HW1

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Exercise 1

Since P is a projector, $P^2 = P$, then

$$||P||_2 = ||P^2||_2 \leq ||P||_2^2$$

Therefore,$||P||_2 \geq 1$ Now, going to show that equality holds iff P is an orthogonal projector.

If P is an orthogonal projector, $P' = P$ and suppose P has SVD $P = U\Sigma V'$

$$\sigma = ||P||_2 = ||P^2||_2 = ||PP'||_2 = ||\Sigma^2||_2 = \sigma^2$$

where σ is the largest eigen value of P. Therefore, $\sigma = 1$

Suppose P is not orthogonal. Then exist a, $a \notin range(P)$ and $a \perp range(I - P)$

$$||Pa||_2 = ||a + (P - I)a||_2 > ||a||_2$$

Then

$$||P||_2 = \sup_{||a||_2=1} ||Pa||_2 > \sup_{||a||_2=1} ||a||_2 = 1$$

By contradiction P is orthogonal.

Exercise 2

Taking derivative wrt x, we got the normal equation

$$A'Ax = A'b \Rightarrow A'r = 0$$

Therefore the problem of minimization is equavalent to

$$
\begin{aligned}
A'r &= 0 \\
Ax + r &= b
\end{aligned}
$$

$$\min_x\{\frac{1}{2}||Ax-b||^2 + \frac{1}{2}\delta^2||x||^2 + c'x\} = \min_x\{\frac{1}{2}(Ax-b)'(Ax-b) + \frac{1}{2}\delta^2 x'x + c'x\}$$

Taking derivative wrt x, and set it to 0:

$$A'Ax - A'b + \delta^2 x + c = 0$$

Therefore the problem is equivalent to

$$
\begin{aligned}
A'r + \delta^2 x &= -c \\
Ax + r &= b
\end{aligned}
$$

$$\begin{pmatrix} I & A \\ A^T & \delta^2 I \end{pmatrix} \begin{pmatrix} r \\ x \end{pmatrix} = \begin{pmatrix} b \\ -c \end{pmatrix}$$

Exercise 3

Suppose that $m > n$ and A has full rank so that A^TA is invertible. $A = U\Sigma V^T$ is the full SVD, ie, U is $m \times m$ square matrix and V is $n \times n$ square matrix. $U^TU = UU^T = I_m$, $V^TV = VV^T = I_n$.And $\Sigma = \begin{pmatrix} L \\ 0 \end{pmatrix}$, where $L = diag(\lambda_1, \cdots, \lambda_n)$, λ_i are singular values of A

1

$$
\begin{aligned}
(A^TA)^{-1} &= (V\Sigma^T U^T U \Sigma V^T)^{-1} \\
&= (V\Sigma^T\Sigma V^T)^{-1} \\
&= (VL^2V^T)^{-1} \\
&= VL^{-2}V^T
\end{aligned}
$$

$L^{-2} = diag(\lambda_1^{-2}, \cdots, \lambda_n^{-2})$

2

$$\begin{aligned}(A^TA)^{-1}A^T &= VL^{-2}V^TV\Sigma^TU^T \\ &= VL^{-2}\Sigma^TU^T \\ &= V\begin{pmatrix} L^{-1} & 0 \end{pmatrix}U^T\end{aligned}$$

3

$$\begin{aligned}A(A^TA)^{-1} &= U\Sigma V^TVL^{-2}V^T \\ &= U\Sigma L^{-2}V^T \\ &= U\begin{pmatrix} L^{-1} \\ 0 \end{pmatrix}V^T\end{aligned}$$

4

$$\begin{aligned}A(A^TA)^{-1}A^T &= U\begin{pmatrix} L^{-1} \\ 0 \end{pmatrix}V^TV\Sigma^TU^T \\ &= U\begin{pmatrix} L^{-1} \\ 0 \end{pmatrix}\begin{pmatrix} L & 0 \end{pmatrix}U^T \\ &= U\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}U^T\end{aligned}$$

Exercise 4

Since $rank(A) = m$ then $dim(range(A)) = m$, this implies $dim(null(A)) = n - m$. So the dimension if solution space is n-m.

Suppose x_0 is a solution of $Ax = b$, and $x \in null(A)$, clearly $x_0 + x$ is a solution of $Ax = b$. Then our problem becomes minimize $||x_0 + x||_2$. Let B be the basis of null(A). Then $x \in range(B)$, exist v, s.t $Bv = x$.

$$\begin{aligned}\min_x ||x_0 + x||_2 &= \min_v ||x_0 + Bv||_2 \\ &= \min_v ||x_0 + Bv||_2\end{aligned}$$

$$= \min_v ||Bv - (-x_0)||_2$$

This is of same form of regular least square problem in overdetermind system. Then we can easily get the solution like overdetermind system.

normal equation

$$B'Bv = -B'x_0$$

QR

Assume $B = QR$, then

$$Rv = -Q^T x_0$$

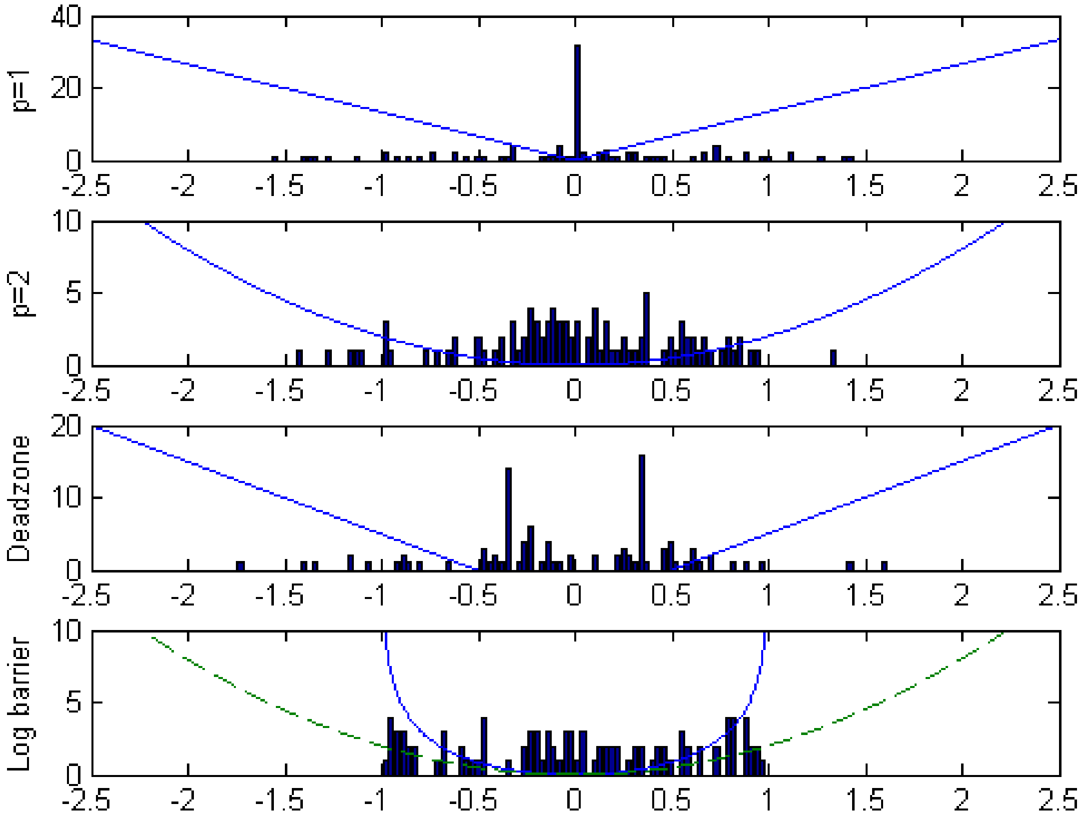
SVD

Assume $B = U\Sigma V^T$ is the full singular value decomposition

$$\Sigma V^T v = -U^T x_0$$

Exercise 5

Declaration of reference: I'm new to MATLAB. The code of log-barrier penalty and generating plots consult `http://cvxr.com/cvx/examples/cvxbook/Ch06_approx_fitting/html/penalty_comp_cvx.html#source`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p=2
Deadzone
Log barrier
r