Computations \In MathML

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1. Arithmetic

Table of Contents

1.1 Number Systems

The Natural Numbers

***Definition* 1.** The set of **natural numbers**

$$\mathbb{N} = \{1, 2, 3, \ldots\}.$$

The set $\mathbb{N}$ of natural numbers is unbounded. For any natural number n, we call m a ***devisor*** or ***factor*** of n, if there is another natural number k such that $n = mk$.

***Definition* 2.** If the only divisors of a natural number p are 1 and p, where $p \neq 1$, then p is said to be **prime**.

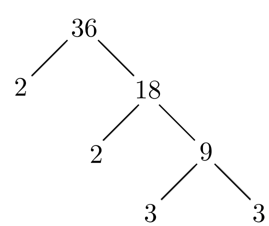
$$\mathcal{P} = \{2, 3, 5, \ldots\}$$

We note that 2 is the only *even* natural number that is prime. Every number that is prime has exactly two factors, namely 1 and itself. If a natural number $n \neq 1$ is not prime, then we say that it is ***composite***.

***Axiom* 1**. Every $\neg 1$ natural number $\in (\mathcal{P} \vee \mathcal{C})$.

Composite numbers may be factored as products of primes, as in

$$36 = 2 \times 2 \times 3 \times 3, \text{ or}$$



***Theorem* 1.** The **fundamental theorem of arithmetic** states that every natural number has a unique *prime factorization.*

Irrespective of the incipient expression, every scheme leads to the same prime factorization. Consider the two distinct approaches for obtaining the prime factorization of 72.

$$\begin{aligned} 72 &= 8 \times 9 \\ &= (2 \times 4)(3 \times 3) \\ &= 2 \times 2 \times 2 \times 3 \times 3 \end{aligned} \qquad \begin{aligned} 72 &= 4 \times 18 \\ &= (2 \times 2)(2 \times 9) \\ &= 2 \times 2 \times 2 \times 3 \times 3 \end{aligned}$$

Ultimately, the result is uniform, $72 = 2 \times 2 \times 2 \times 3 \times 3$.

Zero

If we add the element 0 to the set of natural numbers, we obtain a new set of numbers which are called the *whole numbers.*

***Definition* 3.** The set of **whole numbers**

$$\mathbb{W} = \{0, 1, 2, \ldots\}.$$

***Axiom* 2**. $\dfrac{w}{0} \vdash \mathrm{Lor}_{\mathrm{ipsum}}$

Let's consider it a self-evident notion that devision by zero is meaningless, and therefore undefined.

The Integers

***Axiom* 3.** There exists a whole number zero, denoted by 0, such that

$$a + 0 = a.$$

Additive identity.

***Axiom* 4.** For any whole number a, there exists a unique number $-a$ (*minus* a, whenever $a < 0$, Lang ['71]), such that $a + (-a) = 0$.

Additive inverse.

Fortunes and debts (intuitionistic/constructive logic)

***Definition* 4.** The set of **negative numbers**

$$\mathbb{N}^{-} = \{..., -3, -2, -1\}.$$

A **fortune** subtracted from *zero* is a **debt**: $0 - f = -f = \delta$.

A **debt** subtracted from *zero* is a **fortune**: $0 - \delta = 0 - (-f) = f$.

The product or quotient of two **fortunes** is one **fortune**: $(f_m \times f_n = f) \vee \left(\frac{f_m}{f_n} = f\right)$.

The product or quotient of two **debts** is one **fortune**: $(\delta_m \times \delta_n = -f_m \times -f_n = f) \vee \left(\frac{\delta_m}{\delta_n} = \frac{-f_m}{-f_n} = f\right)$.

The product or quotient of a **debt** and a **fortune** is a **debt**: $[(\delta \times f = -f_m \times f_n = \delta_{mn})$, where $\delta < 0.] \vee \left(\frac{\delta}{f} = \frac{-f_m}{f_n}\right)$, where $\delta < 0$.

The product or quotient of a **fortune** and a **debt** is a **debt**: $[(f \times \delta = f_m \times -f_n = \delta_{mn})$, where $\delta < 0.] \vee \left(\frac{f}{\delta} = \frac{f_n}{-f_m}\right)$, where $\delta < 0$.

If we take the set of natural numbers $\mathbb{N} = \{1, 2, 3, ...\}$, add zero, then add the set of negative numbers, we obtain the set of ***integers***, i.e. $(\mathbb{N}^{-} \wedge 0 \wedge \mathbb{N}) = \{..., -3, -2, -1, 0, 1, 2, 3, ...\}$.

***Definition* 5.** The set of **integers**

$$\mathbb{Z} = \{..., -3, -2, -1, 0, 1, 2, 3, ...\}.$$

That is to say an integer is a whole number, either positive or negative, or zero. Therefore, $-12{,}209$, -52, 0, 387, 210023, are integers, however $-3/7$, 0.125, $\sqrt{2}$, e, and π are not.

Rational Numbers

Every natural number is also a whole number, which is to say that *if* $(n \in \mathbb{N}) \rightarrow (n \in \mathbb{W})$. We may also write $\mathbb{N} \subset \mathbb{W}$, i.e. the set of natural numbers is a subset of the set of whole numbers. Similarly, $(w \in \mathbb{W}) \rightarrow (w \in \mathbb{Z})$. Let us introduce the notion of a rational number to characterize any number that is the ratio of two integers.

***Definition* 6.** The set of **rational numbers**

$$\mathbb{Q} = \left\{\frac{m}{n}, \text{ such that } m, n \text{ are integers, } n \neq 0\right\}.$$

The $n \neq 0$ restriction is required as devision by zero is senseless. The set $\left\{\frac{a}{b}, \frac{a+1}{b}, \frac{-a}{b}\right\}$ containing ratios of integers, is a set of rational numbers. Arithmetically, we may consider integers like 6 elements of correlative sets of synonymous rational-numbers $\left\{\frac{6}{1}, \frac{12}{2}, \frac{24}{4}, 6, ...\right\}$. The integer 6 is also a rational number. Generally, note that $\mathbb{Z} \subset \mathbb{Q}$.

Any decimal that terminates is also a rational number. For instance

$$0.01 = \frac{01}{100},\ \ 0.101 = \frac{101}{1000},\ \ 1.0001 = \frac{10001}{10000}.$$

Decimal Transformation. To convert a terminating decimal to a fraction – count the number of decimal places, then write 1 followed by that number of zeros for the denominator. Any decimal that repeats may also be expressed as the ratio of two integers.

$$0.0\overline{21} = 0.0212121...$$

The sequence of integers under the replica bar repeat indefinitely. In the case of $0.0\overline{21}$, there are tow digits below the replica bar.
Suppose we let $x = 0.0\overline{21}$, then

$$x = 0.0212121...,$$

now multiplying by 100 will move the decimal two places to the right.

$$100x = 2.12121...$$

If we align these two results, and subtract

$$\begin{aligned} 100x &= 2.12121... \\ -x &= 0.0212121... \end{aligned}$$

We obtain the following value

$$\begin{aligned} 99x &= 2.1 \\ x &= \frac{2.1}{99}, \end{aligned}$$

Which is not a ratio of two integers. We can rectify this by multiplying both numerator and denominator by 10.

$$x = \frac{21}{990}$$

This last result may be further reduced by dividing both numerator and denominator by 3, yielding $0.0\overline{21} = 7/330$, a ratio of two integers, and as such a rational number.

‣ Example 1. Show that $0.0\overline{621}$ is a rational number.

In this case we have three digits under the replica bar. If we let $x = 0.0\overline{621}$, then multiply x by 1000, this will shift the decimal three places to the right. By subtraction, and devision we have

$$\begin{aligned} 1000x &= 621.621621... \\ -x &= \quad 0.621621... \\ 99x &= 621 \\ x &= \frac{621}{99}. \end{aligned}$$

If we now divide the numerator and denominator by 27 (or alternately by 9 then 3), we see that $0.0\overline{621} = 23/27$. As such, being the ratio of two integers, $0.0\overline{621}$ is a rational number. ◇

We will use the diamond (alt. lozenge) character above to mark the conclusion of a computation or a calculation (i.e. proof). Let's continue or discussion of number types.

The Irrational Numbers

If a number is not rational, we categorize it as irrational.

***Definition* 7.** Any number that cannot be expressed as the ratio of two integers is regarded an **irrational number**.

***Theorem* 2.** Regarding the **irrationality of $\sqrt{2}$**.
***Proof* 1**. We'll use *reductio ad absurdum*, i.e. proof by contradiction. Although freely wielded in mathematical prose, this (non-constructive) technique is not ubiquitously accepted across mathematics pedagogy as universally valid. The aim is to establish the truth or validity of a proposition by showing that the assumption of its negation in a severally valid context leads to a contradiction.

Suppose $\sqrt{2}$ is indeed rational, which would imply that it may be expressed as the ratio of two integers p and q as follows.

$$\sqrt{2} = \frac{p}{q}$$

(1) Square both sides of the equation, (2) then disjoin the fraction by multiplying both sides by q^2.

$$\left(\sqrt{2}\right)^2 = \left(\frac{p}{q}\right)^2 \tag{1}$$

$$2 = \frac{p^2}{q^2}$$

$$p^2 = 2q^2. \tag{2}$$

The premise $p^2 = 2q^2$ implies that p is even — taking for granted that

$$\left[\left(\frac{x}{2}\ \top\right) \longleftrightarrow \left(\frac{x^2}{2}\ \top\right)\right] \wedge \left[\left(\frac{x}{2}\ \bot\right) \longleftrightarrow \left(\frac{x^2}{2}\ \bot\right)\right].$$

We should also note for context that an integer squared is an integer, which may be written as $(p^2 = 2q^2) \mapsto (x = 2k)$. In the particular case when p and q are mutually prime, q is necessarily odd. Moreover, the square of an even number is divisible by four. It follows that

$$\begin{aligned} p^2 &= 2(4k^2 + 4k + 1) \\ &= 2[4(k^2 + k) + 1] \\ &= 8(k^2 + k) + 2. \end{aligned}$$

Perspicuously, $4 \nmid [8(k^2 + k) + 2]$ leading us to conclude that q must be even — a contradiction of our initial assumption regarding q. ◇

***Proof* 2**. Per Thrm. 1, p and q (up to the order of their factors)[1] each have distinct prime factorizations: $p^2 = \underbrace{(p_1 \cdot p_2 \cdots p_n)}_{n} \underbrace{(p_1 \cdot p_2 \cdots p_n)}_{n}$.

It follows that p^2 has an even number of prime factors and so does q^2.

1. X is true up to Y — Pete L. Clark, StackExchange '13

However, this contradicts (2) as $2q^2$ thereby would have an odd number of factors in its prime factorization. Thus, our assumption that $\sqrt{2}$ is rational must be ejected as invalid. $\diamond$

Lorem Ipsum. $\underbrace{hello, world}_{world}$