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(2) If τ is a term containing no variables, then for some unique $k \in \omega$, $A \vdash \tau = \bar{k}$.

Proof. By induction on τ.

- Case $\tau = \bar{0}$. $A \vdash \bar{0} = \bar{0}$.
- Case $\tau = S\varsigma$. By IH, there is a $k \in \omega$ with $A \vdash \varsigma = \bar{k}$. Therefore, $A \vdash S\varsigma = S\bar{k}$ by logic (no axioms in A necessary), i.e. $A \vdash \tau = \overline{k+1}$.
- Case $\tau = \rho + \varsigma$. By IH, there are $j, k \in \omega$ with $A \vdash \rho = \bar{j}$ and $A \vdash \varsigma = \bar{k}$. Thus $A \vdash \tau = \bar{j} + \bar{k}$, and we have already shown (in a homework long ago) that these axioms give the correct result.
- Case $\tau = \rho \cdot \varsigma$ and $\tau = \rho^{\varsigma}$ proceed similarly to above.

Uniqueness follows trivially, since if $A \vdash \tau = \bar{j}$ and $A \vdash \tau = \bar{k}$ with $j \neq k$, then $A \vdash \bar{j} = \bar{k}$ which we have already shown impossible. $\square$

(14) If $f : \omega^n \mapsto \omega$ is a recursive function then there is a formula $\phi(\vec{x}, y)$ such that if $f(\vec{k}) = r$ then $A \vdash \forall y[\phi(\overline{\vec{k}}, y) \Leftrightarrow y = \bar{r}]$.

Proof. Since f is recursive, it is defined by a (recursive) formula $\theta(\vec{x}, y)$ such that $A \vdash \theta(\vec{x}, y)$ iff $f(\vec{x}) = y$. Consider the formula $\varphi(\vec{x}, y) = \theta(\vec{x}, y) \wedge \forall z < y : \neg\theta(\vec{x}, z)$. Note that φ is recursive, since it is built up only from the recursive θ and bounded quantification.

If $f(\vec{k}) = r$ then $A \vdash \varphi(\vec{k}, r)$, since θ holds there and φ is recursive.

$A \vdash \forall y < r : \neg\varphi(\vec{k}, r)$, since θ does not hold for any $y < r$ and that formula is recursive. $A \vdash \forall y[r < y \Rightarrow \neg\varphi(\vec{k}, r)]$, since we can use the proof of $\varphi(\vec{k}, r)$ as its counterexample. Then using (L3) (trichotomy), $A \vdash \forall y[y \neq r \Rightarrow \neg\varphi(\vec{k}, r)]$, and thus $A \vdash \forall y[\varphi(\vec{k}, y) \Leftrightarrow y = r]$. $\square$