Homology in a nutshell

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The notion of homology is introduced here in a least mathematical way. Mathematical definitions and formulae are replaced by pictures and analogy.

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2. Boundary Operators
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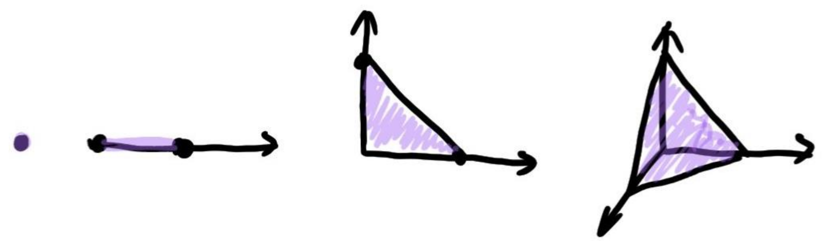
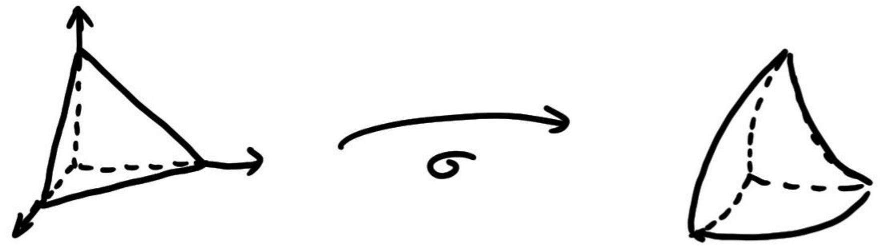
1.Orientations

The idea of homology is to characterize “holes” in space with simplices.

Simplices are things like this.

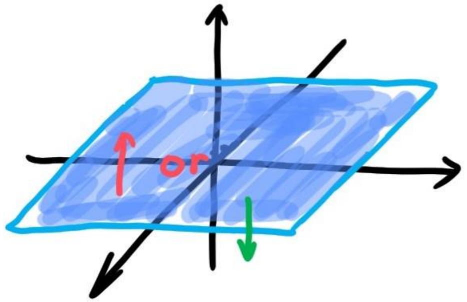


We want our simplices to be flexible, so we define singular simplices to be continuous maps from the purple areas in the picture above into any space.



1.Orientations

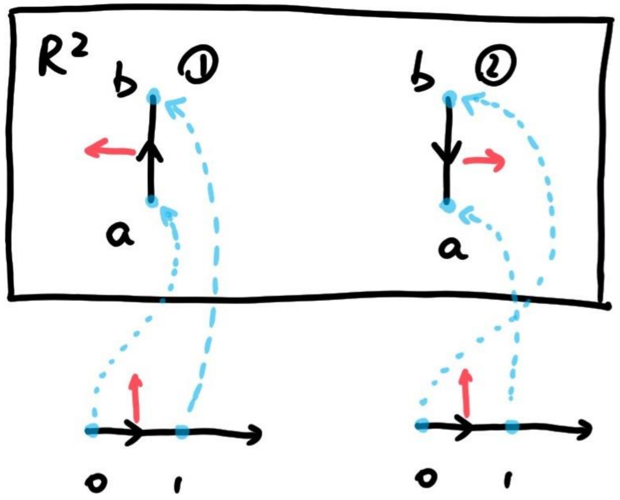
To understand the orientations of simplices, we first need to look at a (n-1)-dimensional surface in a n-dimensional space.

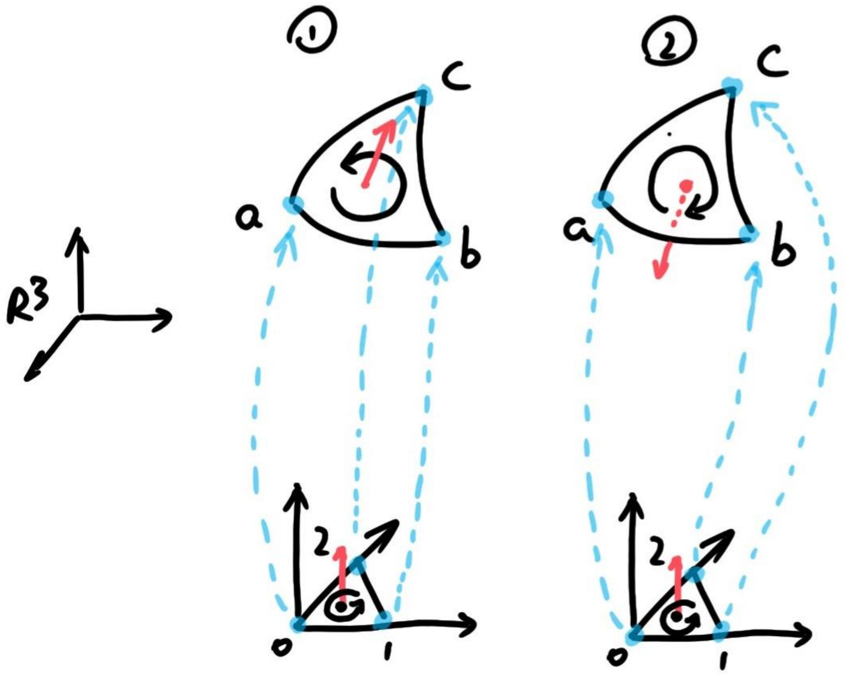


Just like what this picture shows, an orientable surface can only have two orientation.

1.Orientations

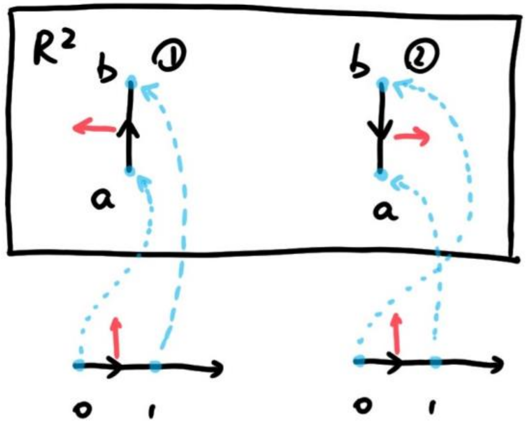
When we embed an (n-1)-dimensional simplex into a n-dimensional space, it becomes a piece of (n-1)-dimensional orientable surface, so it has and can only have two orientations.





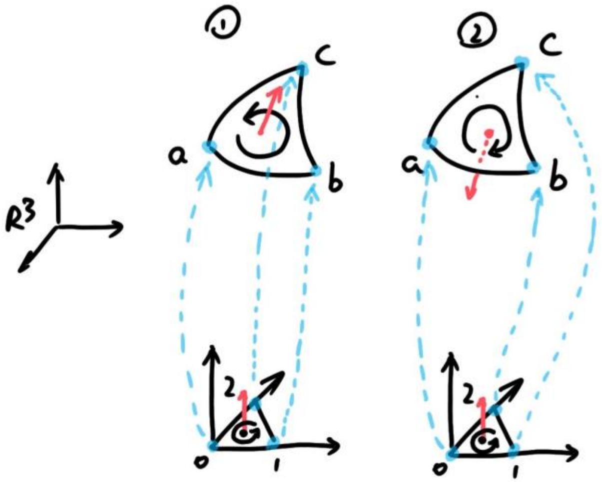
1.Orientations

Actually the orientations of singular simplices are characterized by the order of vertices in which a simplex is mapped to (the image of) a singular simplex.
Two singular simplices have the same orientation if and only if their order of vertices can be transformed into each other by even number of transpositions.



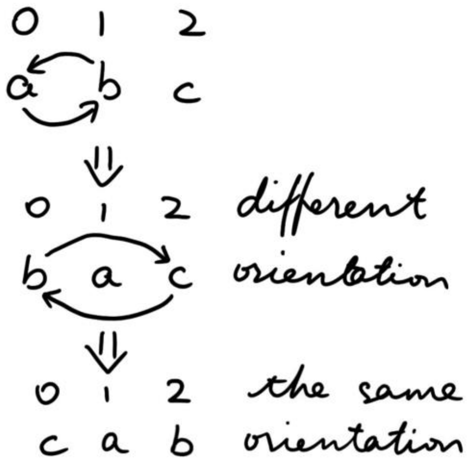
①: $0 \mapsto a$, $1 \mapsto b$

②: $0 \mapsto b$, $1 \mapsto a$

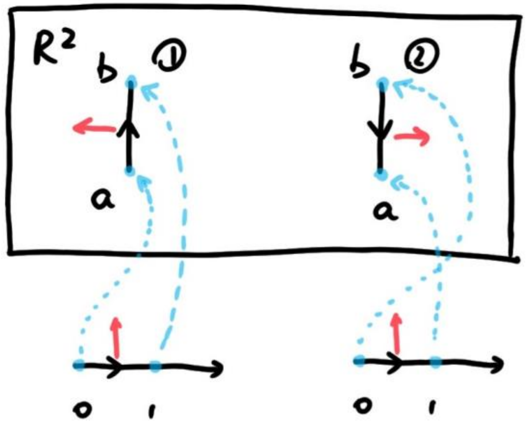


① $0 \mapsto a$, $1 \mapsto b$, $2 \mapsto c$ OR $0 \mapsto b$, $1 \mapsto c$, $2 \mapsto a$ OR $0 \mapsto c$, $1 \mapsto a$, $2 \mapsto b$

② $0 \mapsto a$, $1 \mapsto c$, $2 \mapsto b$ OR $0 \mapsto c$, $1 \mapsto b$, $2 \mapsto a$ OR $0 \mapsto b$, $1 \mapsto a$, $2 \mapsto c$



1.Orientations



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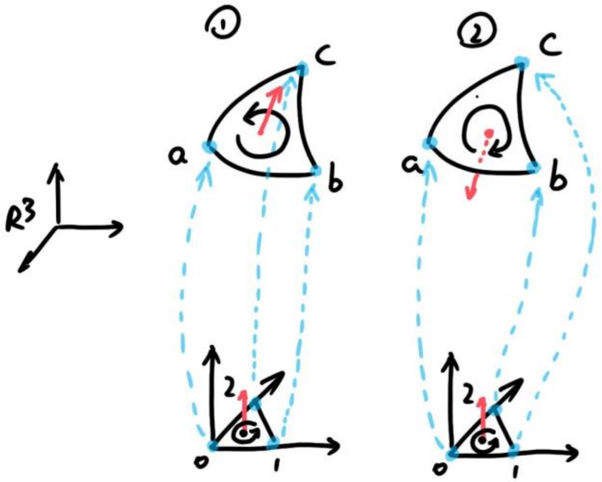

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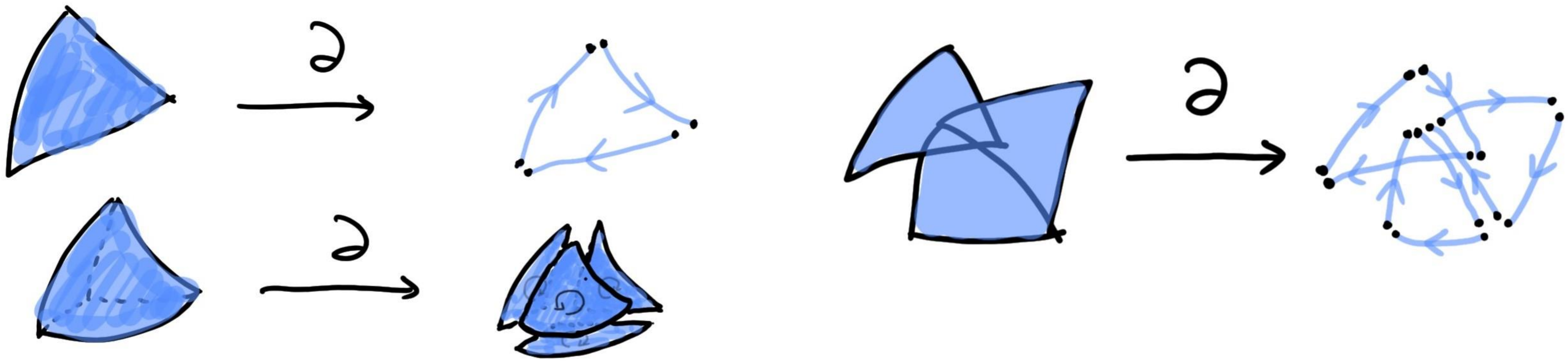
Therefore, we do not need to embed (the image of) a singular simplex into a space just one dimension higher to define its orientation.

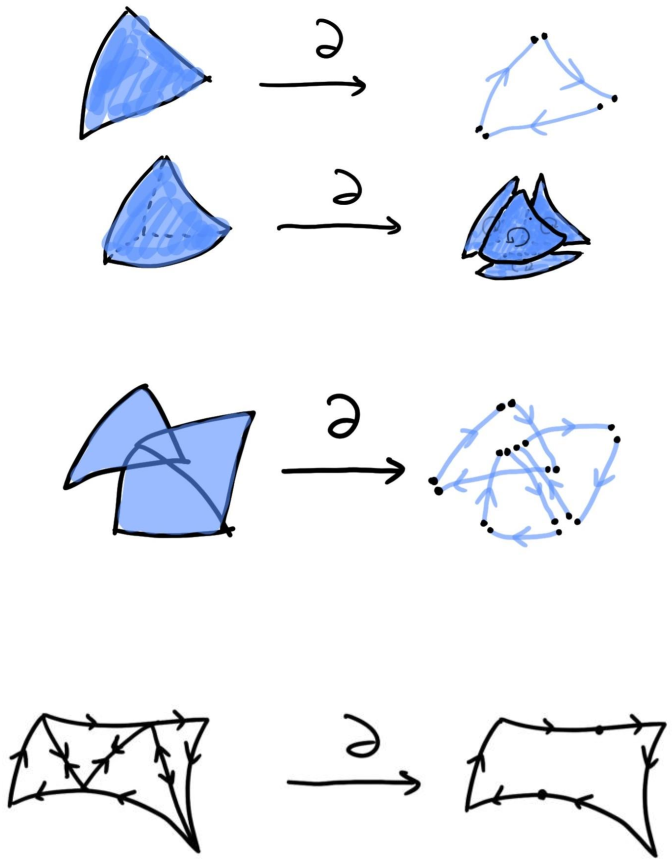
Additionally, we require that two singular simplices with the opposite orientations can cancel each other.

2.Boundary Operators

With boundary operators, we get the boundaries of a bunch of singular simplices of the same dimension. These boundaries consist of singular simplices with one dimension lower.



2.Boundary Operators



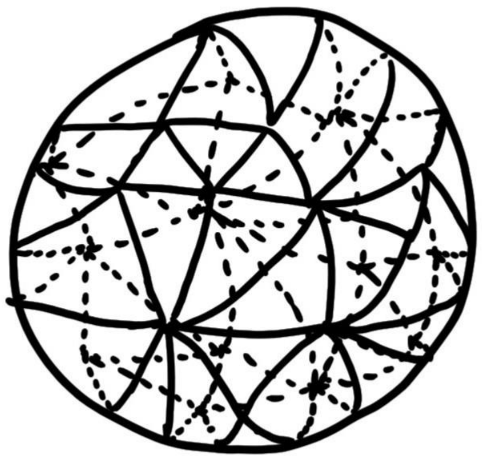
$$\partial\sigma = \sum_{i=0}^{p} (-1)^i \sigma \circ F_{i,p}.$$

The mathematical construction of boundary operators guarantees that all the faces we get, which are simplices with one dimension lower, are all in some sense have the orientation pointing outwards.

Therefore, when we attach simplices of the same dimension together along their faces to get a complex and then apply the boundary operator, all the faces inside get cancelled. We are left with the boundary of the complex.

3.Homology groups

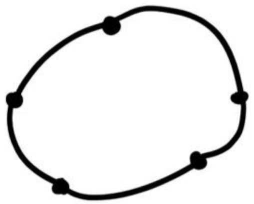
We want to characterize an “n-dimensional hole” in our space, so how can we do that? A good way is to enclose a hole with a “sphere” consisting of singular simplices with one dimension lower. For example, if we want to enclose a 3-dimensional hole, we will need a 2-dimensional complex like this.

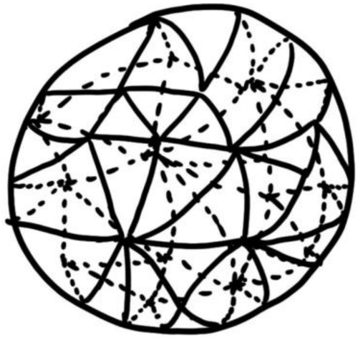


It should be a “closed surface”.

3.Homology groups

How do we know that our complex is a “closed surface”?
The boundary of this complex should be zero!



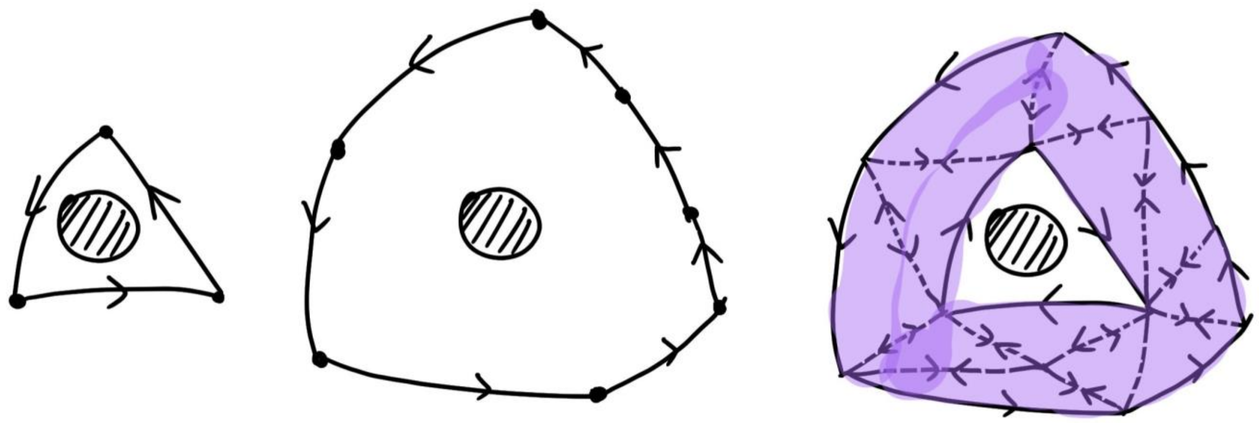


Additionally, because the boundary of any complex is always a “closed surface”.
Applying boundary operators twice will give a zero.

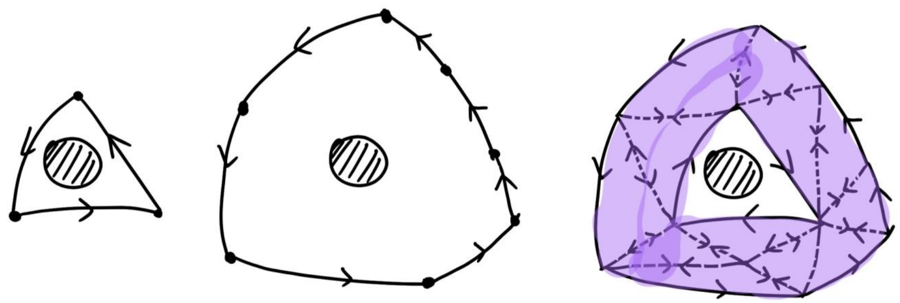
$$\partial(\partial c) = 0.$$

3.Homology groups

The way to enclose a “hole” is not unique.
However, the difference between two “closed surface” consisting of singular simplices is always the boundary of a complex with one dimension higher. This is because there should be no other “hole” between the two “closed surface”.



3.Homology groups



Therefore homology groups have the following definition.

$$H_p(X) = Z_p(X)/B_p(X) = \mathrm{Ker}\,\partial_p/\mathrm{Im}\,\partial_{p+1}.$$

In some sense, the “dimension” of a homology group tells how many “holes” we have in our space. And the basis are equivalence classes of the form

c+{boundaries}

Two “different ways of enclosing a whole” are in the same class.

c1+{boundaries} = c2+{boundaries}