Multivariate Statistical Analysis

Lecture 13

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Outline

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The Characteristic Function of Wishart Distribution

Theorem

Let $\mathbf{A} \sim \mathcal{W}_p(\mathbf{\Sigma}, n)$*, then the characteristic function of*

$$a_{11}, a_{22}, \ldots, a_{pp}, 2a_{12}, \ldots, 2a_{p-1,p},$$

is is given by

$$\mathbb{E}\left[\exp(\mathrm{i}\,\mathrm{tr}(\mathbf{A}\mathbf{\Theta}))\right] = \left(\det\left(\mathbf{I} - 2\mathrm{i}\mathbf{\Theta}\mathbf{\Sigma}\right)\right)^{-\frac{n}{2}},$$

where $\mathbf{\Theta} \in \mathbb{R}^{p\times p}$ *is symmetric.*

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Matrix F-Distribution

The density of F-distribution with m and n degrees of freedom in univariate case is

$$\frac{1}{B\left(\frac{m}{2},\frac{n}{2}\right)}\left(\frac{m}{n}\right)^{\frac{n}{2}}u^{\frac{n}{2}-1}\left(1+\frac{m}{n}\cdot u\right)^{-\frac{m+n}{2}},$$

where

$$B\left(\frac{m}{2},\frac{n}{2}\right)=\frac{\Gamma(\frac{m}{2})\Gamma(\frac{n}{2})}{\Gamma\left(\frac{m+n}{2}\right)}.$$

How to generalized it to multivariate case?

Matrix F-Distribution

Let $\mathbf{A} \sim \mathcal{W}_p(\mathbf{I}, n)$ and $\mathbf{B} \sim \mathcal{W}_p(\boldsymbol{\Sigma}^{-1}, m)$ be independent, then

$$\mathbf{U} = \mathbf{B}^{-1/2}\mathbf{A}\mathbf{B}^{-1/2},$$

has matrix F-distribution with n and m degrees of freedom.

Its density function is

$$f(\mathbf{U}) = \frac{\Gamma_p\left(\frac{m+n}{2}\right)(\det(\boldsymbol{\Sigma}))^{-\frac{n}{2}}}{\Gamma_p(\frac{m}{2})\Gamma_p(\frac{n}{2})} \cdot (\det(\mathbf{U}))^{\frac{n-p-1}{2}} \left(\det(\mathbf{I} + \mathbf{U}\boldsymbol{\Sigma}^{-1})\right)^{-\frac{m+n}{2}}.$$

It is natural to define the multivariate Beta function as

$$B_p(a, b) = \frac{\Gamma_p(a)\Gamma_p(b)}{\Gamma_p(a+b)}.$$

Matrix Beta Distribution

The density of Beta distribution with parameters $m/2$ and $n/2$ in univariate case is

$$f(w) = \frac{1}{B(\frac{m}{2}, \frac{n}{2})} \cdot w^{\frac{n}{2}-1}(1-w)^{\frac{m}{2}-1},$$

where

$$B\left(\frac{m}{2}, \frac{n}{2}\right) = \frac{\Gamma(\frac{m}{2})\Gamma(\frac{n}{2})}{\Gamma\left(\frac{m+n}{2}\right)}.$$

How to generalized it to multivariate case?

Matrix Beta Distribution

Let $\mathbf{A} \sim \mathcal{W}_p(\boldsymbol{\Sigma}, n)$ and $\mathbf{B} \sim \mathcal{W}_p(\boldsymbol{\Sigma}, m)$ be independent, then

$$\mathbf{W} = (\mathbf{A} + \mathbf{B})^{-1/2}\mathbf{A}(\mathbf{A} + \mathbf{B})^{-1/2}$$

has matrix Beta distribution with parameters $n/2$ and $m/2$ if $\mathbf{0} \prec \mathbf{W} \prec \mathbf{I}$ and 0 elsewhere.

Its density function is

$$f(\mathbf{W}) = \frac{1}{B_p(\frac{n}{2}, \frac{m}{2})} \cdot (\det(\mathbf{W}))^{\frac{n-p-1}{2}} (\det(\mathbf{I} - \mathbf{W}))^{\frac{m-p-1}{2}},$$

which does not depend on $\boldsymbol{\Sigma}$.

Outline

Likelihood Ratio Criterion and T^2-Statistic

Let $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N$ constitute a sample from $\mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ with $N > p$.

We shall derive T^2-Statistic

$$T^2 = N(\bar{\mathbf{x}} - \boldsymbol{\mu}_0)^\top \mathbf{S}^{-1}(\bar{\mathbf{x}} - \boldsymbol{\mu}_0)$$

from likelihood ratio criterion

$$\lambda = \frac{\max_{\boldsymbol{\Sigma} \in \mathbb{S}_p^{++}} L(\boldsymbol{\mu}_0, \boldsymbol{\Sigma})}{\max_{\boldsymbol{\mu} \in \mathbb{R}^p, \boldsymbol{\Sigma} \in \mathbb{S}_p^{++}} L(\boldsymbol{\mu}, \boldsymbol{\Sigma})}.$$

We have

$$\lambda^{\frac{2}{N}} = \frac{1}{1 + T^2/(N-1)},$$

where

$$T^2 = N(\bar{\mathbf{x}} - \boldsymbol{\mu}_0)^\top \mathbf{S}^{-1}(\bar{\mathbf{x}} - \boldsymbol{\mu}_0), \qquad \bar{\mathbf{x}} = \frac{1}{N}\sum_{\alpha=1}^{N} \mathbf{x}_\alpha$$

and

$$\mathbf{S} = \frac{1}{N-1}\sum_{\alpha=1}^{N}(\mathbf{x}_\alpha - \bar{\mathbf{x}})(\mathbf{x}_\alpha - \bar{\mathbf{x}})^\top.$$

Likelihood Ratio Criterion and T^2-Statistic

The condition $\lambda^{2/N} > c$ for some $c \in (0,1)$ is equivalent to

$$T^2 < \frac{(N-1)(1-c)}{c}.$$