

# Multivariate Statistical Analysis

## Lecture 13

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# Outline

- 1 The Characteristic Function of Wishart Distribution
- 2 More Matrix Variate Distributions
- 3 Likelihood Ratio Criterion and  $T^2$ -Statistic

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# The Characteristic Function of Wishart Distribution

## Theorem

Let  $\mathbf{A} \sim \mathcal{W}_p(\boldsymbol{\Sigma}, n)$ , then the characteristic function of

$$a_{11}, a_{22}, \dots, a_{pp}, 2a_{12}, \dots, 2a_{p-1,p},$$

is given by

$$\mathbb{E} [\exp(i \operatorname{tr}(\mathbf{A}\boldsymbol{\Theta}))] = (\det(\mathbf{I} - 2i\boldsymbol{\Theta}\boldsymbol{\Sigma}))^{-\frac{n}{2}},$$

where  $\boldsymbol{\Theta} \in \mathbb{R}^{p \times p}$  is symmetric.

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# Matrix $F$ -Distribution

The density of  $F$ -distribution with  $m$  and  $n$  degrees of freedom in univariate case is

$$\frac{1}{B\left(\frac{m}{2}, \frac{n}{2}\right)} \left(\frac{m}{n}\right)^{\frac{n}{2}} u^{\frac{n}{2}-1} \left(1 + \frac{m}{n} \cdot u\right)^{-\frac{m+n}{2}},$$

where

$$B\left(\frac{m}{2}, \frac{n}{2}\right) = \frac{\Gamma(\frac{m}{2})\Gamma(\frac{n}{2})}{\Gamma(\frac{m+n}{2})}.$$

How to generalized it to multivariate case?

# Matrix $F$ -Distribution

Let  $\mathbf{A} \sim \mathcal{W}_p(\mathbf{I}, n)$  and  $\mathbf{B} \sim \mathcal{W}_p(\boldsymbol{\Sigma}^{-1}, m)$  be independent, then

$$\mathbf{U} = \mathbf{B}^{-1/2} \mathbf{A} \mathbf{B}^{-1/2},$$

has matrix  $F$ -distribution with  $n$  and  $m$  degrees of freedom.

Its density function is

$$f(\mathbf{U}) = \frac{\Gamma_p\left(\frac{m+n}{2}\right) (\det(\boldsymbol{\Sigma}))^{-\frac{n}{2}}}{\Gamma_p(\frac{m}{2}) \Gamma_p(\frac{n}{2})} \cdot (\det(\mathbf{U}))^{\frac{n-p-1}{2}} (\det(\mathbf{I} + \mathbf{U} \boldsymbol{\Sigma}^{-1}))^{-\frac{m+n}{2}}.$$

It is natural to define the multivariate Beta function as

$$B_p(a, b) = \frac{\Gamma_p(a)\Gamma_p(b)}{\Gamma_p(a+b)}.$$

# Matrix Beta Distribution

The density of Beta distribution with parameters  $m/2$  and  $n/2$  in univariate case is

$$f(w) = \frac{1}{B\left(\frac{m}{2}, \frac{n}{2}\right)} \cdot w^{\frac{n}{2}-1} (1-w)^{\frac{m}{2}-1},$$

where

$$B\left(\frac{m}{2}, \frac{n}{2}\right) = \frac{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{m+n}{2}\right)}.$$

How to generalize it to multivariate case?

# Matrix Beta Distribution

Let  $\mathbf{A} \sim \mathcal{W}_p(\boldsymbol{\Sigma}, n)$  and  $\mathbf{B} \sim \mathcal{W}_p(\boldsymbol{\Sigma}, m)$  be independent, then

$$\mathbf{W} = (\mathbf{A} + \mathbf{B})^{-1/2} \mathbf{A} (\mathbf{A} + \mathbf{B})^{-1/2}$$

has matrix Beta distribution with parameters  $n/2$  and  $m/2$  if  $\mathbf{0} \prec \mathbf{W} \prec \mathbf{I}$  and 0 elsewhere.

Its density function is

$$f(\mathbf{W}) = \frac{1}{B_p\left(\frac{n}{2}, \frac{m}{2}\right)} \cdot (\det(\mathbf{W}))^{\frac{n-p-1}{2}} (\det(\mathbf{I} - \mathbf{W}))^{\frac{m-p-1}{2}},$$

which does not depend on  $\boldsymbol{\Sigma}$ .

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# Likelihood Ratio Criterion and $T^2$ -Statistic

Let  $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_N$  constitute a sample from  $\mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$  with  $N > p$ .

We shall derive  $T^2$ -Statistic

$$T^2 = N(\bar{\mathbf{x}} - \boldsymbol{\mu}_0)^\top \mathbf{S}^{-1} (\bar{\mathbf{x}} - \boldsymbol{\mu}_0)$$

from likelihood ratio criterion

$$\lambda = \frac{\max_{\boldsymbol{\Sigma} \in \mathbb{S}_p^{++}} L(\boldsymbol{\mu}_0, \boldsymbol{\Sigma})}{\max_{\boldsymbol{\mu} \in \mathbb{R}^p, \boldsymbol{\Sigma} \in \mathbb{S}_p^{++}} L(\boldsymbol{\mu}, \boldsymbol{\Sigma})}.$$

# Likelihood Ratio Criterion and $T^2$ -Statistic

We have

$$\lambda^{\frac{2}{N}} = \frac{1}{1 + T^2/(N - 1)},$$

where

$$T^2 = N(\bar{\mathbf{x}} - \boldsymbol{\mu}_0)^\top \mathbf{S}^{-1}(\bar{\mathbf{x}} - \boldsymbol{\mu}_0), \quad \bar{\mathbf{x}} = \frac{1}{N} \sum_{\alpha=1}^N \mathbf{x}_\alpha$$

and

$$\mathbf{S} = \frac{1}{N - 1} \sum_{\alpha=1}^N (\mathbf{x}_\alpha - \bar{\mathbf{x}})(\mathbf{x}_\alpha - \bar{\mathbf{x}})^\top.$$

# Likelihood Ratio Criterion and $T^2$ -Statistic

The condition  $\lambda^{2/N} > c$  for some  $c \in (0, 1)$  is equivalent to

$$T^2 < \frac{(N-1)(1-c)}{c}.$$