Quantum Information and Quantum Computation

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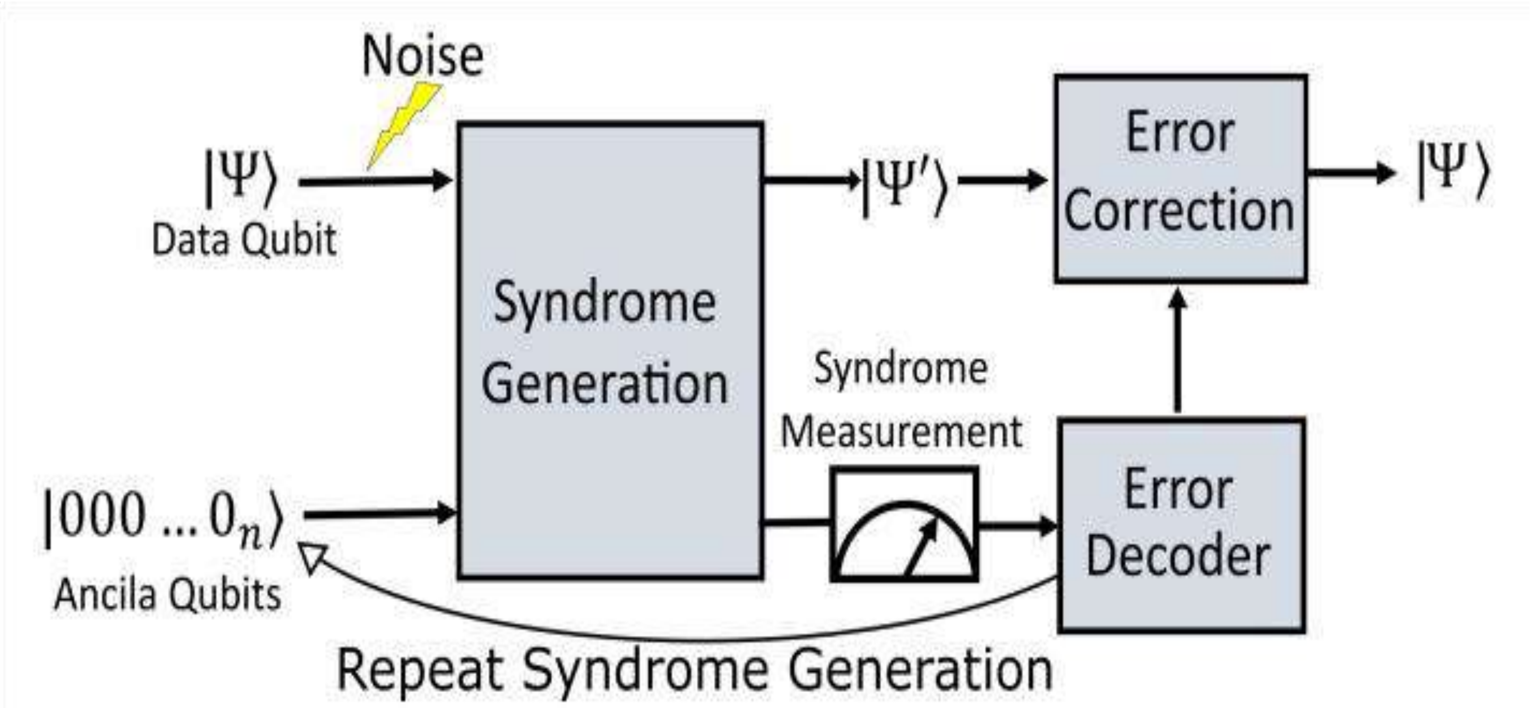
Lecture 13

Quantum error correction



中科院院士潘建伟提出，量子计算分为三个发展阶段：

1. 实现量子计算优越。

2. 实现专用的量子模拟机，可应用于组合优化量子化学、机器学习等特定问题。

3. 在实现量子纠错的基础上，构建可编程通用量子计算机。

Why Quantum Error Correction?

Cause: circuit interacts with the surroundings

- decoherence
- decay of the quantum information stored in the device

Solution: Quantum Error Correcting Codes

- protect quantum information against errors
- perform operations fault-tolerantly on enccoded states

量子相干性?

Spatial and Temporal Coherence

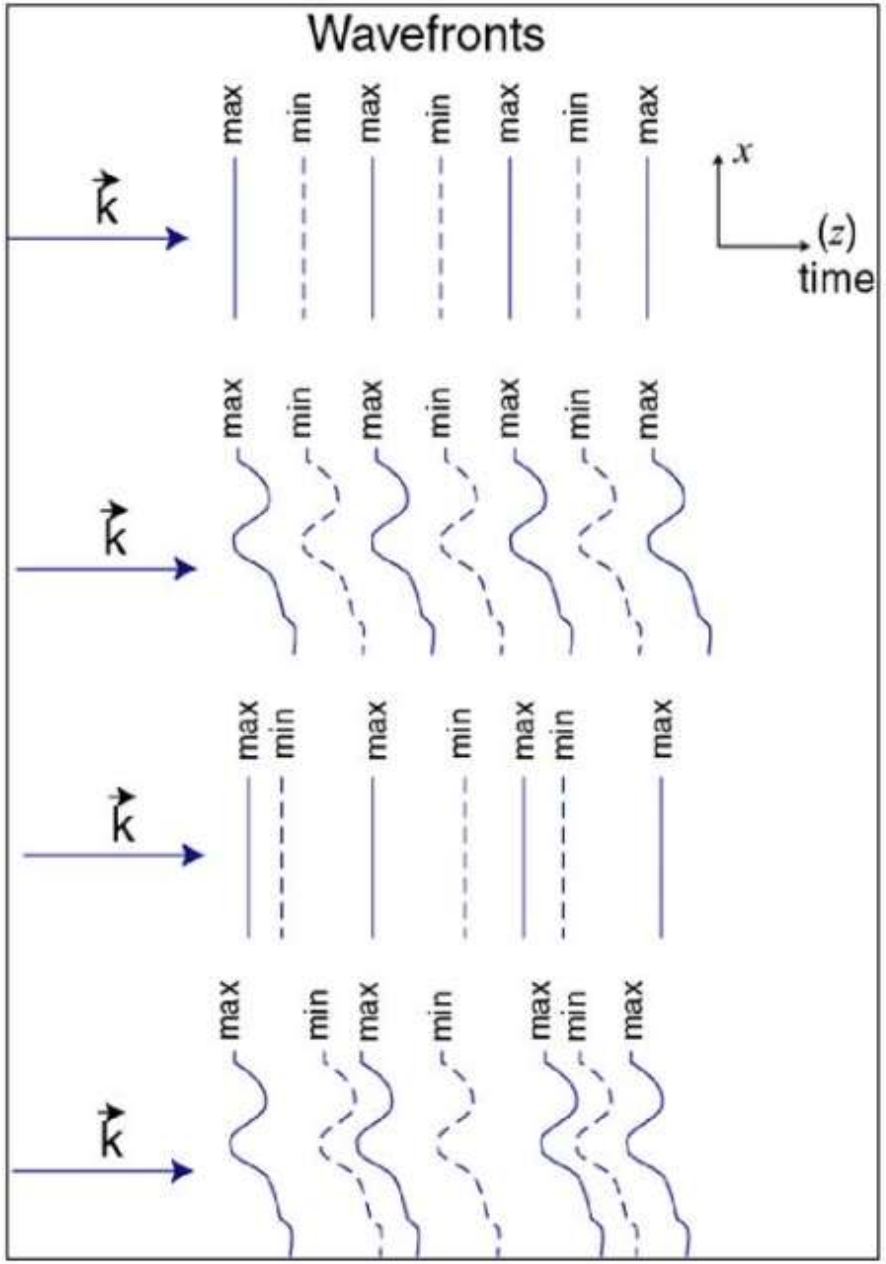
Beams can be coherent or only partially coherent (indeed, even incoherent) in both space and time.

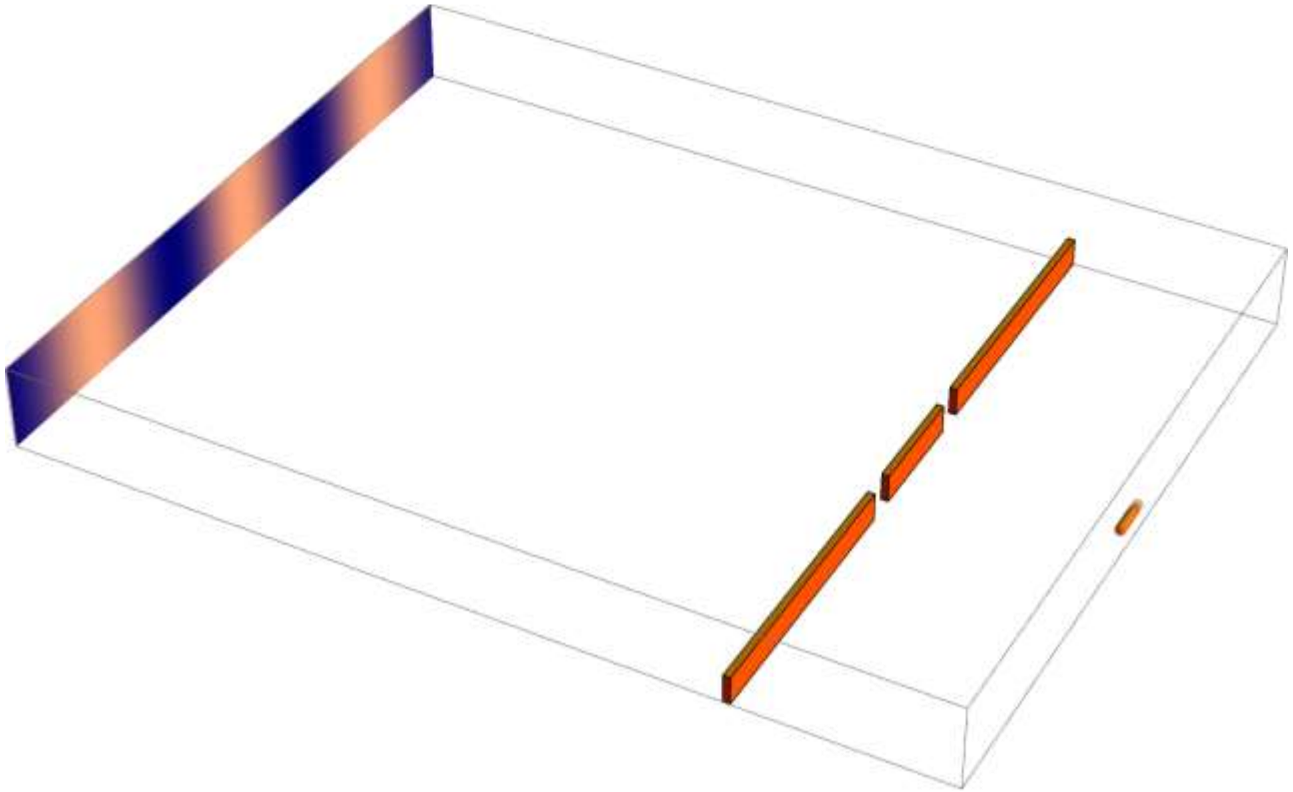
Spatial and Temporal Coherence:

Temporal Coherence; Spatial Incoherence

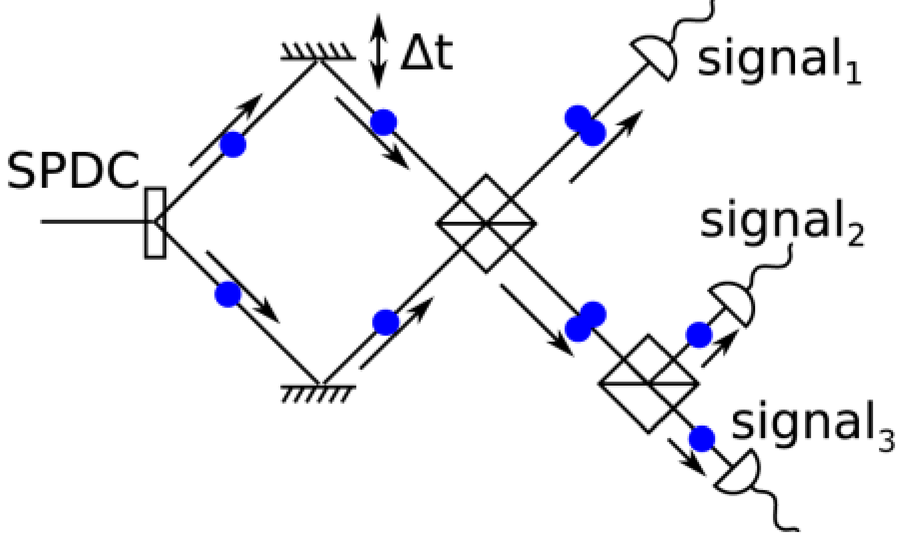
Spatial Coherence; Temporal Incoherence

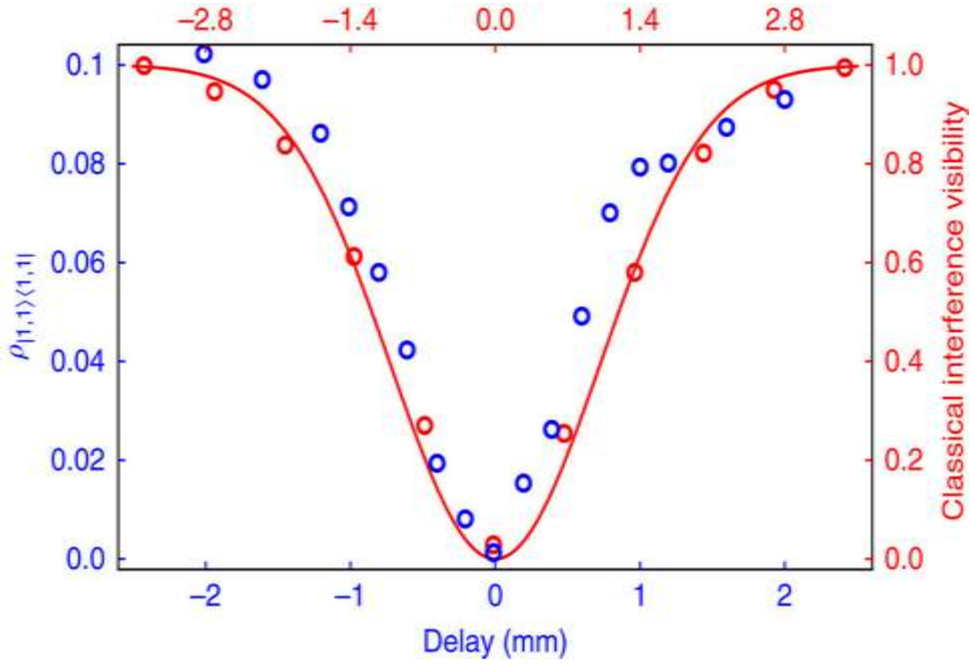
Spatial and Temporal Incoherence



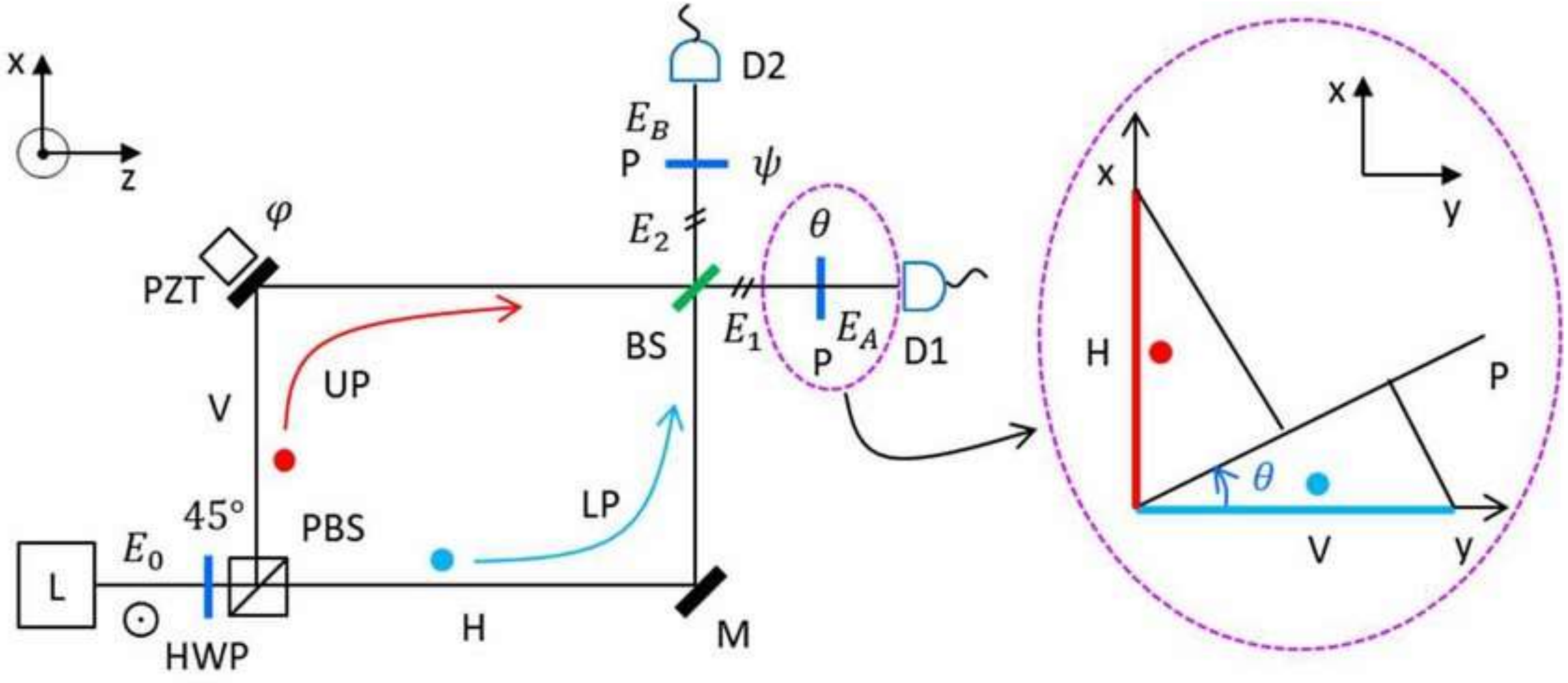


杨氏双缝干涉：空间相干性



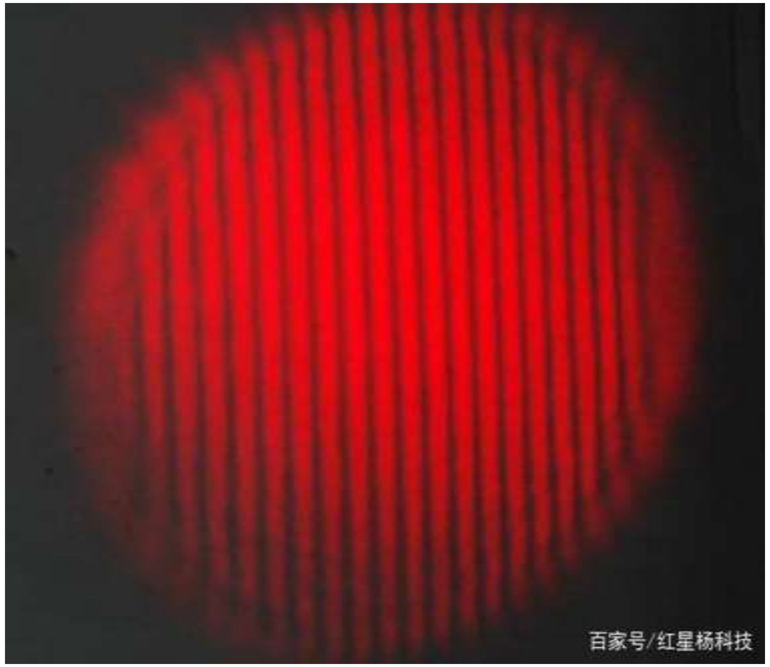


量子HOM干涉：时间相干性



时间相干性：相位随机

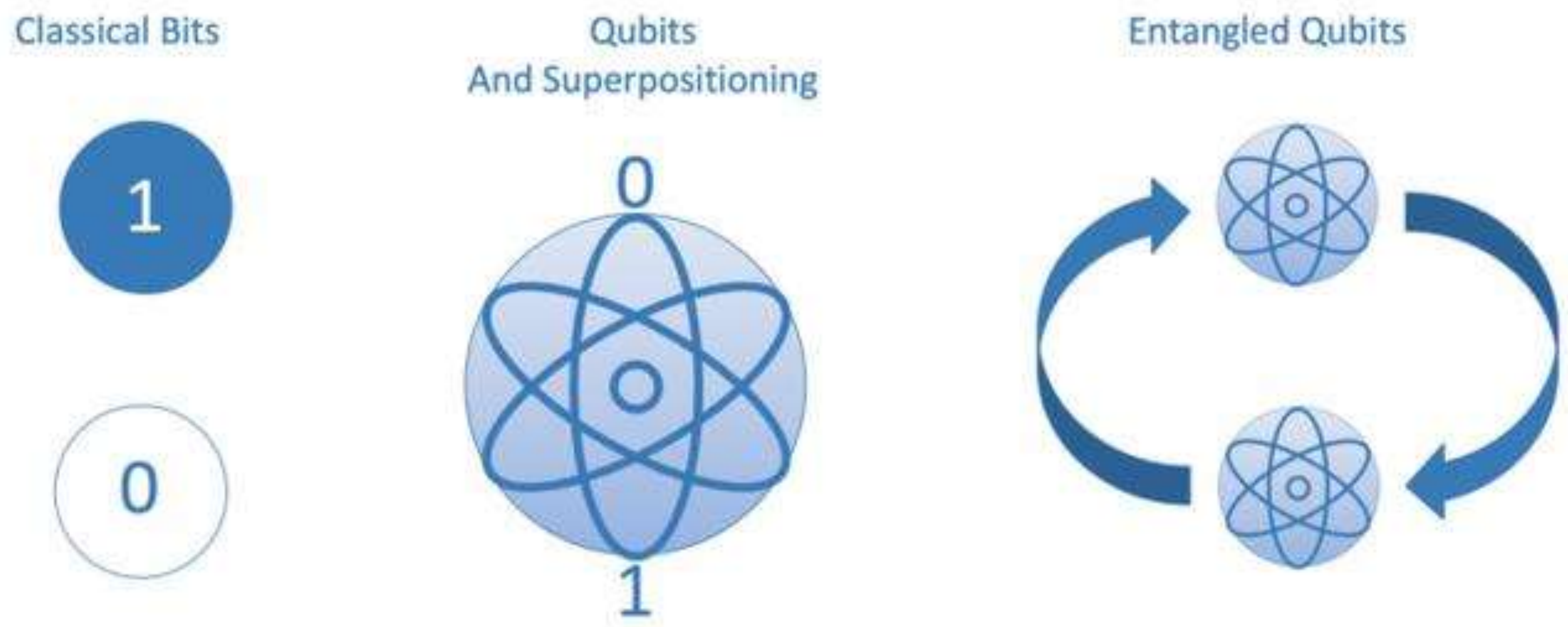
PZT：相位稳定装置



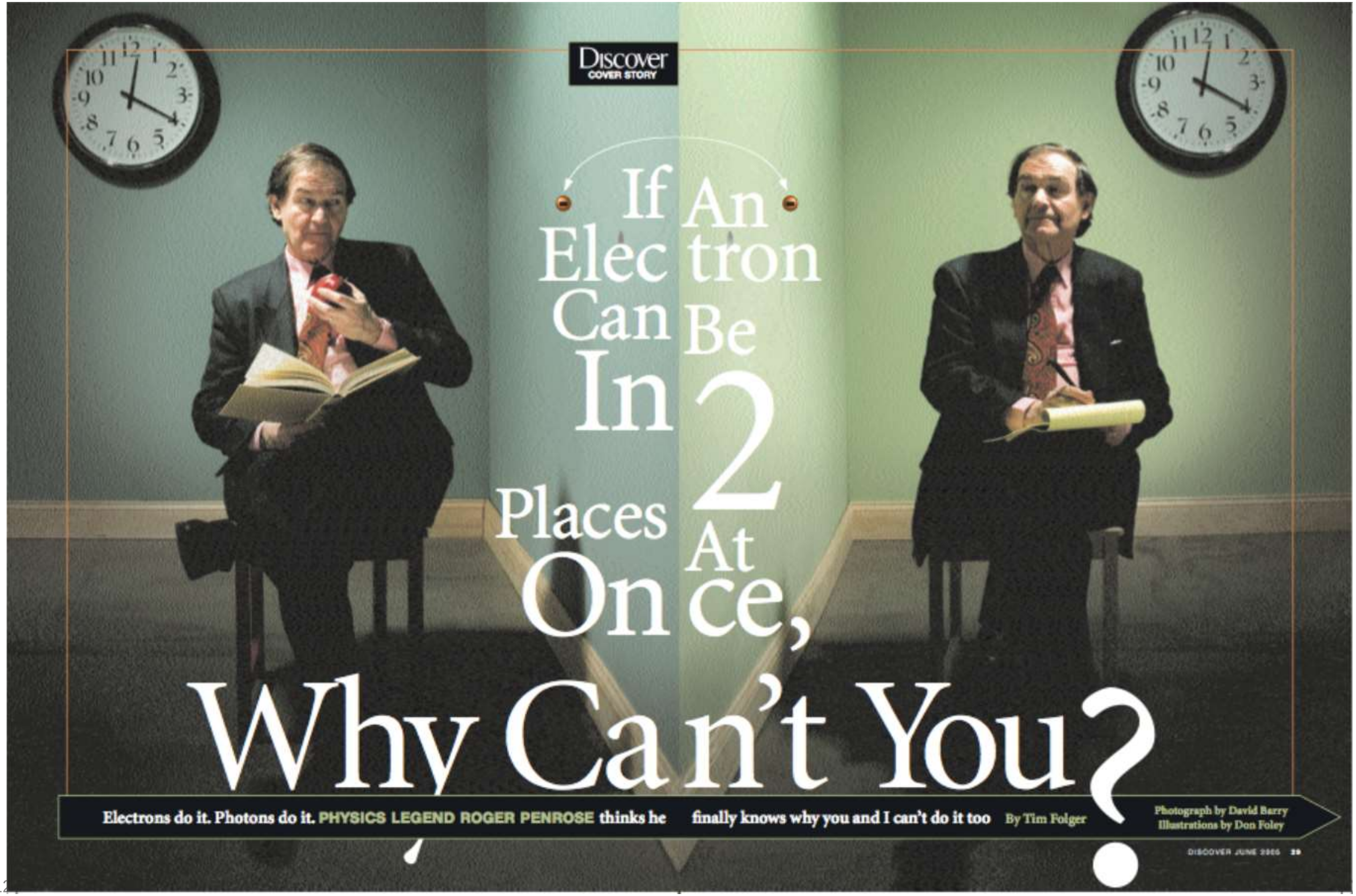
空间相干性：相位确定

不需要相位稳定装置

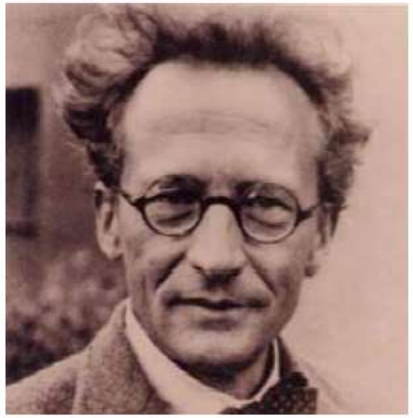
量子叠加态

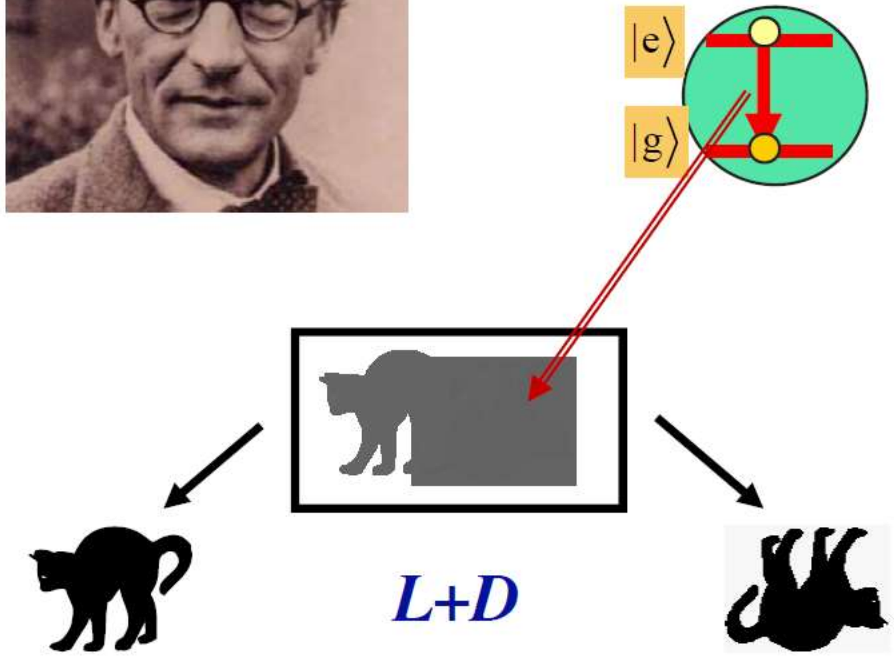


Classical Bits
1
0
Qubits
And Superpositioning
0
1
Entangled Qubits


Discover
COVER STORY
If An Electron Can Be In 2 Places At Once, Why Can't You?
Electrons do it. Photons do it. PHYSICS LEGEND ROGER PENROSE thinks he finally knows why you and I can't do it too
By Tim Folger
Photograph by David Barry
Illustrations by Don Foley

为什么常规条件下不存在宏观量子态?





薛定锷猫

“月亮的客观存在”

薛定谔认为打开黑箱前猫处在既死又活、非死非活的状态这个结论是谬论，许多人也持类似观点。

❶ 那么，应该如何理解薛定谔猫佯谬呢？

目前普遍接受的解释是：

- 猫是一个由大量原子、分子构成的宏观体系，其运动只表现出粒子性、并不表现出明显的波粒二象性 (退相干效应). 因此，猫不宜作为量子力学直接的研究对象。

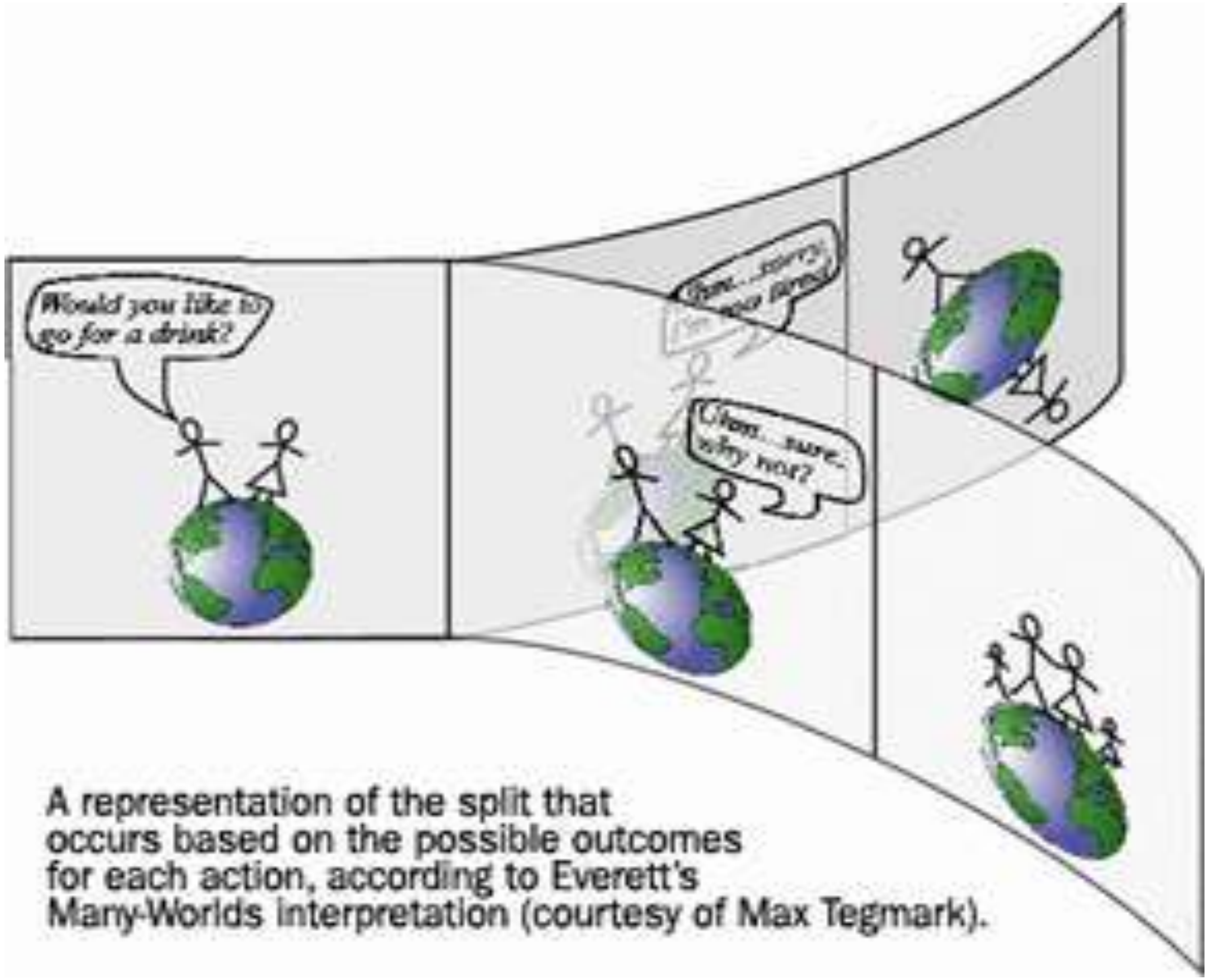
- 由于猫本身不适合作为量子力学的研究对象，所以薛定谔猫佯谬问题中的外部观测者打开黑箱门户的动作并不构成一次测量。

- 使用量子力学研究薛定谔猫佯谬时，合适的研究对象是黑箱内放射性物质中的原子。因此，原子衰变触发盖革计数器构成一次测量。猫不会处在既死又活的叠加态(态的线性组合体现的是所谓的相干性，这是波的性质)，它的死活实际上在外部观测者打开黑箱门户之前就已经确定了。

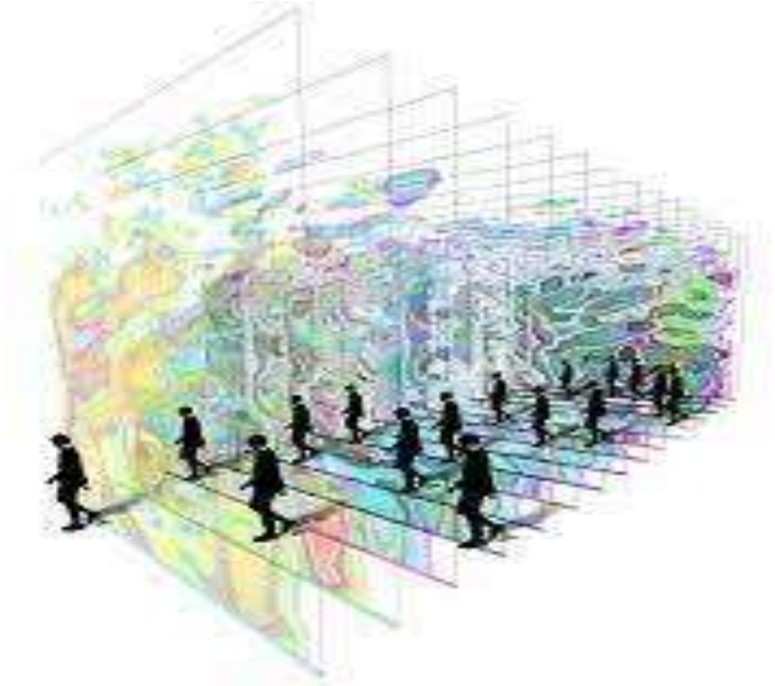
片名原本叫“Coherence”，直译为“相干性”。

互相干预了对方导致的不同的决定，而决定又干预了其他人，这样无尽的衍生发展产生了无限可能。

Parallel universe

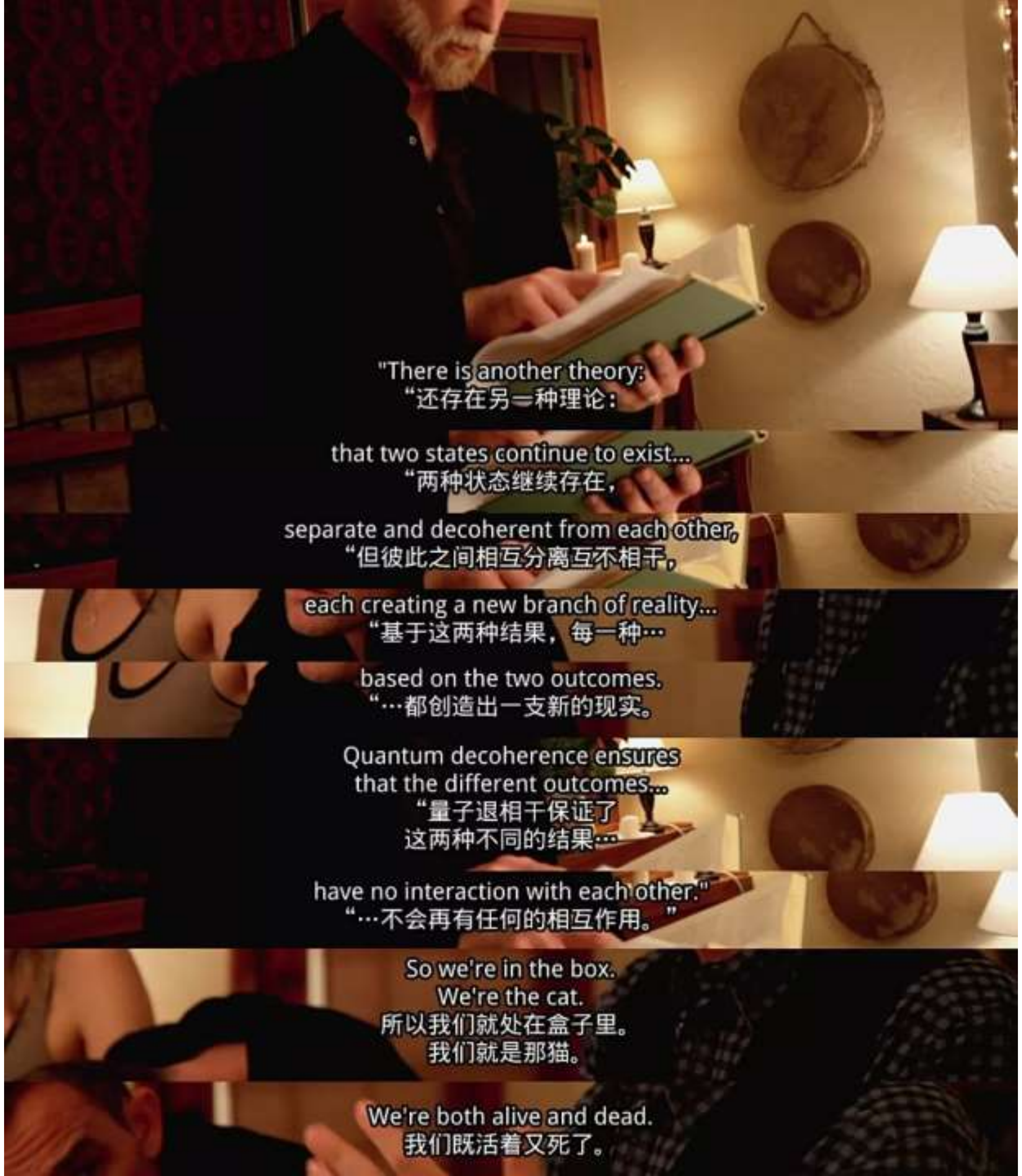


A representation of the split that occurs based on the possible outcomes for each action, according to Everett's Many-Worlds interpretation (courtesy of Max Tegmark).



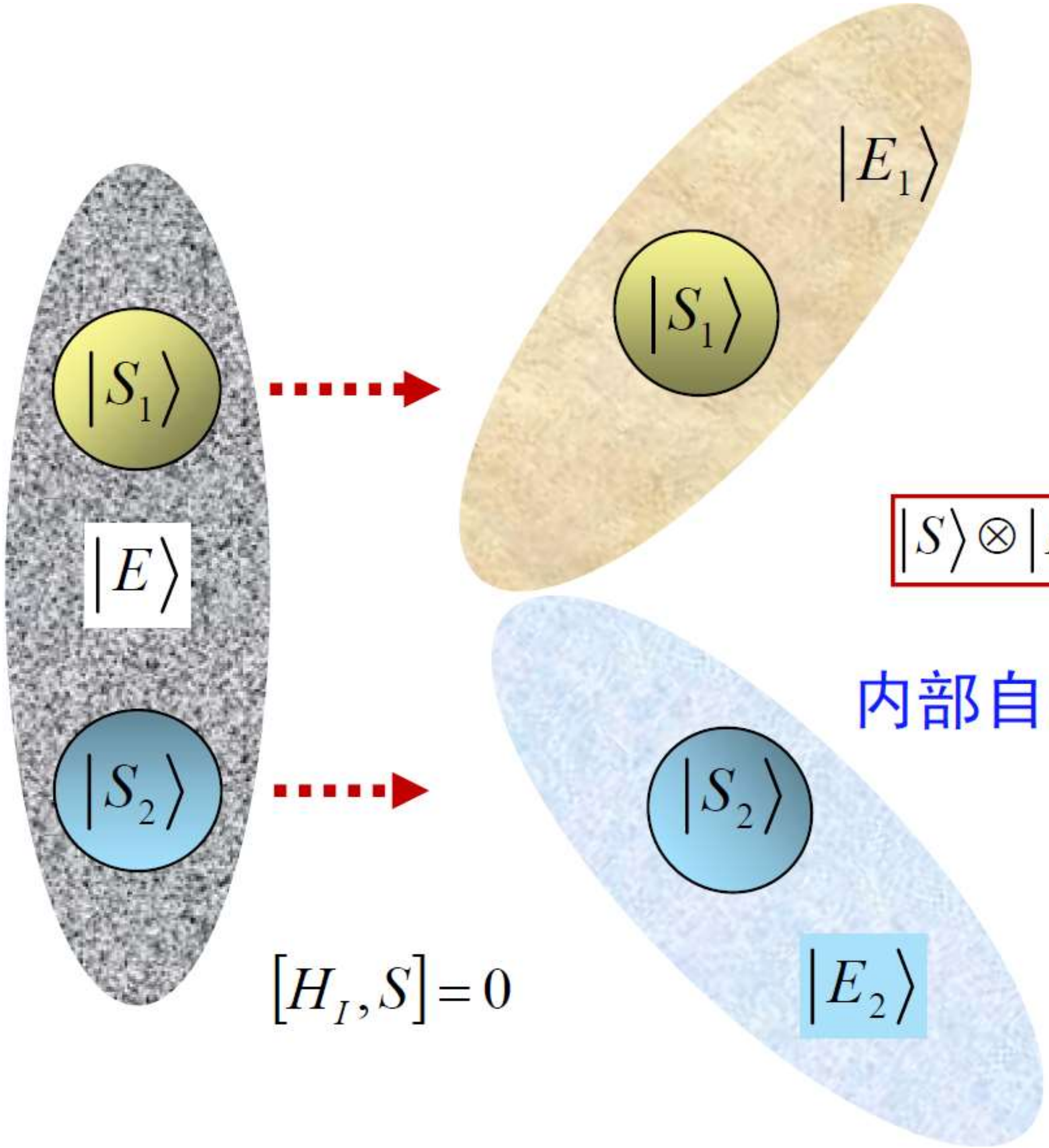
This whole nigth we’ve been worrying there’s some dark version of us out there somewhere.

What if we’re the dark version?



最后天亮，彗星解体，世界看似回到正常的时候，发生的也不是“塌缩”，而是多世界之间“退相干”

与环境的纠缠导致宏观物体退相干



$[H_I, S] = 0$

Wigner 1967；
Joos，Zeh, 1985

外部环境中粒子散射

$$\boxed{|S\rangle \otimes |E\rangle \xrightarrow{H_I} |S\rangle \otimes |E_S\rangle}$$

内部自由度与集体自由度耦合

R. Omnes，2001

孙昌璞， 1993,2001
因子化结构与宏观热力学极限

C.P. Sun, et al 1993-2003,10篇文章

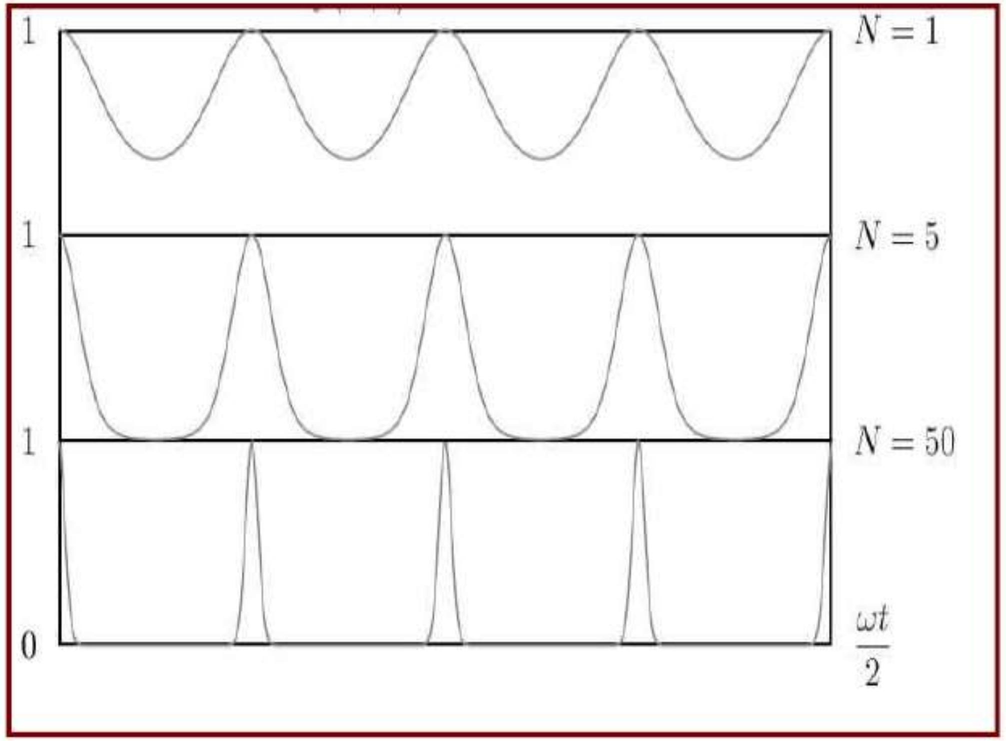
$$\left|d\right\rangle = \prod_{j=1}^{n} \left|d^{[j]}\right\rangle \qquad \left|l\right\rangle = \prod_{j=1}^{n} \left|l^{[j]}\right\rangle$$

$$\left\langle d \middle| l \right\rangle = \prod_{j=1}^{N} \left\langle d^{[j]} \middle| l^{[j]} \right\rangle \xrightarrow{N\to\infty} 0$$

$\Downarrow$

随机相位

$$\left\langle l \middle| d \right\rangle == ?\left(\exp[-i\sum_{k=1}^{N} \theta_{mn}] \right)_{平均} \to 0$$



C.P. Sun, et al，2006

随机位相解释量子退相干 Heisenberg, 1927

测量前 $$|\psi\rangle = c_0|0\rangle + c_1|1\rangle$$

$$\langle x|\phi_n\rangle == \phi_n(x), n = 1,0$$

$$\psi(x) = \langle x|\psi\rangle = c_1\phi_1(x) + c_0\phi_0(x)$$

测量后引入随机位相 $$|\psi'\rangle = c_0|0\rangle + c_1 e^{i\theta}|1\rangle$$

导致密度矩阵非对角项消失

$$\boxed{\rho(x,x) = |c_1|^2|\phi_1(x)|^2 + |c_0|^2|\phi_0(x)|^2 + \underbrace{c^*{}_1 c_0 e^{i\theta}\phi^*{}_1(x)\phi_0(x) + c.c}_{\text{干涉条纹:}}}$$

$$\boxed{\langle \exp(i\theta)\rangle \to 0}$$

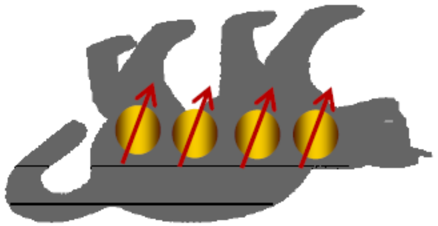
$$<\rho(x,x)> \to |c_1|^2|\phi_1(x)|^2 + |c_0|^2|\phi_0(x)|^2$$

量子几率到经典几率的转换

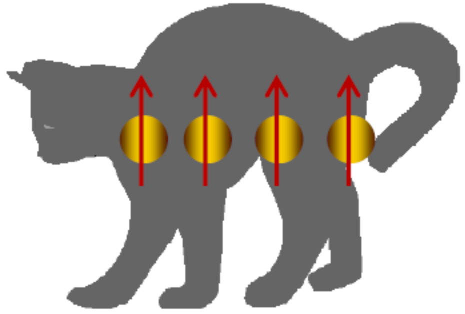
薛定谔猫与宏观物体退相干

Sun，1993, 2001

内部态 ⇔ 内部环境



$$\boxed{|猫\rangle=\frac{1}{\sqrt{2}}\left[|死\rangle\otimes\prod_{j=1}|d_j\rangle+|活\rangle\otimes\prod_{j=1}|l_j\rangle\right]}$$



$$\rho = Tr_{内}\left(|猫\rangle\langle 猫|\right)=\frac{1}{2}|死\rangle\langle 死|+\frac{1}{2}|活\rangle\langle 活|+\underbrace{\prod_j\left\langle d_j\,|\,l_j\right\rangle}_{\to 0}\otimes\cdots$$

$$\boxed{\langle d\,|\,l\rangle=\prod_{j=1}^{N}\left\langle d^{[j]}\middle|l^{[j]}\right\rangle=\xrightarrow{N\to\infty}0}$$

Quantum error: 量子信息因退相干和其他量子噪声所引起的错误。

- 质数分解SHOR量子算法要求的计算结果正确率不能低于99.3%。
- 量子比特是量子计算出错的源头，增加更多量子比特，意味着出现错误的可能也会增加。

根据谷歌的论文结果来看悬铃木的保真度只有0.2%。

容错的量子计算：不仅可以处理存储的量子信息上的噪声，还可以处理错误的量子门、错误的量子准备、错误的测量等。

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Protecting a bosonic qubit with autonomous quantum error correction

Jeffrey M. Gertler, Brian Baker, Juliang Li, Shruti Shirol, Jens Koch & Chen Wang

Nature **590**, 243–248 (2021) | Cite this article

✓ 保护薛定谔猫的最直接的策略是使盒子尽可能紧密。

容错的量子计算：不仅可以处理存储的量子信息上的噪声，还可以处理错误的量子门、错误的量子准备、错误的测量等。

✓ 增加编码系统中的码距。

容错的量子计算：不仅可以处理存储的量子信息上的噪声，还可以处理错误的量子门、错误的量子准备、错误的测量等。

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Suppressing quantum errors by scaling a surface code logical qubit

Google Quantum AI

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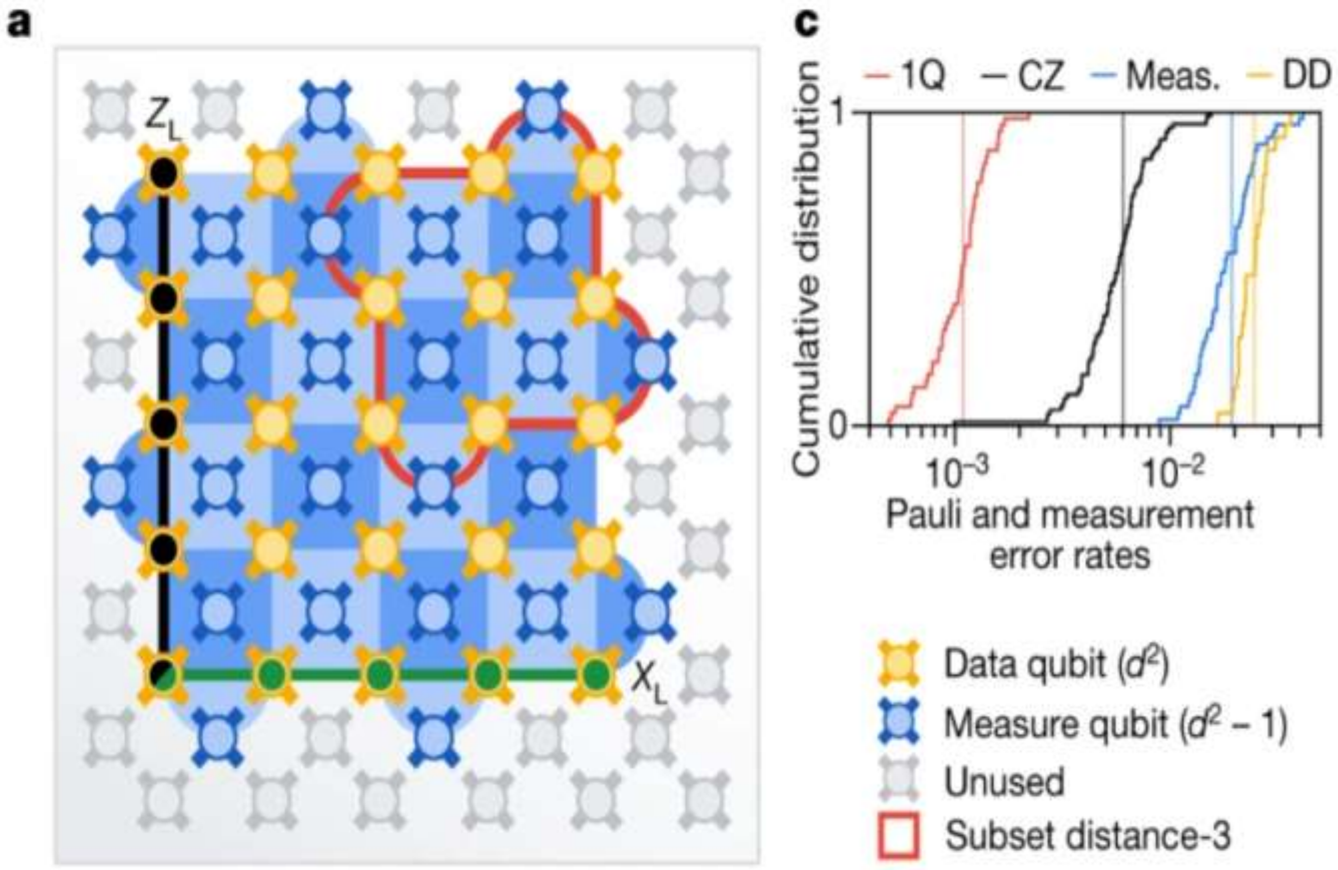


✓ 增加编码系统中的码距。

容错的量子计算：不仅可以处理存储的量子信息上的噪声，还可以处理错误的量子门、错误的量子准备、错误的测量等。

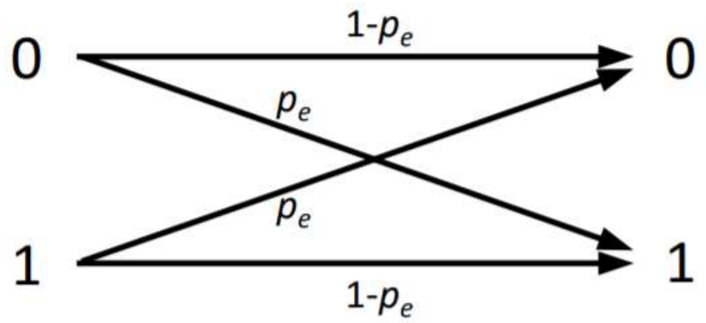
$$0 \rightarrow 000$$

$$1 \rightarrow 111$$

✓ 使用信息冗余，即多次存储信息，如果发现信息不吻合，只需进行多数表决即可。

Classical errors: the binary symmetric channel

One of the simplest models for single-bit (classical) errors is the *binary symmetric channel*, in which each possible state of the bit, 0 and 1 "flips" to the other with some probability p_e:



Note that, without loss of generality we can assume $p_e \leq 0.5$, because if $p_e > 0.5$ then it is more likely than not that a bit-flip has occurred, so we can interpret a received 0 as a 1 and vice-versa. In the case where $p_e = 0.5$ we cannot recover any information from the channel.

Classical error correction: the three-bit repetition code

If we wish to send a single bit over a binary symmetric channel, then we can encode the bit, by simply repeating it three times. That is, if we wish to transmit a 0, we send three bits (sequentially) in the state 0, and likewise for 1. This can be denoted as:

$$0 \to 000$$
$$1 \to 111$$

Once the three bits have been received, they are decoded by a "majority vote". So in order for an error to occur, it is necessary that either two of the three bits have been flipped (which can occur in three different ways), or all three have been, that is:

$$p_e' = 3p_e^2(1 - p_e) + p_e^3$$

Which is less than p_e if $p_e < 0.5$. Typically, p_e is small, and we can describe this as suppressing the error to $\mathcal{O}(p_e^2)$.

经典纠错方法能否直接应用于量子纠错?

- 信息比特的复制，例如比特0需要经过信息复制变成000。
- 离散的错误，例如单个比特或者两个比特反转引发的错误。
- 纠错码的测量确定性。

(1) 量子信息：将单量子比特编码

$$\alpha\,|0\rangle+\beta\,|1\rangle \longrightarrow \alpha\,|000\rangle+\beta\,|111\rangle$$

（原量子比特）（通过3个量子比特编码后）

注：量子比特的编码方式不是克隆而是直接制造，比如上述量子态编码可以通过3光子纠缠来实现。

（2）**差错连续性。**经典信息的错误是分立的，而量子位态取值与一个二维的Hilbert空间的任意态失，所以量子信息的错误可以是连续的。比如在一个量子门的操作下一个量子位态应由

$$\alpha\,|0\rangle+\beta\,|1\rangle \longrightarrow \alpha\,|0\rangle+\beta e^{\varphi i}\,|1\rangle$$

但是由于误差，使得

$$\alpha\,|0\rangle + \beta e^{\varphi i}\,|1\rangle \longrightarrow \alpha\,|0\rangle + \beta e^{i(\varphi+\delta)}\,|1\rangle$$

虽然δ是一个小量，但依然是错误的量子态，随着时间的推移，这些小量会积累起来变成大错。

（3）**不可测量性。**由于不可观测的量子特性，使得纠错过程出现两个难题：

1是否出错？

2出现什么错误？

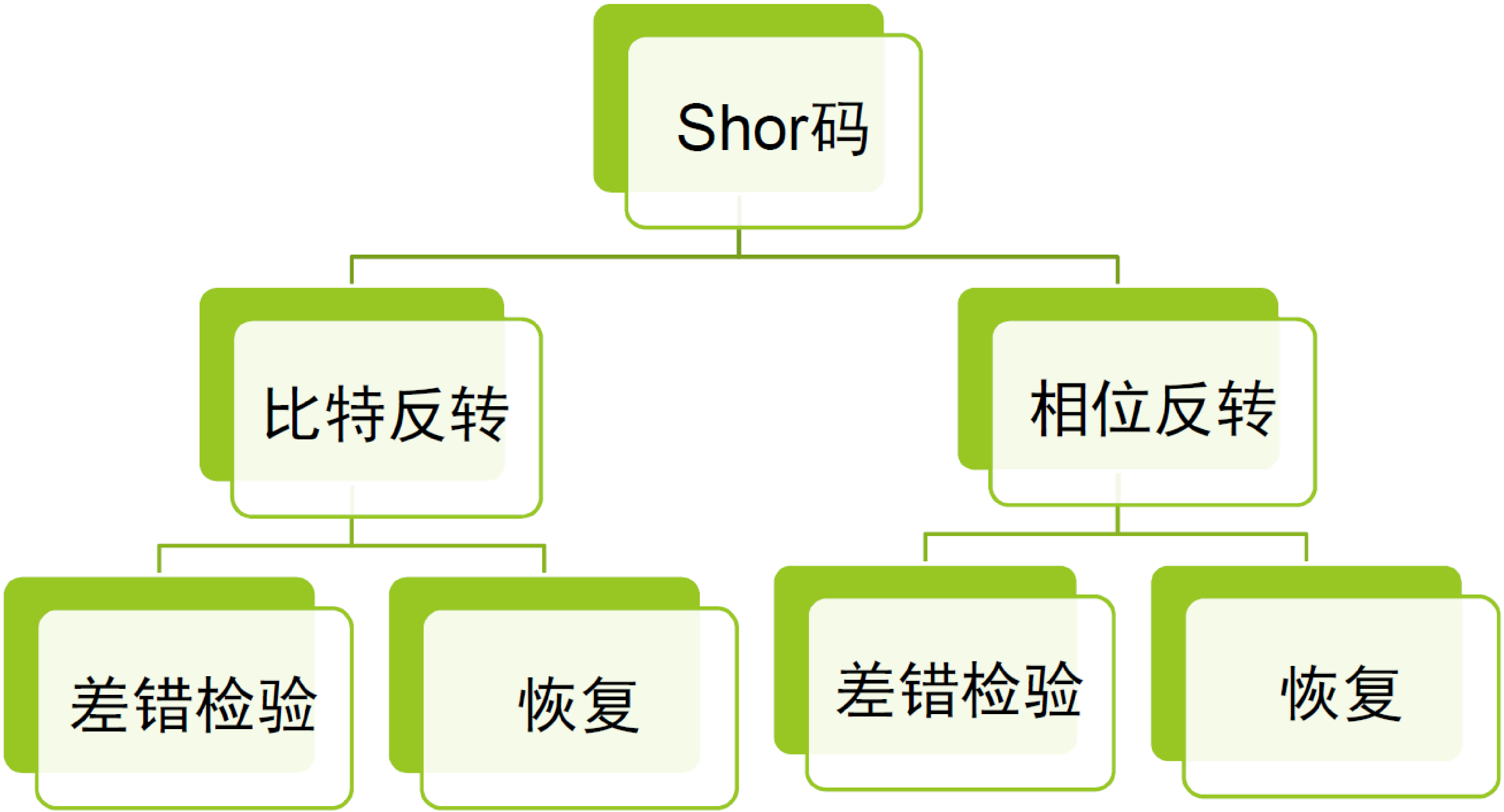
Complicating factors with quantum error correction

Ostensibly, it appears that we cannot directly transfer classical error correction techniques to the problem of quantum error correction for three reasons:

1. The no-cloning principle forbids the copying of quantum states.
2. Measurement destroys quantum information.
3. Quantum states are continuous: $\alpha|0\rangle + \beta|1\rangle$.

Nevertheless, we shall see that with some ingenuity we *can* correct quantum errors.

量子纠错码



比特翻转

方法1

$P_0 = |000\rangle\langle 000| + |111\rangle\langle 111|$无差错

$P_1 = |100\rangle\langle 100| + |011\rangle\langle 011|$第一比特翻转

$P_2 = |010\rangle\langle 010| + |101\rangle\langle 101|$第二比特翻转

$P_3 = |001\rangle\langle 001| + |110\rangle\langle 110|$第三比特翻转

例如第一比特翻转时，比特状态为：

$$\alpha |100\rangle + \beta |011\rangle$$

这时测量状态φ时，$\langle \varphi | P_1 | \varphi \rangle = 1$

测量并不会引起状态的改变，测量前后比特状态都为$\alpha |100\rangle + \beta |011\rangle$，通过测量只得到了φ的差错信息，并不会得知α，β的值

方法2

由Z_1 Z_2与Z_2 Z_3代替 P_0 P_1 P_2 P_3

Z_1Z_2测量目的是比较第1量子比特与第2量子比特

$$Z_1Z_2 = (|00\rangle\langle 00| + |11\rangle\langle 11|)\otimes I - (|01\rangle\langle 01| + |10\rangle\langle 10|)\otimes I,$$

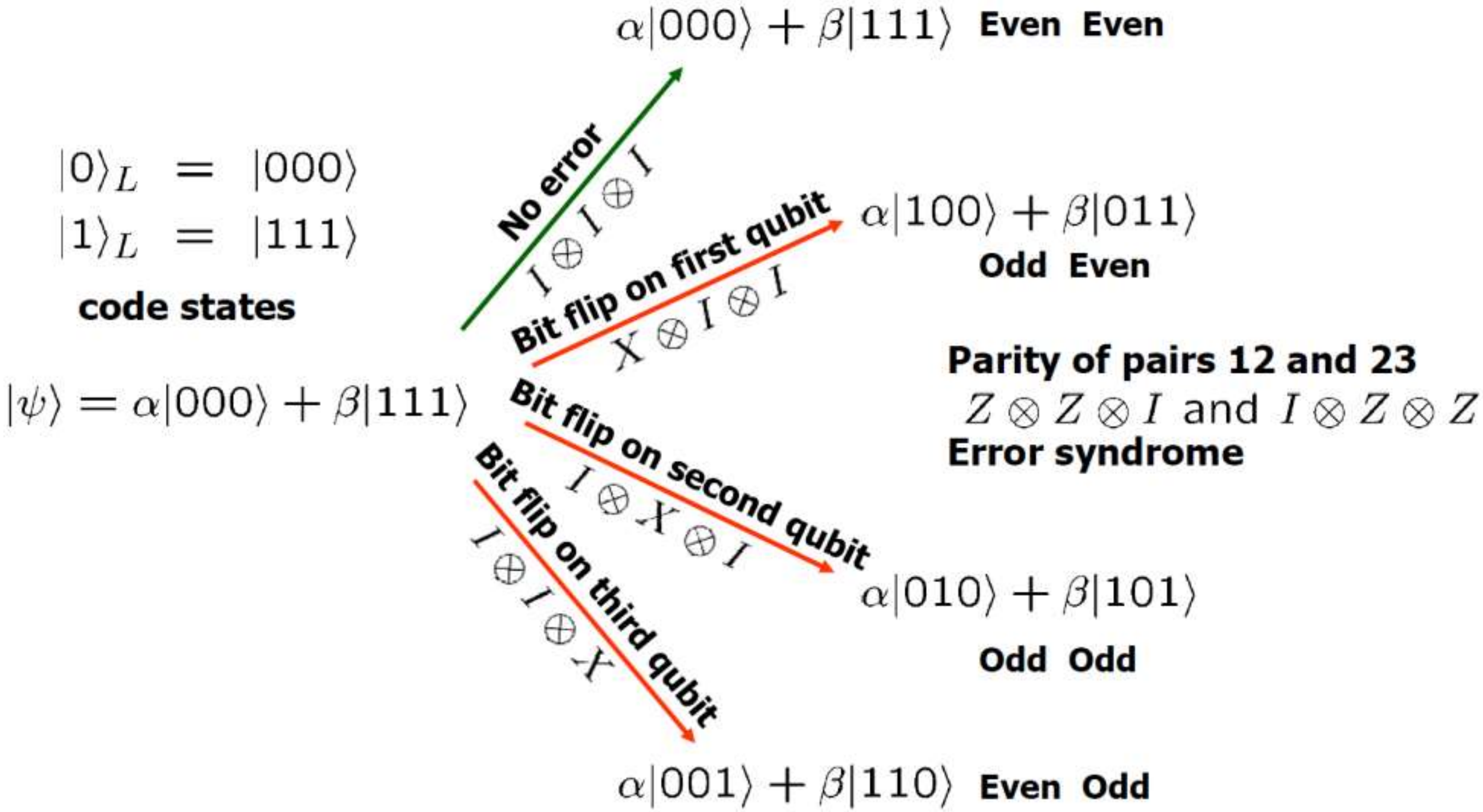
若1与2相同，则给出+1，若不同给出-1

Z_2Z_3测量目的是比较第2量子比特与第3量子比特

若2与3相同，则给出+1，若不同给出-1

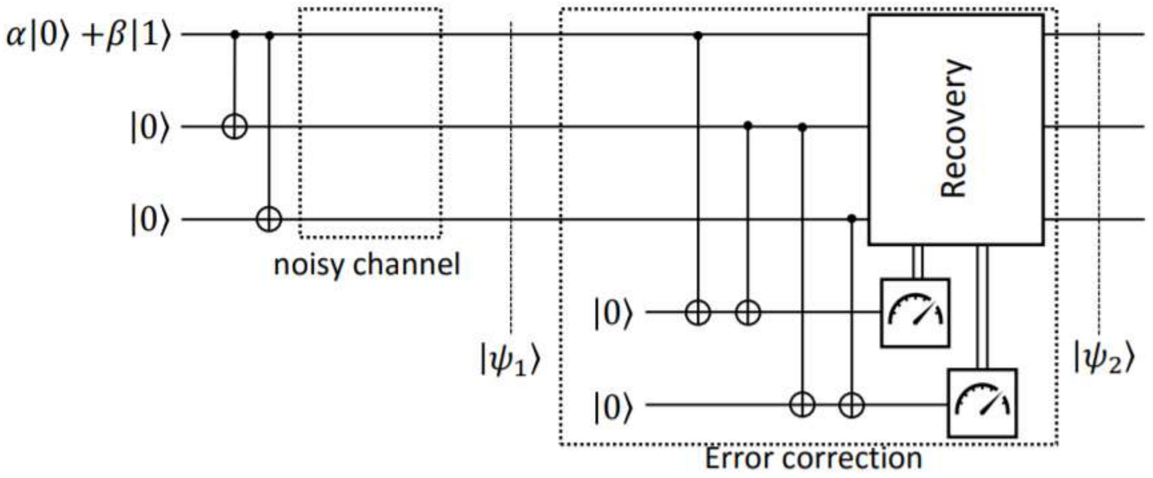
由此，可以判断出错误类型，却不测量有关编码后量子状态的幅值α，β

纠单比特错误量子码


$|0\rangle_L = |000\rangle$
$|1\rangle_L = |111\rangle$
code states
$|\psi\rangle = \alpha|000\rangle + \beta|111\rangle$
No error
$I \oplus I \oplus I$
$\alpha|000\rangle + \beta|111\rangle$ **Even Even**
Bit flip on first qubit
$X \oplus I \oplus I$
$\alpha|100\rangle + \beta|011\rangle$
Odd Even
Bit flip on second qubit
$I \oplus X \oplus I$
$\alpha|010\rangle + \beta|101\rangle$
Odd Odd
Bit flip on third qubit
$I \oplus I \oplus X$
$\alpha|001\rangle + \beta|110\rangle$ **Even Odd**
Parity of pairs 12 and 23
$Z \otimes Z \otimes I$ and $I \otimes Z \otimes Z$
Error syndrome

The three-qubit bit-flip code: error detection and recovery

To detect and recover errors, we supplement the circuit with two ancillas that we use for error detection:



We can thus detect and recover single-qubit bit-flips:

Bit-flip	$\|\psi_1\rangle$	M_1	M_2	**Recovery**	$\|\psi_2\rangle$
-	$\alpha\|000\rangle + \beta\|111\rangle$	0	0	$I \otimes I \otimes I$	$\alpha\|000\rangle + \beta\|111\rangle$
1	$\alpha\|100\rangle + \beta\|011\rangle$	1	0	$X \otimes I \otimes I$	$\alpha\|000\rangle + \beta\|111\rangle$
2	$\alpha\|010\rangle + \beta\|101\rangle$	1	1	$I \otimes X \otimes I$	$\alpha\|000\rangle + \beta\|111\rangle$
3	$\alpha\|001\rangle + \beta\|110\rangle$	0	1	$I \otimes I \otimes X$	$\alpha\|000\rangle + \beta\|111\rangle$

That is, we have made comparative *parity-check* measurements that tell us only about the error and not about the quantum state itself, and so these measurements have not destroyed the quantum state.

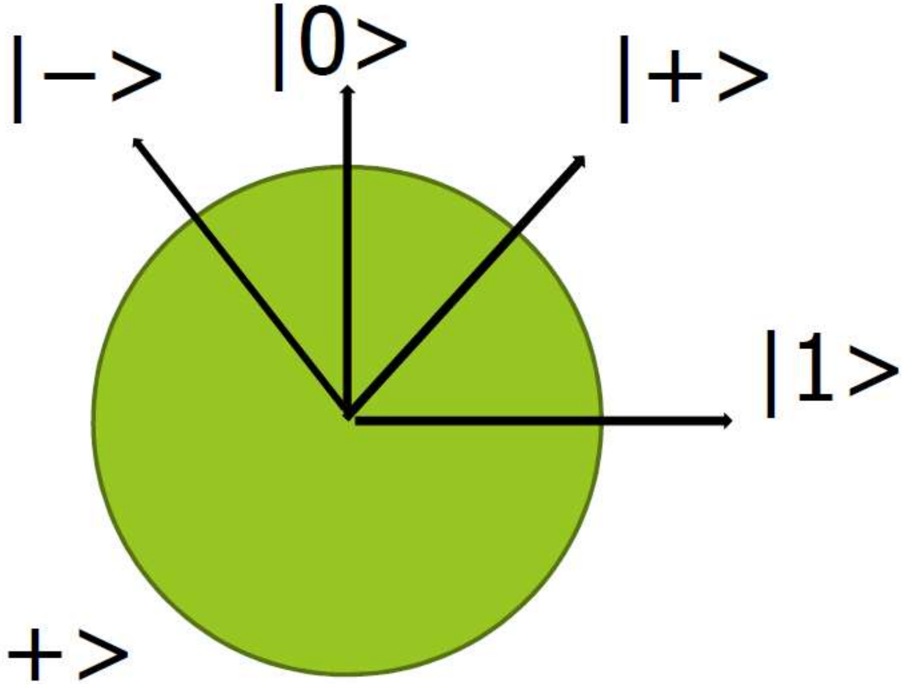
相位翻转

由α |0>+β |1> ⟶ α |0>−β |1>（相位翻转）

处理方式：将相位翻转转化为比特翻转

设 $|+\rangle \equiv (|0\rangle + |1\rangle)/\sqrt{2}$,

$|-\rangle \equiv (|0\rangle - |1\rangle)/\sqrt{2}$.



得出

$|0_L\rangle = |000\rangle$ ⟹ $|0_L\rangle = |{+}{+}{+}\rangle$

$|1_L\rangle = |111\rangle$ ⟹ $|1_L\rangle = |{-}{-}{-}\rangle$

The three-qubit phase-flip code

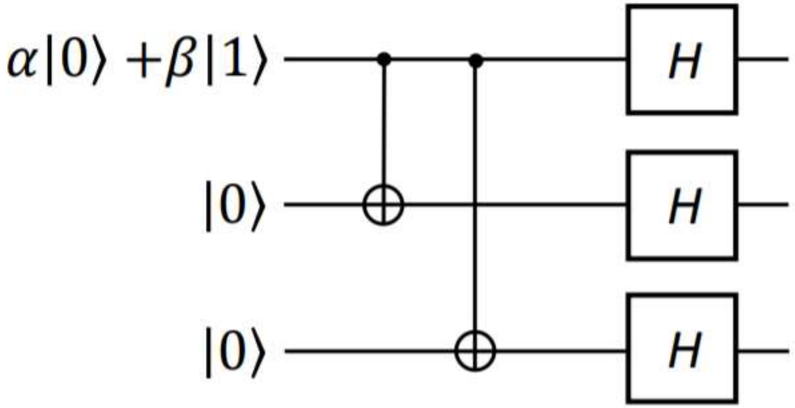
The three-qubit bit-flip code demonstrates how we can overcome two of the possible problems with quantum error correction that we previously identified:

- We can use entanglement to enable repetition.
- We can detect errors using parity-check measurements that do not destroy the quantum information.

However, we still have not addressed the fact that quantum states are continuous. To begin to do this, we'll look at an error correction code for a different type of error. The three-qubit phase-flip code has the following action on an arbitrary single-qubit state:

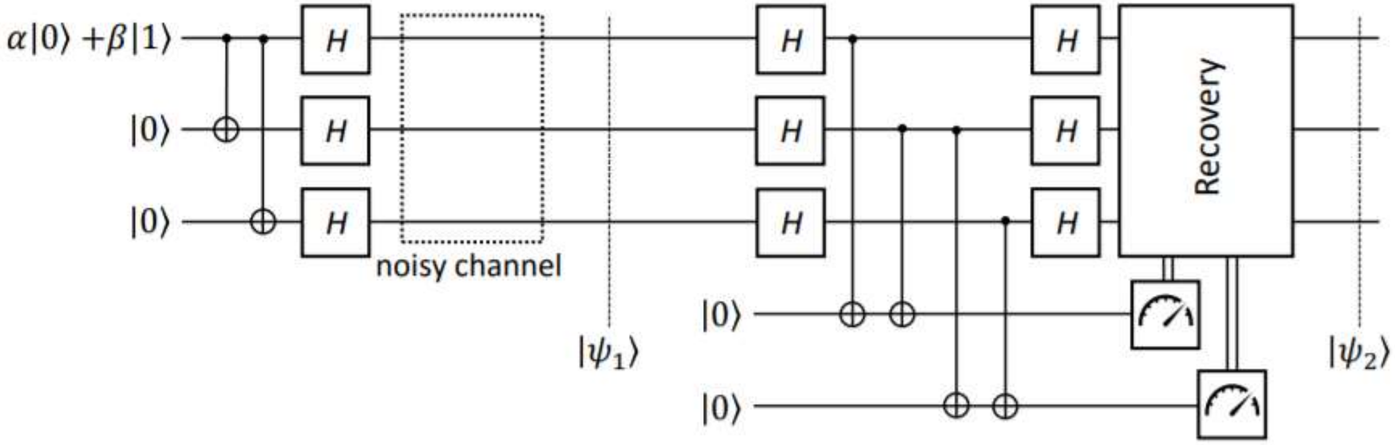
$$(\alpha\,|0\rangle + \beta\,|1\rangle)\,|0\rangle^{\otimes 2} \to \alpha\,|{+}{+}{+}\rangle + \beta\,|{-}{-}{-}\rangle$$

Which is achieved by the following circuit:



Three-qubit phase-flip code: error detection and recovery

Once again, to detect and recover errors, we supplement the circuit with two ancillas that we use for error detection:



By definition, a phase flip sends:

$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \rightarrow \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) = |-\rangle$$

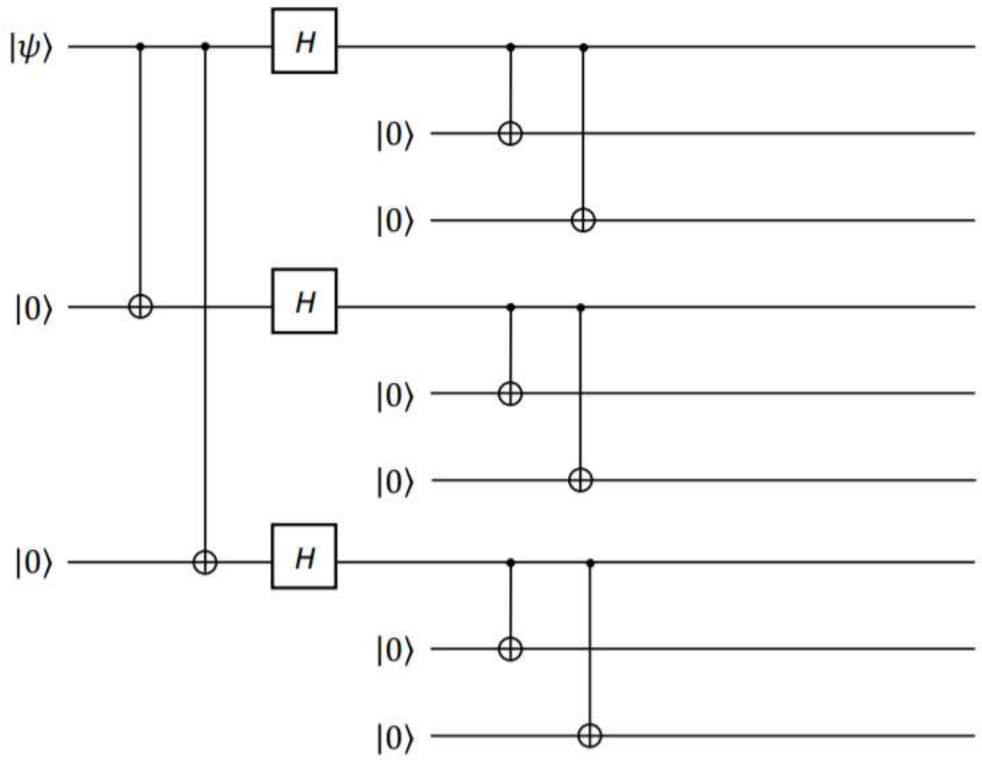
$$|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \rightarrow \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = |+\rangle$$

Thus we have:

Ph-flip	$\lvert\psi_1\rangle$	M_1	M_2	**Recovery**	$\lvert\psi_2\rangle$
-	$\alpha\lvert+++\rangle + \beta\lvert---\rangle$	0	0	$I \otimes I \otimes I$	$\alpha\lvert+++\rangle + \beta\lvert---\rangle$
1	$\alpha\lvert-++\rangle + \beta\lvert+--\rangle$	1	0	$Z \otimes I \otimes I$	$\alpha\lvert+++\rangle + \beta\lvert---\rangle$
2	$\alpha\lvert+-+\rangle + \beta\lvert-+-\rangle$	1	1	$I \otimes Z \otimes I$	$\alpha\lvert+++\rangle + \beta\lvert---\rangle$
3	$\alpha\lvert++-\rangle + \beta\lvert--+\rangle$	0	1	$I \otimes I \otimes Z$	$\alpha\lvert+++\rangle + \beta\lvert---\rangle$, [1]

The Shor code

The Shor code is a 9-qubit code which is constructed by *concatenating the three-qubit bit-flip and three-qubit phase-flip codes*:



This encodes the computational basis states as follows:

$$|0\rangle \rightarrow |0_L\rangle = \frac{1}{2\sqrt{2}}(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)$$

$$|1\rangle \rightarrow |1_L\rangle = \frac{1}{2\sqrt{2}}(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)$$

Correcting bit-flips with the Shor code

The Shor code can detect and correct a bit-flip on any single qubit. For example, suppose we have an arbitrary quantum state $\alpha|0\rangle + \beta|1\rangle$ which we encode with the Shor code as:

$$\frac{1}{2\sqrt{2}}\Big(\alpha(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle) \\ + \beta(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)\Big)$$

If a bit-flip occurs on the first qubit, the state becomes:

$$\frac{1}{2\sqrt{2}}\Big(\alpha(|100\rangle + |011\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle) \\ + \beta(|100\rangle - |011\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)\Big)$$

Which can be detected (and thus recovered from) by parity-check measurements between the first three qubits as in the three-qubit bit-flip code. By symmetry we can see that the same principle applies to all of the nine qubits.

Correcting phase-flips with the Shor code

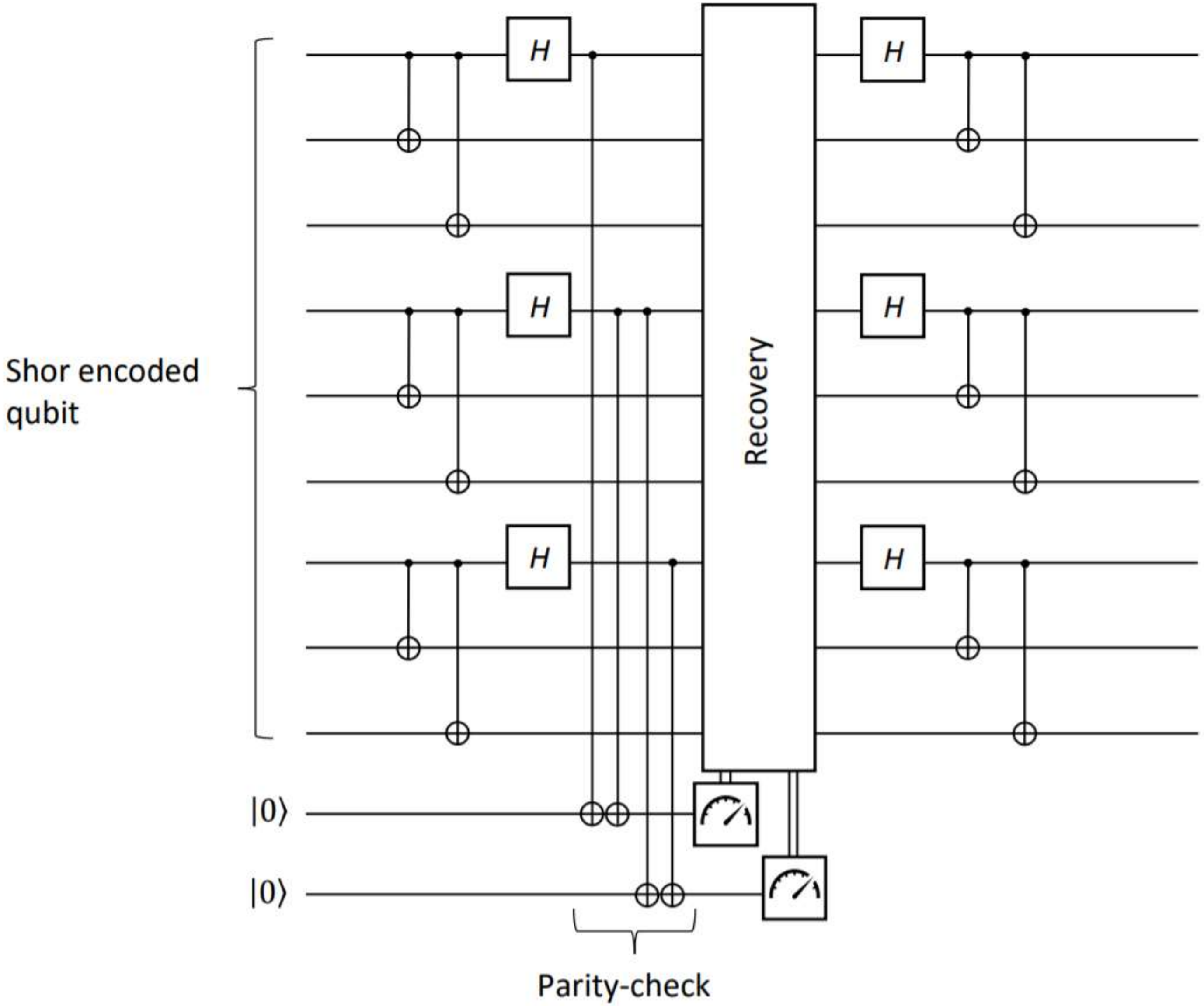
The Shor code can also detect and correct a phase-flip on any single qubit. If a phase-flip occurs on the first qubit, the state becomes:

$$\frac{1}{2\sqrt{2}}\Big(\alpha(|000\rangle - |111\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)$$
$$+ \beta(|000\rangle + |111\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)\Big)$$

The key idea here is to detect which of the three blocks of three qubits has experienced a change of sign. This is achieved using the circuit shown on the following slide.

We can also correct combinations of bit- and phase-flips in this way.

Circuit for correcting phase-flips with the Shor code



Correcting any single qubit error with the Shor code (1)

Suppose the first qubit encounters an error which sends $|0\rangle \to a|0\rangle + b|1\rangle$ and $|1\rangle \to c|0\rangle + d|1\rangle$. We thus have the state:

$$\frac{1}{2\sqrt{2}}\Big(\alpha(a|000\rangle + b|100\rangle + c|011\rangle + d|111\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)$$

$$+ \beta(a|000\rangle + b|100\rangle - c|011\rangle - d|111\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)\Big)$$

Letting $k+m=a$, $k-m=d$, $l+n=b$ and $l-n=c$, we get

$$\frac{1}{2\sqrt{2}}\Bigg(k\Big(\alpha(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)$$

$$+ \beta(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)\Big)$$

$$+l\Big(\alpha(|100\rangle + |011\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)$$

$$+ \beta(|100\rangle - |011\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)\Big)$$

$$+m\Big(\alpha(|000\rangle - |111\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)$$

$$+ \beta(|000\rangle + |111\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)\Big)$$

$$+n\Big(\alpha(|100\rangle - |011\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)$$

$$+ \beta(|100\rangle + |011\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)\Big)\Bigg)$$

Correcting any single qubit error with the Shor code (2)

As before, we first perform parity-check measurements to detect a bit-flip. The parity check for a bit-flip in the first block of three qubits requires two ancillas (the first comparing the first and second qubits, the second comparing the second and third qubits), whose state (after the parity-check CNOTs) we can append to the Shor code state:

$$
\begin{aligned}
\frac{1}{2\sqrt{2}}\Bigg(& k\Big(\alpha(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle) \\
&\qquad + \beta(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)\Big)\,|00\rangle \\
& + l\Big(\alpha(|100\rangle + |011\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle) \\
&\qquad + \beta(|100\rangle - |011\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)\Big)\,|10\rangle \\
& + m\Big(\alpha(|000\rangle - |111\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle) \\
&\qquad + \beta(|000\rangle + |111\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)\Big)\,|00\rangle \\
& + n\Big(\alpha(|100\rangle - |011\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle) \\
&\qquad + \beta(|100\rangle + |011\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)\Big)\,|10\rangle\Bigg)
\end{aligned}
$$

Correcting any single qubit error with the Shor code (3)

If the parity-check measurement outcome is 00, the state collapses to (un-normalised):

$$
\begin{aligned}
&k\Big(\alpha(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle) \\
&\qquad + \beta(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)\Big) \\
&+m\Big(\alpha(|000\rangle - |111\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle) \\
&\qquad + \beta(|000\rangle + |111\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)\Big)
\end{aligned}
$$

In which case there is no bit-flip. Or if the measurement outcome is 10:

$$
\begin{aligned}
&l\Big(\alpha(|100\rangle + |011\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle) \\
&\qquad + \beta(|100\rangle - |011\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)\Big) \\
&+n\Big(\alpha(|100\rangle - |011\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle) \\
&\qquad + \beta(|100\rangle + |011\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)\Big)\Big)
\end{aligned}
$$

i.e., a bit-flip has occurred which we can then correct.

Correcting any error by correcting only bit- and phase-flips

Following the bit-flip parity-check measurement (and correction if necessary) we perform a parity-check measurement to check for a phase flip. Using the same argument as for the bit-flip detection, if we measure 0 the state collapses to:

$$\begin{aligned}&\alpha(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)\\&\quad + \beta(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)\end{aligned}$$

Or if we measure a 1 we get:

$$\begin{aligned}&\alpha(|000\rangle - |111\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)\\&\quad + \beta(|000\rangle + |111\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)\end{aligned}$$

i.e., a phase-flip has occurred which we can then correct. Therefore we have recovered the original state.

Therefore performing bit- and phase-flip parity-check measurements collapses a general state into the case where either the bit / phase flip has occurred or not as per the measurement outcome. This remarkable property allows us to correct a continuum of errors by performing only bit- and phase-flip checks.

What to remember

We have seen that there are three obstacles to applying the techniques and principles of classical error correction directly to quantum error correction, each of which can be worked around:

- The no-cloning principle means that we cannot simply copy quantum states in repetition codes – instead we can use entangling to "copy" the information.
- Measurements destroy quantum information: so instead we design the error correcting codes so that the measurements only tell us whether an error has occurred, and nothing about the quantum state itself.
- Quantum errors are continuous: but we have seen that the process of error correction effectively digitises the errors.

Additionally, we have seen that, in practise, classical error correction codes are typically more sophisticated and efficient than simple repetition codes, and that these can be used to design quantum error correction codes, of which the Steane code is an important example.