Life Analysis

Failure function up to time t

Cumulative density function: it tells the probability that the observed object is likely to fail up to time t.

P(T<=t) = F(t)

Cumulative survival function: it tells the probability that the observed object is likely to survive beyond time t.

P(T>t) = 1-F(t) = S(t)

The survival function has following properties:

$$\begin{aligned} S(0) &= 1 \\ \lim_{t\to\infty} S(t) &= 0 \\ \frac{\partial S}{\partial t} &< 0 \end{aligned}$$

S(t) + F(t)=1

f(t) is defined as the corresponding probability density function of F(t).

Hazard function is defined as below:

$$\theta(t) = \frac{f(t)}{1 - F(t)} = \frac{f(t)}{S(t)}$$

The probability density function $f(t)$ summarizes the concentration of spell lengths (exit times) at each instant of time along the time axis. The hazard function summarizes the same concentration at each point of time, but conditions the expression on survival in the state up to that instant, and so can be thought of as summarizing the instantaneous *transition intensity*.

Integrated hazard function

$$\begin{aligned} H(t) &\equiv \int_0^t \theta(u)du \\ &= -\ln[S(t)]. \end{aligned}$$

The probability of exit within the interval is

$$\Pr(a_{j-1} < T \le a_j) = F(a_j) - F(a_{j-1}) = S(a_{j-1}) - S(a_j)$$

Interval hazard rate, also known as the discrete hazard rate, is the probability of exit in the interval a[j-1]~a[j]

$$\begin{aligned} h(a_j) &= \Pr(a_{j-1} < T \le a_j | T > a_{j-1}) \\ &= \frac{\Pr(a_{j-1} < T \le a_j)}{\Pr(T > a_{j-1})} \\ &= \frac{S(a_{j-1}) - S(a_j)}{S(a_{j-1})} \\ &= 1 - \frac{S(a_j)}{S(a_{j-1})} \end{aligned}$$

The probability of survival until the end of interval j is the product of probabilities of not experiencing event in each of the intervals up to and including the current one.

$$\begin{aligned} S(j) &\equiv S_j = (1-h_1)(1-h_2).....(1-h_{j-1})(1-h_j) \\ &= \prod_{k=1}^{j}(1-h_k) \end{aligned}$$

otherwise

$$\begin{aligned} F_j &= F(j) = 1 - S(j) \\ &= 1 - \prod_{k=1}^{j}(1-h_k). \end{aligned}$$

The probability of exit within the j interval:

$$
\begin{aligned}
f(j) &= \Pr(a_{j-1} < T \leq a_j) \\
&= S(j-1) - S(j) \\
&= \frac{S(j)}{1-h_j} - S(j) \\
&= \left(\frac{1}{1-h_j} - 1\right) S(j) \\
&= \frac{h_j}{1-h_j} \prod_{k=1}^{j} (1-h_k)
\end{aligned}
$$

in further,

$$
f(j) = h_j S_{j-1} = \frac{h_j}{1-h_j} S_j
$$

Weibull model:

probability density function

$$
f(t) = \frac{\beta}{\eta} \left(\frac{t-\gamma}{\eta}\right)^{\beta-1} e^{-\left(\frac{t-\gamma}{\eta}\right)^{\beta}}
$$

Mean or MTTF

$$
\overline{T} = \gamma + \eta \cdot \Gamma\left(\frac{1}{\beta} + 1\right)
$$

The gamma function is defined as:

$$
\Gamma(n) = \int_0^{\infty} e^{-x} x^{n-1} dx
$$

Median:

$$\breve{T} = \gamma + \eta\,(\ln 2)^{\frac{1}{\beta}}$$

Mode:

$$\tilde{T} = \gamma + \eta\left(1 - \frac{1}{\beta}\right)^{\frac{1}{\beta}}$$

Standard deviation:

$$\sigma_T = \eta \cdot \sqrt{\Gamma\left(\frac{2}{\beta} + 1\right) - \Gamma\left(\frac{1}{\beta} + 1\right)^2}$$

Cumulative density function

$$F(t) = 1 - e^{-\left(\frac{t-\gamma}{\eta}\right)^{\beta}}$$

Survival function S(t) also known as reliability function:

$$R(t) = e^{-\left(\frac{t-\gamma}{\eta}\right)^{\beta}}$$

Conditional Reliability Function:

$$R(t|T) = \frac{R(T+t)}{R(T)} = \frac{e^{-\left(\frac{T+t-\gamma}{\eta}\right)^{\beta}}}{e^{-\left(\frac{T-\gamma}{\eta}\right)^{\beta}}}$$

The reliable life:

Given the reliability R, the reliable life of a unit is as below:

$$T_R = \gamma + \eta \cdot \{-\ln(R)\}^{\frac{1}{\beta}}$$

Failure Rate also known as hazard rate:

$$\lambda(t) = \frac{f(t)}{R(t)} = \frac{\beta}{\eta}\left(\frac{t-\gamma}{\eta}\right)^{\beta-1}$$

PdM

Based on all these mathematics, we can then answer the following PdM questions:

	Questions	Validity	Solution
1	Given reliability or survival objective probability, what is the most likely life length?		$T_R = \gamma + \eta \cdot \{-\ln(R)\}^{\frac{1}{\beta}}$
2	Given that the component survives at time t, what is the reliability for another time period delta t?		$R(t\|T) = \frac{R(T+t)}{R(T)} = \frac{e^{-\left(\frac{T+t-\gamma}{\eta}\right)^{\beta}}}{e^{-\left(\frac{T-\gamma}{\eta}\right)^{\beta}}}$
3	Given that a component survives to time t, what is the probability of the failure happening within next time unit?		$h(a_j) = \Pr(a_{j-1} < T \le a_j \| T > a_{j-1}) = \frac{\Pr(a_{j-1} < T \le a_j)}{\Pr(T > a_{j-1})} = \frac{S(a_{j-1}) - S(a_j)}{S(a_{j-1})} = 1 - \frac{S(a_j)}{S(a_{j-1})}$ or 1-R(t|T)
4	How many components are most likely to fail within next time period?		N*H(t)
5	What is the failure probability for k components at next time unit?		b(n, p, k) here n is the total number of the components, p is the hazard probability
6	Given that k components are in stock, what is the probability of stock out?		sum of b(n, p, i) and i is from k to N
7	Given that k components are in stock, what is the mean time between stock-out?		solve H(t)=p and p is from above

8	What is the mean time of a component? Or, what is the life expectation? Or, what is the MTBF?		$\overline{T} = \gamma + \eta \cdot \Gamma\left(\frac{1}{\beta} + 1\right)$
9	What is the median time of component?		$\breve{T} = \gamma + \eta\left(\ln 2\right)^{\frac{1}{\beta}}$
10	What is the mode time of a component?		$\tilde{T} = \gamma + \eta\left(1 - \frac{1}{\beta}\right)^{\frac{1}{\beta}}$

Reference

	subject	source
1	survival analysis	Survival Analysis.pdf
2	weibull distribution	http://reliawiki.org/index.php/The_Weibull_Distribution