

Optimization Theory

Lecture 10

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- 1 Lower Complexity Bound
- 2 Nonsmooth Convex Optimization

1 Lower Complexity Bound

2 Nonsmooth Convex Optimization

Nesterov's Acceleration

Nesterov's acceleration:

$$\begin{cases} \mathbf{y}_t = \mathbf{x}_t + \beta_t(\mathbf{x}_t - \mathbf{x}_{t-1}), \\ \mathbf{x}_{t+1} = \mathbf{y}_t - \eta_t \nabla f(\mathbf{y}_t). \end{cases}$$

Can we further accelerate Nesterov's acceleration?

Let us check our possibilities in minimizing L -smooth convex functions by first-order methods.

Assumption

An iterative method \mathcal{M} generates a sequence of test points $\{\mathbf{x}_t\}$ such that

$$\mathbf{x}_t \in \mathbf{x}_0 + \text{span}\{\nabla f(\mathbf{x}_0), \dots, \nabla f(\mathbf{x}_{t-1})\}.$$

“Worst Functions” and Zero-Chain

Consider the following functions

$$f_t(\mathbf{x}) = \frac{L}{4} \left(\frac{1}{2} \left(x_1^2 + \sum_{i=1}^{t-1} (x_i - x_{i+1})^2 + x_t^2 \right) - x_1 \right),$$

for $t = 1, \dots, d$, where $\mathbf{x} = [x_1, \dots, x_d]^\top$.

Let $\mathbb{R}^{t,d} = \{\mathbf{x} \in \mathbb{R}^d : x_{t+1} = \dots = x_d = 0\}$, that is the subspace of \mathbb{R}^d , in which only the first t components of the point can differ from zero.

Lemma

Let $\mathbf{x}_0 = \mathbf{0}$. Then for any sequence $\{\mathbf{x}_1, \dots, \mathbf{x}_t\}$ satisfying the condition

$$\mathbf{x}_t \in \mathcal{L}_t = \text{span}\{\nabla f_t(\mathbf{x}_0), \dots, \nabla f_t(\mathbf{x}_{t-1})\},$$

we have $\mathcal{L}_t \subseteq \mathbb{R}^{t,d}$.

“Worst Functions”

Consider the following functions

$$f_t(\mathbf{x}) = \frac{L}{4} \left(\frac{1}{2} \left(x_1^2 + \sum_{i=1}^{t-1} (x_i - x_{i+1})^2 + x_t^2 \right) - x_1 \right),$$

for $t = 1, \dots, d$, where $\mathbf{x} = [x_1, \dots, x_d]^\top$. They are L -smooth and convex.

We can verify $\nabla^2 f(\mathbf{x}) = \frac{L}{4} \mathbf{A}_t$ with

$$\mathbf{A}_t = \begin{bmatrix} 2 & -1 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & -1 & 2 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & \cdots & -1 & 2 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

“Worst Functions”

The equation $\nabla f(\mathbf{x}^*) = \mathbf{0}$ leads to

$$x_i^* = \begin{cases} 1 - \frac{i}{t+1}, & i = 1, \dots, t, \\ 0, & i = t+1, \dots, d. \end{cases}$$

Then we have

$$f(\mathbf{x}^*) = -\frac{L}{8} \left(1 - \frac{i}{t+1} \right).$$

and

$$\|\mathbf{x}^*\|_2^2 \leq \frac{t+1}{3}.$$

“Worst Functions”

Lemma

For all $\mathbf{x} \in \mathbb{R}^{t,d}$, we have $f_t(\mathbf{x}) = f_p(\mathbf{x})$ for $p = t, t+1, \dots, d$.

Corollary

For any $\{\mathbf{x}_t\}_{t=1}^P$ with $\mathbf{x}_0 = \mathbf{0}$ and $\mathbf{x}_t \in \mathcal{L}_t$, we have $\mathbf{x}_t \in \mathbb{R}^{t,d}$ and

$$f_p(\mathbf{x}_t) = f_t(\mathbf{x}_t) \geq f_t^*$$

for any $p = t, t+1, \dots, d$.

Lower Complexity Bound (Convex)

Theorem

For any t such that $t \in [1, (d-1)/2]$ and any $\mathbf{x}_0 \in \mathbb{R}^d$, there exists an L -smooth and convex function $f : \mathbb{R}^d \rightarrow \mathbb{R}$ such that for any first-order algorithm \mathcal{M} with

$$\mathbf{x}_t \in \mathbf{x}_0 + \text{span}\{\nabla f(\mathbf{x}_0), \dots, \nabla f(\mathbf{x}_{t-1})\},$$

we have

$$f(\mathbf{x}_t) - f^* \geq \frac{3L \|\mathbf{x}_0 - \mathbf{x}^*\|_2^2}{8(t+1)^2},$$

where \mathbf{x}^ is the minimizer of f and $f^* = f(\mathbf{x}^*)$.*

Lower Complexity Bound

- ① The above theorem is valid when the iteration number is not too large as compared with the dimension the variables.
- ② Without a direct use of finite dimensional arguments, we cannot justify a better complexity of the corresponding numerical scheme.
- ③ Nesterov's acceleration is optimal for minimizing smooth and convex function by first-order methods.

Lower Complexity Bound (Strongly Convex)

Consider the d -dimensional regularized “worst functions”

$$f(\mathbf{x}) = \frac{L - \mu}{4} \left(\frac{1}{2} \left(x_1^2 + \sum_{i=1}^{d-1} (x_i - x_{i+1})^2 + \left(1 - \frac{\sqrt{\kappa} - 1}{\sqrt{\kappa} + 1} \right) x_d^2 \right) - x_1 \right) + \frac{\mu}{2} \|\mathbf{x}\|_2^2,$$

for $t = 1, \dots, d$, where $\mathbf{x} = [x_1, \dots, x_d]^\top$ and $\kappa = L/\mu$.

- 1 The functions are L -smooth and μ -strongly convex.
- 2 The zero-chain property still holds.
- 3 The minimizer is $\mathbf{x}^* = [q, q^2, \dots, q^d]$ with $q = (\sqrt{\kappa} - 1)/(\sqrt{\kappa} + 1)$.

Lower Complexity Bound (Strongly Convex)

Theorem

For any t and d such that $t \leq d/2$, $d \geq 2$ and any $\mathbf{x}_0 \in \mathbb{R}^d$, there exists an L -smooth and μ -strongly convex function $f : \mathbb{R}^d \rightarrow \mathbb{R}$ such that for any first-order algorithm \mathcal{M} with

$$\mathbf{x}_t \in \mathbf{x}_0 + \text{span}\{\nabla f(\mathbf{x}_0), \dots, \nabla f(\mathbf{x}_{t-1})\},$$

we have

$$f(\mathbf{x}_t) - f^* \geq \frac{\mu}{4} \left(\frac{\sqrt{\kappa} - 1}{\sqrt{\kappa} + 1} \right)^{2t}.$$

where \mathbf{x}^ is the minimizer of f and $f^* = f(\mathbf{x}^*)$.*

Lower Complexity Bound (Nonconvex)

The lower complexity bound for finding an ϵ -stationary point of L -smooth nonconvex function are based on

$$f(\mathbf{x}) = \frac{\sqrt{\mu}}{2}(x_1 - 1)^2 + \frac{1}{2} \sum_{i=1}^t (x_{i+1} - x_i)^2 + \mu \sum_{i=1}^T \Gamma_r(x_i),$$

where $\mathbf{x} = [x_1, \dots, x_{T+1}]^\top \in \mathbb{R}^{T+1}$ and $\Gamma_r(x) = 120 \int_1^x \frac{t^2(t-1)}{1 + (t/r)^2} dt$.

- ① Let $r \geq 1$ and $\mu \leq 1$. For any $\mathbf{x} \in \mathbb{R}^{T+1}$ such that $x_T = x_{T+1} = 0$, we have $\|\nabla f(\mathbf{x})\|_2 \geq \mu^{3/4}/4$.
- ② The lower complexity bound is $\Omega(L\epsilon^{-2}(f(\mathbf{x}_0) - f^*))$. See paper:
 - Yair Carmon, John C. Duchi, Oliver Hinder, Aaron Sidford. Lower bounds for finding stationary points II: first-order methods. *Mathematical Programming*. 185(1):315–355, 2021.

Outline

- 1 Lower Complexity Bound
- 2 Nonsmooth Convex Optimization

Nonsmooth Convex Optimization

We consider optimization with a nonsmooth objective function

$$\min_{\mathbf{x} \in \mathcal{C}} f(\mathbf{x}).$$

Here we assume that $f : \mathbb{R}^d \rightarrow \mathbb{R}$ is a Lipschitz convex function defined on a convex and closed set $\mathcal{C} \subseteq \mathbb{R}^d$, but not necessarily smooth.

For constrained optimization, we assume the projection operator

$$\text{proj}_{\mathcal{C}}(\mathbf{y}) = \arg \min_{\mathbf{x} \in \mathcal{C}} \|\mathbf{y} - \mathbf{x}\|_2^2$$

can be efficiently computed for all $\mathbf{y} \in \mathbb{R}^d$.

Subgradient Descent Method

Suppose $\mathcal{C} = \mathbb{R}^d$, we have introduced gradient descent

$$\mathbf{x}_{t+1} = \mathbf{x}_t - \eta_t \nabla f(\mathbf{x})$$

converges by taking $\eta_t = \eta > 0$.

For nonsmooth case, we can replace the gradient by the subgradient and introduce projection step

$$\begin{cases} \tilde{\mathbf{x}}_{t+1} = \mathbf{x}_t - \eta_t \mathbf{g}_t, \\ \mathbf{x}_{t+1} = \text{proj}_{\mathcal{C}}(\tilde{\mathbf{x}}_{t+1}) \end{cases}$$

for $t = 0, 1, \dots$, where $\mathbf{g}_t \in \partial f(\mathbf{x}_t)$ and

$$\lim_{t \rightarrow +\infty} \eta_t = 0.$$

Convergence Analysis (Convex)

Theorem

A convex function f is G -Lipschitz continuous on $\text{dom } f$ if

$$\max_{\mathbf{g} \in \partial f(\mathbf{x})} \{\|\mathbf{g}\|_2\} \leq G$$

for all $\mathbf{x} \in \text{dom } f$.

Theorem

Let $\mathbf{z} = \text{proj}_{\mathcal{C}}(\mathbf{y})$ for some convex and closed $\mathcal{C} \subseteq \mathbb{R}^d$ and $\mathbf{y} \in \mathbb{R}^d$, then

$$\|\mathbf{z} - \mathbf{x}\|_2^2 \leq \|\mathbf{y} - \mathbf{x}\|_2^2$$

for any $\mathbf{x} \in \mathcal{C}$.

Convergence Analysis (Convex)

We assume the convex function $f : \mathbb{R}^d \rightarrow \mathbb{R}$ satisfies

$$\max_{\mathbf{g} \in \partial f(\mathbf{x})} \{\|\mathbf{g}\|_2\} \leq G$$

on domain \mathcal{C} . Then for all $\hat{\mathbf{x}} \in \mathcal{C}$, we have

$$\frac{1}{\sum_{t=0}^{T-1} \eta_t} \sum_{t=0}^{T-1} \eta_t f(\mathbf{x}_t) \leq f(\hat{\mathbf{x}}) + \frac{\|\mathbf{x}_0 - \hat{\mathbf{x}}\|_2^2 + \sum_{t=0}^{T-1} \eta_t^2 G^2}{2 \sum_{t=0}^{T-1} \eta_t}.$$

① Taking $\eta_t = \eta_0 / \sqrt{T}$ leads to

$$\frac{1}{T} \sum_{t=0}^{T-1} f(\mathbf{x}_t) \leq f(\hat{\mathbf{x}}) + \frac{\|\mathbf{x}_0 - \hat{\mathbf{x}}\|_2^2 + \eta_0^2 G^2}{2\eta_0 \sqrt{T}}.$$

② Taking $\eta_t = \eta_0 / (\sqrt{t+1} + \sqrt{t})$ leads to

$$\sum_{t=0}^{T-1} \frac{f(\mathbf{x}_t)}{\sqrt{T(t+1)} + \sqrt{Tt}} \leq f(\hat{\mathbf{x}}) + \frac{\|\mathbf{x}_0 - \hat{\mathbf{x}}\|_2^2 + \eta_0^2 (\ln(2T-1) + 2) G^2 / 2}{2\eta_0 \sqrt{T}}.$$

Convergence Analysis (Strongly Convex)

If we additionally suppose f is μ -strongly convex and set

$$\eta_t = \frac{2}{\mu(t+1)},$$

then

$$\sum_{t=0}^{T-1} \frac{t}{T(T-1)} f(\mathbf{x}_t) \leq f(\hat{\mathbf{x}}) + \frac{2G^2}{\mu(T-1)}.$$