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Effects of nearest neighbors interactions on control of nonlinear vehicular platooning

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***Abstract*— The paper introduces a novel platooning model for vehicles subject to nonlinear drag. A linear parametric controller, taking into account both the actual position and the relative displacement with respect to neighbors, is applied to each vehicle and, then, its effects are investigated via a semi-analytical approach based on partial differential equations. When the control law on the interactions with the neighbors is asymmetric, the platoon can exhibit undesired oscillations around the desired formation. Numerical simulations confirm the predicted results.**

I. INTRODUCTION

In the last decade technological advances have risen a strong interest on unmanned and driving-assisted vehicles, especially thanks to a wide range of possible applications. The recent studies in terrestrial, naval and aerial platoons are settled in this framework (see, e.g., [1], [2], [3] and references therein), and they are aimed to solve a number of different problems deriving from the original goal of moving the vehicles along a certain path while preserving a specific formation. Examples in this regard range from a group of drones monitoring a territory [4] to cars moving on a road avoiding collisions [5]. In the study of platoons the actual number of considered vehicles plays a crucial role, since, as it increases, the complexity of the whole system grows rapidly. For this reason, the common approach consists in using simple models for the single unit. Within this general idea the so-called single- and double-integrator models have been widely used in the recent literature (see, e.g., [6], [7] and references therein).

In this paper we introduce a platoon based on a second order nonlinear unit model, which takes into account possible realistic nonlinear effects on the vehicle motion. Such a representation is meant to cover a wide range of situations, and therefore it can take into account information both about the absolute position of the unit (given, e.g., by GPS devices or by the driver herself), and about the relative displacement with respect to the neighbors (provided, e.g., by proximity sensors or again by the driver). Then, we enforce on each vehicle a similar parametric control law with fixed structure, and we investigate the appearance of propagative phenomena, as the parameters vary, by exploiting a partial differential equation (PDE) formulation of the problem.

In Section II the platooning model is introduced as a nonlinear extension of the double-integrator one. For the sake of simplicity the parametric control law is chosen as a possible local feedback on the absolute position and on the relative displacement from the neighbors. A locally diffeomorphic PDE model of the platoon is finally derived. In Section III the conditions for the emergence of propagative phenomena in the form of traveling waves are investigated through the analysis of the dynamics exhibited by a proper ordinary differential equation (ODE) denoted as *reference ODE*. Then, a numerical method to efficiently compute the linear spectrum associated to the linearized problem is developed in Section IV. Finally, some numerical simulations validating the expected behaviors are presented in Section V, while the conclusions of Section VI ends the paper.

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II. THE PLATOONING MODEL

Let us consider a platoon of N vehicles moving along a line, that can be regarded as the path the units are forced to go through on. Let p_n be the absolute position of the n-th vehicle along this path, $v_n := \dot{p}_n$ its velocity and $a_n := \ddot{p}_n$ its acceleration. The goal of the control strategy is to have the units moving with a certain desired velocity v_d and maintaining a specific formation described by the desired positions $p_{n,d} := v_d t + k_n$, where k_n represents the absolute displacement of the n-th unit along the line. We introduce the relative displacement $z_n := p_n - p_{n,d}$ of the n-th vehicle from its desired position, and also its relative distance $s_n := p_{n+1} - p_n$ with respect to the neighbor in front of it. Denoting $\Delta k_n := k_{n+1} - k_n$, it follows that

$$\begin{aligned} s_n &= p_{n+1} - p_{n+1,d} - p_n + p_{n,d} + p_{n+1,d} - p_{n,d} \\ &= z_{n+1} - z_n + k_{n+1} - k_n = z_{n+1} - z_n + \Delta k_n \ . \end{aligned}$$

In this paper we extend the double-integrator model to a more realistic setting assuming that each vehicle is subjected to a nonlinear friction force $f_n(\dot{p}_n)$, due to the medium where the motion occur and depending nonlinearly on the unit speed (see [7] for a linearized drag model). Regarding to the control law $g_n(z_n, s_n, s_{n-1})$ of the n-vehicle, we assume that it acts on the unit acceleration, and, for the sake of simplicity, that it is made up of three additive linear components: a constant term aimed to compensate the friction at the desired velocity, a linear term depending on the displacement from the desired position, and a linear term taking care of the distance with respect to the neighbors. Therefore, the complete model of the n-th unit reads

$$\begin{aligned}
\ddot{p}_n = \ddot{z}_n &= f_n(\dot{p}_n) + g_n(z_n, s_n, s_{n-1}) \\
&= -\left(a_n\dot{p}_n + b_n\dot{p}_n^3\right) - (u_n + c_n z_n) + \\
&+ \alpha_n(s_n - \Delta k_n) - \beta_n(s_{n-1} - \Delta k_{n-1}) \\
&= -\left(a_n(\dot{z}_n + v_d) + b_n(\dot{z}_n + v_d)^3\right) - (u_n + c_n z_n) + \\
&+ \alpha_n(z_{n+1} - z_n) - \beta_n(z_n - z_{n-1}) \\
&= -c_n z_n - \left(a_n + 3b_n v_d^2\right)\dot{z}_n - 3b_n v_d \dot{z}_n^2 - b_n\dot{z}_n^3 + \\
&+ \alpha_n(z_{n+1} - z_n) - \beta_n(z_n - z_{n-1}) \ , \qquad (1)
\end{aligned}$$

where $u_n = -\left(a_n v_d + b_n v_d^3\right)$ provides the proper acceleration to win over the friction at the speed v_d. In order to provide an actual friction and a reasonable control law (i.e. such that the given acceleration is addressed to move the vehicle closer to the desired position) it is also assumed that b_n, c_n, α_n and β_n are positive. Observe that depending on the values of the parameters c_n, α_n and β_n this model can describe both a platoon driven by an absolute positioning system (provided, for example, by GPS devices), and a group of units provided only with proximity sensors (such as sonars or lasers).

In general, for $n = 2, \ldots, N-1$ each vehicle is governed by a second order ODE system of the form

$$\begin{aligned}
\dot{y}_{n,1} &= y_{n,2} \\
\dot{y}_{n,2} &= -c_n y_{n,1} - (a_n + 3b_n v_d^2)y_{n,2} - 3b_n v_d y_{n,2}^2 \\
&\quad - b_n y_{n,2}^3 + \alpha_n(y_{n+1,1} - y_{n,1}) - \beta_n(y_{n,1} - y_{n-1,1}) \ .
\end{aligned}$$

The first and the last unit of the platoon, instead, have only a single neighbor and need specific models. Here, we use the common practice of introducing a *fictitious leader* and a *fictitious follower* in order to have a uniform model for all the vehicles (see, e.g., [6]). The typical assumption in the *pure neighboring scenario* is that these two new units always stay in the desired position. Anyways, it is worth observing that the first and last vehicle, in order to create the fictitious units, have to be provided with an absolute positioning device, meaning that the formation problem can not be solved without a way to derive the exact positions.

In this paper, we let the fictitious units free to move around their desired positions, the leader behaving as the last vehicle and the follower as the first, i.e.

$$\begin{aligned}
\dot{y}_{1,1} &= y_{1,2} \\
\dot{y}_{1,2} &= -c_1 y_{1,1} - (a_1 + 3b_1 v_d^2)y_{1,2} - 3b_1 v_d y_{1,2}^2 \\
&\quad - b_1 y_{1,2}^3 + \alpha_1(y_{2,1} - y_{1,1}) - \beta_1(y_{1,1} - y_{N,1}) \ , \\
\dot{y}_{N,1} &= y_{N,2} \\
\dot{y}_{N,2} &= -c_N y_{N,1} - (a_N + 3b_N v_d^2)y_{N,2} - 3b_N v_d y_{N,2}^2 \\
&\quad - b_N y_{N,2}^3 + \alpha_N(y_{1,1} - y_{N,1}) - \beta_N(y_{N,1} - y_{N-1,1}) \ .
\end{aligned}$$

Observe that such an assumption is equivalent to enforce a periodic boundary condition on the platoon. Moreover, it requires that the first and last unit are able to communicate, i.e. they still have to be provided with more complex devices than the rest of the vehicles.

Let us now embed the platoon into a continuous space according to the procedure illustrated in [8], [9]. To this aim, consider $\mathcal{X} \subseteq \mathbb{R}$ and choose N equally separated points $x_n \in \mathcal{X}$, so that $x_{n+1} - x_n = \delta x$ for all $n = 1, \ldots, N$, the x_n being associated to the n-th vehicle. Without any loss of generality, here we assume $\delta x > 0$. Then, choose a sufficiently smooth function $\phi(t,x)$ (see [8] for further details) in order to interpolate the spatial coordinate of each unit at their representative point, i.e. such that $\phi(t,x_n) = p_n(t)$. Substituting it into the n-th unit dynamics (1), one obtains

$$\begin{aligned}
&\partial_{tt}\phi(t,x_n) = f_n\left(\partial_t\phi(t,x_n), x_n\right) \\
&+ g_n\left(\phi(t,x_{n+1}), \phi(t,x_n), \phi(t,x_{n-1}), x_n\right) \\
&= -c(x_n)\phi(t,x_n) - \left(a(x_n) + 3v_d^2 b(x_n)\right)\partial_t\phi(t,x_n) \\
&- 3v_d b(x_n)(\partial_t\phi(t,x_n))^2 - b(x_n)(\partial_t\phi(t,x_n))^3 \\
&+ \alpha(x_n)(\phi(t,x_{n+1}) - \phi(t,x_n)) \\
&- \beta(x_n)(\phi(t,x_n) - \phi(t,x_{n-1})) \ .
\end{aligned}$$

The dependence on x_{n+1} and x_{n-1} can be removed by observing that

$$\phi(t,x_{n\pm1}) = \phi(t,x_n) \pm \partial_x\phi(t,x_n)\delta x + O\left(\delta x^2\right) \ ,$$

that leads to

$$\begin{aligned}
&\partial_{tt}\phi(t,x) = -c(x)\phi(t,x) - \left(a(x) + 3v_d^2 b(x)\right)\partial_t\phi(t,x) \\
&- 3v_d b(x)(\partial_t\phi(t,x))^2 - b(x)(\partial_t\phi(t,x))^3 \\
&+ \alpha(x)\partial_x\phi(t,x)\delta x - \beta(x)\partial_x\phi(t,x)\delta x + O(\delta x^2) \ ,
\end{aligned}$$

where the original dynamics on the points x_n has been extended to the whole space $\mathcal{X}$. As illustrated in [8], under suitable choices for $\mathcal{X}$, δx and $\phi(t,x)$ the residual $O\left(\delta x^2\right)$ can be removed obtaining a finite order PDE, whose solutions are diffeomorphic to those of the original platoon.

Remark 2.1: The choice of the method introdued in [8] is motivated by the desire of bulding an actual equivalence between the dynamics of the original platoon (1) and its PDE representation. This property is granted here by a local diffeomorphism, a feature that is not possessed by other PDE approaches in the literature (see, e.g., [10], [11]).

Hereafter, for the sake of simplicity we assume that both the uncontrolled dynamics f_n and the local control law g_n are the same for each vehicle, i.e. that

$$a(x) = a \ , \ b(x) = b \ , \ c(x) = c \ , \ \alpha(x) = \alpha \ , \ \beta(x) = \beta \ .$$

The related PDE, then, assumes the form

$$\begin{aligned}
\partial_{tt}\phi(t,x) = -c\phi(t,x) - \left(a + 3v_d^2 b\right)\partial_t\phi(t,x) \\
- 3v_d b(\partial_t\phi(t,x))^2 - b(\partial_t\phi(t,x))^3 \\
+ \alpha\delta x\partial_x\phi(t,x) - \beta\delta x\partial_x\phi(t,x) \qquad (2)
\end{aligned}$$

and, according to the previous assumptions on the fictitious units, it is also subjected to periodic boundary conditions.

III. Propagating phenomena analysis

In this paper we aim to investigate the existence of propagating phenomena such as fronts or traveling waves in the proposed model, and their relationship with the control law components associated to the actual position and to the neighbors. In order to look for these kind of dynamics, we consider for the PDE model (2) a prototype solution of the form

$$\phi(t,x) = \varphi(Kx + Ct) \ ,$$

where φ is a periodic unknown function, $K \in \mathbb{R}$ is the wave number, and $C \in \mathbb{R}$ is the propagation speed. Substituting such a solution into the above PDE and neglecting the function argument for the sake of notation, one obtains the so-called *reference ODE* [8]

$$\ddot{\varphi} + \frac{c}{C^2}\varphi + \frac{\left(a + 3v_d^2 b\right)C - (\alpha-\beta)K\delta x}{C^2}\dot{\varphi} \\ + 3v_d b\dot{\varphi}^2 + bC\dot{\varphi}^3 = 0 \ ,$$

whose solutions can be regarded to as propagating phenomena representing actual behaviors in the original platoon.

Remark 3.1: The PDE (2) is diffeomorphic to the original platoon, but the reference ODE is not. However, any solution of the reference ODE still corresponds to a traveling wave in the PDE problem. The stability of this latter can not be derived by the analysis of the first one and requires a more sophisticated analysis, as shown in Section IV.

Remark 3.2: The "stability" of the wave solution φ with respect to the reference ODE does not relate to the actual stability of the corresponding $\phi(t,x)$ in the original PDE problem. Indeed, the evaluation of the stability through the reference ODE is equivalent to taking into account only perturbations of the same kind of φ, wave number and propagation speed included. Therefore, a more sophisticated analysis is required, as shown in Section IV.

The reference ODE is now analyzed in order to find its periodic solutions φ, since such trajectories correspond to possible traveling wave solutions of the PDE problem. Their stability, however, will be studied later by using other tools. Thus, let us introduce

$$h_1 = -\frac{c}{C^2} \ , \ h_2 = -\frac{\left(a + 3v_d^2 b\right)C - (\alpha-\beta)K\delta x}{C^2} \ , \\ h_3 = -3v_d b \ , \ h_4 = -bC \ ,$$

and define $\xi_1 = \varphi$, $\xi_2 = \dot{\varphi}$. Then, the reference ODE assumes the convenient form

$$\dot{\xi}_1 = \xi_2 \quad (3)$$

$$\dot{\xi}_2 = h_1\xi_1 + h_2\xi_2 + h_3\xi_2^2 + h_4\xi_2^3 \ . \quad (4)$$

The unique equilibrium point of the above system is located in the origin $\xi_{0,1} = 0$, $\xi_{0,2} = 0$ and it corresponds in the PDE problem to a flat profile of zero value, i.e. it represents the case where all the displacements z_n of the vehicles vanish and the formation goal is reached. Observe that $\varphi \equiv 0$ is an actual periodic solution, even though its period, and then C and K, are undetermined. Moreover, the related zero-flat profile can be regarded to as a wave propagating towards both the directions of the line, depending on the signs of K and C.

Remark 3.3: It is worth observing that, in general, K is not allowed to assume any possible value. Indeed, the periodic boundary conditions enforce the quantization of K to a numerable set of values, such that the condition $T = \sigma K N \delta x$ is satisfied for some integer σ, T being the period of φ. Therefore, the candidate solutions φ have to be narrowed to the ones existing for a proper set of the actually supported K's.

In order to locate other periodic φ's, the trajectories structure in the plane (ξ_1, ξ_2) is studied. To this aim, we investigate the bifurcation diagram of the equlibrium in the origin of the reference ODE problem.
Since the Jacobian of system (3)-(4) is

$$J(\xi_1,\xi_2) = \begin{bmatrix} 0 & 1 \\ h_1 & h_2 + 2h_3\xi_2 + 3h_4\xi_2^2 \end{bmatrix} \ ,$$

the eigenvalues of the equilibrium in the origin can be evaluated as the roots λ of

$$\det(\lambda I - J(0,0)) = \lambda^2 - h_2\lambda - h_1 = 0 \ .$$

Their explicit form is

$$\lambda_{1,2} = \frac{h_2}{2} \pm \frac{1}{2}\sqrt{h_2^2 + 4h_1} \ ,$$

which means that the origin is "stable", if $h_2 < 0$, $h_1 < 0$, that is, if

$$\left(a + 3v_d^2 b\right)C > (\alpha-\beta)K\delta x \ ,$$

being $c > 0$ by assumption.
Since we are primarily interested in oscillations around the desired positions, i.e. around the zeros of the displacements, we look for periodic solutions around the unique equilibrium of the reference ODE. To this aim, the first important observation is that the origin needs to be "unstable", i.e. h_2 has to be positive. Moreover, it is known that the integral of the differential quantity $\dot{\xi}_1 d\xi_1 - \dot{\xi}_2 d\xi_2$ along a closed curve equals the integral of the system *divergence* over the area inside the same curve [12, c. 2]. Therefore, if a limit cycle exists, it has to span regions where the divergence changes its sign. Then, let us also compute the divergence of the reference ODE:

$$\mathrm{div}(\xi_1,\xi_2) = \partial_{\xi_1}\dot{\xi}_1 + \partial_{\xi_2}\dot{\xi}_2 = h_2 + 2h_3\xi_2 + 3h_4\xi_2^2 \ .$$

The regions of the plane (ξ_1, ξ_2) where $\mathrm{div}(\xi_1,\xi_2)$ has different signs are stripes parallel to ξ_1, separated by the critical values (when real)

$$-\frac{h_3}{3h_4} \pm \frac{1}{3h_4}\sqrt{h_3^2 - 3h_2h_4} \ .$$

In order for a limit cycle to exists, the divergence has not to be constant along it and, therefore, the critical condition $h_3^2 - 3h_2h_4 > 0$ has to be satisfied. Two main scenarios can be distinguished. In the first case $h_4 < 0$, corresponding to $C > 0$, the critical condition is always satisfied and the

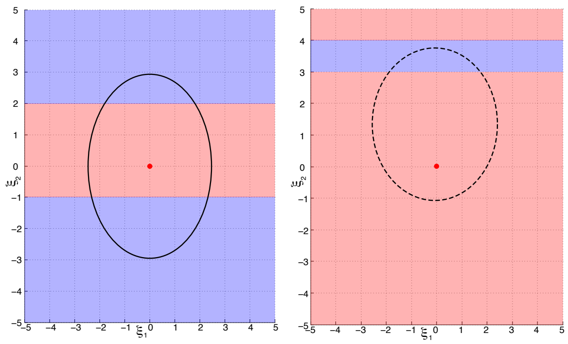

Fig. 1. Possible configurations of the divergence with respect to h_4 along with paths around the origin. Blue/red: negative/positive divergence.

outer stripes are area contracting regions ($\mathrm{div}(\xi_1, \xi_2) < 0$). This situation is compatible with the existence of a periodic solution around the origin, and since as $h_2 > 0$ grows the central stripe widens, it suggests that for increasing values of h_2 the possible cycle enlarges itself around the origin. The second case, instead, corresponds to $h_4 > 0$, i.e. to $C < 0$. In this situation the outer stripes are area expanding regions ($\mathrm{div}(\xi_1, \xi_2) > 0$), while the central one is contracting. In such a case, as h_2 grows, the central stripe distances itself from to the origin and also it shrinks until it vanishes for a sufficient big value of h_2. Under such conditions the more h_2 increases, the more unlikely is that the integral of the divergence can be zero on a path encircling the origin, and therefore a limit cycle may exist. A sketch of the two situations is depicted in Figure 1.

Summing up the above reasoning, the search of periodic solutions φ can be narrowed to the case $h_2 > 0$ and $C > 0$. Indeed, under these conditions there exists a very compatible scenario for the emergence of symmetric oscillations around the origin, that is the Hopf bifurcation [13]. For this phenomenon to occur, the unique equilibrium of the reference ODE has to undergo the change from stable to unstable focus. Then, oscillations arise growing as bigger as the equilibrium becomes more unstable. Such a scenario leads to the conditions $h_2 > 0$, $h_2^2 + 4h_1 < 0$. Notice that, since h_1 is always negative thanks to the assumptions $c > 0$ and $C > 0$, these inequalities boil down to $0 < h_2 < 2\sqrt{-h_1}$, whose explicit form, then, reads

$$0 < (\alpha - \beta)K\delta x - \left(a + 3v_d^2 b\right) C < 2C\sqrt{c} \;, \tag{5}$$

where we have used the previous condition $C > 0$.

The inequalities (5) only represent necessary conditions for a periodic solution φ to be born around the origin via Hopf bifurcation, but they reveal some useful insights about the existence of traveling waves in the original problem. First, notice that, when $\alpha = \beta$, the conditions (5) can be satisfied only for negative values of the propagation speed, but $C < 0$ is prevented by the previous reasoning. Therefore, in the symmetric case the reference ODE can not show any other periodic solution, and thus the platoon is expected to not support traveling waves. Then, observe that when $\alpha \neq \beta$ even positive values of C may be supported, given $(\alpha - \beta)K > 0$, which implies that the direction of propagation of the wave (i.e. the sign of K) depends on the largest between α and β. In general, in the non symmetric case conditions (5) admit infinite admissible pairs (K, C), but, as already explained, they have to be narrowed down to the K's actually supported by the periodic boundary conditions.

Remark 3.4: If (K, C) is a proper couple, then by exchanging the values of α and β it follows that $(-K, C)$ is a proper couple for the new case. This behavior is reasonable and already expected, because of the intrinsic symmetries of the problem with respect to the moving direction of the platoon.

IV. Linear stability of waves

When dealing with the solution of a PDE, a common tool for studying its stability is the computation of the *continuous spectrum* of the functional operator (2), evaluated at the nominal solution. Such a spectrum refers to the so-called *linear stability* of $\phi(t, x)$, that only consider the local point-wise dynamics around it [14].

Remark 4.1: It is worth stressing that the linear stability is a very strict concept that does not take into account a number of behaviors, which are commonly regarded to as stable dynamics. For instance, a perturbation, that converges to a time-shifted version of the nominal wave, enforces in the linear spectrum positive values, denoting an unstable dynamics due to the locally diverging motion (see [14] for further insights on the topic).

In order to study the linear stability of a traveling wave, let us introduce the moving coordinate $\zeta = Kx + Ct$ and then transform $\phi(t, x)$ into $\phi(t, \zeta)$. Hence, the original PDE becomes

$$\begin{aligned}\partial_{tt}\phi(t,\zeta) = &-c\phi(t,\zeta) - \left(a + 3v_d^2 b\right)\partial_t\phi(t,\zeta) \\ &- 3v_d b(\partial_t\phi(t,\zeta))^2 - b(\partial_t\phi(t,\zeta))^3 \\ &+ \alpha\delta x\partial_\zeta\phi(t,\zeta)K - \beta\delta x\partial_\zeta\phi(t,\zeta)K\end{aligned}$$

Defining

$$v(t,\zeta) \coloneqq \phi(t,\zeta) \;,\; w(t,\zeta) \coloneqq \partial_t\phi(t,\zeta) \;, \tag{6}$$

the problem reads

$$\partial_t v(t,\zeta) = w(t,\zeta) \tag{7}$$

$$\begin{aligned}\partial_t w(t,\zeta) = &-cv(t,\zeta) - \left(a + 3v_d^2 b\right) w(t,\zeta) - 3v_d b w^2(t,\zeta) \\ &- bw^3(t,\zeta) + (\alpha - \beta)K\delta x\partial_\zeta v(t,\zeta) \;.\end{aligned} \tag{8}$$

Observe now that a traveling wave is a stationary solution of (7)-(8) and therefore it satisfies

$$\begin{aligned}0 &= w_s(t,\zeta) \\ 0 &= -cv_s(t,\zeta) + (\alpha - \beta)K\delta x\partial_\zeta v_s(t,\zeta) \;.\end{aligned}$$

In order to investigate the linear dynamics of local perturbations around a wave, let us consider a perturbed solution of the form

$$v(t,\zeta) = v_s(t,\zeta) + \nu(\zeta)e^{\lambda t} \;, \tag{9}$$

$$w(t,\zeta) = w_s(t,\zeta) + \mu(\zeta)e^{\lambda t} \;. \tag{10}$$

Substituting (9)-(10) into (7)-(8), and balancing the first order terms $e^{\lambda t}$, i.e. by considering the only linear part of the dynamics, it follows that the perturbation satisfies

$$\lambda\nu(\zeta) = \mu(\zeta) \ , \tag{11}$$
$$\lambda\mu(\zeta) = \nu(\zeta) - \left(a + 3v_d^2 b\right)\mu(\zeta) + (\alpha - \beta)K\delta x \partial_\zeta \nu(\zeta) \ . \tag{12}$$

Notice that problem (11)-(12) can be efficiently solved numerically. Indeed, consider m samples ν_i and μ_i, $i = 1, \dots, m$ along the perturbations of the trajectory, covering the wave in the space interval $(0, N\delta x)$ at a certain fixed instant. Then, (11)-(12) can be discretized to

$$\mu_n = \lambda\nu_n$$
$$\frac{(\alpha-\beta)K}{2}\left(\nu_{n+1} - \nu_{n-1}\right) + \nu_n - \left(a + 3v_d^2 b\right)\mu_n = \lambda\mu_n$$

along with the periodic conditions

$$\nu_{m+1} = \nu_1 \ , \ \mu_{m+1} = \mu_1 \ , \ \nu_{-1} = \nu_m \ , \ \mu_{-1} = \mu_m \ .$$

By defining the following quantities

$$\chi := \begin{bmatrix} \nu_1 & \dots & \nu_m & \mu_1 & \dots & \mu_m \end{bmatrix}^T$$
$$A := \begin{bmatrix} 0 & I \\ I + \Omega\left(\frac{(\alpha-\beta)K}{2}\right) & -\mathrm{diag}\left(a + 3v_d^2 b\right) \end{bmatrix}$$
$$\Omega(\tau) := \begin{bmatrix} 0 & \tau & 0 & \dots & -\tau \\ -\tau & 0 & \tau & \dots & 0 \\ 0 & -\tau & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \tau \\ \tau & 0 & \dots & -\tau & 0 \end{bmatrix}$$

the discretization of the original perturbation problem (11)-(12) boils down to

$$A\chi = \lambda\chi \tag{13}$$

that is a standard eigenvalue problem, whose solutions can be obtained efficiently via numerical techniques. Hence, the λ's represent a discretization of the continuous spectrum of the original operator (2).

It is worth highlighting that (13) does not actually depend on the nominal solution ν_s, μ_s, meaning that the linear stability of each traveling wave, the zero-flat solution included, is exactly the same. For this reason is very unlikely that the corresponding continuous spectrum will provide a stable configuration, i.e. all the λ's located in the left semi-plane of the complex plane. However, we stress again that the linear instability is perfectly compatible with other kinds of stability concept, such as orbital (or nonlinear) stability [14].

V. Numerical simulations

In this section we investigate the parameters space of the proposed model through numerical simulations, in order to verify the reliability of the information obtained by the approach illustrated in Sections III and IV. To this aim, we have considered a platoon of 12 vehicles subjected to fictitious units, which enforce a periodic condition such as

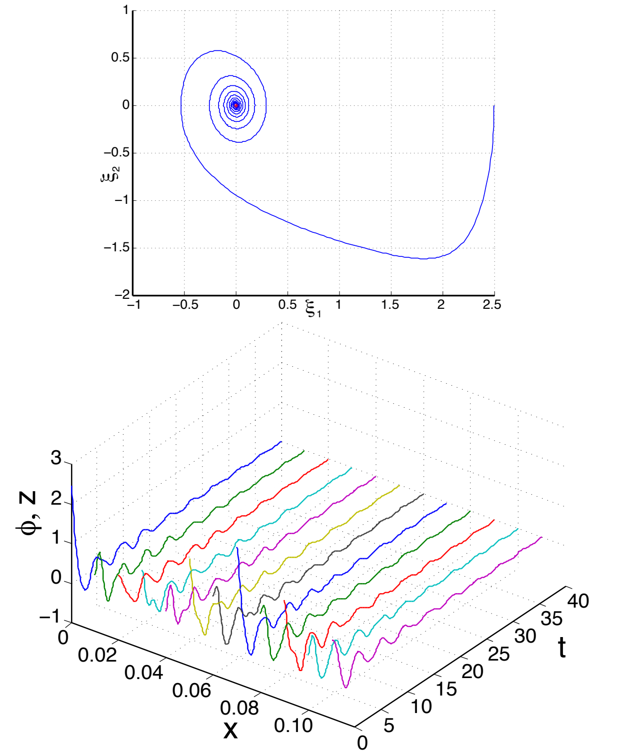

Fig. 2. Top: Stable focus of the reference ODE. Bottom: Platoon displacements converging to zero.

the formation was placed on a closed curve. For the friction dynamics it has been chosen $a = 0.1$ and $b = 1$, favoring the nonlinear effect over the linear one. It is worth mentioning that in certain media, a can assume small negative values enlarging the variety of possible behaviors with respect to the case here presented. The desired speed has been set to 0.1, while the position control has been characterized by $c = 2$.

As a first scenario, we have considered the symmetric case featuring $\alpha = \beta = 2$. Under these assumptions the reference ODE denotes the origin as "stable" solution and, therefore, small periodic orbit around it are prevented. Then, either the resting condition $z_n = 0$, $n = 1, \dots, 12$, is attractive with respect to small perturbations, or they diverge to solutions which can not be close to it. Observe that any regular regime of the system, when bounded, is expected to have the form of a traveling wave, due to the period boundary conditions. In Figure 2 the simulation of a perturbed trajectory of the reference ODE is depicted assuming $\delta x = 0.01$ and $C = 1$. The spiraling of such an orbit, since the state space is $\mathbb{R}^2$, prevents the existence of even unstable limit cycle around the origin. Therefore, no small oscillation is expected around the resting state. Figure 2 also shows the actual stable behavior of the platoon.

In the second scenario the same platoon has been investigated assuming $\alpha = 9$ and $\beta = 6$. At $K = 6.0163$, $C = 0.4056$, $\delta x = 0.01$ the reference ODE has the dynamics illustrated in Figure 3. The equilibrium in the origin is unstable and a periodic solution exists around it. Observe that (5) is

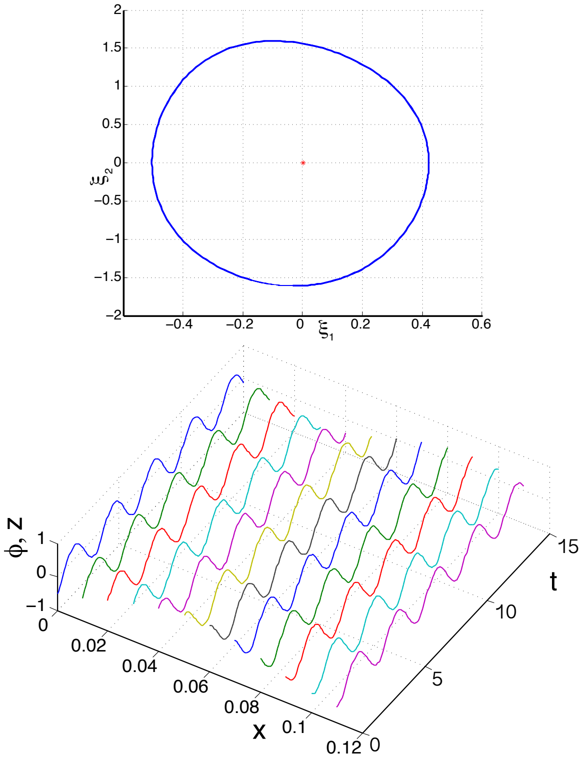

Fig. 3. Top: Unstable equilibrium surrounded by a stable limit cycle in the reference ODE. Bottom: Oscillating displacements in the platoon.

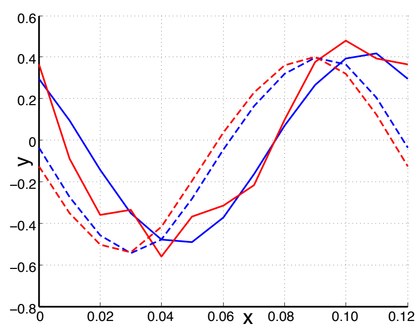

Fig. 4. Solid blue: Starting profile of the platoon as computed by the reference ODE. Solid red: Perturbed initial condition of the platoon. Dotted blue: final profile of the nominal solution after the transient. Dotted red: final profile of the perturbed solution after the same transient. The time-shift between the nominal and perturbed regime is sufficient to enforce positive real-part eigenvalues in the spectrum.

satisfied. The actual dynamics of the platoon is presented in Figure 3, showing that a self-sustained traveling wave exists, and that it is stable with respect to small perturbations. The comparison between the nominal wave and its perturbation reveals that convergence to time-shifted versions of the first one is possible, see Figure 4. Such a result is coherent with the continuous spectrum of the system, reported in Figure 5.

VI. Final remarks

A novel model of platoon subjected to nonlinear drag has been presented. All the units are driven by an identical local parametric control law, which has been chosen in the affine

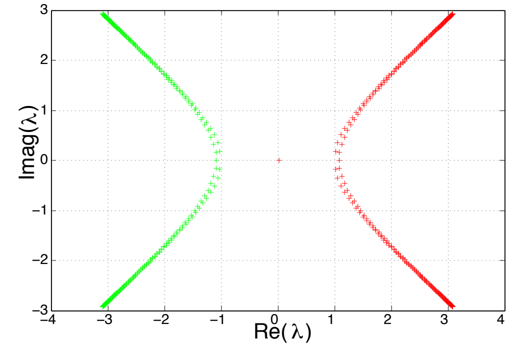

Fig. 5. Discretization of the continuous spectrum.

linear class with the goal of having the vehicles to move along a path with a desired target velocity while maintaining a certain fixed formation. By using an embedding technique, the original system has been reduced to a diffomorphic PDE model, that has been investigated to locate the parametric conditions under which interesting phenomena, such as resting states or self-sustained oscillations, may exist. Finally, numerical analysis tools have also shown the good degree of accuracy of the predictions. In particular, the obtained results highlight that the proposed nonlinear platoon can exhibit traveling waves propagating along the formation when the control input depending on the relative distance with the nearest neighbors is symmetrical.

References

[1] U. Ozguner, T. Acarman, and K. Redmill, *Autonomous Ground Vehicles*. Artech House Publishers, 2012.

[2] G. Antonelli, *Underwater Robots: Motion and Force Control of Vehicle-Manipulator Systems*, 2nd ed. Springer-Verlag, 2010.

[3] R. Austin, *Unmanned Aircraft Systems: UAVS Design, Development and Deployment*. Amer. Inst. of Aeronautics, 2010.

[4] Z. Chao, S.-L. Zhou, L. Ming, and W.-G. Zhang, "UAV formation flight based on nonlinear model predictive control," *Mathematical Problems in Engineering*, no. 261367, 2012.

[5] L. Baskar, B. De Schutter, J. Hellendoorn, and P. Z., "Technical report 09-043 – traffic control and intelligent vehicle highway systems: A survey," Delft University of Technology, Tech. Rep., 2011.

[6] F. Lin, M. Fardad, and M. Jovanovic, "Optimal control of vehicular formations with nearest neighbor interactions," *IEEE Transaction on automatic control*, vol. 57, no. 9, pp. 2203–2218, 2012.

[7] B. Bamieh, M. Jovanovic, P. Mitra, and S. Patterson, "Coherence in large-scale networks: Dimension-dependent limitations of local feedback," *IEEE Transaction on automatic control*, vol. 57, no. 9, pp. 2235–2249, 2012.

[8] G. Innocenti and P. Paoletti, "Embedding dynamical networks into distributed models," *Communications in Nonlinear Science and Numerical Simulation*, vol. 24, no. 1–3, pp. 21–39, 2014.

[9] ——, "A virtual space embedding for the analysis of dynamical networks," in *Proc. of the 52nd IEEE Conference on Decision and Control*, 12 2013, pp. 1331–1336.

[10] P. Frihauf and M. Krstic, "Leader-enabled deployment onto planar curves: A PDE-based approach," *IEEE Transaction on automatic control*, vol. 56, no. 8, pp. 1791–1806, 2011.

[11] P. Barooah, P. Mehta, and J. Hespanha, "Mistuning-based control design to improve closed-loop stability margin of vehicular platoons," *IEEE Transaction on automatic control*, vol. 54, no. 9, pp. 2100–2113, 2009.

[12] H. K. Khalil, *Nonlinear Systems*, 3rd ed. Prentice Hall, 2002.

[13] S. Strogatz, *onlinear Dynamics And Chaos: With Applications To Physics, Biology, Chemistry, And Engineering*. Westview Press, 2001.

[14] B. Sandstede, *Stability of travelling waves*, ser. Handbook of Dynamical Systems. Elsevier, 2002, vol. vol. 2, pp. 983–1055.