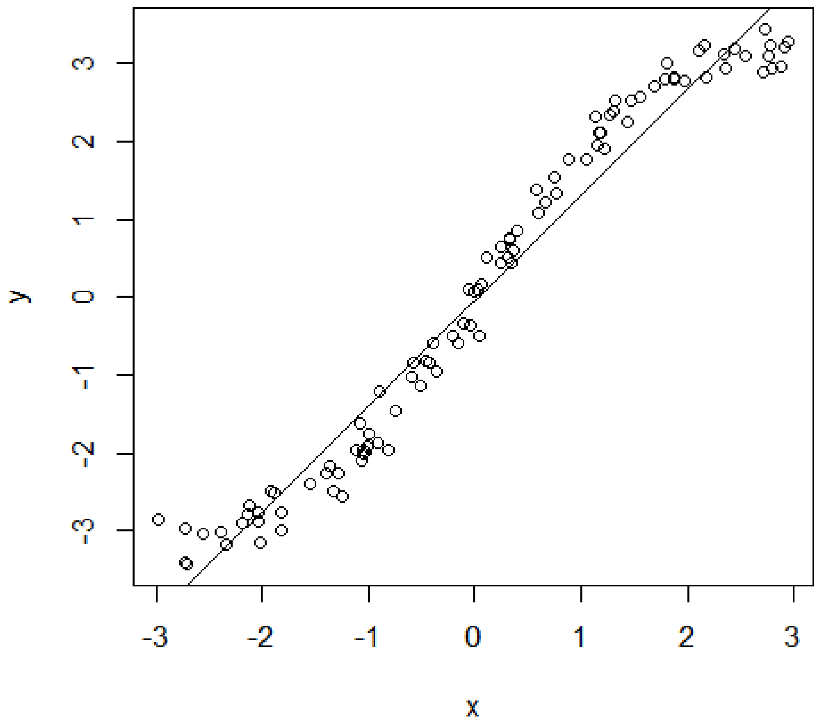


Non-linear data

Daten simulieren: X ist random uniform numbers. Y ist X plus eine Sinusfkt plus etwas Zufall:

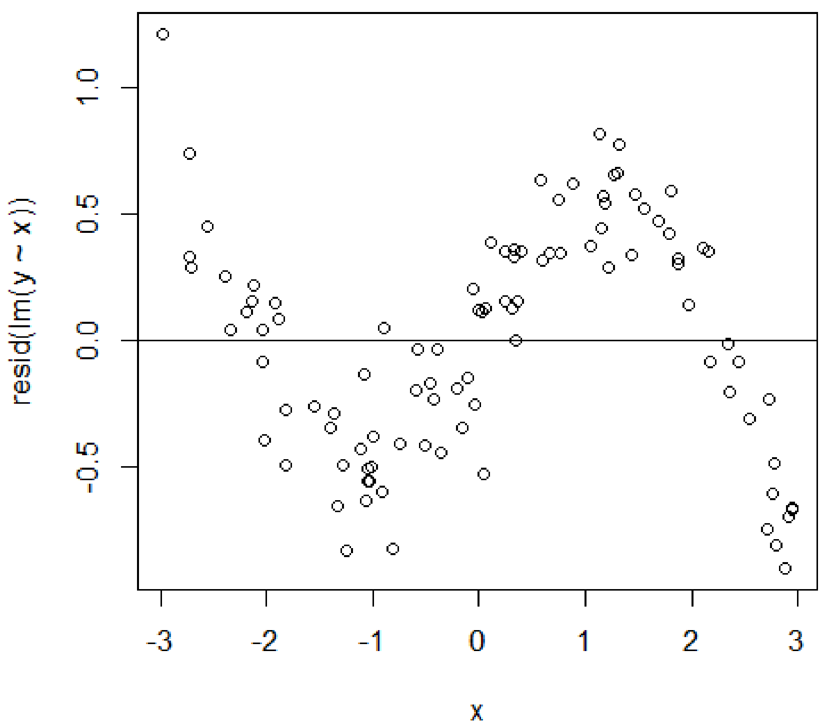
```
x <- runif(100, -3, 3); y <- x + sin(x) + rnorm(100, sd = .2);
plot(x, y); abline(lm(y ~ x))
```

Und diese Fkt (den Synus) sieht man im Plot:



Aber noch viel deutlicher im Residualplot!

```
plot(x, resid(lm(y ~ x)));
abline(h = 0)
```

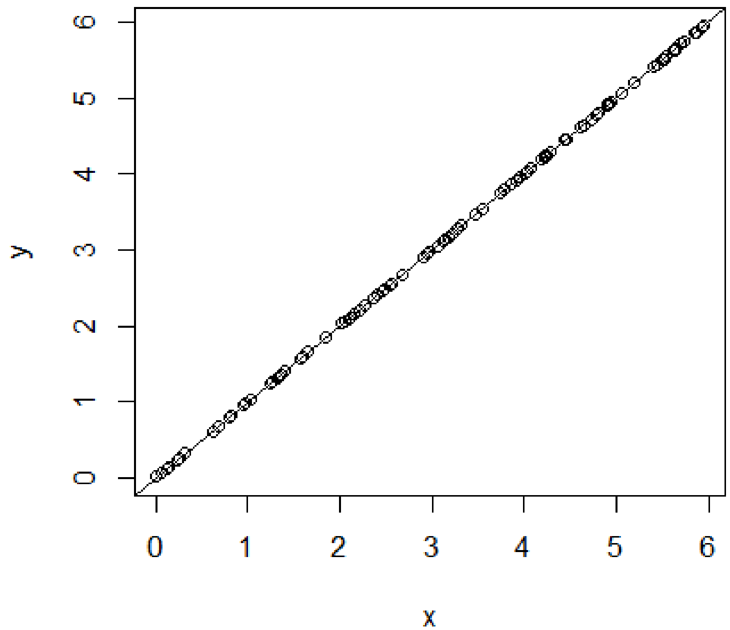


= Varianz nicht ueberall gleich!

Heteroskedasticity

Simulierte Daten: Varianz abhaengig von x!

```
x <- runif(100, 0, 6); y <- x + rnorm(100,  mean = 0, sd = .001 * x);
plot(x, y); abline(lm(y ~ x))
```

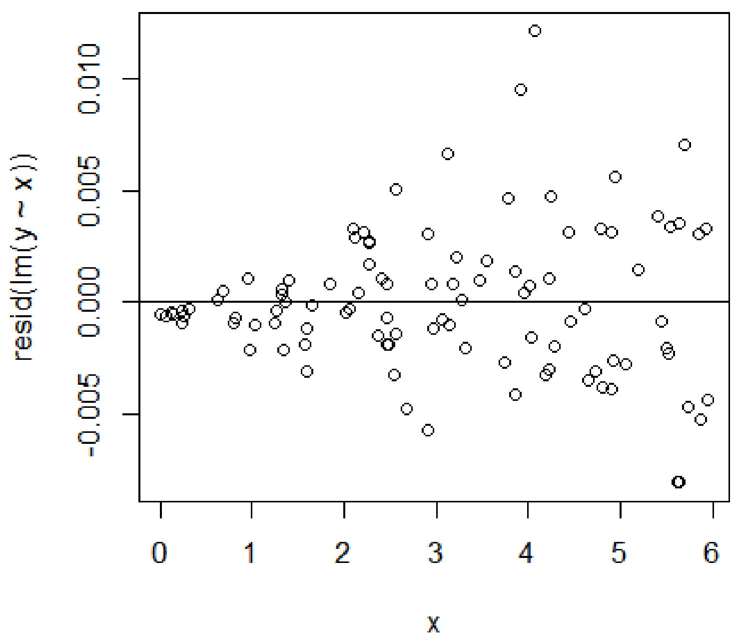


sieht eigentlich
aus
wie die perfekte

…jedoch im Residualplot sieht man, dass die Residuen nicht zufaellig sind!

Getting rid of the blank space can be helpful

```
plot(x, resid(lm(y ~ x)));
abline(h = 0)
```



Leider ist so ein Residualplot nur mit 1 Variable moeglich :-(

Estimating residual variation

- Model $Y_i = \beta_0 + \beta_1 X_i + \epsilon_i$ where $\epsilon_i \sim N(0, \sigma^2)$.

Residual variation ist sigma^2.

- The ML estimate of σ^2 is $\frac{1}{n}\sum_{i=1}^n e_i^2$, the average squared residual.

Max.Likelihood

- Most people use

$$\hat\sigma^2 = \frac{1}{n-2}\sum_{i=1}^n e_i^2.$$

-2: zwei Freiheitsgrade weniger:
eins fuer estimate of beta0,
eins fuer estimate of beta1

- The $n-2$ instead of n is so that $E[\hat\sigma^2] = \sigma^2$

-> unbiased

between friends: the difference is rather small, so forget about it…

Diamond example

```
y <- diamond$price; x <- diamond$carat; n <- length(y)
fit <- lm(y ~ x)
summary(fit)$sigma
```

```
[1] 31.84
```

```
sqrt(sum(resid(fit)^2) / (n - 2))
```

```
[1] 31.84
```

…same

Summarizing variation

Trick: Y^_i addieren und subtrahieren

$$\sum_{i=1}^{n}(Y_i - \bar{Y})^2 = \sum_{i=1}^{n}(Y_i - \hat{Y}_i + \hat{Y}_i - \bar{Y})^2$$

dann Quadrat ausmultiplizieren

$$= \sum_{i=1}^{n}(Y_i - \hat{Y}_i)^2 + 2\boxed{\sum_{i=1}^{n}(Y_i - \hat{Y}_i)(\hat{Y}_i - \bar{Y})} + \sum_{i=1}^{n}(\hat{Y}_i - \bar{Y})^2$$

Scratch work

Y^_i = beta^0 + beta^1*X_i
und beta^0 = Y_ + beta^1*X_. Einsetzen:

Hier immer benuetzt:
sum(X_i - X_) = 0,
sum(Y_i - Y_) = 0

$$(Y_i - \hat{Y}_i) = \{Y_i - (\bar{Y} - \hat{\beta}_1\bar{X}) - \hat{\beta}_1 X_i\} = (Y_i - \bar{Y}) - \hat{\beta}_1(X_i - \bar{X})$$

$$(\hat{Y}_i - \bar{Y}) = (\bar{Y} - \hat{\beta}_1\bar{X} - \hat{\beta}_1 X_i - \bar{Y}) = \hat{\beta}_1(X_i - \bar{X})$$

derselbe Trick bei Y^_i - Y_

$$\textstyle\sum_{i=1}^{n}(Y_i - \hat{Y}_i)(\hat{Y}_i - \bar{Y}) = \sum_{i=1}^{n}\{(Y_i - \bar{Y}) - \hat{\beta}_1(X_i - \bar{X}))\}\{\hat{\beta}_1(X_i - \bar{X})\}$$

Nun diese beiden vereinfachten Terme hier einsetzen.

$$= \hat{\beta}_1 \textstyle\sum_{i=1}^{n}(Y_i - \bar{Y})(X_i - \bar{X}) - \hat{\beta}_1^2 \sum_{i=1}^{n}(X_i - \bar{X})^2$$

$$= \hat{\beta}_1^2 \textstyle\sum_{i=1}^{n}(X_i - \bar{X})^2 - \hat{\beta}_1^2 \sum_{i=1}^{n}(X_i - \bar{X})^2 = 0$$

zweimal dasselbe, voneinander subtrahiert - gibt 0!

Summarizing variation

$$\boxed{\sum_{i=1}^{n}(Y_i - \bar{Y})^2} = \sum_{i=1}^{n}(Y_i - \hat{Y}_i)^2 + \boxed{\sum_{i=1}^{n}(\hat{Y}_i - \bar{Y})^2}$$

Totale Var. = Residual Var + "Model-Variation"

Or

Total Variation = Residual Variation + Regression Variation

weil mittlerer Term oben 0 wird!
(siehe Folie 13)

Define the percent of total varation described by the model as

$$R^2 = \frac{\sum_{i=1}^{n}(\hat{Y}_i - \bar{Y})^2}{\sum_{i=1}^{n}(Y_i - \bar{Y})^2} = 1 - \frac{\sum_{i=1}^{n}(Y_i - \hat{Y}_i)^2}{\sum_{i=1}^{n}(Y_i - \bar{Y})^2}$$

Model-Variation ist immer kleiner als die totale Variation.

Relation between R^2 and r (the corrrelation)

Recall that $(\hat{Y}_i - \bar{Y}) = \hat{\beta}_1 (X_i - \bar{X})$ so that

2 slides ago

$$R^2 = \frac{\sum_{i=1}^{n} (\hat{Y}_i - \bar{Y})^2}{\sum_{i=1}^{n} (Y_i - \bar{Y})^2} = \hat{\beta}_1^2 \frac{\sum_{i=1}^{n} (X_i - \bar{X})}{\sum_{i=1}^{n} (Y_i - \bar{Y})^2} = \mathrm{Cor}(Y, X)^2$$

Since, recall,

$$\hat{\beta}_1 = \mathrm{Cor}(Y, X) \frac{\mathrm{Sd}(Y)}{\mathrm{Sd}(X)}$$

So, R^2 is literally r squared.

Some facts about R^2

- R^2 is the percentage of variation explained by the regression model.
- $0 \leq R^2 \leq 1$
- R^2 is the sample correlation squared.
- R^2 can be a misleading summary of model fit.
  - Deleting data can inflate R^2.
  - (For later.) Adding terms to a regression model always increases R^2.
- Do `example(anscombe)` to see the following data. `illustrates some issues with R^2`
  - Basically same mean and variance of X and Y.
  - Identical correlations (hence same R^2).
  - Same linear regression relationship.

data(anscombe);example(anscombe)

Die haben alle
dasselbe R^2!