

Homework 1

Deadline: ~~March 30, 2022~~ April 6, 2022

1. For any $\mathbf{A} \in \mathbb{S}^n$, $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{y} \in \mathbb{R}^n$, prove that $\mathbf{x}^\top \mathbf{A} \mathbf{y} = \mathbf{y}^\top \mathbf{A} \mathbf{x}$.
2. Prove that for any matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$, the column rank of \mathbf{A} is equal to the row rank of \mathbf{A} .
3. For $\mathbf{A} \in \mathbb{R}^{m \times n}$ and $\mathbf{B} \in \mathbb{R}^{n \times p}$, prove that $\|\mathbf{A}\mathbf{B}\|_F \leq \|\mathbf{A}\|_F \|\mathbf{B}\|_F$.
4. Suppose $\mathbf{A} \in \mathbb{R}^{n \times n}$ has eigenvalues $\lambda_1, \dots, \lambda_n$. Prove the following statements
 - (a) $\text{tr}(\mathbf{A}) = \sum_{i=1}^n \lambda_i$,
 - (b) $\det(\mathbf{A}) = \prod_{i=1}^n \lambda_i$.
5. Prove the SVD always exists for any $\mathbf{A} \in \mathbb{R}^{m \times n}$. (Hint: Using spectral decomposition theorem)
6. Given the symmetric matrix

$$\mathbf{N} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^\top & \mathbf{D} \end{bmatrix}$$

with non-singular \mathbf{D} and let $\mathbf{S} = \mathbf{D} - \mathbf{B}^\top \mathbf{A}^{-1} \mathbf{B}$. Prove that

- (a) $\mathbf{N} \succ \mathbf{0} \iff \mathbf{A} \succ \mathbf{0} \text{ and } \mathbf{S} \succ \mathbf{0}$.
- (b) If $\mathbf{A} \succ \mathbf{0}$, then $\mathbf{N} \succeq \mathbf{0} \iff \mathbf{S} \succeq \mathbf{0}$.