Syntax

Terms	$M ::= \ x \mid \lambda x.M \mid M_1\ M_2 \mid \#op(M) \mid \text{handle } M \text{ with } H \mid (\lambda E)^{\ell}$
Values	$v ::= \ \lambda x.M \mid (\lambda E)^{\ell} \mid \ell$
Handlers	$H ::= \ \{\text{return } x \mapsto M\} \mid \{op(x,k) \mapsto M\} \uplus H$
Evaluation contexts	$E ::= \ [\,] \mid v_1\ E \mid E\ M_2 \mid \#op(E) \mid \text{handle } E \text{ with } H$
Type variables	α, β, γ
Types	$A, B, C, D ::= \ \alpha \mid A \to B \mid \forall \alpha.A$
Typing contexts	$\Gamma ::= \emptyset \mid \Gamma, x : A \mid \Gamma, \alpha$
Runtime error	ε

Reduction rules

Because the first-class continuation $(\lambda E)^{\ell}$ has a memory location ℓ, it will be convenient to recast the small step operational semantics slightly, presenting it as a reduction system for R, defined as follows:
$R ::= M/\sigma \mid \varepsilon$, where M is a term and σ is a store, mapping finitely memory locations to values.

$\boxed{R_1 \rightsquigarrow R_2}$

$$(\lambda x.M)\, v\ /\sigma \rightsquigarrow M[v/x]\ /\sigma \qquad \text{R\_Beta}$$

$$\text{handle } v \text{ with } H\ /\sigma \rightsquigarrow M[v/x]\ \ /\sigma\ \ (\text{where } \{\text{return } x \mapsto M\} \in H) \qquad \text{R\_Return}$$

$$\text{handle } E[\#op(v)] \text{ with } H\ /\sigma \rightsquigarrow M[v/x][(\lambda(\text{handle } E \text{ with } H))^{\ell}/k]\ \ /\sigma[\ell \mapsto \text{true}] \qquad \text{R\_Handle}$$
$$(\text{where } op \notin E,\ \ell \text{ is fresh and } \{op(x,k) \mapsto M\} \in H\ )$$

$$(\lambda E)^{\ell}\ v\ /\sigma[\ell \mapsto \text{true}] \rightsquigarrow E[v]\ /\sigma[\ell \mapsto \text{false}] \qquad \text{R\_Resume}$$

$$(\lambda E)^{\ell}\ v\ /\sigma[\ell \mapsto \text{false}] \rightsquigarrow \varepsilon \qquad \text{R\_Error}$$

We say op $\notin E$, if and only if there is no such E_1, E_2, and H such that $E = E_1[\text{ handle } E_2 \text{ with } H]$, and H contains an operation clause for op.

Evaluation rules

$$\frac{R_1 \rightsquigarrow R_2}{E[R_1] \longrightarrow E[R_2]}\ \text{E\_Eval}$$

$$\frac{R \rightsquigarrow \varepsilon}{E[R] \longrightarrow \varepsilon}\ \text{E\_Error}$$

Term typing

$\boxed{\Gamma \vdash M : A}$

$$\frac{x : A \in \Gamma}{\Gamma \vdash x : A}\ \textsc{T\_Var} \qquad \frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x.M : A \to B}\ \textsc{T\_Abs} \qquad \frac{\Gamma, \alpha \vdash M : A}{\Gamma \vdash M : \forall \alpha.A}\ \textsc{T\_Gen}$$

$$\frac{\Gamma \vdash M_1 : A \to B \qquad \Gamma \vdash M_2 : A}{\Gamma \vdash M_1 M_2 : B}\ \textsc{T\_App} \qquad \frac{ty(op) = \forall \alpha.A \to B \qquad \Gamma \vdash M : A[C/\alpha] \qquad \Gamma \vdash C}{\Gamma \vdash \#op(M) : B[C/\alpha]}\ \textsc{T\_Op}$$

$$\frac{\Gamma \vdash M : A \qquad \Gamma \vdash H : A \Rightarrow B}{\Gamma \vdash \text{handle } M \text{ with } H : B}\ \textsc{T\_Handle} \qquad \frac{\Gamma, x : A \vdash E[x] : B \qquad x\ is\ fresh}{\Gamma \vdash (\lambda E)^{\ell} : A \to B}\ \textsc{T\_EAbs}$$

$\boxed{\Gamma \vdash H : A \Rightarrow B}$

$$\frac{\begin{array}{c} H = \{return\ x \mapsto M\} \uplus \{op_i(x_i, k_i) \mapsto N_i\} \\ \Gamma, x : A \vdash M : B \qquad ty(op_i) = \forall \alpha.C_i \to D_i \qquad \Gamma, \alpha, x_i : C_i, k_i : D_i \to B \vdash N_i : B \end{array}}{\Gamma \vdash H : A \Rightarrow B}\ \textsc{T\_Handler}$$

Progress

If $\Delta \vdash M : A$ (Δ is the type variable environment), then:

1. $M \longrightarrow M'$ for some M';
2. M is a value; or $M = E[\#op(v)]$ for some E, op, and v such that $op \notin E$.
3. $M \longrightarrow \varepsilon$

Proof By induction on the typing derivation for M.
Case (T_Var): Contradictory.
Case (T_Abs): Obvious.
Case (T_Gen): By the IH.
Case (T_App):

$$\frac{\Gamma \vdash M_1 : A \to B \qquad \Gamma \vdash M_2 : A}{\Gamma \vdash M_1 M_2 : B}\ \textsc{T\_App}$$

By case analysis on the behavior of M_1. We have four cases to consider by the IH.
- **Subcase $M_1 \longrightarrow M_1'$ for some M_1':** We have $M \longrightarrow M_1' M_2$.
- **Subcase $M_1 = E_1[\#op(v)]$ for some E_1, op, and v such that op $\notin E_1$:** We have $E = E_1 M_2$.
- **Subcase $M_1 \longrightarrow \varepsilon$:** We have $M \longrightarrow \varepsilon$.
- **Subcase $M_1 = v_1$ for some v_1:** By case analysis on the behavior of M_2 with the IH.
  - **Subcase $M_2 \longrightarrow M_2'$ for some M_2':** We have $M \longrightarrow v_1 M_2'$.
  - **Subcase $M_2 = E_2[\#op(v)]$ for some E_2, op, and v such that op $\notin E_2$:** We have $E = v_1 E_2$.
  - **Subcase $M_2 \longrightarrow \varepsilon$:** We have $M \longrightarrow \varepsilon$.

Subcase $M_2 = v_2$ for some v_2: we have two cases to consider.
Subcase $v_1 = (\lambda E)^\ell$: If $\ell \mapsto true$, then $M = E[v_2] \ /\sigma[\ell \mapsto false]$. Otherwise $M \longrightarrow \varepsilon$.
Subcase $v_1 = \lambda x.M'$: By (R_Beta), $M = (\lambda x.M')\, v_2 \longrightarrow M'[v_2/x]$.
Case (T_Op):

$$\frac{ty(op) = \forall \alpha.A \to B \qquad \Gamma \vdash M : A[C/\alpha] \qquad \Gamma \vdash C}{\Gamma \vdash \#op(M) : B[C/\alpha]}\ \text{T\_Op}$$

Subcase $M \longrightarrow M'$ for some M': We have $\#op(M) \longrightarrow \#op(M')$.
Subcase $M = E'[\#\mathbf{op}'(v)]$ for some E', $\mathbf{op}'$, and v such that $\mathbf{op}' \notin E'$: We have $E = \#\mathrm{op}(E')$.
Subcase $M = v$ for some v: We have $E = [\,]$.
Subcase $M \longrightarrow \varepsilon$: We have $\#op(M) \longrightarrow \varepsilon$.
Case (T_Handle):

$$\frac{\Gamma \vdash M : A \qquad \Gamma \vdash H : A \Rightarrow B}{\Gamma \vdash \text{handle } M \text{ with } H : B}\ \text{T\_Handle}$$

Subcase $M \longrightarrow M'$ for some M': We have handle M with $H \longrightarrow$ handle M' with H.
Subcase $M = E'[\#\mathbf{op}'(v)]$ for some E', $\mathbf{op}'$, and v, such that $\mathbf{op}' \notin E'$: If handler H contains an operation clause $\mathrm{op}'(x, k) \to M'$, then we have handle M with $H \longrightarrow M'[v/x][\lambda y.\text{handle } E'[y] \text{ with } H/k]$. Otherwise, if H contains no operation clause for op', we have $E =$ handle E' with H.
Subcase $M = v$ for some v: By (R_Return).
Subcase $M \longrightarrow \varepsilon$: We have handle M with $H \longrightarrow \varepsilon$.
Case (T_Eabs): Obvious.

Subject Reduction

Lemma 1 (Value substitution) *Suppose that $\Gamma \vdash v : A$.*
1. If $\Gamma, x : A \vdash M : B$, then $\Gamma \vdash M[v/x] : B$
2. If $\Gamma, x : A \vdash H : C \Rightarrow D$, then $\Gamma \vdash H[v/x] : C \Rightarrow D$
Proof*: By mutual induction on the typing derivations.*

Theorem 1 (Subject reduction) *If $\Gamma \vdash M : C$ and $M \longrightarrow N\,(M \not\rightarrow \varepsilon)$, then $\Gamma \vdash N : C$.*

Proof: By induction on the typing derivation $\Gamma \vdash M : C$.
Case (T_Var): No reduction.
Case (T_Abs): No reduction.
Case (T_Gen): By the IH.
Case (T_App): We have two reduction rules to consider.

$$\frac{\Gamma \vdash M_1 : A \to B \qquad \Gamma \vdash M_2 : A}{\Gamma \vdash M_1 M_2 : B}\ \text{T\_App}$$

Subcase: R_Beta implies $M_1 = \lambda x.N$, $M_2 = v$, and $(\lambda x.N)v \ /\sigma \rightsquigarrow N[v/x] \ /\sigma$. Inversion on $\Gamma \vdash M_1 : A \to B$, gives $\Gamma, x : A \vdash N : B$. By value substitution, $\Gamma \vdash N[v/x] : B$.
Subcase: R_Resume implies $M_1 = (\lambda E)^\ell$, $M_2 = v$, and $(\lambda E)^\ell v \ /\sigma[\ell \mapsto true] \rightsquigarrow E[v] \ /\sigma[\ell \mapsto false]$. Inversion on $\Gamma \vdash M_1 : A \to B$, gives $\Gamma, x : A \vdash E[x] : B$. By value substitution, $\Gamma \vdash E[v] : B$.

Case (T_Op): No reduction.
Case (T_Eabs): No reduction.
Case (T_Handle): We have two reduction rules to consider.

$$\frac{\Gamma \vdash M : A \qquad \Gamma \vdash H : A \Rightarrow B}{\Gamma \vdash \text{handle } M \text{ with } H : B} \ \textsc{T\_Handle}$$

Subcase: R_RETURN implies $M = v$, and handle v with $H\ /\sigma \rightsquigarrow N[v/x]/\sigma$. Inversion on $\Gamma \vdash M : A$, gives $\Gamma, \vdash v : A$. By value substitution, $\Gamma \vdash N[v/x] : B$.
Subcase: R_HANDLE