Model based prediction

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Basic idea

1. Assume the data follow a probabilistic model
2. Use Bayes' theorem to identify optimal classifiers

Pros:

- Can take advantage of structure of the data
- May be computationally convenient
- Are reasonably accurate on real problems

Cons:

- Make additional assumptions about the data
- When the model is incorrect you may get reduced accuracy

Model based approach

Wahrscheinlichkeit, dass outcome Y
zur Klasse k gehoert,
gegeben bestimmte
Prediktorvariablen (x)

1. Our goal is to build parametric model for conditional distribution $P(Y = k|X = x)$

2. A typical approach is to apply Bayes theorem:

$$\Pr(Y = k|X = x) = \frac{\Pr(X = x|Y = k)\Pr(Y = k)}{\sum_{\ell=1}^{K} \Pr(X = x|Y = \ell)\Pr(Y = \ell)}$$ total probability

$$\Pr(Y = k|X = x) = \frac{f_k(x)\pi_k}{\sum_{\ell=1}^{K} f_\ell(x)\pi_\ell}$$ pi_k: Prior

3. Typically prior probabilities π_k are set in advance.

4. A common choice for $f_k(x) = \frac{1}{\sigma_k\sqrt{2\pi}} e^{-\frac{(x-\mu_k)^2}{\sigma_k^2}}$, a Gaussian distribution

5. Estimate the parameters (μ_k, σ_k^2) from the data.

6. Classify to the class with the highest value of $P(Y = k|X = x)$

Classifying using the model

A range of models use this approach

- Linear discriminant analysis assumes $f_k(x)$ is multivariate Gaussian with same covariances
- Quadratic discriminant analysis assumes $f_k(x)$ is multivariate Gaussian with different covariances
- Model based prediction assumes more complicated versions for the covariance matrix
- Naive Bayes assumes independence between features for model building

http://statweb.stanford.edu/~tibs/ElemStatLearn/

Why linear discriminant analysis?

$$\log \frac{\Pr(Y = k|X = x)}{\Pr(Y = j|X = x)}$$

Prob(Y gehoert zu Klasse k, gegeben Predikt x)

Prob(Y gehoert zu Klasse j, gegeben Predikt x)

$$= \log \frac{f_k(x)}{f_j(x)} + \log \frac{\pi_k}{\pi_j}$$

$$= \log \frac{\pi_k}{\pi_j} - \frac{1}{2}(\mu_k + \mu_j)^T \Sigma^{-1} (\mu_k + \mu_j)$$

$$+ x^T \Sigma^{-1} (\mu_k - \mu_j)$$

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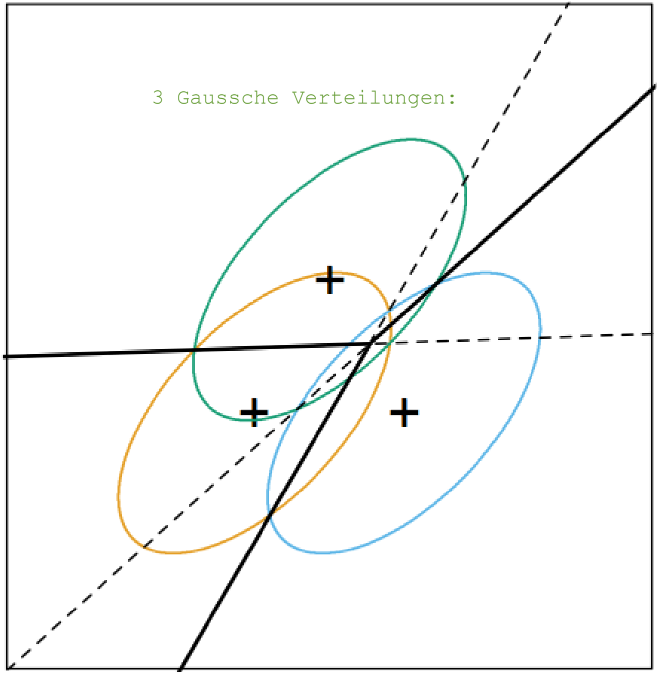
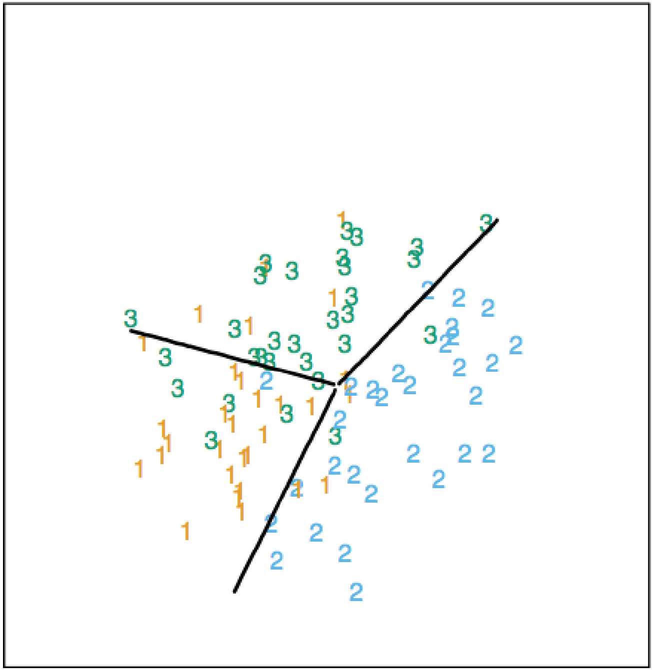
Decision boundaries





Discriminant function

mu_k: mean of class k for all features

$$\delta_k(x) = x^T \Sigma^{-1} \mu_k - \frac{1}{2} \mu_k \Sigma^{-1} \mu_k + \log(\mu_k)$$

sigma: Kovarianzmatrix for class k

- Decide on class based on $\hat{Y}(x) = \text{argmax}_k \delta_k(x)$
- We usually estimate parameters with maximum likelihood

Naive Bayes

Suppose we have many predictors, we would want to model: $P(Y = k|X_1, \dots, X_m)$

We could use Bayes Theorem to get:

$$P(Y = k|X_1, \dots, X_m) = \frac{\pi_k P(X_1, \dots, X_m|Y = k)}{\sum_{\ell=1}^{K} P(X_1, \dots, X_m|Y = k)\pi_\ell}$$

$$\propto \pi_k P(X_1, \dots, X_m|Y = k)$$

proportional, da Nenner eine Konstante ist

This can be written:

$$P(X_1, \dots, X_m, Y = k) = \pi_k P(X_1|Y = k) P(X_2, \dots, X_m|X_1, Y = k)$$

$$= \pi_k P(X_1|Y = k) P(X_2|X_1, Y = k) P(X_3, \dots, X_m|X_1, X_2, Y = k)$$

$$= \pi_k P(X_1|Y = k) P(X_2|X_1, Y = k) \dots P(X_m|X_1 \dots, X_{m-1}, Y = k)$$

We could make an assumption to write this:

naive Annahme:
dass alle Prediktoren unabhaengig sind.

$$\approx \pi_k P(X_1|Y = k) P(X_2|Y = k) \dots P(X_m|, Y = k)$$

Speziell nuetzlich bei ganz vielen
Prediktoren die binaer oder
kategorial sind.

Example: Iris Data

```
data(iris); library(ggplot2)
names(iris)
```

```
[1] "Sepal.Length" "Sepal.Width"  "Petal.Length" "Petal.Width"  "Species"
```

```
table(iris$Species)
```

```

    setosa versicolor  virginica 
        50         50         50 
```

Create training and test sets

```
inTrain <- createDataPartition(y=iris$Species,
                               p=0.7, list=FALSE)
training <- iris[inTrain,]
testing <- iris[-inTrain,]
dim(training); dim(testing)
```

```
[1] 45  5
```

Build predictions

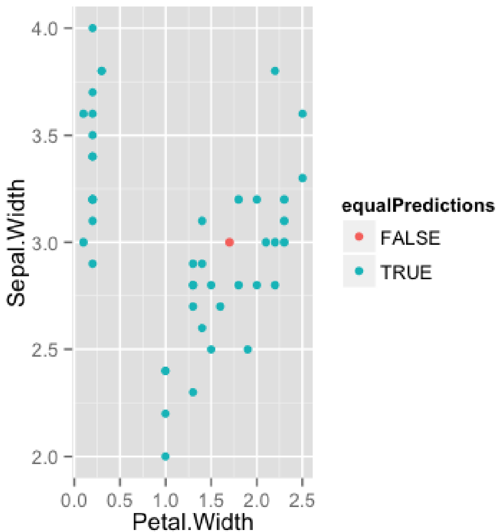
```
modlda = train(Species ~ .,data=training,method="lda") l_inear d_iscriminant a_nalysis
modnb = train(Species ~ ., data=training,method="nb") n_aive b_ayes
plda = predict(modlda,testing); pnb = predict(modnb,testing)
table(plda,pnb)
```

```
            pnb
plda         setosa versicolor virginica
  setosa         15          0         0
  versicolor      0         13         1
  virginica       0          0        16
```

```
Bis auf einen Fall uebereinstimmende Kategorisierung
von beiden Klassifizierungsmethoden.
```

Comparison of results

```
equalPredictions = (plda==pnb)
qplot(Petal.Width,Sepal.Width,colour=equalPredictions,data=testing)
```



Notes and further reading

- Introduction to statistical learning
- Elements of Statistical Learning
- Model based clustering
- Linear Discriminant Analysis
- Quadratic Discriminant Analysis