

Multivariate Statistics

Lecture 08

Fudan University

1 Distribution of T^2 -Statistic

2 Uses of T^2 -Statistic

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Distribution of T^2 -Statistic

Theorem 1

Let $T^2 = \mathbf{y}^\top \mathbf{S}^{-1} \mathbf{y}$, where \mathbf{y} is distributed according to $\mathcal{N}_p(\boldsymbol{\nu}, \boldsymbol{\Sigma})$ and $n\mathbf{S}$ is independently distributed as $\sum_{\alpha=1}^n \mathbf{z}_\alpha \mathbf{z}_\alpha^\top$ with $\mathbf{z}_1, \dots, \mathbf{z}_n$ independent, each with distribution $\mathcal{N}_p(\mathbf{0}, \boldsymbol{\Sigma})$. Then the random variable

$$\frac{T^2}{n} \cdot \frac{n - p + 1}{p}$$

is distributed as a noncentral F -distribution with p and $n - p + 1$ degrees of freedom and noncentrality parameter $\boldsymbol{\nu}^\top \boldsymbol{\Sigma}^{-1} \boldsymbol{\nu}$. If $\boldsymbol{\nu} = \mathbf{0}$, the distribution is central F .

In the example of likelihood ratio criterion, we consider the special case of $\mathbf{y} = \sqrt{N}(\bar{\mathbf{x}} - \boldsymbol{\mu}_0)$, $\boldsymbol{\nu} = \sqrt{N}(\boldsymbol{\mu} - \boldsymbol{\mu}_0)$ and $n = N - 1$.

Distribution of T^2 -Statistic

Corollary 1

Let $\mathbf{x}_1, \dots, \mathbf{x}_N$ be a sample from $\mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ and let

$$T^2 = N(\bar{\mathbf{x}} - \boldsymbol{\mu}_0)^\top \mathbf{S}^{-1}(\bar{\mathbf{x}} - \boldsymbol{\mu}_0).$$

The distribution of

$$\frac{T^2}{N-1} \cdot \frac{N-p}{p}.$$

is noncentral F with p and $N-p$ degrees of freedom and noncentrality parameter $N(\bar{\mathbf{x}} - \boldsymbol{\mu}_0)^\top \boldsymbol{\Sigma}^{-1}(\bar{\mathbf{x}} - \boldsymbol{\mu}_0)$. If $\boldsymbol{\mu} = \boldsymbol{\mu}_0$ then the F -distribution is central.

Distribution of T^2 -Statistic

Theorem 2

Suppose $\mathbf{y}_1, \dots, \mathbf{y}_m$ are independent with \mathbf{y}_α distributed according to $\mathcal{N}(\mathbf{\Gamma}\mathbf{w}_\alpha, \mathbf{\Phi})$, where \mathbf{w}_α is an r -component vector. Let $\mathbf{H} = \sum_{\alpha=1}^m \mathbf{w}_\alpha \mathbf{w}_\alpha^\top$ assumed non-singular, $\mathbf{G} = \sum_{\alpha=1}^m \mathbf{y}_\alpha \mathbf{w}_\alpha^\top \mathbf{H}^{-1}$ and

$$\mathbf{C} = \sum_{\alpha=1}^m (\mathbf{y}_\alpha - \mathbf{G}\mathbf{w}_\alpha)(\mathbf{y}_\alpha - \mathbf{G}\mathbf{w}_\alpha)^\top = \sum_{\alpha=1}^m \mathbf{y}_\alpha \mathbf{y}_\alpha^\top - \mathbf{G}\mathbf{H}\mathbf{G}^\top.$$

Then \mathbf{C} is distributed as

$$\sum_{\alpha=1}^{m-r} \mathbf{u}_\alpha \mathbf{u}_\alpha^\top$$

where $\mathbf{u}_1, \dots, \mathbf{u}_{m-r}$ are independently distributed according to $\mathcal{N}(\mathbf{0}, \mathbf{\Phi})$ independently of \mathbf{G} .

Distribution of T^2 -Statistic

For large samples the distribution of T^2 given this corollary is approximately valid even if the parent distribution is not normal.

Theorem 3

Let x_1, x_2, \dots be a sequence of independently identically distributed random vectors with mean vector μ and covariance matrix Σ . Let

$$\hat{\mathbf{x}}_N = \frac{1}{N} \sum_{\alpha=1}^N \mathbf{x}_\alpha, \quad \hat{\mathbf{S}}_N = \frac{1}{N-1} \sum_{\alpha=1}^N (\mathbf{x}_\alpha - \bar{\mathbf{x}})(\mathbf{x}_\alpha - \bar{\mathbf{x}})^\top$$

and

$$T_N^2 = N(\bar{\mathbf{x}}_N - \mu_0)^\top \mathbf{S}_N^{-1} (\bar{\mathbf{x}}_N - \mu_0).$$

Then the limiting distribution of T_N^2 as $N \rightarrow \infty$ is the χ^2 -distribution with p degrees of freedom if $\mu = \mu_0$.

Distribution of T^2 -Statistic

When the null hypothesis is true ($\mu_0 = \mu$), the likelihood ratio criterion holds that

$$\lambda^{\frac{2}{N}} = \frac{1}{1 + T^2/(N-1)} = \frac{1}{1 + T^2/n},$$

where $T^2 =$ and $n = N - 1$.

Then T^2 is distributed according to central F -distribution with degree of freedom p and $n - 1 - p$:

$$\begin{aligned} \frac{T^2}{n} \cdot \frac{n-p+1}{p} &\sim \frac{\chi^2(p)/p}{\chi^2(n-1-p)/(n-1-p)} \\ \Rightarrow \frac{T^2}{n} &\sim \frac{\chi^2(p)}{\chi^2(n-1-p)} \\ \Rightarrow \lambda^{\frac{2}{N}} &\sim \frac{\chi^2(n-1-p)}{\chi^2(n-1-p) + \chi^2(p)} \end{aligned}$$

Distribution of T^2 -Statistic

Theorem 4

Let u be distributed according to the χ^2 -distribution with a degrees of freedom and w be distributed according to the χ^2 -distribution with b degrees of freedom. The density of $v = u/(u + w)$, when u and w are independent is

$$\frac{1}{B\left(\frac{a}{2}, \frac{b}{2}\right)} v^{\frac{a}{2}-1} (1-v)^{\frac{b}{2}-1}, \quad (1)$$

where $B(\alpha, \beta) = \int_0^1 t^{\alpha-1} (1-t)^{\beta-1} dt$.

The function (1) is the density of beta distribution with parameters $a/2$ and $b/2$.

Outline

1 Distribution of T^2 -Statistic

2 Uses of T^2 -Statistic

Testing the Hypothesis for the Mean

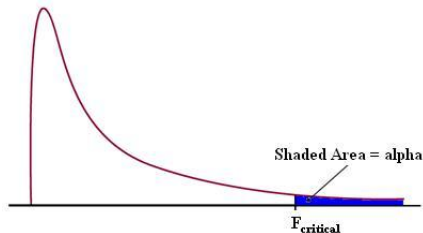
The likelihood ratio test of the hypothesis $\boldsymbol{\mu} = \boldsymbol{\mu}_0$ on the basis of a sample of N from $\mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ is defined by the critical region

$$T^2 \geq T_0^2,$$

where $T^2 = N(\bar{\mathbf{x}} - \boldsymbol{\mu}_0)^\top \mathbf{S}^{-1}(\bar{\mathbf{x}} - \boldsymbol{\mu}_0)$.

If the significance level is α , then

$$T_0^2 = \frac{(N-1)p}{N-p} F_{p, N-p}(\alpha) \triangleq T_{p, N-1}^2(\alpha).$$



Testing the Hypothesis for the Mean

The statistic T^2 is computed from $\bar{\mathbf{x}}$ and \mathbf{A} . The vector

$$\mathbf{b} = \mathbf{A}^{-1}(\bar{\mathbf{x}} - \boldsymbol{\mu}_0)$$

is the solution of $\mathbf{A}\mathbf{b} = \bar{\mathbf{x}} - \boldsymbol{\mu}_0$. Then

$$\begin{aligned} T^2 &= N(\bar{\mathbf{x}} - \boldsymbol{\mu}_0)^\top \mathbf{S}^{-1}(\bar{\mathbf{x}} - \boldsymbol{\mu}_0) \\ &= N(\bar{\mathbf{x}} - \boldsymbol{\mu}_0)^\top \left(\frac{\mathbf{A}}{N-1} \right)^{-1} (\bar{\mathbf{x}} - \boldsymbol{\mu}_0) \\ &= N(N-1)(\bar{\mathbf{x}} - \boldsymbol{\mu}_0)^\top \mathbf{b}. \end{aligned}$$

Note that $T^2/(N-1)$ is the nonzero root of

$$\det \left(N(\bar{\mathbf{x}} - \boldsymbol{\mu}_0)(\bar{\mathbf{x}} - \boldsymbol{\mu}_0)^\top - \lambda \mathbf{A} \right) = 0$$

A Confidence Region for the Mean Vector

The probability of drawing a sample of N from $\mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ with sample mean $\bar{\mathbf{x}}$ and sample covariance matrix \mathbf{S} such that

$$N(\bar{\mathbf{x}} - \boldsymbol{\mu})^\top \mathbf{S}^{-1}(\bar{\mathbf{x}} - \boldsymbol{\mu}) \leq T_{p, N-1}^2(\alpha).$$

is $1 - \alpha$.

The set

$$\left\{ \mathbf{m} : N(\bar{\mathbf{x}} - \mathbf{m})^\top \mathbf{S}^{-1}(\bar{\mathbf{x}} - \mathbf{m}) \leq T_{p, N-1}^2(\alpha) \right\}$$

corresponds to the interior and boundary of an ellipsoid. We state that $\boldsymbol{\mu}$ lies within this ellipsoid with confidence $1 - \alpha$.

Confidence Intervals for Linear Combinations of the Mean

Lemma 2 (Generalized Cauchy-Schwarz Inequality)

For any positive definite matrix $\mathbf{S} \in \mathbb{R}^{p \times p}$ and $\mathbf{y}, \boldsymbol{\gamma} \in \mathbb{R}^p$, we have

$$(\boldsymbol{\gamma}^\top \mathbf{y})^2 \leq (\boldsymbol{\gamma}^\top \mathbf{S} \boldsymbol{\gamma})(\mathbf{y}^\top \mathbf{S}^{-1} \mathbf{y}).$$

Let $\mathbf{y} = \bar{\mathbf{x}} - \boldsymbol{\mu}$, then

$$\begin{aligned} \left| \boldsymbol{\gamma}^\top (\bar{\mathbf{x}} - \boldsymbol{\mu}) \right| &\leq \sqrt{\boldsymbol{\gamma}^\top \mathbf{S} \boldsymbol{\gamma}} \sqrt{(\bar{\mathbf{x}} - \boldsymbol{\mu})^\top \mathbf{S}^{-1} (\bar{\mathbf{x}} - \boldsymbol{\mu})} \\ &\leq \sqrt{\boldsymbol{\gamma}^\top \mathbf{S} \boldsymbol{\gamma}} \sqrt{T_{p, N-1}^2(\alpha)/N} \end{aligned}$$

holds for all $\boldsymbol{\gamma}$ with probability $1 - \alpha$.

Thus we can assert with confidence $1 - \alpha$ that the unknown parameter vector satisfies simultaneously for all $\boldsymbol{\gamma}$ the inequalities

$$\left| \boldsymbol{\gamma}^\top \bar{\mathbf{x}} - \boldsymbol{\gamma}^\top \mathbf{m} \right| \leq \sqrt{\boldsymbol{\gamma}^\top \mathbf{S} \boldsymbol{\gamma}} \sqrt{T_{p, N-1}^2(\alpha)/N}.$$

Two-Sample Problems (Unknown Covariance)

Suppose $\mathbf{y}_1^{(i)}, \dots, \mathbf{y}_{N_i}^{(i)}$ is a sample from $\mathcal{N}(\boldsymbol{\mu}^{(i)}, \boldsymbol{\Sigma})$ for $i = 1, 2$. We wish to test the null hypothesis $\boldsymbol{\mu}^{(1)} = \boldsymbol{\mu}^{(2)}$.

① For $i = 1, 2$, we have

$$\bar{\mathbf{y}}^{(i)} = \frac{1}{N_i} \sum_{\alpha=1}^{N_i} \mathbf{y}_{\alpha}^{(i)} \sim \mathcal{N}\left(\boldsymbol{\mu}^{(i)}, \frac{1}{N_i} \boldsymbol{\Sigma}\right).$$

② Since

$$\bar{\mathbf{y}}^{(1)} - \bar{\mathbf{y}}^{(2)} = \begin{bmatrix} \mathbf{I} & -\mathbf{I} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{y}}^{(1)} \\ \bar{\mathbf{y}}^{(2)} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \bar{\mathbf{y}}^{(1)} \\ \bar{\mathbf{y}}^{(2)} \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} \boldsymbol{\mu}^{(1)} \\ \boldsymbol{\mu}^{(2)} \end{bmatrix}, \begin{bmatrix} \frac{1}{N_1} \boldsymbol{\Sigma} & \mathbf{0} \\ \mathbf{0} & \frac{1}{N_2} \boldsymbol{\Sigma} \end{bmatrix}\right),$$

we have

$$\bar{\mathbf{y}}^{(1)} - \bar{\mathbf{y}}^{(2)} \sim \mathcal{N}\left(\boldsymbol{\mu}^{(1)} - \boldsymbol{\mu}^{(2)}, \left(\frac{1}{N_1} + \frac{1}{N_2}\right) \boldsymbol{\Sigma}\right).$$

Two-Sample Problems (Unknown Covariance)

Under the null hypothesis, we have

$$\sqrt{N_1 N_2 / (N_1 + N_2)} (\bar{\mathbf{y}}^{(1)} - \bar{\mathbf{y}}^{(2)}) \sim \mathcal{N}(\mathbf{0}, \mathbf{\Sigma}).$$

Let

$$\mathbf{S} = \frac{1}{N_1 + N_2 - 2} \left(\sum_{\alpha=1}^{N_1} (\mathbf{y}_{\alpha}^{(1)} - \bar{\mathbf{y}}^{(1)}) (\mathbf{y}_{\alpha}^{(1)} - \bar{\mathbf{y}}^{(1)})^{\top} + \sum_{\alpha=1}^{N_2} (\mathbf{y}_{\alpha}^{(2)} - \bar{\mathbf{y}}^{(2)}) (\mathbf{y}_{\alpha}^{(2)} - \bar{\mathbf{y}}^{(2)})^{\top} \right),$$

then

$$(N_1 + N_2 - 2)\mathbf{S} = \sum_{\alpha=1}^{N_1 + N_2 - 2} \mathbf{z}_{\alpha} \mathbf{z}_{\alpha}^{\top},$$

where \mathbf{z}_{α} are independent and $\mathbf{z}_{\alpha} \sim \mathcal{N}(\mathbf{0}, \mathbf{\Sigma})$.

Two-Sample Problems (Unknown Covariance)

Let

$$T^2 = \frac{N_1 N_2}{N_1 + N_2} (\bar{\mathbf{y}}^{(1)} - \bar{\mathbf{y}}^{(2)})^\top \mathbf{S}^{-1} (\bar{\mathbf{y}}^{(1)} - \bar{\mathbf{y}}^{(2)}),$$

then

$$\frac{T^2}{N_1 + N_2 - 2} \cdot \frac{N_1 + N_2 - p - 1}{p}$$

is distributed according to central F -distribution with p and $N_1 + N_2 - p - 1$ degrees of freedom.

The critical region is

$$T^2 \geq \frac{(N_1 + N_2 - 2)p}{N_1 + N_2 - p - 1} F_{p, N_1 + N_2 - p - 1}(\alpha)$$

with significance level α .

Two-Sample Problems (Unknown Covariance)

The probability of

$$\begin{aligned} T^2 &= \frac{N_1 N_2}{N_1 + N_2} (\bar{\mathbf{y}}^{(1)} - \bar{\mathbf{y}}^{(2)})^\top \mathbf{S}^{-1} (\bar{\mathbf{y}}^{(1)} - \bar{\mathbf{y}}^{(2)}) \\ &\leq \frac{(N_1 + N_2 - 2)p}{N_1 + N_2 - p - 1} F_{p, N_1 + N_2 - p - 1}(\alpha) \end{aligned}$$

is $1 - \alpha$.

A confidence region for $\boldsymbol{\mu}^{(1)} - \boldsymbol{\mu}^{(2)}$ with confidence level $1 - \alpha$ is the set of vectors \mathbf{m} satisfying

$$\begin{aligned} &\frac{N_1 N_2}{N_1 + N_2} (\bar{\mathbf{y}}^{(1)} - \bar{\mathbf{y}}^{(2)} - \mathbf{m})^\top \mathbf{S}^{-1} (\bar{\mathbf{y}}^{(1)} - \bar{\mathbf{y}}^{(2)} - \mathbf{m}) \\ &\leq \frac{(N_1 + N_2 - 2)p}{N_1 + N_2 - p - 1} F_{p, N_1 + N_2 - p - 1}(\alpha). \end{aligned}$$

A Problem of Several Samples

There is a theoretical reason for believing the gene structures of three species of *Iris virginica* to be such that the mean vectors of the three populations are related as

$$3\boldsymbol{\mu}^{(1)} = \boldsymbol{\mu}^{(3)} + 2\boldsymbol{\mu}^{(2)},$$

where $\boldsymbol{\mu}^{(i)}$ is the mean vector of the i -th population.

A Problem of Several Samples

Let $\{\mathbf{x}_\alpha^{(i)}\}$ for $\alpha = 1, \dots, N_i$, $i = 1, \dots, q$ be independent samples from $\mathcal{N}(\boldsymbol{\mu}^{(i)}, \boldsymbol{\Sigma})$, $i = 1, \dots, q$, respectively. Let us test the hypothesis

$$H : \sum_{i=1}^q \beta_i \boldsymbol{\mu}^{(i)} = \boldsymbol{\mu}.$$

where β_1, \dots, β_q are given scalars and $\boldsymbol{\mu}$ is a given vector.

A Problem of Several Samples

The criterion is

$$T^2 = c \left(\sum_{i=1}^q \beta_i \bar{\mathbf{x}}^{(i)} - \boldsymbol{\mu} \right) \mathbf{S}^{-1} \left(\sum_{i=1}^q \beta_i \bar{\mathbf{x}}^{(i)} - \boldsymbol{\mu} \right)^{\top}$$

where

$$\bar{\mathbf{x}}^{(i)} = \frac{1}{N_i} \sum_{\alpha=1}^{N_i} \mathbf{x}_{\alpha}^{(i)}, \quad \frac{1}{c} = \sum_{i=1}^q \frac{\beta_i^2}{N_i}$$

and

$$\left(\sum_{i=1}^q N_i - q \right) S = \sum_{i=1}^q \sum_{\alpha=1}^{N_i} (\mathbf{x}_{\alpha}^{(i)} - \bar{\mathbf{x}}^{(i)}) (\mathbf{x}_{\alpha}^{(i)} - \bar{\mathbf{x}}^{(i)})^{\top}.$$

This T^2 has the T^2 -distribution with $\sum_{i=1}^q N_i - q$ degrees of freedom.

A Problem of Symmetry

Consider testing the hypothesis

$$H : \mu_1 = \mu_2 = \cdots = \mu_p$$

on the basis of sample $\mathbf{x}_1, \dots, \mathbf{x}_N$ from $\mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, where

$$\boldsymbol{\mu} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_p \end{bmatrix}.$$

A Problem of Symmetry

Let \mathbf{C} be any $(p-1) \times p$ matrix of rank $p-1$ such that

$$\mathbf{C}\mathbf{1}_p = \mathbf{0}_{p-1}.$$

Then we have

$$\mathbf{y}_\alpha = \mathbf{C}\mathbf{x}_\alpha \sim \mathcal{N}(\mathbf{C}\boldsymbol{\mu}, \mathbf{C}\boldsymbol{\Sigma}\mathbf{C}^\top)$$

and the hypothesis H is equivalent to $\mathbf{C}\boldsymbol{\mu} = \mathbf{0}_{p-1}$ (why?).

A Problem of Symmetry

The statistic to be used is

$$T^2 = N\bar{\mathbf{y}}^\top \mathbf{S}^{-1} \bar{\mathbf{y}}$$

where

$$\bar{\mathbf{y}} = \frac{1}{N} \sum_{\alpha=1}^N \mathbf{y}_\alpha = \frac{1}{N} \sum_{\alpha=1}^N \mathbf{C} \mathbf{x}_\alpha = \mathbf{C} \bar{\mathbf{x}}$$

$$\mathbf{S} = \frac{1}{N-1} \sum_{\alpha=1}^N (\mathbf{y}_\alpha - \bar{\mathbf{y}})(\mathbf{y}_\alpha - \bar{\mathbf{y}})^\top = \frac{1}{N-1} \sum_{\alpha=1}^N \mathbf{C}(\mathbf{y}_\alpha - \bar{\mathbf{y}})(\mathbf{y}_\alpha - \bar{\mathbf{y}})^\top \mathbf{C}^\top.$$

This statistic has the T^2 -distribution with $N - 1$ degrees of freedom for a $(p - 1)$ -dimensional distribution.

Two-Sample Problems (Unequal Covariance)

Let $\{\mathbf{x}_\alpha^{(i)}\}$ for $\alpha = 1, \dots, N_i$, $i = 1, \dots, q$ be independent samples from $\mathcal{N}(\boldsymbol{\mu}^{(i)}, \boldsymbol{\Sigma}_i)$ for $i = 1, 2$, respectively. We wish to test the hypothesis

$$H : \boldsymbol{\mu}^{(1)} = \boldsymbol{\mu}^{(2)}.$$

We cannot use the technique in the case of equal covariance, because

$$\sum_{\alpha=1}^{N_1} (\mathbf{x}_\alpha^{(1)} - \bar{\mathbf{x}}^{(1)}) (\mathbf{x}_\alpha^{(1)} - \bar{\mathbf{x}}^{(1)})^\top + \sum_{\alpha=1}^{N_2} (\mathbf{x}_\alpha^{(2)} - \bar{\mathbf{x}}^{(2)}) (\mathbf{x}_\alpha^{(2)} - \bar{\mathbf{x}}^{(2)})^\top.$$

does not have the form of a multiple of

$$\frac{1}{N_1} \boldsymbol{\Sigma}_1 + \frac{1}{N_2} \boldsymbol{\Sigma}_2.$$

Two-Sample Problems ($N_1 = N_2$)

If $N_1 = N_2 = N$, we can use the T^2 -test in an obvious way.

- ① Let $\mathbf{y}_\alpha = \mathbf{x}_\alpha^{(1)} - \mathbf{x}_\alpha^{(2)}$, then $\mathbf{y}_1, \dots, \mathbf{y}_N$ are independent and

$$\mathbf{y}_\alpha \sim \mathcal{N}(\boldsymbol{\mu}^{(1)} - \boldsymbol{\mu}^{(2)}, \boldsymbol{\Sigma}_1 + \boldsymbol{\Sigma}_2).$$

- ② Define

$$\begin{aligned}\bar{\mathbf{y}} &= \frac{1}{N} \sum_{\alpha=1}^N \mathbf{y}_\alpha = \bar{\mathbf{x}}_\alpha^{(1)} - \bar{\mathbf{x}}_\alpha^{(2)}, \\ (N-1)\mathbf{S} &= \sum_{\alpha=1}^N (\mathbf{y}_\alpha - \bar{\mathbf{y}})(\mathbf{y}_\alpha - \bar{\mathbf{y}})^\top \\ &= \sum_{\alpha=1}^N (\mathbf{x}_\alpha^{(1)} - \mathbf{x}_\alpha^{(2)} - \bar{\mathbf{x}}_\alpha^{(1)} + \bar{\mathbf{x}}_\alpha^{(2)})(\mathbf{x}_\alpha^{(1)} - \mathbf{x}_\alpha^{(2)} - \bar{\mathbf{x}}_\alpha^{(1)} + \bar{\mathbf{x}}_\alpha^{(2)})^\top.\end{aligned}$$

- ③ Then $T^2 = N\bar{\mathbf{y}}^\top \mathbf{S}^{-1} \bar{\mathbf{y}}$ is suitable for testing the hypothesis $\boldsymbol{\mu}^{(1)} = \boldsymbol{\mu}^{(2)}$ and has the T^2 -distribution with $N-1$ degrees of freedom.

Two-Sample Problems ($N_1 = N_2$)

If we had known $\Sigma_1 = \Sigma_2$, we would have used a T^2 -statistic with $2N - 2$ degrees of freedom; thus we have lost $N - 1$ degrees of freedom in constructing a test which is independent of the two covariance matrices.

Lemma 3

Let $\mathbf{x}_1, \dots, \mathbf{x}_m$ be independent samples from $\mathcal{N}(\mu_\alpha, \Sigma_\alpha)$ for $i = 1, \dots, m$. Define

$$\mathbf{z}_1 = \sum_{\alpha=1}^N a_\alpha \mathbf{x}_\alpha \quad \text{and} \quad \mathbf{z}_2 = \sum_{\alpha=1}^N b_\alpha \mathbf{x}_\alpha,$$

then

$$\text{Cov}(\mathbf{z}_1, \mathbf{z}_2) = \sum_{\alpha=1}^N a_\alpha b_\alpha \Sigma_\alpha.$$

Two-Sample Problems ($N_1 \neq N_2$)

For the case of $N_1 \neq N_2$, we let $N_1 < N_2$ and define

$$\mathbf{y}_\alpha = \mathbf{x}_\alpha^{(1)} - \sqrt{\frac{N_1}{N_2}} \mathbf{x}_\alpha^{(2)} + \frac{1}{\sqrt{N_1 N_2}} \sum_{\beta=1}^{N_1} \mathbf{x}_\beta^{(2)} - \frac{1}{N_2} \sum_{\gamma=1}^{N_2} \mathbf{x}_\gamma^{(2)}$$

for $\alpha = 1, \dots, N_1$. We have

$$\mathbb{E}[\mathbf{y}_\alpha] = \boldsymbol{\mu}^{(1)} - \boldsymbol{\mu}^{(2)}$$

and

$$\text{Cov}(\mathbf{y}_\alpha, \mathbf{y}_{\alpha'}) = \begin{cases} \boldsymbol{\Sigma}_1 + \frac{N_1}{N_2} \boldsymbol{\Sigma}_2, & \alpha = \alpha', \\ \mathbf{0}, & \text{otherwise.} \end{cases}$$

Two-Sample Problems ($N_1 \neq N_2$)

We test $\mu^{(1)} = \mu^{(2)}$ by using

$$T^2 = N_1 \bar{\mathbf{y}}^\top \mathbf{S}^{-1} \bar{\mathbf{y}},$$

which has T^2 -distribution with $N - 1$ degrees of freedom, where

$$\bar{\mathbf{y}} = \frac{1}{N_1} \sum_{\alpha=1}^{N_1} \mathbf{y}_\alpha = \bar{\mathbf{x}}^{(1)} - \bar{\mathbf{x}}^{(2)},$$

$$(N_1 - 1)\mathbf{S} = \sum_{\alpha=1}^{N_1} (\mathbf{y}_\alpha - \bar{\mathbf{y}})(\mathbf{y}_\alpha - \bar{\mathbf{y}})^\top = \sum_{\alpha=1}^{N_1} (\mathbf{u}_\alpha - \bar{\mathbf{u}})(\mathbf{u}_\alpha - \bar{\mathbf{u}})^\top,$$

$$\bar{\mathbf{u}} = \frac{1}{N_1} \sum_{\alpha=1}^{N_1} \mathbf{u}_\alpha,$$

$$\mathbf{u}_\alpha = \mathbf{x}_\alpha^{(1)} - \sqrt{\frac{N_1}{N_2}} \mathbf{x}_\alpha^{(2)}.$$