KPZ Universality Class and Schur Measure

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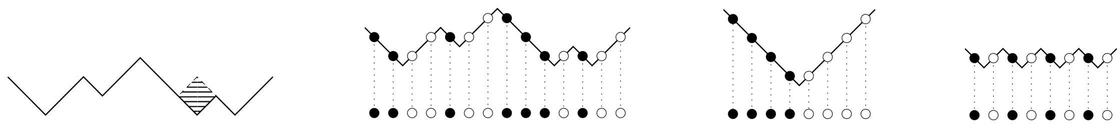
This report briefly summarizes sections 1 to 5 of the article *Lectures on integrable probability* [1]. I will first introduce several models conjectured in the KPZ universality class and then discuss the Schur Measure as a powerful tool to study these models.

1 Models in KPZ universality class

Consider the interface which is a broken line with slopes ± 1, as shown in Figure 1 (a) and suppose that a new unit box is added at each local minimum independently after an exponential waiting time. This model describes a scenario where sand particles are dropped from the sky, and the newly dropped particles always fall to the local minimum.



(a) (b) (c) (d)

Figure 1

There is also an equivalent formulation of this growth model. Project the interface to a straight line and put "particles" (black dots) at projections of unit segments of slope -1 and "holes" (white dots) at projections of segments of slope $+1$, see Figure 1 (b)(c)(d). Under certain constraints, each particle independently jumps to the right after an exponential waiting time. This is a simplified model of the known Totally Asymmetric Simple Exclusion Process.

Theorem 1 *Suppose that at time* 0 *the interface $h(x;t)$ is a wedge ($h(x,0) = |x|$) as shown in Figure 1 (c). Then for every $x \in (-1,1)$*

$$\lim_{t\to\infty} \mathbb{P}\left(\frac{h(t,t\,x) - c_1(x)\,t}{c_2(x)\,t^{1/3}} \geq -s\right) = F_2(s)$$

where $c_1(x)$, $c_2(x)$ are certain explicit functions of x.

Theorem 2 *Suppose that at time* 0 *the interface $h(x;t)$ is flat as shown in Figure 1 (d). Then for every $x \in \mathbb{R}$*

$$\lim_{t\to\infty} \mathbb{P}\left(\frac{h(t,x) - c_3\,t}{c_4\,t^{1/3}} \geq -s\right) = F_1(s)$$

where $c3$, $c4$ are certain explicit positive constants.

Here $F_1(s)$ and $F_2(s)$ are distributions from random matrix theory, known as the Tracy-Widom distributions. These two theorems give the conjectural answer for the whole "universality class" of 2d random growth models, which is usually referred to as the KPZ (Kardar-Parisi-Zhang) universality class.

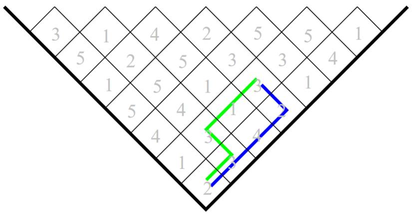


Figure 2

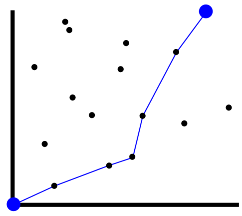


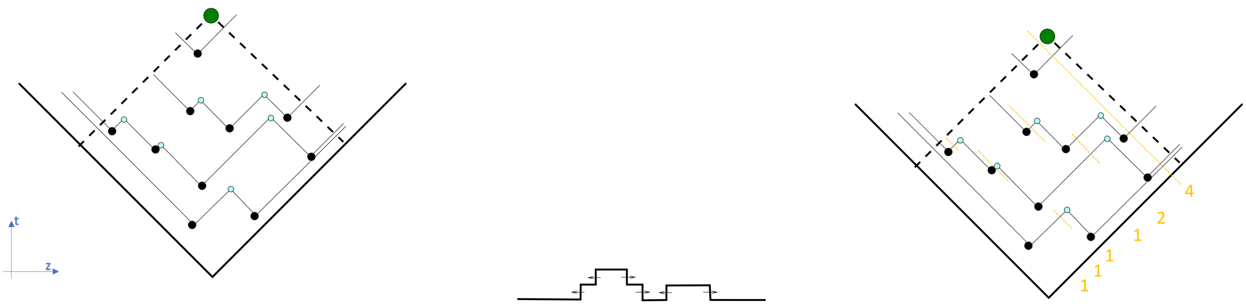
Figure 3

Let us concentrate on wedge initial condition. Give each box (i,j) in the positive quadrant a random "waiting time" $w(i,j)$. Once our random interface reaches the lowest vertex of the box (i,j) it takes time $w(i,j)$ for it to absorb the box. Let $T(i,j)$ be the time it takes from the beginning to absorb the box (i,j) into the growing surface, then by simple argument one can shows that

$$T(i,j) = \underset{(1,1)=b[1]\to b[2]\to \ldots b[i+j-1]=(i,j)}{max} \sum_{k=1}^{i+j-1} w_{b[k]}$$

where the maximum is taken over all the directed paths joining $(1,1)$ and (i,j). This quantity $T(i,j)$ is known as the Last Passage Percolation time. Universality considerations make one believe that the limit behavior of the Last Passage Percolation time should not depend on the distribution of $w(i,j)$, but we are very far from proving this at the moment.

Now consider the homogeneous, density 1 Poisson point process in the first quadrant, and let $L(\theta)$ be the maximal number of points one can collect along a North-East path from $(0,0)$ to (θ,θ), as shown at Figure 3. This quantity can be seen as a limit of the Last Passage Percolation times when $w(i,j)$ takes only two values 0 and 1, and the probability of 1 is very small. Such considerations explain that $L(\theta)$ should be also in the KPZ universality class.



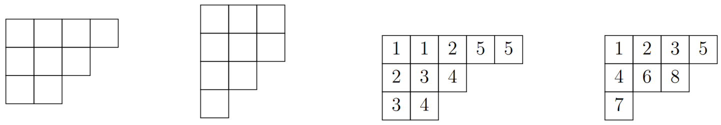
(a) (b) (c)

Figure 4

In the Poisson process in the first quadrant, there is a graphical way to find the value of $L(\theta)$. As shown in Figure 4 (a), for each point of the process draw two rays starting from it and parallel to the axes, and extend each ray till the first intersection with another ray. In this way, we get a collection of broken lines. $L(\theta)$ is equal to the number of broken lines separating (θ,θ) and the origin.

This interpretation is closely related to the Polynuclear Growth Model. Let t be the time and z be the coordinate of space. As shown in Figure 4 (b), the height profile $h(x,t)$ is an integer-valued step function with steps $+1$ and -1, and at time zero $h(x,0)=0$. If at time t a seed is born at position $x=z$, $h(z,t)$ increases by 1. After that the down step starts moving with speed 1 to the right, while the up step starts moving to the left with the same speed. When up and down steps (born by different seeds) meet each other, they disappear. If we give the t and z coordinates as shown to Figure 4 (a), then the broken lines in Figure 4 (a) exactly describe the movement of the steps (discontinuities) in the height profile $h(x,t)$ of the Polynuclear Growth Model. Therefore, the Polynuclear Growth Model is also in the KPZ universality class.

2 From Plancherel measure to Schur measures



(a) (b)

Figure 5

Let λ be a Young diagram of size n or, equivalently, a partition of n. In other words, λ is a sequence $\lambda_1 \geq \lambda_2 \geq$ ...of non-negative integers (which are identified with row lengths of the Young diagram), such that $\Sigma_i \lambda_i = |\lambda| = n$. The length $\ell(\lambda)$ of λ is defined as the number of non-zero numbers λ_i (equivalently, the number of rows in λ). We denote the set of all Young diagrams by $\mathbb{Y}$. We draw Young diagrams as collections of unit boxes, as shown in Figure 5 (a).

Standard Young tableaux of shape λ is a filling of boxes of λ with numbers from 1 to $|\lambda|$ in such a way that the numbers strictly increase both along the columns and along the rows (in particular, this implies that each number appears exactly once). Examples of Young tableaux are given in Figure 5 (b). The number of all standard Young tableau of shape λ is denoted as $dim(\lambda)$

In fact, as shown in Figure 4 (c), we can compute the number of broken lines separating (θ, θ) and $(0, 0)$ at each step and record these numbers to form a Young diagram [2]. The Young diagram in Figure 4 (c) is $(4, 2, 1, 1, 1, 1)$.

Theorem 3 *The distribution of $\lambda(\theta)$ is given by the Poissonized Plancherel measure*

$$\mathbb{P}(\lambda(\theta) = \mu) = e^{-\theta^2} (\frac{\theta^{|\mu|} \, dim(\mu)}{|\mu|!})^2, \mu \in \mathbb{Y}$$

The Poissonized Plancherel measure is a special case of the Schur measure. I will introduce the definition of Schur measure and its asymptotic behavior in the following paragraphs.

Let Λ be the algebra of symmetric functions in infinitely many countable variables. It can be rigorous defined as the projective limit over the symmetric polynomials over finitely many variables.

Definition 1 *The Schur polynomial $s_\lambda(x_1, ..., x_N)$ is a symmetric polynomial in N variables parameterized by Young diagram λ with $\ell(\lambda) \leq N$ and given by*

$$s_\lambda(x_1, ..., x_N) = \frac{det[x_i^{\lambda_j + N - j}]_{i,j=1}^N}{\Pi_{i<j} \, (x_i - x_j)}$$

Any algebra homomorphism $\rho : \Lambda \to \mathbb{C}$ is called a specialization. The substitution map $x_i \mapsto u_i$, $u_i \in \mathbb{C}$ is a specialization. We call a specialization ρ Schur–positive if for every Young diagram λ we have $s_\lambda(\rho) \geq 0$

Definition 2 *Take any two Schur–positive specializations ρ_1, ρ_2. The Schur measure $\mathbb{S}_{\rho_1,\rho_2}$ is the probability measure on the set of ll Young diagrams defined through*

$$\mathbb{P}_{\rho_1,\rho_2}(\lambda) = \frac{s_\lambda(\rho)_1 \, s_\lambda(\rho_2)}{H(\rho_1; \rho_2)}$$

where the normalizing constant $H(\rho_1; \rho_2)$ is given by

$$H(\rho_1; \rho_2) = exp(\sum_{k=1}^{\infty} \frac{p_k(\rho_1) \, p_k(\rho_2)}{k}), \quad p_k = \sum_i x_i^k$$

Proposition 1 *Let ρ_θ be the Schur–positive specialization so that $p_1(\rho_\theta) = \theta$ and $p_k(\rho_\theta) = 0$, $k > 1$, then $\mathbb{P}_{\rho_\theta,\rho_\theta}$ is the Poissonized Plancherel measure.*

One of the most interesting fact about Schur measure is the following theorem, and it allows us to derive the asymptotic behaviors of these random Young diagrams.

Theorem 4 *Suppose that the $\lambda \in \mathbb{Y}$ is distributed according to the Schur measure $\mathbb{S}_{\rho_1,\rho_2}$. Then $X(\lambda)$ is a determinantal point process on $\mathbb{Z}+1/2$ with correlation kernel $K(i,j)$ defined by the generating series*

$$\sum_{i,j\in\mathbb{Z}+1/2} K(i,j)\, v^i\, w^{-j} = \frac{H(\rho_1;v)\, H(\rho_2;w^{-1})}{H(\rho_2;v^{-1})\, H(\rho_1;w)} \sum_{k=\frac{1}{2},\frac{3}{2},\frac{5}{2},\ldots} (\frac{w}{v})^k$$

where

$$H(\rho;z) = \sum_{k=0}^{\infty} h_k(\rho)\, z^k = exp(\sum_{k=1}^{\infty} p_k(\rho)\, \frac{z^k}{k})$$

$$p_k = \sum_i x_i^k, \quad h_k = \sum_{i_1\le i_2\le\ldots\le i_k} x_{i_1}\ldots x_{i_k}$$

Let me explain the terminologies in the theorem above.

In a reasonable "state space" $\mathscr{X}$, such as $\mathbb{R}^d$ or $\mathbb{Z}^d$, a point configuration is a locally finite collection of points of the space $\mathscr{X}$. A random point process is a probability measure on the set of all point configurations in $\mathscr{X}$. The set of all point configurations in $\mathscr{X}$, denoted $Conf(\mathscr{X})$ is typically very large, and a good tool to study it is correlation function.

Definition 3 *Suppose we are given a point process. Let A range over finite subsets of $\mathscr{X}$, the correlation function $\rho(A)$ is defined as*

$$\rho(A) = \mathbb{P}\{X \in Conf(\mathscr{X}) | A \subset X\}$$

If $A = \{x_1, ..., x_n\}$, $\rho(A)$ is also written as $\rho_n(x_1, ..., x_n)$.

Definition 4 *A point process on $\mathscr{X}$ is said to be determinantal if there exists a function $K(x,y)$ on $\mathscr{X}\times\mathscr{X}$ such that the correlation functions are given by the the determinantal formula*

$$\rho_n(x_1, ..., x_n) = det[K(x_i, x_j)]_{i,j=1}^n$$

In the determinantal point process constructed for a Schur measure, given a Young diagram λ, we associate to it a point configuration $X(\lambda) = \{\lambda_i - i + 1/2\} \subset \mathbb{Z}+1/2$. This is similar to the "particles" shown in Figures 1 (b) (c) (d).

In the case of the Poissonized Plancherel measure, the asymptotic behavior of $K(i,j)$ can be understood through complex analysis. $K(i,i)$ is the density of particles of our point process or, looking at Figure 1 (b) (c) (d), the average local slope of the (rotated) Young diagram in the limit.

Letting $u = 2\,cos(\phi)$, $0 < \phi < \pi$, then we have

$$\lim_{\theta\to\infty} K_\theta(u\,\theta, u\,\theta) = \frac{\phi}{\pi}$$

Turning to the original picture, we get the asymptotic behaviors of these Young diagrams. With this

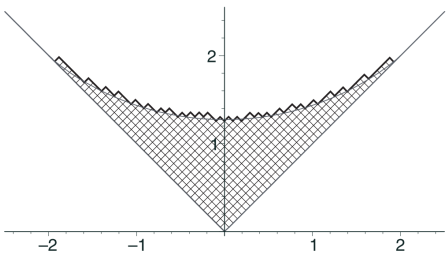


Figure 6

powerful tool, the local behavior of the boundaries of random Young diagrams in the limit can also be studies.

References

[1] Borodin Alexei and Gorin Vadim. Lectures on integrable probability.

[2] Bruce E. Sagan. *The Symmetric Group:Representations, Combinatorial Algorithms, and Symmetric Functions*, volume 203 of *Graduate Texts in Mathematics*. Springer New York.