

Multivariate Statistics

Lecture 11

Fudan University

- 1 Multivariate Linear Regression
- 2 Likelihood Ratio Criterion for Testing Linear Hypotheses
- 3 Testing Equality of Means with Common Covariance
- 4 Testing Equality of Several Covariance Matrices
- 5 Testing that Several Normal Distribution are Identical
- 6 Testing that the Covariance is Proportional to a Given Matrix
- 7 Testing that the Covariance is Equal to a Give Matrix

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Univariate Least Squares

Consider scalar variables x_1, \dots, x_N drawn with expected values $\beta^\top \mathbf{z}_1, \dots, \beta^\top \mathbf{z}_N$ respectively, where each $\mathbf{z}_\alpha \in \mathbb{R}^q$ is known and we shall estimate β .

- ① If the variances of \mathbf{z}_α are the same, the least squares estimator of β is

$$\hat{\beta} = \left(\frac{1}{N} \sum_{i=1}^N \mathbf{z}_\alpha \mathbf{z}_\alpha^\top \right)^{-1} \left(\frac{1}{N} \sum_{i=1}^N x_\alpha \mathbf{z}_\alpha \right).$$

- ② If the populations are normal, the vector $\hat{\beta}$ is the maximum likelihood estimator of β .

- ③ The unbiased estimator of the common variance σ^2 is

$$s^2 = \frac{1}{N - q} \sum_{\alpha=1}^N (x_\alpha - \beta^\top \mathbf{z}_\alpha)^2.$$

- ④ Under the normality assumption, the maximum likelihood estimator of σ^2 is

$$\hat{\sigma}^2 = \frac{(N - q)s^2}{N}.$$

The Estimation in Multivariate Linear Regression

Theorem 1

Suppose \mathbf{x}_α is an observation from $\mathcal{N}_q(\mathbf{B}\mathbf{z}_\alpha, \mathbf{\Sigma})$ for $\alpha = 1, \dots, N$, where $[\mathbf{z}_1, \dots, \mathbf{z}_N] \in \mathbb{R}^{N \times q}$ of rank q is given and $N \geq p + q$, the maximum likelihood estimator of \mathbf{B} is given by

$$\hat{\mathbf{B}} = \mathbf{C}\mathbf{A}^{-1},$$

where

$$\mathbf{C} = \sum_{\alpha=1}^N \mathbf{x}_\alpha \mathbf{z}_\alpha^\top \quad \text{and} \quad \mathbf{A} = \sum_{\alpha=1}^N \mathbf{z}_\alpha \mathbf{z}_\alpha^\top;$$

the maximum likelihood estimator of $\mathbf{\Sigma}$ is give by

$$\hat{\mathbf{\Sigma}} = \frac{1}{N} \sum_{\alpha=1}^N (\mathbf{x}_\alpha - \hat{\mathbf{B}}\mathbf{z}_\alpha)(\mathbf{x}_\alpha - \hat{\mathbf{B}}\mathbf{z}_\alpha)^\top.$$

Properties of the Estimators

The likelihood function is

$$L(\mathbf{B}, \mathbf{\Sigma}) = \frac{1}{(2\pi)^{\frac{Np}{2}} (\det(\mathbf{\Sigma}))^{\frac{N}{2}}} \exp \left(-\frac{1}{2} \sum_{\alpha=1}^N (\mathbf{x}_{\alpha} - \mathbf{B}\mathbf{z}_{\alpha})^{\top} \mathbf{\Sigma}^{-1} (\mathbf{x}_{\alpha} - \mathbf{B}\mathbf{z}_{\alpha}) \right).$$

We shall find $\hat{\mathbf{H}}$ such that

$$\begin{aligned} & \sum_{\alpha=1}^N (\mathbf{x}_{\alpha} - \mathbf{B}\mathbf{z}_{\alpha})(\mathbf{x}_{\alpha} - \mathbf{B}\mathbf{z}_{\alpha})^{\top} \\ &= \sum_{\alpha=1}^N \left((\mathbf{x}_{\alpha} - \hat{\mathbf{H}}\mathbf{z}_{\alpha})(\mathbf{x}_{\alpha} - \hat{\mathbf{H}}\mathbf{z}_{\alpha})^{\top} + (\hat{\mathbf{H}}\mathbf{z}_{\alpha} - \mathbf{B}\mathbf{z}_{\alpha})(\hat{\mathbf{H}}\mathbf{z}_{\alpha} - \mathbf{B}\mathbf{z}_{\alpha})^{\top} \right). \end{aligned}$$

Lemma 1

If $\mathbf{A} \in \mathbb{R}^{p \times p}$ and $\mathbf{G} \in \mathbb{R}^{p \times p}$ are positive definite, then $\text{tr}(\mathbf{F}\mathbf{A}\mathbf{F}^{\top}\mathbf{G}) > 0$ for non-zero $\mathbf{F} \in \mathbb{R}^{p \times p}$.

Properties of the Estimators

The density then can be written as

$$\frac{1}{(2\pi)^{\frac{Np}{2}} (\det(\mathbf{\Sigma}))^{\frac{N}{2}}} \exp \left(-\frac{1}{2} \text{tr} \left(\mathbf{\Sigma}^{-1} \left(N\hat{\mathbf{\Sigma}} + (\hat{\mathbf{B}} - \mathbf{B})\mathbf{A}(\hat{\mathbf{B}} - \mathbf{B})^{\top} \right) \right) \right).$$

Then $\hat{\mathbf{B}}$ and $\hat{\mathbf{\Sigma}}$ form a sufficient set statistics for \mathbf{B} and $\mathbf{\Sigma}$.

Distribution of the Estimators

Let β_{ig} (or $\hat{\beta}_{ig}$) be the (i, g) -th element of \mathbf{B} (or $\hat{\mathbf{B}}$).

- 1 The joint distribution of $\hat{\beta}_{ig}$ is normal since the $\hat{\beta}_{ig}$ are linear combinations of the $x_{i\alpha}$.
- 2 We have $\mathbb{E}[\hat{\mathbf{B}}] = \mathbf{B}$, which means $\hat{\mathbf{B}}$ is an unbiased estimator of \mathbf{B} .
- 3 The covariance between $\hat{\beta}_i^\top$ and $\hat{\beta}_j^\top$ (two rows of $\hat{\mathbf{B}}$) is $\sigma_{ij}\mathbf{A}^{-1}$.

Distribution of the Estimators

It follows that

$$N\hat{\Sigma} = \sum_{\alpha=1}^N \mathbf{x}_{\alpha} \mathbf{x}_{\alpha}^{\top} - \hat{\mathbf{B}} \hat{\mathbf{A}} \hat{\mathbf{B}}^{\top}$$

is distributed according to $\mathcal{W}(\Sigma, N - q)$.

Theorem 2

Suppose $\mathbf{y}_1, \dots, \mathbf{y}_m$ are independent with \mathbf{y}_{α} distributed according to $\mathcal{N}(\Gamma \mathbf{w}_{\alpha}, \Phi)$, where \mathbf{w}_{α} is an r -component vector. Let $\mathbf{H} = \sum_{\alpha=1}^m \mathbf{w}_{\alpha} \mathbf{w}_{\alpha}^{\top}$ assumed non-singular, $\mathbf{G} = \sum_{\alpha=1}^m \mathbf{y}_{\alpha} \mathbf{w}_{\alpha}^{\top} \mathbf{H}^{-1}$ and

$$\mathbf{C} = \sum_{\alpha=1}^m (\mathbf{y}_{\alpha} - \mathbf{G} \mathbf{w}_{\alpha}) (\mathbf{y}_{\alpha} - \mathbf{G} \mathbf{w}_{\alpha})^{\top} = \sum_{\alpha=1}^m \mathbf{y}_{\alpha} \mathbf{y}_{\alpha}^{\top} - \mathbf{G} \mathbf{H} \mathbf{G}^{\top}.$$

Then \mathbf{C} is distributed as $\sum_{\alpha=1}^{m-r} \mathbf{u}_{\alpha} \mathbf{u}_{\alpha}^{\top}$ where $\mathbf{u}_1, \dots, \mathbf{u}_{m-r}$ are independently distributed according to $\mathcal{N}(\mathbf{0}, \Phi)$ independently of \mathbf{G} .

The Best Linear Unbiased Estimator

Let β_{ig} be the (i, g) -th entry of \mathbf{B} .

An estimator F is a linear estimator of β_{ig} if

$$F = \sum_{\alpha=1}^N \mathbf{f}_{\alpha}^{\top} \mathbf{x}_{\alpha}.$$

It is a linear unbiased estimator of β_{ig} if

$$\beta_{ig} = \mathbb{E}[F] = \mathbb{E} \left[\sum_{\alpha=1}^N \mathbf{f}_{\alpha}^{\top} \mathbf{x}_{\alpha} \right] = \sum_{\alpha=1}^N \mathbf{f}_{\alpha}^{\top} \mathbf{B} \mathbf{z}_{\alpha} = \sum_{\alpha=1}^N \sum_{j=1}^p \sum_{h=1}^q f_{j\alpha} \beta_{jh} z_{h\alpha},$$

is an identity in \mathbf{B} , that is, if

$$\sum_{\alpha=1}^N f_{j\alpha} z_{h\alpha} = \begin{cases} 1, & j = i, h = g, \\ 0, & \text{otherwise.} \end{cases}$$

The Best Linear Unbiased Estimator

A linear unbiased estimator F is best if it has minimum variance over all linear unbiased estimators; that is, if $\mathbb{E}[(F - \beta_{ig})^2] \leq \mathbb{E}[(G - \beta_{ig})^2]$ for $G = \sum_{\alpha=1}^N \mathbf{g}_{\alpha}^{\top} \mathbf{x}_{\alpha}$ and $\mathbb{E}[G] = \beta_{ig}$.

The least squares estimator $\hat{\mathbf{B}}$ is the best linear unbiased estimator of \mathbf{B} .

- 1 Let $\tilde{\beta}_{ig} = \sum_{\alpha=1}^N \sum_{j=1}^p f_{j\alpha} x_{j\alpha}$ be arbitrary unbiased estimator of β_{ig} .
- 2 Then we have

$$\begin{aligned} & \mathbb{E} \left[(\tilde{\beta}_{ig} - \beta_{ig})^2 \right] \\ &= \mathbb{E} \left[(\hat{\beta}_{ig} - \beta_{ig})^2 \right] + 2\mathbb{E} \left[(\hat{\beta}_{ig} - \beta_{ig})(\tilde{\beta}_{ig} - \hat{\beta}_{ig}) \right] + \mathbb{E} \left[(\tilde{\beta}_{ig} - \hat{\beta}_{ig})^2 \right] \\ &= \mathbb{E} \left[(\hat{\beta}_{ig} - \beta_{ig})^2 \right] + \mathbb{E} \left[(\tilde{\beta}_{ig} - \hat{\beta}_{ig})^2 \right] \\ &\geq \mathbb{E} \left[(\hat{\beta}_{ig} - \beta_{ig})^2 \right]. \end{aligned}$$

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Likelihood Ratio Criteria

We partition

$$\mathbf{B} = [\mathbf{B}_1 \quad \mathbf{B}_2]$$

so that \mathbf{B}_1 has q_1 columns and \mathbf{B}_2 has q_2 columns.

We shall derive the likelihood ratio criterion for testing the hypothesis

$$H : \mathbf{B}_1 = \mathbf{B}_1^*,$$

where \mathbf{B}_1^* is a given matrix.

Likelihood Ratio Criteria

The maximum of the likelihood function L for the sample $\mathbf{x}_1, \dots, \mathbf{x}_N$ is

$$\max_{\mathbf{B} \in \mathbb{R}^{p \times q}, \mathbf{\Sigma} \in \mathbb{S}_p^{++}} L(\mathbf{B}, \mathbf{\Sigma}) = (2\pi)^{-\frac{pN}{2}} \det(\hat{\mathbf{\Sigma}}_{\Omega})^{-\frac{N}{2}} \exp\left(-\frac{pN}{2}\right),$$

where

$$\hat{\mathbf{\Sigma}}_{\Omega} = \frac{1}{N} \sum_{\alpha=1}^N (\mathbf{x}_{\alpha} - \hat{\mathbf{B}}\mathbf{z}_{\alpha})(\mathbf{x}_{\alpha} - \hat{\mathbf{B}}\mathbf{z}_{\alpha})^{\top}.$$

Likelihood Ratio Criteria

To find the maximum of the likelihood function with restricted to $\mathbf{B}_1 = \mathbf{B}_1^*$, we partition

$$\mathbf{z}_\alpha = \begin{bmatrix} \mathbf{z}_\alpha^{(1)} \\ \mathbf{z}_\alpha^{(2)} \end{bmatrix}.$$

Let $\mathbf{y}_\alpha = \mathbf{x}_\alpha - \mathbf{B}_1^* \mathbf{z}_\alpha^{(1)}$, then $\mathbf{y}_\alpha \sim \mathcal{N}(\mathbf{B}_2 \mathbf{z}_\alpha^{(2)}, \boldsymbol{\Sigma})$.

Similar to the derivation of $\hat{\mathbf{B}}$, the estimator of \mathbf{B}_2 is

$$\hat{\mathbf{B}}_{2\omega} = \sum_{\alpha=1}^N \mathbf{y}_\alpha \mathbf{z}_\alpha^{(2)\top} \mathbf{A}_{22}^{-1} = \sum_{\alpha=1}^N (\mathbf{x}_\alpha - \mathbf{B}_1^* \mathbf{z}_\alpha^{(1)}) \mathbf{z}_\alpha^{(2)\top} \mathbf{A}_{22}^{-1} = (\mathbf{C}_2 - \mathbf{B}_1^* \mathbf{A}_{12}) \mathbf{A}_{22}^{-1},$$

with

$$\mathbf{C} = [\mathbf{C}_1 \quad \mathbf{C}_2] \quad \text{and} \quad \mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix}.$$

Likelihood Ratio Criteria

The estimator of Σ is given by

$$\begin{aligned} N\hat{\Sigma}_{\omega} &= \sum_{\alpha=1}^N (\mathbf{y}_{\alpha} - \hat{\mathbf{B}}_{2\omega} \mathbf{z}_{\alpha}^{(2)}) (\mathbf{y}_{\alpha} - \hat{\mathbf{B}}_{2\omega} \mathbf{z}_{\alpha}^{(2)})^{\top} \\ &= \sum_{\alpha=1}^N \mathbf{y}_{\alpha} \mathbf{y}_{\alpha}^{\top} - \hat{\mathbf{B}}_{2\omega} \mathbf{A}_{22}^{-1} \hat{\mathbf{B}}_{2\omega}^{\top} \\ &= \sum_{\alpha=1}^N (\mathbf{x}_{\alpha} - \mathbf{B}_1^* \mathbf{z}_{\alpha}^{(1)}) (\mathbf{x}_{\alpha} - \mathbf{B}_1^* \mathbf{z}_{\alpha}^{(1)})^{\top} - \hat{\mathbf{B}}_{2\omega} \mathbf{A}_{22}^{-1} \hat{\mathbf{B}}_{2\omega}^{\top}. \end{aligned}$$

Thus the maximum of the likelihood function over ω is

$$(2\pi)^{-\frac{pN}{2}} \det(\hat{\Sigma}_{\omega})^{-\frac{N}{2}} \exp\left(-\frac{pN}{2}\right).$$

The Likelihood Ratio Criterion for Testing

The likelihood ratio criterion for testing H is

$$\lambda = \frac{(\det(\hat{\mathbf{\Sigma}}_{\Omega}))^{\frac{N}{2}}}{(\det(\hat{\mathbf{\Sigma}}_{\omega}))^{\frac{N}{2}}}.$$

In testing H , one rejects the hypothesis if $\lambda < \lambda_0$ where λ_0 is a suitably chosen number.

The likelihood ratio criterion for testing the null hypothesis $\mathbf{B}_1 = \mathbf{0}$ is invariant with respect to transformations $\mathbf{x}_{\alpha}^* = \mathbf{D}\mathbf{x}_{\alpha}$ for $\alpha = 1, \dots, N$ and non-singular \mathbf{D} .

The Likelihood Ratio Criterion for Testing

Let $u = \lambda^{2/N}$. When $\mathbf{B}_1 = \mathbf{B}_1^*$, the criterion u has the distribution of

$$u = \frac{\det(\mathbf{G})}{\det(\mathbf{G} + \mathbf{H})},$$

where $\mathbf{G} \sim \mathcal{W}(\boldsymbol{\Sigma}, N - q)$, $\mathbf{H} \sim \mathcal{W}(\boldsymbol{\Sigma}, q_1)$, and \mathbf{G} and \mathbf{H} are independent; the criterion u also has the distribution of

$$u = \prod_{i=1}^p v_i,$$

where v_1, \dots, v_p are independent and each of them has the beta density

$$B\left(v \mid \frac{n+1-i}{2}, \frac{m-1}{2}\right) = \frac{\Gamma\left(\frac{n+m+1-i}{2}\right)}{\Gamma\left(\frac{n+1-i}{2}\right) \Gamma\left(\frac{m}{2}\right)} v^{\frac{n+1-i}{2}-1} (1-v)^{\frac{1}{2}m-1},$$

where $m = q_1$.

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Testing Equality of Means with Common Covariance

Let $\mathbf{x}_\alpha^{(g)}$ be an observation from the g -th population $\mathcal{N}(\boldsymbol{\mu}^{(g)}, \boldsymbol{\Sigma})$ for $\alpha = 1, \dots, N_g$, $g = 1, \dots, q$.

We wish to test the hypothesis

$$H_0 : \boldsymbol{\mu}_1 = \dots = \boldsymbol{\mu}_g.$$

The likelihood function is

$$L = \prod_{g=1}^q \frac{1}{(2\pi)^{\frac{\rho N_g}{2}} (\det(\boldsymbol{\Sigma}))^{\frac{N_g}{2}}} \exp \left(-\frac{1}{2} \sum_{\alpha=1}^{N_g} (\mathbf{x}_\alpha^{(g)} - \boldsymbol{\mu}^{(g)})^\top \boldsymbol{\Sigma}^{-1} (\mathbf{x}_\alpha^{(g)} - \boldsymbol{\mu}^{(g)}) \right).$$

- 1 The space Ω is the parameter space in which $\boldsymbol{\Sigma}$ is positive definite and each $\boldsymbol{\mu}^{(g)}$ is any vector.
- 2 The space ω is the parameter space in which $\boldsymbol{\mu}_1 = \dots = \boldsymbol{\mu}_g$ (positive definite) and $\boldsymbol{\Sigma}$ is any positive definite matrix.

Testing Equality of Means with Common Covariance

Let

$$N = \sum_{g=1}^q N_g, \quad \mathbf{A}_g = \sum_{\alpha=1}^{N_g} (\mathbf{x}_{\alpha}^{(g)} - \bar{\mathbf{x}}^{(g)}) (\mathbf{x}_{\alpha}^{(g)} - \bar{\mathbf{x}}^{(g)})^{\top}, \quad \mathbf{A} = \sum_{g=1}^q \mathbf{A}_g,$$

and

$$\mathbf{B} = \sum_{g=1}^q \sum_{\alpha=1}^{N_g} (\mathbf{x}_{\alpha}^{(g)} - \bar{\mathbf{x}}) (\mathbf{x}_{\alpha}^{(g)} - \bar{\mathbf{x}})^{\top}.$$

The maximum likelihood estimators of $\boldsymbol{\mu}^{(g)}$ and $\boldsymbol{\Sigma}$ in Ω are given by

$$\hat{\boldsymbol{\mu}}_{\Omega}^{(g)} = \bar{\mathbf{x}}^{(g)} \quad \text{and} \quad \hat{\boldsymbol{\Sigma}}_{\Omega} = \frac{1}{N} \mathbf{A}.$$

The maximum likelihood estimators of $\boldsymbol{\mu}^{(g)}$ and $\boldsymbol{\Sigma}$ in ω are given by

$$\hat{\boldsymbol{\mu}}_{\omega}^{(g)} = \bar{\mathbf{x}} \quad \text{and} \quad \hat{\boldsymbol{\Sigma}}_{\omega} = \frac{1}{N} \mathbf{B}.$$

Testing Equality of Means with Common Covariance

Lemma 2

If $\mathbf{D} \in \mathbb{R}^{p \times p}$ is positive definite, the maximum of

$$f(\mathbf{G}) = -N \ln \det(\mathbf{G}) - \text{tr}(\mathbf{G}^{-1} \mathbf{D})$$

with respect to positive definite matrices \mathbf{G} exists, occurs at $\mathbf{G} = \frac{1}{N} \mathbf{D}$.

Testing Equality of Means with Common Covariance

The likelihood ratio criterion for testing H_0 is

$$\lambda_0 = \frac{(\det(\hat{\boldsymbol{\Sigma}}_{\Omega}))^{\frac{N}{2}}}{(\det(\hat{\boldsymbol{\Sigma}}_{\omega}))^{\frac{N}{2}}} = \frac{(\det(\mathbf{A}))^{\frac{N}{2}}}{(\det(\mathbf{B}))^{\frac{N}{2}}}.$$

The critical region is

$$\lambda_0 \leq \lambda_0(\epsilon),$$

where $\lambda_0(\epsilon)$ is defined so that above inequality holds with probability ϵ when H_0 is true.

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Testing Equality of Several Covariance Matrices

Let $\mathbf{x}_\alpha^{(g)}$ be an observation from the g -th population $\mathcal{N}(\boldsymbol{\mu}^{(g)}, \boldsymbol{\Sigma}_g)$ for $\alpha = 1, \dots, N_g$, $g = 1, \dots, q$.

We wish to test the hypothesis

$$H_1 : \boldsymbol{\Sigma}_1 = \dots = \boldsymbol{\Sigma}_g.$$

The likelihood function is

$$L = \prod_{g=1}^q \frac{1}{(2\pi)^{\frac{\rho N_g}{2}} (\det(\boldsymbol{\Sigma}_g))^{\frac{N_g}{2}}} \exp \left(-\frac{1}{2} \sum_{\alpha=1}^{N_g} (\mathbf{x}_\alpha^{(g)} - \boldsymbol{\mu}^{(g)})^\top \boldsymbol{\Sigma}_g^{-1} (\mathbf{x}_\alpha^{(g)} - \boldsymbol{\mu}^{(g)}) \right).$$

- 1 The space Ω is the parameter space in which each $\boldsymbol{\Sigma}_g$ is positive definite and $\boldsymbol{\mu}^{(g)}$ are any vector.
- 2 The space ω is the parameter space in which $\boldsymbol{\Sigma}_1 = \dots = \boldsymbol{\Sigma}_g$ (positive definite) and $\boldsymbol{\mu}^{(g)}$ are any vector.

Testing Equality of Several Covariance Matrices

Let

$$N = \sum_{g=1}^q N_g, \quad \mathbf{A}_g = \sum_{\alpha=1}^{N_g} (\mathbf{x}_{\alpha}^{(g)} - \bar{\mathbf{x}}^{(g)}) (\mathbf{x}_{\alpha}^{(g)} - \bar{\mathbf{x}}^{(g)})^{\top} \quad \text{and} \quad \mathbf{A} = \sum_{g=1}^q \mathbf{A}_g.$$

The maximum likelihood estimators of $\boldsymbol{\mu}^{(g)}$ and $\boldsymbol{\Sigma}_g$ in Ω are given by

$$\hat{\boldsymbol{\mu}}_{\Omega}^{(g)} = \bar{\mathbf{x}}^{(g)} \quad \text{and} \quad \hat{\boldsymbol{\Sigma}}_{g\Omega} = \frac{1}{N_g} \mathbf{A}_g.$$

The maximum likelihood estimators of $\boldsymbol{\mu}^{(g)}$ and $\boldsymbol{\Sigma}_g$ in ω are given by

$$\hat{\boldsymbol{\mu}}_{\Omega}^{(g)} = \bar{\mathbf{x}}^{(g)} \quad \text{and} \quad \hat{\boldsymbol{\Sigma}}_{g\Omega} = \frac{1}{N} \mathbf{A}.$$

Testing Equality of Several Covariance Matrices

The likelihood ratio criterion for testing H_1 is

$$\lambda_1 = \frac{\prod_{g=1}^q (\det(\hat{\Sigma}_{g\Omega}))^{\frac{N_g}{2}}}{(\det(\hat{\Sigma}_\omega))^{\frac{N}{2}}} = \frac{\prod_{g=1}^q (\det(\mathbf{A}_g))^{\frac{N_g}{2}}}{(\det(\mathbf{A}))^{\frac{N}{2}}} \cdot \frac{N^{\frac{pN}{2}}}{\prod_{g=1}^q N_g^{\frac{pN_g}{2}}}.$$

The critical region is

$$\lambda_1 \leq \lambda_1(\epsilon),$$

where $\lambda_1(\epsilon)$ is defined so that above inequality holds with probability ϵ when H_1 is true.

Testing Equality of Several Covariance Matrices

Bartlett (1937a) has suggested using the numbers of degrees of freedom. Except for constants, the statistic is

$$V_1 = \frac{\prod_{g=1}^q (\det(\mathbf{A}_g))^{\frac{n_g}{2}}}{(\det(\mathbf{A}))^{\frac{n}{2}}},$$

where $n_g = N_g - 1$ and $n = N - q$.

The statistic is invariant with respect to linear transformation

$$\mathbf{x}^{*(g)} = \mathbf{C}\mathbf{x}^{(g)} + \boldsymbol{\nu}^{(g)}.$$

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Testing that Several Normal Distribution are Identical

Let $\mathbf{x}_\alpha^{(g)}$ be an observation from the g -th population $\mathcal{N}(\boldsymbol{\mu}^{(g)}, \boldsymbol{\Sigma}_g)$ for $\alpha = 1, \dots, N_g$, $g = 1, \dots, q$.

We wish to test

$$H_2 : \boldsymbol{\mu}^{(1)} = \dots = \boldsymbol{\mu}^{(q)}, \quad \boldsymbol{\Sigma}_1 = \dots = \boldsymbol{\Sigma}_q. \quad (1)$$

- 1 Let Ω be the unrestricted parameter space of $\{\boldsymbol{\mu}^{(g)}, \boldsymbol{\Sigma}_g\}_{g=1}^q$, where $\boldsymbol{\Sigma}_g$ is positive definite; and ω^* consists of the space restricted by (1).
- 2 The likelihood function is

$$L = \prod_{g=1}^q \frac{1}{(2\pi)^{\frac{pN_g}{2}} (\det(\boldsymbol{\Sigma}_g))^{\frac{N_g}{2}}} \exp\left(-\frac{1}{2} \sum_{\alpha=1}^{N_g} (\mathbf{x}_\alpha^{(g)} - \boldsymbol{\mu}^{(g)})^\top \boldsymbol{\Sigma}_g^{-1} (\mathbf{x}_\alpha^{(g)} - \boldsymbol{\mu}^{(g)})\right).$$

Testing that Several Normal Distribution are Identical

Let \mathbf{y} be an observation with density $f(\mathbf{y}; \boldsymbol{\theta})$, where $\boldsymbol{\theta}$ is a parameter vector in a space Ω .

- 1 Let H_a be the hypothesis $\boldsymbol{\theta} \in \Omega_a \subset \Omega$.
- 2 Let H_b be the hypothesis $\boldsymbol{\theta} \in \Omega_b \subset \Omega_a$ given $\boldsymbol{\theta} \in \Omega_a$.
- 3 Let H_{ab} be the hypothesis $\boldsymbol{\theta} \in \Omega_b$ given $\boldsymbol{\theta} \in \Omega$.

If the likelihood ratio criterion λ_a , λ_b and λ_{ab} for testing H_a , H_b and H_{ab} are uniquely defined for the observation vector \mathbf{y} , then we have

$$\lambda_a = \frac{\max_{\boldsymbol{\theta} \in \Omega_a} f(\mathbf{y}; \boldsymbol{\theta})}{\max_{\boldsymbol{\theta} \in \Omega} f(\mathbf{y}; \boldsymbol{\theta})}, \quad \lambda_b = \frac{\max_{\boldsymbol{\theta} \in \Omega_b} f(\mathbf{y}; \boldsymbol{\theta})}{\max_{\boldsymbol{\theta} \in \Omega_a} f(\mathbf{y}; \boldsymbol{\theta})} \quad \text{and} \quad \lambda_{ab} = \frac{\max_{\boldsymbol{\theta} \in \Omega_b} f(\mathbf{y}; \boldsymbol{\theta})}{\max_{\boldsymbol{\theta} \in \Omega} f(\mathbf{y}; \boldsymbol{\theta})}.$$

Hence, $\lambda_{ab} = \lambda_a \lambda_b$.

Testing that Several Normal Distribution are Identical

Recall that

- ① $H_1 : \boldsymbol{\Sigma}_1 = \cdots = \boldsymbol{\Sigma}_g$;
- ② $H_0 : \boldsymbol{\mu}_1 = \cdots = \boldsymbol{\mu}_g$ (common covariance matrix);
- ③ $H_2 : \boldsymbol{\mu}^{(1)} = \cdots = \boldsymbol{\mu}^{(q)}, \quad \boldsymbol{\Sigma}_1 = \cdots = \boldsymbol{\Sigma}_q$.

Then we have

$$\begin{aligned}\lambda_2 = \lambda_1 \lambda_0 &= \frac{\prod_{g=1}^q (\det(\mathbf{A}_g))^{\frac{N_g}{2}}}{(\det(\mathbf{A}))^{\frac{N}{2}}} \cdot \frac{N^{\frac{pN}{2}}}{\prod_{g=1}^q N_g^{\frac{pN_g}{2}}} \cdot \frac{(\det(\mathbf{A}))^{\frac{N}{2}}}{(\det(\mathbf{B}))^{\frac{N}{2}}} \\ &= \left(\prod_{g=1}^q \frac{(\det(\mathbf{A}_g))^{\frac{N_g}{2}}}{N_g^{\frac{pN_g}{2}}} \right) \frac{N^{\frac{pN}{2}}}{(\det(\mathbf{B}))^{\frac{N}{2}}}.\end{aligned}$$

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Testing that the Covariance is Proportional to \mathbf{I}

We use a sample of p -component vectors $\mathbf{x}_1, \dots, \mathbf{x}_N$ from $\mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ to test the hypothesis

$$H : \boldsymbol{\Sigma} = \sigma^2 \mathbf{I},$$

where σ^2 is not specified.

The hypothesis H is a combination of the hypothesis:

- ① $H_1 : \boldsymbol{\Sigma}$ is diagonal;
- ② $H_2 : \text{The diagonal elements of } \boldsymbol{\Sigma} \text{ are equal given that } \boldsymbol{\Sigma} \text{ is diagonal.}$

Testing that the Covariance is Proportional to \mathbf{I}

The criterion for H_1 is

$$\lambda_1 = \frac{(\det(\mathbf{A}))^{\frac{N}{2}}}{\prod_{i=1}^p a_{ii}^{\frac{N}{2}}},$$

where $\mathbf{A} = \sum_{\alpha=1}^N (\mathbf{x}_{\alpha} - \bar{\mathbf{x}})(\mathbf{x}_{\alpha} - \bar{\mathbf{x}})^{\top}$ and a_{ij} is the (i, j) -th element of \mathbf{A} .

Testing that the Covariance is Proportional to \mathbf{I}

We can find λ_2 by considering test equality of several covariance matrices.

- 1 View the i th component of \mathbf{x}_α as the α -th observation from the i -th population.
- 2 p here is q in the section of testing equality of several covariance matrices; N here is N_g there; pN here is N there.
- 3 Thus, we have

$$\lambda_2 = \frac{\prod_{i=1}^p \left(\sum_{\alpha=1}^N (x_{i\alpha} - \bar{x}_i)^2 \right)^{\frac{N}{2}}}{\left(\sum_{i=1}^p \sum_{\alpha=1}^N (x_{i\alpha} - \bar{x}_i)^2 / p \right)^{\frac{pN}{2}}} = \frac{\prod_{i=1}^p a_{ii}^{\frac{N}{2}}}{(\text{tr}(\mathbf{A})/p)^{\frac{pN}{2}}}.$$

Testing that the Covariance is Proportional to \mathbf{I}

Thus the criterion for H is

$$\lambda_1 \lambda_2 = \frac{(\det(\mathbf{A}))^{\frac{N}{2}}}{\prod_{i=1}^p a_{ii}^{\frac{N}{2}}} \cdot \frac{\prod_{i=1}^p a_{ii}^{\frac{N}{2}}}{(\text{tr}(\mathbf{A})/p)^{\frac{pN}{2}}} = \frac{(\det(\mathbf{A}))^{\frac{N}{2}}}{(\text{tr}(\mathbf{A})/p)^{\frac{pN}{2}}}.$$

Testing that the Covariance is Proportional to Ψ_0

For the hypothesis

$$\Sigma = \sigma^2 \Psi_0,$$

let \mathbf{C} be matrix such that

$$\mathbf{C}\Psi_0\mathbf{C}^\top = \mathbf{I},$$

$$\mathbf{x}_\alpha^* = \mathbf{C}\mathbf{x}, \mu^* = \mathbf{C}\mu \text{ and } \Sigma^* = \mathbf{C}\Sigma\mathbf{C}^\top.$$

Then hypothesis is transformed into $\Sigma^* = \sigma^2 \Psi_0$ and the criterion is

$$\frac{(\det(\mathbf{A}\Psi_0^{-1}))^{\frac{N}{2}}}{(\text{tr}(\mathbf{A}\Psi_0^{-1})/p)^{\frac{pN}{2}}}.$$

- 1 Multivariate Linear Regression
- 2 Likelihood Ratio Criterion for Testing Linear Hypotheses
- 3 Testing Equality of Means with Common Covariance
- 4 Testing Equality of Several Covariance Matrices
- 5 Testing that Several Normal Distribution are Identical
- 6 Testing that the Covariance is Proportional to a Given Matrix
- 7 Testing that the Covariance is Equal to a Give Matrix

Testing that the Covariance is Equal to a Give Matrix

We use a sample of p -component vectors $\mathbf{x}_1, \dots, \mathbf{x}_N$ from $\mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ to test the hypothesis

$$\boldsymbol{\Sigma} = \mathbf{I}.$$

The likelihood ratio criterion is

$$\lambda_1 = \frac{\max_{\boldsymbol{\mu} \in \mathbb{R}^p} L(\boldsymbol{\mu}, \mathbf{I})}{\max_{\boldsymbol{\mu} \in \mathbb{R}^p, \boldsymbol{\Sigma} \in \mathbb{S}_p^{++}} L(\boldsymbol{\mu}, \boldsymbol{\Sigma})},$$

where

$$L(\boldsymbol{\mu}, \boldsymbol{\Sigma}) = \frac{1}{(2\pi)^{\frac{pN}{2}} (\det(\boldsymbol{\Sigma}))^{\frac{N}{2}}} \exp \left(-\frac{1}{2} \sum_{\alpha=1}^N (\mathbf{x}_\alpha - \boldsymbol{\mu})^\top \boldsymbol{\Sigma}^{-1} (\mathbf{x}_\alpha - \boldsymbol{\mu}) \right).$$

Testing that the Covariance is Equal to a Give Matrix

Then we have

$$\begin{aligned}\lambda_1 &= \frac{(2\pi)^{-\frac{pN}{2}} \exp\left(-\frac{1}{2} \sum_{\alpha=1}^N (\mathbf{x}_\alpha - \bar{\mathbf{x}})^\top (\mathbf{x}_\alpha - \bar{\mathbf{x}})\right)}{(2\pi)^{-\frac{pN}{2}} \det\left(\frac{1}{N} \mathbf{A}\right)^{-\frac{N}{2}} \exp\left(-\frac{pN}{2}\right)} \\ &= \left(\frac{e}{N}\right)^{\frac{pN}{2}} (\det(\mathbf{A}))^{\frac{N}{2}} \exp\left(-\frac{\text{tr}(\mathbf{A})}{2}\right),\end{aligned}$$

where $\mathbf{A} = \sum_{\alpha=1}^N (\mathbf{x}_\alpha - \bar{\mathbf{x}})(\mathbf{x}_\alpha - \bar{\mathbf{x}})^\top$.

Testing that the Covariance is Equal to a Give Matrix

To test the hypothesis

$$H_1 : \mathbf{\Sigma} = \mathbf{\Sigma}_0.$$

The likelihood ratio criterion is

$$\lambda_1 = \left(\frac{e}{N}\right)^{\frac{pN}{2}} (\det(\mathbf{A}\mathbf{\Sigma}_0^{-1}))^{\frac{N}{2}} \exp\left(-\frac{\text{tr}(\mathbf{A}\mathbf{\Sigma}_0^{-1})}{2}\right).$$

Testing that the Mean and the Covariance Simultaneously

Theorem 3

Given the p -component observation vectors $\mathbf{x}_1, \dots, \mathbf{x}_N$, from $\mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, the likelihood ratio criterion for testing the hypothesis

$$H : \boldsymbol{\mu} = \boldsymbol{\mu}_0, \quad \boldsymbol{\Sigma} = \boldsymbol{\Sigma}_0$$

is

$$\lambda = \left(\frac{e}{N} \right)^{\frac{pN}{2}} (\det(\mathbf{A}\boldsymbol{\Sigma}_0^{-1}))^{\frac{N}{2}} \exp \left(-\frac{1}{2} \left(\text{tr}(\mathbf{A}\boldsymbol{\Sigma}_0^{-1}) + N(\bar{\mathbf{x}} - \boldsymbol{\mu}_0)^\top \boldsymbol{\Sigma}_0^{-1} (\bar{\mathbf{x}} - \boldsymbol{\mu}_0) \right) \right),$$

where $\mathbf{A} = \sum_{\alpha=1}^N (\mathbf{x}_\alpha - \bar{\mathbf{x}})(\mathbf{x}_\alpha - \bar{\mathbf{x}})^\top$.

We consider hypotheses

- ① $H_1 : \boldsymbol{\Sigma} = \boldsymbol{\Sigma}_0$;
- ② $H_2 : \boldsymbol{\mu} = \boldsymbol{\mu}_0$ given $\boldsymbol{\Sigma} = \boldsymbol{\Sigma}_0$.