Multivariate Statistical Analysis

Lecture 09

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Outline

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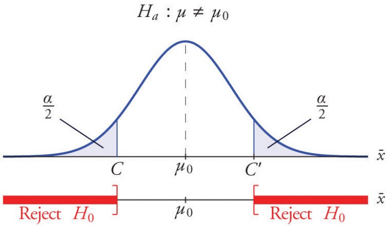
Hypothesis Testing for the Mean

In the univariate case, the difference between the sample mean and the population mean is normally distributed.

We consider

$$z = \frac{\sqrt{N}}{\sigma}(\bar{x} - \mu_0).$$



1. For significance level $\alpha = 0.05$ and $p = 1$, we have $1 - \alpha = 0.95$.
2. What about multivariate case?

Chi-Squared Distribution

If $x_1, \ldots, x_n$ are independent, standard normal random variables, then the sum of their squares,

$$y = \sum_{i=1}^{n} x_i^2,$$

is distributed according to the (central) chi-squared distribution (χ^2-distribution) with n degrees of freedom. One may write $y \sim \chi_n^2$.

We have $\mathbb{E}[y] = n$ and $\mathrm{Var}[y] = 2n$.

Chi-Squared Distribution

The probability density function of the (central) chi-squared distribution is

$$f(y;\, n) = \begin{cases} \dfrac{1}{2^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)} y^{\frac{n}{2}-1} \exp\left(-\dfrac{y}{2}\right), & y > 0; \\ 0, & \text{otherwise}, \end{cases}$$

where

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} \exp(-t)\,\mathrm{d}t.$$

Chi-Squared Distribution

The derivation for the density of Chi-square distribution:

1. Show that $\Gamma(1/2) = \sqrt{\pi}$.
2. For $y_1 = x^2$ with $x \sim \mathcal{N}(0,1)$, the density function of y_1 is

$$\frac{1}{\sqrt{2\pi y_1}} \exp\left(-\frac{1}{2} y_1\right).$$

3. For beta function $B(\alpha, \beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,\mathrm{d}t$, we have

$$B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}.$$

4. Show the density of $y_n = \sum_{i=1}^n x_i^2$ by induction.

Noncentral Chi-Squared Distribution

If $x_1, \dots, x_n$ are independent and each x_i are normally distributed random variables with means μ_i and unit variances, then the sum of their squares,

$$y = \sum_{i=1}^{n} x_i^2,$$

is distributed according to the noncentral Chi-squared distribution with n degrees of freedom and noncentrality parameter

$$\lambda = \sum_{i=1}^{n} \mu_i^2.$$

One may write $y \sim \chi^2_{n,\lambda}$.

We have $\mathbb{E}[y] = n + \lambda$ and $\mathrm{Var}[y] = 2n + 4\lambda$.

Noncentral Chi-Squared Distribution

Theorem

If $y_1, \dots, y_k$ are independent and each y_i is distributed according to the noncentral χ^2-distribution with n_i degrees of freedom and noncentrality parameter λ_i, then

$$\sum_{i=1}^{k} y_i \sim \chi^2_{n,\lambda},$$

where

$$n = \sum_{i=1}^{k} n_i \qquad \text{and} \qquad \lambda = \sum_{i=1}^{k} \lambda_i.$$

Noncentral Chi-Squared Distribution

Theorem

If the n-component random vector $\mathbf{y}$ *is distributed according to* $\mathcal{N}_n(\boldsymbol{\nu}, \mathbf{T})$ *with* $\mathbf{T} \succ \mathbf{0}$, *then*

$$\mathbf{y}^\top \mathbf{T}^{-1} \mathbf{y} \sim \chi^2_{n,\lambda},$$

where

$$\lambda = \boldsymbol{\nu}^\top \mathbf{T}^{-1} \boldsymbol{\nu}.$$

If $\boldsymbol{\nu} = \mathbf{0}$, *the distribution is the central* χ^2*-distribution.*

Noncentral Chi-Squared Distribution

Let $\mathbf{y} \sim \mathcal{N}_p(\boldsymbol{\lambda}, \mathbf{I})$, then

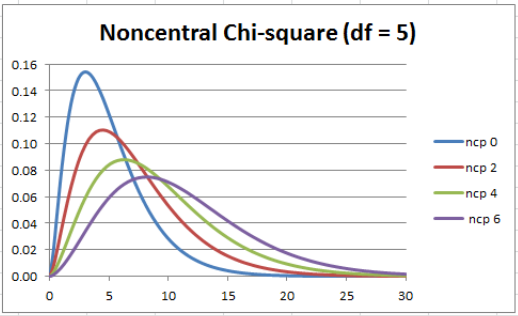
$$v = \mathbf{y}^\top \mathbf{y}$$

is distributed according to the noncentral χ^2-distribution with p degrees of freedom and noncentral parameter $\tau^2 = \boldsymbol{\lambda}^\top \boldsymbol{\lambda}$.

The probability density function is

$$f(v; p, \tau^2) = \begin{cases} \dfrac{\exp\left(-\frac{1}{2}(\tau^2 + v)\right) v^{\frac{p}{2}-1}}{2^{\frac{p}{2}}\sqrt{\pi}} \displaystyle\sum_{\beta=0}^{\infty} \dfrac{\tau^{2\beta} v^{\beta} \Gamma\left(\beta + \frac{1}{2}\right)}{(2\beta)!\, \Gamma\left(\frac{p}{2} + \beta\right)} & v > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Noncentral Chi-Squared Distribution



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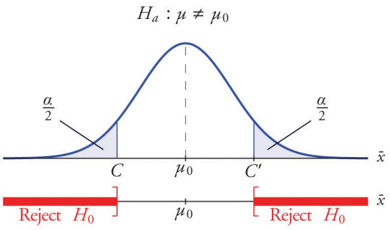
Hypothesis Testing for the Mean (Covariance is Known)

In the univariate case, the difference between the sample mean and the population mean is normally distributed. We consider

$$z = \frac{\sqrt{N}}{\sigma}(\bar{x} - \mu_0).$$



What about multivariate case?

Hypothesis Testing for the Mean (Covariance is Known)

Let $\mathbf{x}_1, \ldots, \mathbf{x}_N$ constitute a sample from $\mathcal{N}_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$.

What about multivariate case to test $\boldsymbol{\mu} = \boldsymbol{\mu}_0$?

$$\frac{\sqrt{N}}{\sigma}(\bar{x} - \mu_0) \implies \frac{N}{\sigma^2}(\bar{x} - \mu_0)^2 \implies N(\bar{\mathbf{x}} - \boldsymbol{\mu}_0)^\top \boldsymbol{\Sigma}^{-1}(\bar{\mathbf{x}} - \boldsymbol{\mu}_0).$$

Rejection Region

Let $\chi^2_p(\alpha)$ be the number such that

$$\Pr\left\{N(\bar{\mathbf{x}}-\boldsymbol{\mu})^\top\boldsymbol{\Sigma}^{-1}(\bar{\mathbf{x}}-\boldsymbol{\mu}) > \chi^2_p(\alpha)\right\} = \alpha.$$

To test the hypothesis that $\boldsymbol{\mu} = \boldsymbol{\mu}_0$ where $\boldsymbol{\mu}_0$ is a specified vector, we use as our rejection region (critical region)

$$N(\bar{\mathbf{x}}-\boldsymbol{\mu}_0)^\top\boldsymbol{\Sigma}^{-1}(\bar{\mathbf{x}}-\boldsymbol{\mu}_0) > \chi^2_p(\alpha).$$

If above inequality is satisfied, we reject the null hypothesis.

Confidence Region

Consider the statement made on the basis of a sample with mean $\bar{\mathbf{x}}$:
"The mean of the distribution satisfies

$$N(\bar{\mathbf{x}} - \boldsymbol{\mu}^*)^\top \boldsymbol{\Sigma}^{-1}(\bar{\mathbf{x}} - \boldsymbol{\mu}^*) \leq \chi_p^2(\alpha).$$

as an inequality on $\boldsymbol{\mu}^*$." This statement is true with probability $1 - \alpha$.

Thus, the set of $\boldsymbol{\mu}^*$ satisfying above inequality is a confidence region for $\boldsymbol{\mu}$ with confidence $1 - \alpha$.

Two-Sample Problems

Suppose there are two samples:

1. $\mathbf{x}_1^{(1)}, \dots, \mathbf{x}_{N_1}^{(1)}$ from $\mathcal{N}(\boldsymbol{\mu}^{(1)}, \boldsymbol{\Sigma})$;
2. $\mathbf{x}_1^{(2)}, \dots, \mathbf{x}_{N_2}^{(2)}$ from $\mathcal{N}(\boldsymbol{\mu}^{(2)}, \boldsymbol{\Sigma})$;

where $\boldsymbol{\Sigma}$ is known.

How to test the hypothesis $\boldsymbol{\mu}^{(1)} = \boldsymbol{\mu}^{(2)}$?

Two-Sample Problems

Then the two sample means

$$\bar{\mathbf{x}}^{(1)} = \frac{1}{N_1}\sum_{\alpha=1}^{N_1}\mathbf{x}_\alpha^{(1)} \sim \mathcal{N}\left(\boldsymbol{\mu}^{(1)}, \frac{1}{N_1}\boldsymbol{\Sigma}\right)$$

and

$$\bar{\mathbf{x}}^{(2)} = \frac{1}{N_2}\sum_{\alpha=1}^{N_2}\mathbf{x}_\alpha^{(2)} \sim \mathcal{N}\left(\boldsymbol{\mu}^{(2)}, \frac{1}{N_2}\boldsymbol{\Sigma}\right).$$

are independent.

Then we have

$$\mathbf{y} = \bar{\mathbf{x}}^{(1)} - \bar{\mathbf{x}}^{(2)} = \begin{bmatrix} \mathbf{I} & -\mathbf{I} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{x}}^{(1)} \\ \bar{\mathbf{x}}^{(2)} \end{bmatrix}, \quad \begin{bmatrix} \bar{\mathbf{x}}^{(1)} \\ \bar{\mathbf{x}}^{(2)} \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} \boldsymbol{\mu}^{(1)} \\ \boldsymbol{\mu}^{(2)} \end{bmatrix}, \begin{bmatrix} \frac{1}{N_1}\boldsymbol{\Sigma} & \mathbf{0} \\ \mathbf{0} & \frac{1}{N_2}\boldsymbol{\Sigma} \end{bmatrix} \right)$$

and

$$\mathbf{y} \sim \mathcal{N}\left(\boldsymbol{\nu}, \left(\frac{1}{N_1} + \frac{1}{N_2} \right) \boldsymbol{\Sigma} \right) \qquad \text{where} \qquad \boldsymbol{\nu} = \boldsymbol{\mu}^{(1)} - \boldsymbol{\mu}^{(2)}.$$

Outline

Sample Correlation Coefficient

Given the sample $\mathbf{x}_1, \dots, \mathbf{x}_N$ from $\mathcal{N}_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, the maximum likelihood estimator of the correlation between the i-th and the j-th components is

$$r_{ij} = \frac{\sum_{\alpha=1}^{N}(x_{i\alpha} - \bar{x}_i)(x_{j\alpha} - \bar{x}_j)}{\sqrt{\sum_{\alpha=1}^{N}(x_{i\alpha} - \bar{x}_i)^2}\sqrt{\sum_{\alpha=1}^{N}(x_{j\alpha} - \bar{x}_j)^2}},$$

where $x_{i\alpha}$ is the i-th component of $\mathbf{x}_\alpha$ and

$$\bar{x}_i = \frac{1}{N}\sum_{\alpha=1}^{N} x_{i\alpha}.$$

We shall find the distribution of r_{ij}.

Sample Correlation Coefficient

If the population correlation

$$\rho_{ij} = \frac{\sigma_{ij}}{\sqrt{\sigma_{ii}\sigma_{jj}}}$$

is zero, then the density of sample correlation r_{ij} is

$$k_N(r) = \frac{\Gamma\left(\frac{N-1}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{N-2}{2}\right)}(1-r^2)^{\frac{N-4}{2}}.$$