Synthetic Volatility Forecasting and Other Aggregation Techniques for Time Series Forecasting

Preliminary Exam

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A seemingly unprecedented event might provoke the questions

1. What does it resemble from the past?
2. What past events are most relevant?
3. Can we incorporate past events in a systematic, principled manner?

When would we ever have to do this?

- Event-driven investing strategies (unscheduled news shock)
- Structural shock to macroeconomic conditions (scheduled news possibly pre-empted by news shock)
- Biomedical panel data subject to exogenous shock or interference

Example: weekend of March 7th and 8th, 2020

Punchline of the paper

Forecasting is possible under structural shocks, so long as we incorporate external information to account for the nonzero errors.

Outline

Setting for the problem

- $y \in \mathbb{R}^n$, a mean-zero, real-valued response to be predicted
- Mean-zero covariate vectors $x \in \mathbb{H}$, a Hilbert space, where $\mathbb{H}$ can be taken to be $\mathbb{R}^p$ for the sake of illustration
- We predict y using a linear function of the covariates X, but this is *not* an assumption about the data generating process. We do not assume $y = f(X) + \epsilon$, but y may contain exogenous noise (called "label noise" by the authors.)

Technical Specifications (many, but familiar)

- We define $\Sigma := \mathbb{E}[xx^T]$
- We define $\theta^* \in \mathbb{H}$ to be s.t. $\mathbb{E}[(y - x^T\theta^*)^2] = \min_\theta \mathbb{E}[(y - x^T\theta)^2]$
  - As defined here, not necessarily unique, which matters for this paper.
- $x = V\Lambda^{1/2}z$, where $\Sigma = V\Lambda V^T$ is the spectral decomposition of Σ and z is a vector of independent components, each subgaussian(σ_x), where $\sigma_x > 0$.
- The conditional noise variance $\mathbb{E}[(y - x^T\theta^*)^2|x]$ is bounded below by $\sigma^2 > 0$.
- $(y - x^T\theta^*)|x$ is subgaussian(σ_y)
- Almost surely, for any eigenvector v of Σ, $Proj_{v^T}(X)$ spans a space of dimension n.
  - Guaranteed when, for example, there exist p linearly independent covariates and $p > n$.

What's the goal here?

In this particular setting, excess risk of an estimator θ has the form

$$\begin{aligned} R(\theta) &= \mathbb{E}_{x,y}[(y - x^T\theta)^2 - (y - x^T\theta^*)^2] \\ &= (\theta - \theta^*)^T \Sigma (\theta - \theta^*) \end{aligned}$$

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$$\min_{\theta \in \mathbb{H}} \|\theta\|$$
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2. Of course, we're concerned with interpolation, i.e. $D = 0$.

Minimum Norm Estimator

Why did we just assume something very technical about $Proj_{v^T}(X)$? This condition implies multiple solutions to the equation $y = X\theta$.

"Almost surely, for any eigenvector v of Σ, $Proj_{v^T}(X)$ spans a space of dimension n."

Since we have more than one choice of θ, we choose the unique $\hat{\theta}$ with minimum norm:

$$\hat{\theta} := (X^T X)^\dagger X^T y$$

We don't have to do this, but the results that follow correspond to the minimum-norm solution. It's most interesting.

Rank of Matrix

We know the rank of a matrix $A \in M_{n \times p}(\mathbb{C})$ is

- The column rank of A (number of linearly independent columns)
- The row rank of A (number of linearly independent rows)
- The dimension of $im(A)$

However, this notion of rank is too rigid. It's integer-valued, and it tells us very little about the distribution of the eigenvalues.

Key Conceptual Innovation: Effective Rank

Definition

For a covariance operator Σ with the decreasing sequence of eigenvalues $\lambda_1, \lambda_2, \ldots$, if $\sum_{i=1}^{\infty} \lambda_i < \infty$ and $\lambda_{k+1} > 0$, then for $k \geq 0$, define

$$r_k = \frac{\sum_{i>k} \lambda_i}{\lambda_{k+1}} \quad R_k = \frac{(\sum_{i>k} \lambda_i)^2}{\sum_{i>k} \lambda_i^2}$$

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4. $r_k(\Sigma^2) \leq r_k(\Sigma) \leq R_k(\Sigma) \leq r_k^2(\Sigma)$
5. For the result we now show, *bigger* values of r_k and R_k are better.

Main Result: Existence Proof, Dichotomy, and Bounds

Theorem

Define $k^* = min\{k \geq 0 : r_k(\Sigma) \geq bn\}$. *For any* σ_x, $\exists b, c, c_1 > 1$ *s.t.* $\forall \delta \in (0,1)$ *s.t.* $\log(1/\delta) < n/c$, *if* $k^* \geq n/c_1$, *then* $\mathbb{E}[R(\hat{\theta})] \geq \sigma^2/c$. *Otherwise*,

- **A** $$R(\hat{\theta}) \leq \underbrace{c\left(\|\theta^*\|^2\|\Sigma\| max\{\sqrt{\frac{r_0(\Sigma)}{n}}, \frac{r_0(\Sigma)}{n}, \sqrt{\frac{\log(1/\delta)}{n}}\}\right)}_{\text{related to the bias}} + \underbrace{c\log(1/\delta)\sigma_y^2\left(\frac{k^*}{n} + \frac{n}{R_{k^*}(\Sigma)}\right)}_{\text{related to the noise}}$$ *with probability at least* $1-\delta$.
- **B** $\mathbb{E}[R(\hat{\theta})] \geq \frac{\sigma^2}{c}\left(\frac{k^*}{n} + \frac{n}{R_{k^*}(\Sigma)}\right)$

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5. Effective rank encodes how far the vectors x are from isotropy. Isotropy implies maximum effective rank, whereas small values of effective rank suggests (just like ordinary matrix rank) that many of the vectors generating Σ are irrelevant to the variation Σ houses.

So what do we want our eigenvalues to look like?

$\Sigma = V\Lambda V^T =$

$$V \begin{pmatrix} \lambda_1 & & & & \\ & \lambda_2 & & \text{\huge 0} & \\ & & \lambda_3 & & \\ & \text{\huge 0} & & \ddots & \\ & & & & \lambda_p \end{pmatrix} V^T$$

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4. The number of non-zero eigenvalues should be large compared to n.

Two Very Simple Examples

Consider $\Sigma = I_p$ (which is induced by isotropy):

$$r_0(\Sigma) = \frac{\sum_{i=1}^{p} \lambda_i}{\lambda_1} = \frac{p}{1} = p = \frac{p^2}{p} = R_0(\Sigma)$$

Next consider infinite-dimensional Σ with spectral decomposition

$$V \begin{pmatrix} \lambda_1 & & & & \\ & \frac{1}{2} & & \mathbf{0} & \\ & & \frac{1}{4} & & \\ & \mathbf{0} & & \frac{1}{8} & \\ & & & & \ddots \end{pmatrix} V^T$$

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Neither!

A more benign example

Example (Coverging at the slowest rate possible)

Fix $\alpha = 1, \beta > 1$. Let $\lambda_i = \frac{1}{i \log^{\beta}(i+1)}$.

What's the point of all this?

- In training, we observe both X and y, but not the noise. We then derive $\hat{\theta}$, a *noisy, imperfect* guess at θ^*.
- In prediction, we do not observe y or the noise; we're given x; we're hostage to the random object $\hat{\theta}$ we've just fit.

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  - $\hat{\theta}$ can be bad by being biased
  - $\hat{\theta}$, as a random object, can be bad by being a noisy stand-in for θ^*

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The Magic

- Not all of the coordinates (directions) in $\hat{\theta}$ matter for prediction.
- In fact, some are nearly irrelevant for prediction.
- Key insight: if there are enough of these unimportant directions, they can store the 'badness' of $\hat{\theta}$ that has been induced by the label noise.
  - $\hat{\theta}$ can be bad by being biased
  - $\hat{\theta}$, as a random object, can be bad by being a noisy stand-in for θ^*

So let's examine the punchline: how exactly can noise by hidden in unimportant directions?

How noise is hidden just right

Recall

$$\hat{\theta} := (X^T X)^\dagger X^T y = (X^T X)^\dagger X^T (\epsilon + f(X)) = (X^T X)^\dagger X^T (noise + signal)$$

Zero-in on the left action of the operator X^T. In any direction $i, 1 \leq i \leq p$, it scales the noise from ϵ by $n\lambda_i$.

Ultimately, for any direction $i, 1 \leq i \leq p$, we can bound the prediction error in direction i by $\frac{n\lambda_i^2}{(\sum_{i>k} \lambda_i)^2}$. If we sum these terms, what do we get?

After all of this waiting, we formalize the notion under discussion.

Definition (Asymptotically Benign)

We say that Σ_n is asymptotically benign iff

$$\lim_{n\to\infty} \left(bias(\theta^*, \Sigma_n, n) + \frac{k_n^*}{n} + \frac{n}{R_{k_n^*}(\Sigma_n)} \right) = 0$$

Some things to think about with papers like this

- Has little practical value, at present. It's a conceptual piece.
- The interpretations that the authors give the phenomenon are a little fuzzy. There's a gap between what the math says and what the authors say it does.
- Theorems change between the Arxiv paper, PNAS paper, early presentations, and later presentations!

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We analyze the real-world example with Brexit included.

Bibliography