Multivariate Statistics

Lecture 02

Fudan University

Outline

1 Joint Distributions

2 Marginal Distributions

3 Transformation of Variables

4 Random Matrix

5 Multivariate Normal Distribution

Outline

1. Joint Distributions
2. Marginal Distributions
3. Transformation of Variables
4. Random Matrix
5. Multivariate Normal Distribution

Joint Distributions (Two Variables)

1. Consider two (real) random variables X and Y. Probabilities of events defined in terms of these variables can be obtained by operations involving the cumulative distribution function (cdf),

$$F(x, y) = \Pr\{X \leq x, Y \leq y\}.$$

   defined for every pair of real numbers (x, y).

2. We are interested in cases where $F(x, y)$ is absolutely continuous; this means the following partial derivative exists almost everywhere:

$$\frac{\partial^2 F(x, y)}{\partial x \partial y} = f(x, y)$$

   and we have

$$F(x, y) = \int_{-\infty}^{y} \int_{-\infty}^{x} f(u, v) \mathrm{d}u \mathrm{d}v$$

3. The nonnegative function $f(x, y)$ is called the probability density function (pdf).

Joint Distributions (Two Variables)

The pair of random variables (X, Y) defines a random point in a plane. The probability that (X, Y) falls in a rectangle is

$$\begin{aligned}
&\Pr\{x \le X \le x + \Delta x,\ y \le Y \le y + \Delta y\} \\
=&F(x + \Delta x, y + \Delta y) - F(x + \Delta x, y) - F(x, y + \Delta y) + F(x, y) \\
=&\int_y^{y+\Delta x} \int_x^{x+\Delta y} f(u, v)\mathrm{d}u\mathrm{d}v,
\end{aligned}$$

where $\Delta x > 0$ and $\Delta y > 0$.

The probability of the random point (X, Y) falling in any set $\mathcal{E}$ for which the following integral is defined (that is, any measurable set $\mathcal{E}$) is

$$\Pr\{(X, Y) \in \mathcal{E}\} = \iint_{\mathcal{E}} f(x, y)\mathrm{d}u\mathrm{d}v.$$

Joint Distributions (Two Variables)

If $f(x,y)$ is continuous in both two variables, the probability element $f(x,y)\Delta x \Delta y$ is approximately the probability that X falls between x and $x+\Delta x$ and Y falls between y and $y+\Delta y$ for small Δx and Δy since

$$\begin{aligned}&\Pr\{x \le X \le x+\Delta x,\ y \le Y \le y+\Delta y\}\\ =&\int_y^{y+\Delta x}\int_x^{x+\Delta y} f(u,v)\mathrm{d}u\mathrm{d}v\\ =&f(x_0,y_0)\Delta x\Delta y\end{aligned}$$

for some x_0, y_0 such that $x \le x_0 \le x+\Delta x$, $y \le y_0 \le y+\Delta y$ by the mean value theorem. The continuity of f means $f(x_0,y_0)\Delta x\Delta y$ is approximately $f(x,y)\Delta x\Delta y$.

Joint Distributions (p Variables)

The cumulative distribution function of p random variables $X_1, \ldots X_p$ is

$$F(x_1, \ldots, x_p) = \Pr\{X_1 \le x_1, \ldots, X_p \le x_p\}.$$

If $F(x_1, \ldots, x_p)$ is absolutely continuous, its density function is

$$\frac{\partial^p F(x_1, \ldots, x_p)}{\partial x_1 \ldots \partial x_p} = f(x_1, \ldots, x_p)$$

(almost everywhere), and

$$F(x_1, \ldots, x_p) = \int_{-\infty}^{x_p} \cdots \int_{-\infty}^{x_1} f(u_1, \ldots, u_p) \mathrm{d}u_1 \ldots \mathrm{d}u_p.$$

Joint Distributions (p Variables)

The probability of falling in any (measurable) set $\mathcal{R}$ in the p-dimensional Euclidean space is

$$\Pr\{(X_1, \ldots, X_p) \in \mathcal{R}\} = \int \cdots \int_{\mathcal{R}} f(x_1, \ldots, x_p) \mathrm{d}x_1 \ldots \mathrm{d}x_p.$$

The probability element

$$f(x_1, \ldots, x_p) \Delta x_1 \ldots \Delta x_p$$

is approximately the probability

$$\Pr\{x_1 \le X_1 \le x_1 + \Delta_1, \ldots, x_p \le X_p \le x_p + \Delta_p\}$$

if $f(x_1, \ldots, x_p)$ is continuous.

Joint Moments

The joint moments of the joint distribution of random variables $X_1, \ldots, X_p$ are defined as integers

$$\mathbb{E}\left[X_1^{h_1} \cdots X_p^{h_p}\right] = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} x_1^{h_1} \cdots x_p^{h_p} f(x_1, \ldots, x_p) \mathrm{d}x_1 \ldots \mathrm{d}x_p.$$

where $k_i \geq 0$ for all $i = 1, \ldots, p$.

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Marginal Distributions (two variables)

Given the cdf of two random variables X, Y as being $F(x, y)$, the marginal cdf of X is

$$F(x) = \Pr\{X \le x\} = \Pr\{X \le x, Y \le \infty\} = F(x, \infty).$$

Clearly, we have

$$F(x) = \int_{-\infty}^{x} \left(\int_{-\infty}^{\infty} f(u, v) \mathrm{d}v \right) \mathrm{d}u.$$

We call

$$f(u) = \int_{-\infty}^{\infty} f(u, v) \mathrm{d}v,$$

say, the marginal density of X. Then

$$F(x) = \int_{-\infty}^{x} f(u) \mathrm{d}u.$$

Marginal Distributions (two variables)

In a similar fashion we define $G(y)$ as the marginal cdf of Y and $g(y)$ as marginal density of Y, that is

$$G(y) = \int_{-\infty}^{y} \left(\int_{-\infty}^{\infty} f(u, v) \mathrm{d}u \right) \mathrm{d}v.$$

and

$$g(v) = \int_{-\infty}^{\infty} f(u, v) \mathrm{d}u.$$

Marginal Distributions (p variables)

Given $F(x_1, \ldots, x_p)$ as the cdf of $X_1, \ldots, X_p$, the marginal cdf of some of $X_1, \ldots, X_p$ say, of $X_1, \ldots, X_r$ $(r < p)$, is

$$\begin{aligned} F(X_1, \ldots, X_r) =& \Pr\{X_1 \leq x_1, \ldots, X_r \leq x_r\} \\ =& \Pr\{X_1 \leq x_1, \ldots, X_r \leq x_r, X_{r+1} \leq \infty, \ldots, X_p \leq \infty\} \\ =& F(x_1, \ldots, x_r, \infty, \ldots, \infty). \end{aligned}$$

The marginal density of $X_1, \ldots, X_r$ is

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(x_1, \ldots, x_r, u_{r+1} \ldots, u_p) \mathrm{d}u_{r+1} \ldots \mathrm{d}u_p.$$

The marginal distribution and density of any other subset of $X_1, \ldots, X_p$ are obtained in the obviously similar fashion.

Joint Moments

The joint moments of a subset of variables can be computed from the marginal distribution; for example,

$$
\begin{aligned}
&\mathbb{E}\left[X_1^{h_1}\cdots X_r^{h_r}\right] \\
=&\mathbb{E}\left[X_1^{h_1}\cdots X_r^{h_r}X_{r+1}^0\ldots X_p^0\right] \\
=&\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} x_1^{h_1}\cdots x_r^{h_r} f(x_1,\ldots,x_p)\mathrm{d}x_1\ldots\mathrm{d}x_p \\
=&\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} x_1^{h_1}\cdots x_r^{h_r}\left[\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} f(x_1\ldots,x_p)\mathrm{d}x_{r+1}\ldots\mathrm{d}x_p\right]\mathrm{d}x_1\ldots\mathrm{d}x_r \\
=&\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} x_1^{h_1}\cdots x_r^{h_r} f(x_1,\ldots,x_r)\mathrm{d}x_1\ldots\mathrm{d}x_r.
\end{aligned}
$$

Statistical Independence

Two random variables X, Y with cdf $F(x, y)$ are said to be independent if

$$F(x, y) = F(x)G(y),$$

where $F(x)$ is the marginal cdf of X and $G(y)$ is the marginal cdf of Y.

This implies the density of X, Y can be written as

$$f(x, y) = f(x)g(y),$$

where $f(x)$ and $g(y)$ are the marginal densities of X and Y respectively.

Conversely, if $f(x, y) = f(x)g(y)$, then $F(x, y) = F(x)G(y)$.

Statistical Independence

The statistical independence of X and Y implies

$$\begin{aligned}
&\Pr\{x_1 \le X \le x_2, y_1 \le Y \le y_2\} \\
=&\int_{y_1}^{y_2} \int_{x_1}^{x_2} f(u,v)\mathrm{d}u\mathrm{d}v \\
=&\int_{y_1}^{y_2} f(u)\mathrm{d}u \int_{x_1}^{x_2} g(v)\mathrm{d}v \\
=&\Pr\{x_1 \le X \le x_2\}\Pr\{y_1 \le Y \le y_2\}.
\end{aligned}$$

Note that we say X and Y are uncorrelated if

$$\begin{aligned}
&\mathrm{Cov}(X,Y) \triangleq \mathbb{E}[(X-\mathbb{E}[X])(Y-\mathbb{E}[Y])] = 0 \\
\iff\ &\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y].
\end{aligned}$$

Independent $\neq$ Uncorrelated

Note that

X are Y are independent implies X are Y uncorrelated.

However,

X are Y are uncorrelated do NOT implies X are Y are independent.

Mutually Independence

If the cdf of $X_1, \ldots, X_p$ is $F(x_1, \ldots, x_p)$, the set of random variables is said to be mutually independent if

$$F(x_1, \ldots, x_p) = F_1(x_1) \ldots F(x_p),$$

where $F_i(x_i)$ is the marginal cdf of X_i, $i = 1, \ldots, p$.

The set $X_1, \ldots, X_r$ is said to be independent of the set $X_{r+1}, \ldots, X_p$ if

$$F(x_1, \ldots, X_p) = F(x_1, \ldots, x_r, \infty, \ldots, \infty) F(\infty, \ldots, \infty, x_{r+1}, \ldots, x_p).$$

Conditional Distributions

If A and B are two events such that the probability of A and B occurring simultaneously is $P(AB)$ and the probability of B occurring is $P(B) > 0$, then the conditional probability of A occurring given that B has occurred is

$$\frac{P(AB)}{P(B)}.$$

Conditional Distributions

Suppose the event A is X falling in the $[x_1, x_2]$ and the event B is Y falling in $[y_1, y_2]$. Then the conditional probability that X falls in $[x_1, x_2]$, given that Y falls in $[y_1, y_2]$, is

$$\begin{aligned}
&\Pr\{x_1 \le X \le x_2 \mid y_1 \le Y \le y_2\} \\
=&\frac{\Pr\{x_1 \le X \le x_2, y_1 \le Y \le y_2\}}{\Pr\{y_1 \le Y \le y_2\}} \\
=&\frac{\int_{x_1}^{x_2} \int_{y_1}^{y_2} f(u, v)\mathrm{d}v\mathrm{d}u}{\int_{y_1}^{y_2} g(v)\mathrm{d}v}.
\end{aligned}$$

Conditional Distributions

For y such that $g(y) > 0$, we define $\Pr\{x_1 \le X \le x_2 \mid Y = y\}$ as the probability that X lies between x_1 and x_2 given that Y is y. Then

$$\Pr\{x_1 \le X \le x_2 \mid Y = y\} = \int_{x_1}^{x_2} f(u \mid y)\mathrm{d}u,$$

where $f(u \mid y) = \dfrac{f(u, y)}{g(y)}$.

For given y, $f(\cdot \mid y)$ is a density function and is called the conditional density of X given y.

If X and Y are independent, we have $f(x \mid y) = f(x)$.

In the general case of $X_1, \ldots, X_p$ with cdf $F(X_1, \ldots, X_p)$, the conditional density of $X_1, \ldots, X_r$, given $X_{r+1} = x_{r+1}, \ldots, X_p = x_p$ is

$$\frac{f(x_1, \ldots, x_p)}{\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(u_1, \ldots, u_r, x_{r+1}, \ldots, x_p)} \mathrm{d}u_1 \cdots \mathrm{d}u_r.$$

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Transformation of Variables

Let the density of p dimensional random vector $\mathbf{x} = [x_1, \dots, x_p]^\top$ be $f(\mathbf{x})$.

Consider the random vector p dimensional random vector $\mathbf{y} = [y_1, \dots, y_p]^\top$ such that $y_i = u_i(\mathbf{x})$ for $i = 1, \dots, p$. Let the density function of $\mathbf{y}$ be $g(\mathbf{y})$.

Assume the transformation $\mathbf{u} = [u_1, \dots, u_p]^\top : \mathbb{R}^p \to \mathbb{R}^p$ from the space of $\mathbf{x}$ to the space of $\mathbf{y}$ is smooth and one-to-one.

Then we have $f(\mathbf{x}) = g(\mathbf{u}(\mathbf{x}))|\det(\mathbf{J}(\mathbf{x}))|$ where

$$\mathbf{J}(\mathbf{x}) = \begin{bmatrix} \frac{u_1(\mathbf{x})}{x_1} & \frac{u_1(\mathbf{x})}{x_2} & \dots & \frac{u_1(\mathbf{x})}{x_p} \\ \frac{u_2(\mathbf{x})}{x_1} & \frac{u_2(\mathbf{x})}{x_2} & \dots & \frac{u_2(\mathbf{x})}{x_p} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{u_p(\mathbf{x})}{x_1} & \frac{u_p(\mathbf{x})}{x_2} & \dots & \frac{u_p(\mathbf{x})}{x_p} \end{bmatrix}.$$

Transformation of Variables

Similarly, we also have $g(\mathbf{y}) = f(\mathbf{u}^{-1}(\mathbf{y}))|\det(\mathbf{J}^{-1}(\mathbf{y}))|$ where

$$\mathbf{J}^{-1}(\mathbf{y}) = \begin{bmatrix} \frac{u_1^{-1}(\mathbf{y})}{y_1} & \frac{u_1^{-1}(\mathbf{y})}{y_2} & \cdots & \frac{u_1^{-1}(\mathbf{y})}{y_p} \\ \frac{u_2^{-1}(\mathbf{y})}{y_1} & \frac{u_2^{-1}(\mathbf{y})}{y_2} & \cdots & \frac{u_2^{-1}(\mathbf{y})}{y_p} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{u_p^{-1}(\mathbf{y})}{y_1} & \frac{u_p^{-1}(\mathbf{y})}{y_2} & \cdots & \frac{u_p^{-1}(\mathbf{y})}{y_p} \end{bmatrix}.$$

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Random Matrix

A random matrix

$$\mathbf{Z} = \begin{bmatrix} z_{11} & z_{12} & \cdots & z_{1n} \\ z_{21} & z_{22} & \cdots & z_{2n} \\ \vdots & \ddots & \ldots & \vdots \\ z_{m1} & z_{m2} & \cdots & z_{mn} \end{bmatrix} \in \mathbb{R}^{m \times n}$$

is a matrix of random variables $z_{11}, \ldots, z_{mn}$.

Random Matrix

If the random variables $z_{11}, \ldots, z_{mn}$ can take on only a finite number of values, the random matrix $\mathbf{Z}$ can be one of a finite number of matrices, say $\mathbf{Z}(l), \ldots, \mathbf{Z}(q)$.

We define

$$\mathbb{E}[\mathbf{Z}] = \sum_{i=1}^{q} \mathbf{Z}(i) p_i = \begin{bmatrix} \mathbb{E}[z_{11}] & \mathbb{E}[z_{12}] & \ldots & \mathbb{E}[z_{1n}] \\ \mathbb{E}[z_{21}] & \mathbb{E}[z_{22}] & \ldots & \mathbb{E}[z_{2n}] \\ \vdots & \ddots & \ldots & \vdots \\ \mathbb{E}[z_{m1}] & \mathbb{E}[z_{m2}] & \ldots & \mathbb{E}[z_{mn}]. \end{bmatrix} \in \mathbb{R}^{m \times n}$$

Random Vector and Mean Vector

For random vector

$$\mathbf{x} == \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{bmatrix} \in \mathbb{R}^p,$$

the expected value

$$\mathbb{E}[\mathbf{x}] = \begin{bmatrix} \mathbb{E}[x_1] \\ \mathbb{E}[x_2] \\ \vdots \\ \mathbb{E}[x_p] \end{bmatrix} \in \mathbb{R}^p,$$

is the mean or mean vector of $\mathbf{x}$.

We shall usually denote the mean vector $\mathbb{E}[\mathbf{x}]$ by $\boldsymbol{\mu}$.

Random Vector and Covariance Matrix

For random vector $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{bmatrix}$ and its mean vector $\boldsymbol{\mu} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_p \end{bmatrix}$, the expected value of the random matrix $(\mathbf{x} - \boldsymbol{\mu})(\mathbf{x} - \boldsymbol{\mu})^\top$ is

$$\mathrm{Cov}(\mathbf{x}) = \mathbb{E}\left[(\mathbf{x} - \boldsymbol{\mu})(\mathbf{x} - \boldsymbol{\mu})^\top\right],$$

the covariance or covariance matrix of $\mathbf{x}$.

1. The i-th diagonal element of this matrix, $\mathbb{E}\left[(x_i - \mu_i)^2\right]$, is the variance of x_i.
2. The i, j-th off-diagonal element $(i \neq j)$, $\mathbb{E}[(x_i - \mu_i)(x_j - \mu_j)]$ is the covariance of x_i and x_j.

Note that

$$
\begin{aligned}
\mathrm{Cov}(\mathbf{x}) =& \mathbb{E}\left[(\mathbf{x}-\boldsymbol{\mu})(\mathbf{x}-\boldsymbol{\mu})^\top\right] \\
=& \mathbb{E}\left[\mathbf{x}\mathbf{x}^\top - \boldsymbol{\mu}\mathbf{x}^\top - \mathbf{x}\boldsymbol{\mu}^\top + \boldsymbol{\mu}\boldsymbol{\mu}^\top\right] \\
=& \mathbb{E}\left[\mathbf{x}\mathbf{x}^\top\right] - \mathbb{E}\left[\boldsymbol{\mu}\mathbf{x}^\top\right] - \mathbb{E}\left[\mathbf{x}\boldsymbol{\mu}^\top\right] + \mathbb{E}[\boldsymbol{\mu}\boldsymbol{\mu}^\top] \\
=& \mathbb{E}\left[\mathbf{x}\mathbf{x}^\top\right] - \boldsymbol{\mu}\mathbb{E}\left[\mathbf{x}^\top\right] - \mathbb{E}\left[\mathbf{x}\right]\boldsymbol{\mu}^\top + \boldsymbol{\mu}\boldsymbol{\mu}^\top \\
=& \mathbb{E}\left[\mathbf{x}\mathbf{x}^\top\right] - \boldsymbol{\mu}\boldsymbol{\mu}^\top - \boldsymbol{\mu}\boldsymbol{\mu}^\top + \boldsymbol{\mu}\boldsymbol{\mu}^\top \\
=& \mathbb{E}\left[\mathbf{x}\mathbf{x}^\top\right] - \boldsymbol{\mu}\boldsymbol{\mu}^\top,
\end{aligned}
$$

where we have used the following lemma:

Lemma

If $\mathbf{Z}$ is an $m \times n$ random matrix, $\mathbf{D}$ is a fixed $l \times m$ real matrix, $\mathbf{E}$ is a fixed $n \times q$ real matrix, and $\mathbf{F}$ is a fixed $l \times q$ real matrix, then

$$\mathbb{E}[\mathbf{DZE}+\mathbf{F}] = \mathbf{D}\mathbb{E}[\mathbf{Z}]\mathbf{E}+\mathbf{F}.$$

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Univariate Normal Distribution

A random variable X is normally distributed with mean μ and standard deviation σ can be written in the following notation

$$X \sim \mathcal{N}(\mu, \sigma).$$

The probability density function of univariate normal distribution is

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right).$$

The standard normal distribution is a normal distribution with a mean of 0 and standard deviation of 1.

The Central Limit Theorem

The sum of many random variables will have an approximately normal distribution.

Let $X_1, \ldots, X_n$ be independent and identically distributed random variables with the same arbitrary distribution, zero mean, and variance σ^2.

Let $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$, then the random variable

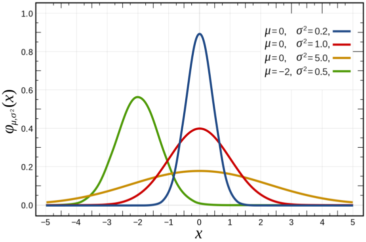
$$Z = \lim_{n\to\infty} \sqrt{n}\left(\frac{\bar{X}_n - \mu}{\sigma}\right)$$

is a standard normal distribution.

What about multivariate case?

Normal Distribution

正态分布



~~**正太分布**~~

词语起源

播报 编辑

“正太”一词最初出现于日本《ファンロード(Fanroad)》杂志中的“Q&A”栏目。在该栏目中，当被问及「喜欢男孩的女性应该被称作什么」时，该杂志的编辑”あるイニシャル・K“回答「喜欢”正太郎”的正太控(ショタコン)」。 [2]

该回答所提及的“正太郎”，源于漫画家横山光辉的作品《铁人28号》主角“金田正太郎”的名字。 [2]

此后，“正太控”一词开始流行。在传播过程中，“正太控”中的“正太”二字逐渐被分离出来，成为了形容”年龄小的男生“的词汇。 [2]

金田正太郎

Multivariate Normal Distribution

The multivariate normal distribution of a p-dimensional random vector $\mathbf{x} = [x_1, \ldots, x_p]^\top$ can be written in the following notation:

$$\mathbf{x} \sim \mathcal{N}_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$$

or to make it explicitly known that $\mathbf{x}$ is p-dimensional.

$$\mathbf{x} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma}),$$

with p-dimensional mean vector

$$\boldsymbol{\mu} = \mathbb{E}[\mathbf{x}] = \begin{bmatrix} \mathbb{E}[x_1] \\ \vdots \\ \mathbb{E}[x_p] \end{bmatrix} \in \mathbb{R}^p$$

and covariance matrix

$$\boldsymbol{\Sigma} = \mathbb{E}\left[(\mathbf{x} - \boldsymbol{\mu})(\mathbf{x} - \boldsymbol{\mu})^\top\right] \in \mathbb{R}^{p \times p}.$$

Multivariate Normal Distribution

The density function of univariate normal distribution is

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right),$$

where μ is the mean and σ^2 is the variance.

The density function of p-dimensional multivariate normal distribution is

$$f(\mathbf{x}) = \frac{1}{\sqrt{(2\pi)^p \det(\boldsymbol{\Sigma})}} \exp\left(-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^\top \boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\right),$$

where $\boldsymbol{\mu} \in \mathbb{R}^p$ is the mean and $\boldsymbol{\Sigma} \succ \mathbf{0}$ is the $p \times p$ covariance matrix.

Multivariate Normal Distribution

The density function of p-dimensional multivariate normal distribution is

$$f(\mathbf{x}) = \frac{1}{\sqrt{(2\pi)^p \det(\boldsymbol{\Sigma})}} \exp\left(-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^\top \boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\right),$$

where $\boldsymbol{\mu} \in \mathbb{R}^p$ is the mean and $\boldsymbol{\Sigma} \succ \mathbf{0}$ is the $p \times p$ covariance matrix.

When the covariance matrix $\boldsymbol{\Sigma}$ is singular, we call the distribution is degenerate normal distribution and we cannot write its density function.

This course will focus on the case of $\boldsymbol{\Sigma} \succ \mathbf{0}$.

How to obtain the pdf of multivariate normal distribution?

We generalize the form of pdf for univariate normal distribution

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right),$$

to the multivariate case

$$f(\mathbf{x}) = K \exp\left(-\frac{1}{2}(\mathbf{x}-\mathbf{b})^\top \mathbf{A}(\mathbf{x}-\mathbf{b})\right),$$

where $\mathbf{A}$ is symmetric positive definite.

We can verify that if $\mathbb{E}[\mathbf{x}] = \boldsymbol{\mu}$ and $\mathrm{Cov}[\mathbf{x}] = \boldsymbol{\Sigma}$, then

$$K = \frac{1}{\sqrt{(2\pi)^p \det(\boldsymbol{\Sigma})}}, \quad \mathbf{b} = \boldsymbol{\mu}, \quad \mathbf{A} = \boldsymbol{\Sigma}^{-1}.$$

How to obtain the pdf of multivariate normal distribution?

We first show

$$K = \frac{1}{\sqrt{(2\pi)^p \det(\mathbf{A})}}$$

by considering the random vector

$$\mathbf{y} = \mathbf{C}^{-1}(\mathbf{x} - \mathbf{b}) \in \mathbb{R}^p,$$

where $\mathbf{C} \in \mathbb{R}^{p \times p}$ satisfies $\mathbf{C}^\top \mathbf{A} \mathbf{C} = \mathbf{I}$.

How to obtain the pdf of multivariate normal distribution?

We show $\mathbf{b} = \boldsymbol{\mu}$ and $\mathbf{A} = \boldsymbol{\Sigma}^{-1}$ by using the following lemma.

Lemma

1. If $\mathbf{Z}$ is an $m \times n$ random matrix, $\mathbf{D}$ is an $l \times m$ real matrix, $\mathbf{E}$ is an $n \times q$ real matrix, and $\mathbf{F}$ is an $l \times q$ real matrix, then

$$\mathbb{E}[\mathbf{DZE} + \mathbf{F}] = \mathbf{D}\mathbb{E}[\mathbf{Z}]\mathbf{E} + \mathbf{F}.$$

2. If $\mathbf{y} = \mathbf{Dx} + \mathbf{f} \in \mathbb{R}^l$, where $\mathbf{D}$ is an $l \times m$ real matrix, $\mathbf{x} \in \mathbb{R}^m$ is a random vector, then

$$\mathbb{E}[\mathbf{y}] = \mathbf{D}\mathbb{E}[\mathbf{x}] + \mathbf{f}$$

and

$$\mathrm{Cov}[\mathbf{y}] = \mathbf{D}\mathrm{Cov}[\mathbf{x}]\mathbf{D}^\top.$$

Multivariate Normal Distribution

If the density of a p-dimensional random vector $\mathbf{x}$ is

$$K \exp\left(-\frac{1}{2}(\mathbf{x}-\mathbf{b})^\top \mathbf{A}(\mathbf{x}-\mathbf{b})\right),$$

where $\mathbf{A} \in \mathbb{R}^{p\times p}$ is symmetric positive definite. Then the expectation of $\mathbf{x}$ is $\mathbf{b}$ and its covariance matrix is $\mathbf{A}^{-1}$.

Conversely, given a vector $\boldsymbol{\mu} \in \mathbb{R}^p$ and a positive definite matrix $\boldsymbol{\Sigma} \in \mathbb{R}^{p\times p}$, there is a multivariate normal density

$$n(\mathbf{x} \mid \boldsymbol{\mu}, \boldsymbol{\Sigma}) = \frac{1}{\sqrt{(2\pi)^p \det(\boldsymbol{\Sigma})}} \exp\left(-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^\top \boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\right).$$