2-Dim case

$$f(x,y) = \frac{1}{\sqrt{(2\pi)^k |\Sigma|_{①}}} \exp\underbrace{\left[-\frac{1}{2}(\bar{x}-\bar{\mu})^T \cdot \Sigma^{-1}_{②}(\bar{x}-\bar{\mu})\right]}_{③}$$

Determinant of covariance matrix:

$$① \quad |\Sigma| = \begin{vmatrix} \sigma_x^2 & \rho\sigma_x\sigma_y \\ \rho\sigma_y\sigma_x & \sigma_y^2 \end{vmatrix} = \sigma_x^2\sigma_y^2 - \rho^2\sigma_x^2\sigma_y^2 = \underline{\sigma_x^2\sigma_y^2(1-\rho^2)}$$

Inverse of the co.var. matrix:

$$② \quad \Sigma^{-1} = \frac{1}{Det(\Sigma)} \Sigma' = \frac{1}{\sigma_x^2\sigma_y^2(1-\rho^2)}\begin{bmatrix} \sigma_y^2 & -\rho\sigma_x\sigma_y \\ -\rho\sigma_y\sigma_x & \sigma_x^2 \end{bmatrix}$$

Calculating exp. part only:

$$③ \quad -\frac{1}{2}\begin{bmatrix} x-\mu_x \\ y-\mu_y \end{bmatrix}^T \cdot \Sigma^{-1}\begin{bmatrix} x-\mu_x \\ y-\mu_y \end{bmatrix} =$$

$$-\frac{1}{2}\begin{bmatrix} x-\mu_x \\ y-\mu_y \end{bmatrix}^T \cdot \underbrace{\frac{1}{\sigma_x^2\sigma_y^2(1-\rho^2)}\begin{bmatrix} \sigma_y^2 & -\rho\sigma_x\sigma_y \\ -\rho\sigma_y\sigma_x & \sigma_x^2 \end{bmatrix}}_{\Sigma^{-1}}\begin{bmatrix} x-\mu_x \\ y-\mu_y \end{bmatrix} =$$

$$= -\frac{1}{2}\begin{bmatrix} x-\mu_x & y-\mu_y \end{bmatrix} \frac{1}{\sigma_x^2\sigma_y^2(1-\rho^2)} \begin{bmatrix} \sigma_y^2(x-\mu_x) - (\rho\sigma_x\sigma_y)(y-\mu_y) \\ \rho\sigma_y\sigma_x(x-\mu_x) + \sigma_x^2(y-\mu_y) \end{bmatrix}$$

multiply

$$= -\frac{1}{2\sigma_x^2\sigma_y^2(1-\rho^2)}\Big[\sigma_y^2(x-\mu_x)^2 - \underline{(\rho\sigma_x\sigma_y)(y-\mu_y)(x-\mu_x)} - \underline{(\rho\sigma_y\sigma_x)(x-\mu_x)(y-\mu_y)} + \sigma_x^2(y-\mu_y)^2\Big]$$

$$= -\frac{1}{2\sigma_x^2\sigma_y^2(1-\rho^2)}\Big[\sigma_y^2(x-\mu_x)^2 + \sigma_x(y-\mu_y)^2 - 2\rho\sigma_x\sigma_y(x-\mu_x)(y-\mu_y)\Big]$$

(3) $$= -\frac{1}{2(1-\rho^2)}\left[\frac{(x-\mu_x)^2}{\sigma_x^2} + \frac{(y-\mu_y)^2}{\sigma_y^2} - \frac{2\rho(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y}\right]$$

Result:

$$f(x,y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \exp\left[-\frac{1}{2(1-\rho^2)}\left(\frac{(x-\mu_x)^2}{\sigma_x^2} + \frac{(y-\mu_y)^2}{\sigma_y^2} - \frac{2\rho(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y}\right)\right]$$