Optimization Theory

Lecture 04

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Outline

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Subgradient and Subdifferential

We say a vector $\mathbf{g} \in \mathbb{R}^d$ is a subgradient of a proper convex function $f : \mathbb{R}^d \to \mathbb{R}$ at $\mathbf{x} \in \operatorname{dom} f$ if

$$f(\mathbf{y}) \geq f(\mathbf{x}) + \langle \mathbf{g}, \mathbf{y} - \mathbf{x} \rangle$$

holds for any $\mathbf{y} \in \mathbb{R}^d$.

The set of subgradients at $\mathbf{x} \in \operatorname{dom} f$ is called the subdifferential of f at $\mathbf{x}$, defined as

$$\partial f(\mathbf{x}) \triangleq \left\{ \mathbf{g} \in \mathbb{R}^d : f(\mathbf{y}) \geq f(\mathbf{x}) + \langle \mathbf{g}, \mathbf{y} - \mathbf{x} \rangle \text{ holds for any } \mathbf{y} \in \mathbb{R}^d \right\}.$$

Examples of Subdifferential

1. The subdifferential of $f(x) = |x|$ at 0 is the set

$$\partial f(x) = [-1, 1].$$

   What about the general norm?

2. The subdifferential of an indicator function $\mathbb{1}_{\mathcal{C}}(\mathbf{x})$ is

$$\partial \mathbb{1}_{\mathcal{C}}(\mathbf{x}) = \mathcal{N}_{\mathcal{C}}(\mathbf{x}),$$

   where

$$\mathcal{N}_{\mathcal{C}}(\mathbf{x}) = \left\{\mathbf{g} \in \mathbb{R}^d : \langle \mathbf{g}, \mathbf{y} - \mathbf{x}\rangle \leq 0 \text{ for all } \mathbf{y} \in \mathcal{C}\right\}$$

   is called the normal cone of $\mathcal{C}$ at $\mathbf{x}$.

3. If a convex function f is differentiable at $\mathbf{x} \in \mathcal{C}$, then

$$\partial f(\mathbf{x}) = \{\nabla f(\mathbf{x})\}.$$

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Subdifferential Calculus

Let f_1 and f_2 be proper convex functions on $\mathbb{R}^d$, then

$$\partial(f_1 + f_2)(\mathbf{x}) \supseteq \partial f_1(\mathbf{x}) + \partial f_2(\mathbf{x}).$$

If the sets $\mathrm{ri}(\mathrm{dom}\, f_1)$ and $\mathrm{ri}(\mathrm{dom}\, f_2)$ have a point in common (overlap sufficiently), we have

$$\partial(f_1 + f_2)(\mathbf{x}) = f_1(\mathbf{x}) + \partial f_2(\mathbf{x}).$$

We define the relative interior $\mathrm{ri}(\mathcal{C})$ for convex $\mathcal{C} \subseteq \mathbb{R}^d$ as

$$\begin{aligned}\mathrm{ri}(\mathcal{C}) = \{\mathbf{z} \in \mathcal{C} : \ & \text{for every } \mathbf{x} \in \mathcal{C} \text{ such that} \\ & \text{there exist a } \mu > 1 \text{ such that } (1-\mu)\mathbf{x} + \mu\mathbf{z} \in \mathcal{C}\}.\end{aligned}$$

It means every line segment in $\mathcal{C}$ having $\mathbf{z}$ as one endpoint can be prolonged beyond $\mathbf{z}$ without leaving $\mathcal{C}$.

Subdifferential Calculus

Nonempty subdifferential and convexity:

1. If any $\mathbf{x} \in \operatorname{dom} f$ satisfies $\partial f(\mathbf{x}) \neq \emptyset$, then f is convex.
2. If $f : \mathbb{R}^d \to \mathbb{R}$ is convex and $\mathbf{x}$ belongs to the interior of $\operatorname{dom} f$, then $\partial f(\mathbf{x}) \neq \emptyset$.

Theorem (Hyperplane Separation Theorem)

Let $\mathcal{X} \subseteq \mathbb{R}^d$ is a convex set and $\mathbf{x}_0$ belongs to its boundary. Then, there exists a nonzero vector $\mathbf{w} \in \mathbb{R}^d$ such that

$$\langle \mathbf{w}, \mathbf{x} \rangle \leq \langle \mathbf{w}, \mathbf{x}_0 \rangle.$$

Subdifferential Calculus

The subgradient of a convex function may not exist at a boundary point of the domain.

As an example, consider the function

$$f(x) = -\sqrt{x}$$

defined on $[0, +\infty)$, where we have $\partial f(0) = \emptyset$.

Subdifferential Calculus

Given matrix $\mathbf{A} \in \mathbb{R}^{d \times m}$ and vector $\mathbf{b} \in \mathbb{R}^d$, define

$$h(\mathbf{x}) = f(\mathbf{A}\mathbf{x} + \mathbf{b}),$$

where f is a proper convex on $\mathbb{R}^d$.

Then the function $h(\mathbf{x})$ is convex and

$$\partial h(\mathbf{x}) \supseteq \mathbf{A}^\top \partial f(\mathbf{A}\mathbf{x} + \mathbf{b}).$$

If the range of $\mathbf{A}$ contains a point of $\mathrm{ri}(\mathrm{dom}\, h)$, then

$$\partial h(\mathbf{x}) = \mathbf{A}^\top \partial f(\mathbf{A}\mathbf{x} + \mathbf{b}).$$

Optimal Condition

Theorem

Consider proper closed convex function f and closed convex set $\mathcal{C} \subseteq (\mathrm{dom}\, f)^\circ$. A point $\mathbf{x}^* \in \mathcal{C}$ is a solution of convex optimization problem

$$\min_{\mathbf{x} \in \mathcal{C}} f(\mathbf{x})$$

if and only if

$$\mathbf{0} \in \partial(f(\mathbf{x}^*) + \mathbb{1}_{\mathcal{C}}(\mathbf{x}^*)).$$

Equivalently, there exists a subgradient $\mathbf{g}^* \in \partial f(\mathbf{x}^*)$, such that any $\mathbf{y} \in \mathcal{C}$ satisfies

$$\langle \mathbf{g}^*, \mathbf{y} - \mathbf{x}^* \rangle \geq 0.$$

In particular, the point $\mathbf{x}^*$ is the solution of the problem in unconstrained case if

$$\mathbf{0} \in \partial f(\mathbf{x}^*).$$

Outline

Regularity Conditions

The following regularity conditions are useful in the convergence analysis of convex optimization problems.

1. We say that a function $f : \mathcal{C} \to \mathbb{R}$ is G-Lipschitz continuous if for all $\mathbf{x}, \mathbf{y} \in \mathcal{C}$, we have

$$|f(\mathbf{x}) - f(\mathbf{y})| \leq G \|\mathbf{x} - \mathbf{y}\|_2 .$$

2. We say a differentiable function $f : \mathbb{R}^d \to \mathbb{R}$ is L-smooth if it has L-Lipschitz continuous gradient. That is, for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$, we have

$$\|\nabla f(\mathbf{x}) - \nabla f(\mathbf{y})\|_2 \leq L \|\mathbf{x} - \mathbf{y}\|_2 .$$

3. If the function

$$g(\mathbf{x}) = f(\mathbf{x}) - \frac{\mu}{2} \|\mathbf{x}\|_2^2$$

is convex for some $\mu > 0$, we say f is μ-strongly convex.

Strong Convexity

Theorem

The function $f : \mathcal{C} \to \mathbb{R}$ defined on convex set $\mathcal{C}$ is μ-strongly-convex if and only if

$$f(\alpha \mathbf{x} + (1-\alpha)\mathbf{y}) \leq \alpha f(\mathbf{x}) + (1-\alpha) f(\mathbf{y}) - \frac{\mu\alpha(1-\alpha)}{2} \|\mathbf{x} - \mathbf{y}\|_2^2$$

for all $\mathbf{x}, \mathbf{y} \in \mathcal{C}$ and $\alpha \in [0, 1]$.

Theorem

If a function f is differentiable on open set $\mathcal{C}$, then it is μ-strongly convex on $\mathcal{C}$ if and only if

$$f(\mathbf{y}) \geq f(\mathbf{x}) + \langle \nabla f(\mathbf{x}), \mathbf{y} - \mathbf{x} \rangle + \frac{\mu}{2} \|\mathbf{y} - \mathbf{x}\|_2^2$$

hols for any $\mathbf{x}, \mathbf{y} \in \mathcal{C}$.

Strong Convexity

If there exists some

$$\mathbf{x}^* = \arg\min_{\mathbf{x} \in \mathcal{C}} f(\mathbf{x}),$$

then it is the unique minimizer.

Moreover, the solution is stable such that any approximate solution $\hat{\mathbf{x}}$ satisfying

$$f(\mathbf{x}) \leq f(\mathbf{x}^*) + \epsilon$$

leads to

$$\|\mathbf{x}^* - \hat{\mathbf{x}}\|_2^2 \leq \frac{2\epsilon}{\mu}.$$

Lipschitz Continuity and Smoothness

Theorem

A convex function f is G-Lipschitz continuous on $(\mathrm{dom}\, f)^\circ$ if and only if

$$\|\mathbf{g}\|_2 \leq G$$

for all $\mathbf{g} \in \partial f(\mathbf{x})$ and $\mathbf{x} \in (\mathrm{dom}\, f)^\circ$.

Theorem

A function $f : \mathbb{R}^d \to \mathbb{R}$ is L-smooth (possibly nonconvex), then it holds

$$|f(\mathbf{y}) - f(\mathbf{x}) - \langle \nabla f(\mathbf{x}), \mathbf{y} - \mathbf{x} \rangle| \leq \frac{L}{2} \|\mathbf{y} - \mathbf{x}\|_2^2$$

holds for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$.

Theorem

A function $f : \mathbb{R}^d \to \mathbb{R}$ is convex and L-smooth, then we have

1. $0 \leq f(\mathbf{y}) - f(\mathbf{x}) - \langle \nabla f(\mathbf{x}), \mathbf{y} - \mathbf{x} \rangle \leq \dfrac{L}{2} \|\mathbf{y} - \mathbf{x}\|_2^2$
2. $f(\mathbf{x}) + \langle \nabla f(\mathbf{x}), \mathbf{y} - \mathbf{x} \rangle + \dfrac{1}{2L} \|\nabla f(\mathbf{y}) - \nabla(\mathbf{x})\|_2^2 \leq f(\mathbf{y})$
3. $\dfrac{1}{L} \|\nabla f(\mathbf{y}) - \nabla(\mathbf{x})\|_2^2 \leq \langle \nabla f(\mathbf{x}) - \nabla f(\mathbf{y}), \mathbf{x} - \mathbf{y} \rangle$

for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$.

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Second-Order Characterization

Theorem (Smoothness and Convexity)

Let $f(\cdot)$ be a twice differentiable function defined on $\mathbb{R}^d$

1. *It is L-smooth if and only if $-L\mathbf{I} \preceq \nabla^2 f(\mathbf{x}) \preceq L\mathbf{I}$ for all $\mathbf{x} \in \mathbb{R}^d$.*
2. *It is convex if and only if $\nabla^2 f(\mathbf{x}) \succeq \mathbf{0}$ for all $\mathbf{x} \in \mathbb{R}^d$.*
3. *It is μ-strongly-convex if and only if $\nabla^2 f(\mathbf{x}) \succeq \mu\mathbf{I}$ for all $\mathbf{x} \in \mathbb{R}^d$.*

Sometimes, we say $f(\cdot)$ is ℓ-weakly convex if the function

$$g(\mathbf{x}) = f(\mathbf{x}) + \frac{\ell}{2}\|\mathbf{x}\|_2^2$$

is convex for some $\ell > 0$.

Second-Order Characterization

Theorem

Let $f : \mathbb{R}^d \to \mathbb{R}$ be a twice differentiable function. Suppose that $\nabla^2 f(\cdot)$ is continuous in an open neighborhood of $\mathbf{x}^* \in \mathbb{R}^d$.

1. *If $\mathbf{x}^*$ is a local minimizer of $f(\cdot)$, then it holds that*

$$\nabla f(\mathbf{x}^*) = \mathbf{0} \qquad \text{and} \qquad \nabla^2 f(\mathbf{x}^*) \succeq \mathbf{0}.$$

2. *If it holds that*

$$\nabla f(\mathbf{x}^*) = \mathbf{0} \qquad \text{and} \qquad \nabla^2 f(\mathbf{x}^*) \succ \mathbf{0},$$

   then the point $\mathbf{x}^*$ is a strict local minimizer of $f(\cdot)$.

Second-Order Characterization

Some examples:

1. For unconstrained quadratic problem

$$\min_{\mathbf{x}\in\mathbb{R}^d} f(\mathbf{x}) \triangleq \frac{1}{2}\mathbf{x}^\top\mathbf{A}\mathbf{x} - \mathbf{b}^\top\mathbf{x},$$

   where $\mathbf{A} \in \mathbb{R}^{d\times d}$. We have

$$\nabla^2 f(\mathbf{x}) = \mathbf{A}.$$

2. For regularized generalized linear model

$$\min_{\mathbf{x}\in\mathbb{R}^d} f(\mathbf{x}) \triangleq \frac{1}{n}\sum_{i=1}^{n}\phi_i(\mathbf{a}^\top\mathbf{x}) + \frac{\lambda}{2}\|\mathbf{x}\|_2^2 .$$

   where $\phi_i : \mathbb{R}^d \to \mathbb{R}$ is twice differentiable. We have

$$\nabla f(\mathbf{x}) = \frac{1}{n}\sum_{i=1}^{n}\phi_i'(\mathbf{a}_i^\top\mathbf{x})\mathbf{a} + \lambda\mathbf{x} \quad \text{and} \quad \nabla^2 f(\mathbf{x}) = \frac{1}{n}\sum_{i=1}^{n}\phi_i''(\mathbf{a}_i^\top\mathbf{x})\mathbf{a}_i\mathbf{a}_i^\top + \lambda\mathbf{I}.$$