1. (a.)

We can compute the probability distribution as follows:

$p(a) = \sum_{b,c} p(a,b,c)$

a	p(a)
0	0.6
1	0.4

$p(b) = \sum_{a,c} p(a,b,c)$

b	p(b)
0	0.592
1	0.408

$p(a,b) = \sum_{c} p(a,b,c)$

a	b	p(a,b)
0	0	0.336
0	1	0.264
1	0	0.256
1	1	0.144

Therefore p(a)p(b) is:

a	b	p(a)p(b)
0	0	0.3552
0	1	0.2448
1	0	0.2368
1	1	0.1632

Since $p(a)p(b) \neq p(a, b)$, a and b are dependent.

$p(c) = \sum_{a,b} p(a,b,c)$

c	p(c)
0	0.48
1	0.52

$p(a \mid c) = \dfrac{\sum_b p(a,b,c)}{p(c)}$

c	a	p(a|c)
0	0	0.5
0	1	0.5
1	0	0.6923
1	1	0.3077

$p(b \mid c) = \dfrac{\sum_a p(a,b,c)}{p(c)}$

c	b	p(b|c)
0	0	0.8
0	1	0.2
1	0	0.4
1	1	0.6

$p(a,b \mid c) = \dfrac{p(a,b,c)}{p(c)}$

c	a	b	p(a, b | c)
0	0	0	0.4
0	0	1	0.1
0	1	0	0.4
0	1	1	0.1
1	0	0	0.2769
1	0	1	0.4154
1	1	0	0.1231
1	1	1	0.1846

$p(a \mid c)\, p(b \mid c)$

c	a	b	$p(a\|c)p(b\|c)$
0	0	0	0.4
0	0	1	0.1
0	1	0	0.4
0	1	1	0.1
1	0	0	0.2769
1	0	1	0.4154
1	1	0	0.1231
1	1	1	0.1846

Therefore $p(a|c)p(b|c) = p(a,b|c)$, hence $a \perp b \mid c$

(b.)

$p(c \mid a) = \dfrac{\sum_b p(a,b,c)}{p(a)}$

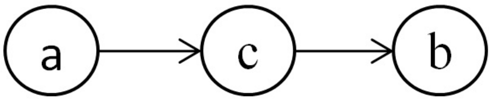
a	c	p(c|a)
0	0	0.4
0	1	0.6
1	0	0.6
1	1	0.4

Then p(a)p(b|c)p(c|a) is

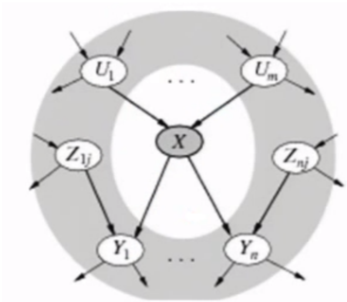
a	b	c	p(a)p(b|c)p(c|a)
0	0	0	0.192
0	0	1	0.144
0	1	0	0.048
0	1	1	0.216
1	0	0	0.192
1	0	1	0.064
1	1	0	0.048
1	1	1	0.096

Hence p(a)p(b|c)p(c|a) = p(a,b,c)

The corresponding Bayes network is:



2.



Consider Figure 14.4 (b) on the textbook:
The grey part is the Markov Blanket. We need to consider all possible paths from the center node X to all possible nodes outside the blanket. There are three kinds of such paths. First, consider $U_1 \ldots U_m$, they are parents of X. For any node U_i among $U_1 \ldots U_n$, the link between X and U_i has its tail on U_i. In this case, the path between X and outer nodes on the parent could be either tail-to-tail or head-to-tail. Once the parent is given, X is independent with all such nodes. Second, consider $Y_1 \ldots Y_n$, they are children of node X. For any node Y_i among $Y_1 \ldots Y_n$, the link between X and Y_i has its head on Y_i. In this case, the path between X and outer nodes which are children of any Y_i is head-to-tail on Y_i. Once Y_i is given, X is independent with all such nodes. The last kind of path comes from $Z_{1j} \ldots Z_{nj}$, they share children with X. The path will be head-to-head on the shared children $Y_1 \ldots Y_n$, thus given Y_i cannot ensure independence of X and nodes on such paths. However, such path will be either tail-to-tail or head-to-tail on Z_{ij} (a node among $Z_{1j} \ldots Z_{nj}$), then, once Z_{ij} and Y_i is given, X will be independent with outer nodes in such path. So, conditioned on all the nodes in the Markov Blanket, X will be independent with all the remaining variables in the graph.

3. (a.)
Independent pairs of variables are:
$(e \perp c)$, $(a \perp c)$, $(b \perp c)$

(b.)
Given a third variables, conditionally independent pairs of variables are:
$(e \perp b \mid a)$, $(e \perp d \mid a)$, $(e \perp c \mid a)$, $(e \perp d \mid b)$, $(e \perp c \mid b)$, $(a \perp d \mid b)$, $(a \perp c \mid b)$

4. (a.)
p(a,b,c) = p(a)p(b|a)p(c|b)

(b.)
$p(c) = \sum_{a,b} p(a,b,c) = \sum_{a,b} p(a)p(b|a)p(c|b)$

$p(a,c) = \sum_b p(a,b,c) = \sum_b p(a)p(b|a)p(c|b) = p(a)\sum_b p(b|a)p(c|b)$

$p(a)p(c) = p(a)\sum_{a,b} p(a)p(b|a)p(c|b) \neq p(a)\sum_b p(b|a)p(c|b) = p(a,c)$

Therefore a and c are dependent.

(c.)
$$p(a|b) = \frac{p(a,b)}{p(b)} = \frac{\sum_c p(a,b,c)}{p(b)} = \frac{\sum_c p(a)p(b|a)p(c|b)}{p(b)} = \frac{p(a)p(b|a)\sum_c p(c|b)}{p(b)}$$

$$p(c|b) = \frac{p(c,b)}{p(b)} = \frac{\sum_a p(a,b,c)}{p(b)} = \frac{\sum_a p(a)p(b|a)p(c|b)}{p(b)} = \frac{p(c|b)\sum_a p(a)p(b|a)}{p(b)}$$

$$\begin{aligned}
p(a|b)p(c|b) &= \frac{p(a)p(b|a)\sum_c p(c|b)}{p(b)} \times \frac{p(c|b)\sum_a p(a)p(b|a)}{p(b)} \\
&= \frac{p(a)p(b|a)p(c|b)\sum_c p(c|b)\sum_a p(a)p(b|a)}{p(b)p(b)} \\
&= \frac{p(a,b,c)\sum_a\sum_c p(a)p(b|a)p(c|b)}{p(b)p(b)} \\
&= \frac{p(a,b,c)\sum_{a,c} p(a,b,c)}{p(b)p(b)} \\
&= \frac{p(a,b,c)}{p(b)} \\
&= p(a,c|b)
\end{aligned}$$

Therefore p(a|b)p(c|b) = p(a,c|b) , a and c are conditionally independent given b.

5. (a.)
p(a,b,c) = p(a)p(b|a,c)p(c)

(b.)
$p(a,c) = \sum_b p(a,b,c) = \sum_b p(a)p(b|a,c)p(c) = p(a)p(c)\sum_b p(b|a,c)$
$= p(a)p(c)$
Therefore a and c are independent.

(c.)
$$p(a|b) = \frac{p(a,b)}{p(b)} = \frac{\sum_c p(a,b,c)}{p(b)} = \frac{\sum_c p(a)p(b|a,c)p(c)}{p(b)} = \frac{p(a)\sum_c p(b|a,c)p(c)}{p(b)}$$

$$p(c|b) = \frac{p(c,b)}{p(b)} = \frac{\sum_a p(a,b,c)}{p(b)} = \frac{\sum_a p(a)p(b|a,c)p(c)}{p(b)} = \frac{p(c)\sum_a p(a)p(b|a,c)}{p(b)}$$

$$
\begin{aligned}
p(a|b)p(c|b) &= \frac{p(a)\sum_c p(b|a,c)p(c)}{p(b)} \times \frac{p(c)\sum_a p(a)p(b|a,c)}{p(b)} \\
&= \frac{p(a)p(c)\sum_c p(b|a,c)p(c)\sum_a p(a)p(b|a,c)}{p(b)p(b)} \\
&= \frac{p(a)p(c)p(b|a,c)}{p(b)} \times \frac{\sum_c p(b|a,c)p(c)\sum_a p(a)p(b|a,c)}{p(b)p(b|a,c)} \\
&= \frac{p(a,b,c)}{p(b)} \times \frac{\sum_c p(b|a,c)p(c)\sum_a p(a)p(b|a,c)}{p(b)p(b|a,c)} \\
&= p(a,c|b) \times \frac{\sum_c p(b|a,c)p(c)\sum_a p(a)p(b|a,c)}{p(b)p(b|a,c)} \\
&\neq p(a,c|b)
\end{aligned}
$$

Therefore a and c are not conditionally independent given b.

6. (a.)

From the given net work, we know that $J \perp I, M, B | G$

Therefore (ii) **P**(J|G)=**P**(J|G,I) and (iii) **P**(M|G,B,I)=**P**(M|G,B,I,J) are asserted.

(b.)

From the given Bayes net, we know that

P(B,I,M,G,J) = **P**(J|G)**P**(G|B,I,M)**P**(I|B,M)**P**(B)**P**(M)

Therefore:

$$
\begin{aligned}
\mathrm{P(b, i, \neg m, g, j)} &= \mathrm{P(j|g)P(g|b, i, \neg m)P(i|b, \neg m)P(b)P(\neg m)} \\
&= 0.9 \times 0.8 \times 0.5 \times 0.9 \times 0.9 \\
&= 0.2916
\end{aligned}
$$

(c.)

P(J|B,I,M) = $\sum_g \mathbf{P}(\mathrm{J|G})\mathbf{P}(\mathrm{G|B, I, M})$, and the problem can be described as P(J|b,i,m)

Therefore, the probability is

$$
\begin{aligned}
\mathrm{P(J|b, i, m)} &= \sum_g \mathbf{P}(\mathrm{J|G})\mathbf{P}(\mathrm{G|b, i, m}) \\
&= [\mathrm{P(j|g)P(g|b, i, m) + P(j|\neg g)P(\neg g|b, i, m)}, \\
&\qquad \mathrm{P(\neg j|g)P(g|b, i, m) + P(\neg j|\neg g)P(\neg g|b, i, m)}] \\
&= [0.9 \times 0.9 + 0 \times 0.1, 0.1 \times 0.9 + 1 \times 0.1] \\
&= [0.81, 0.19]
\end{aligned}
$$

(d.)

In this Bayes net, one will not be FoundGuilty if he/she is not Indicted regardless of whether he/she BrokeElectionLaw, regardless of whether he/she PoliticallyMotivatedProsecutor.

So G is context-specifically independent of B and M given I = false.

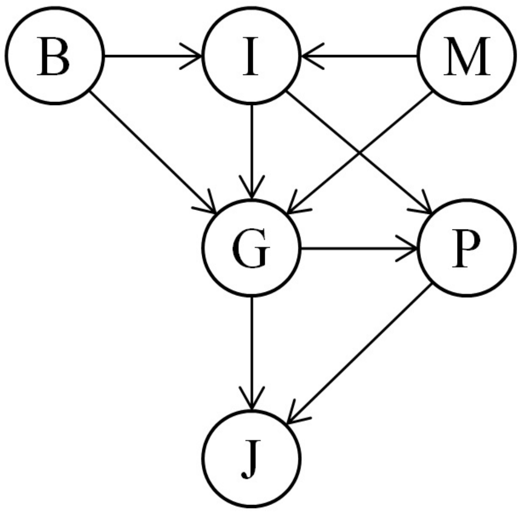
(e.)

Three links added to the net.

I to P : Inducted could result in PresidentPardon

G to P : FoundGuilty could result in PresidentPardon

P to J : PresidentPardon could lead to not Jailed



7. (a.)

Since q1 and q2 are in detailed balance with π

We have: $\pi(x)q1(x \to x') = \pi(x')q1(x' \to x)$

$\pi(x)q2(x \to x') = \pi(x')q2(x' \to x)$

For their combination α q1 + $(1-\alpha)$ q2

We have $\pi(x)\big(\alpha\, \mathrm{q1}(x \to x') + (1-\alpha)\, \mathrm{q2}(x \to x')\big)$

$$= \pi(x)\alpha\, \mathrm{q1}(x \to x') + \pi(x)(1-\alpha)\, \mathrm{q2}(x \to x')$$
$$= \alpha\pi(x)\, \mathrm{q1}(x \to x') + (1-\alpha)\pi(x)\, \mathrm{q2}(x \to x')$$
$$= \alpha\pi(x')\, \mathrm{q1}(x' \to x) + (1-\alpha)\pi(x')\, \mathrm{q2}(x' \to x)$$
$$= \pi(x')(\alpha\, \mathrm{q1}(x' \to x) + (1-\alpha)\, \mathrm{q2}(x' \to x))$$

Therefore the combination is in detailed balance with π

(b.)

$q(x \to x') = (q1 \circ q2)(x \to x') = \sum_{x"} q1(x \to x")q2(x" \to x')$

Therefore:

$$\sum_x \pi(x)q(x \to x') = \sum_x \pi(x) \sum_{x"} q1(x \to x")q2(x" \to x')$$
$$= \sum_{x"} q2(x" \to x') \sum_x \pi(x)\, q1(x \to x")$$
$$= \sum_{x"} q2(x" \to x')\pi(x")$$
$$= \pi(x')$$

Therefore, q = $q1 \circ q2$ also has π as its stationary distribution.