Multivariate Statistical Analysis

Lecture 11

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Outline

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The Density of Wishart Distribution

The density of $\mathbf{A} \sim \mathcal{W}_p(\mathbf{\Sigma}, n)$ is

$$\frac{(\det(\mathbf{A}))^{\frac{n-p-1}{2}} \exp\left(-\frac{1}{2}\mathrm{tr}\left(\mathbf{\Sigma}^{-1}\mathbf{A}\right)\right)}{2^{\frac{np}{2}}\pi^{\frac{p(p-1)}{4}}(\det(\mathbf{\Sigma}))^{\frac{n}{2}}\prod_{i=1}^{p}\Gamma\left(\frac{1}{2}(n+1-i)\right)}.$$

for positive definite $\mathbf{A}$.

Properties of Wishart Distribution

Let $\mathbf{A} \sim \mathcal{W}_p(\mathbf{\Sigma}, n)$ and partition $\mathbf{A}$ and $\mathbf{\Sigma}$ into q and $p-1$ rows and columns as

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} \qquad \text{and} \qquad \mathbf{\Sigma} = \begin{bmatrix} \mathbf{\Sigma}_{11} & \mathbf{\Sigma}_{12} \\ \mathbf{\Sigma}_{21} & \mathbf{\Sigma}_{22} \end{bmatrix},$$

then we have

(a) $\mathbf{A}_{11} \sim \mathcal{W}_q(\mathbf{\Sigma}_{11}, n)$ and $\mathbf{A}_{22} \sim \mathcal{W}_{p-q}(\mathbf{\Sigma}_{22}, n)$;

(b) if $q = 1$, then

$$\mathbf{A}_{21} \mid \mathbf{A}_{22} \sim \mathcal{N}_{p-q}(\mathbf{A}_{22}\mathbf{\Sigma}_{22}^{-1}\mathbf{\Sigma}_{21}, \sigma_{11.2}^2 \mathbf{A}_{22})$$

where $\sigma_{11.2}^2 = \mathbf{\Sigma}_{11} - \mathbf{\Sigma}_{12}\mathbf{\Sigma}_{22}^{-1}\mathbf{\Sigma}_{21}$;

(c) if $n > p - q$, then

$$\mathbf{A}_{11.2} = \mathbf{A}_{11} - \mathbf{A}_{12}\mathbf{A}_{22}^{-1}\mathbf{A}_{21} \sim \mathcal{W}_q(\mathbf{\Sigma}_{11.2}, n-p+q)$$

is independent on $\mathbf{A}_{22}$ and $\mathbf{A}_{12}$, where $\mathbf{\Sigma}_{11.2} = \mathbf{\Sigma}_{11} - \mathbf{\Sigma}_{12}\mathbf{\Sigma}_{22}^{-1}\mathbf{\Sigma}_{21}$.

Quiz

Define $\bar{\mathbb{S}}^p \to \mathbb{R}^{p\times p}$ as

$$\mathbf{F}(\mathbf{X}) = \mathbf{X}^{-1},$$

where $\bar{\mathbb{S}}^p = \{\mathbf{X} \in \mathbb{R}^{p\times p} : \mathbf{X} = \mathbf{X}^\top \text{ and } \mathbf{X} \text{ is non-singular}\}$.

What is the determinant of Jacobian of $\mathbf{F}(\mathbf{X})$?