Proof of the Asymptotic Distribution of the Sharpe Ratio Estimator

1. Sharpe Ratio Estimator

The annualized Sharpe ratio estimator is given by:

$$\hat{S} = \frac{\hat{\mu}}{\hat{\sigma}}\sqrt{T_y},$$

where:

- $\hat{\mu}$ is the sample mean of daily returns,
- $\hat{\sigma}$ is the sample standard deviation of daily returns,
- T_y is the average number of trading days in a year.

2. Assumptions

- The returns r_t are independently and identically distributed (i.i.d.) with mean μ and variance σ^2.
- The returns are normally distributed: $r_t \sim \mathcal{N}(\mu, \sigma^2)$.

3. Distribution of Sample Mean and Variance

- The sample mean $\hat{\mu}$ is normally distributed:

  $$\hat{\mu} \sim \mathcal{N}\left(\mu, \frac{\sigma^2}{T}\right),$$

  where T is the number of observed daily returns.

- The sample variance $\hat{\sigma}^2$ follows a scaled chi-squared distribution:

  $$\hat{\sigma}^2 \sim \frac{\sigma^2}{T}\chi^2_{T-1}.$$

  For large T, $\hat{\sigma}^2$ is approximately normal:

  $$\hat{\sigma}^2 \sim \mathcal{N}\left(\sigma^2, \frac{2\sigma^4}{T}\right).$$

4. Delta Method Application

Let:

$$\hat{S} = \sqrt{T_y} \cdot \frac{\hat{\mu}}{\hat{\sigma}} = \sqrt{T_y} \cdot g(\hat{\mu}, \hat{\sigma}^2),$$

where $g(x, y) = \frac{x}{\sqrt{y}}$.

The gradient of g is:

$$\nabla g = \left(\frac{1}{\sqrt{y}}, -\frac{x}{2y^{3/2}} \right).$$

The asymptotic variance of $g(\hat{\mu}, \hat{\sigma}^2)$ is:

$$\nabla g \cdot \Sigma \cdot \nabla g^T,$$

where Σ is the covariance matrix of $(\hat{\mu}, \hat{\sigma}^2)$. Since $\hat{\mu}$ and $\hat{\sigma}^2$ are independent for normal returns, Σ is diagonal:

$$\Sigma = \begin{pmatrix} \frac{\sigma^2}{T} & 0 \\ 0 & \frac{2\sigma^4}{T} \end{pmatrix}.$$

Plugging in ∇g and Σ:

$$\text{Asymptotic Variance} = \left(\frac{1}{\sigma} \right)^2 \cdot \frac{\sigma^2}{T} + \left(-\frac{\mu}{2\sigma^3} \right)^2 \cdot \frac{2\sigma^4}{T} = \frac{1}{T} + \frac{\mu^2}{2\sigma^2 T} = \frac{1 + \frac{S^2}{2}}{T},$$

where $S = \frac{\mu}{\sigma}$ is the true Sharpe ratio.

5. Asymptotic Distribution

The delta method implies:

$$\sqrt{T} \left(g(\hat{\mu}, \hat{\sigma}^2) - g(\mu, \sigma^2) \right) \xrightarrow{d} \mathcal{N} \left(0, 1 + \frac{S^2}{2} \right).$$

Multiplying by $\sqrt{T_y}$ and rearranging:

$$\sqrt{\frac{T}{T_y}} \left(\hat{S} - S \right) \xrightarrow{d} \mathcal{N} \left(0, 1 + \frac{S^2}{2} \right).$$

6. Conclusion

The asymptotic distribution of the Sharpe ratio estimator $\hat{S}$ is:

$$\sqrt{\frac{T}{T_y}} (\hat{S} - S) \xrightarrow{d} \mathcal{N} \left(0, 1 + \frac{S^2}{2} \right).$$

This shows that the root mean square error of $\hat{S}$ decreases with the square root of the length of the period T.