Syntax

Terms	$M ::= \ x \mid \lambda x.M \mid M_1\ M_2 \mid \#op(M) \mid \text{handle } M \text{ with } H \mid \lambda E^{\ell}$
Values	$v ::= \ \lambda x.M \mid \lambda E^{\ell} \mid \ell$
Handlers	$H ::= \ \{\text{return } x \mapsto M\} \mid \{op(x,k) \mapsto M\}\ ; H$
Evaluation contexts	$E ::= \ [\,] \mid v_1\ E \mid E\ M_2 \mid \#op(E) \mid \text{handle } E \text{ with } H$
Type variables	α, β, γ
Types	$A, B, C, D ::= \ \alpha \mid A \to B \mid \forall \alpha.A$
Typing contexts	$\Gamma ::= \emptyset \mid \Gamma, x : A \mid \Gamma, \alpha$
Runtime error	ε

Reduction rules

$\boxed{M_1 \rightsquigarrow M_2}$

$$(\lambda x.M)\, v\ /\sigma \rightsquigarrow M[v/x]\ /\sigma \qquad \text{R\_BETA}$$

$$\text{handle } v \text{ with } H\ /\sigma \rightsquigarrow M[v/x]\ \ /\sigma\ \ (\text{where}\{\text{ return } x \mapsto M\} \in H) \qquad \text{R\_RETURN}$$

$$\text{handle } E[\#op(v)] \text{ with } H\ /\sigma \rightsquigarrow M[v/x][\lambda E^{\ell}/k]\ \ /\sigma[\ell \mapsto \mathit{true}] \qquad \text{R\_HANDLE}$$

$$(\text{where } op \notin E,\ \{op(x,k) \mapsto M\} \in H, \text{ and } \lambda E^{\ell} = \lambda y.\text{ handle } E[y]\ \text{ with } H)$$

$$\left(\lambda E^{\ell}\right) v\ /\sigma[\ell \mapsto \mathit{true}] \rightsquigarrow\ \text{handle } E[v]\ \text{ with } H\ /\sigma[\ell \mapsto \mathit{false}] \qquad \text{R\_RESUME}$$

$$\left(\lambda E^{\ell}\right) v\ /\sigma[\ell \mapsto \mathit{false}] \rightsquigarrow \varepsilon \qquad \text{R\_ERROR}$$

Term typing

$\boxed{\Gamma \vdash M : A}$

$$\frac{x : A \in \Gamma}{\Gamma \vdash x : A}\ \text{T\_VAR} \qquad \frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x.M : A \to B}\ \text{T\_ABS} \qquad \frac{\Gamma, \alpha \vdash M : A}{\Gamma \vdash M : \forall \alpha.A}\ \text{T\_GEN}$$

$$\frac{\Gamma \vdash M_1 : A \to B \qquad \Gamma \vdash M_2 : A}{\Gamma \vdash M_1 M_2 : B}\ \text{T\_APP} \qquad \frac{ty(op) = \forall \alpha.A \to B \qquad \Gamma \vdash M : A[C/\alpha]}{\Gamma \vdash \#op(M) : B[C/\alpha]}\ \text{T\_OP}$$

$$\frac{\Gamma \vdash M : A \qquad \Gamma \vdash H : A \Rightarrow B}{\Gamma \vdash \text{handle } M \text{ with } H : B}\ \text{T\_HANDLE} \qquad \frac{\Gamma, x : A \vdash \text{handle } E[x] \text{ with } H : B}{\Gamma \vdash \lambda E^{\ell} : A \to B}\ \text{T\_EABS}$$

Progress

If $\Delta \vdash M : A$, then:

1. $M \longrightarrow M'$ for some M';
2. M is a value; or $M = E[\#op(v)]$ for some E, op, and v such that $op \notin E$.
3. $M \longrightarrow \varepsilon$

PROOF By induction on the typing derivation for M.
Case (T_Var): Contradictory.
Case (T_Abs): Obvious.
Case (T_Gen): By the IH.
Case (T_App):

$$\frac{\Gamma \vdash M_1 : A \to B \qquad \Gamma \vdash M_2 : A}{\Gamma \vdash M_1 M_2 : B} \text{ T\_App}$$

By case analysis on the behavior of M_1. We have four cases to consider by the IH.

Subcase $M_1 \longrightarrow M_1'$ for some M_1': We have $M \longrightarrow M_1' M_2$.

Subcase $M_1 = E_1[\#op(v)]$ for some E_1, op, and v such that op $\notin E_1$: We have $E = E_1 M_2$.

Subcase $M_1 \longrightarrow \varepsilon$: We have $M \longrightarrow \varepsilon$.

Subcase $M_1 = v_1$ for some v_1: By case analysis on the behavior of M_2 with the IH.

Subcase $M_2 \longrightarrow M_2'$ for some M_2': We have $M \longrightarrow v_1 M_2'$.

Subcase $M_2 = E_2[\#op(v)]$ for some E_2, op, and v such that op $\notin E_2$: We have $E = v_1 E_2$.

Subcase $M_2 \longrightarrow \varepsilon$: We have $M \longrightarrow \varepsilon$.

Subcase $M_2 = v_2$ for some v_2: we have two cases to consider.

Subcase $v_1 = \lambda E^\ell$: If $\ell \mapsto true$, then $M = (\lambda E^\ell)\, v_2 \longrightarrow$ handle $E[v_2]$ with $H\ /\sigma[\ell \mapsto false]$. Otherwise $M \longrightarrow \varepsilon$.

Subcase $v_1 = \lambda x.M'$: By (R_Beta), $M = (\lambda x.M')\, v_2 \longrightarrow M'[v_2/x]$.

Case (T_Op):

$$\frac{ty(op) = \forall \alpha.A \to B \qquad \Gamma \vdash M : A[C/\alpha]}{\Gamma \vdash \#op(M) : B[C/\alpha]} \text{ T\_Op}$$

Subcase $M \longrightarrow M'$ for some M': We have $\#op(M) \longrightarrow \#op(M')$.

Subcase $M = E'[\#\mathbf{op}'(v)]$ for some E', op$'$, and v such that op$' \notin E'$: We have $E = \#\mathrm{op}(E')$.

Subcase $M = v$ for some v: We have $E = []$.

Subcase $M \longrightarrow \varepsilon$: We have $\#op(M) \longrightarrow \varepsilon$.

Case (T_Handle):

$$\frac{\Gamma \vdash M : A \qquad \Gamma \vdash H : A \Rightarrow B}{\Gamma \vdash \text{handle } M \text{ with } H : B} \text{ T\_Handle}$$

Subcase $M \longrightarrow M'$ for some M': We have handle M with $H \longrightarrow$ handle M' with H.

Subcase $M = E'[\#\mathbf{op}'(v)]$ for some E', $\mathbf{op}'$, and v, such that $\mathbf{op}' \notin E'$: If handler H contains an operation clause $\mathrm{op}'(x, k) \rightarrow M'$, then we have handle M with $H \longrightarrow M'[v/x][\lambda y.\text{handle } E'[y] \text{ with } H/k]$. Otherwise, if H contains no operation clause for op', we have $E =$ handle E' with H.

Subcase $M = v$ for some v: By (R_Return).

Subcase $M \longrightarrow \varepsilon$: We have handle M with $H \longrightarrow \varepsilon$.

Case (T_Eabs): Obvious.