

# Optimization Theory

## Lecture 14

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- 1 Stochastic Variance Reduced Gradient
- 2 Catalyst Acceleration and Direct Acceleration
- 3 Stochastic Recursive Gradient Algorithm

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# Stochastic Variance Reduced Gradient (SVRG)

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**Algorithm 1** Stochastic Variance Reduced Gradient

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1: Input:  $\mathbf{x}_0, \eta, m, S$ 
2:  $\tilde{\mathbf{x}}^{(0)} = \mathbf{x}_0$ 
3: for  $s = 0, \dots, S - 1$ 
4:    $\tilde{\mu} = \nabla f(\tilde{\mathbf{x}}^{(s)})$ 
5:    $\mathbf{x}_0 = \tilde{\mathbf{x}} = \tilde{\mathbf{x}}^{(s)}$ 
6:   for  $t = 0, \dots, m - 1$ 
7:     draw  $t_i$  from  $\{1, \dots, n\}$  uniformly
8:      $\mathbf{x}_{t+1} = \mathbf{x}_t - \eta(\nabla f_{t_i}(\mathbf{x}_t) - \nabla f_{t_i}(\tilde{\mathbf{x}}) + \tilde{\mu}),$ 
9:   end for
10:  Option I:  $\tilde{\mathbf{x}}^{(s+1)} = \mathbf{x}_m$ 
11:  Option II:  $\tilde{\mathbf{x}}^{(s+1)} = \mathbf{x}_t$  for randomly chosen  $t \in \{0, \dots, m - 1\}$ 
12: end for
13: Output:  $\tilde{\mathbf{x}}^{(S)}$ 
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# Stochastic Variance Reduced Gradient (SVRG)

Assume  $\eta = \Theta(1/L)$  and  $m$  is sufficient large so that

$$\rho = \frac{1}{\mu\eta(1-2L\eta)m} + \frac{2L\eta}{1-2L\eta} < 1,$$

then SVRG holds that

$$\mathbb{E}[f(\tilde{\mathbf{x}}^{(s)}) - f(\mathbf{x}^*)] \leq \rho^s(f(\tilde{\mathbf{x}}_0) - f(\mathbf{x}^*)).$$

The incremental first-order oracle complexity to achieve

$$\mathbb{E}[f(\tilde{\mathbf{x}}^{(s)}) - f(\mathbf{x}^*)] \leq \epsilon$$

is at most  $\mathcal{O}((\kappa + n) \log(1/\epsilon))$ .

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## Algorithm 2 L-SVRG

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- 1: **Input:**  $\eta$ ,  $T$  and  $p$ .
  - 2:  $\mathbf{x}_0 = \mathbf{w}_0$
  - 3: **for**  $t = 0, 1, \dots, T$  **do**
  - 4:    $\mathbf{v}_t = \nabla f_{t_i}(\mathbf{x}_t) - \nabla f_{t_i}(\mathbf{w}_t) + \nabla f(\mathbf{w}_t)$
  - 5:    $\mathbf{x}_{t+1} = \mathbf{x}_t - \eta \mathbf{v}_t$
  - 6:    $\mathbf{w}_{t+1} = \begin{cases} \mathbf{x}_t & \text{with probability } p \\ \mathbf{w}_t & \text{with probability } 1 - p \end{cases}$
  - 7: **end for**
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Comparisons on IFO complexities:

- ① SAG/SVRG/SAGA is better than GD.
- ② SAG/SVRG/SAGA is worse than AGD when  $\kappa \geq \Omega(n^2)$ .
- ③ The optimal dependency on condition number should be  $\sqrt{\kappa}$ .

How to accelerate variance reduced methods?



Consider the inexact proximal point iteration

$$\begin{aligned}\mathbf{x}_{t+1} &\approx \text{prox}_{f/\gamma}(\mathbf{x}_t) \\ &= \arg \min_{\mathbf{x} \in \mathbb{R}^d} \left( f(\mathbf{x}) + \frac{\gamma}{2} \|\mathbf{x} - \mathbf{x}_t\|_2^2 \right).\end{aligned}$$

How design the algorithm?

- 1 Select appropriate value of  $\gamma$ .
- 2 Introduce the step of acceleration.

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## Algorithm 3 Catalyst Acceleration

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- 1: **Input:** initial point  $\mathbf{x}_0 \in \mathbb{R}^d$ , iterations number  $T$ , parameters  $\gamma$  and  $\alpha_0 > 0$ , sequence  $\{\epsilon_t\}$ , sub-problem solver  $\mathcal{A}$ .
  - 2:  $q = \mu/(\mu + \gamma)$ ,  $\mathbf{y}_0 = \mathbf{x}_0$
  - 3: **for**  $t = 0, 1, \dots, T$  **do**
  - 4:   Apply  $\mathcal{A}$  to find
$$\mathbf{x}_{t+1} \approx \arg \min_{\mathbf{x} \in \mathbb{R}^d} \left( G_t(\mathbf{x}) \triangleq f(\mathbf{x}) + \frac{\gamma}{2} \|\mathbf{x} - \mathbf{y}_t\|_2^2 \right)$$
such that  $G_t(\mathbf{x}_{t+1}) - G_t^* \leq \epsilon_t$
  - 5:   Compute  $\alpha_t \in (0, 1)$  from equation  $\alpha_{t+1}^2 = (1 - \alpha_{t+1})\alpha_t^2 + q\alpha_{t+1}$
  - 6:   Compute  $\mathbf{y}_{t+1} = \mathbf{x}_{t+1} + \beta_t(\mathbf{x}_{t+1} - \mathbf{x}_t)$ , where  $\beta_t = \frac{\alpha_t(1 - \alpha_t)}{\alpha_t^2 + \alpha_{t+1}}$
  - 7: **end for**
  - 8: **Output:**  $\mathbf{x}_T$
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## Theorem

Let  $\alpha_0 = \sqrt{q}$  with  $q = \mu/(\mu + \beta)$  and

$$\epsilon_t = \frac{2}{9}(f(\mathbf{x}_0) - f^*)(1 - \rho)^{t+1} \quad \text{with} \quad \rho < \sqrt{q}.$$

Then Algorithm 3 generates  $\{\mathbf{x}_t\}$  such that

$$f(\mathbf{x}_t) - f^* \leq \frac{8(1 - \rho)^t}{(\sqrt{q} - \rho)^2}(f(\mathbf{x}_0) - f^*).$$

A generic framework for acceleration:

- ① Let  $\mathcal{A}$  be GD and  $\beta = \Theta(L)$ , then total FO complexity is

$$\tilde{\mathcal{O}}(\sqrt{\kappa} \log(1/\epsilon)).$$

- ② Let  $\mathcal{A}$  be SVRG and  $\beta = \Theta(L/n)$ , then total IFO complexity is

$$\tilde{\mathcal{O}}(\sqrt{\kappa n} \log(1/\epsilon)), \quad \text{where} \quad \kappa \geq \Omega(n).$$

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## Algorithm 4 Katyusha

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```
1: Input:  $\mathbf{x}_0, \eta, m, S, \tau_1, \tau_2$ 
2:  $\mathbf{y}_0 = \mathbf{z}_0 = \tilde{\mathbf{x}}^{(0)} = \mathbf{x}_0$ 
3: for  $s = 0, \dots, S - 1$ 
4:    $\tilde{\boldsymbol{\mu}}^{(s)} = \nabla f(\tilde{\mathbf{x}}^{(s)})$ 
5:   for  $t = 0, \dots, m - 1$ 
6:      $k = sm + t$ 
7:      $\mathbf{x}_{k+1} = \tau_1 \mathbf{z}_k + \tau_2 \tilde{\mathbf{x}}^{(s)} + (1 - \tau_1 - \tau_2) \mathbf{y}_k$ 
8:     draw  $i_k$  from  $\{1, \dots, n\}$  uniformly
9:      $\mathbf{z}_{k+1} = \mathbf{z}_k - \eta(\nabla f_{i_k}(\mathbf{x}_t) - \nabla f_{i_k}(\tilde{\mathbf{x}}) + \tilde{\boldsymbol{\mu}}^{(s)})$ ,
10:     $\mathbf{y}_{k+1} = \mathbf{x}_{k+1} + \tau_1(\mathbf{z}_{k+1} - \mathbf{z}_k)$ ,
11:   end for
12:    $\tilde{\mathbf{x}}^{(s+1)} = \left( \sum_{j=0}^{m-1} (1 + \eta \mu)^j \right)^{-1} \sum_{j=0}^{m-1} (1 + \eta \mu)^j \mathbf{y}_{sm+j+1}$ 
13: end for
14: Output:  $\tilde{\mathbf{x}}^{(S)}$ 
```

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# Direct Acceleration: Katyusha

Katyusha outputs  $\tilde{\mathbf{x}}^{(S)}$  satisfying  $\mathbb{E}[f(\tilde{\mathbf{x}}^{(S)})] - f^* \leq \epsilon$  within

- ①  $\mathcal{O}((n + \sqrt{\kappa n}) \log(1/\epsilon))$  IFO complexity for strongly convex objective;
- ②  $\mathcal{O}(n \log(1/\epsilon) + \sqrt{nL/\epsilon})$  IFO complexity for convex objective.

The above results achieve the near optimal IFO complexities.

- 1 Stochastic Variance Reduced Gradient
- 2 Catalyst Acceleration and Direct Acceleration
- 3 Stochastic Recursive Gradient Algorithm

# Stochastic Recursive Gradient Algorithm

The poster features a central photograph of a man holding a young child. A red arrow points from the word 'SARAH.' to the child. The poster is framed by logos for COR@L (Computational Optimization Research at Lehigh) and ISE (Industrial and Systems Engineering) at the top. Below the central photo are three smaller portraits of Lam Nguyen, Jie Liu, and Katya Scheinberg, each with their name and affiliation (Lehigh). The title 'SARAH.' is prominently displayed in blue, followed by the subtitle 'A Novel Method for Machine Learning Problems Using Stochastic Recursive Gradient Algorithm' in blue and red. The author's name, Martin Takáč, is listed below. The Lehigh University logo is at the bottom center. The date 'August 8, 2017' is on the bottom left, and the poster location 'Poster: Tue Aug 8th @ Gallery #48' is on the bottom right.

**COR@L**  
COMPUTATIONAL OPTIMIZATION RESEARCH AT LEHIGH

**ISE**  
INDUSTRIAL AND SYSTEMS ENGINEERING

Lam Nguyen  
(Lehigh)

Jie Liu  
(Lehigh)

Katya Scheinberg  
(Lehigh)

**SARAH.**

**A Novel Method for Machine Learning Problems Using  
Stochastic Recursive Gradient Algorithm**

Martin Takáč

**LEHIGH**  
UNIVERSITY.

August 8, 2017

**Poster: Tue Aug 8th @ Gallery #48**

# Stochastic Recursive Gradient Algorithm (SARAH)

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**Algorithm 5** Stochastic Variance Reduced Gradient

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```
1: Input:  $\mathbf{x}_0, \eta, m, S$ 
2:  $\tilde{\mathbf{x}}^{(0)} = \mathbf{x}_0$ 
3: for  $s = 0, \dots, S - 1$ 
4:    $\mathbf{v}_0 = \nabla f(\tilde{\mathbf{x}}^{(s)})$ 
5:    $\mathbf{x}_0 = \tilde{\mathbf{x}} = \tilde{\mathbf{x}}^{(s)}$ 
6:   for  $t = 0, \dots, m - 1$ 
7:     draw  $t_i$  from  $\{1, \dots, n\}$  uniformly
8:      $\mathbf{x}_{t+1} = \mathbf{x}_t - \eta \mathbf{v}_t$ 
9:      $\mathbf{v}_{t+1} = \nabla f_{t_i}(\mathbf{x}_{t+1}) - \nabla f_{t_i}(\mathbf{x}_t) + \mathbf{v}_t$ 
10:  end for
11:   $\tilde{\mathbf{x}}^{(s+1)} = \mathbf{x}_t$  for randomly chosen  $t \in \{0, \dots, m - 1\}$ 
12: end for
13: Output:  $\tilde{\mathbf{x}}^{(S)}$ 
```

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# Stochastic Recursive Gradient Algorithm (SARAH)

SARAH outputs  $\tilde{\mathbf{x}}^{(S)}$  satisfying  $\mathbb{E} \|\nabla f(\tilde{\mathbf{x}}^{(S)})\|_2 \leq \epsilon$  within

- ①  $\mathcal{O}((n + \kappa) \log(1/\epsilon))$  IFO complexity for strongly convex objective;
- ②  $\mathcal{O}((n + L/\epsilon) \log(1/\epsilon))$  IFO complexity for convex objective.

The more interesting result is in the nonconvex optimization

- ① Cong Fang, Chris Junchi Li, Zhouchen Lin, Tong Zhang.  
SPIDER: Near-Optimal Non-Convex Optimization via Stochastic  
Path-Integrated Differential Estimator. *NeurIPS* 2018.

# SGD for Nonconvex Optimization

We consider the stochastic optimization problem

$$\min_{\mathbf{x} \in \mathbb{R}^d} f(\mathbf{x}) \triangleq \mathbb{E}_{\xi}[F(\mathbf{x}; \xi)],$$

where  $f(\mathbf{x})$  is  $L$ -smooth and lower bounded, and each  $F(\mathbf{x}; \xi)$  is differentiable.

We suppose there exists  $\sigma > 0$  such that

$$\mathbb{E} \|\nabla F(\mathbf{x}; \xi) - \nabla f(\mathbf{x})\|_2^2 \leq \sigma^2$$

for any  $\mathbf{x} \in \mathbb{R}^d$ . The SGD iteration

$$\mathbf{x}_{t+1} = \mathbf{x}_t - \eta_t \cdot \frac{1}{b} \sum_{i=1}^b \nabla F(\mathbf{x}_t; \xi_{t_i})$$

with  $\xi_{t_i} \stackrel{\text{i.i.d.}}{\sim} \mathcal{D}$  can find an  $\epsilon$ -stationary point of  $f(\cdot)$  within

$$\mathcal{O}(L\sigma^2\epsilon^{-4})$$

stochastic first-order oracle (SFO) complexity in expectation.

# SARAH/SPIDER for Nonconvex Optimization

We consider the  $L$ -average smooth function, i.e. there exists  $L > 0$  such that

$$\mathbb{E} \|\nabla F(\mathbf{x}; \xi) - \nabla F(\mathbf{y}; \xi)\|_2^2 \leq L^2 \|\mathbf{x} - \mathbf{y}\|_2^2$$

for any  $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$ .

The algorithms with stochastic recursive gradient require

$$\mathcal{O}(\sigma^2 \epsilon^{-2} + L \sigma^2 \epsilon^{-3})$$

SFO complexity to find an  $\epsilon$ -stationary point.

# SARAH/SPIDER for Nonconvex Optimization

We consider the finite-sum problem

$$\min_{\mathbf{x} \in \mathbb{R}^d} f(\mathbf{x}) \triangleq \frac{1}{n} \sum_{i=1}^n f_i(\mathbf{x}).$$

Under the  $L$ -average smooth assumption, the algorithms with stochastic recursive gradient require

$$\mathcal{O}(n + L\sqrt{n}\epsilon^{-2})$$

SFO complexity to find an  $\epsilon$ -stationary point.

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**Algorithm 6** ProbAbilistic Gradient Estimator (PAGE)

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- 1: **Input:**  $\eta$ ,  $T$ ,  $b_0$ ,  $b$  and  $p$ .
  - 2:  $\mathcal{S}_0 = \{\xi_1, \dots, \xi_0\}$ , where  $\xi_i \stackrel{\text{i.i.d.}}{\sim}$  sampled from  $\mathcal{D}$
  - 3:  $\mathbf{v}_0 = \frac{1}{b_0} \sum_{\xi \in \mathcal{S}_0} \nabla F(\mathbf{x}_0; \xi)$
  - 4: **for**  $t = 0, 1, \dots, T$  **do**
  - 5:    $\mathbf{x}_{t+1} = \mathbf{x}_t - \eta \mathbf{v}_t$
  - 6:    $\mathcal{S}_{t+1} = \{\xi_1, \dots, \xi_0\}$ , where  $\xi_i \stackrel{\text{i.i.d.}}{\sim}$  sampled from  $\mathcal{D}$
  - 7:   
$$\mathbf{v}_{t+1} = \begin{cases} \frac{1}{b_0} \sum_{\xi \in \mathcal{S}_{t+1}} \nabla F(\mathbf{x}_{t+1}; \xi) & \text{with probability } 1 - p \\ \mathbf{v}_t + \frac{1}{b} \sum_{\xi \in \mathcal{S}_{t+1}} (\nabla F(\mathbf{x}_{t+1}; \xi) - \nabla F(\mathbf{x}_t; \xi)) & \text{with probability } p \end{cases}$$
  - 8: **end for**
  - 9:  $\mathbf{x}_{\text{out}} = \mathbf{x}_t$  for randomly chosen  $t \in \{0, \dots, T-1\}$
-