Formulation of Convolutional Perfectly Matched Layer

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1 Original Formulation

Electric fields

$$j\omega\epsilon_x E_x + \sigma_x^e E_x = \frac{1}{S_{ez}}\frac{\partial H_z}{\partial y} - \frac{1}{S_{ez}}\frac{\partial H_y}{\partial z} \tag{1}$$

$$j\omega\epsilon_y E_y + \sigma_y^e E_y = \frac{1}{S_{ex}}\frac{\partial H_x}{\partial z} - \frac{1}{S_{ex}}\frac{\partial H_z}{\partial x} \tag{2}$$

$$j\omega\epsilon_z E_z + \sigma_z^e E_z = \frac{1}{S_{ey}}\frac{\partial H_y}{\partial x} - \frac{1}{S_{ex}}\frac{\partial H_x}{\partial y} \tag{3}$$

Here

$$S_{ei} = 1 + \frac{\sigma_{pei}}{j\omega\epsilon_0} \tag{4}$$

where σ_{pei}s are PML conductivities.
Magnetic fields

$$j\omega\mu_x H_x + \sigma_x^m H_x = -\frac{1}{S_{mz}}\frac{\partial E_z}{\partial y} + \frac{1}{S_{mz}}\frac{\partial E_y}{\partial z} \tag{5}$$

$$j\omega\mu_y H_y + \sigma_y^m H_y = -\frac{1}{S_{mx}}\frac{\partial E_x}{\partial z} + \frac{1}{S_{mx}}\frac{\partial E_z}{\partial x} \tag{6}$$

$$j\omega\mu_z H_z + \sigma_z^m H_z = -\frac{1}{S_{my}}\frac{\partial E_y}{\partial x} + \frac{1}{S_{mx}}\frac{\partial E_x}{\partial y} \tag{7}$$

Here

$$S_{mi} = 1 + \frac{\sigma_{pmi}}{j\omega\mu_0} \tag{8}$$

2 Equations in Time Domain

Take (**??**) for example

$$\epsilon_x\frac{\partial E_x}{\partial t} + \sigma_x^e E_x = \frac{1}{\kappa_{ey}}\frac{\partial H_z}{\partial y} - \frac{1}{\kappa_{ez}}\frac{\partial H_y}{\partial z} + \xi_{ey}(t) * \frac{\partial H_z}{\partial y} - \xi_{ez}(t) * \frac{\partial H_y}{\partial z} \tag{9}$$

3 Discrete Formulation

Equation (??) can be written in discrete form as follow

$$
\begin{aligned}
&\epsilon_x(i,j,k)\frac{E_x^{n+1}(i,j,k)-E_x^{n}(i,j,k)}{\Delta t}+\sigma_x^e(i,j,k)\frac{E_x^{n+1}(i,j,k)+E_x^{n}(i,j,k)}{2}\\
&=\frac{1}{\kappa_{ey}(i,j,k)}\frac{H_z^{n+1/2}(i,j,k)-H_z^{n}(i,j-1,k)}{\Delta y}\\
&-\frac{1}{\kappa_{ez}(i,j,k)}\frac{H_y^{n+1/2}(i,j,k)-H_y^{n}(i,j,k-1)}{\Delta z}\\
&+\psi_{exy}^{n+1/2}(i,j,k)-\psi_{exz}^{n+1/2}(i,j,k)
\end{aligned}
\tag{10}
$$

Here

$$
\psi_{exy}^{n+1/2}(i,j,k)=\sum_{m=0}^{m=n-1}Z_{0ey}(m)\left(H_z^{n-m+1/2}(i,j,k)-H_z^{n-m+1/2}(i,j-1,k)\right)
\tag{11}
$$

$$
\begin{aligned}
Z_{0ey}(m)&=\frac{1}{\Delta y}\int_{m\Delta t}^{(m+1)\Delta t}\xi_{ey}(\tau)d\tau\\
&=-\frac{\sigma_{pey}}{\Delta y\epsilon_0\kappa_{ey}^2}\int_{m\Delta t}^{(m+1)\Delta t}e^{-\left(\frac{\sigma_{pei}}{\epsilon_0\kappa_{ei}}+\frac{\alpha_{ey}}{\epsilon_0}\right)\tau}d\tau\\
&=a_{ey}e^{-\left(\frac{\sigma_{pei}}{\epsilon_0\kappa_{ei}}+\frac{\alpha_{ey}}{\epsilon_0}\right)\frac{m\Delta t}{\epsilon_0}}
\end{aligned}
\tag{12}
$$

and

$$
a_{ey}=\frac{\sigma_{pey}}{\Delta y\left(\sigma_{pey}\kappa_{ey}+\alpha_{ey}\kappa_{ey}^2\right)}\left[e^{-\left(\frac{\sigma_{pei}}{\epsilon_0\kappa_{ei}}+\frac{\alpha_{ey}}{\epsilon_0}\right)\frac{\Delta t}{\epsilon_0}}-1\right]
\tag{13}
$$

4 Computing $\psi(n)=\sum_{m=0}^{m=n-1}Ae^{mT}B(n-m)$

This equation can be written in recursive method

$$
\psi(n)=AB(n)+e^{T}\psi(n-1)
\tag{14}
$$

thus equation (??) can be written as follow

$$
\psi_{exy}^{n+1/2}(i,j,k)=b_{ey}\psi_{exy}^{n-1/2}(i,j,k)+a_{ey}\left(H_z^{n+1/2}(i,j,k)-H_z^{n+1/2}(i,j-1,k)\right)
\tag{15}
$$

where

$$
a_{ey}=\frac{\sigma_{pey}}{\Delta y\left(\sigma_{pey}\kappa_{ey}+\alpha_{ey}\kappa_{ey}^2\right)}\left[b_{ey}-1\right]
\tag{16}
$$

$$
b_{ey}=e^{-\left(\frac{\sigma_{pey}}{\kappa_{ey}}+\alpha_{ey}\right)\frac{\Delta t}{\epsilon_0}}
\tag{17}
$$

5 Final Updating Formulations

In CPML region

$$
\begin{aligned}
&E_x^{n+1}(i,j,k) = C_{exe}(i,j,k)E_x^n(i,j,k)\\
&+(1/\kappa_{ey}(i,j,k))C_{exhz}(i,j,k)\left(H_z^{n+1/2}(i,j,k) - H_z^{n+1/2}(i,j-1,k)\right)\\
&+(1/\kappa_{ez}(i,j,k))C_{exhy}(i,j,k)\left(H_y^{n+1/2}(i,j,k) - H_y^{n+1/2}(i,j,k-1)\right)\\
&\Delta y C_{exhz}(i,j,k)\psi_{exy}^{n+1/2}(i,j,k) + \Delta y C_{exhy}(i,j,k)\psi_{exz}^{n+1/2}(i,j,k)
\end{aligned}
\tag{18}
$$

or written in simple format

$$
\begin{aligned}
&E_x^{n+1}(i,j,k) = C_{exe}(i,j,k)E_x^n(i,j,k)\\
&+C_{exhz}(i,j,k)\left(H_z^{n+1/2}(i,j,k) - H_z^{n+1/2}(i,j-1,k)\right)\\
&+C_{exhy}(i,j,k)\left(H_y^{n+1/2}(i,j,k) - H_y^{n+1/2}(i,j,k-1)\right)\\
&C_{\psi_{exhz}}(i,j,k)\psi_{exy}^{n+1/2}(i,j,k) + C_{\psi_{exhy}}(i,j,k)\psi_{exz}^{n+1/2}(i,j,k)
\end{aligned}
\tag{19}
$$