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MATH 5000
Homework 3

(1) Let $(A, <)$ be a well ordering. For each $a \in A$ let $f(a)$ be the unique ordinal α such that $\alpha \cong A_{<a}$. Then $\operatorname{rng} f$ is an ordinal.

Proof. Clearly, $\operatorname{rng} f$ is a set of ordinals, so it suffices to show that $\operatorname{rng} f$ is transitive.

Suppose $\beta \in \alpha \in \operatorname{rng} f$. Then there is some $a \in A$ such that $\alpha \cong A_{<a}$. β is an ordinal $< \alpha$, so there must be a $b \in A$ with $\beta \cong A_{<b}$. But $\beta = f(b)$, so $\beta \in \operatorname{rng} f$. □

(2) ω is an ordinal.

Proof. ω is a set of ordinals, so it suffices to show that ω is transitive.

If ω weren't transitive, then we could take the least $\alpha \in \omega$ where $\alpha \not\subseteq \omega$. α couldn't be $\emptyset$, so there must be some $\beta \in \omega$ such that $\alpha = S\beta = \beta \cup \{\beta\}$. Then either $\beta \not\subseteq \omega$ (impossible because $\alpha > \beta$ and α was the least) or $\beta \notin \omega$ (also impossible). □

(3) Given a functional $\varphi(x, y)$, the $f : \alpha \mapsto B$ defined by recursion by φ is unique.

Proof. Trivial. f ends up being a union of approximation functions; there can't be two different values for that union (read: I apparently don't understand what I'm proving and it is too late to ask `:-/`). □

(4) **(i)** $m + 1 = Sm$

Proof. $1 = S0$, so $m + S0 = S(m + 0) = Sm$ □

(ii) $0 + m = m$

Proof. $0+0 = 0$. Suppose $0+m = m$: $0+Sm = Sm$ iff $S(0+m) = Sm$ iff $Sm = Sm$. □

(iii) $Sm + n = S(m + n)$

Proof. $Sm + 0 = S(m+0) = Sm$. Suppose $Sm + n = S(m+n)$: $Sm + Sn = S(Sm+n) = SS(m+n) = S(m+Sn)$. □

(iv) $m + n = n + m$

Proof. $m + 0 = 0 + m = m$. Suppose $m + n = n + m$: $m + Sn = S(m+n) = S(n+m) = Sn + m$. □

(v) $m + (n + p) = (m + n) + p$

Proof. $m+(n+0) = (m+n)+0 = m+n$. Suppose $m+(n+p) = m + (n + p)$:

$$\begin{aligned} m + (n + Sp) &= m + S(n+p) \\ &= S(m + (n+p)) \\ &= S((m+n)+p) \\ &= (m+n) + Sp \end{aligned}$$

□

(5) **(i)** $m \cdot 1 = m$

Proof. $1 = S0$ so $m \cdot S0 = m \cdot 0 + m = m$. □

(ii) $0 \cdot m = 0$

Proof. $0 \cdot 0 = 0$. Suppose $0m = 0$: $0 \cdot Sm = 0m + 0 = 0$. □

(iii) $Sm \cdot n = mn + n$

Proof. $Sm \cdot 0 = m \cdot 0 + 0 = 0$. Suppose $Sm \cdot n = mn + n$:

$$\begin{aligned} Sm \cdot Sn &= Sm \cdot n + Sm \\ &= mn + n + Sm \\ &= mn + m + Sn \\ &= m \cdot Sn + Sn \end{aligned}$$

□

(iv) $mn = nm$

Proof. $m \cdot 0 = 0m = 0$. Suppose $mn = nm$: $m \cdot Sn = mn + m = nm + m = Sn \cdot m$. $\square$

(v) $m(n+p) = mn + mp$

Proof. $m(n+0) = mn + m \cdot 0 = mn$. Suppose $m(n+p) = mn + mp$:

$$\begin{aligned} m(n+Sp) &= m \cdot S(n+p) \\ &= m(n+p) + m \\ &= mn + mp + m \\ &= mn + m \cdot Sp \end{aligned}$$

$\square$

(vi) $m(np) = (mn)p$

Proof. $m(n \cdot 0) = (mn) \cdot 0 = 0$. Suppose $m(np) = (mn)p$:

$$\begin{aligned} m(n \cdot Sp) &= m(np + n) \\ &= m(np) + mn \\ &= (mn)p + mn \\ &= (mn) \cdot Sp \end{aligned}$$

$\square$

(6) There is a unique function $f : \omega \times \omega \mapsto \omega$ such that for all $m, n \in \omega$: $f(m, 0) = 1$ and $f(m, Sn) = f(m, n) \cdot m$.

Proof. Given any $m \in \omega$, we can construct the unique function $g : \omega \mapsto \omega$ by recursion: $g(0) = 1$ and $g(Sn) = g(n) \cdot m$. Let $f' : \omega \mapsto (\omega \mapsto \omega)$ be the function that takes this m to this g. Then let $f = \{((m, n), g) | (m, (n, g)) \in f'\}$. $\square$

(7) Let $A = \{a_1, a_2, \cdots, a_n\}$ be a finite set. $x \in \triangle A$ iff $\{a \in A | x \in a\}$ has an odd number of elements.

Proof. If $n = 1$ then $\{a \in A | x \in a\}$ has an odd number of elements iff x is in the only member of A.

Suppose $b \notin A$. $x \in \triangle(A \cup \{b\})$ iff $x \in (\triangle A)\triangle b$. Case 1: If $x \in \triangle A$ and $x \notin b$, then x is in an odd number of the elements of A, so x is in an odd number of the elements of $A \cup \{b\}$ (converse similar). Case 2: If $x \notin \triangle A$ and $x \in b$, then x is in an even number of the elements of A, so x is in an odd number of the elements of $A \cup \{b\}$ (converse similar). $\square$