Optimization Theory

Lecture 01

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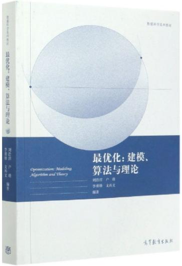
Outline

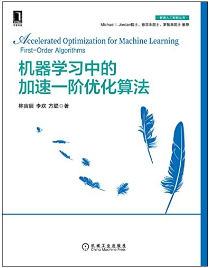
Outline

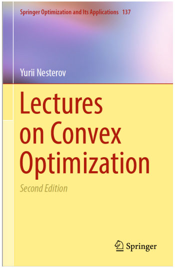
Course Overview

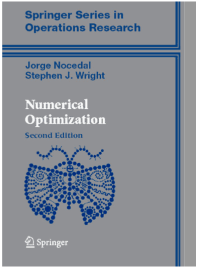
Homepage: `https://luoluo-sds.github.io/`

Recommended reading:

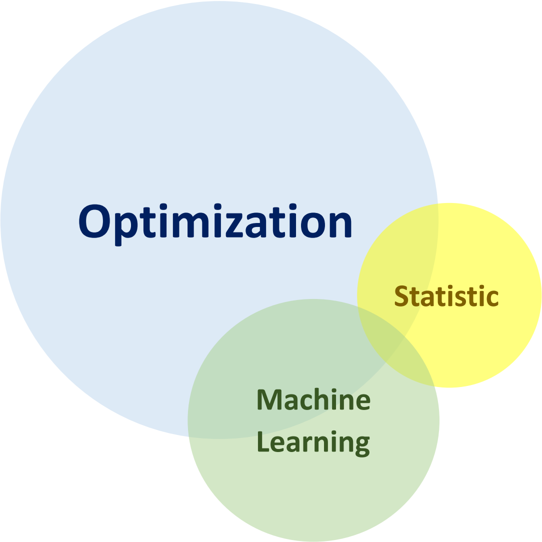








Course Overview



Grading Policy

Homework, 40%

Final Exam, 60%

or

Homework + Project?

The Forms of Optimization Problem

1. Minimization problem

$$\min_{\mathbf{x}\in\mathcal{X}} f(\mathbf{x})$$

2. Minimax problem

$$\min_{\mathbf{x}\in\mathcal{X}} \max_{\mathbf{y}\in\mathcal{Y}} f(\mathbf{x},\mathbf{y})$$

3. Bilevel problem

$$\begin{aligned}\min_{\mathbf{x}\in\mathcal{X}}\ & \Phi(\mathbf{x}) \triangleq f(\mathbf{x},\mathbf{y}^*(\mathbf{x})) \\ \text{s.t. }\ & \mathbf{y}^*(\mathbf{x}) \in \arg\min_{\mathbf{x}\in\mathcal{X}} g(\mathbf{x},\mathbf{y})\end{aligned}$$

The Classification of Optimization Problems

The description of the feasible set:

1. unconstrained vs. constrained
2. continuous vs. discrete

The properties of the objective function:

1. linear vs. nonlinear
2. smooth vs. nonsmooth
3. convex vs. nonconvex

The settings in real application:

1. deterministic vs. stochastic
2. non-distributed vs. distributed

Course Overview

We focus on algorithms and theory for continuous optimization.

Some popular topics in machine learning:

1. convex/nonconvex optimization
2. minimax optimization
3. stochastic optimization
4. distributed optimization

Should I quit this course?

The course is good for you if you

1. are interested in the mathematics behind optimization
2. use theory to design better optimization algorithms in practice
3. do research in optimization theory

The course may not be good for you if you

1. want to learn how to train deep neural networks
2. are not interested in mathematical principle

Prerequisite course: calculus, linear algebra, probability and statistics.

Outline

Supervised Learning

Prediction problem

1. input $\mathbf{a} \in \mathcal{A}$: known information
2. output $b \in \mathcal{B}$: unknown information
3. goal: to predict b based on $\mathbf{a}$
4. observe training data $(\mathbf{a}_1, b_1),\ldots,(\mathbf{a}_n, b_n)$
5. learning/training:
    - find prediction function from $\mathcal{A}$ to $\mathcal{B}$
    - model with parameter $\mathbf{x}$ that relates $\mathbf{a}$ to b
    - training: learn $\mathbf{x}$ that fits the training data

Examples: Binary Classification

Predict whether the price of a stock will go up or down tomorrow.

1. Create feature vector $\mathbf{a} \in \mathbb{R}^d$ containing information that are potentially correlated with its price.
2. Desired response variable (unknown)

$$b = \begin{cases} 1, & \text{if stock goes up,} \\ -1, & \text{if goes down.} \end{cases}$$

3. Find a linear predictor $\mathbf{x} \in \mathbb{R}^d$ and we hope that

$$b = \begin{cases} 1 & \text{if } \mathbf{a}^\top \mathbf{x} \geq 0, \\ -1 & \text{if } \mathbf{a}^\top \mathbf{x} < 0. \end{cases}$$

Examples: Binary Classification

Construct the optimization problem

$$\min_{\mathbf{x}\in\mathbb{R}^d} f(\mathbf{x}) \triangleq \frac{1}{n}\sum_{i=1}^{n} l(b_i\mathbf{a}_i^\top\mathbf{x}).$$

We consider the following loss functions.

1. 0-1 loss (not continuous):

$$l(z) = 1 - \mathrm{sign}(z)$$

2. hinge loss (convex but nonsmooth):

$$l(z) = \max\{1 - z, 0\}$$

3. logistic loss (convex and smooth):

$$l(z) = \ln(1 + \exp(-z))$$

Examples: Binary Classification

We typically introduce the regularization term

$$\min_{\mathbf{x}\in\mathbb{R}^d} f(\mathbf{x}) \triangleq \frac{1}{n}\sum_{i=1}^{n} l(b_i\mathbf{a}_i^\top \mathbf{x}) + \lambda R(\mathbf{x}), \quad \text{where } \lambda > 0.$$

Some popular regularization terms in statistics.

1. ridge regularization (smooth and convex)

$$R(\mathbf{x}) \triangleq \|\mathbf{x}\|_2^2$$

2. Lasso regularization (nonsmooth and nonconvex)

$$R(\mathbf{x}) \triangleq \|\mathbf{x}\|_1$$

3. capped-ℓ_1 regularization (nonsmooth and convex)

$$R(\mathbf{x}) \triangleq \sum_{j=1}^{d} \min\{|x_j|, \alpha\} \quad \text{with} \quad \alpha > 0$$

Examples: Binary Classification

We can use more general loss function and formulate

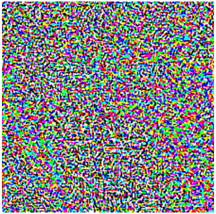
$$\min_{\mathbf{x}\in\mathbb{R}^d} f(\mathbf{x}) \triangleq \frac{1}{n}\sum_{i=1}^{n} l(\mathbf{x}; \mathbf{a}_i, b_i) + \lambda R(\mathbf{x}), \quad \text{where } \lambda > 0.$$

For example, we select $l(\mathbf{x}; \mathbf{a}_i, b_i)$ by the architecture of neural networks.

Examples: Adversarial Learning

"panda"
57.7% confidence

$+ .007 \times$



noise

$=$

"gibbon"
99.3 % confidence

Examples: Adversarial Learning

In normal training, we consider

$$\min_{\mathbf{x}\in\mathbb{R}^d} f(\mathbf{x}) \triangleq \frac{1}{n}\sum_{i=1}^{n} l(\mathbf{x}; \mathbf{a}_i, b_i) + \lambda R(\mathbf{x}).$$

In adversarial training, we allow a perturbed $\mathbf{y}_i$ for each $\mathbf{a}_i$.

It leads to the following minimax optimization problem

$$\min_{\mathbf{x}\in\mathbb{R}^d} \max_{\mathbf{y}_i\in\mathcal{Y}_i, i=1,\dots,n} \tilde{f}(\mathbf{x}, \mathbf{y}_1, \dots, \mathbf{y}_n) \triangleq \frac{1}{n}\sum_{i=1}^{n} l(\mathbf{x}; \mathbf{y}_i, b_i) + \lambda R(\mathbf{x}),$$

where $\mathcal{Y}_i = \{\mathbf{y} : \|\mathbf{y} - \mathbf{a}_i\| \leq \delta\}$ for some small $\delta > 0$.

Examples: Generative Adversarial Network (GAN)

Given n data samples $\mathbf{a}_1, \dots, \mathbf{a}_n \in \mathbb{R}^d$ from an unknown distribution, GAN aims to generate additional sample with the same distribution as the observed samples.

We formulate the minimax optimization problem

$$\min_{\mathbf{w}\in\mathcal{W}} \max_{\boldsymbol{\theta}\in\Theta} \; \frac{1}{n}\sum_{i=1}^{n} \ln D(\boldsymbol{\theta}, \mathbf{a}_i) + \mathbb{E}_{\mathbf{z}\sim\mathcal{N}(\mathbf{0},\mathbf{I})}\big[\ln(1 - D(\boldsymbol{\theta}, G(\mathbf{w}, \mathbf{z})))\big].$$

1. $D(\boldsymbol{\theta}, \cdot)$ is the discriminator that tries to separate the generated data $G(\mathbf{w}; \mathbf{z})$ from the real data samples $\mathbf{a}_i$
2. $G(\mathbf{w}, \cdot)$ is the generator that tries to make $D(\boldsymbol{\theta}, \cdot)$ cannot separate the distributions of $G(\mathbf{w}; \mathbf{z})$ and $\mathbf{a}_i$

Examples: Hyperparameter Tuning

Consider the formulation of supervised learning

$$\min_{\mathbf{x}\in\mathbb{R}^d} f(\mathbf{x}) \triangleq \frac{1}{n}\sum_{i=1}^{n} l(\mathbf{x};\mathbf{a}_i, b_i) + \lambda R(\mathbf{x}), \quad \text{where } \lambda > 0.$$

How to select the value of λ?

Use the validation sets $\{(\hat{\mathbf{a}}_1, \hat{b}_1), \dots, (\hat{\mathbf{a}}_m, \hat{b}_m)\}$.

1. do grid search on $\{\lambda_1, \dots, \lambda_q\}$
2. formulate the bilevel optimization

Examples: Hyperparameter Tuning

The bilevel formulation of hyperparameter tuning

$$\min_{\lambda\in\mathbb{R}^+} f(\lambda, \mathbf{x}^*(\lambda)) \triangleq \frac{1}{m}\sum_{i=1}^{m} l(\mathbf{x}^*(\lambda); \hat{\mathbf{a}}_i, \hat{b}_i),$$

$$\text{where} \quad \mathbf{x}^*(\lambda) \in \arg\min_{\mathbf{x}\in\mathbb{R}^d} g(\mathbf{x}) \triangleq \frac{1}{n}\sum_{i=1}^{n} l(\mathbf{x}; \mathbf{a}_i, b_i) + \lambda R(\mathbf{x}).$$

Outline

Stochastic Optimization

We consider the optimization problem

$$\min_{\mathbf{x}\in\mathbb{R}^d} f(\mathbf{x}) \triangleq \frac{1}{n}\sum_{i=1}^{n} f_i(\mathbf{x}), \quad \text{where } n \text{ is extremely large.}$$

Stochastic optimization

1. Accessing the exact information of $f(\mathbf{x})$ is expensive.
2. We design the algorithms by using the mini-batch

$$\frac{1}{b}\sum_{j=1}^{b} f_{\xi_j}(\mathbf{x}),$$

   where each ξ_j is randomly sampled from $\{1,\dots,n\}$ and $b \ll n$.
3. We allow $n = +\infty$, which leads to the online problem

$$\min_{\mathbf{x}\in\mathbb{R}^d} f(\mathbf{x}) \triangleq \mathbb{E}_\xi[F(\mathbf{x};\xi)].$$

Distributed Optimization

We consider the optimization problem

$$\min_{\mathbf{x}\in\mathbb{R}^d} f(\mathbf{x}) \triangleq \frac{1}{n}\sum_{i=1}^{n} f_i(\mathbf{x}),$$

where the information of component functions f_i are distributed on different machines.

Distributed optimization

1. centralized vs. decentralized
2. synchronized vs. asynchronous
3. federated learning

Convex Optimization

"In fact the great watershed in optimization isn't between linearity and nonlinearity, but convexity and nonconvexity." by R. T. Rockfeller

We start from addressing the convex optimization problem

$$\min_{\mathbf{x}\in\mathcal{X}} f(\mathbf{x}),$$

which requires the basics of linear algebra, topology and convex analysis.

Outline

Notations

We use x_i to denote the entry of the n-dimensional vector $\mathbf{x}$ such that

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n.$$

We use a_{ij} to denote the entry of matrix $\mathbf{A}$ with dimension $m \times n$ such that

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \in \mathbb{R}^{m\times n}.$$

Notations

We can also present the matrix as

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} & \cdots & \mathbf{A}_{1q} \\ \mathbf{A}_{21} & \mathbf{A}_{22} & \cdots & \mathbf{A}_{2q} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{A}_{p1} & \mathbf{A}_{p2} & \cdots & \mathbf{A}_{pq} \end{bmatrix} \in \mathbb{R}^{m \times n}.$$

if the sub-matrices are compatible with the partition.

We define

$$\mathbf{0} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \in \mathbb{R}^{m \times n}, \quad \mathbf{I} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \in \mathbb{R}^{n \times n}.$$

Matrix Operations: Transpose

The transpose of a matrix results from flipping the rows and columns. Given a matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$ such that

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \in \mathbb{R}^{m\times n},$$

then its transpose, written $\mathbf{A}^\top \in \mathbb{R}^{n\times m}$, is an $n \times m$ matrix such that

$$\mathbf{A}^\top = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix} \in \mathbb{R}^{n\times m}.$$

Vector Norms

A norm of a vector $\mathbf{x} \in \mathbb{R}^n$ written by $\|\mathbf{x}\|$, is informally a measure of the length of the vector. For example, we have the commonly-used Euclidean norm (or ℓ_2 norm),

$$\|\mathbf{x}\|_2 = \sqrt{\mathbf{x}^\top \mathbf{x}} = \sqrt{\sum_{i=1}^{n} x_i^2}.$$

Formally, a norm is any function $\mathbb{R}^n \to \mathbb{R}$ that satisfies four properties:

1. For all $\mathbf{x} \in \mathbb{R}^n$, we have $\|\mathbf{x}\| \geq 0$ (non-negativity).
2. $\|\mathbf{x}\| = 0$ if and only if $\mathbf{x} = \mathbf{0}$ (definiteness).
3. For all $\mathbf{x} \in \mathbb{R}^n$ and $t \in \mathbb{R}$, we have $\|t\mathbf{x}\| = |t| \, \|\mathbf{x}\|$ (homogeneity).
4. For all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, we have $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$ (triangle inequality).

Vector Norms

There are some examples for $\mathbf{x} \in \mathbb{R}^n$:

1. The ℓ_1-norm: $\|\mathbf{x}\|_1 = \sum_{i=1}^n |x_i|$
2. The ℓ_2-norm: $\|\mathbf{x}\|_2 = \sqrt{\sum_{i=1}^n x_i^2}$
3. The ℓ_∞-norm: $\|\mathbf{x}\|_\infty = \max_i |x_i|$
4. The ℓ_p-norm: $\|\mathbf{x}\|_p = \left(\sum_{i=1}^n |x_i|^p\right)^{1/p}$ for $p > 1$

Vector Norms

Given a norm $\|\cdot\|$ on $\mathbb{R}^d$, its dual norm $\|\cdot\|_*$ on $\mathbb{R}^d$ is defined as follows:

$$\|\mathbf{u}\|_* = \sup_{\|\mathbf{v}\|=1} \mathbf{u}^\top \mathbf{v}.$$

The definition leads to inequality $\mathbf{u}^\top \mathbf{v} \leq \|\mathbf{u}\|_* \|\mathbf{v}\|$ for any $\mathbf{u}, \mathbf{v} \in \mathbb{R}^d$.

Some norms are commonly used in machine learning:

1. ℓ_p-norm vs. ℓ_q-norm, where $0 \leq p \leq +\infty$ and $1/p + 1/q = 1$
2. $\mathbf{H}$-norm vs. $\mathbf{H}^{-1}$-norm, where $\mathbf{H}$ is positive definite (see definition later).

Matrix Norms

Given vector norm $\|\cdot\|$, the corresponding induced matrix norm of $\mathbf{A} \in \mathbb{R}^{m \times n}$ is defined as

$$\|\mathbf{A}\| = \sup_{\mathbf{x}\in\mathbb{R}^n, \mathbf{x}\neq\mathbf{0}} \frac{\|\mathbf{A}\mathbf{x}\|}{\|\mathbf{x}\|} = \sup_{\mathbf{x}\in\mathbb{R}^n, \|\mathbf{x}\|=1} \|\mathbf{A}\mathbf{x}\|.$$

For example, we define

$$\|\mathbf{A}\|_1 = \sup_{\mathbf{x}\in\mathbb{R}^n, \|\mathbf{x}\|_1=1} \|\mathbf{A}\mathbf{x}\|_1$$

and

$$\|\mathbf{A}\|_\infty = \sup_{\mathbf{x}\in\mathbb{R}^n, \|\mathbf{x}\|_\infty=1} \|\mathbf{A}\mathbf{x}\|_\infty.$$

Matrix Norms

General matrix norm norm is any function $\mathbb{R}^{m\times n} \to \mathbb{R}$ that satisfies

1. For all $\mathbf{A} \in \mathbb{R}^{m\times n}$, we have $\|\mathbf{A}\| \geq 0$ (non-negativity).
2. $\|\mathbf{A}\| = 0$ if and only if $\mathbf{A} = \mathbf{0}$ (definiteness).
3. For all $\mathbf{A} \in \mathbb{R}^{m\times n}$ and $t \in \mathbb{R}$, we have $\|t\mathbf{A}\| = |t|\,\|\mathbf{A}\|$ (homogeneity).
4. For all $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{m\times n}$, we have $\|\mathbf{A} + \mathbf{B}\| \leq \|\mathbf{A}\| + \|\mathbf{B}\|$ (triangle inequality).

Singular Value Decomposition

The singular value decomposition (SVD) of $\mathbf{A} \in \mathbb{R}^{m \times n}$ matrix is

$$\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^\top,$$

where $\mathbf{U} \in \mathbb{R}^{m \times m}$ is orthogonal, $\mathbf{\Sigma} \in \mathbb{R}^{m \times n}$ is rectangular diagonal matrix with non-negative real numbers on the diagonal and $\mathbf{V} \in \mathbb{R}^{n \times n}$ is orthogonal.

Singular Value Decomposition

The SVD is not unique. It is always possible to choose the decomposition so that the singular values σ_i are in descending order.

The term sometimes refers to the compact SVD, a similar decomposition

$$\mathbf{A} = \mathbf{U}_r \mathbf{\Sigma}_r \mathbf{V}_r^\top$$

in which $\mathbf{\Sigma}_r$ is square diagonal of size $r \times r$, where $r \leq \min\{m, n\}$ is the rank of $\mathbf{A}$, and has only the non-zero singular values. In this variant, $\mathbf{U}_r$ is an $m \times r$ column orthogonal matrix and $\mathbf{V}_r$ is an $n \times r$ column orthogonal matrix such that $\mathbf{U}_r^\top \mathbf{U}_r = \mathbf{V}_r^\top \mathbf{V}_r = \mathbf{I}$.

Quadratic Forms

Given a square matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$ and a vector $\mathbf{x} \in \mathbb{R}^n$, the scalar $\mathbf{x}^\top \mathbf{A}\mathbf{x}$ is called a quadratic form and we have

$$\mathbf{x}^\top \mathbf{A}\mathbf{x} = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_i x_j.$$

We often implicitly assume that the matrices appearing in a quadratic form are symmetric.

Definiteness

1. A symmetric matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$ is positive definite (PD) if for all non-zero vectors $\mathbf{x} \in \mathbb{R}^n$ holds that $\mathbf{x}^\top \mathbf{A}\mathbf{x} > 0$. This is usually denoted by $\mathbf{A} \succ \mathbf{0}$.
2. A symmetric matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$ is positive semi-definite (PSD) if for all vectors $\mathbf{x} \in \mathbb{R}^n$ holds that $\mathbf{x}^\top \mathbf{A}\mathbf{x} \geq 0$. This is usually denoted by $\mathbf{A} \succeq \mathbf{0}$.
3. A symmetric matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$ is negative definite (ND) if for all non-zero vectors $\mathbf{x} \in \mathbb{R}^n$ holds that $\mathbf{x}^\top \mathbf{A}\mathbf{x} < 0$. This is usually denoted by $\mathbf{A} \prec \mathbf{0}$.
4. A symmetric matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$ is negative semi-definite (NSD) if for all vectors $\mathbf{x} \in \mathbb{R}^n$ holds that $\mathbf{x}^\top \mathbf{A}\mathbf{x} \leq 0$. This is usually denoted by $\mathbf{A} \preceq \mathbf{0}$.
5. A symmetric matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$ is indefinite if it is neither positive semi-definite nor negative semi-definite i.e., if there exist $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^n$ such that $\mathbf{x}_1^\top \mathbf{A}\mathbf{x}_1 > 0$ and $\mathbf{x}_2^\top \mathbf{A}\mathbf{x}_2 < 0$.

Matrix Calculus

Suppose that $f : \mathbb{R}^{m\times n} \to \mathbb{R}$ is a smooth function that takes as input a matrix $\mathbf{X}$ of size $m \times n$ and returns a real value. Then the gradient of f with respect to $\mathbf{X}$ is

$$\nabla f(\mathbf{X}) = \begin{bmatrix} \dfrac{\partial f(\mathbf{X})}{\partial x_{11}} & \cdots & \dfrac{\partial f(\mathbf{X})}{\partial x_{1n}} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial f(\mathbf{X})}{\partial x_{m1}} & \cdots & \dfrac{\partial f(\mathbf{X})}{\partial x_{mn}} \end{bmatrix} \in \mathbb{R}^{m\times n}.$$

Some Basic Results

1. For $\mathbf{X} \in \mathbb{R}^{m\times n}$, we have $\dfrac{\partial (f(\mathbf{X}) + g(\mathbf{X}))}{\partial \mathbf{X}} = \dfrac{\partial f(\mathbf{X})}{\partial \mathbf{X}} + \dfrac{\partial g(\mathbf{X})}{\partial \mathbf{X}}$.
2. For $\mathbf{X} \in \mathbb{R}^{m\times n}$ and $t \in \mathbb{R}$, we have $\dfrac{\partial t f(\mathbf{X})}{\partial \mathbf{X}} = t\dfrac{\partial f(\mathbf{X})}{\partial \mathbf{X}}$.
3. For $\mathbf{A}, \mathbf{X} \in \mathbb{R}^{m\times n}$, we have $\dfrac{\partial \mathrm{tr}(\mathbf{A}^\top \mathbf{X})}{\partial \mathbf{X}} = \mathbf{A}$.
4. For $\mathbf{A} \in \mathbb{R}^{n\times n}$ and $\mathbf{x} \in \mathbb{R}^{n}$, we have $\dfrac{\partial \mathbf{x}^\top \mathbf{A}\mathbf{x}}{\partial \mathbf{x}} = (\mathbf{A} + \mathbf{A}^\top)\mathbf{x}$.

   If $\mathbf{A}$ is symmetric, we have $\dfrac{\partial \mathbf{x}^\top \mathbf{A}\mathbf{x}}{\partial \mathbf{x}} = 2\mathbf{A}\mathbf{x}$.

We can find more results in the matrix cookbook:
https://www.math.uwaterloo.ca/~hwolkowi/matrixcookbook.pdf

The Hessian Matrix

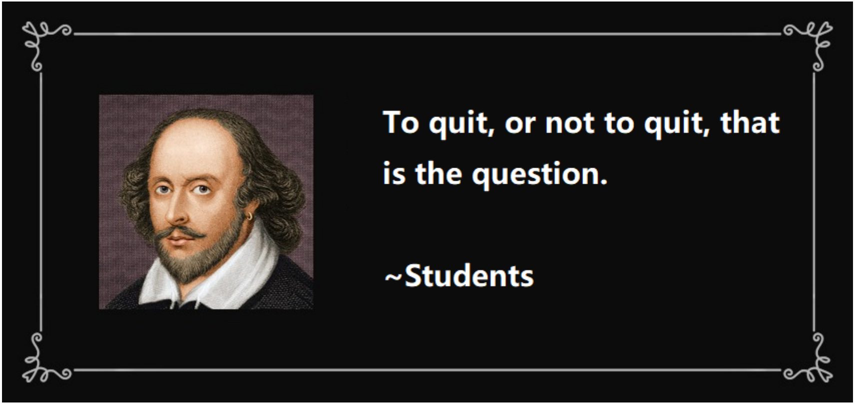
Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is a smooth function that takes as input a matrix $\mathbf{x} \in \mathbb{R}^n$ and returns a real value. Then the Hessian matrix with respect to $\mathbf{x}$, written as $\nabla^2 f(\mathbf{x})$, which is defined as

$$\nabla^2 f(\mathbf{x}) = \begin{bmatrix} \dfrac{\partial^2 f(\mathbf{x})}{\partial x_1 \partial x_1} & \cdots & \dfrac{\partial^2 f(\mathbf{x})}{\partial x_1 \partial x_n} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial^2 f(\mathbf{x})}{\partial x_n \partial x_1} & \cdots & \dfrac{\partial^2 f(\mathbf{x})}{\partial x_n \partial x_n} \end{bmatrix} \in \mathbb{R}^{n \times n}.$$

Taylor's expansion for multivariable function $f : \mathbb{R}^n \to \mathbb{R}$

$$f(\mathbf{x}) \approx f(\mathbf{a}) + \nabla f(\mathbf{a})^\top (\mathbf{x} - \mathbf{a}) + \frac{1}{2}(\mathbf{x} - \mathbf{a})^\top \nabla^2 f(\mathbf{a})(\mathbf{x} - \mathbf{a})$$

Outline



You can make the decision after this section.

Topology in Euclidean Space

Open set, closed set, bounded set and compact set:

1. A subset $\mathcal{C}$ of $\mathbb{R}^d$ is called open, if for every $\mathbf{x} \in \mathcal{C}$ there exists $\delta > 0$ such that the ball $\mathcal{B}_\delta(\mathbf{x}) = \{\mathbf{y} : \|\mathbf{y} - \mathbf{x}\|_2 \leq \delta\}$ is included in $\mathcal{C}$.
2. A subset $\mathcal{C}$ of $\mathbb{R}^d$ is called closed, if its complement $\mathcal{C}^{\mathrm{c}} = \mathbb{R}^n \backslash \mathcal{C}$ is open.
3. A subset $\mathcal{C}$ of $\mathbb{R}^d$ is called bounded, if there exists $r > 0$ such that $\|\mathbf{x}\|_2 < r$ for all $\mathbf{x} \in \mathcal{C}$.
4. A subset $\mathcal{C}$ of $\mathbb{R}^d$ is called compact, if it is both bounded and closed.

Is there any subset of $\mathbb{R}^d$ that is both open and closed?

Interior, closure and boundary:

1. The interior of $\mathcal{C} \in \mathbb{R}^n$ is defined as

$$\mathcal{C}^\circ = \{\mathbf{y} : \text{there exist } \varepsilon > 0 \text{ such that } \mathcal{B}_\varepsilon(\mathbf{y}) \subset \mathcal{C}\}$$

2. The closure of $\mathcal{C} \in \mathbb{R}^n$ is defined as

$$\bar{\mathcal{C}} = \mathbb{R}^n \backslash (\mathbb{R}^n \backslash \mathcal{C})^\circ.$$

3. The boundary of $\mathcal{C} \in \mathbb{R}^n$ is defined as $\bar{\mathcal{C}} \backslash \mathcal{C}^\circ$.

Topology in General Case

In a metric space, an open set is a set that, along with every point $\mathbf{x}$, contains all points that are sufficiently near to $\mathbf{x}$.

The other concept also can be generalized in the similar way.

For example, the positive-definite matrix on $\mathbb{R}^{d\times d}$ with distance under spectral norm is open.

Convergence Rates

Assume the sequence $\{\mathbf{x}_k\}$ converges to $\mathbf{x}^*$. We define the errors

$$z_k = \|\mathbf{x}_k - \mathbf{x}^*\|$$

and suppose

$$\lim_{k \to +\infty} \frac{z_{k+1}}{z_k^r} = C \quad \text{for some } C \in \mathbb{R}.$$

Q-convergence rates.

1. linear: $r = 1$, $0 < C < 1$;
2. sublinear: $r = 1$, $C = 1$;
3. superlinear: $r = 1$, $C = 0$;
4. quadratic: $r = 2$.

Convergence Rates

Consider the example

$$x_k = \begin{cases} 1 + 2^{-k}, & \text{if } k \text{ is even,} \\ 1, & \text{if } k \text{ is odd.} \end{cases}$$

It should converge to $x^* = 1$ linearly, however,

$$\lim_{k \to +\infty} \frac{|x_{k+1} - x^*|}{|x_k - x^*|}$$

does not exist.

Convergence Rates

Suppose that $\{\mathbf{x}_k\}$ converges to $\mathbf{x}^*$. The sequence is said to converge R-linearly to $\mathbf{x}^*$ if there exists a sequence $\{\mathbf{y}_k\}$ such that

$$\|\mathbf{x}_k - \mathbf{y}_k\|_2 \leq \varepsilon_k$$

for all k and $\{\varepsilon_k\}$ converges Q-linearly to zero.

Convex Set

We say a set $\mathcal{C} \subseteq \mathbb{R}^n$ is convex if for all $\mathbf{x}, \mathbf{y} \in \mathcal{C}$ and $\alpha \in [0, 1]$, it holds that

$$\alpha \mathbf{x} + (1 - \alpha)\mathbf{y} \in \mathcal{C}.$$

Geometrically, a set $\mathcal{C}$ is convex means that the line-segment connecting any two points in $\mathcal{C}$ also belongs to $\mathcal{C}$.

Given any collection of convex sets (finite, countable or uncountable), their intersection is itself a convex set.

Projection

Given a closed and convex set $\mathcal{C} \subseteq \mathbb{R}^n$ and any point $\mathbf{y} \in \mathbb{R}^d$, we define the projection of $\mathbf{y}$ onto $\mathcal{C}$ in Euclidean norm as the point in $\mathcal{C}$ that is closest to $\mathbf{y}$ as

$$\mathrm{proj}_{\mathcal{C}}(\mathbf{y}) = \operatorname*{arg\,min}_{\mathbf{x} \in \mathcal{C}} \|\mathbf{y} - \mathbf{x}\|_2^2 .$$

Projection

Some properties of the prjection:

1. The projection $\mathrm{proj}_{\mathcal{C}}(\mathbf{y})$ is uniquely defined.
2. If $\mathbf{y} \notin \mathcal{C}$, then $\mathbf{z} = \mathrm{proj}_{\mathcal{C}}(\mathbf{y})$ lies on the boundary of $\mathcal{C}$. The hyperplane

$$\{\mathbf{x} : \langle \mathbf{y} - \mathbf{z}, \mathbf{x} - \mathbf{z} \rangle = 0\}$$

separates $\mathbf{y}$ and $\mathcal{C}$ in that they lie on different sides, that is

$$\langle \mathbf{y} - \mathbf{z}, \mathbf{y} - \mathbf{z} \rangle > 0 \quad \text{and} \quad \langle \mathbf{y} - \mathbf{z}, \mathbf{x} - \mathbf{z} \rangle \leq 0$$

It implies

$$\|\mathbf{x} - \mathbf{z}\|_2^2 \leq \|\mathbf{x} - \mathbf{y}\|_2^2$$

for any $\mathbf{x} \in \mathcal{C}$.

Convex Function

A function $f : \mathcal{C} \to \mathbb{R}$, defined on a convex set $\mathcal{C}$, is convex if it holds

$$f(\alpha\mathbf{x} + (1-\alpha)\mathbf{y}) \leq \alpha f(\mathbf{x}) + (1-\alpha)f(\mathbf{y})$$

for all $\mathbf{x}, \mathbf{y} \in \mathcal{C}$ and $\alpha \in [0, 1]$.

Convex Function and epigraph

The epigraph of a function $f : \mathcal{C} \to \mathbb{R}$ is defined as the set

$$\operatorname{epi} f \triangleq \{(\mathbf{x}, u) \in \mathcal{C} \times \mathbb{R} : f(\mathbf{x}) \leq u\}.$$

We say a function $f(\mathbf{x})$ is closed if its epigraph is closed.

A function $f(\mathbf{x})$ is convex if and only if its epigraph is a convex set.

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Convex Function

One may extend a convex function with domain $\mathcal{C} \subset \mathbb{R}^d$ to a proper convex function

$$f_{\mathcal{C}}(\mathbf{x}) = \begin{cases} f(\mathbf{x}), & \text{if } \mathbf{x} \in \mathcal{C}, \\ +\infty, & \text{if } \mathbf{x} \notin \mathcal{C}. \end{cases}$$

We define

$$\mathrm{dom}\, f \triangleq \{\mathbf{x} : f(\mathbf{x}) < +\infty\}.$$

We say a convex function is proper if its domain is non-empty and its values are all larger than $-\infty$.

We say a function $f(\mathbf{x})$ on $\mathbb{R}^d$ is concave if $-f(\mathbf{x})$ is convex. Linear functions are both convex and concave.

Convex Function

Properties of convex functions:

1. Given any $\mathbf{g} : \mathbb{R}^d \to \mathbb{R}^k$ such that each component $g_j(\mathbf{x})$ is convex, then the set $\mathcal{C} = \{\mathbf{x} : \mathbf{g}(\mathbf{x}) \leq \mathbf{0}\}$ is convex.
2. The sup over a family of convex functions is convex.
3. The positively weighted sum of convex functions is convex.
4. The partial minimization of a convex function is convex.

Convex Function

Given a closed convex set $\mathcal{C} \in \mathbb{R}^d$, we can define a convex function $\mathbb{1}_{\mathcal{C}}(\mathbf{x})$ on $\mathbb{R}^d$, called the indicator function of $\mathcal{C}$ on $\mathbb{R}^d$, as

$$\mathbb{1}_{\mathcal{C}}(\mathbf{x}) \triangleq \begin{cases} 0, & \text{if } \mathbf{x} \in \mathcal{C}, \\ +\infty, & \text{if } \mathbf{x} \notin \mathcal{C}. \end{cases}$$

We may write $f_{\mathcal{C}}(\mathbf{x}) = f(\mathbf{x}) + \mathbb{1}_{\mathcal{C}}(\mathbf{x})$ and the problem

$$\min_{\mathbf{x} \in \mathcal{C}} f(\mathbf{x})$$

is equivalent to

$$\min_{\mathbf{x} \in \mathbb{R}^d} f(\mathbf{x}) + \mathbb{1}_{\mathcal{C}}(\mathbf{x}).$$

Convex Function

We shall only consider closed convex functions in our problems.

1. All convex functions can be made closed by taking the closure of its epigraph.
2. In some pessimistic case, a closed convex function may not be continuous at the boundary of its domain. Consider the function

$$f(x, y) = \begin{cases} \dfrac{x^2}{y}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

   with domain $\{(x, y) : y > 0\} \cup \{(0, 0)\}$.
3. We will only consider problems where the optimal solution can be achieved at a point that is continuous.

Why do we love convex optimization?

1. Let $f(\mathbf{x})$ be a convex function defined on a convex set $\mathcal{C}$.
2. Let $\mathbf{x}^*$ be a local solution of

$$\min_{\mathbf{x}\in\mathcal{C}} f(\mathbf{x}).$$

   That is, there exist some $\delta > 0$ such that any $\hat{\mathbf{x}} \in \mathcal{B}_\delta(\mathbf{x}^*)$ holds

$$f(\mathbf{x}^*) \leq f(\hat{\mathbf{x}}).$$

3. Then the local solution $\mathbf{x}^*$ is a global solution!

First-Order Condition

If a function f is differentiable on open set $\mathcal{C}$, then it is convex on $\mathcal{C}$ if and only if

$$f(\mathbf{y}) \geq f(\mathbf{x}) + \langle \nabla f(\mathbf{x}), \mathbf{y} - \mathbf{x} \rangle$$

hols for any $\mathbf{x}, \mathbf{y} \in \mathcal{C}$.

However, the gradient may not exist in general case.

Subgradient

We say a vector $\mathbf{g} \in \mathbb{R}^d$ is a subgradient of a proper convex function $f : \mathbb{R}^d \to \mathbb{R}$ at $\mathbf{x} \in \mathrm{dom}\, f$ if

$$f(\mathbf{y}) \geq f(\mathbf{x}) + \langle \mathbf{g}, \mathbf{y} - \mathbf{x} \rangle$$

holds for any $\mathbf{y} \in \mathrm{dom}$.

The set of subgradients at $\mathbf{x} \in \mathrm{dom}\, f$ is called the subdifferential of f at $\mathbf{x}$, defined as

$$\partial f(\mathbf{x}) \triangleq \left\{ \mathbf{g} : f(\mathbf{y}) \geq f(\mathbf{x}) + \langle \mathbf{g}, \mathbf{y} - \mathbf{x} \rangle \text{ holds for any } \mathbf{y} \in \mathbb{R}^d \right\}.$$