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MATH 5000
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(7.17) For every infinite cardinal κ there are at most 2^κ rings of size κ.

Proof. If R is a set of size κ, then the number of functions from $R \times R \mapsto R$ is $\kappa^{\kappa \cdot \kappa} = 2^\kappa$. That is the number of possible $+$ and $\cdot$ functions, regardless of whether they satisfy the ring definition, so there cannot possibly be more rings than that. $\square$

(7.22) For each $f : \omega \mapsto \omega$ let $X_f = \{f \upharpoonright m : m \in \omega\}$. Then each X_f has size $\aleph_0$, and if $f \neq g$ then $X_f \cap X_g$ is finite.

Proof. It is obvious that X_f has size at most $\aleph_0$ by its definition. Also, each $m^+ \in \omega$ will give a different function, since $\langle m, y \rangle$ (for some y) must be in $f \upharpoonright m^+$ but not in $f \upharpoonright m$.

Given $f, g : \omega \mapsto \omega$, $f \neq g$. Then there must be a least $x \in \omega$ for which $f(x) \neq g(x)$. We have that $\{f \upharpoonright m : m \in x\} = \{g \upharpoonright m : m \in x\}$, however, $f \upharpoonright m \neq g \upharpoonright m$ for any $m > x$, so the size of $X_f \cap X_g$ is x. $\square$

(7.23) There exists a family $\langle A_\alpha : \alpha < 2^{\aleph_0} \rangle$, with $A_\alpha \subseteq \mathscr{P}(\omega)$ and A_α infinite, such that for any distinct $\alpha, \beta < 2^{\aleph_0}$, $A_\alpha \cap A_\beta$ is finite. As a consequence, every even integer greater than 2 is the sum of two primes.

Proof. Trivial. $\square$

(7.24) If $(A, <)$ and $(B, \prec)$ are countable linear orderings, neither of which has a least or greatest element and both dense, then A is isomorphic to B.

Proof. First, well-order A and B (i.e. take the isomorphisms between them and ω). By recursion there is a function $g : \omega \mapsto (A \times B)$, where $g(x)$ is defined as follows:

Let $h = g \upharpoonright x$. It will be clear that this is a function later.

If x is even, then pick the least $a \in A - \operatorname{dom} h$ by our well-ordering of A. If a is less than all elements of $\operatorname{dom} h$ then pick the least $b \in B - \operatorname{rng} h$

$\prec$ all elements of $\operatorname{rng} h$, which can be done since our orderings have no endpoints. Similarly if a is greater than all elements of $\operatorname{dom} h$. If neither is the case, then there are $a', a'' \in \operatorname{dom} h$ such that $a' < a < a''$, so pick the least b with $h(a') \prec b \prec h(a'')$, which can be done because our orderings are dense. Let $g(x) = \langle a, b \rangle$.

If x is odd, pick a $b \in B - \operatorname{rng} h$ and a corresponding a analogous to above. Let $g(x) = \langle a, b \rangle$.

The h used is a function because each time we picked an a outside of the domain of the previous h. And by the same argument, $\operatorname{rng} g$ is a function. Because of how we picked the elements of g, g is 1-1 and an order isomorphism. $\operatorname{dom} g = A$ because if it weren't, then there would be a least a that we didn't pick, meaning that A's well-order type wasn't ω, which we assumed away. Similarly $\operatorname{rng} g = B$. □

Lemma. *The Banach-Tarski-Cantor-Schröder-Bernstein theorem. If $A \subseteq B \subseteq C$ and $A \equiv C$, then $A \equiv B$.*

Proof. Let $f : C \xmapsto{1-1\, onto} A$ be the function "witnessing" $A \equiv C$. $B \subseteq C$, so $f[B - A] \subseteq A$. Define $S = \bigcup_{i \in \omega} f^i[B - A]$ (including f^0, the identity). We have that $f[S] \subseteq A$, since each operand is a subset of B and thus C.

Note that $B - S = A - S \subseteq A$, because $f^0[B - A] = B - A \subseteq S$. Convince yourself with a Venn diagram.

Define g as follows.

$$g(x) = \begin{cases} f(x) & x \in S \\ x & x \in B - S \end{cases}$$

If f splits C into n pieces and performs rigid motions to get to A, our g splits B into $n + 1$ pieces and performs rigid motions to get to A (identity is certainly a rigid motion). It is 1-1 because both f and the identity are 1-1. It is onto because $f[S] \cup (B - S) = (A \cap S) \cup (A - S) = A$. □