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(1) If $\mathcal{A}, \mathcal{B} \prec \mathcal{C}$ and $\mathcal{A} \leq \mathcal{B}$, then $\mathcal{A} \prec \mathcal{B}$.

Proof. Given a vector $\bar{a}$ of elements of A and a formula $\varphi(\bar{a})$. We have that $\mathcal{A} \models \varphi(\bar{a})$ iff $\mathcal{C} \models \varphi(\bar{a})$, since $\mathcal{A} \prec \mathcal{C}$. However, since $A \subseteq B$ and $\mathcal{B} \prec \mathcal{C}$, $\mathcal{B} \models \varphi(\bar{a})$ iff $\mathcal{C} \models \varphi(\bar{a})$. Therefore $\mathcal{A} \models \varphi(\bar{a})$ iff $\mathcal{B} \models \varphi(\bar{a})$; i.e. $\mathcal{A} \prec \mathcal{B}$. □

(3) A set of setences Γ is complete iff any two models of Γ are elementarily equivalent.

Proof. ($\Rightarrow$) Suppose $\mathcal{A}$ and $\mathcal{B}$ are models of Γ and $\mathcal{A} \not\equiv \mathcal{B}$. Then there must be some sentence φ such that $\mathcal{A} \models \varphi$ and $\mathcal{B} \not\models \varphi$. But if $\Gamma \models \varphi$, then $\mathcal{B} \not\models \Gamma$, and similarly if $\Gamma \not\models \varphi$ then $\mathcal{A} \not\models \Gamma$, so Γ must not be complete.

($\Leftarrow$) Suppose Γ is not complete. Then there is some φ such that neither $\Gamma \models \varphi$ nor $\Gamma \models \neg\varphi$. Thus there must be a model $\mathcal{A} \models \Gamma \cup \{\varphi\}$ and a model $\mathcal{B} \models \Gamma \cup \{\neg\varphi\}$. Clearly $\mathcal{A} \not\equiv \mathcal{B}$. □

(4a) A set of sentences $\Sigma \models \varphi$ iff $\Sigma \cup \{\neg\varphi\}$ is inconsistent.

Proof. ($\Leftarrow$) Suppose $\Sigma \cup \{\neg\varphi\}$ has no models. Then either Σ has no models or it has a model $\mathcal{A}$. If it has none, then we're done. It must be the case that $\mathcal{A} \models \varphi$ because of our assumption.

($\Rightarrow$) Suppose a structure $\mathcal{A} \models \Sigma$ and $\Sigma \models \varphi$. By definition $\mathcal{A} \models \varphi$, so it is impossible that $\mathcal{A} \models \neg\varphi$. □

(5) If $\mathcal{U}$ is an ultrafilter, $Y \in \mathcal{U}$, and $Y = \bigcup_{i<n} Y_i$, then $Y_i \in \mathcal{U}$ for some i.

Proof. Suppose not. Then $Y \in \mathcal{U}$ and $\bar{Y}_i \in \mathcal{U}$ for every $i < n$. Then the intersection $Y \cap \bigcap_{i<n} \bar{Y}_i$ would be empty. □