Quantum Information and Quantum Computation

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Lecture 2

Introduction to quantum mechanics

Bits

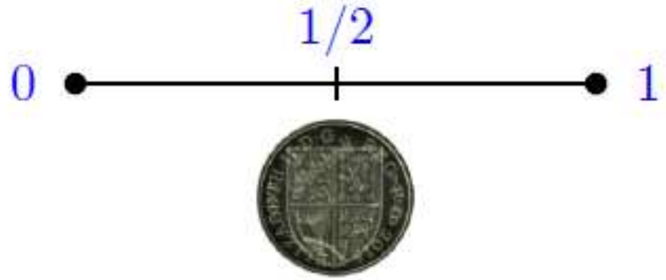
A building block of classical computational devices is a two-state system or a classical bit:

$$0 \bullet \qquad\qquad\qquad \bullet 1$$

Indeed, any system with a finite set of discrete and stable states, with controlled transitions between them, will do:

Probabilistic bits

When you don't know the state of a system exactly but only have partial information, you can use probabilities to describe it:



It is convenient to represent system's state using vectors:

 $= \begin{pmatrix} 1 \\ 0 \end{pmatrix}$  $= \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Then a uniformly random bit is represented by

 $= \frac{1}{2}$  $+ \frac{1}{2}$  $= \frac{1}{2}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Using probabilities to represent information (or lack of it...) is more useful than you might think!

Quantum superposition. . .

In nature, the state of an actual physical system is more uncertain than we are used to in our daily lives. . .

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That's why complex amplitudes rather than probabilities are used in quantum mechanics!

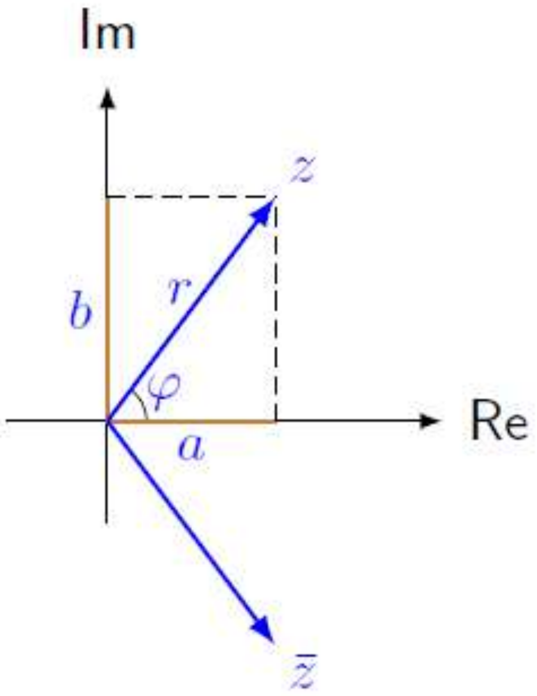
Complex numbers ($i^2 = -1$)

Representations:

- algebraic: $z = a + ib$
- exponential: $z = re^{i\varphi} = r(\cos\varphi + i\sin\varphi)$

Operations:

- addition and subtraction:
  $(a+ib) \pm (c+id) = (a \pm c) + i(b \pm d)$
- multiplication:
  $(a+ib) \cdot (c+id) = (ac - bd) + i(ad + bc)$
  $re^{i\varphi} \cdot r'e^{i\varphi'} = rr'e^{i(\varphi+\varphi')}$
- complex conjugate: $z^* = \bar{z} = a - ib = re^{-i\varphi}$
- absolute value:
  $|z| = \sqrt{a^2+b^2} = r$, $|z_1 \cdot z_2| = |z_1| \cdot |z_2|$
- absolute value squared: $|z|^2 = a^2 + b^2 = r^2$
  important: $|z|^2 = z\bar{z}$
- inverse: $1/z = \bar{z}/|z|^2$



Classical vs quantum bits

Classical

Recall that a random bit can be described by a probability vector:

p  $+\, q$  $= p \begin{pmatrix} 1 \\ 0 \end{pmatrix} + q \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix}$

where $p, q \in \mathbb{R}$ such that $p, q \geq 0$ and $p + q = 1$.

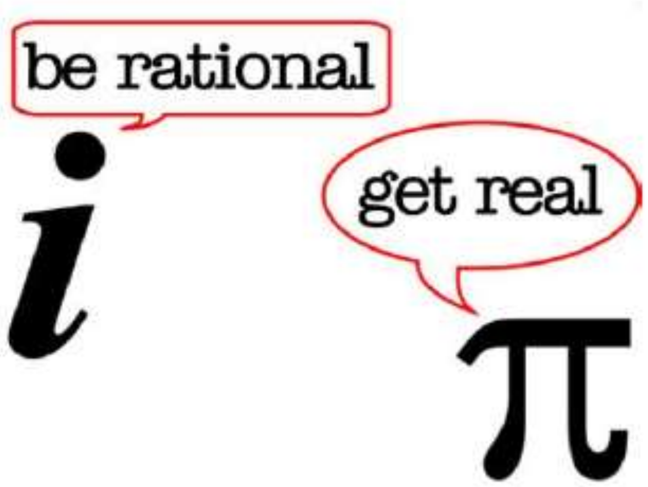
Quantum

A quantum bit (or qubit for short) is described by a quantum state:

$$\alpha|0\rangle + \beta|1\rangle = \alpha \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

where $\alpha, \beta \in \mathbb{C}$ are called amplitudes and satisfy $|\alpha|^2 + |\beta|^2 = 1$. Here $|0\rangle, |1\rangle$ are used as place-holders for the two discernible states of a coin (or any other physical system for that matter).

Any system that can exist in states $|0\rangle$ and $|1\rangle$ can also exist in a superposition $\alpha|0\rangle + \beta|1\rangle$, according to quantum mechanics!



Can I buy $4.1 + 2.8i$ bottles of wine?

Measurement

Classical

Observing a random coin

p  $+\ q$ 

results in heads with probability p and tails with probability q.

Quantum

Measuring the quantum state

$$\alpha|0\rangle + \beta|1\rangle$$

results in $|0\rangle$ with probability $|\alpha|^2$ and $|1\rangle$ with probability $|\beta|^2$.

Important:

- After the measurement, the system is in the measured state, so repeating the measurement will always yield the same value!
- We can only extract one bit of information from a single copy of a random bit or a qubit!

Global and relative phases

Phase

If $re^{i\varphi}$ is a complex number, $e^{i\varphi}$ is called phase.

Global phase

The following states differ only by a global phase:

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \qquad e^{i\varphi}|\psi\rangle = e^{i\varphi}\alpha|0\rangle + e^{i\varphi}\beta|1\rangle$$

These states are indistinguishable! Why? Because $|\alpha|^2 = |e^{i\varphi}\alpha|^2$ and $|\beta|^2 = |e^{i\varphi}\beta|^2$ so it makes no difference during measurements.

Relative phase

These states differ by a relative phase:

$$|+\rangle := \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \qquad |-\rangle := \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

Global and relative phases

Phase

If $re^{i\varphi}$ is a complex number, $e^{i\varphi}$ is called phase.

Global phase

The following states differ only by a global phase:

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Relative phase

These states differ by a relative phase:

$$|+\rangle := \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \qquad |-\rangle := \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

Are they also indistinguishable? No! (Measure in a *different basis*.)

Remember: global phase does not matter, relative phase matters!

Qubit states: the Bloch sphere

Any qubit state can be written as

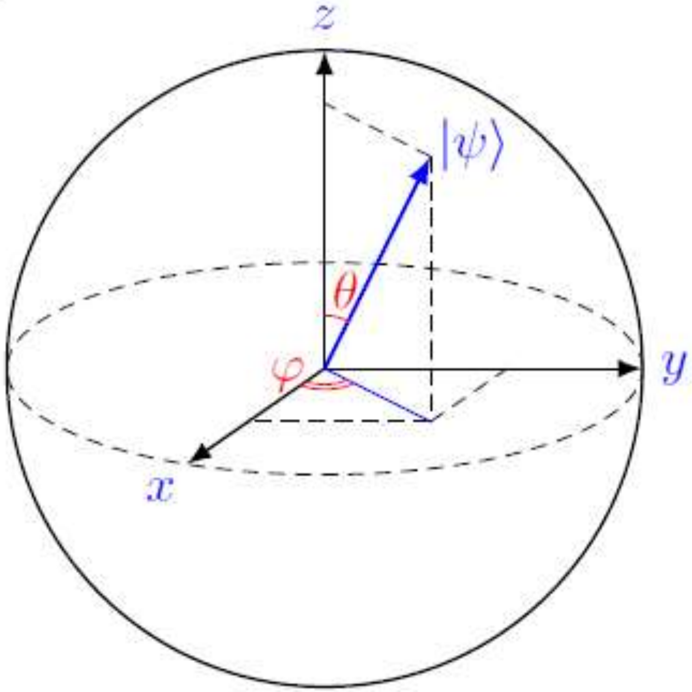
$$|\psi\rangle = \underbrace{\cos\frac{\theta}{2}}_{\alpha}|0\rangle + \underbrace{e^{i\varphi}\sin\frac{\theta}{2}}_{\beta}|1\rangle$$

for some angles $\theta \in [0,\pi]$ and $\varphi \in [0,2\pi)$.

There is a one-to-one correspondence between qubit states and points on a unit sphere (also called Bloch sphere):

Bloch vector of $|\psi\rangle$ in spherical coordinates:

$$\begin{cases} x = \sin\theta\cos\varphi \\ y = \sin\theta\sin\varphi \\ z = \cos\theta \end{cases}$$



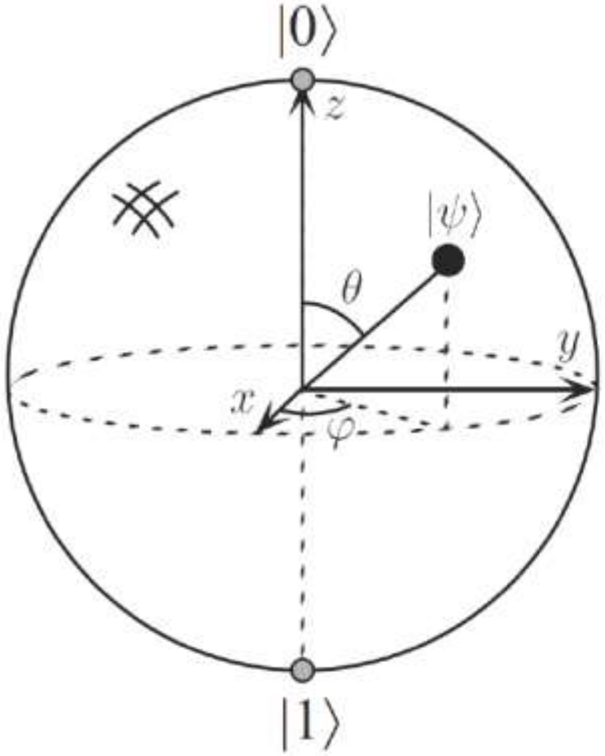
Measurement probabilities:

$$|\alpha|^2 = (\cos\tfrac{\theta}{2})^2 = \tfrac{1}{2} + \tfrac{1}{2}\cos\theta$$

$$|\beta|^2 = (\sin\tfrac{\theta}{2})^2 = \tfrac{1}{2} - \tfrac{1}{2}\cos\theta$$

Bloch sphere (some physical meaning)

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle = \underbrace{e^{i\gamma}}_{\text{irrelevant}} \left(\cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle\right)$$



θ: zenith (polar) angle, $0 \le \theta \le \pi$

φ: azimuth angle, $\mod 2\pi$

(often $|0\rangle$ at the bottom, $|1\rangle$ at the top)

Corresponds to direction of a spin in real space

Z axis $\to$ $|0\rangle$

$-Z$ axis $\to$ $|1\rangle$

X axis $\to$ $\frac{|0\rangle + |1\rangle}{\sqrt{2}}$

$-X$ axis $\to$ $\frac{|0\rangle - |1\rangle}{\sqrt{2}}$

Y axis $\to$ $\frac{|0\rangle + i|1\rangle}{\sqrt{2}}$

$-Y$ axis $\to$ $\frac{|0\rangle - i|1\rangle}{\sqrt{2}}$

equator $\to$ $\frac{|0\rangle + e^{i\varphi}|1\rangle}{\sqrt{2}}$

XZ plane $\to$ $\cos\frac{\theta}{2}|0\rangle \pm \sin\frac{\theta}{2}|1\rangle$

YZ plane $\to$ $\cos\frac{\theta}{2}|0\rangle \pm i\sin\frac{\theta}{2}|1\rangle$

Orthogonal vectors in Hilbert space correspond to opposite directions on Bloch sphere

Summary

- **Quantum computing** = quantum physics + computers + math
- **Complex numbers:** $i^2 = -1$, if $z = a + ib$ then $\bar{z} = a - ib$ and $|z|^2 = z\bar{z} = a^2 + b^2$, Euler's identity: $e^{i\varphi} = \cos\varphi + i\sin\varphi$
- **Classical probabilities:** $p, q \geq 0$ and $p + q = 1$
- **Quantum amplitudes:** $\alpha, \beta \in \mathbb{C}$ and $|\alpha|^2 + |\beta|^2 = 1$
- **Qubit state:** $\binom{\alpha}{\beta} = \alpha|0\rangle + \beta|1\rangle$ where α, β are as above
- **Measurement:** get 0 with probability $|\alpha|^2$ and 1 with prob. $|\beta|^2$
- **Phases:** global phase $e^{i\varphi}|\psi\rangle$ does not matter, relative phase matters
- **Bloch sphere:** $|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle$

Superposition, Measurements and Decoherence

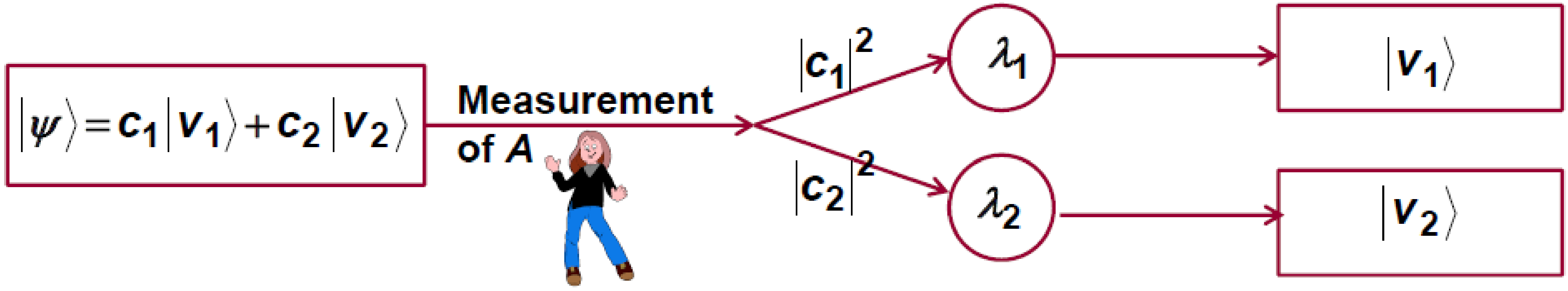
Consider the following linear superposition state of a particle made from eigenkets of A:

$$\hat{A}|v_k\rangle = \lambda_k |v_k\rangle$$

$$|\psi\rangle = c_1|v_1\rangle + c_2|v_2\rangle$$

An observer makes a measurement to see if the particle was in state $|v_1\rangle$ or $|v_2\rangle$

Depending on the result the state collapses:



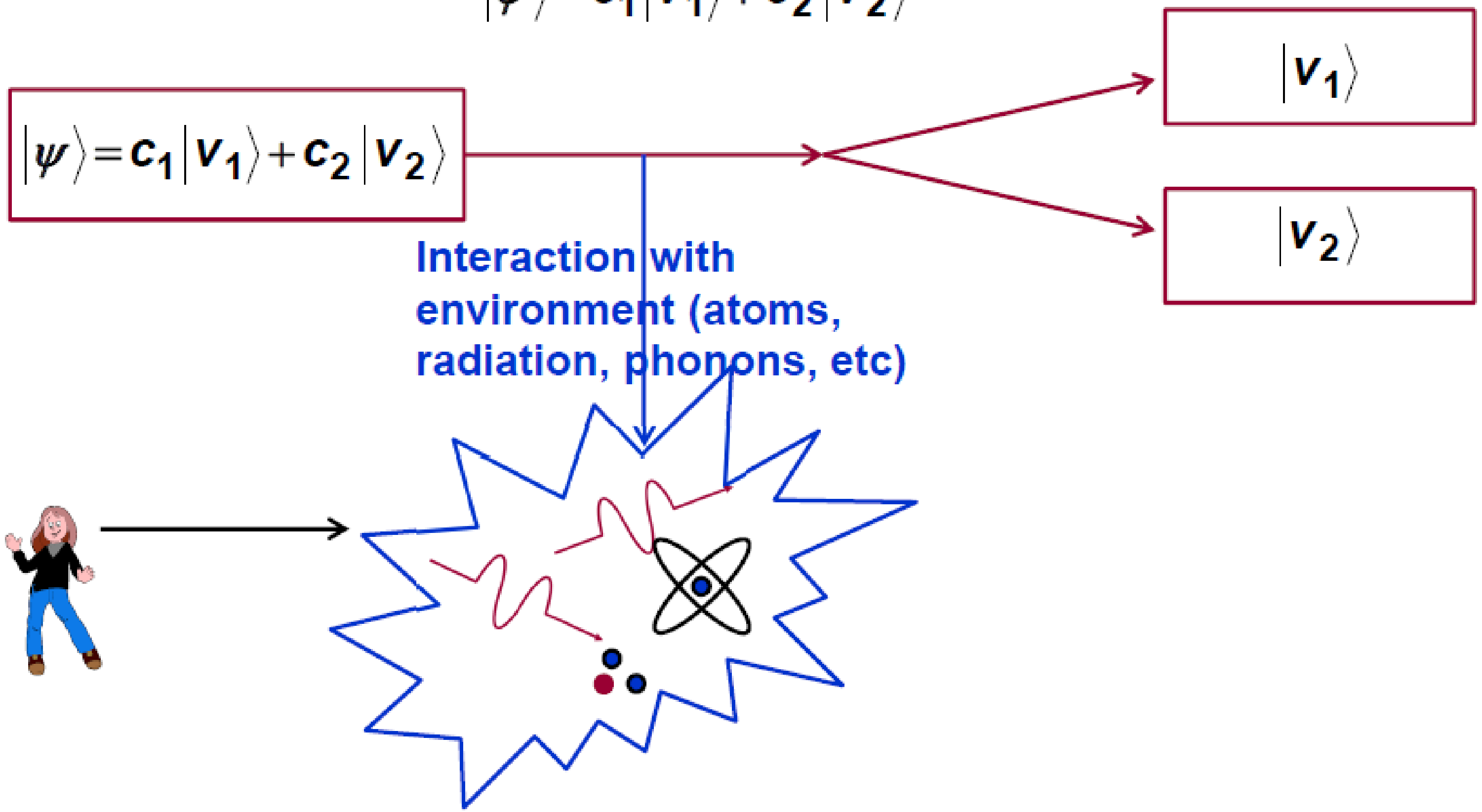
LESSON: If an observer "looks" at a quantum state, he destroys the linear superposition structure of the quantum state and collapses it

Superposition, Interaction, and Decoherence

Consider again the following linear superposition state of a particle made from eigenkets of *A*:

$$\hat{A}|v_k\rangle = \lambda_k |v_k\rangle$$

$$|\psi\rangle = c_1|v_1\rangle + c_2|v_2\rangle$$



LESSON: If environment degrees of freedom are changed by the interaction in a way that can let an observer determine the value of *A* by looking at the environment, then this is equivalent to a direct measurement of *A* and the quantum state collapses

Decoherence

$$|\psi\rangle = c_1|v_1\rangle + c_2|v_2\rangle$$

Any interaction with the environment can destroy the linear superposition and collapse the quantum state

The products,

$$c_2^* c_1 \qquad \text{and} \qquad c_1^* c_2$$

present a good measure of the degree of superposition in a quantum state

These products are generated by the operators $\hat{\sigma}_+ = |v_2\rangle\langle v_1|$ and $\hat{\sigma}_- = |v_1\rangle\langle v_2|$:

$$\langle\psi|\hat{\sigma}_-|\psi\rangle = c_1^* c_2 \qquad \langle\psi|\hat{\sigma}_+|\psi\rangle = c_2^* c_1$$

One can expect that as time goes by, interaction with the environment can make these products go to zero:

$$\langle\psi(t)|\hat{\sigma}_-|\psi(t)\rangle = c_1^*(t)c_2(t) \xrightarrow{t\to\infty} 0 \qquad \langle\psi(t)|\hat{\sigma}_+|\psi(t)\rangle = c_2^*(t)c_1(t) \xrightarrow{t\to\infty} 0$$

This phenomenon which results in the destruction of quantum mechanical superpositions is called quantum mechanical decoherence

Pure States and Statistical Mixtures

$$\hat{O}|v_k\rangle = \lambda_k |v_k\rangle$$

Consider two very different sets of states:

Set A

A large number of identical copies of:

$$|\psi\rangle = c_1|v_1\rangle + c_2|v_2\rangle$$

Linear superposition states (pure states)

Measurement of O over the entire set

Mean value obtained:

$$\langle\psi|\hat{O}|\psi\rangle = \lambda_1|c_1|^2 + \lambda_2|c_2|^2$$

Set B

A large number of states $|v_1\rangle$ and $|v_2\rangle$ in the ratio of:

$$|c_1|^2 : |c_2|^2$$

Statistical mixture of states

Measurement of O over the entire set

Mean value obtained:

$$\lambda_1|c_1|^2 + \lambda_2|c_2|^2$$

How does one represent and/or distinguish these two states then??

Density Operator in Quantum Mechanics

Density operators are a useful way to represent quantum states
Most generally, a quantum state is not represented by a state vector $|\psi\rangle$ but by a density operator $\hat{\rho}$

Set A

A large number of identical copies of:

$$|\psi\rangle = c_1|v_1\rangle + c_2|v_2\rangle$$

Linear superposition states (pure states)

Density Operator for Pure States (Set A):

For pure states $|\psi\rangle$ the density operator is:

$$\hat{\rho} = |\psi\rangle\langle\psi|$$

Example:

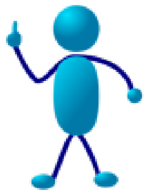
$$|\psi\rangle = c_1|v_1\rangle + c_2|v_2\rangle$$

$$\Rightarrow \hat{\rho} = |\psi\rangle\langle\psi| = |c_1|^2\,|v_1\rangle\langle v_1| + |c_2|^2\,|v_2\rangle\langle v_2| + c_1^* c_2\,|v_2\rangle\langle v_1| + c_2^* c_1\,|v_1\rangle\langle v_2|$$

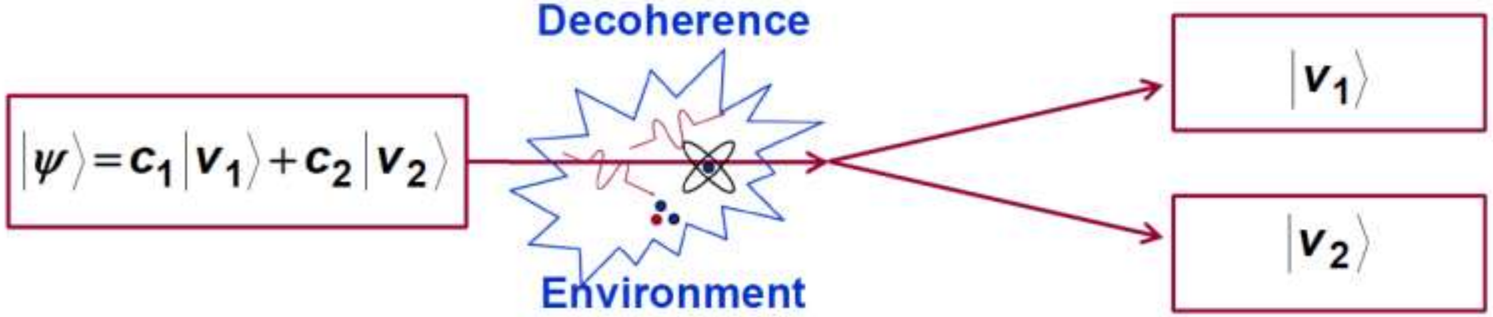
In matrix representation: $|v_1\rangle \rightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\quad |v_2\rangle \rightarrow \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

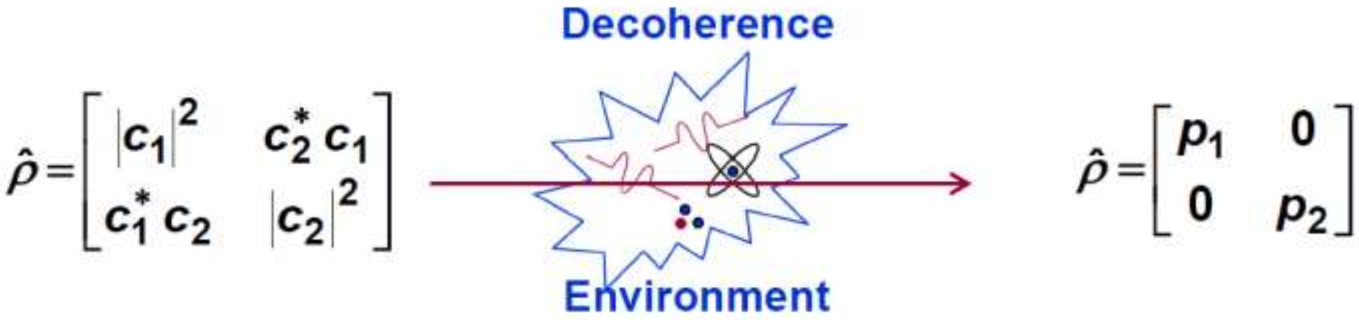
$$\hat{\rho} = \begin{bmatrix} |c_1|^2 & c_2^* c_1 \\ c_1^* c_2 & |c_2|^2 \end{bmatrix}$$

The diagonal elements indicate the occupation probabilities, and the off-diagonal elements represent coherences



Decoherence and the Density Matrix





Decoherence makes the off-diagonal components of the density matrix go to zero with time!

Why tensor product?

Imagine you have two random coins:

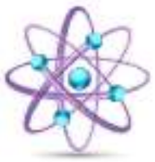
$$P = \begin{pmatrix} p_0 \\ p_1 \end{pmatrix}$$

$$Q = \begin{pmatrix} q_0 \\ q_1 \end{pmatrix}$$

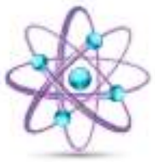
What is their joint probability distribution?

$$\begin{array}{c} 00: \\ 01: \\ 10: \\ 11: \end{array} \begin{pmatrix} p_0 q_0 \\ p_0 q_1 \\ p_1 q_0 \\ p_1 q_1 \end{pmatrix} = \begin{pmatrix} p_0 \begin{pmatrix} q_0 \\ q_1 \end{pmatrix} \\ p_1 \begin{pmatrix} q_0 \\ q_1 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} p_0 \\ p_1 \end{pmatrix} \otimes \begin{pmatrix} q_0 \\ q_1 \end{pmatrix} = P \otimes Q$$

Similarly, if you have to qubit states



$$|\psi\rangle = \begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix}$$

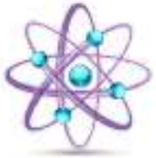


$$|\varphi\rangle = \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix}$$

their joint state is $|\psi\rangle \otimes |\varphi\rangle$. Note that $\| |\psi\rangle \otimes |\varphi\rangle \| = \| |\psi\rangle \| \| |\varphi\rangle \| = 1$.

Computational basis: notation

 $|\psi\rangle = \begin{array}{c} \begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix} \end{array} \begin{array}{c} |0\rangle \\ |1\rangle \end{array}$

 $|\varphi\rangle = \begin{array}{c} \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix} \end{array} \begin{array}{c} |0\rangle \\ |1\rangle \end{array}$

$$|\psi\rangle \otimes |\varphi\rangle = \begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix} \otimes \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix} = \begin{pmatrix} \alpha_0\beta_0 \\ \alpha_0\beta_1 \\ \alpha_1\beta_0 \\ \alpha_1\beta_1 \end{pmatrix} \begin{array}{c} |00\rangle \\ |01\rangle \\ |10\rangle \\ |11\rangle \end{array}$$

Standard basis notation for the joint system: $|i\rangle \otimes |j\rangle \equiv |i, j\rangle \equiv |ij\rangle$.
For example:

$$|00\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad |01\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad |10\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad |11\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Product and entangled states

A state $|\Psi\rangle \in \mathbb{C}^n \otimes \mathbb{C}^m$ of a combined system is product if it can be expressed as $|\Psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$ for some $|\psi_1\rangle \in \mathbb{C}^n$ and $|\psi_2\rangle \in \mathbb{C}^m$. Otherwise it is called entangled.

Example: This two-qubit state is a product state:

$$\frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle) = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

Example: Neither of the following two-qubit states can be written as a product of single-qubit states, hence they are both entangled:

$$\frac{1}{\sqrt{2}}(|10\rangle + |01\rangle) \quad \text{and} \quad \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

Note: Physical separation does not imply that the joint state must be product! Just like two distant random coins can still be correlated, two physically separated particles can also be entangled.

Unentangled States

States belonging to a combined Hilbert space of two systems, “a” and “b”, are of two types:

1) Unentangled states
2) Entangled states

Unentangled States:

These states can be written as:

$$\left|\text{a unique state of system "a"}\right\rangle_a \otimes \left|\text{a unique state of system "b"}\right\rangle_b$$

Examples:

$$\text{i)}\ |\psi\rangle = |e_1\rangle_a \otimes |e_2\rangle_b$$

$$\text{ii)}\ |\psi\rangle = \left[\frac{1}{\sqrt{2}}\left(|e_1\rangle_a + |e_2\rangle_a\right)\right] \otimes \left[|e_1\rangle_b\right] = \frac{1}{\sqrt{2}}\left\{|e_1\rangle_a \otimes |e_1\rangle_b + |e_2\rangle_a \otimes |e_1\rangle_b\right\}$$

$$\text{iii)}\ |\psi\rangle = |e_1\rangle_a \otimes \left[\frac{1}{\sqrt{2}}\left(|e_1\rangle_b - |e_2\rangle_b\right)\right] = \frac{1}{\sqrt{2}}\left\{|e_1\rangle_a \otimes |e_1\rangle_b - |e_1\rangle_a \otimes |e_2\rangle_b\right\}$$

$$\text{iv)}\ |\psi\rangle = \left[\frac{1}{\sqrt{2}}\left(|e_1\rangle_a + |e_2\rangle_a\right)\right] \otimes \left[\frac{1}{\sqrt{2}}\left(|e_1\rangle_b + |e_2\rangle_b\right)\right]$$

Entangled States

Entangled States:

Entangled states cannot be factorized or separated in the same fashion, for example:

$$\frac{1}{\sqrt{2}}\left[|e_1\rangle_a \otimes |e_2\rangle_b - |e_2\rangle_a \otimes |e_1\rangle_b\right]$$

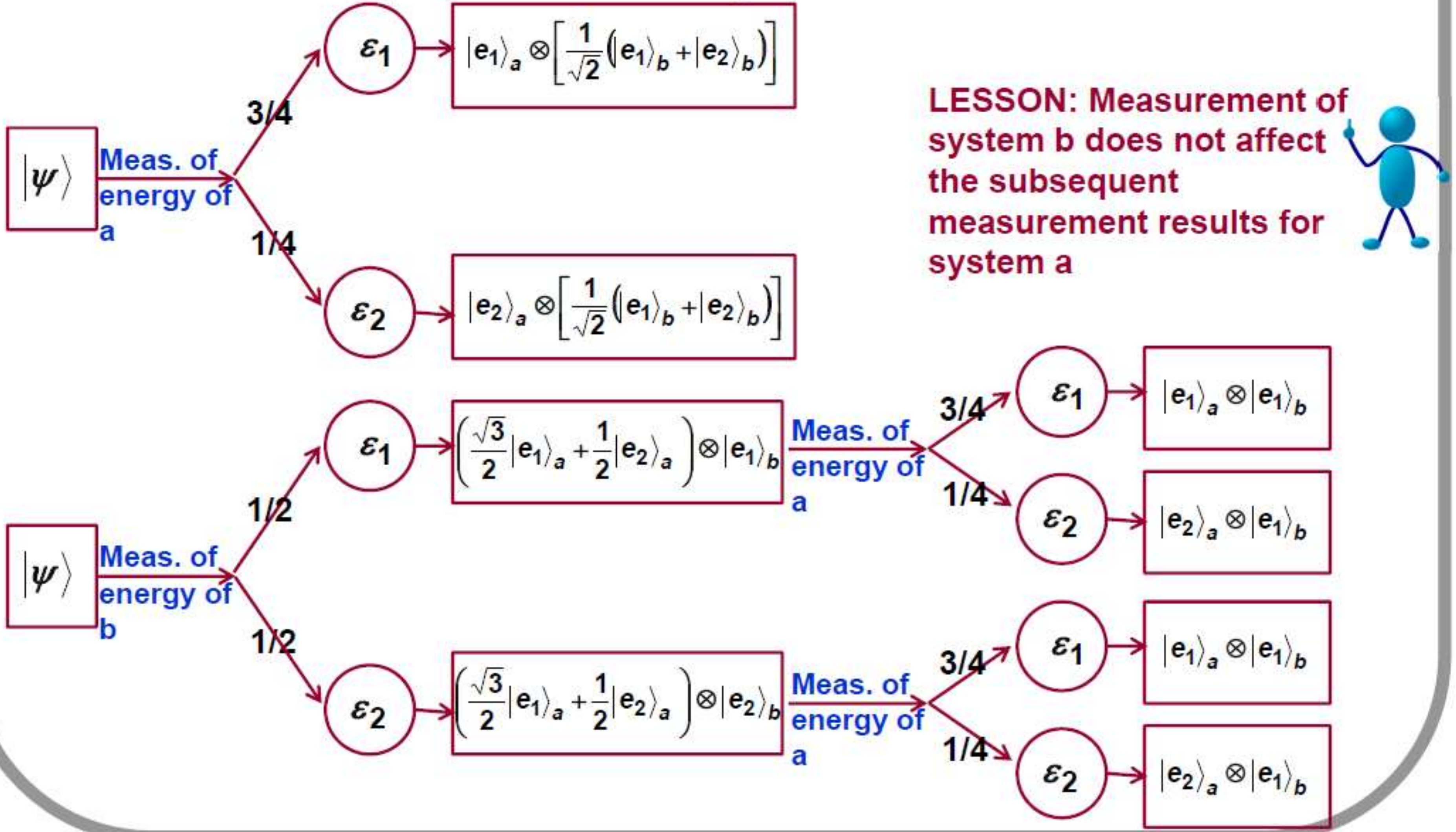
is an entangled state and it cannot be written as:

$$|\phi\rangle_a \otimes |x\rangle_b$$

Unentangled States and Measurements

Consider the complicated un-entangled state of two different two-level systems:

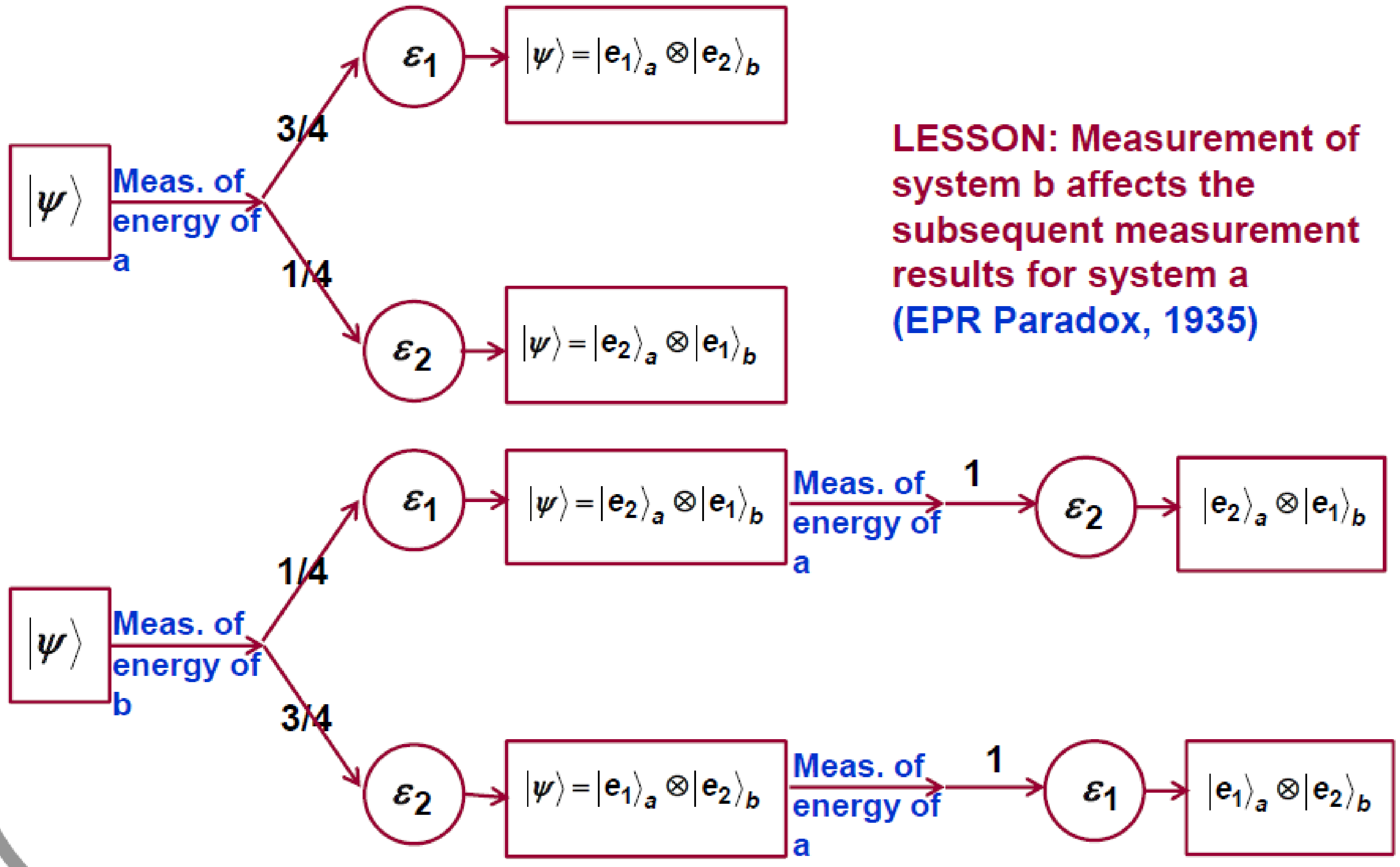
$$|\psi\rangle = \left[\left(\frac{\sqrt{3}}{2}|e_1\rangle_a + \frac{1}{2}|e_2\rangle_a\right)\right] \otimes \left[\frac{1}{\sqrt{2}}\left(|e_1\rangle_b + |e_2\rangle_b\right)\right]$$



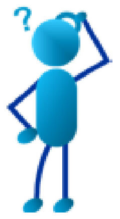
Entangled States and Measurements

Consider the entangled state of two different two-level systems:

$$\frac{\sqrt{3}}{2}|e_1\rangle_a \otimes |e_2\rangle_b - \frac{1}{2}|e_2\rangle_a \otimes |e_1\rangle_b$$



LESSON: Measurement of system b affects the subsequent measurement results for system a
(EPR Paradox, 1935)



Density Operators for Joint Hilbert Spaces

Unentangled States:

If the quantum state of a system consisting of two subsystems “a” and “b” is an unentangled state:

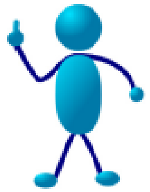
$$|\psi\rangle = |\phi\rangle_a \otimes |x\rangle_b$$

then the density operator is: $\hat{\rho} = |\psi\rangle\langle\psi| = \{|\phi\rangle_a \otimes |x\rangle_b\}\{{}_a\langle\phi| \otimes {}_b\langle x|\}$

$$= |\phi\rangle_a\, {}_a\langle\phi| \otimes |x\rangle_b\, {}_b\langle x|$$

$$= \hat{\rho}_a \otimes \hat{\rho}_b$$

Therefore, the density operator can be written as a tensor product of the density operators of the subsystems



Example:

$$|\psi\rangle = \left[\frac{1}{\sqrt{2}}\left(|e_1\rangle_a + |e_2\rangle_a\right)\right] \otimes \left[\frac{1}{\sqrt{2}}\left(|e_1\rangle_b - |e_2\rangle_b\right)\right]$$

$$\hat{\rho} = \hat{\rho}_a \otimes \hat{\rho}_b$$

Where:

$$\hat{\rho}_a = \frac{1}{2}\left\{|e_1\rangle_a\, {}_a\langle e_1| + |e_1\rangle_a\, {}_a\langle e_2| + |e_2\rangle_a\, {}_a\langle e_1| + |e_2\rangle_a\, {}_a\langle e_2|\right\}$$

$$\hat{\rho}_b = \frac{1}{2}\left\{|e_1\rangle_b\, {}_b\langle e_1| - |e_1\rangle_b\, {}_b\langle e_2| - |e_2\rangle_b\, {}_b\langle e_1| + |e_2\rangle_b\, {}_b\langle e_2|\right\}$$

Density Operators for Joint Hilbert Spaces

Entangled States:

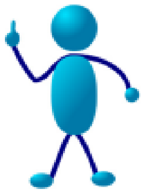
Consider the entangled state:

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left\{|e_1\rangle_a \otimes |e_2\rangle_b - |e_2\rangle_a \otimes |e_1\rangle_b\right\}$$

The density operator is:

$$\begin{aligned}\hat{\rho} &= |\psi\rangle\langle\psi| \\ &= \frac{1}{2}\Big\{|e_1\rangle_a\, {}_a\langle e_1 \otimes |e_2\rangle_b\, {}_b\langle e_2| + |e_2\rangle_a\, {}_a\langle e_2| \otimes |e_1\rangle_b\, {}_b\langle e_1| \\ &\qquad - |e_1\rangle_a\, {}_a\langle e_2| \otimes |e_2\rangle_b\, {}_b\langle e_1| - |e_2\rangle_a\, {}_a\langle e_1| \otimes |e_1\rangle_b\, {}_b\langle e_2| \Big\}\end{aligned}$$

The density operator for entangled states cannot be written as a tensor product of the density operators of the subsystems:



$$\hat{\rho} \neq \hat{\rho}_a \otimes \hat{\rho}_b$$

Entanglement and Decoherence

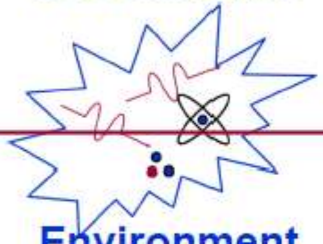
There is an intimate connection between entanglement and decoherence

A Brief Review:

$$|\psi\rangle = c_1|v_1\rangle + c_2|v_2\rangle$$

$$|v_1\rangle \qquad |v_2\rangle$$

$$\hat{\rho} = \begin{bmatrix} |c_1|^2 & c_2^* c_1 \\ c_1^* c_2 & |c_2|^2 \end{bmatrix}$$



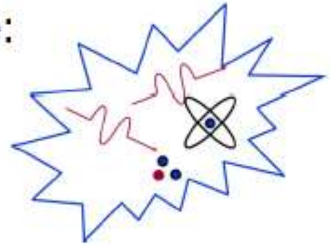
$$\hat{\rho} = \begin{bmatrix} p_1 & 0 \\ 0 & p_2 \end{bmatrix}$$

Decoherence makes the off-diagonal components of the density matrix go to zero with time!

Entanglement and Decoherence

First, we need to make a model of the environment
Suppose the (mutually orthogonal) environment states are:

$$|E_0\rangle \quad |E_1\rangle \quad |E_2\rangle$$



The initial quantum state of the system is:

$$|\psi\rangle = c_1 |v_1\rangle + c_2 |v_2\rangle$$

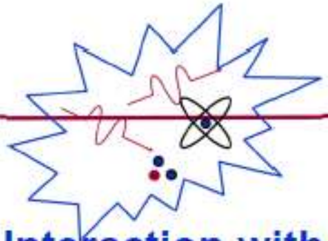
The initial joint state of the "system + environment" is:

$$|\phi(t=0)\rangle = |\psi\rangle \otimes |E_0\rangle = c_1 |v_1\rangle \otimes |E_0\rangle + c_2 |v_2\rangle \otimes |E_0\rangle$$

The environment "measures" the state of the system such that at some later time the environment state reflects the system state as follows:

$$|\phi(t)\rangle = c_1 |v_1\rangle \otimes |E_1\rangle + c_2 |v_2\rangle \otimes |E_2\rangle \longrightarrow$$ This is an entangled state

$$\boxed{|\phi(t=0)\rangle = |\psi\rangle \otimes |E_0\rangle} \longrightarrow \boxed{|\phi(t)\rangle = b_1 |v_1\rangle \otimes |E_1\rangle + b_2 |v_2\rangle \otimes |E_2\rangle}$$



Interaction with the environment

There is no state collapse!

Entanglement and Decoherence

$$|\phi(t=0)\rangle = |\psi\rangle \otimes |E_0\rangle$$

Interaction with the environment

$$|\phi(t)\rangle = b_1|v_1\rangle \otimes |E_1\rangle + b_2|v_2\rangle \otimes |E_2\rangle$$

There is no state collapse!

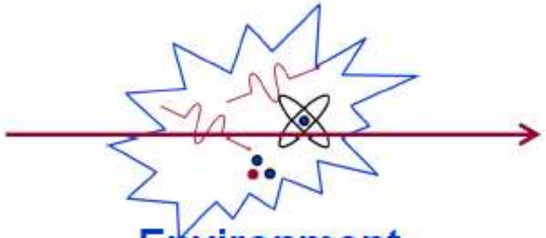
Now we find the density operator for the system by taking the partial trace of the full density operator :

$$\hat{\rho}_{\text{full}}(t) = |\varphi(t)\rangle\langle\varphi(t)|$$

$$\Rightarrow \hat{\rho}(t) = \text{Trace}\{\ \hat{\rho}_{\text{full}}(t)\ \} = \langle E_0|\hat{\rho}_{full}(t)|E_0\rangle + \langle E_1|\hat{\rho}_{full}(t)|E_1\rangle + \langle E_2|\hat{\rho}_{full}(t)|E_2\rangle$$

$$= |b_1|^2|v_1\rangle\langle v_1| + |b_2|^2|v_2\rangle\langle v_2|$$

$$\hat{\rho} = \begin{bmatrix} |c_1|^2 & c_2^* c_1 \\ c_1^* c_2 & |c_2|^2 \end{bmatrix}$$



Environment

$$\hat{\rho} = \begin{bmatrix} |b_1|^2 & 0 \\ 0 & |b_2|^2 \end{bmatrix}$$

Decoherence!!

Interaction with the environment made the off-diagonal components of the system density operator go to zero