Multivariate Statistical Analysis

Lecture 09

Fudan University

luoluo@fudan.edu.cn

Outline

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The Biased Estimator

The sample mean $\bar{\mathbf{x}}$ seems the natural estimator of the population mean $\boldsymbol{\mu}$.

However, Stein (1956) showed $\bar{\mathbf{x}}$ is not admissible with respect to the mean squared loss when $p \geq 3$.

James–Stein Estimator

Consider the loss function

$$L(\boldsymbol{\mu}, \mathbf{m}) = \|\mathbf{m} - \boldsymbol{\mu}\|_2^2\,,$$

where $\mathbf{m}$ is an estimator of the mean $\boldsymbol{\mu}$.

The estimator proposed by James and Stein is

$$\mathbf{m}(\bar{\mathbf{x}}) = \left(1 - \frac{p-2}{\|\bar{\mathbf{x}} - \boldsymbol{\nu}\|_2^2}\right)(\bar{\mathbf{x}} - \boldsymbol{\nu}) + \boldsymbol{\nu},$$

where $\boldsymbol{\nu} \in \mathbb{R}^p$ is an arbitrary fixed vector and $p \geq 3$.

Bayesian Estimation View

Consider $\mathbf{x}_\alpha \sim \mathcal{N}(\boldsymbol{\mu}, N\mathbf{I})$ for $\alpha = 1, \dots, N$, we additionally suppose

$$\boldsymbol{\mu} \sim \mathcal{N}(\boldsymbol{\nu}, \tau^2\mathbf{I}).$$

Then the posterior distribution of $\boldsymbol{\mu}$ given $\mathbf{x}_1, \dots, \mathbf{x}_N$ has mean

$$\left(1 - \mathbb{E}\left[\frac{p-2}{\|\bar{\mathbf{x}} - \boldsymbol{\nu}\|_2^2}\right]\right)(\bar{\mathbf{x}} - \boldsymbol{\nu}) + \boldsymbol{\nu}.$$

James–Stein Estimator

Interestingly, we have

$$\mathbb{E}\left[\|\mathbf{m}(\bar{\mathbf{x}}) - \boldsymbol{\mu}\|_2^2\right] < \mathbb{E}\left[\|\bar{\mathbf{x}} - \boldsymbol{\mu}\|_2^2\right]$$

by only suppose $\mathbf{x}_\alpha \sim \mathcal{N}(\boldsymbol{\mu}, N\mathbf{I})$ without prior on $\boldsymbol{\mu}$, where

$$\mathbf{m}(\bar{\mathbf{x}}) = \left(1 - \frac{p-2}{\|\bar{\mathbf{x}} - \boldsymbol{\nu}\|_2^2}\right)(\bar{\mathbf{x}} - \boldsymbol{\nu}) + \boldsymbol{\nu}.$$

Improved Biased Estimator

The James–Stein estimator is

$$\mathbf{m}(\bar{\mathbf{x}}) = \left(1 - \frac{p-2}{\|\bar{\mathbf{x}} - \boldsymbol{\nu}\|_2^2}\right)(\bar{\mathbf{x}} - \boldsymbol{\nu}) + \boldsymbol{\nu}.$$

For small values of $\|\bar{\mathbf{x}} - \boldsymbol{\nu}\|_2$, the multiplier of $(\bar{\mathbf{x}} - \boldsymbol{\nu})$ is negative; that is, the estimator $\mathbf{m}(\bar{\mathbf{x}})$ is in the direction from $\boldsymbol{\nu}$ opposite to that of $\bar{\mathbf{x}}$.

We can improve $\mathbf{m}(\bar{\mathbf{x}})$ by using

$$\tilde{\mathbf{m}}(\bar{\mathbf{x}}) = \left(1 - \frac{p-2}{\|\bar{\mathbf{x}} - \boldsymbol{\nu}\|_2^2}\right)^+ (\bar{\mathbf{x}} - \boldsymbol{\nu}) + \boldsymbol{\nu},$$

which holds that $\mathbb{E}\left[\|\tilde{\mathbf{m}}(\bar{\mathbf{x}}) - \boldsymbol{\mu}\|_2^2\right] \leq \mathbb{E}\left[\|\mathbf{m}(\bar{\mathbf{x}}) - \boldsymbol{\mu}\|_2^2\right]$.

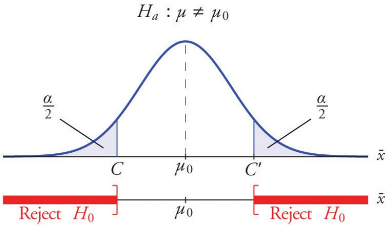
Outline

Hypothesis Testing for the Mean

In the univariate case, the difference between the sample mean and the population mean is normally distributed.

We consider

$$z = \frac{\sqrt{N}}{\sigma}(\bar{x} - \mu_0).$$



1. For significance level $\alpha = 0.05$ and $p = 1$, we have $1 - \alpha = 0.95$.
2. What about multivariate case?

Chi-Squared Distribution

If $x_1, \ldots, x_n$ are independent, standard normal random variables, then the sum of their squares,

$$y = \sum_{i=1}^{n} x_i^2,$$

is distributed according to the (central) chi-squared distribution (χ^2-distribution) with n degrees of freedom. One may write $y \sim \chi^2_n$.

We have $\mathbb{E}[y] = n$ and $\mathrm{Var}[y] = 2n$.

Chi-Squared Distribution

The probability density function of the (central) chi-squared distribution is

$$f(y;\, n) = \begin{cases} \dfrac{1}{2^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)} y^{\frac{n}{2}-1} \exp\left(-\dfrac{y}{2}\right), & y > 0; \\ 0, & \text{otherwise}, \end{cases}$$

where

$$\Gamma(\alpha) = \int_0^{+\infty} t^{\alpha-1} \exp(-t)\, \mathrm{d}t.$$

Chi-Squared Distribution

The derivation for the density of Chi-square distribution:

1. Show that $\Gamma(1/2) = \sqrt{\pi}$.
2. For $y_1 = x^2$ with $x \sim \mathcal{N}(0,1)$, the density function of y_1 is
$$\frac{1}{\sqrt{2\pi y_1}} \exp\left(-\frac{1}{2} y_1\right).$$
3. For beta function $B(\alpha, \beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,\mathrm{d}t$, we have
$$B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}.$$
4. Show the density of $y_n = \sum_{i=1}^{n} x_i^2$ by induction.

Noncentral Chi-Squared Distribution

If $x_1, \ldots, x_n$ are independent and each x_i are normally distributed random variables with means μ_i and unit variances, then the sum of their squares,

$$y = \sum_{i=1}^{n} x_i^2,$$

is distributed according to the noncentral Chi-squared distribution with n degrees of freedom and noncentrality parameter

$$\lambda = \sum_{i=1}^{n} \mu_i^2.$$

One may write $y \sim \chi^2_{n,\lambda}$.

We have $\mathbb{E}[y] = n + \lambda$ and $\mathrm{Var}[y] = 2n + 4\lambda$.

Noncentral Chi-Squared Distribution

Theorem

If $y_1, \ldots, y_k$ *are independent and each* y_i *is distributed according to the noncentral* χ^2*-distribution with* n_i *degrees of freedom and noncentrality parameter* λ_i*, then*

$$\sum_{i=1}^{k} y_i \sim \chi^2_{n,\lambda},$$

where

$$n = \sum_{i=1}^{k} n_i \qquad \textit{and} \qquad \lambda = \sum_{i=1}^{k} \lambda_i.$$

Noncentral Chi-Squared Distribution

Theorem

If the n-component random vector $\mathbf{y}$ *is distributed according to* $\mathcal{N}_n(\boldsymbol{\nu}, \mathbf{T})$ *with* $\mathbf{T} \succ \mathbf{0}$*, then*

$$\mathbf{y}^\top \mathbf{T}^{-1} \mathbf{y} \sim \chi^2_{n,\lambda},$$

where

$$\lambda = \boldsymbol{\nu}^\top \mathbf{T}^{-1} \boldsymbol{\nu}.$$

If $\boldsymbol{\nu} = \mathbf{0}$*, the distribution is the central* χ^2_n*-distribution.*

Noncentral Chi-Squared Distribution

Let $\mathbf{y} \sim \mathcal{N}_p(\boldsymbol{\lambda}, \mathbf{I})$, then

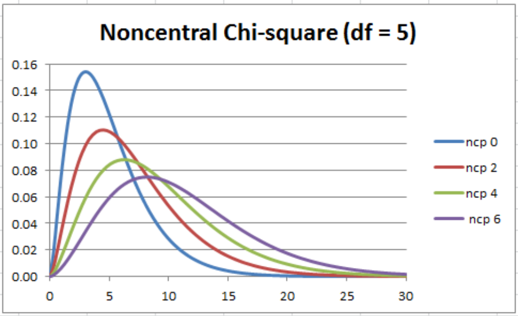
$$v = \mathbf{y}^\top \mathbf{y}$$

is distributed according to the noncentral χ^2-distribution with p degrees of freedom and noncentral parameter $\lambda = \boldsymbol{\lambda}^\top \boldsymbol{\lambda}$.

The probability density function is

$$\begin{aligned} & f(v; p, \lambda) \\ = & \begin{cases} \displaystyle\sum_{\beta=0}^{\infty} \frac{(\lambda/2)^\beta \exp\left(-(\lambda/2)\right)}{\beta!} \cdot \frac{1}{2^{\frac{p+2\beta}{2}} \Gamma\left(\frac{p}{2} + \beta\right)} y^{\frac{p}{2}+\beta-1} \exp\left(-\frac{v}{2}\right) & v > 0, \\ 0, & v \leq 0. \end{cases} \end{aligned}$$

Noncentral Chi-Squared Distribution



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Hypothesis Testing for the Mean (Covariance is Known)

In the univariate case, the difference between the sample mean and the population mean is normally distributed. We consider

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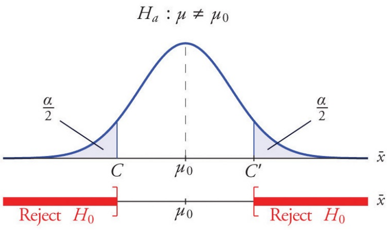

What about multivariate case?

Hypothesis Testing for the Mean (Covariance is Known)

Let $\mathbf{x}_1, \ldots, \mathbf{x}_N$ constitute a sample from $\mathcal{N}_p(\boldsymbol{\mu}, \mathbf{\Sigma})$.

What about multivariate case to test $\boldsymbol{\mu} = \boldsymbol{\mu}_0$?

$$\frac{\sqrt{N}}{\sigma}(\bar{x} - \mu_0) \implies \frac{N}{\sigma^2}(\bar{x} - \mu_0)^2 \implies N(\bar{\mathbf{x}} - \boldsymbol{\mu}_0)^\top \mathbf{\Sigma}^{-1}(\bar{\mathbf{x}} - \boldsymbol{\mu}_0).$$

Rejection Region

Let $\chi_p^2(\alpha)$ be the number such that

$$\Pr\left\{N(\bar{\mathbf{x}}-\boldsymbol{\mu})^\top \boldsymbol{\Sigma}^{-1}(\bar{\mathbf{x}}-\boldsymbol{\mu}) > \chi_p^2(\alpha)\right\} = \alpha.$$

To test the hypothesis that $\boldsymbol{\mu} = \boldsymbol{\mu}_0$ where $\boldsymbol{\mu}_0$ is a specified vector, we use as our rejection region (critical region)

$$N(\bar{\mathbf{x}}-\boldsymbol{\mu}_0)^\top \boldsymbol{\Sigma}^{-1}(\bar{\mathbf{x}}-\boldsymbol{\mu}_0) > \chi_p^2(\alpha).$$

If above inequality is satisfied, we reject the null hypothesis.

Confidence Region

Consider the statement made on the basis of a sample with mean $\bar{\mathbf{x}}$:

"The mean of the distribution satisfies

$$N(\bar{\mathbf{x}} - \boldsymbol{\mu}^*)^\top \boldsymbol{\Sigma}^{-1}(\bar{\mathbf{x}} - \boldsymbol{\mu}^*) \leq \chi_p^2(\alpha).$$

as an inequality on $\boldsymbol{\mu}^*$." This statement is true with probability $1 - \alpha$.

Thus, the set of $\boldsymbol{\mu}^*$ satisfying above inequality is a confidence region for $\boldsymbol{\mu}$ with confidence $1 - \alpha$.

Two-Sample Problems

Suppose there are two samples:

1. $\mathbf{x}_1^{(1)}, \ldots, \mathbf{x}_{N_1}^{(1)}$ from $\mathcal{N}(\boldsymbol{\mu}^{(1)}, \boldsymbol{\Sigma})$;
2. $\mathbf{x}_1^{(2)}, \ldots, \mathbf{x}_{N_2}^{(2)}$ from $\mathcal{N}(\boldsymbol{\mu}^{(2)}, \boldsymbol{\Sigma})$;

where $\boldsymbol{\Sigma}$ is known.

How to test the hypothesis $\boldsymbol{\mu}^{(1)} = \boldsymbol{\mu}^{(2)}$?