Multivariate Statistical Analysis

Lecture 14

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Outline

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Likelihood Ratio Criterion and T^2-Statistic

Let $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N$ constitute a sample from $\mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ with $N > p$.

We shall derive T^2-Statistic

$$T^2 = N(\bar{\mathbf{x}} - \boldsymbol{\mu}_0)^\top \mathbf{S}^{-1}(\bar{\mathbf{x}} - \boldsymbol{\mu}_0)$$

from likelihood ratio criterion

$$\lambda = \frac{\max_{\boldsymbol{\Sigma} \in \mathbb{S}_p^{++}} L(\boldsymbol{\mu}_0, \boldsymbol{\Sigma})}{\max_{\boldsymbol{\mu} \in \mathbb{R}^p, \boldsymbol{\Sigma} \in \mathbb{S}_p^{++}} L(\boldsymbol{\mu}, \boldsymbol{\Sigma})}.$$

Likelihood Ratio Criterion and T^2-Statistic

We have

$$\lambda^{\frac{2}{N}} = \frac{1}{1 + T^2/(N-1)},$$

where

$$T^2 = N(\bar{\mathbf{x}} - \boldsymbol{\mu}_0)^\top \mathbf{S}^{-1}(\bar{\mathbf{x}} - \boldsymbol{\mu}_0), \qquad \bar{\mathbf{x}} = \frac{1}{N}\sum_{\alpha=1}^{N} \mathbf{x}_\alpha$$

and

$$\mathbf{S} = \frac{1}{N-1}\sum_{\alpha=1}^{N}(\mathbf{x}_\alpha - \bar{\mathbf{x}})(\mathbf{x}_\alpha - \bar{\mathbf{x}})^\top.$$

The condition $\lambda^{2/N} > c$ for some $c \in (0,1)$ is equivalent to

$$T^2 < \frac{(N-1)(1-c)}{c}.$$

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Multivariate Analysis of Variance

We consider testing the equality of means with common covariance.

Let $\mathbf{x}_\alpha^{(g)}$ be an observation from the g-th population $\mathcal{N}_p(\boldsymbol{\mu}^{(g)}, \boldsymbol{\Sigma})$ for $\alpha = 1, \dots, N_g$ and $g = 1, \dots, q$. We wish to test the hypothesis

$$H_0 : \boldsymbol{\mu}_1 = \cdots = \boldsymbol{\mu}_g.$$

The likelihood function is

$$L(\boldsymbol{\mu}^{(1)}, \dots, \boldsymbol{\mu}^{(g)}, \boldsymbol{\Sigma})$$
$$= \prod_{g=1}^{q} \frac{1}{(2\pi)^{\frac{pN_g}{2}} (\det(\boldsymbol{\Sigma}))^{\frac{N_g}{2}}} \exp\left(-\frac{1}{2} \sum_{\alpha=1}^{N_g} \left(\mathbf{x}_\alpha^{(g)} - \boldsymbol{\mu}^{(g)}\right)^\top \boldsymbol{\Sigma}^{-1} \left(\mathbf{x}_\alpha^{(g)} - \boldsymbol{\mu}^{(g)}\right) \right).$$

1. We let $\boldsymbol{\theta} = \{\boldsymbol{\mu}^{(1)}, \dots, \boldsymbol{\mu}^{(g)}, \boldsymbol{\Sigma}\}$ be the parameters.
2. The set Ω is the space in which $\boldsymbol{\Sigma}$ is positive definite and each $\boldsymbol{\mu}^{(g)}$ is any p-dimensional vector.
3. The set ω is the space in which $\boldsymbol{\mu}^{(1)} = \cdots = \boldsymbol{\mu}^{(g)}$ (p-dimensional vectors) and $\boldsymbol{\Sigma}$ is positive definite matrix.

The likelihood ratio criterion is

$$\lambda = \frac{\sup_{\boldsymbol{\theta}\in\omega} L(\boldsymbol{\theta})}{\sup_{\boldsymbol{\theta}\in\Omega} L(\boldsymbol{\theta})} = \frac{(\det(\hat{\boldsymbol{\Sigma}}_\Omega))^{\frac{N}{2}}}{(\det(\hat{\boldsymbol{\Sigma}}_\omega))^{\frac{N}{2}}},$$

where

$$\hat{\boldsymbol{\Sigma}}_\Omega = \frac{1}{N}\sum_{g=1}^{q}\sum_{\alpha=1}^{N_g}\left(\mathbf{x}_\alpha^{(g)} - \bar{\mathbf{x}}^{(g)}\right)\left(\mathbf{x}_\alpha^{(g)} - \bar{\mathbf{x}}^{(g)}\right)^\top$$

and

$$\hat{\boldsymbol{\Sigma}}_\omega = \frac{1}{N}\sum_{g=1}^{q}\sum_{\alpha=1}^{N_g}\left(\mathbf{x}_\alpha^{(g)} - \bar{\mathbf{x}}\right)\left(\mathbf{x}_\alpha^{(g)} - \bar{\mathbf{x}}\right)^\top.$$

We can write

$$N\hat{\boldsymbol{\Sigma}}_{\omega} = \mathbf{A} + \mathbf{B},$$

where

$$\mathbf{A} = N\hat{\boldsymbol{\Sigma}}_{\Omega} = \sum_{g=1}^{q}\sum_{\alpha=1}^{N_g}\left(\mathbf{x}_{\alpha}^{(g)} - \bar{\mathbf{x}}^{(g)}\right)\left(\mathbf{x}_{\alpha}^{(g)} - \bar{\mathbf{x}}^{(g)}\right)^{\top} \sim \mathcal{W}_p(\boldsymbol{\Sigma}, N-q)$$

and

$$\mathbf{B} = \sum_{g=1}^{q} N_g(\bar{\mathbf{x}}^{(g)} - \bar{\mathbf{x}})(\bar{\mathbf{x}}^{(g)} - \bar{\mathbf{x}})^{\top} \sim \mathcal{W}_p(\boldsymbol{\Sigma}, q-1)$$

are independent.

Wilks' Lambda distribution

For two independent random matrices $\mathbf{A} \sim \mathcal{W}_p(\mathbf{\Sigma}, n)$ and $\mathbf{B} \sim \mathcal{W}_p(\mathbf{\Sigma}, m)$ with $n \geq p$, the ratio

$$\frac{\det(\mathbf{A})}{\det(\mathbf{A} + \mathbf{B})}$$

has Wilks' Lambda distribution with degrees of freedom n and m, which is typically written as

$$\frac{\det(\mathbf{A})}{\det(\mathbf{A} + \mathbf{B})} \sim \Lambda_{p,n,m}.$$

Wilks' Lambda distribution

Theorem

Let $\mathbf{A} \sim \mathcal{W}_p(\mathbf{\Sigma}, n)$ *and* $\mathbf{B} \sim \mathcal{W}_p(\mathbf{\Sigma}, m)$ *be two independent Wishart distributed variables, then we can write*

$$\frac{\det(\mathbf{A})}{\det(\mathbf{A}+\mathbf{B})} = \prod_{i=1}^{p} u_i \sim \Lambda_{p,n,m},$$

where $u_1, \dots, u_p$ *are independent distributed as*

$$u_i \sim \mathrm{Beta}\left(\frac{n+1-i}{2}, \frac{m}{2}\right).$$

Quiz

Let $\mathbf{A} \sim \mathcal{W}_p(\mathbf{\Sigma}, n)$, we can follow above theorem to show

$$\det(\mathbf{A}) = \det(\mathbf{\Sigma}) \prod_{i=1}^{p} v_i$$

with some independent random variables $v_1, \dots, v_p$?

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