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Determination of the Damping of a Pendulum with Time of Flight

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Abstract

In this lab we measured the decay of energy in a pendulum, both from friction and air drag. We used a laser *time of flight* (ToF) method to determine the velocity–and thereby the energy—of a pendulum; in order to examine the decay of energy against time. We fitted the measured decay in energy to an exponential decay model and a novel model. The measurements with increased air drag qualitatively confirmed the theoretical expectation, however no quantitative statement could be made.

The measured data did not fit the prediction that a heavier pendulum is damped faster, so there was an unknown factor that we could not take into account. The unknown factor of the pendulum we believe was caused by the pendulum not being completely constrained to move around one axis. Without this issue, the loss of energy due to friction could be better characterized. We also didn't fit our model to a more complicated model that we derived.

Non-Invasive Measurement Techniques
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1 Introduction

The main issue in this lab is how to measure the velocity of a moving object in a non-invasive manner. This can be solved by using a laser *time of flight* (ToF) method where two light sensors are aimed at by two different laser beams which are separated by a known distance. When an object crosses these laser beams; light sensors will register these events. Knowing the difference in time between these events for both sensors and the spatial distance between them—you can calculate the velocity.

The accuracy of the laser ToF method is largely limited by the sensitivity of the light sensor and the sampling frequency of the *analog to digital converter* (ADC). It is also limited by how small the distance between the laser beams is, and by the calibration method that is used to determine this distance.

The goal of this lab was to measure the decay of energy of a pendulum using this laser ToF method. The pendulum has two sources of energy loss: friction and air drag. We examined the characteristics of both these contributions by a comparison of the measured data with a theoretical model.

2 Theory

2.1 Time of Flight Methods

In a time of flight measurement the time $\Delta t = t_2 - t_1$ is measured that an object needs to travel from one position x_1 to another position x_2. If the distance between the two positions $\Delta x = x_2 - x_1$ is known, the mean velocity between the two positions is

$$v_{mean} = \frac{\Delta x}{\Delta t}. \tag{1}$$

2.2 The Compound Pendulum

A compound pendulum consists of a rigid body with a moment of inertia I that is swinging around a pivot. Its movement can be completely described by considering the angle φ between the vector from the pivot to the center of mass $\vec{R}$ and the gravitational force $\vec{F}_g$ (in case a visualisation is needed see fig. 1 in the following section). The force $\vec{F}_g$ exerts a back driving torque $\tau = (\vec{R} \times \vec{F}_g)_z = -RF_g \sin\varphi$ on the body. For small angles φ one can approximate $sin\varphi \approx \varphi$. Hence the angular equation of motion is

$$I\ddot{\varphi} = -RF_g\varphi. \tag{2}$$

This is the equation for a simple harmonic oscillator with the angular frequency $\omega = \sqrt{\frac{RF_g}{I}}$ considering $F_g = Mg$ where M is the mass of the body and g is the gravitational constant. The solution to this equation is

$$\varphi(t) = \varphi_0 \cos(\omega t + \delta). \tag{3}$$

The assumption $\delta = 0 \Rightarrow \varphi(t = 0) = \varphi_0$ is made in the following discussion since the phase is not important in this context.

Now one can ask which energy is stored in such an oscillation. Assuming that at $\varphi = 0$ the energy is only kinetic energy all the energy of the system at turning point $\varphi = \varphi_0$ should only consist of potential energy in the gravitational field. This potential energy is equal to the negative of the work that has to be performed [1] against the back driving torque τ if one moves the pendulum from the position $\varphi = 0$ to $\varphi = \varphi_0$ on get

$$W = \int_0^{\varphi_0} \tau(\varphi) d\varphi = -\int_0^{\varphi_0} RF_g \varphi d\varphi = -\frac{1}{2} RF_g \varphi_0^2. \tag{4}$$

So the relation between the energy of the oscillation $E = -W$ and its amplitude φ_0 is

$$\varphi_0 = \sqrt{\frac{2E}{RF_g}}. \tag{5}$$

This can now be plugged into eq. (3). Due to the damping described in the following sections the amplitude will decay over time so that there is a time dependence $\varphi_0(t)$ and thus also the energy $E(t)$ will flow out of the system with time t. The assumption that the angular frequency ω stays constant during this process is empirically applicable for weakly damped oscillations; which gives that

$$\varphi(t) = \sqrt{\frac{2E(t)}{RF_g}} \cos(\omega t). \tag{6}$$

If one differentiates this term it can be assumed that $E(t)$ varies much slower with time than $\cos(\omega t)$ which corresponds to the observation that φ changes rapidly from $-\varphi_0$ to $+\varphi_0$ due to the oscillation with ω whereas the decrease $\varphi(t) > \varphi(t+T)$ is much smaller. So one can write that

$$\dot{\varphi}(t) \approx -\omega \sqrt{\frac{2E(t)}{RF_g}} \sin(\omega t). \tag{7}$$

2.3 The damping torque τ_f due to friction at the suspension point

At the surface of the rod where the pendulum is suspended a friction force F_f occurs during the oscillation. In Coulomb's model of friction [2] the friction force is considered as being proportional to the normal force F_N that keeps the pendulum on its track. Since the pendulum is continuously moving, the proportionality constant is given by the coefficient of kinetic friction μ_k that depends on the used materials[1]. So one gets the equation

$$F_f = \mu_k F_N. \tag{8}$$

Regarding fig. 1 it becomes clear that the normal force consists of two parts $F_N = F_{N1} + F_{N2}$ where:

[1] Actually the pendulum stops in the turning points for a very short time. But as long as the coefficient of static friction μ_s is not much larger than μ_k this should not have a visible effect. For most materials μ_s and μ_k are in the same order of magnitude.

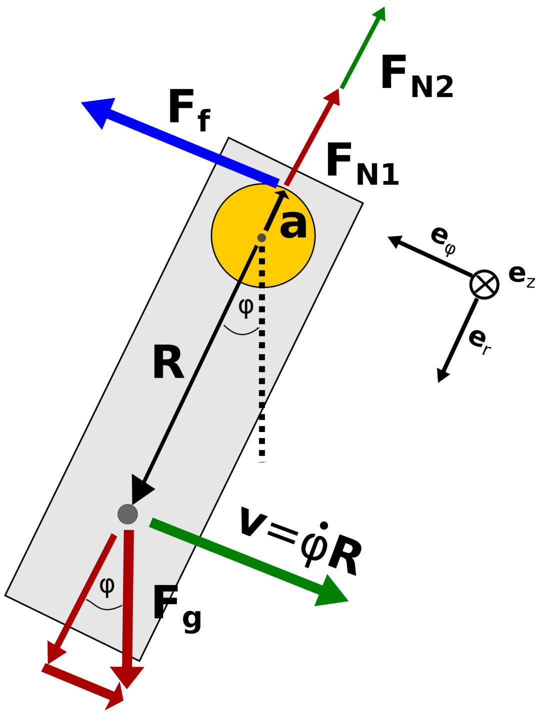

Figure 1: Geometry of the pendulum and forces acting during the oscillation with corresponding cylindrical coordinate system. (Vectors are bold)

- The force $F_{N1} = F_g \cos\varphi$ counters the component of the gravitational force that is acting in the direction of $\vec{R}$.
- The force $F_{N2} = M\dot{\varphi}^2 R$ acts as a centripetal force to keep the pendulum on its circular path. [3]

If a is the radius of the rod where the pendulum is fixed (see again figure 1), then the resulting torque due to the friction force is

$$\tau_f = aF_f = a\mu_k M\left(g\cos\varphi + \dot{\varphi}^2 R\right) \approx a\mu_k M\left(g - \frac{g}{2}\varphi^2 + \dot{\varphi}^2 R\right). \tag{9}$$

2.4 The damping torque τ_d due to the air drag

Since the pendulum is moving quite fast and its shape is approximately circular so the air flowing around it should give rise to a lot of curls. In this case one can apply the drag equation [4]

$$F_d = \frac{1}{2}\rho v^2 C_D A. \tag{10}$$

In this equation the following parameters occur:

- ρ is the density of the fluid (here air).

- v is the velocity of the floating object.
- A is the cross-sectional area of the object.
- c_d is the *air drag coefficient* that depends on the shape of the object.

However, this equation cannot be applied directly to the pendulum since each infinitesimal cross-sectional area element $dA = drdz$, see fig. 1, of it moves with a different velocity $v = \dot{\varphi} r$ depending on the distance r from the pivot. So the infinitesimal force acting on dA is given by $dF_d = \frac{1}{2}\rho C_d \dot{\phi}^2 r^2 drdz$. It gives rise to an infinitesimal torque $d\tau = rdF_d$.

If L is the length of the pendulum ($\neq R$) and d is its width the total torque due to the drag force is

$$\tau_d = \int d\tau_d = \frac{1}{2}\rho\dot{\phi}^2 C_d \int_0^L r^3 dr \int_0^d dz = \frac{1}{8}\rho\dot{\varphi}^2 C_d L^4 d. \tag{11}$$

2.5 Determining the Time dependence of Energy $E(t)$

Due to the frictional torques τ_D and τ_f derived in the sections above (eq. (9) and (11)) the energy of the oscillating System decreases. The ratio of this decrease dE/dt is equal to the power $P = (\tau_D + \tau_f)\,\dot{\varphi}$ that the torques take out of the system. At this point it is crucial to consider that the equations 9 and 11 only contain information about the absolute value of the torques. Since these torques describe friction, they should always act against the direction of $\dot{\varphi}$. So one has to add this information to get the correct result

$$P = -\text{sign}\left(\dot{\varphi}\right)\left(\tau_{\text{D}} + \tau_{\text{f}}\right)\dot{\varphi} = -\left(\tau_{\text{D}} + \tau_{\text{f}}\right)\left|\dot{\varphi}\right|. \tag{12}$$

Now one can plug in τ_D and τ_f respectively. One gets the differential equation

$$\frac{dE}{dt} = -\left(\frac{1}{8}\rho\dot{\varphi}^2 C_D L^4 d + a\mu_k M\left(g - \frac{g}{2}\varphi^2 + \dot{\varphi}^2 R\right)\right)\left|\dot{\varphi}\right|. \tag{13}$$

Defining the constants

$$A_1 = \omega^3\left(\frac{2}{RF_g}\right)^{3/2}\left(\frac{1}{8}\rho C_D L^4 d + a\mu_k M R\right)$$
$$A_2 = \omega\left(\frac{2}{RF_g}\right)^{3/2} a\mu_k M\frac{g}{2}$$
$$A_3 = \omega\left(\frac{2}{RF_g}\right)^{1/2} a\mu_k M g$$

and plugging in eq. (7) and (6) one gets

$$\frac{dE}{dt} = -A_1 E^{\frac{3}{2}}\left|\cos^3(\omega t)\right| - A_2 E^{\frac{3}{2}}\left|\cos(\omega t)\right|\sin^2(\omega t) - A_3 E^{\frac{1}{2}}\left|\cos(\omega t)\right|. \tag{14}$$

This equation cannot be solved analytically but the program Mathematica can solve it numerically. In fig. 2 the result is visible. The right graph was calculated with with a higher angular frequency omega than the left one.

```
In[93]:= s1[omega_] :=
  NDSolve[{En'[t] == -En[t]^(3/2) Abs[Cos[ω*t]^3] - 1/ω^2 En[t]^(3/2) Abs[Cos[ω*t]] Sin[ω*t]^2 - 1/ω^2 En[t]^(1/2) Abs[Cos[ω*t]] /.
     {ω → omega}, En[0] == 10}, En, {t, 0, 10}];
 GraphicsRow[{Plot[Evaluate[En[t] /. s1[3]], {t, 0, 10}, PlotRange → All], Plot[Evaluate[En[t] /. s1[10]], {t, 0, 10}, PlotRange → All]}]
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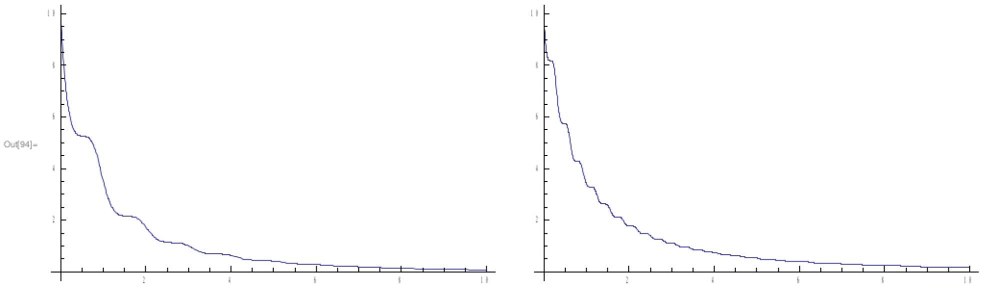


Figure 2: Equation 14 solved numerically by Mathematica for two different values of ω.

One can observe that the irregularities seem to vanish with a higher value for ω. This is understandable from the fact that oscillations with high frequency do only produce very small fluctuations in the solution that are not visible. To get the solution without fluctuations one could take the mean value over time $< |\cos(\omega t)| >=< |\sin(\omega t)| >= \frac{1}{2}$. Defining new constants $B_1 = \frac{1}{8}(A_1 + A_2)$ and $B_2 = \frac{1}{2}A_3$; then eq. (14) becomes

$$\frac{dE}{dt} = -B_1 E^{\frac{3}{2}} - B_2 E^{\frac{1}{2}} \tag{15}$$

and have the analytical solution

$$E(t) = \frac{B_2}{B_1} \tan^2\left(\frac{\sqrt{B_1 B_2}}{2} t + C\right). \tag{16}$$

This analytical function can be used later to fit the measured values.

In case that the damping due to air drag is much bigger than the damping due to friction we can assume that $B_1 \gg B_2$. So eq. (15) simplifies to

$$\frac{dE}{dt} = -B_1 E^{\frac{3}{2}} \tag{17}$$

and has the solution

$$E(t) = \frac{4}{(B_1 t + C)^2}. \tag{18}$$

3 Experimental

The experimental set-up consists of the following components:

- laser.
- focusing lens.

- beam splitter.
- pendulum.
- two photodiodes.
- circuit, ADC, PC.

All these items are arranged according to the schematic in fig. 3 and fig. 4 shows the real system set-up. The two laser beams near the pendulum are covered

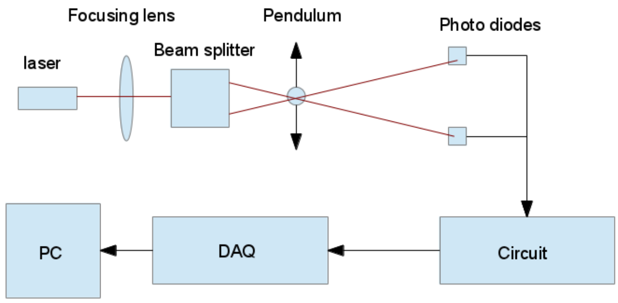


Figure 3: Schematic diagram of the experimental set-up. The laser is focused onto the pendulum by the focusing lens. Between focusing lens and the pendulum there is a beam splitter that splits the beam in to two beams that at the pendulum position are very narrowly separated. The photo diodes converts the beam into an electrical signal that goes in to the PC through an ADC.

by the pendulum at the different times meaning that there will be valleys in the output signal from the photo diodes at different times. The velocity of the pendulum will be inversely proportional to the time difference of these two valleys measured by the photo diodes.

The two laser beams are directed at the point where the pendulum has its lowest potential energy or highest kinetic energy; and to get a good value for this velocity the two beams are at this point only separated by a distance of order of a millimetre in length. Every time the pendulum swings by this point its total energy—which is the same as its kinetic energy—can be calculated and it will decay over time because of energy loss due to air drag and friction.

Since the separation of the two beams is very small we need a high sampling frequency to measure the velocity accurately; we used a sampling frequency of 100 kHz which means we can measure velocities in the order of $\approx 1\ \text{mm} \cdot 100\ \text{kHz} = 100\text{m/s}$. But since there is always some noise in the system the velocities which you can distinguish between is lower than this.

To see how the decay in energy of the pendulum depends on the friction, more mass can be added to the pendulum nearly without affecting the air drag, compare fig. 5a and fig. 5b. Similarly to examine the energy loss from air drag we changed the geometry of the pendulum, compare fig. 5a and fig. 5c.

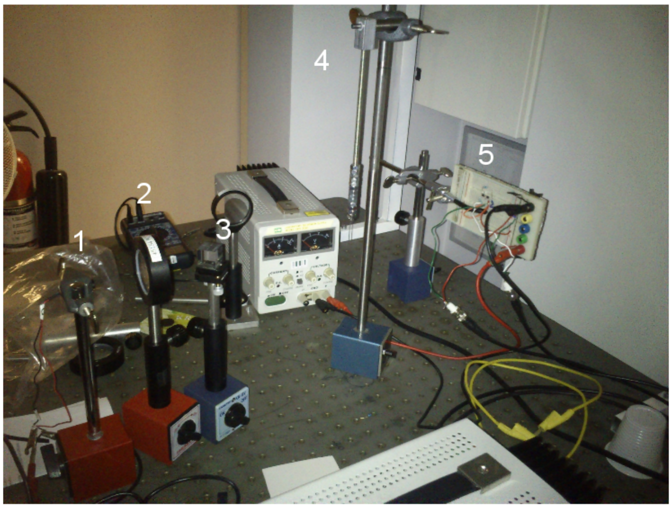


Figure 4: Real experimental setup. 1. Laser, 2. focusing lens, 3. beam splitter, 4. pendulum, 5. photo diodes with circuit.

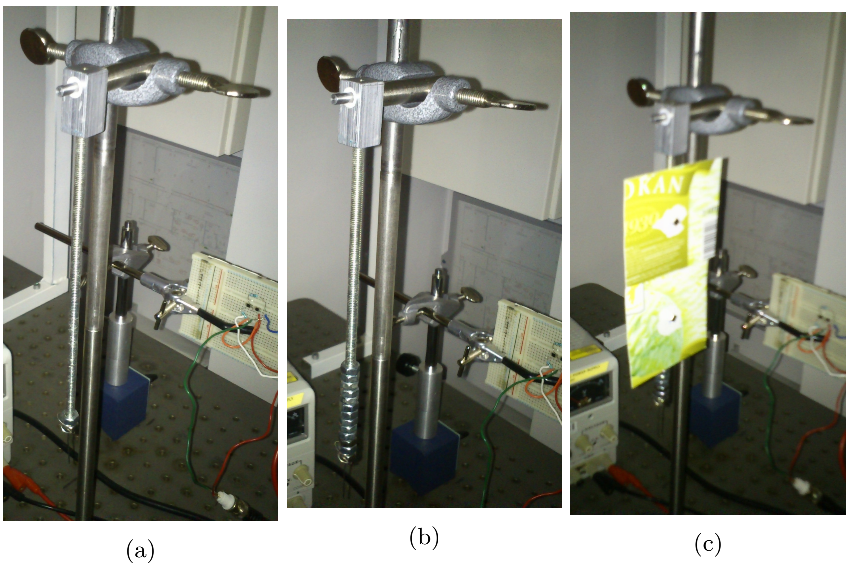
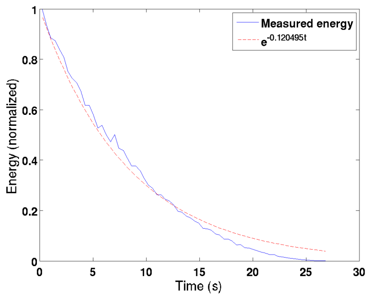


Figure 5: (a) Normal weight and geometry of the pendulum. (b) Pendulum with added mass in the form of screw nuts. (c) Pendulum with increased air drag due to changed geometry.

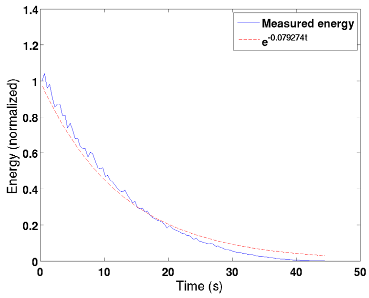
4 Results

4.1 Data Fit to Model of Damped Harmonic Oscillator

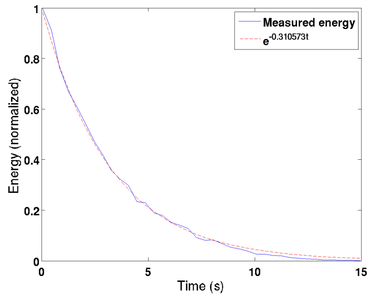
We tried to fit the data obtained from measurements of the *normal* pendulum, the *heavy* pendulum and the the pendulum with *more air drag*. The result can be seen below in fig. 6.



(a)



(b)



(c)

Figure 6: Exponential decay fit to (a) normal weight and geometry of the pendulum. (b) pendulum with added mass in the form of screw nuts. (c) pendulum with increased air drag due to changed geometry.

4.2 Data Fit to Novel Model Derived in Theory.

We then used eq. (15) and fit the data obtained from measurements of the *normal* pendulum, the *heavy* pendulum and the the pendulum with *more air drag*. The results can be seen in fig. 7.

4.3 Analysis of Air Drag

Examining the pendulum with *more air drag* we can assume that the friction due to the movement is neglectable. As in the theory suggested we can then apply fit function $E(t) = 4/\left(at+b\right)^2$ (eq. (18)) on the data. As already mentioned in section 3 three measurements with two different pieces of cardboard where made. The results can be seen in fig. 8.

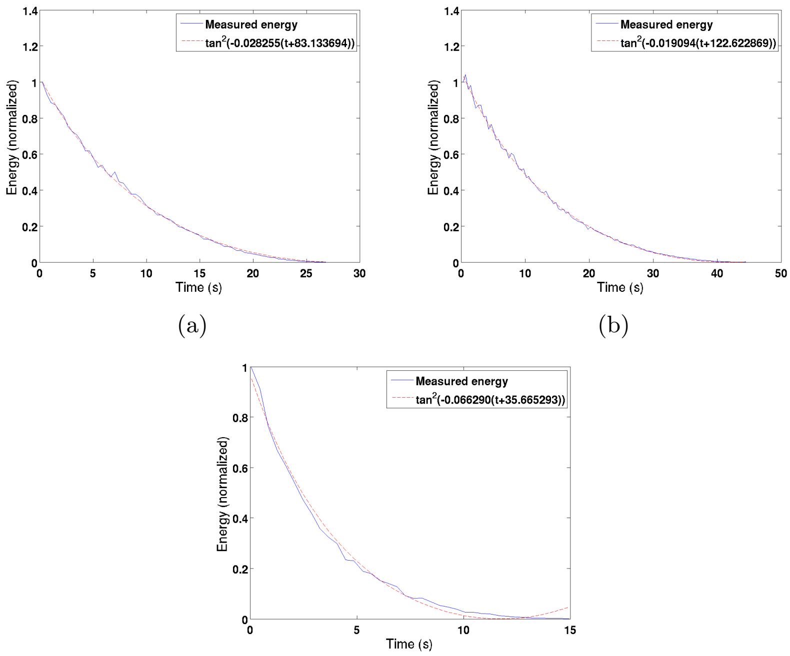
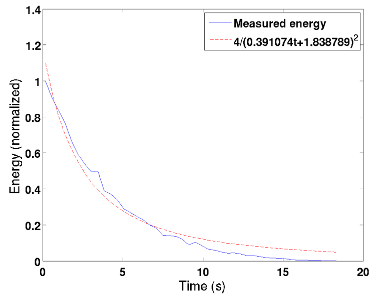
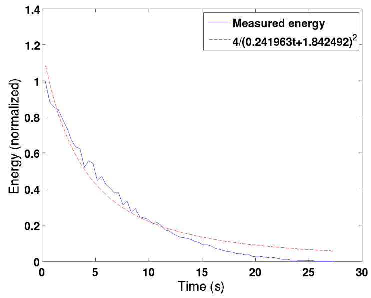


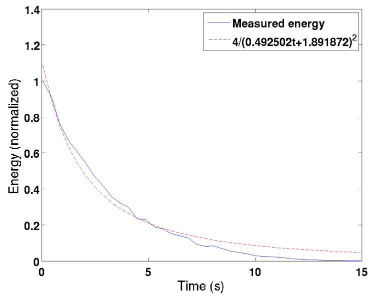
Figure 7: Exponential decay fit to (a) normal weight and geometry of the pendulum. (b) pendulum with added mass in the form of screw nuts. (c) pendulum with increased air drag due to changed geometry.



(a)



(b)



(c)

Figure 8: Model fit to (a) geometry 1. (b) geometry 2. (c) geometry 1+2.

From this figure we get

$$a_{1+2} = 0.49 \tag{19}$$
$$a_1 = 0.39 \tag{20}$$
$$a_2 = 0.24 \tag{21}$$

As we can learn from the derivation of eq. (11) in the theory it is not only important how large the cardboard is but also where it is fixed. So we have to consider which geometry was chosen during the measurement. This can be seen fig. 9. The position of the cardboards was never changed, the were just fixed and unfixed. The results confirm qualitatively the expectation from the theory on the first view: The largest damping is of course observed in the measurement with the biggest cross section (1+2). Since $v = \dot{\varphi} r$ we observe in measurement 1 a higher damping than in measurement 2.

It is now interesting if we can compare the measurements also quantitatively with the theory. In equation 11 we would have to change the integration boundaries to consider the geometry in the right way. (Now we will just neglect the part of the pendulum that is not covered with cardboard.) So the theory suggests that:

$$a_{1+2} \propto \left[(r_0 + L_2 + L_1)^4 - r_0^4 \right] a_1 \propto \left[(r_0 + L_2 + L_1)^4 - (r_0 + L_2)^4 \right] a_2 \propto \left[(r_0 + L_2)^4 - r_0^4 \right] \tag{22}$$

Since no other parameters were modified the proportionality constants should be the same. So they can be determined by dividing the left-hand-sides by the

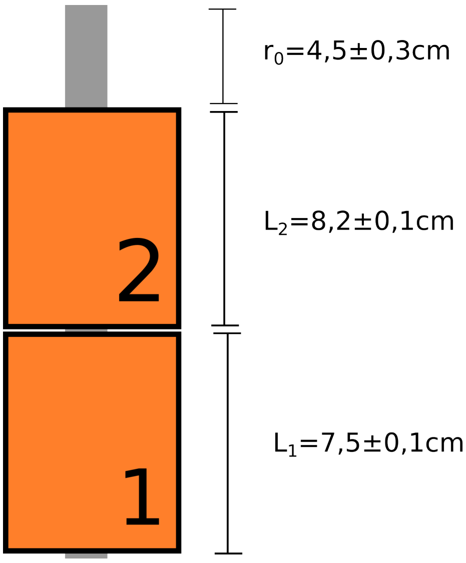


Figure 9: Illustration of how cardboard were added to the pendulum to increase air drag.

right-hand-sides of the equations above. One gets we the values of fig ?? and the values form the fit-curves:

$$C_{1+2} = 2,96 \cdot 10^{-6} C_1 = 2,78 \cdot 10^{-6} C_2 = 9,45 \cdot 10^{-6} \tag{23}$$

This does not look like the theory fits to the measured values. More about this in the discussion.

4.4 calibration of distance between laser beams

Through the use of simple trigonometry and fig. 10 we get the relation for the distance d between the beams at the point where the pendulum swings by to be

$$d = \frac{aB - A(D - B)}{D} \tag{24}$$

For the measured distances and error estimates given in fig. 10 we get that $d = 1.2689 \pm 1.3019$mm. The error estimate was calculated using *Gauss formula for uncertainty propagation.*

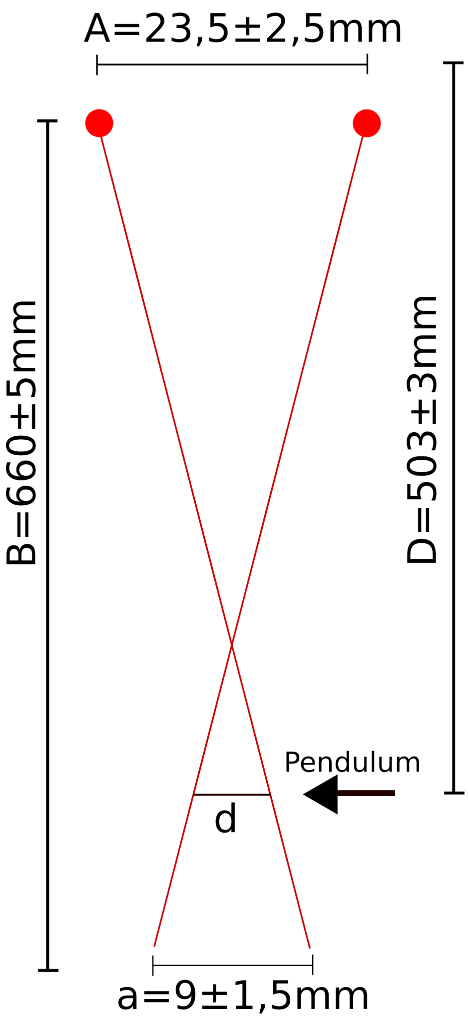


Figure 10: A is the distance between the photo diodes. B is the distance between the beam splitter and the photo diodes. D is the distance between the pendulum and the photo diodes. a is distance between the laser beams where they originate from and d is the unknown separation between the laser beams where the pendulum crosses.

5 Discussion

The results of sections 4.2 and 4.1 show that the new model derived in the theory fits much better to the measured values than the simple model of the damped harmonic oscillator whose amplitude decays exponentially. Comparing fig. 6 and 7 it is visible that the deviations are less particularly for large values of t. However, it was not possible to get quantitative results from the theory neither for the pendulum *with more air drag* nor for the *heavy* pendulum.

An increase of mass should theoretically increase the friction if all other parameters are unchanged[2]. But this could not be observed. There was even a tendency that the damping decreased with an increase of mass. The reason for this could be that the pendulum motion is not entirely constrained to be around the joint axis (z-axis in fig. 1) but it wobbles along this direction as well. This may cause an energy loss that is much higher that the ones describes in the theory. Adding mass straightens the plane of oscillation so that the wobbling becomes fewer and thus the energy loss decreases.

As shown in section 4.3 the results for the pendulum *with more air drag* are qualitatively good but again we cannot make a quantitative statement because the measured damping does not change as expected with the change of the cross sectional area. In the case the deviation might be caused by the theoretical assumptions. For the air drag eq. (10) is only applicable for a rigid body moving through a fluid. The idea to apply it on infinitesimal area elements dA was therefore comparable to dividing the cardboard in many rigid bodies. This might not have been a good assumption if one takes into consideration that for the total air flow always the 3-dimensional shape of the whole body matters. The fluid dynamics of the pendulum might be much more complex than the theory takes into consideration.

The calibration that was made in section 4.4 did not give a satisfying result. The uncertainty is even higher than the value itself. This is obvious regarding fig. 10 since a small change in A produces a large change in d comparable with the behaviour of a scissor. Another calibration method would be to move the pendulum to $\varphi = \pi/2$ and then calculate its initial energy as $E_{init} = RMg$. The first measured energy could then be associated with the initial energy since the loss for a quarter period is neglectable. Due to time constraints we did not apply this method.

6 Conclusions

Eq. (15) and (18) worked very well as a model for the damping of the pendulum, but we also believe that the numerical solution of eq. (14) would fit the data even better. However due to time and knowledge constraints we did not try to fit our data to this model.

To eliminate the unknown factor of the pendulum we could simply use a better pendulum that is more constrained to movement around one axis.

[2]To be sure that R is unchanged we fixed the screws around the center of mass of the pendulum.

References

[1] Torque, http://en.wikipedia.org/wiki/Torque, (December 13, 2013).

[2] Dry friction, http://en.wikipedia.org/wiki/Friction#Dry_friction, (December 13, 2013).

[3] Centripetal force, http://en.wikipedia.org/wiki/Centripetal_force, (December 13, 2013).

[4] Air drag, http://en.wikipedia.org/wiki/Air_drag, (December 13, 2013).