lecture_01

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1 Lecture 1. Course Introduction

(First, the boring but important adminstrative stuff!)

1.0.1 Teachers

Teacher
Teacher Assistant
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1.0.2 Important dates

- **January 2, 2017**: Last day for registration to the exam
- **January 12, 2017**: Examination
- **January 15, 2017** Deadline for submission of last lab report
- **January 29, 2016** Deadline for resubmission of last lab report
- **Februrary 15, 2017**: Last day for registration to the re-exam
- **February 25, 2017**: Reexamination
- Registration to exam always at least **10** days prior to exam day

1.0.3 Overall organization

- 18 Lectures in total
- Two last lectures in week 2: Revision of important concepts, Q&A answers
- 5 Labs consisting of theoretical and practical problems
  - Practical problems can be solved in either Matlab, Octave or Python
- 3 half-day lab reservations a week, 2 of them will be supervised by Juan Carlos.
- Live demos and example code in the lectures will use python packages from SciPy ecosystem and presented using Jupyter/IPython notebooks.
- A crash course guiding you into the scientific Python world will be given in one of the supervised labs.

```
In [2]: %%HTML
        <iframe src="https://se.timeedit.net/web/umu/db1/public1/riq53Q03621Z6YQy7(
        </iframe>
```

```
<IPython.core.display.HTML object>
```

1.1 Reading material

- Lectures, lecture notes, and notebooks for the part "Introduction to Finite Difference Methods"
- Book "The Finite Element Method. Theory, Implementation and Applications" by Mats G. Larson and Frederik Bengzon
- Additionally (not required): "Partial Differential Equations with Numerical Methods" by Stig Larsson and Vidar Thomée
- Collection of course material on "Numerical Methods for Partial Differential Equations" by Hans-Petter Langtangen

Beginning of next slide we add some LaTeXmacros inside a math environment
What are Partial Differential Equations?

Definition: A partial differential equation is an equation involving certaint partial derivatives of an unknown function $u : \Omega \subset \mathbb{R}^d \to \mathbb{R}$ with $d \geqslant 2$.

Recall that **first order** partial derivatives

$$
\begin{aligned}
D_i u(x) = \partial_i u(x) = \partial_{x_i} u(x) = \frac{\partial}{\partial x_i} u(x) = u_{x_i}(x_0) = \lim_{h \to 0} \frac{u(x_0 + h e_i) - u(x_0)}{h} \\
= \lim_{h \to 0} \frac{u(x_1, \ldots, x_{i-1}, x_i + h, x_{i+1}, \ldots x_d) - u(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots x_d)}{h}
\end{aligned}
$$

Second and higher order partial derivatives

$$
\partial_i \partial_j u = \frac{\partial^2}{\partial x_i \partial x_j} u = D_i D_j u
$$

$$
D^\alpha u = \frac{\partial^{|\alpha|}}{\partial_1^{\alpha_1} \cdots \partial_d^{\alpha_d}} u
$$

for a multiindex $\alpha = (\alpha_1, \ldots, \alpha_d)$ and $|\alpha| := \alpha_1 + \ldots + \alpha_d$

1.2 Standard PDE operators

- **Gradient** of scalar function $u : \Omega \subset \mathbb{R}^d \to \mathbb{R}$

$$
\operatorname{grad} u = \nabla u = (\partial_1 u, \ldots, \partial_d u)
$$

- **Divergence** of a vector field $\boldsymbol{u} : \Omega \subset \mathbb{R}^d \to \mathbb{R}^d$

$$
\operatorname{div} \boldsymbol{u} = \nabla \cdot \boldsymbol{u} = \sum_{i=1}^{d} \partial_i \boldsymbol{u}_i
$$

- **Curl** of a vector field $\boldsymbol{u}$

$$
\operatorname{curl} \boldsymbol{u} = \nabla \times \boldsymbol{u} = (\partial_y \boldsymbol{u}_z - \partial_z \boldsymbol{u}_y, \partial_z \boldsymbol{u}_x - \partial_x \boldsymbol{u}_z, \partial_x \boldsymbol{u}_y - \partial_y \boldsymbol{u}_x)
$$

- **Laplace** of u

$$\Delta u = \nabla \cdot (\nabla u) = \sum_{i=1}^{d} \partial_i u$$

1.3 Examples for PDEs: Linear equations

Transport equation Simplest "real" PDE for a function $u(x,t)$ of two variables $x \in \mathbb{R}$, $ t > 0$

$$\partial_t u + b \partial_x u = f$$

or more generally for $u(x,t)$ with $x \in \mathbb{R}^d$

$$\partial_t u + \boldsymbol{b} \cdot \nabla u = f$$

Laplace's and Poisson's equation

$$-\Delta u = f$$

which often models equilibrium states in physics, e.g. electrostatistics, stationary heat distribution

Heat equation

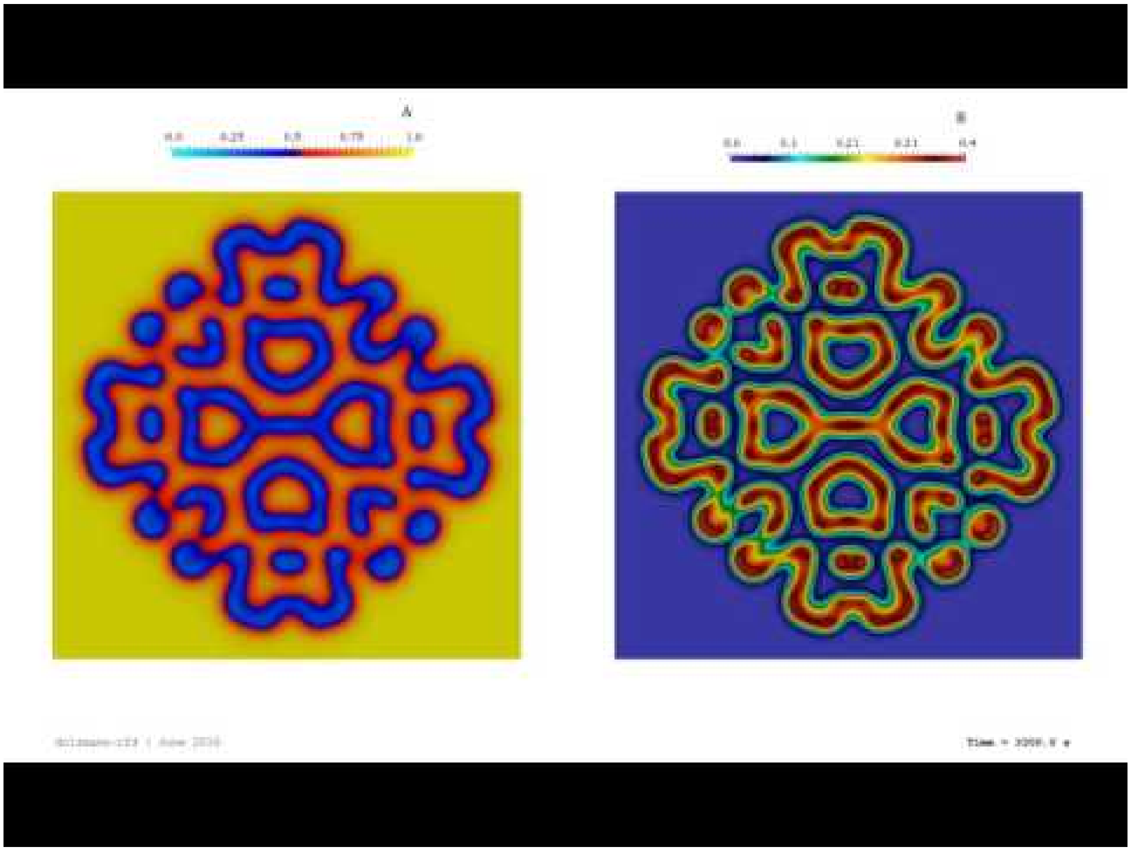
$$\partial_t u - \Delta u = f$$

Wave equation

$$\partial_t^2 u - \Delta u = f$$

In [3]:
```
from IPython.display import YouTubeVideo, HTML
YouTubeVideo('nw2bPnhtxN8', width=1000, height=500)
# Alternative and more general way to embed general webpages
#HTML('<iframe width="1000" height="500" src="//www.youtube.com/embed/oWFSs
```

Out[3]:



1.4 Nonlinear equations

Korteweg-de Fries equation The propagation of waves in shallow waters is modeled by a 3rd order, non-linear (actually, quasi-linear) PDE of the form

$$u_t - 6uu_x + u_{xxx} = 0$$

Minimal surface equation A minimal surface $\Gamma \subset \mathbb{R}^3$ is a surface which locally minimizes its surface area. Expressing the surface locally as the graph of a function $u(x, y)$, Γ being a minimal surface is equivalent to satisfying the non-linear (elliptic) equation

$$\nabla \cdot \left(\frac{\nabla u}{(1 + |\nabla u|^2)^{\frac{1}{2}}} \right) = 0$$

1.5 Linear systems of PDEs

Linear Elasticity The displacement $\boldsymbol{u}$ of a deformable medium at equilibrium subject to an extern load $\boldsymbol{u}$ can be modeled by

$$\mu \Delta \boldsymbol{u} + (\lambda + \mu) \nabla (\nabla \cdot \boldsymbol{u}) = f$$

The instationary case (basicially resembling Newton's second law $F = ma$) is described through

$$\boldsymbol{u}_{tt} - \mu\Delta\boldsymbol{u} - (\lambda + \mu)\nabla(\nabla \cdot \boldsymbol{u}) = f$$

Maxwell's equations In the presence of a charge density ρ and a current density $\boldsymbol{j}$, the electric field $\boldsymbol{E}$ and magnetic field $\boldsymbol{B}$ satisfy

$$\operatorname{div} \boldsymbol{B} = 0 \tag{1}$$
$$\boldsymbol{B}_t + \operatorname{curl} \boldsymbol{E} = 0 \tag{2}$$
$$\operatorname{div} \boldsymbol{E} = 4\pi\rho \tag{3}$$
$$\boldsymbol{E}_t - \operatorname{curl} \boldsymbol{E} = -4\pi\boldsymbol{j} \tag{4}$$

1.6 Nonlinear systems of PDEs

Reaction-Diffusion systems

$$\partial_t\boldsymbol{u} - \Delta\boldsymbol{u} = f(\boldsymbol{u})$$

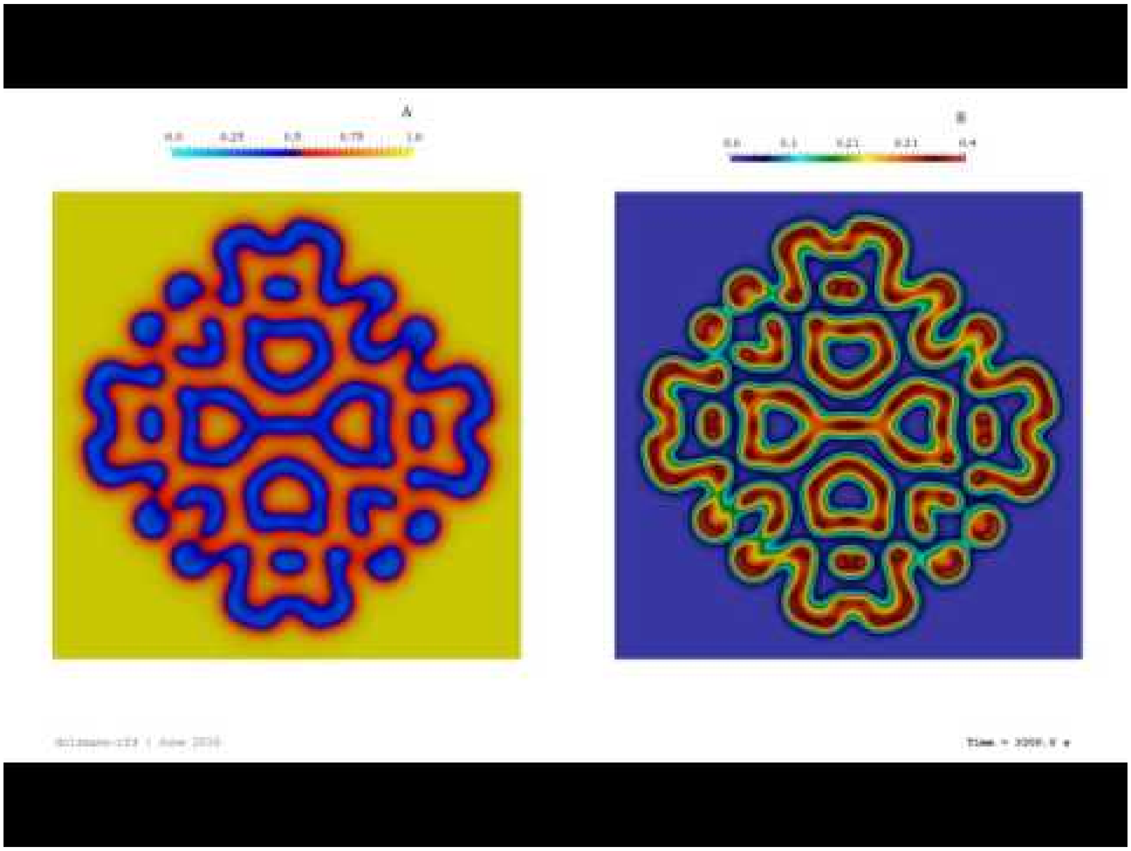
for exampel the Gray-Scott equations describing

$$\frac{\partial u}{\partial t} = D_u\Delta u - uv^2 + F(1-u) \qquad \text{in } \Omega, \tag{5}$$
$$\frac{\partial v}{\partial t} = D_v\Delta v + uv^2 - (F+k)v \qquad \text{in } \Omega, \tag{6}$$

modeling the reaction and diffusion of chemical species U and V described by their concentration u and v. D_u and D_v are the diffusion coefficients, k is the rate constant of the second reaction and F the feed rate.

In [4]:
```
from IPython.display import YouTubeVideo, HTML
YouTubeVideo('nw2bPnhtxN8', width=1000, height=500)
```

Out[4]:



Navier-Stokes equations describing incompressible, viscous flow in terms of the fluid velocity $\boldsymbol{u}$ and the fluid pressure p

$$\partial_t \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} - \nu\Delta\boldsymbol{u} + \nabla p = \boldsymbol{f} \nabla \cdot \boldsymbol{u} = 0$$

In [5]:
```
%%HTML
<center><video width="800" controls><source src="flow-around-a-cylinder.ogg
```

```
<IPython.core.display.HTML object>
```

In [6]:
```
from IPython.core.display import HTML
def css_styling():
    styles = open("../styles/custom.css", "r").read()
    return HTML(styles)

# Comment out next line and execute this cell to restore the default notebo
css_styling()
```

Out[6]:
```
<IPython.core.display.HTML object>
```