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MATH 5000
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(A) Every group which has elements of arbitrarily large finite order is elementarily equivalent to a group which has an element of infinite order.

Proof. Given a model $\mathcal{G}$ of a group $(G, \cdot^{\mathcal{G}})$ which has elements of arbitrarily large finite order. Let $\Sigma = \operatorname{Th}\mathcal{G} \cup \{cc \neq c, ccc \neq c, ...\}$ be a set of statements in the language $(\cdot, c)$.

Claim: every finite subset of Σ has a model. Given a finite $F \subseteq \Sigma$. Wlg, $F = \{\textit{some statements in } \operatorname{Th}\mathcal{G}\} \cup \{c^2 \neq c, ..., c^n \neq c\}$. Pick $g \in G$ of order greater than n. Then $(G, \cdot^G, g)$ is a model of F (it satisfies any subset of $\operatorname{Th}\mathcal{G}$ because $\mathcal{G}$ is still our model, and we chose g to satisfy the rest of the statements).

Therefore, there is a $\mathcal{G}' = (G', \cdot^{\mathcal{G}'}, c^{\mathcal{G}'}) \models \Sigma$. $c^{\mathcal{G}'}$ is an element of infinite order, and the group axioms were in $\operatorname{Th}\mathcal{G}$, so $\mathcal{G}'$ is a group. $\square$

(B) Every model of ZFC is elementarily equivalent to a model of the form (V, ε) in which there is a sequence $\langle a_i : i \in \omega\rangle$, where $a_{i+1} \,\varepsilon\, a_i$ for all i; i.e. an infinitely descending ε-sequence.

Proof. Given a model $\mathcal{Z} = (Z, \in^{\mathcal{Z}})$ of ZFC (language $\mathcal{L} = (\in)$). Define $\mathcal{L}' = (\in, c_0, c_1, ...)$. Let $\Sigma = \operatorname{Th}\mathcal{Z} \cup \{c_1 \in c_0, c_2 \in c_1, ...\}$. Any $\{c_1 \in c_0, ..., c_{n+1} \in c_n\}$ together with a subset of $\operatorname{Th}\mathcal{Z}$ has a model (let c_i in the model be i in our ω). The rest of the argumentation is very similar to **(A)** above. $\square$

(C) Any infinite well-ordering $\mathcal{A} = (A, <)$ is elementarily equivalent to $\mathcal{B} = (B, <)$ such that B has a subset isomorphic to the rationals.

Proof. Do the "standard compactness argument" to get $\mathcal{B}$, using the following: $\mathcal{L}' = (<, \langle c_i : i \in \mathbb{Q}\rangle)$, $\Sigma = \operatorname{Th}\mathcal{A} \cup \{c_i \neq c_j : i \neq j \in \mathbb{Q}\} \cup \{(\exists x\ x < c_i \wedge \exists x\ c_i < x) : i \in \mathbb{Q}\} \cup \{\exists x\ (c_i < x \wedge x < c_j) : i \in \mathbb{Q}\}$. Finite subsets of Σ will have models, because finite subsets of the sentences involving c_i will only demand the existence of finite linear-orderings, which will definitely exist in an infinite well-ordering.

By construction, the c_i in $\mathcal{B}$ will be isomorphic to $\mathbb{Q}$. $\square$

(D) If $\mathcal{A} = (A, +, \cdot)$ is a countable nonstandard model of $\mathrm{Th}\,(\omega, +, \cdot)$, then $\mathcal{A}$ has order type $\omega + (\omega^* + \omega) \cdot \eta$.

Proof. There exists an $\chi \in A$ greater than all standard numbers by the "standard compactness argument" with $\Sigma = \mathrm{Th}\,(\omega, +, \cdot) \cup \{c \neq 0, c \neq 1, c \neq 1+1, ...\}$, but there is no nonstandard number less than or between any of the standard numbers because their nonexistence can be asserted in first-order logic ($\neg\exists x\ x < 0$, $\forall x \neg\exists y(x < y \wedge y < x + 1)$, with $<$, 0, and 1 defined appropriately).

A nonstandard $\chi + a$ for every standard a exists, since if $\chi + a$ were standard, say n, then the sentence $\forall x(x + a = n \Rightarrow x < n)$ would be false in $\mathcal{A}$ but true in $(\omega, +, \cdot)$.

Similarly $\chi - a$ for every standard a exists. If not, then let take a to be the least counterexample. That would imply that $\chi = a$, and thus not nonstandard. This constructs an $\omega^* + \omega$ around χ, and in fact around every nonstandard number.

Given nonstandard κ, λ, with $\lambda < \kappa$ and $\kappa \neq a + \lambda$ for any standard a. There exists a μ such that either $2\mu = \kappa + \lambda$ or $2\mu = \kappa + \lambda + 1$. It can be shown that $\lambda < \mu < \kappa$. There is no standard a such that $\lambda + a = \mu$ because:

$$\begin{aligned} \lambda + a &= \mu \\ 2\lambda + 2a &= 2\mu \\ 2\lambda + 2a &= \lambda + \kappa \\ \lambda + 2a &= \kappa \end{aligned}$$

But $2a$ is standard if a is (the same argument works if $2\mu = \kappa+\lambda+1$). A similar argument shows that there is no standard a such that $\mu+a = \kappa$. This shows density among the $(\omega^* + \omega)$ sections.

Given a nonstandard κ, 2κ is nonstandard and there is no standard a such that $\kappa + a = 2\kappa$, since that would imply that $a = \kappa$, which is nonstandard. Also, there is a nonstandard λ such that $2\lambda = \kappa$ (if λ were standard then so, too, would be κ). Also, there is no standard a such that $\lambda + a = \kappa$ because then $2\lambda + 2a = 2\kappa$, so $\kappa + 2a = 2\kappa$, so $2a = \kappa$ (again, κ would be standard). This shows lack of endpoints among the $(\omega^* + \omega)$ sections. $\square$

(E) Every infinite graph $G = (V, E)$ is four-colorable, as long as every finite subgraph is planar.

Proof. Let $\mathcal{L} = (E^2, R^1, G^1, B^1, Y^1, \rangle c_v : v \in V\langle)$. Let $\Sigma = \{\forall x$ "exactly one of $R(x), G(x), B(x), Y(x)$" $, \forall x \forall y (E(x, y) \Rightarrow (\neg(R(x) \wedge R(y)) \wedge ...))\} \cup \{E(c_i, c_j) : \langle i, j\rangle \in E\}$. Every finite subset F of Σ has a model by the four-color theorem (trivially if either of the first two sentences of Σ are not in F). Therefore there is a model $\mathcal{G} \models \Sigma$.

To color G using $\mathcal{G}$, simply say that a vertex v is red iff $\mathcal{G} \models R(c_v)$, etc. This is a coloring because $\mathcal{G}$ must obey the first two sentences of Σ, and it represents G because of the rest of Σ. $\square$