Multivariate Statistics

Lecture 07

Fudan University

Outline

Outline

1. Noncentral Chi-Squared Distribution

2. Hypothesis Testing for the Mean (Covariance is Known)

3. The Generalized T^2-Statistic

Noncentral Chi-Squared Distribution

If $y_1, \dots, y_k$ are independent and each y_i is distributed according to the noncentral chi-squared distribution with n_i degrees of freedom and noncentrality parameter λ_i, then

$$\sum_{i=1}^{k} y_i \sim \chi^2_{n_1+\cdots+n_k}\left(\sum_{i=1}^{k} \lambda_i\right).$$

Theorem 1

If the n-component vector $\mathbf{y}$ is distributed according to $\mathcal{N}(\boldsymbol{\nu}, \mathbf{T})$ with $\mathbf{T} \succ \mathbf{0}$, then

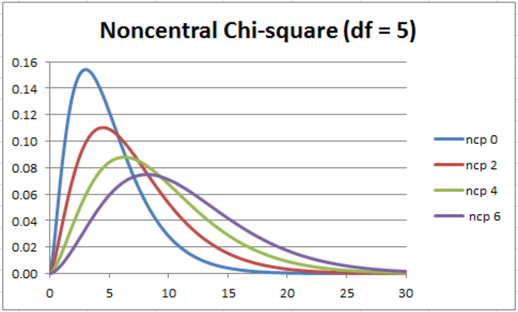
$$\mathbf{y}^\top \mathbf{T}^{-1} \mathbf{y}$$

is distributed according to the noncentral χ^2-distribution with n degrees of freedom and noncentral parameter $\boldsymbol{\nu}^\top \mathbf{T}^{-1} \boldsymbol{\nu}$. If $\boldsymbol{\nu} = \mathbf{0}$, the distribution is the central χ^2-distribution.

Noncentral Chi-Squared Distribution

Let $\mathbf{y} \sim \mathcal{N}_p(\boldsymbol{\lambda}, \mathbf{I})$, then $v = \mathbf{y}^\top \mathbf{y}$ is distributed according to the noncentral χ^2-distribution with p degrees of freedom and noncentral parameter $\tau^2 = \boldsymbol{\lambda}^\top \boldsymbol{\lambda}$. The probability density function is

$$f(v; p, \tau^2) = \begin{cases} \dfrac{\exp\left(-\frac{1}{2}(\tau^2 + v)\right) v^{\frac{p}{2}-1}}{2^{\frac{p}{2}}\sqrt{\pi}} \displaystyle\sum_{\beta=0}^{\infty} \dfrac{\tau^{2\beta} v^{\beta} \Gamma\left(\beta + \frac{1}{2}\right)}{(2\beta)!\,\Gamma\left(\frac{p}{2} + \beta\right)} & v > 0, \\ 0, & \text{otherwise.} \end{cases}$$



Noncentral Chi-Squared Distribution

Theorem 1

If the n-component vector $\mathbf{y}$ is distributed according to $\mathcal{N}(\boldsymbol{\nu}, \mathbf{T})$ with $\mathbf{T} \succ \mathbf{0}$, then

$$\mathbf{y}^\top \mathbf{T}^{-1} \mathbf{y}$$

is distributed according to the noncentral χ^2-distribution with n degrees of freedom and noncentral parameter $\boldsymbol{\nu}^\top \mathbf{T}^{-1} \boldsymbol{\nu}$. If $\boldsymbol{\nu} = \mathbf{0}$, the distribution is the central χ^2-distribution.

For the sample mean $\bar{\mathbf{x}} \sim \mathcal{N}_p\left(\boldsymbol{\mu}, \frac{1}{N}\boldsymbol{\Sigma}\right)$, we have $\sqrt{N}(\bar{\mathbf{x}} - \boldsymbol{\mu}) \sim \mathcal{N}_p(\mathbf{0}, \boldsymbol{\Sigma})$.

It follows from the theorem that

$$N(\bar{\mathbf{x}} - \boldsymbol{\mu})^\top \boldsymbol{\Sigma}^{-1} (\bar{\mathbf{x}} - \boldsymbol{\mu})$$

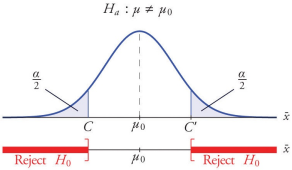
has a (central) χ^2-distribution with p degrees of freedom.

Outline

Hypothesis Testing for the Mean (Covariance is Known)

In the univariate case, the difference between the sample mean and the population mean is normally distributed. We consider

$$z = \frac{\sqrt{N}}{\sigma}(\bar{x} - \mu_0).$$



What about multivariate case?

1. For $\alpha = 0.05$ and $p = 100$, we have $(1 - \alpha)^p \approx 0.006$.
2. For $\alpha \approx 0.0005$ and $p = 100$, we have $(1 - \alpha)^p > 0.95$.

Hypothesis Testing for the Mean (Covariance is Known)

What about multivariate case?

$$\frac{\sqrt{N}}{\sigma}(\bar{x} - \mu_0) \implies \frac{N}{\sigma^2}(\bar{x} - \mu_0)^2 \implies N(\bar{x} - \mu_0)^\top \mathbf{\Sigma}^{-1}(\bar{x} - \mu_0)$$

Noncentral Chi-Squared Distribution

Theorem 1

If the n-component vector $\mathbf{y}$ is distributed according to $\mathcal{N}(\boldsymbol{\nu}, \mathbf{T})$ with $\mathbf{T} \succ \mathbf{0}$, then

$$\mathbf{y}^\top \mathbf{T}^{-1} \mathbf{y}$$

is distributed according to the noncentral χ^2-distribution with n degrees of freedom and noncentral parameter $\boldsymbol{\nu}^\top \mathbf{T}^{-1} \boldsymbol{\nu}$. If $\boldsymbol{\nu} = \mathbf{0}$, the distribution is the central χ^2-distribution.

For the sample mean $\bar{\mathbf{x}} \sim \mathcal{N}_p\left(\boldsymbol{\mu}, \frac{1}{N}\boldsymbol{\Sigma}\right)$, we have $\sqrt{N}(\bar{\mathbf{x}} - \boldsymbol{\mu}) \sim \mathcal{N}_p(\mathbf{0}, \boldsymbol{\Sigma})$.

It follows from the theorem that

$$N(\bar{\mathbf{x}} - \boldsymbol{\mu})^\top \boldsymbol{\Sigma}^{-1} (\bar{\mathbf{x}} - \boldsymbol{\mu})$$

has a (central) χ^2-distribution with p degrees of freedom.

Hypothesis Testing for the Mean (Covariance is Known)

Let $\chi_p^2(\alpha)$ be the number such that

$$\Pr\left\{N(\bar{\mathbf{x}}-\boldsymbol{\mu})^\top \boldsymbol{\Sigma}^{-1}(\bar{\mathbf{x}}-\boldsymbol{\mu}) > \chi_p^2(\alpha)\right\} = \alpha.$$

To test the hypothesis that $\boldsymbol{\mu} = \boldsymbol{\mu}_0$ where $\boldsymbol{\mu}_0$ is a specified vector, we use as our rejection region (critical region)

$$N(\bar{\mathbf{x}}-\boldsymbol{\mu})^\top \boldsymbol{\Sigma}^{-1}(\bar{\mathbf{x}}-\boldsymbol{\mu}) > \chi_p^2(\alpha).$$

Hypothesis Testing for the Mean (Covariance is Known)

Consider the following statement made on the basis of a sample with mean $\bar{\mathbf{x}}$: "The mean of the distribution satisfies

$$N(\bar{\mathbf{x}} - \boldsymbol{\mu}^*)^\top \boldsymbol{\Sigma}^{-1}(\bar{\mathbf{x}} - \boldsymbol{\mu}^*) \leq \chi_p^2(\alpha).$$

as an inequality on $\boldsymbol{\mu}^*$." This statement is true with probability $1 - \alpha$.

Thus, the set of $\boldsymbol{\mu}^*$ satisfying above inequality is a confidence region for $\boldsymbol{\mu}$ with confidence $1 - \alpha$.

Two-Sample Problems

We suppose

1. a sample $\{\mathbf{x}_\alpha^{(1)}\}$, $i = 1, \dots, N_1$ from the distribution $\mathcal{N}(\boldsymbol{\mu}^{(1)}, \boldsymbol{\Sigma})$;
2. a sample $\{\mathbf{x}_\alpha^{(2)}\}$, $i = 1, \dots, N_2$ from the distribution $\mathcal{N}(\boldsymbol{\mu}^{(2)}, \boldsymbol{\Sigma})$.

Then the two sample means

$$\bar{\mathbf{x}}^{(1)} = \frac{1}{N_1} \sum_{\alpha=1}^{N_1} \mathbf{x}_\alpha^{(1)} \sim \mathcal{N}\left(\boldsymbol{\mu}^{(1)}, \frac{1}{N_1}\boldsymbol{\Sigma}\right)$$

and

$$\bar{\mathbf{x}}^{(2)} = \frac{1}{N_2} \sum_{\alpha=1}^{N_2} \mathbf{x}_\alpha^{(2)} \sim \mathcal{N}\left(\boldsymbol{\mu}^{(2)}, \frac{1}{N_2}\boldsymbol{\Sigma}\right).$$

are independent.

Two-Sample Problems

Then we have

$$\mathbf{y} = \bar{\mathbf{x}}^{(1)} - \bar{\mathbf{x}}^{(2)} = \begin{bmatrix} \mathbf{I} & -\mathbf{I} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{x}}^{(1)} \\ \bar{\mathbf{x}}^{(2)} \end{bmatrix}, \quad \begin{bmatrix} \bar{\mathbf{x}}^{(1)} \\ \bar{\mathbf{x}}^{(2)} \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} \boldsymbol{\mu}^{(1)} \\ \boldsymbol{\mu}^{(2)} \end{bmatrix}, \begin{bmatrix} \frac{1}{N_1}\boldsymbol{\Sigma} & \mathbf{0} \\ \mathbf{0} & \frac{1}{N_2}\boldsymbol{\Sigma} \end{bmatrix} \right)$$

and

$$\mathbf{y} \sim \mathcal{N}\left(\boldsymbol{\nu}, \left(\frac{1}{N_1} + \frac{1}{N_2} \right) \boldsymbol{\Sigma} \right) \qquad \text{where} \qquad \boldsymbol{\nu} = \boldsymbol{\mu}^{(1)} - \boldsymbol{\mu}^{(2)}.$$

Thus

$$\frac{N_1 N_2}{N_1 + N_2} (\mathbf{y} - \boldsymbol{\nu})^\top \boldsymbol{\Sigma}^{-1} (\mathbf{y} - \boldsymbol{\nu}) \le \chi_p^2(\alpha).$$

is a confidence region for the difference $\boldsymbol{\nu}$ of the two mean vectors, vectors, and a critical region for testing the hypothesis $\boldsymbol{\mu}^{(1)} = \boldsymbol{\mu}^{(2)}$ is given by

$$\frac{N_1 N_2}{N_1 + N_2} (\boldsymbol{\mu}^{(1)} - \boldsymbol{\mu}^{(2)})^\top \boldsymbol{\Sigma}^{-1} (\boldsymbol{\mu}^{(1)} - \boldsymbol{\mu}^{(2)}) > \chi_p^2(\alpha).$$

Outline

Student t-Distribution

Let $x_1, \ldots, x_N$ be independently and identically drawn from the distribution $\mathcal{N}(\mu, \sigma^2)$, then the random variable

$$t = \frac{\bar{x} - \mu}{s/\sqrt{N}}$$

has student t-distribution with $N - 1$ degrees of freedom, where

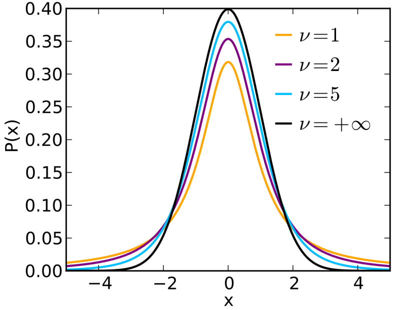
$$\bar{x} = \frac{1}{N} \sum_{\alpha=1}^{N} x_\alpha \qquad \text{and} \qquad s^2 = \frac{1}{N-1} \sum_{\alpha=1}^{N} (x_\alpha - \bar{x})^2.$$

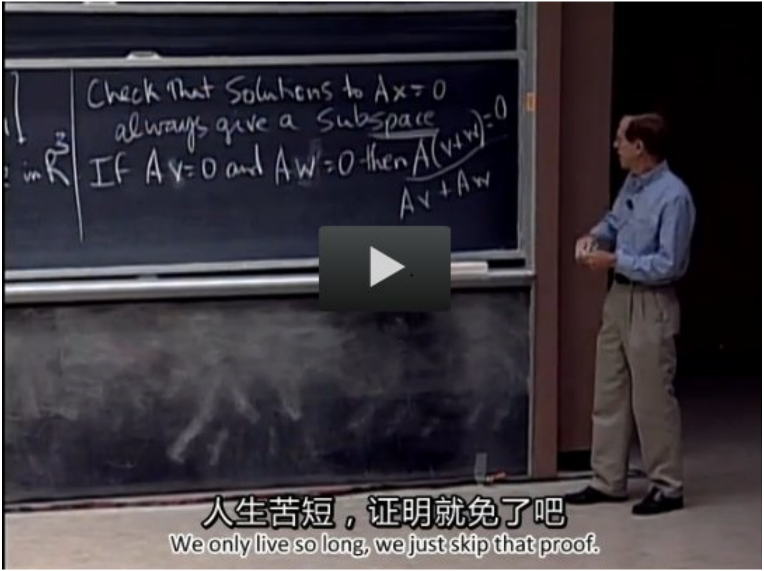
Student t-Distribution

Student's t-distribution has the probability density function given by

$$f(t;\nu) = \frac{\Gamma(\frac{\nu+1}{2})}{\sqrt{\nu\pi}\,\Gamma(\frac{\nu}{2})}\left(1+\frac{t^2}{\nu}\right)^{-\frac{\nu+1}{2}}.$$

where ν is the number of degrees of freedom and Γ is the gamma function.




Check That Solutions to Ax=0
always give a Subspace
If Av=0 and Aw=0 then A(v+w)=0
Av+Aw
人生苦短，证明就免了吧
We only live so long, we just skip that proof.

The multivariate analog of t^2 is

$$T^2 = N(\bar{\mathbf{x}} - \boldsymbol{\mu})^\top \mathbf{S}^{-1}(\bar{\mathbf{x}} - \boldsymbol{\mu}),$$

where

$$\bar{\mathbf{x}} = \frac{1}{N}\sum_{\alpha=1}^{N}\mathbf{x}_\alpha \qquad \text{and} \qquad \mathbf{S} = \frac{1}{N-1}\sum_{\alpha=1}^{N}(\mathbf{x}_\alpha - \bar{\mathbf{x}})(\mathbf{x}_\alpha - \bar{\mathbf{x}})^\top.$$

T^2-Statistic and Likelihood Ratio Criterion

We consider MLE for normal distribution. The likelihood function is

$$L(\boldsymbol{\mu}, \boldsymbol{\Sigma}) = (2\pi)^{-\frac{pN}{2}} (\det(\boldsymbol{\Sigma}))^{-\frac{N}{2}} \exp\left(-\frac{1}{2}\sum_{\alpha=1}^{N}(\mathbf{x}_\alpha - \boldsymbol{\mu})^\top \boldsymbol{\Sigma}^{-1}(\mathbf{x}_\alpha - \boldsymbol{\mu})\right).$$

The likelihood ratio criterion is

$$\lambda = \frac{\max_{\boldsymbol{\Sigma}\in\mathbb{S}_p^{++}} L(\boldsymbol{\mu}_0, \boldsymbol{\Sigma})}{\max_{\boldsymbol{\mu}\in\mathbb{R}^p, \boldsymbol{\Sigma}\in\mathbb{S}_p^{++}} L(\boldsymbol{\mu}, \boldsymbol{\Sigma})}.$$

1. The denominator is the maximum over the entire parameter space.
2. The numerator is the maximum in the space restricted by the null hypothesis.
3. The likelihood ratio test is the procedure of rejecting the null hypothesis when λ is less than a predetermined constant.

T^2-Statistic and Likelihood Ratio Criterion

We have

$$\lambda^{\frac{2}{N}} = \frac{1}{1 + T^2/(N-1)},$$

where $T^2 = N(\bar{\mathbf{x}} - \boldsymbol{\mu}_0)^\top \mathbf{S}^{-1}(\bar{\mathbf{x}} - \boldsymbol{\mu}_0)$.

The likelihood ratio test is defined by the critical region (region of rejection)

$$\lambda \le \lambda_0, \tag{1}$$

where λ_0 is chosen so that the probability of (1) when the null hypothesis is true is equal to the significance level.

The inequality (1) also equivalent to

$$T^2 \ge T_0^2,$$

where $T_0^2 = (N-1)(\lambda_0^{-2/N} - 1)$.

Invariant Property of t^2-Test

The Student t-test is invariant w.r.t scale transformations if $\mu = 0$

1. If $x \sim \mathcal{N}(\mu, \sigma^2)$, then $x^* = cx \sim \mathcal{N}(c\mu, c^2\sigma^2)$ for $c > 0$.
2. The hypothesis $\mathbb{E}[x] = 0$ is equivalent to $\mathbb{E}[cx] = 0$.
3. If observations x_α are transformed to $x_\alpha^* = cx_\alpha$, then

$$t^* = \frac{\bar{x}^* - 0}{s^*/\sqrt{N}} = \frac{\bar{x} - 0}{s/\sqrt{N}} = t.$$

Invariant Property of T^2-Test

The generalized T^2-test has a similar property.

1. If $x \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma}^2)$, then $\mathbf{x}^* = \mathbf{C}\mathbf{x} \sim \mathcal{N}(\mathbf{C}\boldsymbol{\mu}, \mathbf{C}\boldsymbol{\Sigma}\mathbf{C}^\top)$, where $\mathbf{C} \succeq \mathbf{0}$.
2. The hypothesis $\mathbb{E}[\mathbf{x}] = \mathbf{0}$ is equivalent to the hypothesis $\mathbb{E}[\mathbf{x}^*] = \mathbb{E}[\mathbf{C}\mathbf{x}] = \mathbf{0}$.
3. If observations $\mathbf{x}_\alpha$ are transformed to $\mathbf{x}_\alpha^* = \mathbf{C}\mathbf{x}_\alpha$, then T^{*2} computed on $\mathbf{x}_\alpha^*$ is the same as T^2 computed on $\mathbf{x}_\alpha$.

This follows from $\bar{\mathbf{x}}^* = \mathbf{C}\bar{\mathbf{x}}$, $\mathbf{S}^* = \mathbf{C}\mathbf{S}\mathbf{C}^\top$ and the following lemma.

Lemma 1

For any $p \times p$ non-singular matrices $\mathbf{C}$ and $\mathbf{H}$ and any vector $\mathbf{k}$, we have

$$\mathbf{k}^\top \mathbf{H}^{-1} \mathbf{k} = (\mathbf{C}\mathbf{k})^\top (\mathbf{C}\mathbf{H}\mathbf{C}^\top)^{-1} (\mathbf{C}\mathbf{k}).$$

F-Distribution

The F-distribution with d_1 and d_2 degrees of freedom is the distribution of

$$x = \frac{y_1/d_1}{y_2/d_2} = \frac{d_2 y_1}{d_1 y_2}$$

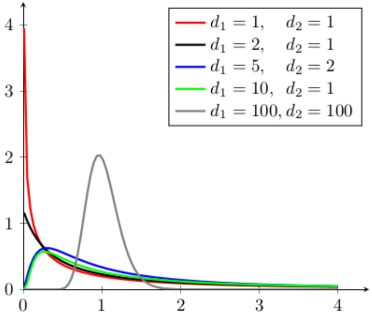
where y_1 and y_2 are independent random variables with Chi-square distributions with respective degrees of freedom d_1 and d_2.

F-Distribution

The probability density function (pdf) for F-Distribution is

$$f(x; d_1, d_2) = \frac{1}{B\left(\frac{d_1}{2}, \frac{d_2}{2}\right)} \left(\frac{d_1}{d_2}\right)^{\frac{d_1}{2}} x^{\frac{d_1}{2}-1} \left(1 + \frac{d_1}{d_2}\, x\right)^{-\frac{d_1+d_2}{2}}$$

where $B(\alpha, \beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,\mathrm{d}t$.



Noncentral F-Distribution

If y_1 is a noncentral Chi-squared random variable with noncentrality parameter λ and d_1 degrees of freedom, and y_2 is a (central) Chi-squared random variable with d_2 degrees of freedom that is independent of y_1, then

$$x = \frac{y_1/d_1}{y_2/d_2}$$

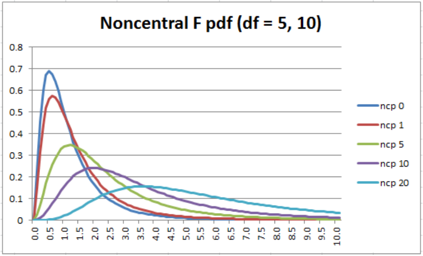
is a noncentral F-distributed random variable.

Noncentral F-Distribution

The probability density function (pdf) for the noncentral F-distribution is

$$
f(x; d_1, d_2, \lambda)
= \begin{cases} \displaystyle\sum_{k=0}^{\infty} \frac{\exp(-\frac{\lambda}{2})(\frac{\lambda}{2})^k}{B\left(\frac{d_2}{2}, \frac{d_1}{2}+k\right) k!} \left(\frac{d_1}{d_2}\right)^{\frac{d_1}{2}+k} \left(\frac{d_2}{d_2+d_1 x}\right)^{\frac{d_1+d_2}{2}+k} x^{\frac{d_1}{2}-1+k}, & x \geq 0, \\ 0, & \text{otherwise}, \end{cases}
$$

where $B(\alpha, \beta) = \displaystyle\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,\mathrm{d}t.$



Distribution of T^2-Statistics

Theorem 2

Let $T^2 = \mathbf{y}^\top \mathbf{S}^{-1}\mathbf{y}$, where $\mathbf{y}$ is distributed according to $\mathcal{N}_p(\boldsymbol{\nu}, \boldsymbol{\Sigma})$ and $n\mathbf{S}$ is independently distributed as $\sum_{\alpha=1}^{n} \mathbf{z}_\alpha \mathbf{z}_\alpha^\top$ with $\mathbf{z}_1, \ldots, \mathbf{z}_n$ independent, each with distribution $\mathcal{N}_p(\mathbf{0}, \boldsymbol{\Sigma})$. Then the random variable

$$\frac{T^2/n}{(n-p+1)/p}$$

is distributed as a noncentral F-distribution with p and $n-p+1$ degrees of freedom and noncentrality parameter $\boldsymbol{\nu}^\top \boldsymbol{\Sigma}^{-1}\boldsymbol{\nu}$. If $\boldsymbol{\nu} = \mathbf{0}$, the distribution is central F.

In the example of likelihood ratio criterion, we consider the special case of $\mathbf{y} = \sqrt{N}(\bar{\mathbf{x}} - \boldsymbol{\mu}_0)^\top$, $\boldsymbol{\nu} = \sqrt{N}(\boldsymbol{\mu} - \boldsymbol{\mu}_0)^\top$ and $n = N - 1$.

Distribution of T^2-Statistics

Corollary 2

Let $\mathbf{x}_1, \dots, \mathbf{x}_N$ be a sample from $\mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ and let

$$T^2 = N(\bar{\mathbf{x}} - \boldsymbol{\mu}_0)^\top \mathbf{S}^{-1} (\bar{\mathbf{x}} - \boldsymbol{\mu}_0).$$

The distribution of

$$\frac{T^2/(N-1)}{(N-p)p}$$

is noncentral F with p and $N-p$ degrees of freedom and noncentrality parameter $N(\bar{\mathbf{x}} - \boldsymbol{\mu})^\top \boldsymbol{\Sigma}^{-1} (\bar{\mathbf{x}} - \boldsymbol{\mu})$ If $\boldsymbol{\mu} = \boldsymbol{\mu}_0$ then the F-distribution is central.

For large samples the distribution of T^2 given this corollary is approximately valid even if the parent distribution is not normal.

Distribution of T^2-Statistics

Theorem 3

Let $x_1, x_2, \ldots$ be a sequence of independently identically distributed random vectors with mean vector $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$. Let

$$\hat{\mathbf{x}}_N = \frac{1}{N}\sum_{\alpha=1}^{N}\mathbf{x}_\alpha, \qquad \hat{\mathbf{S}}_N = \frac{1}{N-1}\sum_{\alpha=1}^{N}(\mathbf{x}_\alpha - \bar{\mathbf{x}})(\mathbf{x}_\alpha - \bar{\mathbf{x}})^\top$$

and

$$T_N^2 = N(\bar{\mathbf{x}}_N - \boldsymbol{\mu}_0)^\top \mathbf{S}_N^{-1}(\bar{\mathbf{x}}_N - \boldsymbol{\mu}_0).$$

Then the limiting distribution of T_N^2 as $N \to \infty$ is the χ^2-distribution with p degrees of freedom if $\boldsymbol{\mu} = \boldsymbol{\mu}_0$.

Theorem 4

Suppose $\mathbf{y}_1, \ldots, \mathbf{y}_m$ are independent with $\mathbf{y}_\alpha$ distributed according to $\mathcal{N}(\mathbf{\Gamma}\mathbf{w}_\alpha, \mathbf{\Phi})$, where $\mathbf{w}_\alpha$ is an r-component vector. Let $\mathbf{H} = \sum_{\alpha=1}^{m} \mathbf{w}_\alpha \mathbf{w}_\alpha^\top$ assumed non-singular, $\mathbf{G} = \sum_{\alpha=1}^{m} \mathbf{y}_\alpha \mathbf{w}_\alpha^\top \mathbf{H}^{-1}$ and

$$\mathbf{C} = \sum_{\alpha=1}^{m} (\mathbf{y}_\alpha - \mathbf{G}\mathbf{w}_\alpha)(\mathbf{y}_\alpha - \mathbf{G}\mathbf{w}_\alpha)^\top = \sum_{\alpha=1}^{m} \mathbf{y}_\alpha \mathbf{y}_\alpha^\top - \mathbf{G}\mathbf{H}\mathbf{G}^\top.$$

Then $\mathbf{C}$ is distributed as

$$\sum_{\alpha=1}^{m-r} \mathbf{u}_\alpha \mathbf{u}_\alpha^\top$$

where $\mathbf{u}_1, \ldots, \mathbf{u}_{m-r}$ are independently distributed according to $\mathcal{N}(\mathbf{0}, \mathbf{\Phi})$ independently of $\mathbf{G}$.