CALIFORNIA INSTITUTE OF TECHNOLOGY
Computing and Mathematical Sciences

CDS 110

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Fall 2024 | | Due: 13 Nov 2024

Problem 1. Trajectory Tracking (100 pts)
In Problem Set 5, you analyzed and implemented the linearized model in (1), as well as developed a controller for the linear velocity.

$$
\begin{aligned}
\dot{p}_x^{\mathcal{I}} &= v_x^{\mathcal{B}} \cos\theta - v_y^{\mathcal{B}} \sin\theta \\
\dot{p}_y^{\mathcal{I}} &= v_x^{\mathcal{B}} \sin\theta + v_y^{\mathcal{B}} \cos\theta \\
\dot{\theta} &= \omega \\
\dot{v}_x^{\mathcal{B}} &= -\frac{1}{\tau} v_x^{\mathcal{B}} + \frac{1}{\tau} u_v \\
\dot{v}_y^{\mathcal{B}} &= -\frac{C_y}{m v_x^{\mathcal{B}}} v_y^{\mathcal{B}} - v_x^{\mathcal{B}} \omega + \frac{C_y}{m} u_\delta \\
\dot{\omega} &= -\frac{L^2 C_y}{2 I_z v_x^{\mathcal{B}}} \omega + \frac{L C_y}{2 I_z} u_\delta
\end{aligned}
\tag{1}
$$

In this assignment, we will design a steering controller and combine it with the velocity controller from Problem Set 5 to create a complete, functional car.
One key aspect when designing robust controllers is defining suitable error terms with respect to the desired trajectories. Then, those errors are shown to converge to 0 exponentially fast using the designed controller. For a typical car-like system, we define the following error terms:

- e_1 - the distance of the center of gravity of the vehicle from the desired trajectory, with its derivative being defined as in
$$\dot{e}_1 = v_y^{\mathcal{B}} + v_x^{\mathcal{B}}(\theta - \theta_d)$$
- e_2 - the orientation error as a difference between the actual orientation θ and the desired orientation θ_d.
$$e_2 = \theta - \theta_d$$

With these error definitions, we can write our error dynamics as follows

$$
\dot{\mathbf{e}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -\frac{C_y}{m v_x^{\mathcal{B}}} & \frac{C_y}{m} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -\frac{L^2 C_y}{2 I_z v_x^{\mathcal{B}}} \end{bmatrix} \mathbf{e} + \begin{bmatrix} 0 \\ \frac{C_y}{m} \\ 0 \\ \frac{C_y L}{2 I_z} \end{bmatrix} u_\delta + \begin{bmatrix} 0 \\ -v_x^{\mathcal{B}} \\ 0 \\ -\frac{L^2 C_y}{2 I_z v_x^{\mathcal{B}}} \end{bmatrix} \omega_d
\tag{2}
$$

where $\mathbf{e} := [e_1, \dot{e}_1, e_2, \dot{e}_2]$ and ω_d is the desired angular velocity of the trajectory.

(a) Using the equations in (1) and the error definitions, show that the resulting system matches the one described in (2). Note that $v_x^{\mathcal{B}}$ and $\dot{\omega}_d$ can be treated as constants.

Solution. Solution from Alexander Vazquez.

First we will show that $\ddot{e}_1 = -\frac{C_y}{mv_x^\beta}\dot{e}_1 + \frac{C_y}{m}e_2 + \frac{C_y}{m}u_\delta - v_x^\beta\omega_d$ as stated in the system (2) given in the problem. By definition of $\dot{e}_1$ we have:

$$\ddot{e}_1 = \frac{d}{dt}(\dot{e}_1) = \frac{d}{dt}\left(v_y^\beta + v_x^\beta(\theta - \theta_d)\right)$$

Treating v_x^β as a constant we have:

$$\ddot{e}_1 = \dot{v}_y^\beta + v_x^\beta(\dot{\theta} - \dot{\theta}_d) = \dot{v}_y^\beta + v_x^\beta(\omega - \omega_d)$$

Substituting $\dot{v}_y^\beta$ we have:

$$\ddot{e}_1 = -\frac{C_y}{mv_x^\beta}v_y^\beta - v_x^\beta\omega + \frac{C_y}{m}u_\delta + v_x^\beta(\omega - \omega_d)$$

Simplifying we have:

$$\ddot{e}_1 = -\frac{C_y}{mv_x^\beta}v_y^\beta + \frac{C_y}{m}u_\delta - v_x^\beta\omega_d$$

Now we compare the equation above with the equation given in system (2) by substituting $\dot{e}_1$ and e_2:

$$\ddot{e}_1 = -\frac{C_y}{mv_x^\beta}\dot{e}_1 + \frac{C_y}{m}e_2 + \frac{C_y}{m}u_\delta - v_x^\beta\omega_d$$

$$\ddot{e}_1 = -\frac{C_y}{mv_x^\beta}(v_y^\beta + v_x^\beta(\theta - \theta_d)) + \frac{C_y}{m}(\theta - \theta_d) + \frac{C_y}{m}u_\delta - v_x^\beta\omega_d$$

$$\ddot{e}_1 = -\frac{C_y}{mv_x^\beta}v_y^\beta - \frac{C_y}{mv_x^\beta}v_x^\beta(\theta - \theta_d) + \frac{C_y}{m}(\theta - \theta_d) + \frac{C_y}{m}u_\delta - v_x^\beta\omega_d$$

Thus because $-\frac{C_y}{mv_x^\beta}v_x^\beta(\theta - \theta_d) + \frac{C_y}{m}(\theta - \theta_d) = 0$ then we have:

$$\ddot{e}_1 = -\frac{C_y}{mv_x^\beta}\dot{e}_1 + \frac{C_y}{m}e_2 + \frac{C_y}{m}u_\delta - v_x^\beta\omega_d = -\frac{C_y}{mv_x^\beta}v_y^\beta + \frac{C_y}{m}u_\delta - v_x^\beta\omega_d$$

Secondly we will show that $\ddot{e}_2 = -\frac{L^2C_y}{2I_zv_x^\beta}(\omega - \omega_d) + \frac{LC_y}{2I_z}u_\delta - \frac{L^2C_y}{2I_zv_x^\beta}\omega_d$ as stated in the system (2) given in the problem. By definition of e_2 we have:

$$\dot{e}_2 = \dot{\theta} - \dot{\theta}_d = \omega - \omega_d$$

$$\ddot{e}_2 = \frac{d}{dt}(\dot{e}_2) = \dot{\omega} - \dot{\omega}_d$$

Assuming $\dot{\omega}_d = 0$ Then we have $\ddot{e}_2 = \dot{\omega} = -\frac{L^2C_y}{2I_zv_x^\beta}\omega + \frac{LC_y}{2I_z}u_\delta$. Simplifying the equitation for $\ddot{e}_2$ as given by (2) we have:

$$\ddot{e}_2 = -\frac{L^2C_y}{2I_zv_x^\beta}(\omega - \omega_d) + \frac{LC_y}{2I_z}u_\delta - \frac{L^2C_y}{2I_zv_x^\beta}\omega_d = -\frac{L^2C_y}{2I_zv_x^\beta}\omega + \frac{LC_y}{2I_z}u_\delta - \frac{L^2C_y}{2I_zv_x^\beta}\omega_d + \frac{L^2C_y}{2I_zv_x^\beta}\omega_d$$

Thus we have:

$$\ddot{e}_2 = -\frac{L^2 C_y}{2 I_z v_x^\beta}(\omega - \omega_d) + \frac{L C_y}{2 I_z} u_\delta - \frac{L^2 C_y}{2 I_z v_x^\beta}\omega_d = -\frac{L^2 C_y}{2 I_z v_x^\beta}\omega + \frac{L C_y}{2 I_z} u_\delta = \dot{\omega}$$

Thus we have shown that the equations provided for $\ddot{e}_1$ and $\ddot{e}_2$ in (2) are correct. Lastly, the equations for $\dot{e}_1$ and $\dot{e}_2$ from (2) are $\dot{e}_1 = \dot{e}_1$ and $\dot{e}_2 = \dot{e}_2$ which are correct by definition.

(b) For the system in (2), analyze the system's reachability and report the rank of the reachability matrix. You may perform the computation symbolically using Python's SymPy library or any other symbolic library. No need to share the code.
Solution. The reachability matrix has rank 4.

(c) Implement a feedback controller with no feedforward term for the system in (2) and simulate the system following a circular trajectory. You could start the car on the trajectory (meaning that the initial conditions should match the initial trajectory point). Plot the performance of the system (position tracking (actual vs. desired), the error vector **e**, and the forward velocity tracking error). What can you say about the performance? Feel free to use this code `https://github.com/lupusorina/cds_110_hw6` as a starting point. For sharing the code, you can choose to fork this repo and put the Github link on Gradescope or use Google colab.
Solution. The plot is presented in Fig. 1.

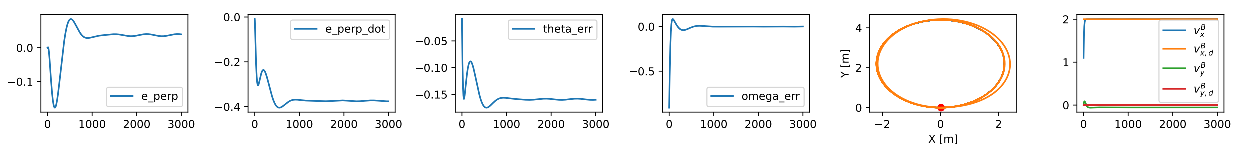

Figure 1: Part C.

(d) Implement a steering controller using LQR for the system in (2). The velocity controller can be kept the same as in the Problem Set 5. Plot the performance of the system (position tracking (actual vs. desired), the error vector **e**, and the forward velocity tracking error).
Solution. The plot is presented in Fig. 2.

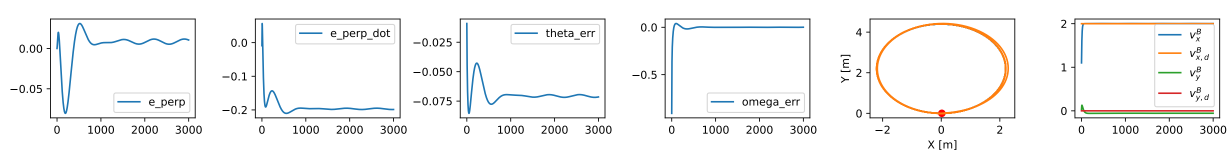

Figure 2: Part D.

We notice that the error is lower compared to part C.

(e) Due to the presence of the steady state term in (2), the tracking errors do not all converge to 0, even if the closed loop matrix is stable. The steady state values of e_1 and e_2 are non-zero because the input due to road curvature $\dot{\theta}_d$ is nonzero. Take a moment to think about this intuitively. If all the errors would converge to 0 and the car is driving a circle, than, u_δ would be 0 as well and the car would not be able to drive the circle, but drive straight. Therefore, we will use adaptive controller (integral gain) to compensate for this. Implement an integral controller on top of the feedback controller developed in part (c).

Solution. The plot is presented in Fig. 3.

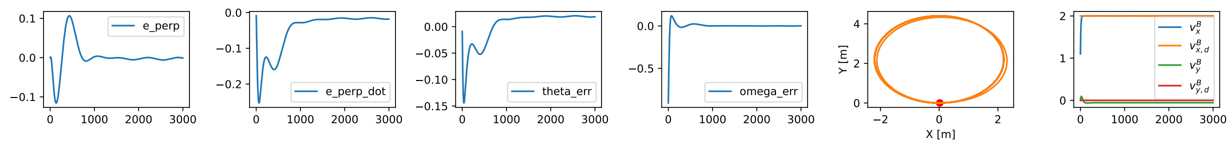

Figure 3: Part E.

(f) You can achieve the same performance as in (e) by adding a feedforward term coming from the curvature of the road. Modify the feedback controller (c) with a feedforward term to compensate for the road curvature.

Hint: Use geometry and the radius of the trajectory to compute the feedforward term.

Solution. From geometry, the feedforward term is

$$u_{\text{ff}} = \arctan\left(\frac{L/2}{R}\right)$$

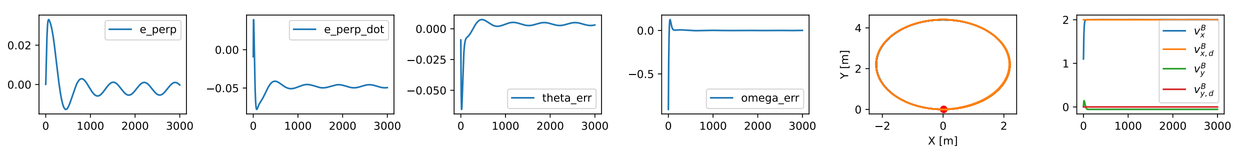

Figure 4: Part F.