Multivariate Statistical Analysis

Lecture 05

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Outline

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Characteristic Function

The characteristic function of a p-dimensional random vector $\mathbf{x}$ is

$$\phi(\mathbf{t}) = \mathbb{E}\left[\exp(\mathrm{i}\,\mathbf{t}^\top \mathbf{x})\right]$$

defined for every real vector $\mathbf{t} \in \mathbb{R}^p$.

For the complex-valued function $g(z)$ be written as

$$g(z) = g_1(z) + \mathrm{i}\, g_2(z),$$

where $g_1(z)$ and $g_2(z)$ are real-valued, the expected value of $g(z)$ is

$$\mathbb{E}[g(z)] = \mathbb{E}[g_1(z)] + \mathrm{i}\,\mathbb{E}[g_2(z)].$$

Characteristic Function

Theorem

If the p-dimensional random vector $\mathbf{x}$ *has the density* $f(\mathbf{x})$ *and the characteristic function* $\phi(\mathbf{t})$, *then*

$$f(\mathbf{x}) = \frac{1}{(2\pi)^p} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \exp(-\mathrm{i}\,\mathbf{t}^\top \mathbf{x})\, \phi(\mathbf{t})\, \mathrm{d}t_1 \ldots \mathrm{d}t_p.$$

- If the random variable have a density, the characteristic function determines the density function uniquely.
- If the random variable does not have a density, the characteristic function uniquely defines the probability of any continuity interval.

Characteristic Function

Theorem

The characteristic function of $\mathbf{x}$ *distributed according to* $\mathcal{N}_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ *is*

$$\phi(\mathbf{t}) = \exp\left(\mathrm{i}\,\mathbf{t}^\top \boldsymbol{\mu} - \frac{1}{2}\mathbf{t}^\top \boldsymbol{\Sigma} \mathbf{t}\right).$$

for every $\mathbf{t} \in \mathbb{R}^p$*.*

Sketch of the proof:

1. The characteristic function of $\mathbf{y} \sim \mathcal{N}_p(\mathbf{0}, \mathbf{I})$ is $\phi_0(\mathbf{t}) = \exp\left(-\mathbf{t}^\top \mathbf{t}/2\right)$.
2. For $\mathbf{x} \sim \mathcal{N}_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, we have $\mathbf{x} = \mathbf{A}\mathbf{y} + \boldsymbol{\mu}$ such that $\boldsymbol{\Sigma} = \mathbf{A}\mathbf{A}^\top$.
3. Using $\phi_0(\mathbf{t})$ to present the characteristic function of $\mathbf{x} \sim \mathcal{N}_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$.

Characteristic Function

Theorem

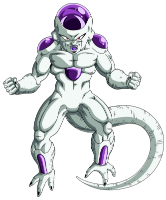
The characteristic function of $\mathbf{x}$ *distributed according to* $\mathcal{N}_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ *is*

$$\phi(\mathbf{t}) = \exp\left(\mathrm{i}\,\mathbf{t}^\top\boldsymbol{\mu} - \frac{1}{2}\mathbf{t}^\top\boldsymbol{\Sigma}\mathbf{t}\right).$$

for every $\mathbf{t} \in \mathbb{R}^p$.

This theorem directly implies $\mathbf{x} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ leads to $\mathbf{C}\mathbf{x} \sim \mathcal{N}(\mathbf{C}\boldsymbol{\mu}, \mathbf{C}\boldsymbol{\Sigma}\mathbf{C}^\top)$.

characteristic function



trick of matrix

Characteristic Function

Theorem

If every linear combination of the components of a random vector $\mathbf{y}$ *is normally distributed, then* $\mathbf{y}$ *is normally distributed.*

In other words, if the p-dimensional random vector $\mathbf{y}$ leads to the univariate random variable

$$\mathbf{u}^\top \mathbf{y}$$

is normally distributed for any fixed $\mathbf{u} \in \mathbb{R}^p$, then $\mathbf{y}$ is normally distributed.

This is another definition of multivariate normal distribution.

Example

Theorem

We let

$$\mathbf{x} \sim \mathcal{N}_p(\boldsymbol{\mu}_1, \boldsymbol{\Sigma}_1), \qquad \mathbf{y} \sim \mathcal{N}_p(\boldsymbol{\mu}_2, \boldsymbol{\Sigma}_2) \qquad \textit{and} \qquad \mathbf{z} = \mathbf{x} + \mathbf{y}.$$

Suppose that $\mathbf{x}$ *and* $\mathbf{y}$ *are independent, then we have*

$$\mathbf{z} \sim \mathcal{N}_p(\boldsymbol{\mu}_1 + \boldsymbol{\mu}_2, \boldsymbol{\Sigma}_1 + \boldsymbol{\Sigma}_2).$$

characteristic function

this result

Outline

The Maximum Likelihood Estimators

Theorem

If $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N$ constitute a sample from $\mathcal{N}_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ with $N > p$, the maximum likelihood estimators of $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ are

$$\hat{\boldsymbol{\mu}} = \bar{\mathbf{x}} = \frac{1}{N}\sum_{\alpha=1}^{N}\mathbf{x}_\alpha \quad \text{and} \quad \hat{\boldsymbol{\Sigma}} = \frac{1}{N}\sum_{\alpha=1}^{N}(\mathbf{x}_\alpha - \bar{\mathbf{x}})(\mathbf{x}_\alpha - \bar{\mathbf{x}})^\top$$

respectively.

The Maximum Likelihood Estimators

The likelihood function is

$$L = \frac{1}{(2\pi)^{\frac{pN}{2}} (\det(\boldsymbol{\Sigma}))^{\frac{N}{2}}} \exp\left[-\frac{1}{2}\sum_{\alpha=1}^{N}(\mathbf{x}_\alpha - \boldsymbol{\mu})^\top \boldsymbol{\Sigma}^{-1}(\mathbf{x}_\alpha - \boldsymbol{\mu})\right].$$

The vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N$ are fixed at the sample values and L is a function of $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$.

The logarithm of the likelihood function is

$$\ln L = -\frac{pN}{2}\ln 2\pi - \frac{N}{2}\ln(\det(\boldsymbol{\Sigma})) - \frac{1}{2}\sum_{\alpha=1}^{N}(\mathbf{x}_\alpha - \boldsymbol{\mu})^\top \boldsymbol{\Sigma}^{-1}(\mathbf{x}_\alpha - \boldsymbol{\mu}).$$

The Maximum Likelihood Estimators

There are some results for estimating the covariance.

Theorem

The function $h : \mathbb{S}^p_{++} \to \mathbb{R}$ *such that*

$$h(\mathbf{X}) = -\log\det(\mathbf{X})$$

is convex, where $\mathbb{S}^p_{++} = \{\mathbf{X} \in \mathbb{R}^{p\times p} : \mathbf{X} \succ \mathbf{0}\}$.

Theorem

If $\mathbf{D} \in \mathbb{R}^{p\times p}$ *is positive definite, the maximum of*

$$f(\mathbf{G}) = -N \ln\det(\mathbf{G}) - \mathrm{tr}(\mathbf{G}^{-1}\mathbf{D})$$

with respect to positive definite matrices $\mathbf{G}$ *exists, occurs at* $\mathbf{G} = \frac{1}{N}\mathbf{D}$.

The Maximum Likelihood Estimators

If $\mathbf{x}_1, \ldots, \mathbf{x}_N$ constitutes a sample from $\mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ with $N > p$ and define

$$\rho_{ij} = \frac{\sigma_{ij}}{\sqrt{\sigma_{ii}\sigma_{jj}}},$$

then what is the maximum likelihood estimator of ρ_{ij}?

We can replace σ_{ii}, σ_{jj} and σ_{ii} with

$$\begin{cases} \hat{\sigma}_{ii} = \dfrac{1}{N}\displaystyle\sum_{\alpha=1}^{N}(x_{i\alpha} - \bar{x}_i)^2, \\ \hat{\sigma}_{ij} = \dfrac{1}{N}\displaystyle\sum_{\alpha=1}^{N}(x_{i\alpha} - \bar{x}_i)(x_{j\alpha} - \mu_j), \\ \hat{\sigma}_{jj} = \dfrac{1}{N}\displaystyle\sum_{\alpha=1}^{N}(x_{j\alpha} - \bar{x}_j)^2, \end{cases}$$

leading to

$$\hat{\rho}_{ij} = \frac{\sum_{\alpha=1}^{N}(x_{i\alpha} - \bar{x}_i)(x_{j\alpha} - \bar{x}_j)}{\sqrt{\sum_{\alpha=1}^{N}(x_{i\alpha} - \bar{x}_i)^2}\sqrt{\sum_{\alpha=1}^{N}(x_{j\alpha} - \bar{x}_j)^2}}.$$

The Maximum Likelihood Estimators

Theorem

On the basis of a given sample, if

$$\hat{\theta}_1, \ldots, \hat{\theta}_m$$

are maximum likelihood estimators of the parameters

$$\theta_1, \ldots, \theta_m$$

of a distribution, then

$$\phi_1(\hat{\theta}_1, \ldots, \hat{\theta}_m), \ldots, \phi_m(\hat{\theta}_1, \ldots, \hat{\theta}_m)$$

are maximum likelihood estimator of

$$\phi_1(\theta_1, \ldots, \theta_m), \ldots, \phi_m(\theta_1, \ldots, \theta_m)$$

if the transformation from $\theta_1, \ldots, \theta_m$ *to* $\phi_1, \ldots, \phi_m$ *is one-to-one.*

The Maximum Likelihood Estimators

If $\phi : \mathcal{S} \to \mathcal{S}^*$ is not one-to-one, we let

$$\phi^{-1}(\boldsymbol{\theta}^*) = \{\boldsymbol{\theta} : \boldsymbol{\theta}^* = \phi(\boldsymbol{\theta})\}.$$

and define (the induced likelihood function)

$$g(\boldsymbol{\theta}^*) = \sup\{f(\boldsymbol{\theta}) : \boldsymbol{\theta}^* = \phi(\boldsymbol{\theta})\}.$$

If $\boldsymbol{\theta} = \hat{\boldsymbol{\theta}}$ maximize $f(\boldsymbol{\theta})$, then $\boldsymbol{\theta}^* = \phi(\hat{\boldsymbol{\theta}})$ also maximize $g(\boldsymbol{\theta}^*)$.

The Maximum Likelihood Estimators

The maximum likelihood estimator of ρ_{ij} is indeed

$$\hat{\rho}_{ij} = \frac{\sum_{\alpha=1}^{N}(x_{i\alpha} - \bar{x}_i)(x_{j\alpha} - \bar{x}_j)}{\sqrt{\sum_{\alpha=1}^{N}(x_{i\alpha} - \bar{x}_i)^2}\sqrt{\sum_{\alpha=1}^{N}(x_{j\alpha} - \bar{x}_j)^2}}.$$

Outline

Distribution Theory

Theorem

Let $\mathbf{x}_1, \ldots, \mathbf{x}_N$ *be independent, each distributed according to* $\mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$. *Then the mean of the sample*

$$\hat{\boldsymbol{\mu}} = \bar{\mathbf{x}} = \frac{1}{N}\sum_{\alpha=1}^{N} \mathbf{x}_\alpha$$

is distributed according to $\mathcal{N}(\boldsymbol{\mu}, \frac{1}{N}\boldsymbol{\Sigma})$ *and independent of*

$$\hat{\boldsymbol{\Sigma}} = \frac{1}{N}\sum_{\alpha=1}^{N} (\mathbf{x}_\alpha - \bar{\mathbf{x}})(\mathbf{x}_\alpha - \bar{\mathbf{x}})^\top.$$

Additionally, we have

$$N\hat{\boldsymbol{\Sigma}} = \sum_{\alpha=1}^{N-1} \mathbf{z}_\alpha \mathbf{z}_\alpha^\top,$$

where $\mathbf{z}_\alpha \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\Sigma})$ *for* $\alpha = 1, \ldots, N-1$, *and* $\mathbf{z}_1, \ldots, \mathbf{z}_{N-1}$ *are independent.*

Distribution Theory

Lemma

Suppose $\mathbf{x}_1, \ldots, \mathbf{x}_N$ *are independent, where* $\mathbf{x}_\alpha \sim \mathcal{N}_p(\boldsymbol{\mu}_\alpha, \boldsymbol{\Sigma})$*. Let* $\mathbf{C} \in \mathbb{R}^{N\times N}$ *be an orthogonal matrix, then*

$$\mathbf{y}_\alpha = \sum_{\beta=1}^{N} c_{\alpha\beta}\mathbf{x}_\beta \sim \mathcal{N}_p(\boldsymbol{\nu}_\alpha, \boldsymbol{\Sigma}),$$

where $\boldsymbol{\nu} = \sum_{\beta=1}^{N} c_{\alpha\beta}\boldsymbol{\mu}_\beta$ *for* $\alpha = 1, \ldots, N$ *and* $\mathbf{y}_1, \ldots, \mathbf{y}_N$ *are independent.*

Lemma

If $\mathbf{C} = \begin{bmatrix} c_{11} & c_{12} & \ldots & c_{1p} \\ c_{21} & c_{22} & \ldots & c_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ c_{p1} & c_{p2} & \ldots & c_{pp} \end{bmatrix} = \begin{bmatrix} c_1^\top \\ c_2^\top \\ \vdots \\ c_p^\top \end{bmatrix} \in \mathbb{R}^{p\times p}$ *is orthogonal,*

then $\sum_{\alpha=1}^{N} \mathbf{x}_\alpha\mathbf{x}_\alpha^\top = \sum_{\beta=1}^{N} \mathbf{y}_\alpha\mathbf{y}_\alpha^\top$ *where* $\mathbf{y}_\alpha = \sum_{\beta=1}^{N} c_{\alpha\beta}\mathbf{x}_\beta$ *for* $\alpha = 1, \ldots, N$*.*

Distribution Theory

Theorem

If $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N$ constitute a sample from $\mathcal{N}_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ with $N > p$, the estimator

$$\hat{\boldsymbol{\Sigma}} = \frac{1}{N} \sum_{\alpha=1}^{N} (\mathbf{x}_\alpha - \bar{\mathbf{x}})(\mathbf{x}_\alpha - \bar{\mathbf{x}})^\top$$

is positive definite with probability is 1.

1. The matrix $\hat{\boldsymbol{\Sigma}}$ be must singular if $N \leq p$.
2. The proof indicates $\mathbf{U}^\top \mathbf{U}$ is non-singular with probability 1 for $\mathbf{U} \in \mathbb{R}^{d \times k}$ with $k \leq d$ and $u_{ij} \overset{i.i.d}{\sim} \mathcal{N}(0, 1)$.