

# Multivariate Statistical Analysis

## Lecture 12

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- 1 The Density of Wishart Distribution
- 2  $T^2$ -Statistic and  $F$ -Distribution
- 3 The Inverted Wishart Distribution

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# The Density of Wishart Distribution

## Theorem

The density of  $\mathbf{A} \sim \mathcal{W}_p(\mathbf{\Sigma}, n)$  is

$$w_p(\mathbf{A} \mid \mathbf{\Sigma}, n) = \frac{(\det(\mathbf{A}))^{\frac{n-p-1}{2}} \exp\left(-\frac{1}{2}\text{tr}(\mathbf{\Sigma}^{-1}\mathbf{A})\right)}{2^{\frac{np}{2}} \pi^{\frac{p(p-1)}{4}} (\det(\mathbf{\Sigma}))^{\frac{n}{2}} \prod_{i=1}^p \Gamma\left(\frac{1}{2}(n+1-i)\right)}.$$

for positive definite  $\mathbf{A}$  and 0 elsewhere.

Sketch of the proof:

- 1 Observe that  $\mathbf{B} = \mathbf{\Sigma}^{-1/2}\mathbf{A}\mathbf{\Sigma}^{-1/2} \sim \mathcal{W}_p(\mathbf{I}_p, n)$ .
- 2 Find the density of  $\mathbf{B} \sim \mathcal{W}_p(\mathbf{I}_p, n)$  by induction.
- 3 Recall that the Jacobian of transform from  $\mathbf{A}$  to  $\mathbf{B}$  has determinant

$$(\det(\mathbf{\Sigma}^{-1/2}))^{p+1} = (\det(\mathbf{\Sigma}))^{-\frac{p+1}{2}}.$$

- 4 Achieve the density of  $\mathbf{A} \sim \mathcal{W}_p(\mathbf{\Sigma}, n)$ .

# The Wishart Distribution

The multivariate gamma function is defined as

$$\Gamma_p(t) = \pi^{\frac{p(p-1)}{4}} \prod_{i=1}^p \Gamma\left(t - \frac{1}{2}(i-1)\right).$$

We also write the density function of Wishart distribution as

$$w_p(\mathbf{A} \mid \mathbf{\Sigma}, n) = \frac{(\det(\mathbf{A}))^{\frac{n-p-1}{2}} \exp\left(-\frac{1}{2}\text{tr}(\mathbf{\Sigma}^{-1}\mathbf{A})\right)}{2^{\frac{np}{2}} \Gamma_p\left(\frac{n}{2}\right) (\det(\mathbf{\Sigma}))^{\frac{n}{2}}}.$$

# Properties of Wishart Distribution

Let  $\mathbf{A} \sim \mathcal{W}_p(\boldsymbol{\Sigma}, n)$  and partition  $\mathbf{A}$  and  $\boldsymbol{\Sigma}$  into  $q$  and  $p - q$  rows and columns as

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} \quad \text{and} \quad \boldsymbol{\Sigma} = \begin{bmatrix} \boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} \\ \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22} \end{bmatrix},$$

then we have

(a)  $\mathbf{A}_{11} \sim \mathcal{W}_q(\boldsymbol{\Sigma}_{11}, n)$  and  $\mathbf{A}_{22} \sim \mathcal{W}_{p-q}(\boldsymbol{\Sigma}_{22}, n)$ ;

(b) if  $q = 1$ , then

$$\mathbf{a}_{21} \mid \mathbf{A}_{22} \sim \mathcal{N}_{p-q}(\mathbf{A}_{22} \boldsymbol{\Sigma}_{22}^{-1} \boldsymbol{\sigma}_{21}, \sigma_{11.2}^2 \mathbf{A}_{22})$$

where  $\sigma_{11.2}^2 = \sigma_{11} - \boldsymbol{\sigma}_{21}^\top \boldsymbol{\Sigma}_{22}^{-1} \boldsymbol{\sigma}_{21}$ ;

(c) if  $n > p - q$ , then

$$\mathbf{A}_{11.2} = \mathbf{A}_{11} - \mathbf{A}_{12} \mathbf{A}_{22}^{-1} \mathbf{A}_{21} \sim \mathcal{W}_q(\boldsymbol{\Sigma}_{11.2}, n - p + q)$$

is independent on  $\mathbf{A}_{22}$  and  $\mathbf{A}_{12}$ , where  $\boldsymbol{\Sigma}_{11.2} = \boldsymbol{\Sigma}_{11} - \boldsymbol{\Sigma}_{12} \boldsymbol{\Sigma}_{22}^{-1} \boldsymbol{\Sigma}_{21}$ .

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# The Distribution of Sample Covariance

Recall that we define

$$\mathbf{A} = \sum_{\alpha=1}^N (\mathbf{x}_{\alpha} - \bar{\mathbf{x}})(\mathbf{x}_{\alpha} - \bar{\mathbf{x}})^{\top} \quad \text{and} \quad \mathbf{S} = \frac{1}{n} \sum_{\alpha=1}^N (\mathbf{x}_{\alpha} - \bar{\mathbf{x}})(\mathbf{x}_{\alpha} - \bar{\mathbf{x}})^{\top}$$

where  $\mathbf{x}_1, \dots, \mathbf{x}_N$  are independent, each with the distribution  $\mathcal{N}_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ , and  $n = N - 1 \geq p$ .

We have  $\mathbf{A} \sim \mathcal{W}_p(\boldsymbol{\Sigma}, n)$ , then

$$\mathbf{S} \sim \mathcal{W}_p\left(\frac{1}{n}\boldsymbol{\Sigma}, n\right)$$



In hypothesis testing for the mean with unknown variance, we consider the  $t$ -student variable

$$t = \frac{\bar{x} - \mu}{s/\sqrt{N}},$$

$$\text{where } \bar{x} = \frac{1}{N} \sum_{\alpha=1}^N x_{\alpha} \text{ and } s^2 = \frac{1}{N-1} \sum_{\alpha=1}^N (x_{\alpha} - \bar{x})^2.$$

We have  $t^2 = \frac{N(\bar{x} - \mu)^2}{s^2}$  and its multivariate analog is

$$T^2 = N(\bar{\mathbf{x}} - \boldsymbol{\mu})^{\top} \mathbf{S}^{-1} (\bar{\mathbf{x}} - \boldsymbol{\mu}),$$

$$\text{where } \bar{\mathbf{x}} = \frac{1}{N} \sum_{\alpha=1}^N \mathbf{x}_{\alpha} \text{ and } \mathbf{S} = \frac{1}{N-1} \sum_{\alpha=1}^N (\mathbf{x}_{\alpha} - \bar{\mathbf{x}})(\mathbf{x}_{\alpha} - \bar{\mathbf{x}})^{\top}.$$

The  $F$ -distribution with  $d_1$  and  $d_2$  degrees of freedom is the distribution of

$$z = \frac{y_1/d_1}{y_2/d_2} = \frac{d_2 y_1}{d_1 y_2},$$

where  $y_1 \sim \chi_{d_1}^2$  and  $y_2 \sim \chi_{d_2}^2$  are independent, written as

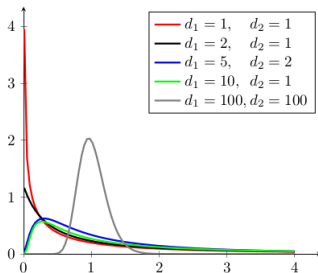
$$z \sim F_{d_1, d_2}.$$

# F-Distribution

The density function of  $F$ -distribution is

$$f(z; d_1, d_2) = \frac{1}{B\left(\frac{d_1}{2}, \frac{d_2}{2}\right)} \left(\frac{d_1}{d_2}\right)^{\frac{d_1}{2}} z^{\frac{d_1}{2}-1} \left(1 + \frac{d_1 z}{d_2}\right)^{-\frac{d_1+d_2}{2}}$$

for  $z > 0$ , where  $B(\alpha, \beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1} dt$ .



# $T^2$ -Statistic and $F$ -Distribution

## Theorem

Let  $\mathbf{A} \sim \mathcal{W}_p(\boldsymbol{\Sigma}, n)$  and  $\mathbf{y} \sim \mathcal{N}_p(\mathbf{0}, \boldsymbol{\Sigma})$  be independent with  $n \geq p$ , then

$$\frac{n-p+1}{p} \cdot \mathbf{y}^\top \mathbf{A}^{-1} \mathbf{y} \sim F_{p, n-p+1}.$$

For  $T^2$ -statistic

$$T^2 = N(\bar{\mathbf{x}} - \boldsymbol{\mu})^\top \mathbf{S}^{-1}(\bar{\mathbf{x}} - \boldsymbol{\mu}),$$

we have

$$\frac{N-p}{(N-1)p} \cdot T^2 \sim F_{p, n-p+1}.$$

# Outline

1 The Density of Wishart Distribution

2  $T^2$ -Statistic and  $F$ -Distribution

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# The Inverted Wishart Distribution

If  $\mathbf{A} \sim \mathcal{W}_p(\mathbf{\Sigma}, m)$ , then  $\mathbf{B} = \mathbf{A}^{-1}$  has the inverted Wishart distribution with  $m$  degrees of freedom and scale parameter  $\mathbf{\Psi} = \mathbf{\Sigma}^{-1}$ , written as

$$\mathbf{B} \sim \mathcal{W}_p^{-1}(\mathbf{\Psi}, m).$$

The density function of  $\mathbf{B}$  is

$$w^{-1}(\mathbf{B} \mid \mathbf{\Psi}, m) = \frac{(\det(\mathbf{\Psi}))^{\frac{m}{2}} (\det(\mathbf{B}))^{-\frac{m+p+1}{2}} \exp\left(-\frac{1}{2}\text{tr}(\mathbf{\Psi}\mathbf{B}^{-1})\right)}{2^{\frac{mp}{2}} \Gamma_p\left(\frac{m}{2}\right)},$$

where

$$\Gamma_p(t) = \pi^{\frac{p(p-1)}{4}} \prod_{i=1}^p \Gamma\left(t - \frac{1}{2}(i-1)\right).$$

Define  $\bar{\mathbb{S}}^p \rightarrow \mathbb{R}^{p \times p}$  as

$$\mathbf{F}(\mathbf{X}) = \mathbf{X}^{-1},$$

where  $\bar{\mathbb{S}}^p = \{\mathbf{X} \in \mathbb{R}^{p \times p} : \mathbf{X} = \mathbf{X}^\top \text{ and } \mathbf{X} \text{ is non-singular}\}$ .

What is the determinant of Jacobian of  $\mathbf{F}(\mathbf{X})$ ?

Let  $\mathbf{x} \sim \mathcal{N}_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ . We define  $\boldsymbol{\Psi} = \boldsymbol{\Sigma}^{-1}$  as the precision matrix.

- 1 It is well-known that

$$\sigma_{ij} = 0 \text{ if and only if } x_i \text{ and } x_j \text{ are independent.}$$

- 2 What is the meaning of  $\psi_{ij} = 0$ ?



# The Conjugate Prior for the Covariance

## Theorem

If  $\mathbf{A} \sim \mathcal{W}_p(\mathbf{\Sigma}, n)$  and  $\mathbf{\Sigma}$  has a prior distribution  $\mathcal{W}^{-1}(\mathbf{\Psi}, m)$ , then the conditional distribution of  $\mathbf{\Sigma}$  given  $\mathbf{A}$  is the inverted Wishart distribution

$$\mathcal{W}^{-1}(\mathbf{A} + \mathbf{\Psi}, n + m).$$

Let each of  $\mathbf{x}_1, \dots, \mathbf{x}_N$  has distribution  $\mathcal{N}_p(\mathbf{0}, \mathbf{\Sigma})$  independently and  $n = N - 1$ , then the sample covariance

$$\mathbf{S} = \frac{1}{n} \sum_{i=1}^n (\mathbf{x}_i - \bar{\mathbf{x}})(\mathbf{x}_i - \bar{\mathbf{x}})^\top \sim \mathcal{W}_p(\mathbf{\Sigma}, n).$$

If  $\mathbf{\Sigma} \sim \mathcal{W}_p^{-1}(\mathbf{\Psi}, m)$ , then we have

$$\mathbf{\Sigma} | \mathbf{S} \sim \mathcal{W}^{-1}(n\mathbf{S} + \mathbf{\Psi}, n + m).$$

# The Inverted Wishart Distribution

## Theorem

Let  $\mathbf{x}_1, \dots, \mathbf{x}_N$  be observations from  $\mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ . Suppose  $\boldsymbol{\mu}$  and  $\boldsymbol{\Sigma}$  have prior densities

$$n \left( \boldsymbol{\mu} \mid \boldsymbol{\nu}, \frac{\boldsymbol{\Sigma}}{K} \right) \quad \text{and} \quad w^{-1}(\boldsymbol{\Sigma} \mid \boldsymbol{\Psi}, m)$$

respectively, where  $n = N - 1$ . Then the posterior density of  $\boldsymbol{\mu}$  and  $\boldsymbol{\Sigma}$  given

$$\bar{\mathbf{x}} = \frac{1}{N} \sum_{\alpha=1}^N \mathbf{x}_{\alpha} \quad \text{and} \quad \mathbf{S} = \frac{1}{N-1} \sum_{\alpha=1}^N (\mathbf{x}_{\alpha} - \bar{\mathbf{x}})(\mathbf{x}_{\alpha} - \bar{\mathbf{x}})^{\top}$$

is

$$n \left( \boldsymbol{\mu} \mid \frac{N\bar{\mathbf{x}} + K\boldsymbol{\nu}}{N + K}, \frac{\boldsymbol{\Sigma}}{N + K} \right) \cdot w^{-1} \left( \boldsymbol{\Sigma} \mid \boldsymbol{\Psi} + n\mathbf{S} + \frac{NK(\bar{\mathbf{x}} - \boldsymbol{\nu})(\bar{\mathbf{x}} - \boldsymbol{\nu})^{\top}}{N + K}, N + m \right).$$