UMEÅ UNIVERSITY **July 1, 2014**
Department of Mathematics and Mathematical Statistics
Lab report

Laboratory 2

Numerical methods for PDE 7.5hp

Hyperbolic Equations

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1 Theory

In this lab I used five different methods to solve the one dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad 0 \leq x \leq L \tag{1}$$

where c is the propagation speed of the wave, $u = u(x, t)$ is the displacement from equilibrium. The initial and boundary conditions to be solved for are

$$\begin{aligned} u(0,t) &= u(L,t) = 0 \\ u(x,0) &= f(x) \\ \dot{u}(x,0) &= 0 \end{aligned} \tag{2}$$

where $f(x)$ is a known function.

From now on we discretize u in both time, t, and space, x. So that

$$\begin{aligned} u(j\Delta x, n\Delta t) &= u_j^n \\ j &= 0, 1, \ldots, J = 10/\Delta x \\ n &= 0, 1, \ldots, N = T/\Delta t \end{aligned} \tag{3}$$

where Δx and Δt are constants chosen so that N and J are integers.

1.1 Simple Explicit Method

This method is also known as *Forward in time, centred in space* (FTCS). Eq. (1) can be rewritten in discretized form as

$$\frac{u_j^{n+1} - 2u_j^n + u_j^{n-1}}{\Delta t^2} = c^2 \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2}. \tag{4}$$

Solving this for u_j^{n+1} gives the updating scheme.

1.2 Fully Implicit Method

For the implicit method we discretize eq. (1) having the spacial derivative calculated forward in time like

$$\frac{u_j^{n+1} - 2u_j^n + u_j^{n-1}}{\Delta t^2} = c^2 \frac{u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}}{\Delta x^2}. \tag{5}$$

Solving this for u_j^{n+1} gives the updating scheme.

1.3 The Crank-Nicholson Method

The *Crank-Nicholson* method is similar to the implicit method except that the average in time in both LHS and RHS of eq. (5) should be the same. In order to accomplish this, take the average of the spacial derivative forwards and backwards in time. So that

$$\frac{u_j^{n+1} - 2u_j^n + u_j^{n-1}}{\Delta t^2} = \frac{c^2}{2} \frac{(u_{j+1}^{n+1} + u_{j+1}^{n-1}) - 2(u_j^{n+1} + u_j^{n-1}) + (u_{j-1}^{n+1} + u_{j-1}^{n-1})}{\Delta x^2}. \tag{6}$$

Solving this for u_j^{n+1} gives the updating scheme.

1.4 FFT Time-step Method in Normal Space

The updating scheme for this problem was derived in [1] for time-stepping in normal space.

1.5 Immediate FFT Method

The numerical solution for this method was derived in [1].

2 Results

To test the different methods to solve eq.(1). I solved it with parameters $c = 10$ m/s, $L = 10$ m and $t \in [0, T]$, where T is an arbitrarily chosen constant. In the implementation I also always choose $\Delta x = L/127$. The time step will be chosen to be $\Delta t = \Delta x/c$ except for the method in section 1.4 which needs lower time-step to converge to a solution.

I solved eq.(1) and (2) for three different known functions $f(x)$ given as

$$f(x) = e^{-(x-L/2)^2} \quad (7)$$

$$f(x) = sin(0.2\pi x) \quad (8)$$

$$f(x) = \begin{cases} 1, & L/3 < x < 2L/3 \\ 0, & else \end{cases} \quad (9)$$

The solutions are very similar for all methods except for a few exceptions. When it came to speed and amplitude of the wave, the implicit method (see section 1.2) damped the solution and also moved slower than the solutions of the other methods, see fig. 1 and 2.

For the method described in section 1.4, I had to decrease the time-step by about a fifth of what the other methods used, otherwise the solution diverged very fast.

2.1 Stability with respect to Time Step

The *explicit* methods, which include the simeple explicit method and the FFT time step method, are unstable for time steps above the *Courant-Lewy-Friedrich* condition (CLF-condition) where the time step is set to $\Delta t = \Delta x/c$. Smaller time-steps give better results, especially for discontinuous initial function. Which is to be expected.

The *implicit* methods which include the fully implicit method and the Crank-Nicholson method are stable for any time step. However a higher time step means higher damping. which according to [1] is to be expected.

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The methods that utilizes the fourier transform handled discontinuous initial functions like eq.(9) much better than the other methods, see fig. 3 and 4.

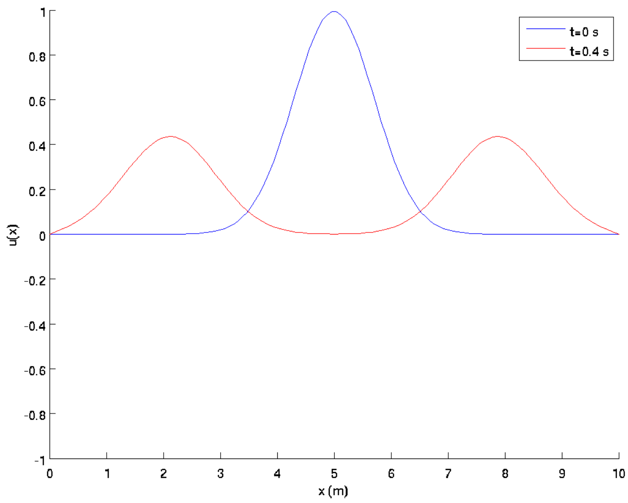

Figure 1: The solution of the wave equation for a normal initial function using an explicit method.

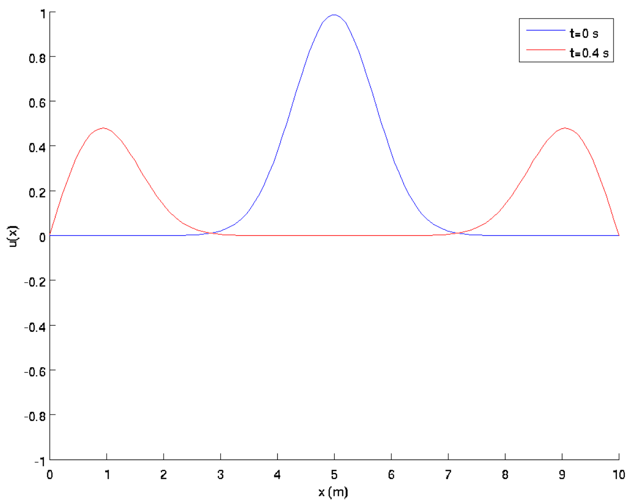

Figure 2: The solution of the wave equation for a normal initial function using an implicit method.

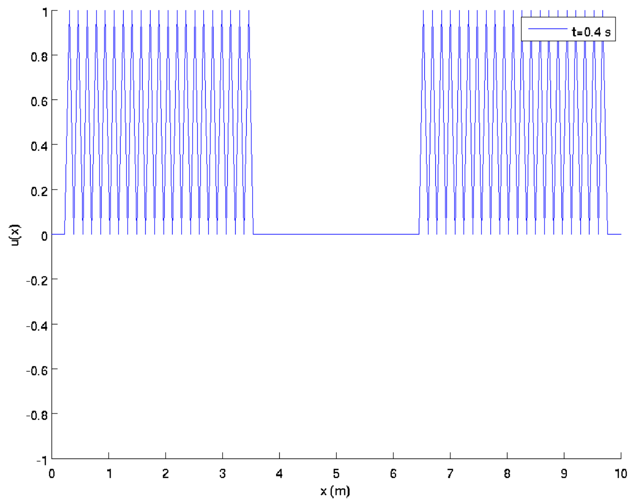

Figure 3: The solution of the wave equation for a discontinuous initial function using an explicit method.

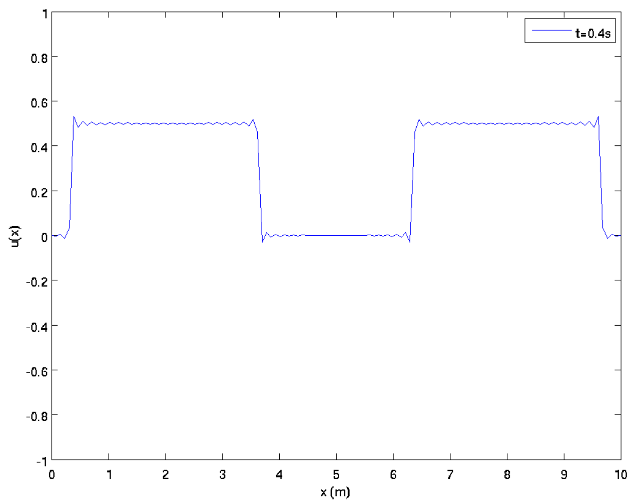

Figure 4: The solution of the wave equation for a discontinuous initial function using the fourier transform method without time-step.

References

[1] "Solving PDE with Finite difference methods," *Lecture notes to Numerical methods for partial differential equations*, Umeå University.