Definition 1. *The minimum load of τ_i in Z_k is the minimum units of execution time that must be assigned to τ_i in Z_k to meet the deadline.*

$$Lt_{i,k} = MAX(0, ZWidth_k - L_{i,k})$$

Definition 2. *The maximum load of τ_i in Z_k is the maximum units of execution time that can be assigned to τ_i in Z_k considering the WCET of a job and the width of a Zone(to ensure a task can only be executed in a processor at a time).*

$$Mt_{i,k} = MIN(R_{i,k}, ZWidth_k)$$

Definition 3. *The execution assignment is **rational** in Z_k if execution time assigned to each task is between minimum load and maximum load, and the sum of execution time of tasks in Γ doesn't exceed the capacity of Z_k.*

$$\forall i \in [0, n) : Lt_{i,k} \leq alloc_{i,k} \leq Mt_{i,k}$$

$$\sum_{i=0}^{n-1} alloc_{i,k} \leq m \cdot ZWidth_k$$

Definition 4. ***Overload effect** occurs if the total minimum load of tasks in Γ is more than the capacity of Z_k.*

$$\sum_{i=0}^{n-1} Lt_{i,k} > m \cdot ZWidth_k$$

Definition 5. *Zone Z_p **zone-connects** to zone Z_q in $J_{i,j}$, denoted as $Z_p \to Z_q$, if*

$$Z_p \in J_{i,j}, Z_q \in J_{i,j}$$

and

$$alloc_{i,p} > 0, alloc_{i,q} < ZWidth_q.$$

*The connection between Z_p and Z_q is called a **zone-connection**.*

Definition 6. *Zone Z_p **connects** to zone Z_q denoted as $Z_p \Rightarrow Z_q$ if*

$$Z_p \to Z_q$$

or

$$\exists Z_k : Z_p \Rightarrow Z_k, Z_k \Rightarrow Z_q.$$

Definition 7. *During scheduling, the overload effect occurs in the current zone Z_k. A zone Z_d is called a **destination zone** if*
(1) $\exists \tau_i \in \Gamma : R_{i,s} < C_i, L_{i,s} > ZWidth_s$, $J_{i,j}$ is the current job of τ_i, $Z_d \in J_{i,j}$ and $alloc_{i,d} > 0$. or
(2) $Rem_d > 0$.
*The element $alloc_{i,d}$ in scheduling matrix is called a **destination point**.*

Lemma 1. *When overload effect occurs in Z_k, there exist a destination zone Z_d, if $Rem_d = 0$, there exist a destination point $alloc_{i,d}$.*

Proof. (1)$\exists i \in [0, k), Rem_i > 0$, Z_i is a destination zone.
(2)$\forall i \in [0, k), Rem_i = 0$. We assume that $\forall \tau_i \in \Gamma : alloc_{i,k} > 0$, then:
for $\tau_i \in \Gamma$, $J_{i,j}$ is the current job. If $JEnd_{i,j} = ZEnd_k$, then

$$\frac{\sum\limits_{t=k+1}^{znum-1} alloc_{i,t}}{\sum\limits_{t=k+1}^{znum-1} ZWidth_t} = \frac{C_i}{T_i}$$

If $JEnd_{i,j} > ZEnd_k$, then

$$\frac{\sum\limits_{t=k+1}^{znum-1} alloc_{i,t}}{\sum\limits_{t=k+1}^{znum-1} ZWidth_t} = \frac{C_i \cdot rcj_i + JEnd_{i,j} - ZEnd_k}{T_i \cdot rcj_i + JEnd_{i,j} - ZEnd_k} > \frac{C_i}{T_i}$$

(rcj_i is the number of remaining complete job of τ_i).
So the total remaining execution units needed in the zones after Z_k is

$$\sum_{i=0}^{n-1} \sum_{t=k+1}^{znum-1} alloc_{i,t} \geq \sum_{i=0}^{n-1} \frac{C_i}{T_i} \cdot \sum_{t=k+1}^{znum-1} ZWidth_t = U_\Gamma \cdot \sum_{t=k+1}^{znum-1} ZWidth_t$$

This contratict to

$$\sum_{i=0}^{n-1} \sum_{t=k}^{znum-1} alloc_{i,t} < U_\Gamma \cdot \sum_{t=0}^{znum-1} ZWidth_t - m \cdot \sum_{t=0}^{k} < U_\Gamma \cdot \sum_{t=k+1}^{znum-1} ZWidth_t.$$

So $\exists \tau_i \in \Gamma : alloc_{i,k} = 0$, which means $L_{i,s} > ZWidth_s$. $\Gamma_e = \{\tau_i : \tau_i \in \Gamma, alloc_{i,k} = 0\}$.

We assume that $\forall \tau_i \in \Gamma_e, R_{i,k} = C_i$. For $\tau_i \in \Gamma_e$,

$$\frac{\sum\limits_{t=k+1}^{znum-1} alloc_{i,t}}{\sum\limits_{t=k+1}^{znum-1} ZWidth_t} = \frac{C_i + C_i \cdot rcj}{T_i - (ZEnd_k - JStart_{i,j_i}) + T_i \cdot rcj} \leq \frac{C_i}{T_i}.$$

For $\tau_i \notin \Gamma_e$,

$$\frac{\sum\limits_{t=k+1}^{znum-1} alloc_{i,t}}{\sum\limits_{t=k+1}^{znum-1} ZWidth_t} = \frac{T_i - (ZEnd_k - JStart_{i,j_i} + C_i \cdot rcj}{T_i - (ZEnd_k - JStart_{i,j_i}) + T_i \cdot rcj} \leq \frac{C_i}{T_i}.$$

So the total remaining execution units needed in the zones after Z_k is

$$\sum_{i=0}^{n-1}\sum_{t=k+1}^{znum-1} alloc_{i,t} \geq \sum_{i=0}^{n-1}\frac{C_i}{T_i}\cdot\sum_{t=k+1}^{znum-1} ZWidth_t = m\cdot\sum_{t=k+1}^{znum-1} ZWidth_t$$

This contratict to

$$\sum_{i=0}^{n-1}\sum_{t=0}^{znum-1} alloc_{i,t} = m * \sum_{t=0}^{znum-1} ZWidth_t.$$

So $\exists\tau_i \in \Gamma_e : R_{i,k} < C_i$. In conclusion, $\exists\tau_i \in \Gamma : R_{i,k} < C_i, L_{i,s} > ZWidth_s$, there exist a destination zone.

$\square$

Lemma 2. *For a taskset Γ with a total utilization of $m * ut$ $(0 < ut \leq 1)$, if execution assignment is rational in each zone Z_i $(i \in [0,k), k \leq znum)$, and overload effect occurs in zone Z_k, then $\exists Z_d$ is a destination zone, $d \neq k$, $Z_k \Rightarrow Z_d$.*

Proof. We consider interval $[ZEnd_k, hp)$ as a special big zone Z_B, $ZStart_B = ZEnd_k$; $ZEnd_B = hp$; $ZWidth_B = hp - ZEnd_k$; $Cap_B = ZWidth_B \cdot m$.
$Zones = \{Z_i : i = 0, 1, 2, ..., k, B\}$; $S = \{Z : Z \in Zones, Z_k \Rightarrow Z\}$; $\bar{S} = \{Z : Z \in Zones, Z \notin S, Z \neq Z_k\}$.
$overflow = \sum_{i=0}^{n-1} alloc_{i,k} - m \cdot ZWidth_k$; $tr = \sum_{i=0}^{k-1} Rem_i$.
For a job $J_{i,j}$, $S_1 = J_{i,j} \cap S$, $S_2 = J_{i,j} \cap \bar{S} \neq \emptyset$.

$$\sum_{Z_l \in S_2} alloc_{i,l} = MIN(C_i, \sum_{Z_l \in S_2} ZWidth_l) \geq \frac{C_i}{T_i}\sum_{Z_l \in S_2} ZWidth_l$$

The equality holds only if $S_1 = \emptyset$. So for all jobs having an intersection with $\bar{S}$, the total executing time in $\bar{S}$:

$$E(\bar{S}) = \sum_{J_{i,j}\cap\bar{S}\neq\emptyset}\sum_{Z_l \in J_{i,j}\cap\bar{S}} alloc_{i,l} \geq \sum_{Z_l\in\bar{S}} ZWidth_l \sum_{i=0}^{n-1}\frac{C_i}{T_i} = m\cdot ut\sum_{Z_l\in\bar{S}} ZWidth_l.$$

The equality holds only if for each task $\tau_i \in \Gamma$, any job of τ_i locates complete in either S or $\bar{S}$. It is known that only time 0 and hp can be such boundaries, so we can get that

$$E(\bar{S}) > m \cdot ut \sum_{Z_l\in\bar{S}} ZWidth_l. \tag{1}$$

On the other hand, the assigned time units of Z_l is

$$A(Z_l) = \begin{cases} m \cdot ZWidth_l - Rem_l & l = 0, 1, 2, ..., k-1 \\ m \cdot ut \cdot hp - m\sum_{i=0}^{k} ZWidth_i + tr - overflow & l = B \end{cases}$$

case1: $Z_B \in S$
In this case, $Z_k \Rightarrow Z_B$. Assuming that $Z_k \rightarrow Z_B$, then $\exists \tau_i \in \Gamma$, the current job $J_{i,j}$ satisfys the conditions that $JStart_{i,j} \leq ZStart_k$, $JEnd_{i,j} \geq ZEnd_l$, $alloc_{i,k} > 0$, and $alloc_{i,l} < ZWidth_l$. This contradict to the definition of overload effect. So $\exists Z_{o1}, Z_{o2}, ..., Z_{oz} \in Zones : Z_k \rightarrow Z_{o1}, Z_{o1} \rightarrow Z_{o2}, ..., Z_{oz} \rightarrow Z_B$, $Z_{oz} \neq Z_k, Z_{oz} \neq Z_B$.
$\exists J_{i,j}$ is a current job, $alloc_{oz} > 0$ and $L_{i,k} > ZWidth_k$, which means Z_{oz} is a destination zone.
case2: $Z_B \in \bar{S}$, $\bar{S}' = \bar{S} - Z_B$

$$\begin{aligned} A(\bar{S}) &= m \cdot ut \cdot hp - m \sum_{Z_l \in \bar{S}' \cup Z_k} ZWidth_l + tr - \sum_{Z_l \in \bar{S}'} Rem_l - overflow \\ &\leq m \cdot ut \sum_{Z_l \in \bar{S}} ZWidth_l + tr - \sum_{Z_l \in \bar{S}'} Rem_l - overflow \end{aligned}$$

If $\exists Z_d \in S : Rem_d > 0$, Z_d is the destination zone. Assume that $\forall Z_l \in S, Rem_l = 0$, we can get that $tr = \sum_{Z_l \in \bar{S}'} Rem_l$, $A(\bar{S}) \leq m \cdot ut \sum_{Z_l \in \bar{S}} ZWidth_l$. This is contradict to inequation 1.
In conclusion, $\exists Z_d$ is a destination zone, $d \neq k$, $Z_k \Rightarrow Z_d$. $\square$