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MATH 5000
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(1) To define the transitive closure $TC(x)$, define recursively $g(0) = x$, $g(n+1) = \cup g(n)$ and then take $TC(x) = \bigcup_{n<\omega} g(n)$.

Theorem. *$TC(x)$ is transitive.*

Proof. Given $b \in a \in TC(x)$. Then there must be a least n where $a \in g(n)$. $b \in \cup g(n) = g(n+1)$, so $b \in \bigcup_{n<\omega} g(n) = TC(x)$. □

Theorem. *If Y is a transitive set and $X \subseteq Y$, then $TC(X) \subseteq Y$.*

Proof. $g(0) = X \subseteq Y$. Suppose $g(n) \subseteq Y$. Let $x \in g(n+1)$. Since $g(n+1) = \cup g(n)$, there is a $x' \in g(n)$ such that $x \in x'$. However, $x' \in Y$, and since Y is transitive, $x \in Y$.

So all $g(n) \subseteq Y$, so their union $\subseteq Y$. □

(2) For every ordinal α, $\alpha \in V_{\alpha+1} - V_\alpha$.

Proof. Suppose that for every ordinal $\gamma < \alpha$, $\gamma \in V_{\gamma+1} - V_\gamma$.

Case 0: $\alpha = 0$. Obvious.

Case 1: $\alpha = \beta + 1 = \beta \cup \{\beta\}$. $\beta \in V_{\beta+1} - V_\beta$. $\alpha \in V_{\alpha+1}$ because $\beta, \{\beta\} \in V_\alpha \cup \mathscr{P}(V_\alpha) = V_{\alpha+1}$.
$\alpha \notin V_\alpha$ because $\beta \notin V_\beta$, so $\{\beta\} \notin V_{\beta+1} = V_\alpha$.

Case 2: α is a limit ordinal. $V_\alpha = \bigcup_{\gamma<\alpha} V_\gamma$, and and each γ is in $V_{\gamma+1}$, so $\alpha \subseteq V_\alpha$. Therefore, $\alpha \in V_{\alpha+1}$.
Suppose $\alpha \in V_\alpha$. Then there must be some $\gamma < \alpha$ such that $\alpha \in V_\gamma$. However, $\gamma \notin V_\gamma$, contradicting $\gamma \in \alpha$. So $\alpha \notin V_\alpha$.

□

(3)

Theorem. *$AC \Leftrightarrow$ Given P a partition of A, there is a subset X of A such that each $x \cap X$ for $x \in P$ is a singleton.*

Proof. $\Rightarrow$ Pick a choice function f on P. Then define $X = \operatorname{rng} f$. It is easy to see that this satifies the conditions.

$\Leftarrow$ Given a collection of nonempty sets $\mathscr{A}$. *Wlg* assume they are disjoint. Then $\mathscr{A}$ is a partition of $\cup\mathscr{A}$, so there is an $X \subseteq \cup\mathscr{A}$ such that each $x \cap X$ for $x \in \mathscr{A}$ is a singleton. Then $f(x) = \cup(x \cap X)$ is a choice function on $\mathscr{A}$.

$\square$

Theorem. *For any two sets A and B, the following are equivalent to the axiom of choice:*

(i) Either there is a function $A \xmapsto{1-1} B$ or there is a function $B \xmapsto{1-1} A$.

(ii) Either there is a function $A \xmapsto{onto} B$ or there is a function $B \xmapsto{onto} A$.

Proof. **(i) $\Rightarrow$ (ii):** Assume *wlg* that there is an $f : A \xmapsto{1-1} B$. Since f is 1-1, there is a function f^{-1}, which is obviously onto A, because A was f's domain.

(ii) $\wedge$ $AC \Rightarrow$ **(i):** Assume *wlg* that there is an $f : A \xmapsto{onto} B$. Then f^{-1} is a relation with domain B. Pick a choice function c on $\mathscr{P}(A) - \{0\}$. Define $g : B \mapsto A$ by $g(x) = c(f^{-1}[\{x\}])$. $g(b) = g(b') \Rightarrow f(g(b)) = f(g(b')) \Rightarrow b = b'$, so g is 1-1.

$AC \Rightarrow$ **(ii):** Fix a well-ordering $<_A$ for A and $<_B$ for B. Let $a : A \mapsto \alpha$ be the isomorphism from $(A, <_A)$ to its unique ordinal α. Likewise $b : B \mapsto \beta$. Assume *wlg* $\alpha \leq \beta$. Define $f : B \mapsto A$ by $f(x) = a^{-1}(b(x))$ whenever $b(x) < \alpha$, and $f(x) = a'$ otherwise, where a' some constant member of A. f is onto A.

(ii) $\Rightarrow AC$**:** Given any set A, let α be its Hartog ordinal. So there is no map from A onto α, so there must (by *(ii)*) be a function $f : \alpha \xmapsto{onto} A$. Now define $g : A \mapsto \alpha$ by $g(x) =$ the least ordinal β such that $f(\beta) = x$. Then $(A, <_A)$ where $a <_A b$ iff $f(a) < f(b)$ is a well-ordering of A.

$\square$

Theorem. *$AC \Leftrightarrow$ Every linearly ordered subset L of a poset P can be extended to a maximal $L' \supseteq L$ subset of P which is linearly ordered.*

Proof. $\Rightarrow$ Let $\mathscr{L}$ be the set of all linear orderings of P, and order the orderings by $\subset$. Every linearly-ordered (by $\subset$) $L \subseteq \mathscr{L}$ has an upper bound, namely $\cup L$ (which also must be linearly ordered, thus in $\mathscr{L}$). By Zorn's lemma, there is a maximal linear ordering L' in $\mathscr{L}$, i.e. $L' \supseteq L$ for all $L \in \mathscr{L}$.

$\Leftarrow$ (By showing $\Rightarrow$ (i) above) Given A and B, let P be the set of all 1-1 function subsets of $A \times B$, with $f < g$ iff $f \subset g$. Then, since the empty set is a linear ordering, there is a maximal linearly-ordered subset $L' \subset P$. $f = \cup L'$ is a 1-1 function either from A to B or vice versa. If its domain weren't (*wlg*) A, then we could pick $a \in A - \operatorname{dom} f$ and $b \in B - \operatorname{rng} f$ and add $\langle a, b \rangle$ to the ordering, contradicting L''s maximality.[1]

□

Theorem. *$AC \Leftrightarrow$ for any A, there is a function f with domain $\mathscr{P}(A) - \{0\}$ such that $f(x) \in x$.*

Proof. $\Rightarrow$ Let f be a choice function on $\mathscr{P}(A) - \{0\}$

$\Leftarrow$ Given a collection of nonempty sets $\mathscr{A}$. Let $A = \cup\mathscr{A}$, so that $\mathscr{A} \subseteq \mathscr{P}(A)$. Then there is an f from above, so we can construct a choice function on $\mathscr{A}$ by $f \restriction \mathscr{A}$.

□

[1] This one is very fuzzy, and I'm not sure if I have succeeded in proving this. I think the idea of what to do with the "maximal linear ordering" has me confused.