A Brief Overview of Me Relearning Mechanics

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Taken from my reading of *Introduction to Mechanics* by Daniel Kleppner and Robert J. Kolenkow as well as the teachings of my professor, Dr. Wesley Campbell. It will also be supplemented by things I learned from *Introduction To Classical Mechancis* by David Morin which I believe is much clearer in its explanations.

Preface

These notes are a compilation of new things I learned in my freshman honors mechanics course. Therefore, this will not cover knowledge that should already be cemented through high school physics (AP Physics 1, 2, C). For example, I will not be including the equation of momentum and its conservation since I take it as common knowledge.

For the sake of brevity and since I just haven't gotten to it, there won't be any examples in the main body of this text, but I am looking to add more at the end of the notes at a later date. Also, I really want to add more figures and diagrams and will try to get around to it.

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1 Newton's Laws

1.1 The First Law

The first law states that in inertial reference frames we will observe objects to remain in their current state of motion unless acted upon by an acceleration. Now keep in mind it is very important to establish that we are in a inertial reference frame. Without knowing this fact, an object at rest could easily seem like it is undergoing acceleration.

1.2 The Second Law

The second law is most commonly just known as $\mathbf{F} = \mathbf{ma}$. However, we should first think about what is mass and what is force. Kleppner proposes us to conduct an experiment where we have an object moving along a frictionless track that is propelled by a stretched rubber band. When we pull that rubber band and allow it to hit the object it will accelerate with a certain magnitude. Mass is the ratio of the acceleration when we pull back the rubber band the same amount and place different objects. We then use the definitions of mass and accelerations with their units to create how we define force, leading us to $F = ma$. It is also purely a coincidence that this empirical definition of mass is the same as the gravitational proportionality that is defined as mass.

1.3 The Third Law

The third law is very important since without it many of the conservation laws that govern our world would not exist. It also allows us to know if an object is accelerating because of a force or simply because we are in an non-inertial reference frame. And maybe most importantly it tells us that all forces come from a source.

1.4 Applying The Laws

1. Try to treat everything as a point mass
2. Identify a coordinate system. This may just be tilting the normal cartesian coordinate system.
3. Write an equation describing the net force
4. Write a constraint equation if the body is constrained to move along a certain path (e.g. circle)
5. Solve
6. Do a sanity check. Plug in numbers and extremes. Do they make sense?

2 Forces and Equations of Motion

2.1 The Fundamental Forces

1. Gravitational Force
2. Electromagnetic Force
3. Strong Force
4. Weak Force

The two most familiar fundamental forces will be the gravitational and electromagnetic force. Both of these are able to act over a long distance where the strength of both is inversely proportional to the square of the distances between two particles. The key difference between these two is that gravitational force can only attract while the electromagnetic force can attract and repel. The other two forces are less obvious, but still very important. The strong force ($\lambda_{QCD} = 10^{-15}\ m$) keeps protons and neutrons together, while the weak force plays a role in processes that I am still not familiar with. However, these two forces only act across very short distances so in almost all cases they can be ignored.

*Because the rest of solving with forces is just using Newton's laws I won't be adding more here.

3 Momentum

Newton's law is commonly known as the form $F = ma$ but the more accurate representation of the second law would be $F = \frac{d\vec{P}}{dt}$.

3.1 Momentum Flow and Force

Imagine a stream of water falling onto your hand, it exerts a force that is just as real as if a steal rod was pushing on your hand. How can we represent this continuous stream of droplets and give an equation for the force. We can start by imagining a single droplet. We don't know the instantaneous force of each droplet but we can calculate the impulse.

$$\begin{aligned}
I_{droplet} &= \int_{1\ collision}^{F\ dt} \\
&= \Delta p \\
&= m(v_f - v) \\
&= -mv
\end{aligned}$$

Then by Newton's 3rd Law the force felt by your hand will be

$$I_{hand} = mv$$

Now what if there are many collisions per second. You will feel the average force F_{av} rather than the quick shock of individual drops. Then setting the average time between collisions to be T, the area under a graph of F_{av} over time T is identitcal to the impulse due to one droplet.

$$F_{av}T = \int_{1\ collision}^{F\ dt} = mv$$

Then using the average distance between droplets which we can define as $l = vT$ the average force can then be written as

$$F_{av} = \frac{mv}{T} = \frac{mv^2}{l}$$

This momentum transfer by a stream causes the underlying forces behind lift. Here we have demonstrated how this can be applied to a stream of water but the same principles can be used to model and stream of any type of particles.

3.2 Momentum Flux

Earlier we found the average force exerted by a stream of particles. What if we now wanted to do this, but consider the stream of particles to have an area. We define the momentum flux as the amount of momentum flowing through a certain area per unit time. To derive this we first start with the density ρ_m (kg/m^3). The mass per unit length is then $\rho_m A$ and the momentum per unit length is $\rho_m Av$. The rate at which momentum flows through a certain area (momentum per unit time), the flux, is then

$$\dot{\mathbf{P}} = \rho_m v^2 A\hat{\mathbf{v}}.$$

The force that then hits a surface is the change in momentum and is thus also $\dot{P}$. We can also consider if the force is not directed perpendicularly to the surface and is tilted at angle θ. The force is then

$$\dot{P} = \rho_m v^2 A\cos\theta.$$

At this point, it's useful for us to define a quantity that will be known as **flux density**

$$\mathbf{J} = \rho_m v^2 \hat{v}.$$

The momentum flux can then be defined as the dot product of the flux density and area

$$\dot{P} = (J \cdot A)\hat{\mathbf{v}}.$$

We can also look at surfaces where a stream of particles is not perfectly stopped. The force can then be defined as

$$\mathbf{F_{tot}} = \dot{\mathbf{P}}_{\mathbf{in}} - \dot{\mathbf{P}}_{\mathbf{out}}$$

which is very similar to electric flux through a closed surface.

4 Energy

4.1 Intro

The point of mechanics is often to find how an object moves. When looking at Newton's laws at first glance they may seem sufficient to do just this. With a known force we can find acceleration and integrate to find velocity and position. However, the problem arises when we consider forces that are no longer a function of time. The view of energy serves very useful for solving for equations in the form of

$$\frac{dv(t)}{dt} = \mathbf{F}(\mathbf{r}).$$

Of course we could take a mathematical approach and solve this using various numerical methods, but this wouldn't give us much of a tangible understanding of the physical world. Thus we introduce the concepts of energy and work.

4.2 Derivation

We start with solving the equation for one dimensional motion

$$m\frac{d^2x}{dt^2} = F(x)$$

$$m\int_{x_a}^{x_b} \frac{dv}{dt}dx = \int_{x_b}^{x_a} F(x)dx$$

Examining the left side and substituting $dx = vdt$

$$\begin{aligned} m\int_{x_a}^{x_b} \frac{dv}{dt}dx &= m\int_{t_a}^{t_b} \frac{dv}{dt}vdt \\ &= m\int_{t_a}^{t_b} \frac{1}{2}\frac{d}{dt}(v^2)dt \\ &= \frac{1}{2}mv^2\Big|_{t_a}^{t_b} \\ &= \frac{1}{2}mv_b^2 - \frac{1}{2}mv_a^2 \end{aligned}$$

Thus the change in kinetic energy in 1D motion equals the force applied over a distance

$$\frac{1}{2}mv_b^2 - \frac{1}{2}mv_a^2 = \int_{x_a}^{x_b} F(x)dx$$

4.3 Conservation

I think this section doesn't need much elaboration. Energy is conserved within a closed system.

5 Topics in Dynamics

5.1 Intro

This section will describe some more specific topics regarding Newton's laws that have not been covered in the other sections.

5.2 Oscillations and Normal Modes

Any system can exhibit simple harmonic motion if perturbed from from its equilibrium a small amount. Everything can be described with 3 types of equilibriums

1. Stable Equilibrium: $\frac{dU}{dr} < 0$.
   Imagine an energy graph like that of well where a particle wants to sit in the well of low energy
2. Unstable Equilibrium: $\frac{dU}{dr} > 0$
   This is the case where that well becomes upside down.
3. Neutral Equilibrium: $\frac{dU}{dr} = 0$

When we look at objects in this stable equilibrium state, which is when they will exhibit SHM, we can approximate any energy well to be that of a parabola. This approximation gives us the very important relationship

$$\frac{d^2U}{dr^2} = k = m\omega_o^2$$

5.3 Collisions

6 Angular Momentum and Rotation

6.1 Intro

$$F_{ext} = \frac{dP}{dt}$$

For a point particle we define angular momentum about a point $\vec{r_o}$ to be

$$\vec{\mathbf{L}} = (\vec{r} - \vec{r_o}) \times \vec{p}.$$

In the case that $\vec{r_o} = 0$

$$L = \vec{r} \times \vec{p}.$$

The cross product can also take the form of

$$|\vec{r} \times \vec{p}| = |\vec{r}||\vec{p}| \sin\phi$$

when $\vec{r}$ and $\vec{p}$ are placed on the same plane. Conveniently, we can also write it in the form of

$$|\vec{r} \times \vec{p}| = r_\perp p = r p_\perp.$$

If we expand the momentum term we see

$$\vec{L} = \vec{r} \times m\vec{v}$$

and

$$|\vec{v}| = \omega\rho.$$

Where ρ is defined as the shortest distance from the particle to the axis of rotation. In the case where the rotation axis is just the z-axis, which we commonly define it be, $\rho = \sqrt{x^2 + y^2}$.

6.2 Moment of Inertia

The general definition for angular momentum can be stated as

$$\begin{aligned} L &= (\sum_i m_i \rho_i^2)\omega \\ &= I\omega \end{aligned}$$

We can redefine this over a continuous distance and mass

$$\begin{aligned} I &= \sum_i m_i \rho_i^2 \\ &= \int dV \rho^2 w \\ &= \int dM \rho^2 \end{aligned}$$

Where w is the density. The moment of inertia can in other words be called a weighted sum.

6.3 Parallel Axis Theorem

For any rigid object, the moment of inertia with an axis parallel to that of the axis at the center of mass can be evaluated using the parallel axis theorem. Formally, if I_o is the moment of inertia of an object about the axis that contains its center of mass then the moment of inertia about any parallel axis displaced by a distance L is given by

$$I = I_o + ML^2$$

Derivation
First we will define the quantity $|\vec{\rho}_{com}| = l$. Then consider a plane containing m_j and $\perp$ to the rotation axis. We then define a vector

$$\vec{\rho_j}' = \vec{\rho_j} - \vec{\rho}_{com}$$

Then just moving the terms around

$$\vec{\rho_j} = \vec{\rho_j}' + \vec{\rho}_{com}$$

Now we can plug this quantity into the equation for the moment of inertia.

$$
\begin{aligned}
I &= \sum_j m_j(\vec{\rho_j})^2 \\
&= \sum_j m_j(\vec{\rho_j}' + \vec{\rho}_{com}) \\
&= \sum_j m_j(\vec{\rho_j}')^2 + 2\sum_j m_j(\vec{\rho_j}' \cdot \vec{\rho}_{com}) + \sum_j m_j(\vec{\rho}_{com})^2 \\
&= I_o + 2\sum_j m_j(\vec{\rho_j}' \cdot \vec{\rho}_{com}) + ml^2 \\
&= I_o + ml^2
\end{aligned}
$$

The term $2\sum_j m_j(\vec{\rho_j}' \cdot \vec{\rho}_{com})$ is 0 simply because it is the formula for the center of mass of an object centered about the center of mass.

*This does not work in the general case and is only the special case for the theorem when looking at the center of mass.

6.4 Torque and Dynamics

The torque for a particle is defined as

$$\vec{\tau} = \vec{r} \times \vec{F}.$$

The net torque is also simply just

$$\vec{\tau}_{net} = \sum_i \vec{\tau}_i.$$

It is important to note that just because the net force is 0 the net torque doesn't necessarily have to be and the vice versa is also true. Analogous to Newton's second law torque is also defined as

$$\vec{\tau} = \frac{d\vec{L}}{dt}.$$

Which then also means

$$
\begin{aligned}
\vec{\tau} &= \frac{d\vec{L}}{dt} \\
&= \frac{d}{dt} I\vec{\omega} \\
&= I\frac{d\vec{\omega}}{dt} \\
&= I\vec{\alpha}.
\end{aligned}
$$

Energy
To examine energy we first look at the scenario of a mass rotating about a fixed axis.

$$K = \sum_i \frac{1}{2} m_i v_i^2$$

Then knowing that $v = \omega\rho$.

$$
\begin{aligned}
K &= \sum_i \frac{1}{2} m_i (\rho_i \omega)^2 \\
&= \sum_i \frac{1}{2} m_i \rho_i^2 \omega^2 \\
&= \frac{1}{2} I \omega^2
\end{aligned}
$$

Chasles' Theorem
The motion of any rigid body can be written as a combination of translation of its COM and rotation about its COM, which we call spin. Now why would we care to call the quantity spin? If we were to talk about the rotation of the earth how would we specify if we were talking about the Earth rotating about its center of mass or its rotation around the sun. That's why we have defined the word spin to define the rotation about an objects COM. Chasles' Theorem then also tells us the total kinetic energy is the sum of the rotational and translational kinetic energies.

$$K = \frac{1}{2} M v^2 + \frac{1}{2} I \omega^2$$

7 Harmonic Oscillators

7.1 Intro

8 Rigid Body Motion: It's all in the $\frac{\mathbf{dL}}{\mathbf{dt}}$

8.1 Intro

When looking at rotational motion, we have an analogous for almost every quantity in translation. However, we reach a problem when trying to represent the orientation of an object. When looking at translations we are able to represent the position of objects with a vector commonly denoted with $\vec{r}$. Now, thinking about the rotational motion, this isn't possible because the order that we rotate objects is important. Instead, the orientation must be represented by something that is closer to an ordered list.

Despite this, all hope is not lost, because the velocity can indeed be modeled as a vector something along the form of

$$\vec{\omega} = \omega_x \hat{\imath} + \omega_y \hat{\jmath} + \omega_z \hat{k}$$

Another very important thing to note when analyzing rigid body motion in the general case is that the angular momentum $\vec{L}$ is not necessarily parallel to $\vec{\omega}$. Though, in the special case that we have fixed axis rotation it is.

8.2 Gyroscopes!

My Attempt at an Intuitive Explanation
Before we consider the typical mind blowing demonstration of a bicycle wheel not falling over when its attached to a rope simply because it's spinning we should first examine a top. In my opinion, it is just as much of a surprise that a top does not tip over compared to the bicycle wheel not tipping over.

Consider when the top is not spinning and is attempted to be placed at an upright position (of course if it was perfectly upright it would not topple over but that isn't practically possible). The torque is always applied parallel to the plane created by the top face of the top. Under the circumstance that the top is not spinning it is trivial to see that top would tip over in the direction corresponding with the right hand rule. Now, when the top is already spinning, it has some non zero angular momentum. The angular momentum points perpendicular from the plane that is the top face of the top and the torque is still parallel to it. In this situation the torque, instead of making it tip over, will start to change the angular momentum in accordance to the equation $\tau = \frac{d\vec{L}}{dt}$. I think that all of this makes pretty good sense, however I still always wondered why would the torque now change the direction of the angular momentum instead of still making it tip over.

To answer this, think about the angular momentum vector when the top is not spinning. Well of course its 0 and there is no vector. So now the torque will begin to create a vector for the angular momentum that is in the same direction of the torque. I hope in your mind you can imagine this angular momentum vector growing as torque is applied to it over time. Then, think about the vector when the top is already rotating. The vector already exists and the torque is still doing the exact same thing, however now, since it's applied perpendicular to the angular momentum vector it is just changing the direction of that vector similar to how centripetal acceleration only changes the direction of tangential velocity and not the magnitude.

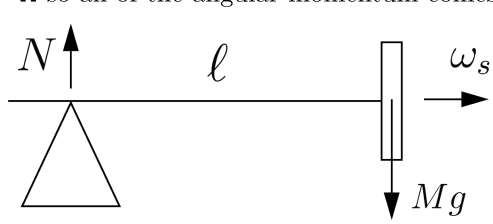
After the case of the top becomes clear, think about the bike wheel. It is essentially the exact same thing, but the top is now rotated 90 degrees. The torque changes the direction of the angular momentum vector because it's simply being applied perpendicular to it.

The precession (the orbital rotation of the object), is then just a result of the object maintaining an angular momentum that is perpendicular to the torque.

The Math

$$\tau = \frac{d\vec{L}}{dt}$$

A flywheel spins at speed ω_s, and has a precession rate of Ω as shown in the figure below. We also consider $\omega_s >> \Omega$ so all of the angular momentum comes from the spin.



Placing the pivot point at where the normal force is exerted we see the torque is

$$\tau = Mgl$$

Since we the rate of spin of the flywheel is much greater than the rate of precession we can simply just consider the angular momentum of the flywheel

$$\vec{L}_s = I\omega(-\sin(\Omega t + \phi)\hat{i} + \cos(\Omega t + \phi)\hat{j})$$
$$\frac{d}{dt}\vec{L}_s = \Omega I\omega(-\cos(\Omega t + \phi)\hat{i} - \sin(\Omega t + \phi)\hat{j})$$

at $\phi = 0, t = 0$

$$\vec{L}_s \parallel \hat{j}$$
$$\frac{d\vec{L}_s}{dt}\Big|_{0,0} = -\Omega I\omega_s\hat{i}$$
$$\tau\Big|_{0,0} = -Mgl\hat{i} = -\Omega I\omega_s\hat{i}$$
$$\Omega = \frac{Mgl}{I\omega_s}$$

Thus we have found the rate of precession and can clearly confirm the flywheel does precess instead of falling down.

8.3 The Principal Axes

Objects commonly don't just spin around an axes that it is perfectly symmetrical about. However for any object we can find 3 principal axes that make our calculations much simpler. If we want to think of it with the inertia tensor, using the principal axes will give us a diagonalized matrix with 3 moments of inertia. They form an orthonormal basis and give us the very important property

$$I\hat{\omega}_k = I_k\hat{\omega}_k.$$

The angular momentum of any object can also be represented by the 3 principal axes which we'll define the basis vectors as $\hat{e}_a, \hat{e}_b, \hat{e}_c$.

$$\vec{\omega} = \omega_a\hat{e}_a + \omega_b\hat{e}_b + \omega\hat{e}_c$$
$$\vec{L} = L_a\hat{e}_a + L_b\hat{e}_b + L_c\hat{e}_c$$
$$KE_{spin} = \frac{1}{2}\vec{\omega}\cdot\vec{L}$$

Note that this is not the lab frame and is non inertial. We call this the body fixed frame.

This also tells us that just because there is no torque it doesn't necessarily mean the angular velocity isn't changing. The angular momentum must be conserved since $\frac{dL}{dt} = 0$, however to keep this true $\vec{\omega}$ must be changing. This can be justified by doing the calculations of an object at two different orientations with the principal axes. In addition, this concludes that in the general case $\tau = I\alpha$ does not apply.

9 Non Inertial Reference Frames

9.1 Uniformly Accelerating Systems

Consider two people in different reference frames. Alice stands still, while Bob flies away in a rocket with acceleration $\vec{a}$ as as measured by Alice. r' is the non inertial frame.

$$\begin{aligned} \vec{r'} &= \vec{r} - (s_0 + \vec{v}t + \frac{1}{2}\vec{a}t^2) \\ \dot{\vec{r'}} &= \dot{\vec{r}} - (\vec{v} + \vec{a}t) \\ \ddot{\vec{r'}} &= \ddot{\vec{r}} - \vec{a} \end{aligned}$$

Therefore the force is

$$\begin{aligned} \vec{F'} &= m(\ddot{\vec{r'}} - \vec{a}) \\ &= \vec{F} - m\vec{a} \\ \vec{F'} &= \vec{F} + F_{fict}. \end{aligned}$$

With any object in an acceleration frame we can simply view it as having a fictitious force acted upon it.

9.2 Rotating Systems

For any fixed length vector $\vec{B}$ that is rotating about an axis $\hat{\Omega}$ at constant rate Ω can be described by

$$\frac{d\vec{B}}{dt} = \vec{\Omega} \times \vec{B}$$

Consider two coordinate systems $\hat{\imath}, \hat{\jmath}, \hat{k}$ the normal cartesian axes we are familiar with and $\hat{\imath}', \hat{\jmath}', \hat{k}'$ that is rotated about $\vec{\Omega}$. We will call the force in the inertial frame F_{in} and in the rotating frame F_{rot}. Then consider the vector $\vec{C} = C_x\hat{\imath} + C_y\hat{\jmath} + C_x\hat{k}$ that describes the position in the inertial frame.

$$\frac{d\vec{C}}{dt}_{in} = \frac{dC_x}{dt}\hat{\imath} + \frac{dC_y}{dt}\hat{\jmath} + \frac{dC_z}{dt}\hat{k}$$

Then the same vector but in terms of unit vectors in the rotating frame

$$\begin{aligned} \vec{C} &= C'_x\hat{\imath}' + C'_y\hat{\jmath}' + C'_z\hat{k}' \\ \frac{d\vec{C}}{dt}_{in} &= \frac{dC'_x}{dt}\hat{\imath}' + \frac{dC'_y}{dt}\hat{\jmath}' + \frac{dC'_z}{dt}\hat{k}' \\ &\quad + C'_x\frac{d\hat{\imath}'}{dt} + C'_y\frac{d\hat{\jmath}'}{dt} + C'_z\frac{d\hat{k}'}{dt} \end{aligned}$$

What someone in the rotating frame would measure

$$\frac{d\vec{C}}{dt}_{rot} = \hat{\imath}'\frac{d\vec{C}'_x}{dt} + \hat{\jmath}\frac{d\vec{C}'_y}{dt} + \hat{k}\frac{d\vec{C}'_z}{dt}.$$

We can see this matches the first three terms of the derivative of the position vector in the inertial frame.

For the next 3 terms

$$\begin{aligned}
C'_x\frac{d\hat{i}'}{dt} + C'_y\frac{d\hat{j}'}{dt} + C'_z\frac{d\hat{k}'}{dt} &= C'_x\vec{\Omega}\times\hat{i}' + C'_y\vec{\Omega}\times\hat{j}' + C'_z\vec{\Omega}\times\hat{k}' \\
&= \vec{\Omega}\times(C'_x\hat{i}' + C'_y\hat{j}' + C'_z\hat{k}') \\
&= \vec{\Omega}\times\vec{C}.
\end{aligned}$$

Now to describe the motion of something measured in the inertial frame in quantities of what someone in a rotating frame would measure we can put it together

$$\frac{d\vec{C}}{dt}_{in} = \frac{d\vec{C}}{dt}_{rot} + \vec{\Omega}\times\vec{C}$$

Applying to this to the position vector and taking the derivative we will see the appearance of 2 more pseudo forces the centrifugal and coriolis.

$$\begin{aligned}
\vec{v}_{in} &= \vec{v}_{rot} + \vec{\Omega}\times\vec{r} \\
\vec{a}_{in} &= \vec{a}_{rot} + 2\vec{\Omega}\times\vec{v}_{rot} + \vec{\Omega}\times(\vec{\Omega}\times\vec{r}) \\
\vec{a}_{rot} &= \vec{a}_{in} - 2\vec{\Omega}\times\vec{v}_{rot} - \vec{\Omega}\times(\vec{\Omega}\times\vec{r})
\end{aligned}$$

Multiply the accelerations with mass we have our 2 pseudo forces

$$\begin{aligned}
\vec{F}_{coriolis} &= -2m\vec{\Omega}\times\vec{r}_{rot} \\
\vec{F}_{centrifugal} &= -m\vec{\Omega}\times(\vec{\Omega}\times\vec{r}).
\end{aligned}$$