Representation Theory Seminar 1 LECTURE NOTES

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Note 0.1 This is only an **outline** of the lecture, which is to say that there are no detailed explanation or proof in the note, which will be completed at the lecture.

1. Basic Concepts of Representation & Preliminaries

1.1. Representation and Examples

Informally, a representation of a group is a collection of invertible linear transformations of a $\mathbb{C}$-linear space that multiply together in the same way the group elements in G behave. (It is just like G acts on a linear space V.)

Let G denote a finite group and V be a $\mathbb{C}$-linear space.

Definition 1.1 (linear representation) A ***linear representation*** of G is a group homomorphism

$$\rho : G \to GL(V),$$

which often be denoted by (V, ρ).

Now here is some explanation of the "odd" definition. The notation $GL(V)$ is denoted by the group all the linear transformation of V i.e. the general linear group. The group homomorphism sends $g \in G$ to a linear transformation $\rho(g) : V \to V$. Try to answer the following questions by yourself:

- What is the significance that ρ must be a group **homomorphism**, not only a map?
- How to understand the first paragraph of this section?

Hence you can get a picture of the definition of the representation.

If there exists a basis B of V, we can get a group homomorphism $G \to GL_n(\mathbb{C})$, where $n = \dim_{\mathbb{C}} V$. The dimension of V, i.e. n, is called the **degree** of the representation, denoted by $\deg \rho \coloneqq n$. Now we can consider the following examples of representations, whose properties we will pay attention to later.

Example 1.2 (trivial representation) Take $V = \mathbb{C}$ and $\dim V = 1$. Hence $GL(\mathbb{C}) = \mathbb{C}^{\times}$ and the representation is given by $\rho : G \to \mathbb{C}^{\times}$. Take $\rho(g) = 1$ for all $g \in G$ to be the identity on V. Hence we get the **trivial representation**. It is really important in the following theory, but so simple that it presents nothing but abstract.

Example 1.3 (permutation representation) Let G acts on a finite set X. Let V be a $\mathbb{C}$ $-$linear space with dimension $|X|$ and basis $\{e_x \mid x \in X\}$. Take $ge_x = e_{gx}$, then (V, ρ)

is a representation. This representation is direct from the group acts and is called the **permutation representation**.
Why the representation is called the permutation representation? It can be explained since it is induced by a group action $G \to S_n$, where S_n is the permutation group of order n, and the matrix of each $g \in G$ is the permutation matrix.
Example 1.4 (regular representation) Take G as a basis of a linear space and let $g.g' = gg'$, then it forms a representation, which is called the **regular representation** with degree $\deg \rho = |G|$. Regular representation is a special case of permutation representation.
Now let us pay attention to some paticular interesting cases — 3 different representations of the group S_3. First of all, it has the trivial representation.
Example 1.5 (sign representation) Take $\rho : S_n \to \{-1, 1\}$ be the sign map of permutation. Hence we get the **sign representation** of S_n.
Example 1.6 Replace $\mathbb{C}$ by $\mathbb{R}$. Take $V = \mathbb{R}^2$ and $G = S_3$, which is isisomorphic to the group of symmetries of an equilateral triangle. The symmetries are the three reflections in the lines that bisect the equilateral triangle, together with three rotations. Positioning the center of the triangle at the origin of V and labeling the three vertices of the triangle as 1, 2, and 3, we then get a representation.
Note 1.7 In fact, **Example 1.6** is also a $\mathbb{C}$ −representation of G.
Note 1.8 We can also regard representation of a group acting on a linear space.
One important way to discover the properties of representations is the correspondence between group representations and group algebras.
Definition 1.9 (group algebra) We define the **group algebra**

$$\mathbb{C}[G] = \left\{ \sum_{g \in G} a_g g : a_g \in \mathbb{C} \right\},$$

as a linear space over $\mathbb{C}$ and the multiplication is given by

$$\left(\sum_{g \in G} a_g g \right) \left(\sum_{h \in G} b_h h \right) := \sum_{k \in G} \left(\sum_{gh = k} a_g b_h \right) k.$$

Example 1.10 Take the notation $k[G]$ as in **Definition 1.9** and take $G = \mathbb{Z}_4$, generated by s. Hence $k[G] \simeq k[x]/(x^4 - 1)$ as k −algebra.
Having defined the group algebra, now we can reveal the correspondence between group algebra and representations. Before that we need the definition of $\mathbb{C}[G]$ −module. The definition will be completed in the lecture.
Theorem 1.11 A representation of G over $\mathbb{C}$ has the structure of a $\mathbb{C}[G]$ −module. Conversely, every unital $\mathbb{C}[G]$ −module provides a representation of G over $\mathbb{C}$.
Is there any relationship between the group algebra and the regular representations? In fact, $\mathbb{C}[G]$ itself has a $\mathbb{C}[G]$ −module structure.

Regarding representations of G as $\mathbb{C}[G]$ −modules has the advantage that many definitions can be borrowed from module theory. The concepts, such as **subrepresentation, morphism of representations, direct sum of representations** can all be described by the relevant concepts in modules. Hence a little module theory is needed.

1.2. Commutative Algebra Preliminaries

We will use *Commutative Algebra* by Atiyah & Macdonald as textbook for this section to introduce basic module theory that we need.

In particular, we will cover the following sections:

- 2.1 modules and module homomorphisms
- 2.2 submodules and quotient modules
- 2.4 direct sum and product
- 2.8 restriction and extension on scalars
- 2.11 algebras

1.3. Subrepresentations & Morphism

Now we will regard a representation of G as a $\mathbb{C}[G]$ −module since **Theorem 1.11** holds.

To introduce the concept **subrepresentation**, we only need to pay attention to $\mathbb{C}[G]$ −submodule of a $\mathbb{C}[G]$ −submodule V. To specify a $\mathbb{C}[G]$ −submodule of V, it is necessary to specify an R −submodule W of V that is closed under the action of $\mathbb{C}[G]$, i.e. W is an invariant subspace of $\rho(g)$ for any $g \in G$. To be precise,

Definition 1.12 (subrepresentation) Let (V, ρ) be a representation of G and W be a subspace of V such that W is invariant under $\rho(g)$, for any $g \in G$. Hence we can get $\rho_W : g \to GL(W)$ and (ρ_W, W) is a **subrepresentation** of (V, ρ).

We make use of the notions of a homomorphism and an isomorphism of $\mathbb{C}[G]$ −modules. Since $\mathbb{C}[G]$ has as a basis the elements of G, to check that an $\mathbb{C}$ −linear homomorphism $f : V \to W$ is in fact a homomorphism of $\mathbb{C}[G]$ −modules, it suffices to check that $f(gv) = gf(v)$ for all $g \in G$ —— we do not need to check for every $x \in \mathbb{C}[G]$.

By means of the identification of $\mathbb{C}[G]$ −modules with representations of G (in **Theorem 1.11**) we may refer to homomorphisms and isomorphisms of group representations. In many books the algebraic condition on the representations that these notions entail is written out explicitly, and two representations that are isomorphic a real so said to be equivalent.

Given two $\mathbb{C}[G]$ −modules V and W, we may form their **direct sum** $V \oplus W$. We write $U = V \oplus W$ to mean that U has $\mathbb{C}[G]$ −submodules such that $U = V + W$ and $U \cap W = \{0\}$. In this situation,we also say that V and W are **direct summands** of U.

2. Irreducible Representation & Maschke's Theorem

2.1. Maschke's Theorem

We come now to our first nontrivial result, one that is fundamental to the study of representations over fields of characteristic zero, such as $\mathbb{C}$. This surprising result says that in this situation representations always break apart as direct sums of smaller representations. We need to mention that if we replace $\mathbb{C}$ by different field k, we need $|G|$ to be invertible in k to make sure the following theorem holds.

Theorem 2.1 (Maschke's Theorem) Let W be an invariant subspace of V over a field k such that $|G|$ is invertible in k, then there exists an invariant subspace W' of V such that $V = W \oplus W'$ as representations. In particular, the theorem holds for $k = \mathbb{C}$.

We can give the proof of the theorem using the properties of $\mathbb{C}$, but in this lecture, we will prove the general theorem.

Proof. Since W is a subspace of V, then there exists a complementary subspace W_1 such that $V = W \oplus W_1$ as linear spaces, and take $\pi : V \to W$ be the projection map. Then we get $V = W \oplus \ker \pi$ as linear spaces.

We need to emphasize that this does NOT prove the theorem, since $\ker \pi$ is not necessary to be invariant. Consider the map

$$\pi' := \frac{1}{|G|} \sum_{g \in G} g \pi g^{-1},$$

then π' is linear. Since

$$\pi'(w) = \frac{1}{|G|} \sum_{g \in G} g \pi (g^{-1} w) = w$$

for $w \in W$ and $\pi'(v) \in W$; hence $V = W \oplus \ker \pi'$. Now we need to prove that $\ker \pi'$ is invariant, since $\pi'(hv) = h\pi'(v)$ (need to be carefully verified in the lecture). $\square$

2.2. Irreducible Representations & Simple Modules

Because the next results apply more generally than to group representations, we let A be a ring and consider its modules. A nonzero A −module V is said to be **simple** or **irreducible** if V has no A −submodules other than 0 and V.