UMEÅ UNIVERSITY
Department of Mathematics and Mathematical Statistics

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Lab report

Laboratory 1

Numerical methods for PDE 7.5hp

Elliptic and Parabolic Equations

Name	Robin Lundberg
E-mail	rolu0008@student.umu.se

Supervisor
Patrik Norqvist, Christian Engström and Juan Carlos Araujo-Cabarcas.

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1 Exercise 1

In this exercise we begin by solving Laplace's equation $\nabla^2\phi(\mathbf{r}) = 0$ numerically in the square $-1 \leq x \leq 1$ and $-1 \leq y \leq 1$ with grid size $h = 0.1$. We are going to use the relaxational method to solve this and to start the iteration off we initialize ϕ to 0.1 except at the origin and boundaries where a constant value is imposed, $\phi(1,y) = \phi(-1,y) = \phi(x,1) = \phi(x,-1) = 0$ and $\phi(0,0) = -1$ The result for iteration 100 and 1000 can be seen in table 1.

Table 1: Values of ϕ at $y = 0$ for different iterations.

iteration	100	1000
x=0	-1.0000	-1.0000
x=0.1	-0.5668	-0.6067
x=0.2	-0.3729	-0.4284
x=0.3	-0.2605	-0.3230
x=0.4	-0.1871	-0.2492
x=0.5	-0.1332	-0.1921
x=0.6	-0.0941	-0.1449
x=0.7	-0.0630	-0.1040
x=0.8	-0.0391	-0.0673
x=0.9	-0.0184	-0.0331
x=1	0	0

As can be seen in table 1 the values at different points vary by almost a factor of two at some points. So I decided to stop iterating when the *relative error*—defined as the 2-norm of the difference between the vector ϕ and it's previous values, divided by the absolute maximum value of *phi*—is less than a *relative tolerance* that is much less than unity. In our case, $relativetolerance = 1e-6$. The results can be seen in fig. 1.

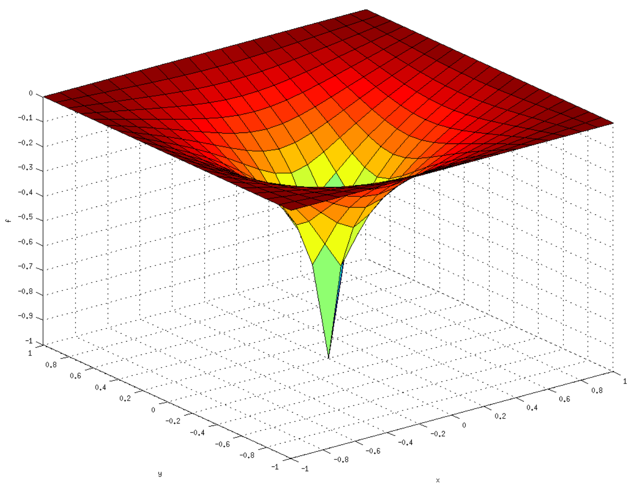


Figure 1: $\phi(x,y)$ after 1931 iterations.

The electric field is defined as $\mathbf{E}(x,y) = -\nabla\phi(x,y)$. I calculated this for the line $y = 0 and 0.1 \leq x \leq 0.9$ with the results in fig. 2. The loglog plot in fig. 3

could show that $log(|\mathbf{E}|) \propto log(1/x)$ which would mean that $|\mathbf{E}| \propto 1/x$ on this line. Deriving $\mathbf{E}$ analytically we get

$$\nabla^2 \phi = 0 \implies \nabla \cdot \mathbf{E} = 0$$

assuming radial symmetry, so that $\mathbf{E} = E\hat{r}$ we get

$$\frac{1}{r}\frac{\partial}{\partial r}\left(E + r\frac{\partial E}{\partial r}\right) \iff r\frac{\partial E}{\partial r} + E = 0 \qquad for\ r \neq 0.$$

The solution for this is

$$E = ar^{-1},\ a = constant, \qquad r \neq 0 \tag{1}$$

and for the line that was plotted in fig. 3 this is equivalent to $E \propto 1/x$ and so the analytical solution agrees with the numerical solution.

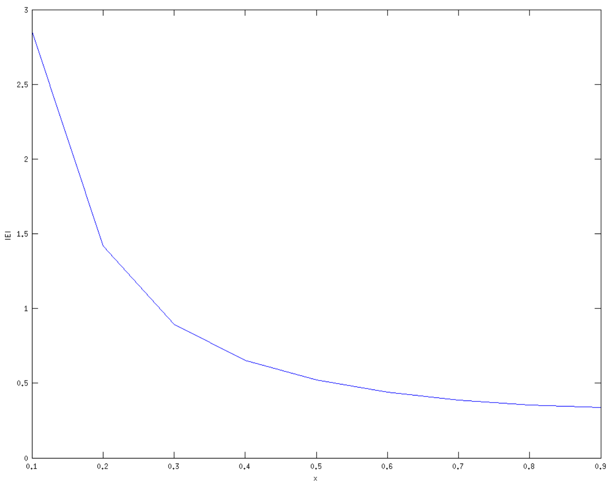


Figure 2: $\phi(x, y)$ after 1931 iterations.

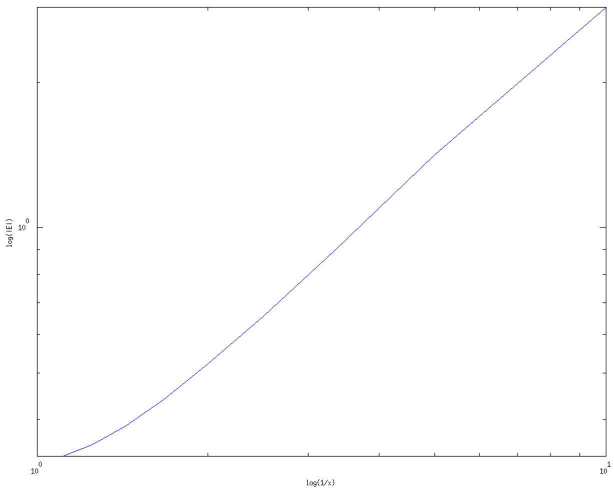


Figure 3: $\phi(x, y)$ after 1931 iterations.

In general the Laplace equation can be extended to Poisson's equation

$$\nabla^2\phi(\mathbf{r}) = -4\pi\rho(\mathbf{r}). \tag{2}$$

To solve this numerically we make it discrete; the laplacian can be written as

$$\nabla^2\phi(x, y) \approx \frac{\phi(x+h, y) + \phi(x-h, y) - 2\phi(x, y)}{h^2} + \frac{\phi(x, y+h) + \phi(x, y-h) - 2\phi(x, y)}{h^2}$$

and so we get

$$\frac{\phi(x+h, y) + \phi(x-h, y) - 2\phi(x, y)}{h^2} + \frac{\phi(x, y+h) + \phi(x, y-h) - 2\phi(x, y)}{h^2} = -4\pi\rho(x, y)$$

which can be solved for $\phi(x, y)$. Using the relaxational method to iteratively get the function $\phi(x, y)$ we get the updating scheme

$$\phi_{n+1}(x, y) = \frac{\phi_n(x+h, y) + \phi_n(x-h, y) + \phi_n(x, y+h)}{4} + \pi h^2\rho(x, y). \tag{3}$$

I solved this for $\rho(x, y) = 1/h^2\delta_{x,0}\delta_{0,y}$ and the results can be seen in fig. 4

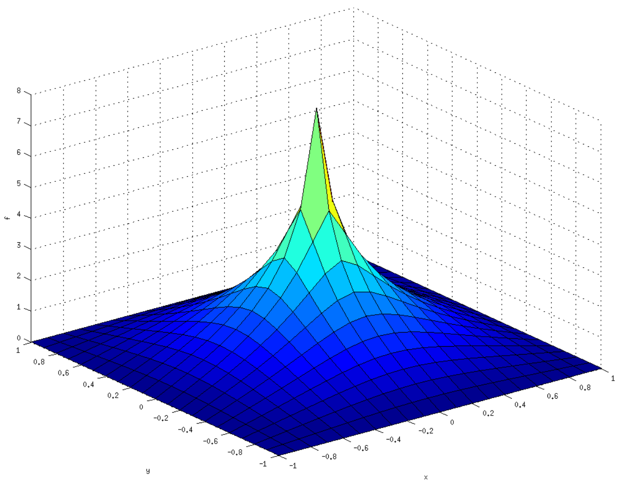


Figure 4: $\phi(x,y)$ after 1931 iterations.

2 Exercise 2

Burger's equation can be solved numerically using the Lax and Lax-Wendroff scheme. The Burger equation is given by

$$\frac{\partial \rho}{\partial t} = \frac{\partial (v\rho)}{\partial x} \tag{4}$$

where

$$v(\rho) = v_m \left(1 - \frac{\rho}{\rho_m}\right) \tag{5}$$

where v_m and ρ_m are constants. The Lax iteration scheme is as follows

$$\rho_j^{n+1} = \frac{1}{2}\left(\rho_{j+1}^n + \rho_{j-1}^n\right) - \frac{\Delta t}{2\Delta x}\left(F_{j+1}^n - F_{j-1}^n\right) \tag{6}$$

with superscript as time index and subscript as position index and

$$F(\rho) = \rho v(\rho). \tag{7}$$

The Lax-Wendroff iteration scheme is given by

$$\begin{aligned} \rho_j^{n+1} = \rho_j^n &- \frac{\Delta t}{2\Delta x}\left(F_{j+1}^n - F_{j-1}^n\right) \\ &+ 2\left(\frac{\Delta t}{2\Delta x}\right)^2 \left[q_{j+1/2}^n (F_{j+1}^n - F_j^n) - q_{j-1/2}^n (F_j^n - F_{j-1}^n)\right] \end{aligned} \tag{8}$$

where

$$q_{j+1/2}^n = \left(\frac{dF}{d\rho}\right)_{j+1/2}^n \approx v_m \left(1 - \frac{\rho_{j+1}^n + \rho_j^n}{\rho_m}\right). \tag{9}$$

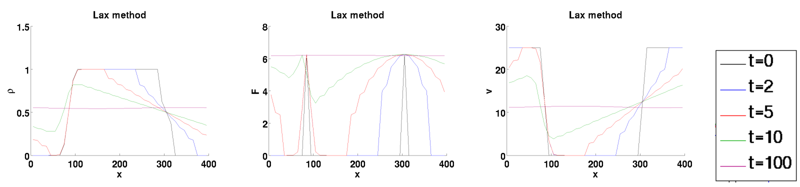
I studied these methods with the initial condition

$$\rho = \begin{cases} \rho_m, & \text{if } L/4 \le x \le 3L/4. \\ 0, & \text{otherwise.} \end{cases} \tag{10}$$

with $L = 400m$, $v_m = 25m/s$, $\rho_m = 1$ arbitrary units and the number of grid points $NG = 40$. so that the spatial step length is $dx = L/NG$ and—using the CLF condition—the time step is $dt = dx/v_m$; since in the analytical solution, the speed of the wave is v_m. However, this initial condition makes the wave in the Lax-Wendroff solution suddenly stop. The reason for this is that the value for F in eq. 7 will become zero, and at this point the updated value for ρ will be the same as the previous, see eq. 7 and 8. I think this is because of the discontinuity of the derivatives of ρ at the points $x = L/4$ and $x = 3L/4$. To fix this I changed the initial conditions to

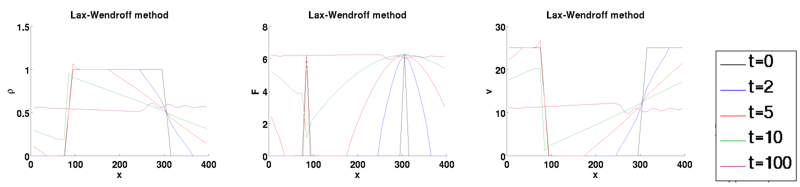
$$\rho = \begin{cases} \rho_m, & \text{if } L/4 \leq x \leq 3L/4. \\ \rho_m/2, & \text{if } x = L/4 - dx. \\ \rho_m/2, & \text{if } x = 3L/4 + dx. \\ 0, & \text{otherwise.} \end{cases} \tag{11}$$

The results for both schemes at different times can be seen in fig. 5 and 6. Comparing the figures for both methods the curves do not always look similar at similar times; also the curves looks more smooth for the Lax scheme compared to the Lax-Wendroff scheme. Otherwise they do evolve in the same manner looking at how the wave propagates, and it is similar to how the analytical solution evolve.



(a) (b) (c)

Figure 5: Results of Burger's equation using the Lax scheme. (a) Plot of $\rho(x)$ versus position for different times. (b) Plot of $F(\rho(x))$ versus position for different times. (c) Plot of $v(\rho(x))$ versus position for different times.



(a) (b) (c)

Figure 6: Results of Burger's equation using the Lax-Wendroff scheme. (a) Plot of $\rho(x)$ versus position for different times. (b) Plot of $F(\rho(x))$ versus position for different times. (c) Plot of $v(\rho(x))$ versus position for different times.

If increasing the time step dt and keeping dx fixed. For small enough times and not to large dt; the solution will no longer have this straight line between the last point where $\rho = \rho_m$ and $\rho = 0$ which is seen in the analytical solution, although otherwise evolving the right way. For larger dt and larger times—the solution will not converge. If decreasing the spatial step dx and keeping dt fixed. The errors are very similar to increasing dt and keeping dx fixed. Comparing the two schemes, the Lax scheme seems to handle this error better than the Lax-Wendroff scheme.

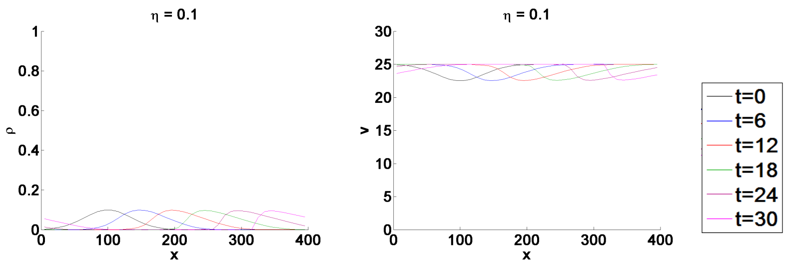
3 Exercise 3

In the previous exercise, the solution for Burger's equation could model the density of cars around a traffic light that turned green on $t = 0$. Another traffic model—but for high-way traffic flow in one lane where a slip road is feeding traffic into the lane—can be modeled using the initial condition

$$\rho(x, t = 0) = \eta \rho_m e^{-\left(\frac{x - L/4}{L/8}\right)^2}. \tag{12}$$

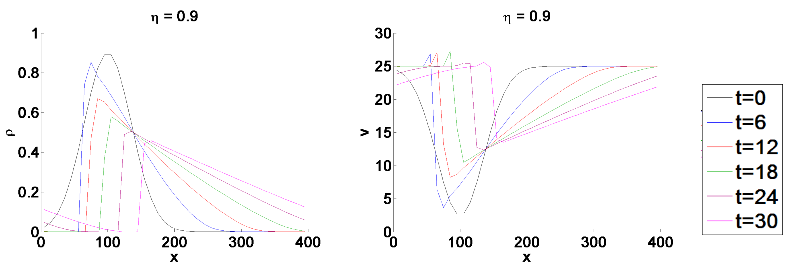
where $0 < \eta \leq 1$ is a constant

To investigate the solution of this model. The simulation will be run for 30 time steps (equivalent to 12 seconds); the same parameters values as in exercise 2 are used but with $\eta = 0.1$ as presented in 7, here the traffic flows rather smoothly. If we instead increase η to 0.9, adding higher density of traffic as presented in 8 we instead get congestion, increasing the density of traffic at one point at the same time decreasing the velocity at that point.



(a) (b)

Figure 7: The result of Burger's equation using initial condition $\rho(x, t = 0) = \eta\rho_m \exp(-((x - L/4)/(L/8))^2)$ with $\eta = 0.1$. In (a) is $\rho(x)$ for different times t and in (b) is $v(\rho(x))$ for different times t.



(a) (b)

Figure 8: The result of Burger's equation using initial condition $\rho(x, t = 0) = \eta\rho_m \exp(-((x - L/4)/(L/8))^2)$ with $\eta = 0.9$. In (a) is $\rho(x)$ for different times t and in (b) is $v(\rho(x))$ for different times t.