Multivariate Statistical Analysis

Lecture 05

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Outline

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Singular Normal Distributions

In previous section, we focus on non-singular normal normally distributed variate $\mathbf{x} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ with $\boldsymbol{\Sigma} \succ \mathbf{0}$ whose density function is

$$n(\mathbf{x} \mid \boldsymbol{\mu}, \boldsymbol{\Sigma}) = \frac{1}{\sqrt{(2\pi)^p \det(\boldsymbol{\Sigma})}} \exp\left(-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^\top \boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\right).$$

What about the case of singular $\boldsymbol{\Sigma}$?

General Linear Transformation

1. Let $\mathbf{x} \sim \mathcal{N}_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, with $\boldsymbol{\Sigma} \succ \mathbf{0}$. Then

$$\mathbf{y} = \mathbf{C}\mathbf{x}$$

   is distributed according to $\mathcal{N}_p(\mathbf{C}\boldsymbol{\mu}, \mathbf{C}\boldsymbol{\Sigma}\mathbf{C}^\top)$ for non-singular $\mathbf{C} \in \mathbb{R}^{p \times p}$.

2. Let $\mathbf{x} \sim \mathcal{N}_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, with $\boldsymbol{\Sigma} \succ \mathbf{0}$. Then

$$\mathbf{y} = \mathbf{C}\mathbf{x}$$

   is distributed according to $\mathcal{N}_q(\mathbf{C}\boldsymbol{\mu}, \mathbf{C}\boldsymbol{\Sigma}\mathbf{C}^\top)$ for $\mathbf{C} \in \mathbb{R}^{q \times p}$ of rank $q \leq p$.

3. Let $\mathbf{x} \sim \mathcal{N}_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$. Then

$$\mathbf{y} = \mathbf{C}\mathbf{x}$$

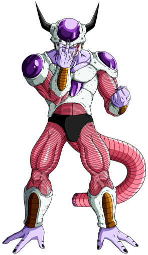
   is distributed according to $\mathcal{N}_q(\mathbf{C}\boldsymbol{\mu}, \mathbf{C}\boldsymbol{\Sigma}\mathbf{C}^\top)$ for any $\mathbf{C} \in \mathbb{R}^{q \times p}$.

Transformation

5.3×10^5

$c \neq 0$

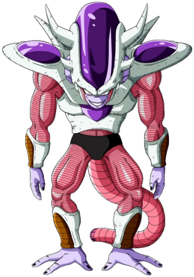
$\sigma^2 > 0$



$> 1.0 \times 10^6$

$\mathbf{C} \in \mathbb{R}^{p \times p}$ is non-singular

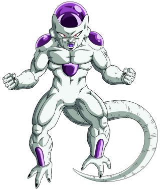
$\mathbf{\Sigma} \succ \mathbf{0}$



$2.0 \times 10^6 \sim 3.0 \times 10^6$

$\mathbf{C} \in \mathbb{R}^{q \times p}$ of rank $q \leq p$

$\mathbf{\Sigma} \succ \mathbf{0}$



$> 3.0 \times 10^7$

$\mathbf{C} \in \mathbb{R}^{q \times p}$

$\mathbf{\Sigma} \succeq \mathbf{0}$

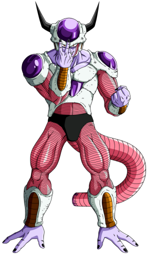
General Linear Transformation

Theorem

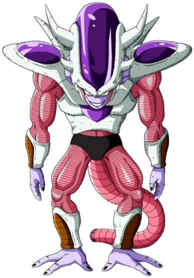
Let $\mathbf{x} \sim \mathcal{N}_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$*, with* $\boldsymbol{\Sigma} \succ \mathbf{0}$*. Then*

$$\mathbf{z} = \mathbf{D}\mathbf{x}$$

is distributed according to $\mathcal{N}_q(\mathbf{D}\boldsymbol{\mu}, \mathbf{D}\boldsymbol{\Sigma}\mathbf{D}^\top)$ *for* $\mathbf{D} \in \mathbb{R}^{q\times p}$ *of rank* $q \leq p$*.*



$\Longrightarrow$



non-singular

full-rank

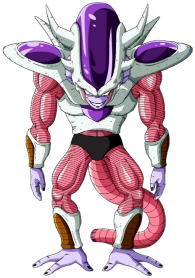
General Linear Transformation

Theorem

Let $\mathbf{x} \sim \mathcal{N}_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$. *Then*

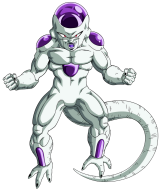
$$\mathbf{z} = \mathbf{D}\mathbf{x}$$

is distributed according to $\mathcal{N}_q(\mathbf{D}\boldsymbol{\mu}, \mathbf{D}\boldsymbol{\Sigma}\mathbf{D}^\top)$ *for any* $\mathbf{D} \in \mathbb{R}^{q \times p}$.



full-rank

$$\xrightarrow[]{\text{understand the singular normal distribution}} \Longrightarrow$$



no limitation

Singular Normal Distribution

Singular normal distribution:

1. The mass is concentrated on a given lower dimensional set.
2. The probability associated with any set that does not intersecting the given low-dimensional set is 0.

For example, consider that

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right).$$

1. Probability of any set that does not intersecting the x_2-axis is 0.
2. The measure of x_2-axis in the space of $\mathbb{R}^2$ is zero.
3. The random vector $\mathbf{x}$ has no density, but its distribution exists.

Singular Normal Distributions

Suppose that $\mathbf{y} \sim \mathcal{N}_q(\boldsymbol{\nu}, \mathbf{T})$, $\mathbf{A} \in \mathbb{R}^{p \times q}$ with $p > q$ and $\boldsymbol{\lambda} \in \mathbb{R}^p$; then we say that

$$\mathbf{x} = \mathbf{A}\mathbf{y} + \boldsymbol{\lambda}$$

has a singular (degenerate) normal distribution in p-space.

We have $\boldsymbol{\mu} = \mathbb{E}[\mathbf{x}] = \mathbf{A}\boldsymbol{\nu} + \boldsymbol{\lambda}$ and

$$\boldsymbol{\mu} = \mathbb{E}[\mathbf{x}] = \mathbf{A}\boldsymbol{\nu} + \boldsymbol{\lambda} \quad \text{and} \quad \boldsymbol{\Sigma} = \mathrm{Cov}(\mathbf{x}) = \mathbf{A}\mathbf{T}\mathbf{A}^\top.$$

The matrix $\boldsymbol{\Sigma}$ is singular and we cannot write density for $\mathbf{x}$.

Singular Normal Distributions

Now we give a formal definition of a normal distribution that includes the singular distribution.

Definition

A p-dimensional random vector $\mathbf{x}$ with $\mathbb{E}[\mathbf{x}] = \boldsymbol{\mu}$ and $\mathrm{Cov}[\mathbf{x}] = \boldsymbol{\Sigma}$ is said to be normally distributed if there is a transformation

$$\mathbf{x} = \mathbf{A}\mathbf{y} + \boldsymbol{\lambda},$$

where $\mathbf{A} \in \mathbb{R}^{p \times r}$, $\boldsymbol{\lambda} \in \mathbb{R}^p$, r is the rank of $\boldsymbol{\Sigma}$ and $\mathbf{y}$ has r-dimensional non-singular normal distribution, e.g., $\mathbf{y} \sim \mathcal{N}_r(\boldsymbol{\nu}, \mathbf{T})$ with $\mathbf{T} \succ \mathbf{0}$.

We also use the notation $\mathcal{N}_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ even if $\boldsymbol{\Sigma}$ is singular.

If $\boldsymbol{\Sigma}$ has rank p, we can take $\mathbf{A} = \mathbf{I}$ and $\boldsymbol{\lambda} = \mathbf{0}$.

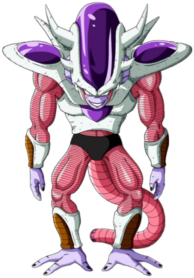
General Linear Transformation

Theorem

Let $\mathbf{x} \sim \mathcal{N}_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$. *Then*

$$\mathbf{z} = \mathbf{D}\mathbf{x}$$

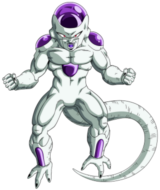
is distributed according to $\mathcal{N}_q(\mathbf{D}\boldsymbol{\mu}, \mathbf{D}\boldsymbol{\Sigma}\mathbf{D}^\top)$ *for any* $\mathbf{D} \in \mathbb{R}^{q \times p}$.



full-rank

only use the definition (without density)

$\Longrightarrow$



no limitation

Examples

Theorem

Let $\mathbf{U}$ *be a* $d \times k$ *random matrix* $(k \leq d)$ *and each of its entry is independent distributed according to* $\mathcal{N}(0,1)$*, then it holds that*

$$\mathbb{E}\left[\mathbf{U}(\mathbf{U}^\top\mathbf{U})^{-1}\mathbf{U}^\top\right] = \frac{k}{d}\mathbf{I}_d.$$

Lemma

Assume $\mathbf{P} \in \mathbb{R}^{d\times k}$ *is column orthonormal* $(k \leq d)$ *and* $\mathbf{v} \sim \mathcal{N}_d(\mathbf{0}, \mathbf{P}\mathbf{P}^\top)$ *is a d-dimensional multivariate normal distributed vector. Then we have*

$$\mathbb{E}\left[\frac{\mathbf{v}\mathbf{v}^\top}{\mathbf{v}^\top\mathbf{v}}\right] = \frac{1}{k}\mathbf{P}\mathbf{P}^\top.$$

Outline

Conditional Distribution

Let $\mathbf{x}$ be distributed according to $\mathcal{N}_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ with $\boldsymbol{\Sigma} \succ \mathbf{0}$.

We partition

$$\mathbf{x} = \begin{bmatrix} \mathbf{x}^{(1)} \\ \mathbf{x}^{(2)} \end{bmatrix} \quad \text{with } \mathbf{x}^{(1)} \in \mathbb{R}^q \text{ and } \mathbf{x}^{(2)} \in \mathbb{R}^{p-q},$$

$$\boldsymbol{\mu} = \begin{bmatrix} \boldsymbol{\mu}^{(1)} \\ \boldsymbol{\mu}^{(2)} \end{bmatrix} \quad \text{with } \boldsymbol{\mu}^{(1)} \in \mathbb{R}^q \text{ and } \boldsymbol{\mu}^{(2)} \in \mathbb{R}^{p-q},$$

and

$$\boldsymbol{\Sigma} = \begin{bmatrix} \boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} \\ \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22} \end{bmatrix}$$

with $\boldsymbol{\Sigma}_{11} \in \mathbb{R}^{q \times q}$, $\boldsymbol{\Sigma}_{12} \in \mathbb{R}^{q \times (p-q)}$, $\boldsymbol{\Sigma}_{21} \in \mathbb{R}^{(p-q) \times q}$ and $\boldsymbol{\Sigma}_{22} \in \mathbb{R}^{(p-q) \times (p-q)}$.

Conditional Distribution

The conditional density of $\mathbf{x}^{(1)}$ given that $\mathbf{x}^{(2)}$ is

$$
\begin{aligned}
& f(\mathbf{x}^{(1)} \mid \mathbf{x}^{(2)}) = \frac{f(\mathbf{x}^{(1)}, \mathbf{x}^{(2)})}{f(\mathbf{x}^{(2)})} \\
=& \frac{1}{\sqrt{(2\pi)^q \det(\mathbf{\Sigma}_{11.2})}} \exp\left(-\frac{1}{2}(\mathbf{x}_{11.2} - \boldsymbol{\mu}_{11.2})^\top \mathbf{\Sigma}_{11.2}^{-1} (\mathbf{x}_{11.2} - \boldsymbol{\mu}_{11.2})\right),
\end{aligned}
$$

where

$$
\begin{aligned}
\mathbf{x}_{11.2} =& \mathbf{x}^{(1)} - \mathbf{\Sigma}_{12}\mathbf{\Sigma}_{22}^{-1}\mathbf{x}^{(2)}, \\
\boldsymbol{\mu}_{11.2} =& \boldsymbol{\mu}^{(1)} - \mathbf{\Sigma}_{12}\mathbf{\Sigma}_{22}^{-1}\boldsymbol{\mu}^{(2)},
\end{aligned}
$$

and

$$
\mathbf{\Sigma}_{11.2} = \mathbf{\Sigma}_{11} - \mathbf{\Sigma}_{12}\mathbf{\Sigma}_{22}^{-1}\mathbf{\Sigma}_{21}.
$$

Outline

Characteristic Function

The characteristic function of a p-dimensional random vector $\mathbf{x}$ is

$$\phi(\mathbf{t}) = \mathbb{E}\left[\exp(\mathrm{i}\,\mathbf{t}^\top \mathbf{x})\right]$$

defined for every real vector $\mathbf{t} \in \mathbb{R}^p$.

For the complex-valued function $g(z)$ be written as

$$g(z) = g_1(z) + \mathrm{i}\, g_2(z),$$

where $g_1(z)$ and $g_2(z)$ are real-valued, the expected value of $g(z)$ is

$$\mathbb{E}[g(z)] = \mathbb{E}[g_1(z)] + \mathrm{i}\,\mathbb{E}[g_2(z)].$$

Characteristic Function

Theorem

If the p-dimensional random vector $\mathbf{x}$ *has the density* $f(\mathbf{x})$ *and the characteristic function* $\phi(\mathbf{t})$, *then*

$$f(\mathbf{x}) = \frac{1}{(2\pi)^p} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \exp(-\mathrm{i}\,\mathbf{t}^\top \mathbf{x})\, \phi(\mathbf{t})\, \mathrm{d}t_1 \ldots \mathrm{d}t_p.$$

- If the random variable have a density, the characteristic function determines the density function uniquely.
- If the random variable does not have a density, the characteristic function uniquely defines the probability of any continuity interval.

Characteristic Function

Theorem

The characteristic function of $\mathbf{x}$ *distributed according to* $\mathcal{N}_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ *is*

$$\phi(\mathbf{t}) = \exp\left(\mathrm{i}\,\mathbf{t}^\top \boldsymbol{\mu} - \frac{1}{2}\mathbf{t}^\top \boldsymbol{\Sigma} \mathbf{t}\right).$$

for every $\mathbf{t} \in \mathbb{R}^p$*.*

Sketch of the proof

1. The characteristic function of $\mathbf{y} \sim \mathcal{N}_p(\mathbf{0}, \mathbf{I})$ is $\phi_0(\mathbf{t}) = \exp\left(-\frac{1}{2}\mathbf{t}^\top \mathbf{t}\right)$.
2. For $\mathbf{x} \sim \mathcal{N}_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, we have $\mathbf{x} = \mathbf{A}\mathbf{y} + \boldsymbol{\mu}$ such that $\boldsymbol{\Sigma} = \mathbf{A}\mathbf{A}^\top$.
3. Using $\phi_0(\mathbf{t})$ to present the characteristic function of $\mathbf{x} \sim \mathcal{N}_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$.

Characteristic Function

Theorem

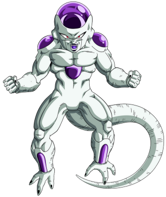
The characteristic function of $\mathbf{x}$ *distributed according to* $\mathcal{N}_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ *is*

$$\phi(\mathbf{t}) = \exp\left(\mathrm{i}\,\mathbf{t}^\top\boldsymbol{\mu} - \frac{1}{2}\mathbf{t}^\top\boldsymbol{\Sigma}\mathbf{t}\right).$$

for every $\mathbf{t} \in \mathbb{R}^p$.

This theorem directly implies $\mathbf{x} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ leads to $\mathbf{C}\mathbf{x} \sim \mathcal{N}(\mathbf{C}\boldsymbol{\mu}, \mathbf{C}\boldsymbol{\Sigma}\mathbf{C}^\top)$.

characteristic function



trick of matrix

Characteristic Function

Theorem

If every linear combination of the components of a random vector $\mathbf{y}$ *is normally distributed, then* $\mathbf{y}$ *is normally distributed.*

In other words, if the p-dimensional random vector $\mathbf{y}$ leads to the univariate random variable

$$\mathbf{u}^\top \mathbf{y}$$

is normally distributed for any fixed $\mathbf{u} \in \mathbb{R}^p$, then $\mathbf{y}$ is normally distributed.

This is another definition of multivariate normal distribution.

Example

Theorem

We let

$$\mathbf{x} \sim \mathcal{N}_p(\boldsymbol{\mu}_1, \boldsymbol{\Sigma}_1), \qquad \mathbf{y} \sim \mathcal{N}_p(\boldsymbol{\mu}_2, \boldsymbol{\Sigma}_2) \qquad \textit{and} \qquad \mathbf{z} = \mathbf{x} + \mathbf{y}.$$

Suppose that $\mathbf{x}$ *and* $\mathbf{y}$ *are independent, then we have*

$$\mathbf{z} \sim \mathcal{N}_p(\boldsymbol{\mu}_1 + \boldsymbol{\mu}_2, \boldsymbol{\Sigma}_1 + \boldsymbol{\Sigma}_2).$$

characteristic function

this result