

# Multivariate Statistical Analysis

## Lecture 16

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# Outline

- 1 Factor Analysis
- 2 Probabilistic Principle Component Analysis
- 3 The Expectation-Maximization Algorithm
- 4 Course Summary

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# Factor Analysis

Let the observable vector  $\mathbf{y} \in \mathbb{R}^p$  be written as

$$\mathbf{y} = \mathbf{W}\mathbf{x} + \boldsymbol{\mu} + \boldsymbol{\epsilon},$$

where

- ①  $\mathbf{W} \in \mathbb{R}^{p \times q}$  is the loading matrix (parameter),
- ②  $\mathbf{x} \in \mathbb{R}^q$  is the common factor (parameter/random vector),
- ③  $\boldsymbol{\mu} \in \mathbb{R}^p$  is the mean vector (parameter),
- ④  $\boldsymbol{\epsilon} \in \mathbb{R}^p$  is the specific factor (random vector).

The model is similar to regression, but  $\mathbf{x}$  is unobserved.

# Factor Analysis

Example of sports games:

$$\mathbf{y} = \mathbf{W}\mathbf{x} + \mu + \epsilon.$$

- 1 **y**: performance in real-world
- 2 **W**: system of the game
- 3 **x**: attributes in the game
- 4  $\mu$ : average attributes
- 5  $\epsilon$ : noise/exception

## Yao Ming's Stats

57 Games	82 Games	80 Games	48 Games
22.3 Points	17.5 Points	18.3 Points	25.0 Points
51.9% FG	52.2% FG	55.2% FG	51.6% FG
0.0% 3PT	0.0% 3PT	0.0% 3PT	0.0% 3PT
85.3% FT	80.9% FT	86.6% FT	86.2% FT
10.2 Rebounds	9.6 Rebounds	8.4 Rebounds	9.4 Rebounds
1.5 Assists	1.5 Assists	1.5 Assists	2.0 Assists
0.5 Steals	0.5 Steals	0.5 Steals	0.4 Steals
1.6 Blocks	1.6 Blocks	1.6 Blocks	2.0 Blocks
2.6 Turnovers	2.5 Turnovers	2.6 Turnovers	3.5 Turnovers
25.6 PER	21.9 PER	22.5 PER	26.5 PER
0.211 WS/48	0.202 WS/48	0.218 WS/48	0.220 WS/48



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# Probabilistic Principle Component Analysis

Let  $\mathbf{y}_1, \dots, \mathbf{y}_N \in \mathbb{R}^p$  be  $N$  independent observations and we have

$$\mathbf{y}_\alpha = \mathbf{W}\mathbf{x}_\alpha + \boldsymbol{\mu} + \epsilon_\alpha,$$

where

$$\mathbf{x}_\alpha \sim \mathcal{N}_q(\mathbf{0}, \mathbf{I}) \quad \text{and} \quad \epsilon_\alpha \sim \mathcal{N}_p(\mathbf{0}, \sigma^2 \mathbf{I})$$

are independent for some  $\sigma^2 > 0$  and  $q < p$ .

We target to estimate parameters

$$\mathbf{W} \in \mathbb{R}^{p \times q}, \quad \boldsymbol{\mu} \in \mathbb{R}^p \quad \text{and} \quad \sigma \in (0, +\infty)$$

by maximum likelihood estimation for given  $\mathbf{y}_1, \dots, \mathbf{y}_N$ .

# Probabilistic Principle Component Analysis

Consider that

$$\mathbf{y}_\alpha \sim \mathcal{N}_p(\boldsymbol{\mu}, \mathbf{W}\mathbf{W}^\top + \sigma^2\mathbf{I}).$$

We construct the likelihood function

$$\begin{aligned} & L(\boldsymbol{\mu}, \mathbf{W}, \sigma^2) \\ &= \prod_{\alpha=1}^N \frac{1}{\sqrt{(2\pi)^p \det(\boldsymbol{\Sigma})}} \exp\left(-\frac{1}{2}(\mathbf{y}_\alpha - \boldsymbol{\mu})^\top (\mathbf{W}\mathbf{W}^\top + \sigma^2\mathbf{I})^{-1}(\mathbf{y}_\alpha - \boldsymbol{\mu})\right), \end{aligned}$$

then we have

$$\begin{aligned} & \ln L(\boldsymbol{\mu}, \mathbf{W}, \sigma^2) \\ & \propto -\frac{N}{2} \ln \det(\mathbf{W}\mathbf{W}^\top + \sigma^2\mathbf{I}) - \frac{1}{2} \sum_{\alpha=1}^N (\mathbf{y}_\alpha - \boldsymbol{\mu})^\top (\mathbf{W}\mathbf{W}^\top + \sigma^2\mathbf{I})^{-1}(\mathbf{y}_\alpha - \boldsymbol{\mu}). \end{aligned}$$



# The Maximum Likelihood Estimators

The maximum likelihood estimators of  $\mu$ ,  $\mathbf{W}$  and  $\sigma^2$  are

$$\hat{\mu} = \bar{\mathbf{y}} = \frac{1}{N} \sum_{\alpha=1}^N \mathbf{y}_{\alpha}, \quad \hat{\mathbf{W}} = \mathbf{U}_q (\mathbf{\Lambda}_q - \hat{\sigma}^2 \mathbf{I}) \mathbf{R} \quad \text{and} \quad \hat{\sigma}^2 = \frac{1}{d-q} \sum_{j=q+1}^d \lambda_j,$$

where

- 1  $\mathbf{U}_q \in \mathbb{R}^{p \times q}$  is orthogonal column consisting of principal eigenvectors of

$$\hat{\mathbf{\Sigma}} = \frac{1}{N} \sum_{\alpha=1}^N (\mathbf{y}_{\alpha} - \bar{\mathbf{y}})(\mathbf{y}_{\alpha} - \bar{\mathbf{y}})^{\top},$$

- 2  $\mathbf{\Lambda}_q \in \mathbb{R}^{q \times q}$  is diagonal with corresponding eigenvalues  $\lambda_1, \dots, \lambda_q$ ,
- 3  $\mathbf{R} \in \mathbb{R}^{q \times q}$  is any orthogonal matrix.

# The Maximum Likelihood Estimators

The maximum likelihood estimators also minimize the error with respect to Frobenius norm

$$(\hat{\mathbf{W}}, \hat{\sigma}^2) = \arg \min_{\mathbf{W} \in \mathbb{R}^{p \times q}, \sigma^2 \in \mathbb{R}^+} \left\| \hat{\mathbf{\Sigma}} - (\mathbf{W}\mathbf{W}^\top + \sigma^2 \mathbf{I}) \right\|_F.$$

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# The Expectation-Maximization Algorithm

For the model

$$\mathbf{y} = \mathbf{W}\mathbf{x} + \boldsymbol{\mu} + \epsilon,$$

where  $\mathbf{x} \sim \mathcal{N}_q(\mathbf{0}, \mathbf{I})$  and  $\epsilon \sim \mathcal{N}_p(\mathbf{0}, \sigma^2 \mathbf{I})$  are independent.

We regard  $\{\mathbf{x}_\alpha\}_{\alpha=1}^N$  as missing data and  $\{\mathbf{x}_\alpha, \mathbf{y}_\alpha\}_{\alpha=1}^N$  as the complete data, then we can achieve

$$\mathbf{y} | \mathbf{x} \sim \mathcal{N}_d(\mathbf{W}\mathbf{x} + \boldsymbol{\mu}, \sigma^2 \mathbf{I})$$

and

$$\mathbf{x} | \mathbf{y} \sim \mathcal{N}_q(\mathbf{M}^{-1} \mathbf{W}^\top (\mathbf{y} - \boldsymbol{\mu}), \sigma^2 \mathbf{M}^{-1}),$$

where  $\mathbf{M} = \mathbf{W}^\top \mathbf{W} + \sigma^2 \mathbf{I}$ .

# The Expectation-Maximization Algorithm

The update of the EM algorithm

- 1 In E-step, we take the expectation

$$l_C = \mathbb{E} \left[ \ln \left( \prod_{\alpha=1}^N p(\mathbf{x}_\alpha | \mathbf{y}_\alpha) \right) \right].$$

- 2 In the M-step, we maximized  $l_C$  with respect to  $\mathbf{W}$  and  $\sigma^2$ :

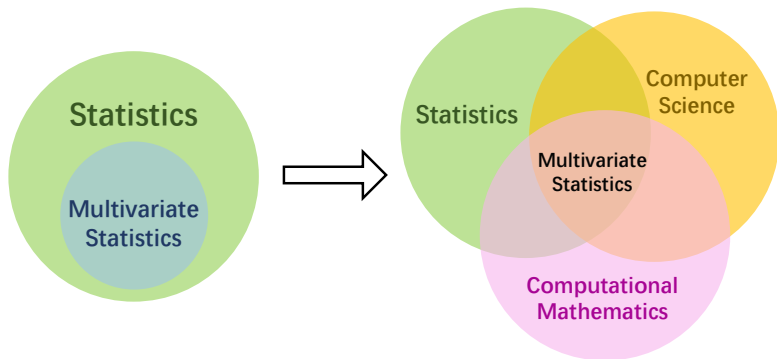
$$\begin{aligned} \mathbf{W}_+ &= \hat{\Sigma} \mathbf{W} (\sigma^2 \mathbf{I} + \mathbf{M}^{-1} \mathbf{W}^\top \hat{\Sigma} \mathbf{W})^{-1}, \\ \sigma_+^2 &= \frac{1}{d} \text{tr} \left( \hat{\Sigma} - \hat{\Sigma} \mathbf{W} \mathbf{M}^{-1} \mathbf{W}_+^\top \right). \end{aligned}$$

Note that the computational complexity of EM is  $\mathcal{O}(Ndq)$ , while the spectral decomposition in MLE requires  $\mathcal{O}(Nd^2 + d^3)$ .

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# Multivariate Statistics



# Good Luck on Finals!

Final Exam



Matrix Calculus



Linear Algebra



Multivariate Statistical Analysis  
(DATA 13004)



Machine Learning



Statistics



Optimization

