Multivariate Statistical Analysis

Lecture 11

Fudan University

luoluo@fudan.edu.cn

Outline

1. The Likelihood Ratio Criterion

Outline

1 The Likelihood Ratio Criterion

The Likelihood Ratio Criterion

The likelihood ratio criterion:

1. Let $L(\mathbf{x}, \boldsymbol{\theta})$ be the likelihood function of the observation $\mathbf{x}$ and the parameter vector $\boldsymbol{\theta} \in \Omega$.
2. Let a null hypothesis be defined by a proper subset ω of Ω. The likelihood ratio criterion is

$$\lambda(\mathbf{x}) = \frac{\sup_{\boldsymbol{\theta} \in \omega} L(\mathbf{x}, \boldsymbol{\theta})}{\sup_{\boldsymbol{\theta} \in \Omega} L(\mathbf{x}, \boldsymbol{\theta})}.$$

3. The likelihood ratio test is the procedure of rejecting the null hypothesis when $\lambda(\mathbf{x})$ is less than a predetermined constant.

Test $\rho = \rho_0$ by the Likelihood Ratio Criterion

We consider the likelihood ratio test of the hypothesis that $\rho = \rho_0$ based on a sample $\mathbf{x}_1, \dots, \mathbf{x}_N$ from

$$\mathcal{N}_2\left(\begin{bmatrix}\mu_1\\ \mu_2\end{bmatrix}, \begin{bmatrix}\sigma_1^2 & \sigma_1\sigma_2\rho\\ \sigma_1\sigma_2\rho & \sigma_2^2\end{bmatrix}\right).$$

Define the set

$$\Omega = \big\{(\mu_1, \mu_2, \sigma_1, \sigma_2, \rho) : \mu_1 \in \mathbb{R}, \mu_2 \in \mathbb{R}, \sigma_1 > 0, \sigma_2 > 0, \rho \in (-1, 1)\big\}$$

and its subset

$$\omega = \big\{(\mu_1, \mu_2, \sigma_1, \sigma_2, \rho) : \mu_1 \in \mathbb{R}, \mu_2 \in \mathbb{R}, \sigma_1 > 0, \sigma_2 > 0, \rho = \rho_0\big\}.$$

We also follow the notation

$$r = \frac{a_{12}}{\sqrt{a_{11}}\sqrt{a_{22}}}, \quad \mathbf{A} = \sum_{\alpha=1}^{N}(\mathbf{x}_\alpha - \bar{\mathbf{x}})(\mathbf{x}_\alpha - \bar{\mathbf{x}})^\top \quad \text{and} \quad \bar{\mathbf{x}} = \frac{1}{N}\sum_{\alpha=1}^{N}\mathbf{x}_\alpha.$$

Test $\rho = \rho_0$ by the Likelihood Ratio Criterion

The likelihood ratio criterion is

$$\frac{\sup_{\omega} L(\mathbf{x}, \boldsymbol{\theta})}{\sup_{\Omega} L(\mathbf{x}, \boldsymbol{\theta})} = \left(\frac{(1-\rho_0^2)(1-r^2)}{(1-\rho_0 r)^2} \right)^{\frac{N}{2}}.$$

The likelihood ratio test is

$$\frac{(1-\rho_0^2)(1-r^2)}{(1-\rho_0 r)^2} \leq c$$

where c is chosen by the prescribed significance level.

Test $\rho = \rho_0$ by the Likelihood Ratio Criterion

The critical region can be written equivalently as

$$(\rho_0^2 c - \rho_0^2 + 1)r^2 - 2\rho_0 c r + c - 1 + \rho_0^2 \geq 0,$$

that is,

$$r > \frac{\rho_0 c + (1-\rho_0^2)\sqrt{1-c}}{\rho_0^2 c - \rho_0^2 + 1} \quad \text{and} \quad r < \frac{\rho_0 c - (1-\rho_0^2)\sqrt{1-c}}{\rho_0^2 c - \rho_0^2 + 1}.$$

Thus the likelihood ratio test of $H : \rho = \rho_0$ against alternatives $\rho \neq \rho_0$ has a rejection region of the form $r > r_1$ and $r < r_2$.