In European alternating current (AC) systems, the polarity of the line voltage cycles 50 times per second (50 Hz) following a sinusoidal curve. This means that each cycle takes 20 ms.

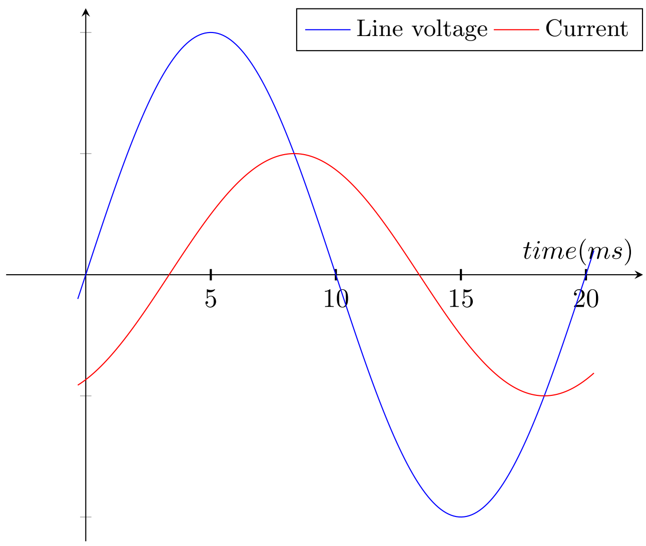
When a load is connected to an AC line, the current will in general also be sinusoidal, but it may be shifted in phase by an angle θ (We'll get back to this later in this document as in the most general case, current and voltage may deviate from a pure sinusoidal form).

With respect to the phase angle, three cases can be distinguished:

- $\theta = 0$ for a resistive load. Examples of this is an electrical heater system or an incandescent lightbulb.
- $\theta > 0$ for inductive loads. The current lags the voltage. This is the case for electrical motors or coils.
- $\theta < 0$ for capacitive loads. The current precedes the voltage. This is the case for capacitors and battery chargers.

In the most general case, a load acts as a combination of a resistive and a reactive part, where the latter is either capacitive or inductive.

The following graph shows a line voltage and its current plotted over time. In this case, the current lags the voltage by a 60 degrees ($\pi/3$ radians) angle, the so-called phase angle.



At any one point in time, the instantaneous power P_t, is the product of the voltage U_t and current I_t at that time:

$$P(t) = U(t) * I(t)$$

with:

$$U(t) = U_{\text{peak}} * sin(2\pi ft)$$
$$I(t) = I_{\text{peak}} * sin(2\pi ft - \theta)$$

This gives:

$$P(t) = U_{\text{peak}} * I_{\text{peak}} * sin(2\pi ft) * sin(2\pi ft - \theta)$$

We can now calculate the average (real) power by integrating P(t) over one cycle T and dividing it by T:

$$P_{\text{real}} = U_{\text{peak}} * I_{\text{peak}} * \frac{1}{T} \int_0^T sin(2\pi ft) * sin(2\pi ft - \theta)dt$$

For simplicity, we can reason about the power by ignoring the fact that we are working with voltages and currents. We can even abstract away the frequency (or cycle period) and consider simple sine curves in a single cycle of 2π radians.

In figures 1 and 2, we plot the curves for the cases where $\theta = 0$, a purely "resistive load", and $\theta = \pi/2$, a purely "inductive load". The instantaneous power in the former case is always positive, whereas in the capacitive case it's alternately positive and negative. The physical interpretation is that for a resistive load, the direction of energy transfer is from the generation side to the consumption side. For the purely capacitive load case, energy flows in equal amounts from and to the producer and the load. The effect is that, on balance, no net energy is transferred between the two and there's just energy "bouncing" between opposite ends.

$$U(t) = U_{\text{peak}} * sin(2\pi ft)$$
$$I(t) = I_{\text{peak}} * sin(2\pi ft + \pi/3)$$

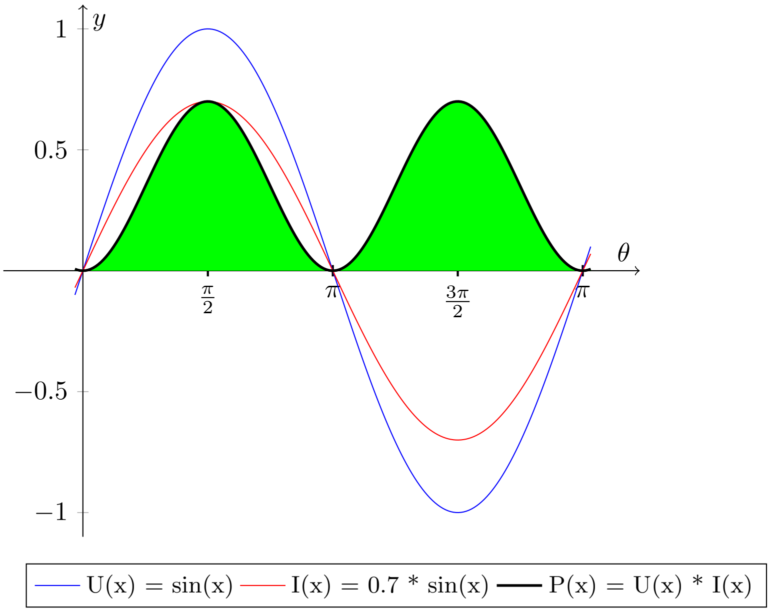


Figure 1: Perfect resistive load case

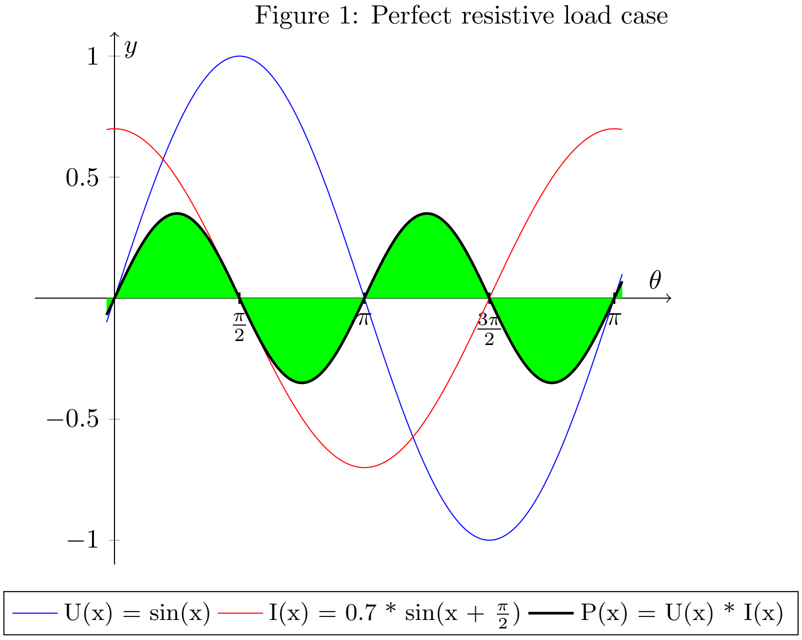


Figure 2: Perfect capacitive load case

We want to calculate the average effective power of two sine waves that are out of phase by an angle of θ.

This can be expressed as:

$$P_{real} = \frac{1}{2\pi} \int_0^{2\pi} \sin x * \sin(x - \theta) dx$$

We transform the integral using these base trigonometric identities:

$$\cos(\alpha - \beta) = \cos(\alpha) cos(\beta) + \sin(\alpha) sin(\beta)$$
$$\cos(\alpha + \beta) = \cos(\alpha) cos(\beta) - \sin(\alpha) sin(\beta)$$

Subtracting these gives:

$$\cos(\alpha - \beta) - \cos(\alpha + \beta) = 2\sin(\alpha) sin(\beta)$$

or:

$$\sin(\alpha) sin(\beta) = \frac{1}{2}(\cos(\alpha - \beta) - \cos(\alpha + \beta))$$

Hence:

$$\sin(x) sin(x - \theta) = \frac{1}{2}(\cos(\theta) - \cos(2x - \theta))$$

Applying this to $\mathrm{P_{real}}$, we get:

$$
\begin{aligned}
P_{\text{real}} &= \frac{1}{2\pi}\int_0^{2\pi} \sin(x) * \sin(x-\theta)dx \\
&= \frac{1}{4\pi}\int_0^{2\pi} \cos(\theta)dx - \frac{1}{4\pi}\cancelto{0}{\int_0^{2\pi} \cos(2x-\theta)dx} \\
&= \frac{1}{4\pi}\cos(\theta)\int_0^{2\pi} dx \\
&= \frac{1}{4\pi}\cos(\theta)\Big[x\Big]_0^{2\pi} dx \\
&= \frac{1}{4\cancel{\pi}}\cos(\theta)2\cancel{\pi} \\
&= \frac{\cos(\theta)}{2}
\end{aligned}
$$

We get the final result in which we observe the appearance of the infamous $\cos(\phi)$ factor:

$$
P_{\text{real}} = \frac{1}{2\pi}\int_0^{2\pi} \sin x * \sin(x-\theta)dx = \frac{\cos\theta}{2}
$$

If we bring this into the domain of the calculation of the real power, we get:

$$
\begin{aligned}
P_{\text{real}} &= U_{\text{peak}} * I_{\text{peak}} * \frac{1}{T}\int_0^{T} sin(2\pi ft) * sin(2\pi ft - \theta)dt \\
&= U_{\text{peak}} * I_{\text{peak}} * \frac{\cos\theta}{2}
\end{aligned}
$$

Some readers may have heard about (AC) RMS voltages and currents. If you've ever measured, or saw anyone measuring voltages or currents using a multimeter or current probes, the devices would carry a label "True RMS Meter". So, what does RMS stand for and what is its definition?

RMS is the acronym for "Root Mean Square", and for sinusoidal waves (voltage or current), it's calculated as follows:

$$
V_{\text{RMS}} = \sqrt{\frac{1}{T}\int_0^{T} V^2(t)dt}
$$

Again, doing this for a pure sinusoidal waveform, this gives:

$$\begin{aligned} V_{\text{RMS}} &= \sqrt{\frac{1}{2\pi}\int_0^{2\pi}(V_{\text{peak}} * sin(\theta))^2 d\theta} \\ &= \sqrt{\frac{V_{\text{peak}}^2}{2\pi}\int_0^{2\pi} sin^2(\theta)d\theta} \\ &= V_{\text{peak}} * \sqrt{\frac{1}{2\pi}\int_0^{2\pi} sin^2(\theta)d\theta} \end{aligned}$$

We already encountered the integral of $\sin^2(\theta)$ in this article. It evaluates to π so that we obtain:

$$\begin{aligned} V_{\text{RMS}} &= V_{\text{peak}} * \sqrt{\frac{\pi}{2\pi}} \\ &= \frac{V_{\text{peak}}}{\sqrt{2}} \end{aligned}$$

Similarly, we get a formula for I_{RMS}:

$$I_{\text{RMS}} = \frac{I_{\text{peak}}}{\sqrt{2}}$$

Substituting the RMS values for the peak values in the formula for P_{real}, we get the final, very simple and important result:

$$\begin{aligned} P_{\text{real}} &= V_{\text{peak}} * I_{\text{peak}} * \frac{\cos\theta}{2} \\ &= V_{\text{RMS}} * I_{\text{RMS}} * \cos\theta \end{aligned}$$

With $V_{\text{RMS}} * I_{\text{RMS}}$ being P_{apparent}, we see that:

$$P_{\text{real}} = P_{\text{apparent}} * \cos\theta$$

If we define P_{reactive} to be:

$$P_{\text{reactive}} = P_{\text{apparent}} * \sin\theta$$

We also see that:

$$P_{\text{apparent}}^2 = P_{\text{real}}^2 + P_{\text{reactive}}^2$$