Homework 1

Deadline: March 30, 2022

1. For any $\mathbf{A} \in \mathbb{S}^n$, $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{y} \in \mathbb{R}^n$, prove that $\mathbf{x}^\top \mathbf{A}\mathbf{y} = \mathbf{y}^\top \mathbf{A}\mathbf{x}$.

2. Prove that for any matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$, it turns out that the column rank of $\mathbf{A}$ is equal to the row rank of $\mathbf{A}$.

3. For $\mathbf{A} \in \mathbb{R}^{m\times n}$ and $\mathbf{B} \in \mathbb{R}^{n\times p}$, prove that $\|\mathbf{AB}\|_F \le \|\mathbf{A}\|_F \|\mathbf{B}\|_F$.

4. Suppose $\mathbf{A} \in \mathbb{R}^{n\times n}$ has eigenvalues $\lambda_1, \dots, \lambda_n$ associated with eigenvectors $\mathbf{x}_1, \dots, \mathbf{x}_n$ respectively. Prove the following statements

   (a) $\mathrm{tr}(\mathbf{A}) = \sum_{i=1}^n \lambda_i$

   (b) $\det(\mathbf{A}) = \prod_{i=1}^n \lambda_i$

5. Prove the SVD always exists for any $\mathbf{A} \in \mathbb{R}^{m\times n}$. (Hint: Using spectral decomposition theorem)

6. Given the symmetric matrix

$$\mathbf{N} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^\top & \mathbf{D} \end{bmatrix}$$

   with non-singular $\mathbf{D}$ and let $\mathbf{S} = \mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{B}^\top$. Prove that

   (a) $\mathbf{N} \succ \mathbf{0} \Longleftrightarrow \mathbf{A} \succ \mathbf{0}$ and $\mathbf{S} \succ \mathbf{0}$.

   (b) If $\mathbf{A} \succ \mathbf{0}$, then $\mathbf{N} \succ \mathbf{0} \Longleftrightarrow \mathbf{S} \succ \mathbf{0}$.