5 LU decomposition, Matrix inversion and Gauss-Seidel

Exercises adapted from:

Kiusalaas. (2013). Numerical Methods in Engineering with Python 3. Third Edition. Ch 2. Systems of Linear Algebraic Equations.

1. Given LU decomposition $A = LU$, determine A

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 5/3 & 1 \end{bmatrix} \quad U = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 3 & 21 \\ 0 & 0 & 0 \end{bmatrix}$$

2. Use the results of LU decomposition

$$A = LU = \begin{bmatrix} 1 & 0 & 0 \\ 3/2 & 1 & 0 \\ 1/2 & 11/13 & 1 \end{bmatrix} \begin{bmatrix} 2 & -3 & -1 \\ 0 & 13/2 & -7/2 \\ 0 & 0 & 32/13 \end{bmatrix}$$

   to solve $Ax = b$, where $b^T = [1, -1, 2]$.

3. Write a Python program that solves $Ax = b$ using LU decomposition. Use the functions *LUdecomp* and *LUsolve*.

$$A = \begin{bmatrix} 1 & 4 & 1 \\ 1 & 6 & -1 \\ 2 & -1 & 2 \end{bmatrix} \quad b = \begin{bmatrix} 7 \\ 13 \\ 5 \end{bmatrix}$$

   What are values for the lower and upper triangular matrices L and U?

4. Write a function that inverts a matrix. Test the function by inverting

$$A = \begin{bmatrix} 0.6 & -0.4 & 1.0 \\ -0.3 & 0.2 & 0.5 \\ 0.6 & -1.0 & 0.5 \end{bmatrix}$$

5. Invert the triangular matrices

$$A = \begin{bmatrix} 2 & 4 & 3 \\ 0 & 6 & 5 \\ 0 & 0 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 0 & 0 \\ 3 & 4 & 0 \\ 4 & 5 & 6 \end{bmatrix}$$

   What do you observe?

6. Invert the following matrices with any method

$$A = \begin{bmatrix} 5 & -3 & -1 & 0 \\ -2 & 1 & 1 & 1 \\ 3 & -5 & 1 & 2 \\ 0 & 8 & -4 & -3 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 3 & -9 & 6 & 4 \\ 2 & -1 & 6 & 7 & 1 \\ 3 & 2 & -3 & 15 & 5 \\ 8 & -1 & 1 & 4 & 2 \\ 11 & 1 & -2 & 18 & 7 \end{bmatrix}$$

Comment on the reliability of the results.

7. Write a program for inverting $n \times n$ lower triangular matrix. The inversion procedure should contain only forward substitution. Test the program by inverting the matrix

$$A = \begin{bmatrix} 36 & 0 & 0 & 0 \\ 18 & 36 & 0 & 0 \\ 9 & 12 & 36 & 0 \\ 5 & 4 & 9 & 36 \end{bmatrix}$$

8. Use the Gauss-Seidel iterative method to solve

$$\begin{bmatrix} -2 & 5 & 9 \\ 7 & 1 & 1 \\ -3 & 7 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ -26 \end{bmatrix}$$

9. Use the Gauss-Seidel with relaxation to solve $Ax = b$, where

$$A = \begin{bmatrix} 4 & -1 & 0 & 0 \\ -1 & 4 & -1 & 0 \\ 0 & -1 & 4 & -1 \\ 0 & 0 & -1 & 3 \end{bmatrix} \quad b = \begin{bmatrix} 15 \\ 10 \\ 10 \\ 10 \end{bmatrix}$$

Take $x_i = b_i/A_{ii}$ as the starting vector, and use $\omega = 1.1$ for the relaxation factor.

10. Write a program for solving $Ax = b$ by the Gauss-Seidel method based on the function *gaussSeidel*. Input should consists of the matrix A and the vector b. Test the program with

$$A = \begin{bmatrix} 3 & -2 & 1 & 0 & 0 & 1 \\ -2 & 4 & -2 & 1 & 0 & 0 \\ 1 & -2 & 4 & -2 & 1 & 0 \\ 0 & 1 & -2 & 4 & -2 & 1 \\ 0 & 0 & 1 & -2 & 4 & -2 \\ 1 & 0 & 0 & 1 & -2 & 3 \end{bmatrix} \quad b = \begin{bmatrix} 10 \\ -8 \\ 10 \\ 10 \\ -8 \\ 10 \end{bmatrix}$$

Source codes are from: Kiusalaas. (2013). Numerical Methods in Engineering with Python 3. Third Edition.

```
import numpy as np
import math

''' a = LUdecomp(a)
    LUdecomposition: [L][U] = [a]

    x = LUsolve(a,b)
    Solution phase: solves [L][U]{x} = {b}
'''

def LUdecomp(a):
    n = len(a)
    for k in range(0,n-1):
        for i in range(k+1,n):
            if a[i,k] != 0.0:
                lam = a [i,k]/a[k,k]
                a[i,k+1:n] = a[i,k+1:n] - lam*a[k,k+1:n]
                a[i,k] = lam
    return a

def LUsolve(a,b):
    n = len(a)
    for k in range(1,n):
        b[k] = b[k] - np.dot(a[k,0:k],b[0:k])
    b[n-1] = b[n-1]/a[n-1,n-1]
    for k in range(n-2,-1,-1):
       b[k] = (b[k] - np.dot(a[k,k+1:n],b[k+1:n]))/a[k,k]
    return b

def matInv(a):
    n = len(a[0])
    aInv = np.identity(n)
    a,seq = LUdecomp(a)
    for i in range(n):
        aInv[:,i] = LUsolve(a,aInv[:,i],seq)
    return aInv
```

```
''' x, numIter, omega = gaussSeidel(iterEqs,x,tol = 1.0e-9)
    Gauss-Seidel method for solving [A]{x} = {b}.
    The matrix [A] should be sparse. User must supply the
    function iterEqs(x,omega) that returns the improved {x},
    given the current {x} (omega is the relaxation factor).
'''

def gaussSeidel(iterEqs,x,tol = 1.0e-9):
    omega = 1.0
    k = 10
    p = 1
    for i in range(1,501):
        xOld = x.copy()
        x = iterEqs(x,omega)
        dx = math.sqrt(np.dot(x-xOld,x-xOld))
        if dx < tol: return x,i,omega
      # Compute relaxation factor after k+p iterations
        if i == k: dx1 = dx
        if i == k + p:
            dx2 = dx
            omega = 2.0/(1.0 + math.sqrt(1.0  \
                  - (dx2/dx1)**(1.0/p)))
    print('Gauss-Seidel failed to converge')
```