Multivariate Statistical Analysis

Lecture 13

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Outline

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The Conjugate Prior for the Covariance

Theorem

If $\mathbf{A} \sim \mathcal{W}_p(\mathbf{\Sigma}, n)$ and $\mathbf{\Sigma}$ has a prior distribution $\mathcal{W}^{-1}(\mathbf{\Psi}, m)$, then the conditional distribution of $\mathbf{\Sigma}$ given $\mathbf{A}$ is the inverted Wishart distribution

$$\mathcal{W}^{-1}(\mathbf{A} + \mathbf{\Psi}, n + m).$$

Let each of $\mathbf{x}_1, \ldots, \mathbf{x}_N$ has distribution $\mathcal{N}_p(\mathbf{0}, \mathbf{\Sigma})$ independently and $n = N - 1$, then the sample covariance

$$\mathbf{S} = \frac{1}{n} \sum_{i=1}^{n} (\mathbf{x}_\alpha - \bar{\mathbf{x}})(\mathbf{x}_\alpha - \bar{\mathbf{x}})^\top \sim \mathcal{W}_p(\mathbf{\Sigma}, n).$$

If $\mathbf{\Sigma} \sim \mathcal{W}_p^{-1}(\mathbf{\Psi}, m)$, then we have

$$\mathbf{\Sigma} \,|\, \mathbf{S} \sim \mathcal{W}^{-1}(n\mathbf{S} + \mathbf{\Psi}, n + m).$$

The Inverted Wishart Distribution

Theorem

Let $\mathbf{x}_1, \ldots, \mathbf{x}_N$ *be observations from* $\mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$*. Suppose* $\boldsymbol{\mu}$ *and* $\boldsymbol{\Sigma}$ *have prior densities*

$$n\left(\boldsymbol{\mu} \,\middle|\, \boldsymbol{\nu}, \frac{\boldsymbol{\Sigma}}{K}\right) \qquad \text{and} \qquad w^{-1}(\boldsymbol{\Sigma} \mid \boldsymbol{\Psi}, m)$$

respectively, where $n = N - 1$*. Then the posterior density of* $\boldsymbol{\mu}$ *and* $\boldsymbol{\Sigma}$ *given*

$$\bar{\mathbf{x}} = \frac{1}{N}\sum_{\alpha=1}^{N} \mathbf{x}_\alpha \quad \text{and} \quad \mathbf{S} = \frac{1}{N-1}\sum_{\alpha=1}^{N} (\mathbf{x}_\alpha - \bar{\mathbf{x}})(\mathbf{x}_\alpha - \bar{\mathbf{x}})^\top$$

is

$$n\left(\boldsymbol{\mu} \,\middle|\, \frac{N\bar{\mathbf{x}} + K\boldsymbol{\nu}}{N+K}, \frac{\boldsymbol{\Sigma}}{N+K}\right) \cdot w^{-1}\left(\boldsymbol{\Sigma} \,\middle|\, \boldsymbol{\Psi} + n\mathbf{S} + \frac{NK(\bar{\mathbf{x}} - \boldsymbol{\nu})(\bar{\mathbf{x}} - \boldsymbol{\nu})^\top}{N+K}, N + m\right).$$

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The Characteristic Function of Wishart Distribution

Theorem

Let $\mathbf{A} \sim \mathcal{W}_p(\mathbf{\Sigma}, n)$*, then the characteristic function of*

$$a_{11}, a_{22}, \ldots, a_{pp}, 2a_{12}, \ldots, 2a_{p-1,p},$$

is

$$\mathbb{E}\left[\exp(\mathrm{i}\,\mathrm{tr}(\mathbf{A}\mathbf{\Theta}))\right] = \left(\det\left(\mathbf{I} - 2\mathrm{i}\mathbf{\Theta}\mathbf{\Sigma}\right)\right)^{-\frac{n}{2}}.$$