

# Homework 1

Deadline: ~~March 30, 2022~~ April 6, 2022

1. For any  $\mathbf{A} \in \mathbb{S}^n$ ,  $\mathbf{x} \in \mathbb{R}^n$  and  $\mathbf{y} \in \mathbb{R}^n$ , prove that  $\mathbf{x}^\top \mathbf{A} \mathbf{y} = \mathbf{y}^\top \mathbf{A} \mathbf{x}$ .
2. Prove that for any matrix  $\mathbf{A} \in \mathbb{R}^{m \times n}$ , the column rank of  $\mathbf{A}$  is equal to the row rank of  $\mathbf{A}$ .
3. For  $\mathbf{A} \in \mathbb{R}^{m \times n}$  and  $\mathbf{B} \in \mathbb{R}^{n \times p}$ , prove that  $\|\mathbf{A}\mathbf{B}\|_F \leq \|\mathbf{A}\|_F \|\mathbf{B}\|_F$ .
4. Suppose  $\mathbf{A} \in \mathbb{R}^{n \times n}$  has eigenvalues  $\lambda_1, \dots, \lambda_n$ . Prove the following statements
  - (a)  $\text{tr}(\mathbf{A}) = \sum_{i=1}^n \lambda_i$ ,
  - (b)  $\det(\mathbf{A}) = \prod_{i=1}^n \lambda_i$ .
5. Prove the SVD always exists for any  $\mathbf{A} \in \mathbb{R}^{m \times n}$ . (Hint: Using spectral decomposition theorem)
6. Given the symmetric matrix

$$\mathbf{N} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^\top & \mathbf{D} \end{bmatrix}$$

with non-singular  $\mathbf{A}$  and let  $\mathbf{S} = \mathbf{D} - \mathbf{B}^\top \mathbf{A}^{-1} \mathbf{B}$ . Prove that

- (a)  $\mathbf{N} \succ \mathbf{0} \iff \mathbf{A} \succ \mathbf{0} \text{ and } \mathbf{S} \succ \mathbf{0}$ .
- (b) If  $\mathbf{A} \succ \mathbf{0}$ , then  $\mathbf{N} \succeq \mathbf{0} \iff \mathbf{S} \succeq \mathbf{0}$ .