

# Multivariate Statistics

## Lecture 05

Fudan University

- 1 Properties of the Maximum Likelihood Estimators
- 2 Sufficiency
- 3 Completeness

1 Properties of the Maximum Likelihood Estimators

2 Sufficiency

3 Completeness

# The Maximum Likelihood Estimators

## Theorem 1

If  $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_N$  constitute a sample from  $\mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$  with  $p < N$ , the maximum likelihood estimators of  $\boldsymbol{\mu}$  and  $\boldsymbol{\Sigma}$  are

$$\hat{\boldsymbol{\mu}} = \bar{\mathbf{x}} = \frac{1}{N} \sum_{\alpha=1}^N \mathbf{x}_{\alpha} \quad \text{and} \quad \hat{\boldsymbol{\Sigma}} = \frac{1}{N} \sum_{\alpha=1}^N (\mathbf{x}_{\alpha} - \bar{\mathbf{x}})(\mathbf{x}_{\alpha} - \bar{\mathbf{x}})^{\top}$$

respectively.

## Lemma 1

If  $\mathbf{D} \in \mathbb{R}^{p \times p}$  is positive definite, the maximum of

$$f(\mathbf{G}) = -N \ln \det(\mathbf{G}) - \text{tr}(\mathbf{G}^{-1} \mathbf{D})$$

with respect to positive definite matrices  $\mathbf{G}$  exists, occurs at  $\mathbf{G} = \frac{1}{N} \mathbf{D}$ .

# The Maximum Likelihood Estimators

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respectively.

Can we guarantee  $\hat{\boldsymbol{\Sigma}} = \frac{1}{N} \sum_{\alpha=1}^N (\mathbf{x}_{\alpha} - \bar{\mathbf{x}})(\mathbf{x}_{\alpha} - \bar{\mathbf{x}})^{\top}$  is positive definite?

# Distribution Theory

In the univariate case, the mean of a sample is distributed normally and independently of the sample variance.

In the multivariate case, the sample mean  $\hat{\boldsymbol{\mu}}$  is also distributed normally and independently of  $\hat{\boldsymbol{\Sigma}}$ .

## Lemma 1

Suppose  $\mathbf{x}_1, \dots, \mathbf{x}_N$  are independent, where  $\mathbf{x}_\alpha \sim \mathcal{N}_p(\boldsymbol{\mu}_\alpha, \boldsymbol{\Sigma})$ . Let  $\mathbf{C} \in \mathbb{R}^{N \times N}$  be an orthogonal matrix, then

$$\mathbf{y}_\alpha = \sum_{\beta=1}^N c_{\alpha\beta} \mathbf{x}_\beta \sim \mathcal{N}_p(\boldsymbol{\nu}_\alpha, \boldsymbol{\Sigma}),$$

where  $\boldsymbol{\nu}_\alpha = \sum_{\beta=1}^N c_{\alpha\beta} \boldsymbol{\mu}_\beta$  for  $\alpha = 1, \dots, N$  and  $\mathbf{y}_1, \dots, \mathbf{y}_N$  are independent.

## Lemma 2

If  $\mathbf{C} = \begin{bmatrix} c_{11} & c_{12} & \dots & c_{1p} \\ c_{21} & c_{22} & \dots & c_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ c_{p1} & c_{p2} & \dots & c_{pp} \end{bmatrix} = \begin{bmatrix} c_1^\top \\ c_2^\top \\ \vdots \\ c_p^\top \end{bmatrix} \in \mathbb{R}^{p \times p}$  is orthogonal, then

$\sum_{\alpha=1}^N \mathbf{x}_\alpha \mathbf{x}_\alpha^\top = \sum_{\beta=1}^N \mathbf{y}_\beta \mathbf{y}_\beta^\top$  where  $\mathbf{y}_\alpha = \sum_{\beta=1}^p c_{\alpha\beta} \mathbf{x}_\beta$  for  $\alpha = 1, \dots, N$ .

Let  $\mathbf{X} = \begin{bmatrix} \mathbf{x}_1^\top \\ \mathbf{x}_2^\top \\ \vdots \\ \mathbf{x}_p^\top \end{bmatrix}$  and  $\mathbf{Y} = \begin{bmatrix} \mathbf{y}_1^\top \\ \mathbf{y}_2^\top \\ \vdots \\ \mathbf{y}_p^\top \end{bmatrix}$ , then  $\mathbf{y}_\alpha = \mathbf{X}^\top \mathbf{c}_\alpha$  and  $\mathbf{Y} = \mathbf{C}\mathbf{X}$ .

## Theorem 2

Let  $\mathbf{x}_1, \dots, \mathbf{x}_N$  be independent, each distributed according to  $\mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ . Then the mean of the sample

$$\hat{\boldsymbol{\mu}} = \bar{\mathbf{x}} = \frac{1}{N} \sum_{\alpha=1}^N \mathbf{x}_{\alpha}$$

is distributed according to  $\mathcal{N}(\boldsymbol{\mu}, \frac{1}{N} \boldsymbol{\Sigma})$  and independent of

$$\hat{\boldsymbol{\Sigma}} = \frac{1}{N} \sum_{\alpha=1}^N (\mathbf{x}_{\alpha} - \bar{\mathbf{x}})(\mathbf{x}_{\alpha} - \bar{\mathbf{x}})^{\top}.$$

Additionally, we have  $N\hat{\boldsymbol{\Sigma}} = \sum_{\alpha=1}^{N-1} \mathbf{z}_{\alpha} \mathbf{z}_{\alpha}^{\top}$ , where  $\mathbf{z}_{\alpha} \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\Sigma})$  for  $\alpha = 1, \dots, N-1$ , and  $\mathbf{z}_1, \dots, \mathbf{z}_{N-1}$  are independent.



## Theorem 1

If  $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_N$  constitute a sample from  $\mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$  with  $p < N$ , the maximum likelihood estimators of  $\boldsymbol{\mu}$  and  $\boldsymbol{\Sigma}$  are

$$\hat{\boldsymbol{\mu}} = \bar{\mathbf{x}} = \frac{1}{N} \sum_{\alpha=1}^N \mathbf{x}_{\alpha} \quad \text{and} \quad \hat{\boldsymbol{\Sigma}} = \frac{1}{N} \sum_{\alpha=1}^N (\mathbf{x}_{\alpha} - \bar{\mathbf{x}})(\mathbf{x}_{\alpha} - \bar{\mathbf{x}})^{\top}$$

respectively.

## Theorem 3

Using the notation of Theorem 1, if  $N > p$ , the probability is 1 of drawing a sample so that

$$\hat{\boldsymbol{\Sigma}} = \frac{1}{N} \sum_{\alpha=1}^N (\mathbf{x}_{\alpha} - \bar{\mathbf{x}})(\mathbf{x}_{\alpha} - \bar{\mathbf{x}})^{\top}$$

is positive definite.

An estimator  $\mathbf{t}$  of a parameter vector  $\boldsymbol{\theta}$  is unbiased if and only if

$$\mathbb{E}[\mathbf{t}] = \boldsymbol{\theta}.$$

For the estimators obtain from MLE for normal distribution, the vector  $\hat{\boldsymbol{\mu}}$  is an unbiased estimator of  $\boldsymbol{\mu}$  and  $\hat{\boldsymbol{\Sigma}}$  is a biased estimator of  $\boldsymbol{\Sigma}$ .

# Distribution Theory

Consider the result of MLE for normal distribution:

① We have

$$\mathbb{E}[\hat{\boldsymbol{\mu}}] = \mathbb{E}[\bar{\mathbf{x}}] = \mathbb{E}\left[\sum_{\alpha=1}^N \mathbf{x}_{\alpha}\right] = \boldsymbol{\mu}$$

and (not limited to normal distribution)

$$\mathbb{E}[\hat{\boldsymbol{\Sigma}}] = \mathbb{E}\left[\frac{1}{N} \sum_{\alpha=1}^{N-1} \mathbf{z}_{\alpha} \mathbf{z}_{\alpha}^{\top}\right] = \frac{N-1}{N} \boldsymbol{\Sigma}.$$

② The sample covariance

$$\mathbf{S} = \frac{1}{N-1} \sum_{\alpha=1}^N (\mathbf{x}_{\alpha} - \bar{\mathbf{x}})(\mathbf{x}_{\alpha} - \bar{\mathbf{x}})^{\top}$$

is an unbiased estimator of  $\boldsymbol{\Sigma}$ .

# Outline

- 1 Properties of the Maximum Likelihood Estimators
- 2 Sufficiency
- 3 Completeness

# Properties of Statistics

Let

$$\bar{\mathbf{x}} = \frac{1}{N} \sum_{\alpha=1}^N \mathbf{x}_{\alpha} \quad \text{and} \quad \mathbf{S} = \frac{1}{N-1} \sum_{\alpha=1}^N (\mathbf{x}_{\alpha} - \bar{\mathbf{x}})(\mathbf{x}_{\alpha} - \bar{\mathbf{x}})^{\top}.$$

We shall show that  $\bar{\mathbf{x}}$  and  $\mathbf{S}$  are sufficient statistics and are complete.

# Sufficiency

A statistic  $\mathbf{t}(\mathbf{y})$  is sufficient for a family of distributions of random variable  $\mathbf{y}$  with parameter  $\theta$ , if the conditional distribution of  $\mathbf{y}$  given  $\mathbf{t}(\mathbf{y}) = \mathbf{t}_0$  does not depend on  $\theta$ .

The statistic  $\mathbf{t}$  gives as much information about  $\theta$  as the entire sample  $\mathbf{y}$ .

## Theorem 4

A statistic  $\mathbf{t}(\mathbf{y})$  is sufficient for  $\theta$  if and only if the density  $f(\mathbf{y} \mid \theta)$  can be factored as

$$f(\mathbf{y} \mid \theta) = g(\mathbf{t}(\mathbf{y}), \theta)h(\mathbf{y})$$

where  $g(\mathbf{t}(\mathbf{y}), \theta)$  and  $h(\mathbf{y})$  are nonnegative and  $h(\mathbf{y})$  does not depend on  $\theta$ .

For the MLE of normal distribution, we apply this theorem with

$$\theta = \{\mu, \Sigma\}, \quad \mathbf{y} = \{\mathbf{x}_1, \dots, \mathbf{x}_N\} \quad \text{and} \quad \mathbf{t}(\mathbf{y}) = \{\bar{\mathbf{x}}, \mathbf{S}\}.$$

## Theorem 5

If  $\mathbf{x}_1, \dots, \mathbf{x}_N$  are observations from  $\mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ , then  $\bar{\mathbf{x}}$  and  $\mathbf{S}$  are sufficient for  $\boldsymbol{\mu}$  and  $\boldsymbol{\Sigma}$ .

If  $\boldsymbol{\Sigma}$  is given,  $\bar{\mathbf{x}}$  is sufficient for  $\boldsymbol{\mu}$ . However, if  $\boldsymbol{\mu}$  is given,  $\mathbf{S}$  is not sufficient for  $\boldsymbol{\Sigma}$ ;

# Outline

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A family of distributions of  $\mathbf{y}$  indexed by  $\theta$  is complete if for every real-valued function  $g(\mathbf{y})$ , we have

$$\mathbb{E}[g(\mathbf{y})] = 0$$

identically in  $\theta$  implies  $g(\mathbf{y}) = 0$  except for a set of  $\mathbf{y}$  of probability 0 for every  $\theta$ .

If the family of distributions of a sufficient set of statistics is complete, the set is called a complete sufficient set.

## Theorem 6

The sufficient set of statistics  $\bar{\mathbf{x}}$ ,  $\mathbf{S}$  is complete for  $\boldsymbol{\mu}$ ,  $\boldsymbol{\Sigma}$  when the sample is drawn from  $\mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ .

Sketch of the proof:

- ① We have  $N\hat{\boldsymbol{\Sigma}} = \sum_{\alpha=1}^{N-1} \mathbf{z}_{\alpha} \mathbf{z}_{\alpha}^{\top}$ , where  $\mathbf{z}_{\alpha} = \sum_{\beta=1}^N b_{\alpha\beta} \mathbf{x}_{\beta}$  and

$$\mathbf{B} = \begin{bmatrix} \times & \cdots & \times \\ \vdots & \ddots & \vdots \\ \frac{1}{\sqrt{N}} & \cdots & \frac{1}{\sqrt{N}} \end{bmatrix}$$

- ② The condition  $\mathbb{E}[g(\bar{\mathbf{x}}, n\mathbf{S})] \equiv 0$  implies the Laplace transform of  $g(\bar{\mathbf{x}}, \mathbf{B} - N\bar{\mathbf{x}}\bar{\mathbf{x}}^{\top}) h(\bar{\mathbf{x}}, \mathbf{B})$  is zero, where  $\mathbf{B} = \sum_{\alpha=1}^{N-1} \mathbf{z}_{\alpha} \mathbf{z}_{\alpha}^{\top} + N\bar{\mathbf{x}}\bar{\mathbf{x}}^{\top}$  and  $h(\bar{\mathbf{x}}, \mathbf{B})$  is the joint density of  $\bar{\mathbf{x}}$  and  $\mathbf{B}$ .