Derivation of the Probability of Loss Formula

1. Assumptions

We assume that daily returns r_t are independent and identically distributed (i.i.d.) and normally distributed:

$$r_t \sim \mathcal{N}(\mu_d, \sigma_d^2)$$

Over T trading days, the cumulative return is:

$$R_T = \sum_{t=1}^{T} r_t$$

Since the sum of i.i.d. normal random variables is also normal, we have:

$$R_T \sim \mathcal{N}(T \cdot \mu_d, T \cdot \sigma_d^2)$$

2. Annualized Parameters

Define the annualized expected return μ and annualized volatility σ, and let T_y be the number of trading days in a year (typically 252). Then:

$$\mu_d = \frac{\mu}{T_y}, \quad \sigma_d = \frac{\sigma}{\sqrt{T_y}}$$

So over T days:

- Mean:

$$T \cdot \mu_d = \mu \cdot \frac{T}{T_y}$$

- Variance:

$$T \cdot \sigma_d^2 = \sigma^2 \cdot \frac{T}{T_y}$$

- Standard deviation:

$$\sigma_T = \sigma \cdot \sqrt{\frac{T}{T_y}}$$

3. Probability of Loss

We want to calculate the probability that the cumulative return over T days is negative:

$$P_{\text{loss}} = \mathbb{P}(R_T < 0)$$

Since $R_T \sim \mathcal{N}\left(\mu \cdot \frac{T}{T_y}, \sigma^2 \cdot \frac{T}{T_y}\right)$, we standardize:

$$P_{\text{loss}} = \mathbb{P}\left(\frac{R_T - \mu \cdot \frac{T}{T_y}}{\sigma \cdot \sqrt{\frac{T}{T_y}}} < \frac{0 - \mu \cdot \frac{T}{T_y}}{\sigma \cdot \sqrt{\frac{T}{T_y}}}\right)$$

This simplifies to:

$$P_{\text{loss}} = \Phi\left(\frac{-\mu \cdot \frac{T}{T_y}}{\sigma \cdot \sqrt{\frac{T}{T_y}}}\right) = \Phi\left(-\frac{\mu}{\sigma} \cdot \sqrt{\frac{T}{T_y}}\right)$$

4. Final Formula

Thus, the probability of loss is:

$$P_{\text{loss}}(T, S) = \Phi\left(-S \cdot \sqrt{\frac{T}{T_y}}\right)$$

where $S = \frac{\mu}{\sigma}$ is the Sharpe ratio, and $\Phi(\cdot)$ is the cumulative distribution function (CDF) of the standard normal distribution.

5. Intuition

- A higher Sharpe ratio S implies a lower probability of loss.
- The term $\sqrt{T/T_y}$ adjusts for the time period being less than or greater than a full year.
- The formula quantifies how likely the strategy is to lose money over a specific horizon under normality assumptions.