Multivariate Statistical Analysis

Lecture 13

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Outline

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The Conjugate Prior for the Covariance

Theorem

If $\mathbf{A} \sim \mathcal{W}_p(\boldsymbol{\Sigma}, n)$ *and* $\boldsymbol{\Sigma}$ *has a prior distribution* $\mathcal{W}^{-1}(\boldsymbol{\Psi}, m)$*, then the conditional distribution of* $\boldsymbol{\Sigma}$ *given* $\mathbf{A}$ *is the inverted Wishart distribution*

$$\mathcal{W}^{-1}(\mathbf{A} + \boldsymbol{\Psi}, n + m).$$

Let each of $\mathbf{x}_1, \dots, \mathbf{x}_N$ has distribution $\mathcal{N}_p(\mathbf{0}, \boldsymbol{\Sigma})$ independently and $n = N - 1$, then the sample covariance

$$\mathbf{S} = \frac{1}{n}\sum_{i=1}^{n}(\mathbf{x}_\alpha - \bar{\mathbf{x}})(\mathbf{x}_\alpha - \bar{\mathbf{x}})^\top \sim \mathcal{W}_p(\boldsymbol{\Sigma}, n).$$

If $\boldsymbol{\Sigma} \sim \mathcal{W}_p^{-1}(\boldsymbol{\Psi}, m)$, then we have

$$\boldsymbol{\Sigma} \,|\, \mathbf{S} \sim \mathcal{W}^{-1}(n\mathbf{S} + \boldsymbol{\Psi}, n + m).$$

The Inverted Wishart Distribution

Theorem

Let $\mathbf{x}_1, \ldots, \mathbf{x}_N$ *be observations from* $\mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$. *Suppose* $\boldsymbol{\mu}$ *and* $\boldsymbol{\Sigma}$ *have prior densities*

$$n\left(\boldsymbol{\mu} \;\middle|\; \boldsymbol{\nu}, \frac{\boldsymbol{\Sigma}}{K}\right) \qquad \textit{and} \qquad w^{-1}(\boldsymbol{\Sigma} \mid \boldsymbol{\Psi}, m)$$

respectively, where $n = N - 1$. *Then the posterior density of* $\boldsymbol{\mu}$ *and* $\boldsymbol{\Sigma}$ *given*

$$\bar{\mathbf{x}} = \frac{1}{N}\sum_{\alpha=1}^{N} \mathbf{x}_\alpha \quad \textit{and} \quad \mathbf{S} = \frac{1}{N-1}\sum_{\alpha=1}^{N} (\mathbf{x}_\alpha - \bar{\mathbf{x}})(\mathbf{x}_\alpha - \bar{\mathbf{x}})^\top$$

is

$$n\left(\boldsymbol{\mu} \;\middle|\; \frac{N\bar{\mathbf{x}} + K\boldsymbol{\nu}}{N+K}, \frac{\boldsymbol{\Sigma}}{N+K}\right) \cdot w^{-1}\left(\boldsymbol{\Sigma} \mid \boldsymbol{\Psi} + n\mathbf{S} + \frac{NK(\bar{\mathbf{x}} - \boldsymbol{\nu})(\bar{\mathbf{x}} - \boldsymbol{\nu})^\top}{N+K}, N + m\right).$$

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The Characteristic Function of Wishart Distribution

Theorem

Let $\mathbf{A} \sim \mathcal{W}_p(\mathbf{\Sigma}, n)$*, then the characteristic function of*

$$a_{11}, a_{22}, \ldots, a_{pp}, 2a_{12}, \ldots, 2a_{p-1,p},$$

is is given by

$$\mathbb{E}\left[\exp(\mathrm{i}\,\mathrm{tr}(\mathbf{A}\mathbf{\Theta}))\right] = \left(\det\left(\mathbf{I} - 2\mathrm{i}\mathbf{\Theta}\mathbf{\Sigma}\right)\right)^{-\frac{n}{2}},$$

where $\mathbf{\Theta} \in \mathbb{R}^{p \times p}$ *is symmetric.*

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Matrix F-Distribution

The density of F-distribution with m and n degrees of freedom in univariate case is

$$\frac{1}{B\left(\frac{m}{2},\frac{n}{2}\right)}\left(\frac{m}{n}\right)^{\frac{n}{2}} u^{\frac{n}{2}-1}\left(1+\frac{m}{n}\cdot u\right)^{-\frac{m+n}{2}},$$

where

$$B\left(\frac{m}{2},\frac{n}{2}\right)=\frac{\Gamma(\frac{m}{2})\Gamma(\frac{n}{2})}{\Gamma\left(\frac{m+n}{2}\right)}.$$

How to generalized it to multivariate case?

Matrix Beta Distribution

The density of Beta distribution with parameters $m/2$ and $n/2$ in univariate case is

$$f(w) = \frac{1}{B(\frac{m}{2}, \frac{n}{2})} \cdot w^{\frac{n}{2}-1}(1-w)^{\frac{m}{2}-1},$$

where

$$B\left(\frac{m}{2}, \frac{n}{2}\right) = \frac{\Gamma(\frac{m}{2})\Gamma(\frac{n}{2})}{\Gamma\left(\frac{m+n}{2}\right)}.$$

How to generalized it to multivariate case?