

Multivariate Statistics

Lecture 05

Fudan University

- 1 Properties of the Maximum Likelihood Estimators
- 2 Sufficiency
- 3 Completeness

1 Properties of the Maximum Likelihood Estimators

2 Sufficiency

3 Completeness

The Maximum Likelihood Estimators

Theorem 1

If $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_N$ constitute a sample from $\mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ with $p < N$, the maximum likelihood estimators of $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ are

$$\hat{\boldsymbol{\mu}} = \bar{\mathbf{x}} = \frac{1}{N} \sum_{\alpha=1}^N \mathbf{x}_{\alpha} \quad \text{and} \quad \hat{\boldsymbol{\Sigma}} = \frac{1}{N} \sum_{\alpha=1}^N (\mathbf{x}_{\alpha} - \bar{\mathbf{x}})(\mathbf{x}_{\alpha} - \bar{\mathbf{x}})^{\top}$$

respectively.

Lemma 1

If $\mathbf{D} \in \mathbb{R}^{p \times p}$ is positive definite, the maximum of

$$f(\mathbf{G}) = -N \ln \det(\mathbf{G}) - \text{tr}(\mathbf{G}^{-1} \mathbf{D})$$

with respect to positive definite matrices \mathbf{G} exists, occurs at $\mathbf{G} = \frac{1}{N} \mathbf{D}$.

The Maximum Likelihood Estimators

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respectively.

Can we guarantee $\hat{\boldsymbol{\Sigma}} = \frac{1}{N} \sum_{\alpha=1}^N (\mathbf{x}_{\alpha} - \bar{\mathbf{x}})(\mathbf{x}_{\alpha} - \bar{\mathbf{x}})^{\top}$ is positive definite?

Distribution Theory

In the univariate case, the mean of a sample is distributed normally and independently of the sample variance.

In the multivariate case, the sample mean $\hat{\boldsymbol{\mu}}$ is also distributed normally and independently of $\hat{\boldsymbol{\Sigma}}$.

Lemma 1

Suppose $\mathbf{x}_1, \dots, \mathbf{x}_N$ are independent, where $\mathbf{x}_\alpha \sim \mathcal{N}_p(\boldsymbol{\mu}_\alpha, \boldsymbol{\Sigma})$. Let $\mathbf{C} \in \mathbb{R}^{N \times N}$ be an orthogonal matrix, then

$$\mathbf{y}_\alpha = \sum_{\beta=1}^N c_{\alpha\beta} \mathbf{x}_\beta \sim \mathcal{N}_p(\boldsymbol{\nu}_\alpha, \boldsymbol{\Sigma}),$$

where $\boldsymbol{\nu} = \sum_{\beta=1}^N c_{\alpha\beta} \boldsymbol{\mu}_\beta$ for $\alpha = 1, \dots, N$ and $\mathbf{y}_1, \dots, \mathbf{y}_N$ are independent.

Lemma 2

If $\mathbf{C} = \begin{bmatrix} c_{11} & c_{12} & \dots & c_{1p} \\ c_{21} & c_{22} & \dots & c_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ c_{p1} & c_{p2} & \dots & c_{pp} \end{bmatrix} = \begin{bmatrix} c_1^\top \\ c_2^\top \\ \vdots \\ c_p^\top \end{bmatrix} \in \mathbb{R}^{p \times p}$ is orthogonal, then

$\sum_{\alpha=1}^N \mathbf{x}_\alpha \mathbf{x}_\alpha^\top = \sum_{\beta=1}^N \mathbf{y}_\beta \mathbf{y}_\beta^\top$ where $\mathbf{y}_\alpha = \sum_{\beta=1}^p c_{\alpha\beta} \mathbf{x}_\beta$ for $\alpha = 1, \dots, N$.

Let $\mathbf{X} = \begin{bmatrix} \mathbf{x}_1^\top \\ \mathbf{x}_2^\top \\ \vdots \\ \mathbf{x}_p^\top \end{bmatrix}$ and $\mathbf{Y} = \begin{bmatrix} \mathbf{y}_1^\top \\ \mathbf{y}_2^\top \\ \vdots \\ \mathbf{y}_p^\top \end{bmatrix}$, then $\mathbf{y}_\alpha = \mathbf{X}^\top \mathbf{c}_\alpha$ and $\mathbf{Y} = \mathbf{C}\mathbf{X}$.

Theorem 2

Let $\mathbf{x}_1, \dots, \mathbf{x}_N$ be independent, each distributed according to $\mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$. Then the mean of the sample

$$\hat{\boldsymbol{\mu}} = \bar{\mathbf{x}} = \frac{1}{N} \sum_{\alpha=1}^N \mathbf{x}_{\alpha}$$

is distributed according to $\mathcal{N}(\boldsymbol{\mu}, \frac{1}{N} \boldsymbol{\Sigma})$ and independent of

$$\hat{\boldsymbol{\Sigma}} = \frac{1}{N} \sum_{\alpha=1}^N (\mathbf{x}_{\alpha} - \bar{\mathbf{x}})(\mathbf{x}_{\alpha} - \bar{\mathbf{x}})^{\top}.$$

Additionally, we have $N\hat{\boldsymbol{\Sigma}} = \sum_{\alpha=1}^{N-1} \mathbf{z}_{\alpha} \mathbf{z}_{\alpha}^{\top}$, where $\mathbf{z}_{\alpha} \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\Sigma})$ for $\alpha = 1, \dots, N-1$, and $\mathbf{z}_1, \dots, \mathbf{z}_{N-1}$ are independent.

Theorem 1

If $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_N$ constitute a sample from $\mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ with $p < N$, the maximum likelihood estimators of $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ are

$$\hat{\boldsymbol{\mu}} = \bar{\mathbf{x}} = \frac{1}{N} \sum_{\alpha=1}^N \mathbf{x}_{\alpha} \quad \text{and} \quad \hat{\boldsymbol{\Sigma}} = \frac{1}{N} \sum_{\alpha=1}^N (\mathbf{x}_{\alpha} - \bar{\mathbf{x}})(\mathbf{x}_{\alpha} - \bar{\mathbf{x}})^{\top}$$

respectively.

Theorem 3

Using the notation of Theorem 1, if $N > p$, the probability is 1 of drawing a sample so that

$$\hat{\boldsymbol{\Sigma}} = \frac{1}{N} \sum_{\alpha=1}^N (\mathbf{x}_{\alpha} - \bar{\mathbf{x}})(\mathbf{x}_{\alpha} - \bar{\mathbf{x}})^{\top}$$

is positive definite.

An estimator \mathbf{t} of a parameter vector $\boldsymbol{\theta}$ is unbiased if and only if

$$\mathbb{E}[\mathbf{t}] = \boldsymbol{\theta}.$$

For the estimators obtain from MLE for normal distribution, the vector $\hat{\boldsymbol{\mu}}$ is an unbiased estimator of $\boldsymbol{\mu}$ and $\hat{\boldsymbol{\Sigma}}$ is a biased estimator of $\boldsymbol{\Sigma}$.

Distribution Theory

Consider the result of MLE for normal distribution:

① We have

$$\mathbb{E}[\hat{\mu}] = \mathbb{E}[\bar{\mathbf{x}}] = \mathbb{E}\left[\sum_{\alpha=1}^N \mathbf{x}_{\alpha}\right] = \boldsymbol{\mu}$$

and (not limited to normal distribution)

$$\mathbb{E}[\hat{\boldsymbol{\Sigma}}] = \mathbb{E}\left[\frac{1}{N} \sum_{\alpha=1}^{N-1} \mathbf{z}_{\alpha} \mathbf{z}_{\alpha}^{\top}\right] = \frac{N-1}{N} \boldsymbol{\Sigma}.$$

② The sample covariance

$$\mathbf{S} = \frac{1}{N-1} \sum_{\alpha=1}^N (\mathbf{x}_{\alpha} - \bar{\mathbf{x}})(\mathbf{x}_{\alpha} - \bar{\mathbf{x}})^{\top}$$

is an unbiased estimator of $\boldsymbol{\Sigma}$.

Outline

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Properties of Statistics

Let

$$\bar{\mathbf{x}} = \frac{1}{N} \sum_{\alpha=1}^N \mathbf{x}_{\alpha} \quad \text{and} \quad \mathbf{S} = \frac{1}{N-1} \sum_{\alpha=1}^N (\mathbf{x}_{\alpha} - \bar{\mathbf{x}})(\mathbf{x}_{\alpha} - \bar{\mathbf{x}})^{\top}.$$

We shall show that $\bar{\mathbf{x}}$ and \mathbf{S} are sufficient statistics and are complete.

Sufficiency

A statistic \mathbf{t} is sufficient for a family of distributions of \mathbf{y} or for a parameter θ if the conditional distribution of \mathbf{y} given \mathbf{t} does not depend on θ .

The statistic \mathbf{t} gives as much information about θ as the entire sample \mathbf{y} .

Theorem 4

A statistic $\mathbf{t}(\mathbf{y})$ is sufficient for θ if and only if the density $f(\mathbf{y} \mid \theta)$ can be factored as

$$f(\mathbf{y} \mid \theta) = g(\mathbf{t}(\mathbf{y}), \theta)h(\mathbf{y})$$

where $g(\mathbf{t}(\mathbf{y}), \theta)$ and $h(\mathbf{y})$ are nonnegative and $h(\mathbf{y})$ does not depend on θ .

For the MLE of normal distribution, we apply this theorem with

$$\theta = \{\boldsymbol{\mu}, \boldsymbol{\Sigma}\}, \quad \mathbf{y} = \{\mathbf{x}_1, \dots, \mathbf{x}_N\} \quad \text{and} \quad \mathbf{t}(\mathbf{y}) = \{\bar{\mathbf{x}}, \mathbf{S}\}.$$

Theorem 5

If $\mathbf{x}_1, \dots, \mathbf{x}_N$ are observations from $\mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, then

- ① $\bar{\mathbf{x}}$ and \mathbf{S} are sufficient for $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$;
- ② if $\boldsymbol{\mu}$ is given, $\sum_{\alpha=1}^N (\mathbf{x}_{\alpha} - \boldsymbol{\mu})(\mathbf{x}_{\alpha} - \boldsymbol{\mu})^{\top}$ is sufficient for $\boldsymbol{\Sigma}$;
- ③ if $\boldsymbol{\Sigma}$ is given, $\bar{\mathbf{x}}$ is sufficient for $\boldsymbol{\mu}$.

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A family of distributions of \mathbf{y} indexed by θ is complete if for every real-valued function $g(\mathbf{y})$, we have

$$\mathbb{E}[g(\mathbf{y})] = 0$$

identically in θ implies $g(\mathbf{y}) = 0$ except for a set of \mathbf{y} of probability 0 for every θ .

If the family of distributions of a sufficient set of statistics is complete, the set is called a complete sufficient set.

Theorem 6

The sufficient set of statistics $\bar{\mathbf{x}}$, \mathbf{S} is complete for $\boldsymbol{\mu}$, $\boldsymbol{\Sigma}$ when the sample is drawn from $\mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$.

Sketch of the proof:

- ① We have $N\hat{\boldsymbol{\Sigma}} = \sum_{\alpha=1}^{N-1} \mathbf{z}_{\alpha} \mathbf{z}_{\alpha}^{\top}$, where $\mathbf{z}_{\alpha} = \sum_{\beta=1}^N b_{\alpha\beta} \mathbf{x}_{\beta}$ and

$$\mathbf{B} = \begin{bmatrix} \times & \cdots & \times \\ \vdots & \ddots & \vdots \\ \frac{1}{\sqrt{N}} & \cdots & \frac{1}{\sqrt{N}} \end{bmatrix}$$

- ② The condition $\mathbb{E}[g(\bar{\mathbf{x}}, n\mathbf{S})] \equiv 0$ implies the Laplace transform of $g(\bar{\mathbf{x}}, \mathbf{B} - N\bar{\mathbf{x}}\bar{\mathbf{x}}^{\top}) h(\bar{\mathbf{x}}, \mathbf{B})$ is zero, where $\mathbf{B} = \sum_{\alpha=1}^{N-1} \mathbf{z}_{\alpha} \mathbf{z}_{\alpha}^{\top} + N\bar{\mathbf{x}}\bar{\mathbf{x}}^{\top}$ and $h(\bar{\mathbf{x}}, \mathbf{B})$ is the joint density of $\bar{\mathbf{x}}$ and \mathbf{B} .