Homework 4

Deadline: May 31, 2022

1. Let $\mathbf{x}_1, \ldots, \mathbf{x}_N$ sampled from

$$\mathcal{N}\left(\begin{bmatrix}\mu_1 \\ \mu_2\end{bmatrix}, \begin{bmatrix}\sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2\end{bmatrix}\right)$$

with $\sigma_1, \sigma_2 > 0$. Define

$$r = \frac{\sum_{\alpha=1}^{N}(x_{1\alpha} - \bar{x}_1)(x_{2\alpha} - \bar{x}_2)}{\sqrt{\sum_{\alpha=1}^{N}(x_{1\alpha} - \bar{x}_1)^2}\sqrt{\sum_{\alpha=1}^{N}(x_{2\alpha} - \bar{x}_2)^2}}.$$

for $i, j = 1, \ldots, p$.

(a) Use Fisher's z to test the hypothesis $\rho = \rho_0$ against the alternatives $\rho \neq \rho_0$ at the 0.01 level with $r = 0.5$ and $N = 50$.

(b) Use Fisher's z to obtain a confidence interval for ρ with confidence 0.95 based on $r = 0.65$ and $N = 25$.

(c) Prove that when $N = 2$ and $\rho = 0$, we have $\Pr(r = 1) = \Pr(r = -1) = 0.5$.

2. Let $\mathbf{z}_1, \ldots, \mathbf{z}_n$ be independently distributed according to $\mathcal{N}(\mathbf{0}, \mathbf{I})$ and

$$\mathbf{W} = \sum_{\alpha=1}^{n}\sum_{\beta=1}^{n} b_{\alpha\beta}\mathbf{z}_\alpha\mathbf{z}_\beta^\top.$$

Prove that if $\mathbf{a}^\top\mathbf{W}\mathbf{a}$ is distributed according to χ^2-distribution with m degrees of freedom for all $\mathbf{a}$ such that $\|\mathbf{a}\|_2 = 1$, then $\mathbf{W} \sim \mathcal{W}(\mathbf{I}, m)$.

3. Suppose $\mathbf{x}_\alpha$ is an observation from $\mathcal{N}_q(\mathbf{B}\mathbf{z}_\alpha, \mathbf{\Sigma})$ for $\alpha = 1, \ldots, N$, where $[\mathbf{z}_1, \ldots, \mathbf{z}_N] \in \mathbb{R}^{N\times q}$ of rank q is given and $N \geq p + q$, the maximum likelihood estimator of $\mathbf{B}$ is given by

$$\hat{\mathbf{B}} = \mathbf{C}\mathbf{A}^{-1},$$

where

$$\mathbf{C} = \sum_{\alpha=1}^{N}\mathbf{x}_\alpha\mathbf{z}_\alpha^\top \qquad \text{and} \qquad \mathbf{A} = \sum_{\alpha=1}^{N}\mathbf{z}_\alpha\mathbf{z}_\alpha^\top.$$

Show that $\mathbf{B} = \hat{\mathbf{B}}$ minimizes the generalized variance

$$\det\left(\sum_{\alpha=1}^{N}(\mathbf{x}_\alpha - \mathbf{B}\mathbf{z}_\alpha)(\mathbf{x}_\alpha - \mathbf{B}\mathbf{z}_\alpha)^\top\right).$$

4. Let $\mathbf{y}^{(1)}, \ldots, \mathbf{y}^{(q)}$ be a set of random vectors each with p components. Suppose that

$$\mathbb{E}\left[\mathbf{y}^{(g)}\right] = \mathbf{0} \qquad \text{and} \qquad \mathbb{E}\left[\mathbf{y}^{(g)}\mathbf{y}^{(h)\top}\right] = \delta_{gh}\mathbf{\Sigma}_g$$

for $g, h = 1, \ldots, q$, where $\delta_{gh} = 1$ for $g = h$ and $\delta_{gh} = 0$ otherwise. Let $\mathbf{C}$ be an $q \times q$ orthogonal matrix such that each element of the last row is $1/\sqrt{q}$. Define

$$\mathbf{z}^{(g)} = \sum_{h=1}^{q} c_{gh} \mathbf{y}^{(h)}$$

for $g = 1, \ldots, q$. Prove that

$$\mathbb{E}\big[\mathbf{z}^{(q)} \mathbf{z}^{(g)\top}\big] = \mathbf{0}$$

for $g = 1, \ldots, q-1$ if and only if $\mathbf{\Sigma}_1 = \cdots = \mathbf{\Sigma}_q$.