Homework 2

Deadline: April 20, 2022

1. Let the joint density of random variable x and y be

$$f(x,y) = \begin{cases} C, & x^2+y^2 \le k^2, \\ 0, & \text{otherwise.} \end{cases}$$

   Prove $C = 1/(\pi k^2)$, $\mathbb{E}[x] = \mathbb{E}[y] = 0$, $\mathbb{E}[x^2] = \mathbb{E}[y^2] = k^2/4$ and $\mathbb{E}[xy] = 0$. Are x and y independent?

2. Suppose the scalar random variables $x_1, \ldots, x_N$ are independent and have a density which is a function only of $x_1^2, \ldots, x_N^2$. Prove that the x_i are normally distributed with mean 0 and common variance. Indicate the mildest conditions on the density for your proof.

3. Let $x_1, \ldots, x_N$ be independently distributed with $x_i \sim \mathcal{N}(\beta+\gamma z_i, \sigma^2)$ for $i = 1, \ldots, N$ and $\sum_{i=1}^N z_i = 0$. Find the distribution of

$$g = \frac{\sum_{i=1}^N x_i z_i}{\sum_{i=1}^N z_i^2}$$

   for $\sum_{i=1}^N z_i^2 > 0$.

4. Let $\mathbf{x}$ have a (singular) normal distribution with mean $\mathbf{0}$ and covariance matrix

$$\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}$$

   Find $\mathbf{a}$ so $\mathbf{x} = \mathbf{a}^\top \mathbf{y}$ and $\mathbf{y}$ has a non-singular normal distribution, and give the density of $\mathbf{y}$.

5. Let $\mathbf{x} = [x_1, x_2]^\top$ have the density $n(\mathbf{x} \mid \mathbf{0}, \mathbf{I}) = f(x_1, x_2)$. Let the density of x_2 given x_1 be $f(x_2 \mid x_1)$. Let the joint density of x_1, x_2 and x_3 be $f(x_1,x_2) f(x_3 \mid x_1)$. Find the covariance matrix of x_1, x_2 and x_3 and the partial correlation between x_2 and x_3 for given x_1.

6. Show that for any joint distribution of

$$\mathbf{x} = \begin{bmatrix} \mathbf{x}^{(1)} \\ \mathbf{x}^{(2)} \end{bmatrix}$$

   for which the expectations exist and any function $h(\mathbf{x}^{(2)})$ that

$$\mathbb{E}\left[x_i - \mathbb{E}[x_i \mid \mathbf{x}^{(2)}]\right] h(\mathbf{x}^{(2)}) = 0.$$

7. Let the density of (x,y) be

$$f(x,y) = \begin{cases} 2n(x \mid 0,1)\, n(y \mid 0,1), & 0 \le y \le x < +\infty, \quad 0 \le -x \le y < +\infty, \\ & 0 \le -y \le -x < +\infty, \quad 0 \le x \le -y < +\infty, \\ 0, & \text{otherwise.} \end{cases}$$

   Show that x, y, $x+y$, $x-y$ each have a marginal normal distribution.