$L\exists\wedge G\forall$

A game of logic

Rules

Each player is dealt some number of cards face up, call it *n*. Typical values for *n* are 2 or 3. The goal of the game is to win, either by getting some number of points (call it *w*, typically 3) or by winning by a condition described below. Play rotates clockwise. The game proceeds by progressively logically describing a set of objects L. A player's turn proceeds in two phases: Scoring and Writing in that order.

I'll discuss the writing phase first, because the scoring phase will be more obvious after you know what is written. In the writing phase, a player picks one or more cards from his hand and plays them according to the instructions on the card. This will usually result in writing a statement of logic on the board.

In the scoring phase, which comes *before* the writing phase, the player looks at the board to see if he can make any derivations. If he can determine (and convince the other players) that one or more statements is a consequence of the others, i.e. it is redundant, then the player gets one point for each such statement. If the player determines that the statements on the board are inconsistent, i.e. they lead to an absurdity like "p and not p" for some statement p, then the player instantly wins the game.

After the player's turn, he should refill his stock of cards from the deck so that he has *n* again.

Nonlogical Symbols

blonde – predicate (true or false for each object). blonde(x) is read as "x is blonde".
happy – predicate. happy(x) is read as "x is happy".
likes – binary relation (true or false for each pair of objects). likes(x,y) is read as "x likes y".
smarter – binary relation. smarter(x,y) is read as "x is smarter than y".
father – unary function (L → L) (associates a single output value for each object input value). father(x) is read as "x's father".
+ - binary function (L x L → L) (associates a single output value for each pair of input values). x + y is read as "x plus y".

Logical Symbols

∀ – for all. Indicates that all objects in the set L have some property. For example, ∀*x* happy(*x*) means every object in L has the property *happy*. Note that one of the logical axioms of this game is that there is at least one object, so for-alls which state contradictions are themselves contradictions (not vacuously true).
∃ - there exists. Indicates that at least one object in the set L has some property. Example: ∀*x*∃*y* (*x*+*y*=*x*) means for every object in L, there is another (possibly the same) object in L which you can add to it to get the original back.
∃! - there uniquely exists. Indicates that exactly one object satisfies the given property in terms of the previous variables. Example: ∀*x*∃!*y* (*x*+*y*=*0*) means for every object, there is a unique object which, when added to the original, sums to zero (unique inverses). Notice that this object may be different for different x's; it is simply saying that not more than one object satisfies the condition with respect to x. If you use a quantifier card to play this quantifier, you are allowed to name the object or function you are identifying, as explained on the card. For example, in the statement above, it is well-formed to make that y a function of x, so we could call that y, say, inv(x).
Λ – and. Indicates that both operands need to be true for the expression to be true. For example: ∀*x* (happy(*x*) Λ x+x=x) means that every object is both happy and that when you add it to itself, you get itself back.
V – or. Indicates that one *or more* operands needs to be true for the expression to be true. For example: happy(0) V blonde(0) means that either 0 is happy, 0 is blonde, *or both*.
¬ - not. Indicates that the expression is true if the operand is false. For example: ¬happy(0) indicates that the object called 0 is not happy.
→ - implies / if-then. The expression is *false* only when the left side is true and the right side is false. Essentially, asserts that if the left side is true, then the right side must be true. Example: ∀*x* happy(x) → ¬blonde(x) means anything that is happy is not blonde, or "for every object, if it is happy then it is not blonde".
↔ - if and only if. The expression is true exactly when the left side and the right side have the same truth value. That is, either the left and right are both true, or they are both false.
() - for grouping. (A Λ B) V C is not the same as A Λ (B V C), as can be seen by supposing A is false and C is true. In this case, the former is true and the latter is false.
= - equality. This expression is true if the two sides are exactly the same object. When two objects are the same, they are completely substitutible for each other in all contexts.

Examples

Predicates

“Blonde” and “happy” are the two predicates (properties) of the game. For every object x, blonde(x) is either true or false. Same for happy. The following are legal uses of predicates:

$\forall$x blonde(x) (everyone is blonde)
$\exists$x happy(x) $\wedge$ ¬blonde(x) (at least one person is happy and not blonde)

The following are common illegal uses of predicates.

$\exists$x x = blonde (ILLEGAL: blonde is a property, not an object)
$\forall$x$\exists$y happy(x) = y (ILLEGAL: happy(x) is either true or false, and can't be equal to an object)
$\forall$x$\forall$y happy(x) = blonde(x) (ILLEGAL: properties cannot be equal (but this one is sensical, you probably meant happy(x) <=> blonde(x)))
$\forall$x blonde(happy(x)) (ILLEGAL: happy(x) is true or false, which cannot be given to blonde, which takes an object)

Relations

“smarter” and “likes” are the two relations of the game. Each of them takes two objects, separated by a comma, and maps them in to a true or false value. They are essentially two-argument predicates.

A subtle point is that even though “smarter” has a meaning and some properties in the real world, they must be thrown out for the game. For example, it is not necessarily true that an object can't be smarter than itself. Also, just because x is smarter than y and y is smarter than z, that doesn't necessarily imply x is smarter than z. These properties must be estabilshed by writing axioms.

The following are legal uses of relations:

$\forall$x ¬smarter(x,x) (nobody is smarter than himself)
$\forall$x$\exists$y smarter(y,x) (no matter who you are, there's always somebody smarter)

The following are common illegal uses of relations:

$\forall$x smarter(x) (ILLEGAL: smarter takes two arguments; x is smarter than whom?)
$\forall$x$\forall$y smarter(x,y) = y (ILLEGAL: smarter(x,y) is true or false, which cannot be equal to an object)

Functions

There are two functions in the game: father and +. Unlike predicates and relations, which take objects and return true or false, functions take objects and return objects. Father takes one object (has domain L), + takes two (has domain L x L).

The following are legal uses of functions:

$\forall$x blonde(father(x)) (everybody's father is blonde)
$\forall$x$\exists$y father(y) = x (everybody has a father)
$\forall$x$\forall$y x + y = y + x (order doesn't matter when adding)

The following are common illegal uses of functions:

$\forall$x father(x) (ILLEGAL: father(x) is an object, and this statement is expecting true or false; it is like saying “everybody's father”)

A subtle point is that even though + generally has a mathematical meaning, it does not in this game. It is just a two-argument function without any associated properties (until you define them). That is, a + b is not necessarily the same as b + a, and (a + b) + c is not necessarily the same as a + (b + c). Also note that this means that a + b + c is *logically* ambiguous, so you must say either (a + b) + c or a + (b + c) to clarify what you mean.

Unique Quantifier

A tricky part of this game is how to use the unique quantifier ($\exists$!) cards. This quantifier states that exacly one object has the following property. So:

$\exists$!x blonde(x) (there is exactly one blonde object)
$\forall$x$\exists$!y likes(y,x) (there is exactly one object that likes every object, *but it need not be the same one*)

In the latter example, there can be a different y for every x. This wouldn't be the case if the quantifiers had been reversed (to $\exists$!y$\forall$x) – it would have to be the same y.

The card states that you are allowed to name the object or function you create if you wish. So, you would say:

$\exists$!x blonde(x) | blondie := x

Which states the unique existence of a blonde object. So we can unambiguously refer to it from now on as 'blondie'. 'blondie' may be referred to from now on without restriction, even when using P,R,F cards that say no other nonlogical symbols may be referred to.

When the unique quantifier comes after some for-all quantifiers, you are allowed to name a function, with one argument per for-all quantifier. For example:

$\forall$x$\exists$!y likes(y,x) | liker(x) := y

Which, again, may be used without restriction. Other incantations of the unique quantifier (for example, if it is after another existence quantifier) may not be named, except by experienced mathematicians who know how the subtleties of how the domain is restricted and whatnot.

P

blonde

This symbol is a **predicate**, or property, which for each object can be either true or false.

When you play this card, you may write any statement involving symbols of pure logic and this symbol, but no other nonlogical symbols. You may play this card along with other P,F, and R cards to mention more symbols in the same statement.

F

father

This symbol is a **function**, which relates every input object to exactly one output object.

When you play this card, you may write any statement involving symbols of pure logic and this symbol, but no other nonlogical symbols. You may play this card along with other P,F, and R cards to mention more symbols in the same statement.

R

likes

This symbol is a **binary relation,** which for any two objects (left and right) given to it, can either be true or false.

When you play this card, you may write any statement involving symbols of pure logic and this symbol, but no other nonlogical symbols. You may play this card along with other P,F, and R cards to mention more symbols in the same statement.

P

happy

This symbol is a **predicate**, or property, which for each object can be either true or false.

When you play this card, you may write any statement involving symbols of pure logic and this symbol, but no other nonlogical symbols. You may play this card along with other P,F, and R cards to mention more symbols in the same statement.

F

+

This symbol is a **binary function**, which relates every two input objects to exactly one output object.

When you play this card, you may write any statement involving symbols of pure logic and this symbol, but no other nonlogical symbols. You may play this card along with other P,F, and R cards to mention more symbols in the same statement.

R

smarter

This symbol is a **binary relation,** which for any two objects (left and right) given to it, can either be true or false.

When you play this card, you may write any statement involving symbols of pure logic and this symbol, but no other nonlogical symbols. You may play this card along with other P,F, and R cards to mention more symbols in the same statement.

Q

$\exists x$

This is the existential quantifier, which asserts the existence of at least one object with a given property.

When you play this card, you may write any statement, as long as it only uses one variable, and it is quantified as above. You may play this card along with other Q cards to chain quantifiers together (renaming variables appropriately).

Q

$\forall x$

This is the universal quantifier, which states a property which holds for all objects.

When you play this card, you may write any statement, as long as it only uses one variable, and it is quantified as above. You may play this card along with other Q cards to chain quantifiers together (renaming variables appropriately).

Q

$\exists x \forall y$

This asserts that at least one object x exists such that for every y some property (in terms of x and y) holds.

When you play this card, you may write any statement, as long as it uses exactly two variables, and they are quantified as above. You may play this card along with other Q cards to chain quantifiers together (renaming variables appropriately).

P

blonde

This symbol is a **predicate**, or property, which for each object can be either true or false.

When you play this card, you may write any statement involving symbols of pure logic and this symbol, but no other nonlogical symbols. You may play this card along with other P,F, and R cards to mention more symbols in the same statement.

F

father

This symbol is a **function**, which relates every input object to exactly one output object.

When you play this card, you may write any statement involving symbols of pure logic and this symbol, but no other nonlogical symbols. You may play this card along with other P,F, and R cards to mention more symbols in the same statement.

R

likes

This symbol is a **binary relation,** which for any two objects (left and right) given to it, can either be true or false.

When you play this card, you may write any statement involving symbols of pure logic and this symbol, but no other nonlogical symbols. You may play this card along with other P,F, and R cards to mention more symbols in the same statement.

P

happy

This symbol is a **predicate**, or property, which for each object can be either true or false.

When you play this card, you may write any statement involving symbols of pure logic and this symbol, but no other nonlogical symbols. You may play this card along with other P,F, and R cards to mention more symbols in the same statement.

F

+

This symbol is a **binary function**, which relates every two input objects to exactly one output object.

When you play this card, you may write any statement involving symbols of pure logic and this symbol, but no other nonlogical symbols. You may play this card along with other P,F, and R cards to mention more symbols in the same statement.

R

smarter

This symbol is a **binary relation,** which for any two objects (left and right) given to it, can either be true or false.

When you play this card, you may write any statement involving symbols of pure logic and this symbol, but no other nonlogical symbols. You may play this card along with other P,F, and R cards to mention more symbols in the same statement.

Q

$\forall x \exists y$

This asserts that for every object x, there is at least one y so that the property following (in terms of x and y) holds.

When you play this card, you may write any statement, as long as it uses exactly two variables, and they are quantified as above. You may play this card along with other Q cards to chain quantifiers together (renaming variables appropriately).

Q

$\forall x \forall y$

This asserts that for all objects x and y, the property following (in terms of x and y) holds.

When you play this card, you may write any statement, as long as it uses exactly two variables, and they are quantified as above. You may play this card along with other Q cards to chain quantifiers together (renaming variables appropriately).

Q

$\forall x \forall y \forall z$

This asserts that for all objects x, y, and z, the property following (in terms of those variables) holds.

When you play this card, you may write any statement, as long as it uses exactly three variables, and they are quantified as above. You may play this card along with other Q cards to chain quantifiers together (renaming variables appropriately).

P blonde This symbol is a **predicate**, or property, which for each object can be either true or false. When you play this card, you may write any statement involving symbols of pure logic and this symbol, but no other nonlogical symbols. You may play this card along with other P,F, and R cards to mention more symbols in the same statement.	**F** father This symbol is a **function**, which relates every input object to exactly one output object. When you play this card, you may write any statement involving symbols of pure logic and this symbol, but no other nonlogical symbols. You may play this card along with other P,F, and R cards to mention more symbols in the same statement.	**R** likes This symbol is a **binary relation,** which for any two objects (left and right) given to it, can either be true or false. When you play this card, you may write any statement involving symbols of pure logic and this symbol, but no other nonlogical symbols. You may play this card along with other P,F, and R cards to mention more symbols in the same statement.
P happy This symbol is a **predicate**, or property, which for each object can be either true or false. When you play this card, you may write any statement involving symbols of pure logic and this symbol, but no other nonlogical symbols. You may play this card along with other P,F, and R cards to mention more symbols in the same statement.	**F** + This symbol is a **binary function**, which relates every two input objects to exactly one output object. When you play this card, you may write any statement involving symbols of pure logic and this symbol, but no other nonlogical symbols. You may play this card along with other P,F, and R cards to mention more symbols in the same statement.	**R** smarter This symbol is a **binary relation,** which for any two objects (left and right) given to it, can either be true or false. When you play this card, you may write any statement involving symbols of pure logic and this symbol, but no other nonlogical symbols. You may play this card along with other P,F, and R cards to mention more symbols in the same statement.
Q $\exists! x$ This asserts the existence of a *unique* object x with a given property. The player should give a name to the object whose existence is being asserted, which may be used later by anyone when writing down a statement (including P,F,R statements). When you play this card, you may write any statement, as long as it uses exactly one variable, and it is quantified as above. You may play this card along with other Q cards to chain quantifiers together (renaming variables and definitions appropriately).	**Q** $\forall x \forall y \exists z$ This asserts that for all objects x and y, there is at least one object z satisfying the following property (in terms of x, y, and z). When you play this card, you may write any statement, as long as it uses exactly three variables, and they are quantified as above. You may play this card along with other Q cards to chain quantifiers together (renaming variables appropriately).	**Q** $\forall x \exists! y$ This asserts that for any object x, there is a *unique* object y satisfying the following property (in terms of x and y). The player should name the function (in terms of x) that is being defined by this statement, which may be used later by anyone when writing a statement (including P,F,R statements). When you play this card, you may write any statement, as long as it uses exactly two variables, and they are quantified as above. You may play this card along with other Q cards to chain quantifiers together (renaming variables and definitions appropriately).

P blonde This symbol is a **predicate**, or property, which for each object can be either true or false. When you play this card, you may write any statement involving symbols of pure logic and this symbol, but no other nonlogical symbols. You may play this card along with other P,F, and R cards to mention more symbols in the same statement.	**F** father This symbol is a **function**, which relates every input object to exactly one output object. When you play this card, you may write any statement involving symbols of pure logic and this symbol, but no other nonlogical symbols. You may play this card along with other P,F, and R cards to mention more symbols in the same statement.	**R** likes This symbol is a **binary relation,** which for any two objects (left and right) given to it, can either be true or false. When you play this card, you may write any statement involving symbols of pure logic and this symbol, but no other nonlogical symbols. You may play this card along with other P,F, and R cards to mention more symbols in the same statement.
P happy This symbol is a **predicate**, or property, which for each object can be either true or false. When you play this card, you may write any statement involving symbols of pure logic and this symbol, but no other nonlogical symbols. You may play this card along with other P,F, and R cards to mention more symbols in the same statement.	**F** + This symbol is a **binary function**, which relates every two input objects to exactly one output object. When you play this card, you may write any statement involving symbols of pure logic and this symbol, but no other nonlogical symbols. You may play this card along with other P,F, and R cards to mention more symbols in the same statement.	**R** smarter This symbol is a **binary relation,** which for any two objects (left and right) given to it, can either be true or false. When you play this card, you may write any statement involving symbols of pure logic and this symbol, but no other nonlogical symbols. You may play this card along with other P,F, and R cards to mention more symbols in the same statement.
Q $\forall x$ This asserts that for every object x, the property following (in terms of x) holds. When you play this card, you may write any statement, as long as it uses exactly one variable, and it is quantified as above. You may play this card along with other Q cards to chain quantifiers together (renaming variables appropriately).	**Q** $\forall x \forall y$ This asserts that for all objects x and y, the property following (in terms of x and y) holds. When you play this card, you may write any statement, as long as it uses exactly two variables, and they are quantified as above. You may play this card along with other Q cards to chain quantifiers together (renaming variables appropriately).	**Q** $\forall x \forall y \forall z$ This asserts that for all objects x, y, and z, the property following (in terms of those variables) holds. When you play this card, you may write any statement, as long as it uses exactly three variables, and they are quantified as above. You may play this card along with other Q cards to chain quantifiers together (renaming variables appropriately).

S

$$\forall x\, \text{blonde}(x) \Rightarrow \text{happy}(x)$$

"Anyone who is blonde is also happy."

When you play this card, simply write the statement shown on the board.

S

$$\forall x \forall y (\text{happy}(x) \wedge \text{likes}(x, y)) \Rightarrow \text{happy}(y)$$

"Anyone who is liked by a happy person is also happy."

When you play this card, simply write the statement shown on the board.

S

$$\forall x\, \text{smarter}(\text{father}(x), x)$$

"Everyone's father is smarter than they are."

When you play this card, simply write the statement shown on the board.

S

$$\forall x \neg \text{likes}(x, \text{father}(x)) \wedge \text{likes}(\text{father}(x), x)$$

"Nobody likes his father, but every father likes his son."

When you play this card, simply write the statement shown on the board.

S

$$\forall x \forall y\, \text{smarter(x,y)} \Rightarrow \neg \text{likes}(y, x)$$

"Nobody likes anybody smarter than him."

When you play this card, simply write the statement shown on the board.

S

$$\forall x \forall y\, \text{smarter}(x+y, x) \wedge \text{smarter}(x+y, y)$$

"The sum of two objects is smarter than each of the summands."

When you play this card, simply write the statement shown on the board.

S

$$\forall x \forall y \forall z\, (\text{smarter}(x, y) \wedge \text{smarter}(y, z)) \Rightarrow \text{smarter}(x, z)$$

"If I am smarter than you, and you are smarter than Bob, then I am smarter than Bob (for all values of I, you, and Bob)."

When you play this card, simply write the statement shown on the board.

S

$$\forall x \forall y\, x \neq y \Rightarrow (\text{smarter}(x, y) \Leftrightarrow \neg \text{smarter}(y, x))$$

"I'm smarter than you if and only if you're not smarter than me (as long as we're not the same person)."

When you play this card, simply write the statement shown on the board.

S

$$\forall x \forall y (x+y=y+x)$$

"Addition is commutative."

When you play this card, simply write the statement shown on the board.

S

$$\forall x \forall y (x + y = \text{father}(y + x))$$

“Addition is commutative 'through father'. Look at the logic if you don't know what that means (which you probably shouldn't).”

When you play this card, simply write the statement shown on the board.

S

$$\forall x \neg \text{blonde}(\text{father}(x))$$

“Nobody's father is blonde.”

When you play this card, simply write the statement shown on the board.

S

$$|L| \geq n$$

This statement asserts that there are at least n objects, where n is a positive integer you choose at the time the card is played.

When you play this card, simply write the statement shown on the board, substituting n appropriately.

S

$$|L| < n$$

This statement asserts that there are fewer than n objects, where n is a positive integer you choose at the time the card is played.

When you play this card, simply write the statement shown on the board, substituting n appropriately.

S

$$\forall x \forall y \neg \text{smarter}(x, y)$$

“Nobody is smarter than anybody else.”

When you play this card, simply write the statement shown on the board.

X

$$?$$

When you play this card, you may write any statement on the board.

X

$$\neg$$

When you play this card, insert a negation sign in any valid position in any statement written on the board.

X

$$\neg$$

When you play this card, insert a negation sign in any valid position in any statement written on the board.

X

$$\neg$$

When you play this card, insert a negation sign in any valid position in any statement written on the board.