Show that $c K_1(\underset{\sim}{u}, \underset{\sim}{v})$ is a valid kernel.

Let ϕ_1 be the feature map for K_1.

$$c K_1(\underset{\sim}{u}, \underset{\sim}{v}) = \langle \sqrt{c}\, \phi_1(\underset{\sim}{u}), \sqrt{c}\, \phi_2(\underset{\sim}{v}) \rangle$$

(inner product axiom and using $c > 0$)

$$= \langle \phi(\underset{\sim}{u}), \phi(\underset{\sim}{v}) \rangle$$

Show that $f(\underset{\sim}{u}) K_1(\underset{\sim}{u}, \underset{\sim}{v}) f(\underset{\sim}{v})$ is a valid kernel.

Let ϕ_1 be the feature map for K_1. Then

$$f(\underset{\sim}{u}) f(\underset{\sim}{v}) K_1(\underset{\sim}{u}, \underset{\sim}{v}) = \langle f(\underset{\sim}{u}) \phi_1(\underset{\sim}{u}), f(\underset{\sim}{v}) \phi_1(\underset{\sim}{v}) \rangle$$

(inner product axiom)

$$= \langle \phi(\underset{\sim}{u}), \phi(\underset{\sim}{v}) \rangle$$

Show that $K_1(\underset{\sim}{u}, \underset{\sim}{v}) + K_2(\underset{\sim}{u}, \underset{\sim}{v})$ is a valid kernel.

<u>Proof by Mercer's theorem</u>

Given a set of instances $\{\underset{\sim}{u}_1, \dots, \underset{\sim}{u}_n\}$, let $\mathbb{K}_1$ be the $n \times n$ Gram matrix associated with $K_1(\cdot, \cdot)$ (and sim. for $K_2(\cdot, \cdot)$).

The Gram matrix associated with $K = K_1 + K_2$ is $\mathbb{K} = \mathbb{K}_1 + \mathbb{K}_2$.

Now $\mathbb{K}$ is PSD since for any $\underset{\sim}{x} \in \mathbb{R}^n$

$$\underset{\sim}{x}^T \mathbb{K} \underset{\sim}{x} = \underset{\sim}{x}^T \mathbb{K}_1 \underset{\sim}{x} + \underset{\sim}{x}^T \mathbb{K}_2 \underset{\sim}{x} \geq 0$$

(since $\underset{\sim}{x}^T \mathbb{K}_1 \underset{\sim}{x} \geq 0$ and $\underset{\sim}{x}^T \mathbb{K}_2 \underset{\sim}{x} \geq 0$)

<u>Alternate proof</u>

Assume the implicit feature space is finite-dimensional, and let ϕ_1, ϕ_2 be the feature maps for K_1, K_2 respectively.

Consider the feature map ϕ which is the concatenation of ϕ_1, ϕ_2:

$$\phi(\underset{\sim}{u}) = \begin{bmatrix} \phi_1(\underset{\sim}{u}) \\ \phi_2(\underset{\sim}{u}) \end{bmatrix}$$

Now observe that

$$K(\underset{\sim}{u}, \underset{\sim}{v}) := \langle \phi(\underset{\sim}{u}), \phi(\underset{\sim}{v}) \rangle$$

$$= \langle \phi_1(\underset{\sim}{u}), \phi_1(\underset{\sim}{v}) \rangle + \langle \phi_2(\underset{\sim}{u}), \phi_2(\underset{\sim}{v}) \rangle$$

(here we're defining the inner product on the concatenated feature space – should show that this is a valid inner product)

$$= K_1(\underset{\sim}{u}) + K_2(\underset{\sim}{v})$$

Show that $\exp(K_1(\underset{\sim}{u}, \underset{\sim}{v}))$ is a valid kernel.

We'll need a couple of extra identities:

(i) if K_1, K_2 are valid kernels, so is

$$K(\underset{\sim}{u}, \underset{\sim}{v}) = K_1(\underset{\sim}{u}, \underset{\sim}{v})\, K_2(\underset{\sim}{u}, \underset{\sim}{v})$$

(ii) if $K_1, K_2, \ldots$ is a sequence of valid kernels and $K(\underset{\sim}{u}, \underset{\sim}{v}) = \lim_{n\to\infty} K_n(\underset{\sim}{u}, \underset{\sim}{v})$ exists for all $\underset{\sim}{u}, \underset{\sim}{v} \in \mathcal{X}$ then $K(\underset{\sim}{u}, \underset{\sim}{v})$ is also valid.

Now we have that

$$\exp(K_1(\underset{\sim}{u}, \underset{\sim}{v})) = \lim_{n\to\infty} K_n^*(\underset{\sim}{u}, \underset{\sim}{v})$$

where $K_n^*(\underset{\sim}{u}, \underset{\sim}{v}) := \sum_{j=0}^{n} \frac{1}{j!}[K_1(\underset{\sim}{u}, \underset{\sim}{v})]^j$.

K_n^* is a valid kernel by the "scalar multiplication", "element-wise product" (i) and "sum" identities.

Hence by the "limit" identity (ii),

$\exp(K_1(\underset{\sim}{u}, \underset{\sim}{v}))$ is a valid kernel.