Topological Nonlinear Field Theory and Tensor Networks

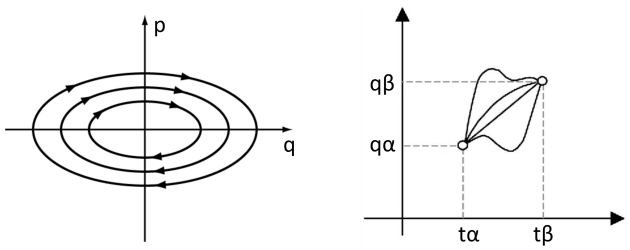
Haoyuan Luo

Wednesday 13th December, 2023

This report is based on the article "Topological nonlinear σ-model, higher gauge theory, and a realization of all $3+1D$ topological orders for boson systems" written by XiaoGang Wen and others in 2019 [1]. I will first introduce the physical background and related mathematical research, then describe tensor networks and finally discuss their discoveries and how cohomology theory and Steenrod squares are applied.

1 Physical Background

1.1 From classical mechanics to field theory



(a) (b)

Figure 1

In classical mechanics, the relation between force and motion is describe by the well known formula $\mathbf{F} = m\,\mathbf{a}$, where $\mathbf{F}$ and $\mathbf{a}$ are vectors that represent force and acceleration respectively. It is later discovered that dynamics of matter can also be formulated in two other ways. One is Hamiltonian mechanics.

$$\frac{\partial}{\partial p_i} H(q_1(t), ..., q_n(t), p_1(t), ..., p_n(t), t) \;=\; \frac{d}{dt}\, q_i(t)$$
$$\frac{\partial}{\partial q_i} H(q_1(t), ..., q_n(t), p_1(t), ..., p_n(t), t) \;=\; -\frac{d}{dt}\, p_i(t)$$

where q_i are called generalized coordinates, p_i are called generalized momentum and H is called a Hamiltonian. A Hamiltonian determines a flow in the phase space, as shown in Figure 1(a). Another one is Lagrangian mechanics.

$$S = \int_{t_\beta}^{t_\alpha} L(q_1(t), ..., q_n(t), \frac{d}{dt}q_1(t), ..., \frac{d}{dt}q_n(t), t)\, dt$$

where S is called an action, and L is called a Lagrangian. q_i are also generalized coordinates. The dynamics of matter is given by the Stationary-action Principle. Take the scenario in Figure 1(b) for example. For fixed initial and final configurations, that is a fixed (t_α, q_α) and a fixed (t_β, q_β), the path that describes the actual motion of the matter is the one that minimizes the action S. This extremum of the functional is solved by the Euler-Lagrange equation, and one can show that the two formulations of mechanics are equivalent.

Since fields, such as electromagnetic fields and sound fields, are also matter, the dynamics of fields can also be described by Hamiltonian and Lagrangian mechanics. Here we focus on Lagrangian mechanics only. The actions for vector fields is given by the following general formula [2].

$$S[\phi_i] = \int \mathscr{L}(\phi_i(\mathbf{s}), \{\frac{\partial \phi_i(\mathbf{s})}{\partial \mathbf{s}^\eta}\}, \mathbf{s}^\xi)\, \mathbf{d}^n\mathbf{s}$$

where η and ξ are set of indices, $\mathbf{s}$ includes both time and space and $\mathscr{L}$ is the Lagrangian density depending on field components ϕ_i and their partial derivatives with respect to time and space. When physicists refer to the term "a field theory", they essentially mean a Lagrangian. Given the initial and final conditions, the evolution of fields should also minimize their actions.

Now, consider a field $\phi(x, y)$ with periodic boundary condition on $[0, 1] \times [0, 1]$. Because $\phi(0, y) = \phi(1, y)$ and $\phi(x, 0) = \phi(x, 1)$, it can be regarded as a section of a vector bundle over a torus. Therefore, it is natural to define actions on vector bundles over a manifold.

Mathematically, field theories on n-dimensional manifolds are differential n-forms, the most famous one of which is the Yang–Mills theory defined on a 4-manifold called spacetime. It plays a key role in the formulation of the Standard Model. The action of Yang–Mills theory is given by [3]

$$S[F] = c \int_M tr(F \wedge \star F)$$

where $\star$ is the Hodge star operator and $\wedge$ is the wedge product of differential forms. F is the curvature matrix whose elements are differential 2-forms [4].

$$F = i \begin{pmatrix} 0 & -E_1\, dx^1 \wedge dx^2 & -E_2\, dx^1 \wedge dx^3 & -E_3\, dx^1 \wedge dx^4 \\ E_1\, dx^2 \wedge dx^1 & 0 & -B^3\, dx^2 \wedge dx^3 & B^2\, dx^2 \wedge dx^4 \\ E_2\, dx^3 \wedge dx^1 & B^3\, dx^3 \wedge dx^2 & 0 & -B^1\, dx^3 \wedge dx^4 \\ E_3\, dx^4 \wedge dx^1 & -B^2\, dx^4 \wedge dx^2 & B^1\, dx^4 \wedge dx^3 & 0 \end{pmatrix}$$

E_i and B^i are components of electric field and magnetic field respectively. The presence of Hodge star operator in the action of Yang-Mills theory indicates that a metric is involved. However, in even space–time dimensions, we may easily define integrands without reference to $\star$.

$$S^{(2)}_{top} = c \int_M tr(F), \quad S^{(4)}_{top} = c \int_M tr(F \wedge F)$$

The F here is 2×2 and 4×4 matrix of differential 2-forms respectively. Actions defined without reference to a metric are called topological gauge field actions. Actually they are two examples of the action $S^{(2l)}_{top}$ on $2l$-dimensional manifolds, which are topological invariants defining the so-called l-th Chern class of the underlying bundle [3]. For dimension 1 and 3, Chern–Simons actions can be defined by considering the base manifold as the boundary of a manifold one dimension higher.

I would like to especially note here an exciting achievement by Edward Witten in 1989 [5]. He associated the $2+1$-dimensional quantum Yang-Mills theory with the Jones polynomial of knot theory. His work stimulated a large amount of research, and earn himself a Fields Medal in 1990.

A variation of the actions defined above occurs in the nonlinear σ-model. Let T be a target manifold equipped with a Riemannian metric g, and let Σ be a differentiable map from a smooth manifold M to T. The Lagrangian density is defined by

$$\mathscr{L} = \frac{1}{2}\, g(\nabla\Sigma, \nabla\Sigma) - V(\Sigma)$$

with respect to any charts in M. The V here is a map from T to $\mathbb{R}$ (or $\mathbb{C}$). The usual vector spaces in each fiber as targets are now replaced with Riemannian manifolds. The article this report is introducing studies a completely topological version of nonlinear σ-model. In the article, actions are expressed in Steenrod squares and cup products of the cohomology groups of the target manifold.

But before we can get into the main topic, let me first introduce some basic notions in quantum mechanics and statistical mechanics.

1.2 Quantum mechanics

To summarize the difference between classical mechanics and quantum mechanics, one would likely state that the observables in classical mechanics are replaced by a Hermitian operator over a Hilbert space. However, the rigorous way to turn a classical Hamiltonian or Lagrangian into a quantum one may still be considered an open question.

Quantum states are elements in a Hilbert space over $\mathbb{C}$ whose norm is 1. They are expressed in bra-ket notation. $\langle\psi_1|$ is a bra vector and $|\psi_2\rangle$ is a ket vector. Bra vectors can be regarded as a row vector and ket vectors can be regarded as a column vector. $\langle\psi_1|\psi_2\rangle$ is their inner product. Quantum states can be written as (possibly infinite) linear combinations of others, and this fact is called quantum superposition. Given a system in $|\psi_2\rangle$, $|\langle\psi_1|\psi_2\rangle|^2$ is the probability that it would fall into the state $|\psi_1\rangle$. For an eigenvector $|\psi\rangle$ of a Hermitian operator A corresponding to an observable a in classical mechanics, the eigenvalue λ_a represents the quantity of this observable the quantum state $|\psi\rangle$ carries. Schrodinger equation describes the evolution of Quantum states [6]

$$i\hbar\frac{\partial}{\partial t}|\psi,t\rangle = H|\psi,t\rangle$$

Here, $|\psi,t\rangle$ is an element in a Hilbert space over $\mathbb{C}$ that evolves over time. $\hbar$ is a constant and i is the imaginary unit. H is the Hamiltonian operator, which is a Hermitian operator over this Hilbert space. Schrodinger equation has the following solution [6]

$$|\psi,t\rangle = e^{\frac{-i\,H\,(t-t_0)}{\hbar}}|\psi,t_0\rangle, \quad e^A = 1 + A + \frac{A^2}{2!} + ...$$

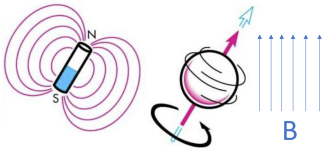


Figure 2

Example 1 *As described in Figure 2. Let B be a magnetic field along z-axis. A spinning electron behaves just like a small magnet in magnetic field. In classical mechanics, let α be the coordinate that describes the rotation of the electron along the axis around which it spins and let ρ and θ be the Euler angles that describe the position of its spin axis. The the Hamiltonian in classical mechanics, is given by*

$$H = k\vec{B}\cdot\vec{L}(\alpha,\rho,\theta,\dot{\alpha},\dot{\rho},\dot{\theta})$$

k is a constant and $\vec{L}$ is the angular momentum expressed in α,ρ,θ. However, in quantum mechanics, the Hilbert space is spanned by an orthonomral basis $\{|\uparrow\rangle,|\downarrow\rangle\}$, which are the quantum states representing spin up and spin down along z-axis. (For electrons there are only two elements in this orthonomral basis, but for others the angular momentum operators can have more eigenvalues and eigenvectors.) The spin up and down along x-axis are linear combinations of this basis, $\frac{1}{\sqrt{2}}|\uparrow\rangle+\frac{1}{\sqrt{2}}|\downarrow\rangle$ and $\frac{1}{\sqrt{2}}|\uparrow\rangle-\frac{1}{\sqrt{2}}|\downarrow\rangle$. That along y-axis are $\frac{1}{\sqrt{2}}|\uparrow\rangle+\frac{i}{\sqrt{2}}|\downarrow\rangle$ and $\frac{1}{\sqrt{2}}|\uparrow\rangle-\frac{i}{\sqrt{2}}|\downarrow\rangle$. The angular momentum in three directions are three Hermitian operators on this Hilbert space

$$L_x = \frac{\hbar}{2}\begin{pmatrix}0&1\\1&0\end{pmatrix}, \quad L_y = \frac{\hbar}{2}\begin{pmatrix}0&-i\\i&0\end{pmatrix}, \quad L_z = \frac{\hbar}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

The Hamiltonian is still

$$H = k\vec{B}\cdot\vec{L}$$

However, the three components of $\vec{L}$ are operators. One can also discover that $|\uparrow\rangle$ and $|\downarrow\rangle$ are eigenvector of L_z with eigenvalues $\frac{\hbar}{2}$ and $-\frac{\hbar}{2}$. And $\frac{1}{\sqrt{2}}|\uparrow\rangle + \frac{1}{\sqrt{2}}|\downarrow\rangle$ and $\frac{1}{\sqrt{2}}|\uparrow\rangle - \frac{1}{\sqrt{2}}|\downarrow\rangle$ are eigenvectors of L_x with the same eigenvalues. Therefore, $\frac{\hbar}{2}$ is the angular momentum in z direction carried by $|\uparrow\rangle$, and similar interpretations hold for all other eigenvectors and eigenvalues.

Quantum mechanics can also be formulated with path integral, which is the quantum version of Lagrangian mechanics. Especially in the field theory we have [7]

$$\langle F|e^{\frac{-i\,H\,T}{\hbar}}|I\rangle \;=\; \int D\phi\, e^{\frac{i}{\hbar}S(\phi)}$$

where S is the action of a field, $|F\rangle$ and $|I\rangle$ are the quantum states representing the initial and final configurations of the field. $\int D\phi$ means to take integral over all the possible configurations of the field from the initial to the final state. This integral is ill-defined, but physicists still manage to work with such definition with loose arguments.

In the article we are introducing, the author tries to solve this problem by defining this integral as a limit over finer and finer triangulations of the target manifold.

1.3 Statistical mechanics and partition function

In reality, it is extremely difficult to figure out the actual configuration of a quantum field. They are fluctuating wildly and measurement taken at precisely the same time at every point in space is almost impossible. Therefore, statistical mechanics is involved.

It is assumed that when a system is isolated, with no energy and particle exchange with the world outside, it will finally explore all the possible states it can have. Such assumption is called ergodicity. With this assumption, physicists believe that all the possible states of a system have the equal probability to occur. This is the premise of statistical mechanics [8].

Given an isolated system, we consider it as two subsystems A_1 and A_2 with energy E_1 and E_2. The question we are wondering is the condition for these two subsystems to be in equilibrium. Because it is isolated, the total energy $E = E_1 + E_2$ is always a constant. We assume that the number of possible states of systems depends only on their energies. Let $\Omega_1(E_1)$ be the number of possible states of A_1, and let $\Omega_2(E - E_1)$ be that of A_2. Intuitively, the total number of possible states of this system is given by

$$\Omega(E) = \int dE_1\, \Omega_1(E_1)\, \Omega_2(E - E_1)$$

By the assumption that all the possible states of a system have the equal probability to occur, the probability density of A_1 to have energy E_1 is therefore

$$\rho(E_1) = \Omega_1(E_1)\, \Omega_2(E - E_1)/\Omega(E)$$

If the two subsystems have a large number of particles then it turns out that $\rho(E_1)$ is a very sharply peaked function near its maximum at E_1^*. When these two subsystems are in equilibrium, E_1 will be around this sharp peak. Hence in equilibrium the energy in subsystem A_1 is given by the condition

$$\frac{1}{\Omega_1}\frac{d\Omega_1}{dE_1}\bigg|_{E_1^*} = \frac{1}{\Omega_2}\frac{d\Omega_2}{dE_2}\bigg|_{E-E_1^*}$$

Defining the equilibrium entropy as $S = k_B\, log(\Omega(E))$, where k_B is a constant, the condition above can be written as $\frac{dS_1}{dE_1} = \frac{dS_2}{dE_2}$. Then we define the temperature as

$$\frac{1}{T} = \frac{dS}{dE}$$

Now assume system A_2 is large enough, so that what happens in A_1 can hardly change the thermodynamical properties of A_2. Such an A_2 is called a heat bath. From the previous argument we

know that

$$\begin{aligned}\rho(s) &\propto \Omega_2(E - E_1, N - N_1) \\ &= Exp\,(S_2(E - E_1, N - N_1)/k_B) \\ &\approx Exp\,((-E_1\frac{\partial S_2}{\partial E} - N_1\frac{\partial S_2}{\partial N})/k_B) \\ &= Exp\,(-(E_1 - \mu N_1)/(k_B\,T))\end{aligned}$$

where $\rho(s)$ is the probability density of finding system A_1 in the state s, N_1 is the number of particles in A_1 and μ is a thermodynamical quantity called chemical potential.

The partition function of the canonical ensemble has the following form

$$Z = \sum_n Exp(\,(-E_n)/(k_B\,T))$$

And that of the grand canonical ensemble has the following form

$$\Xi = \sum_n Exp(\,(-E_n - \mu N_n)/(k_B\,T))$$

The sums are taken over all possible states of a system connected to a heat bath. In the canonical ensemble, the system we are studying cannot have particle exchange with the heat bath, but in the grand canonical ensemble it is allowed. By taking appropriate derivatives to the partition function, quantities we want to study are summed up according the weight given by these exponentials.

2 Tensor network

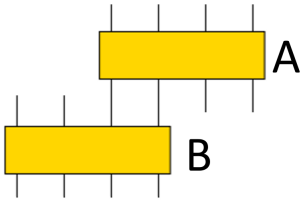


Figure 3

Let R be a $\mathbb{R}$ or $\mathbb{C}$ and let $M_1, ..., M_m$ and $N1, ..., N_n$ be R-vector space of finite dimensions. $A : M_1 \otimes ... \otimes M_m \to N_1 \otimes ... \otimes N_n$ is an R-linear map. We know that

$$\{m_{i_1} \otimes ... \otimes m_{i_m} | 1 \le i_k \ \le dim\, M_k\}, \quad \{m_{i_k}\}_{i_k=1}^{dim\, M_k} \text{ is a basis for } M_k$$

is a basis for $M_1 \otimes ... \otimes M_m$, and

$$\{n_{j_1} \otimes ... \otimes n_{j_n} | 1 \le j_k \ \le dim\, N_k\}, \quad \{n_{j_k}\}_{j_k=1}^{dim\, N_k} \text{ is a basis for } N_k$$

is a basis for $N_1 \otimes ... \otimes N_n$. Map A is determined by its value on basis. Therefore, A is given by $A^{i_1,...,i_m}_{j_1,...,j_n}$. Let $B : G_1 \otimes ... \otimes G_g \to H_1 \otimes ... \otimes H_h$ also be an R-linear map, and there is

$$H_{p(r)} = M_{q(r)},\ 1 \le r \le r_{max} \le min(m, h)$$

Then there is a linear map $(\otimes_{i \ne q(r)} M_i) \otimes (\otimes_{k=1}^{g} G_k) \to (\otimes_{j=1}^{n} N_j) \otimes (\otimes_{l \ne p(r)} H_l)$ determined by the contraction

$$\sum_{i_{q(1)}, l_{p(1)}} ... \sum_{i_{q(r_{max})}, l_{p(r_{max})}} A^{i_1,...,i_m}_{j_1,...,j_n}\, B^{k_1,...,k_g}_{l_1,...,l_h}\, \delta^{l_{p(1)}}_{i_{q(1)}} ... \delta^{l_{p(r_{max})}}_{i_{q(r_{max})}}$$

Here the δ^x_y is the Kronecker delta, which gives 1 only when the upper and lower indices are equal.

These expression are complicated indeed. Pictorial way to express them are invented [9]. As is shown in Figure 3, the boxes are linear maps between tensor spaces. The legs represent the component spaces of the tensor spaces. They can also be regarded as indices labeling the basis in the component spaces. Indices contracted together are represented by legs connected to each other.

3 Topological Nonlinear Field Theory

In the quantum field theory, partition functions are slightly different from classical ones. Because probabilities in quantum mechanics are given by squares of norms of complex numbers, in the quantum field theory, index in partition functions may also be a complex number.

The partition function of a bosonic systems described by the non-linear σ-models over a $d+1$ dimensional space time is

$$Z(M^{d+1};K,\mathscr{L})=\sum_{\phi(x)} e^{-\int_{M^{d+1}} d^{d+1}x\ \mathscr{L}(\phi(x),\partial\phi(x),...)}$$

where $\mathscr{L}$ the Lagrangian density, which can be a complex number, and K is the target manifold. The sum is taken over all the allowed maps from M^{d+1} to K.

However, in the formula above, the sum over all the allowed maps from M^{d+1} to K is ill-defined. The author proposed to consider a discrete version of this model and then take the limit.

$$Z(M^{d+1};K,\omega_{d+1})=\sum_{\phi} e^{2\pi i\int_{M^{d+1}}\phi^*\omega_{d+1}}$$

Here, ω_{d+1} is a real-valued $(d+1)$-cochain on the target manifold. Both M^{d+1} and K are triangulated in a way that preserves the attachment structure of simplices described by the simplicial face maps. Particularly, each n-skeleton in M^{d+1} is mapped to the n-skeleton in target manifold K. Take the $(2+1)$ dimensional case for example. Consider two real and one complex tensors $\mathbb{T}=(w_{v_0},w_{l_{01}}^{v_0v_1},C_{v_0v_1v_2v_3;t_{023}t_{013}t_{123}}^{l_{01}l_{02}l_{03}l_{12}l_{13}l_{23};t_{012}})$, which can be pictorially describe as Figure 4. In Figure 4 each vertex, edge and triangle, represents an index of these tensors. v_i is the index on vertex v_i, l_{ij} is the index on edges l_{ij} and t_{ijk} is the index on triangle t_{ijk}. They are just like legs in tensor networks. Two real w are called weight tensors and the complex C is called top tensor. This tetrahedron is not

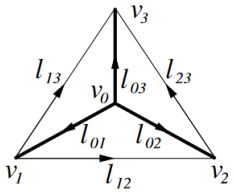


Figure 4

associated with any particular simplex in the target space, but represents tetrahedrons in general in the target space. The author requires that the indices of C corresponding to vertices take the value of all the vertices in the target manifold, that corresponding to edges take the value of all the edges in the target manifold, and that corresponding to triangles take the value of all the triangles in the target manifold. The author also requires that C can only take nonzero values when all the vertices, edges and triangles happen to make up a tetrahedron in K. Therefore, defining on 3-complex that has no boundary

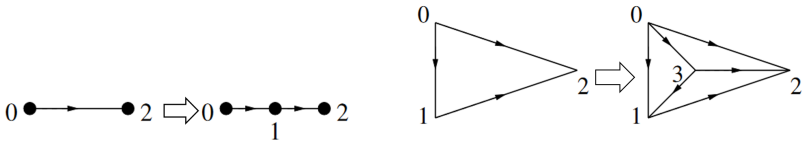
$$Z(M^{2+1};\mathbb{T})=\sum_{v_i,...;l_{ij},...;t_{ijk}}\prod_i w_{v_i}\prod_{(ij)} w_{l_{ij}}^{v_iv_j}\prod_{(ijkm)}(C_{v_iv_jv_kv_m;t_{ikm}t_{ijm}t_{jkm}}^{l_{ij}l_{ik}l_{im}l_{jk}l_{jm}l_{km};t_{ijk}})^{s_{ijkm}}$$

where the sum is taken over all the vertices, edges and triangles in M^{2+1}, which are also mapped to vertices, edges and triangles in K, the author claims that this summation corresponds to a summation $\sum_\phi$ over all the homomorphisms $\phi: M^3\to K$ in a discrete way. The s_{ijkm} is determined by the orientation of the tetrahedron. It takes 1 or *, the conjugation. Using this tensor expression, the author is able to investigate how partition function would change when he takes a finer retriangulation [10].

As shown in Figure 5 (a), if for every 1-simplex, a point is added in the middle splitting it into two, to get the same partition function, one has to require that

$$A^{v_0}_{v_2} = \sum_{v_1} A^{v_0}_{v_1} A^{v_1}_{v_2}$$

As shown in Figure 5 (b), the 2-simplex case is a bit more complicated, but one could also require that



(a) (b)

Figure 5

$$B^{e_{01}e_{12}}_{v_0v_1v_2;e_{02}} = \sum_{e_{03}e_{13}e_{23},v_3} w_{v_3} B^{v_0v_3v_1;e_{01}}_{e_{03}e_{31}} B^{e_{31}e_{12}}_{v_3v_1v_2;e_{32}} B^{e_{03}e_{32}}_{v_0v_3v_2;e_{02}}$$

Here, v_i are indices on the vertices and e_{ij} are those on the edges.

In this way, the author is able to rigorously define such a partition function. He then studies several examples.

When the target manifold is a Eilenberg-MacLane space BG with only non-trivial first homotopy group $\pi_1(K(G)) = G$, where G is finite. Assume the space-time dimension to be $d+1$. Let $\omega_{d+1} \in H^{d+1}(BG, \mathbb{R}/\mathbb{Z})$. Then let the top tensor (the tensor like C) equal to $\omega_{d+1}(W)$ when the vertices, edges, triangles, ... happen to form the boundary of W, where W is a simplex in the target manifold. The partition function is given by

$$Z = \sum_{\phi} [\prod_{n=0}^{d+1} (w_n)^{N_n}] \, e^{i2\pi \int_{M^{d+1}} \phi^* \omega_{d+1}}$$

Because ω_{d+1} is a cocycle, $e^{i2\pi \int_{M^{d+1}} \phi^* \omega_{d+1}}$ is independent of triangulation. By careful chosen of the weight tensor w_n this partition function can be well defined. It is also shown that for two maps $\phi_1, \phi_2 : M^{d+1} \to K$, if they are in the same homotopy class, then there is $e^{i2\pi \int_{M^{d+1}} \phi_1^* \omega_{d+1}} = e^{i2\pi \int_{M^{d+1}} \phi_2^* \omega_{d+1}}$. Based on this observation, further calculation is performed in this article.

When the target manifold is a Eilenberg-MacLane space $B(\Pi_2, 2)$ with only non-trivial second homotopy group $\pi_2(K(G)) = \Pi_2$, where Π_2 is finite. It is found that when n is odd, $H^4(B(\mathbb{Z}_n, 2), \mathbb{R}/\mathbb{Z}) = \mathbb{Z}_n$ is generated by $\omega_4 = \frac{1}{n} Sq^2 b$. When n is even, $H^4(B(\mathbb{Z}_n, 2), \mathbb{R}/\mathbb{Z}) = \mathbb{Z}_{2n}$ is generated by $\omega_4 = \frac{1}{2n} Sq^2 b$. Here b is a Π_2-valued 2-cocycle on $B(\Pi_2, 2)$. Therefore, the partition function take the form

$$Z(M^4; B(\mathbb{Z}_n, 2), k) = \sum_{db=0} e^{2\pi i \int_{M^4} \frac{k}{n} b^2}, \quad n \text{ is odd}$$

$$Z(M^4; B(\mathbb{Z}_n, 2), m) = \sum_{db=0} e^{2\pi i \int_{M^4} \frac{m}{2n} Sq^2 b}, \quad n \text{ is even}$$

The expressions above pretty much resemble those in other field theories. There are also interesting topological gauge transformations. Replacing b by $b + d\lambda_1$ will not change the partition function. (I am not sure of this part. There is not enough time.)

There are many other interesting results in this article. The purpose of this article is to realize a physical concept called topological order, which is believed important in theory of high temperature superconductivity and other novel physical phenomena. Using this topological nonlinear σ model, pointlike and stringlike excitations are analyzed. Due to limited time, I can not cover these topics in this report.

References

[1] Chenchang Zhu, Tian Lan, and Xiao-Gang Wen. Topological non-linear σ-model, higher gauge theory, and a realization of all 3+1d topological orders for boson systems. 100(4):045105.

[2] Herbert Goldstein, Charles Poole, and John Safko. *Classical Mechanics*. 3 edition.

[3] Alexander Altland and Ben Simons. *Condensed Matter Field Theory*. Cambridge University Press, 3 edition.

[4] Jurgen Jost. *Riemannian Geometry and Geometric Analysis*. 6 edition.

[5] Edward Witten. Quantum field theory and the jones polynomial.

[6] J.J. Sakurai and Jim Napolitano. *Modern Quantum Mechanics*. 2 edition.

[7] A. Zee. *Quantum Field Theory in a Nutshell*. 2 edition.

[8] James P Sethna. Entropy, order parameters, and complexity.

[9] Jacob Biamonte and Ville Bergholm. Tensor networks in a nutshell.

[10] Liang Kong and Xiao-Gang Wen. Braided fusion categories, gravitational anomalies, and the mathematical framework for topological orders in any dimensions.