Homework 5

Deadline: June 18, 2022

1. Suppose that each element of the covariance matrix

$$\boldsymbol{\Sigma} = \begin{bmatrix} \sigma_{11} & \dots & \sigma_{1p} \\ \vdots & \dots & \vdots \\ \sigma_{p1} & \dots & \sigma_{pp} \end{bmatrix}$$

   is positive. Show that

   (a) The coefficients of the first principal component are all of the same sign.

   (b) The coefficients of each other principal component cannot be all of the same sign.

2. Let

$$\mathbf{x} = \begin{bmatrix} \mathbf{x}^{(1)} \\ \mathbf{x}^{(2)} \end{bmatrix}, \quad u = \boldsymbol{\alpha}^\top \mathbf{x}^{(1)} \quad \text{and} \quad v = \boldsymbol{\gamma}^\top \mathbf{x}^{(2)},$$

   such that

$$\mathbb{E}[\mathbf{x}] = \mathbf{0}, \quad \mathbb{E}[\mathbf{x}\mathbf{x}^\top] = \begin{bmatrix} \boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} \\ \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22} \end{bmatrix} \quad \text{and} \quad \mathbb{E}[u^2] = \mathbb{E}[v^2] = 1,$$

   where $\boldsymbol{\alpha}$ and $\boldsymbol{\gamma}$ are non-random vectors. Show that choosing $\boldsymbol{\alpha}$ and $\boldsymbol{\gamma}$ to maximize $\mathbb{E}[uv]$ is equivalent to choosing $\boldsymbol{\alpha}$ and $\boldsymbol{\gamma}$ to minimize the generalized variance of $[u, v]^\top$.

3. Find the solution of the following problem

$$\min_{\mathbf{W} \in \mathbb{R}^{d \times q}, \sigma^2 \in \mathbb{R}^+} \left\| \hat{\boldsymbol{\Sigma}} - \left(\mathbf{W}\mathbf{W}^\top + \sigma^2 \mathbf{I} \right) \right\|_2,$$

   where $\|\cdot\|_2$ is the spectral norm, $\hat{\Sigma} \in \mathbb{R}^{d \times d}$ is symmetric positive semi-definite and $d > q$.