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MATH 4730: Set Theory

## 9.12

$\operatorname{cf} 0 = 0$, and $0 = \operatorname{ssup} \emptyset$; $\operatorname{cf}(\kappa + 1) = 1$, and $\kappa + 1 = \operatorname{ssup}\{\kappa\}$, which is clearly of minimal cardinality.

Suppose $\lambda$ is a limit ordinal. Let $S$ be a set of ordinals smaller than $\lambda$ of cardinality strictly smaller than $\operatorname{cf} \lambda$. Then there exists a $\beta < \lambda$ which is strictly greater than every element of $S$. But then $\operatorname{ssup} S$ is at most $\beta < \lambda$. Therefore a set with strict supremum $\lambda$ must have cardinality at least $\operatorname{cf} \lambda$. A set with size exactly $\operatorname{cf} \lambda$ exists by the definition of cofinality. $\square$

## 9.17

Suppose that there exists an $f : \bigcup_{i \in I} A_i \xrightarrow{\text{onto}} \bigotimes_{i \in I} B_i$. Let's view the elements of $\operatorname{rng} f$ as functions $I \mapsto B_i$. Create by the axiom of choice a function $h$ where $h(i) \in B_i \setminus f[A_i]$ (which is always nonempty because $\bar{\bar{A}}_i < \bar{\bar{B}}_i$). There is no element of $\bigcup_{i \in I} A_i$ which maps to $h$, so $f$ must not have been onto. $\square$

## 9.19

Since $\kappa$ is a regular cardinal, $\cup S < \kappa$ because $\bar{\bar{S}} < \kappa$. Since $\kappa$ is a cardinal and thus a limit ordinal, $V_\kappa = \bigcup_{\theta \in \kappa} V_\theta$. Therefore there exists a $\theta < \kappa$ where $S \in V_\theta$, so $S \in V_\kappa$. $\square$

# Plus

**(i)** A series of informal proofs, because most of the results are fairly obvious.

- $5 \ncong [1, 5) \cong 4$.
- *omega* is indecomposable, because any smaller number is finite, and there are $\aleph_0$-many numbers between any finite number and $\omega$.

- $\omega + \omega \ncong [\omega, \omega + \omega) \cong \omega$.
- $\omega^2$ is indecomposable, for any smaller number is of the form $\omega \cdot m + n$ for finite numbers $m$ and $n$. Attempting to decompose, we find $\omega \cdot m + n + \omega^2 = \omega \cdot m + \omega^2 = \omega^2$.
- $\omega_1$ is indecomposable, because for any smaller number $\kappa$, $[0, \kappa) = \kappa$ has cardinality at most $\aleph_0$, so $[\kappa, \omega_1)$ must have cardinality $\aleph_1$, and the order type of this range is certainly no larger than $\omega_1$.
- $\omega_1 + \omega^2$ is nonindecomposable, because it is not isomorphic to $[\omega_1, \omega_1 + \omega^2) \cong \omega^2$.

**(ii)** Consider the ordinal $\omega_1 \cdot \omega$. This is indecomposable for a similar reason that $\omega^2$ is above, and has all the other desired properties.