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MATH 5000
2006-12-04

(A) If $\mathcal{A} = (A, +, \cdot)$ is a nonstandard model of $\mathrm{Th}\,(\omega, +, \cdot)$, then the twin prime conjecture is true in $(\omega, +, \cdot)$ if and only if there is a nonstandard $a \in A$ such that $\mathcal{A} \models$ "a and $a + 2$ are prime".

Proof. We have seen in an earlier homework how to define $<$ and *prime* in $(\omega, +, \cdot)$. Let $twin(x)$ be shorthand for $prime(x) \wedge prime(x + 2)$. Then we can state the twin prime conjecture as follows: $\forall x \exists y\, (x < y \wedge twin(y))$.

$(\Leftarrow)$ Suppose there is a nonstandard $a \in A$ with $\mathcal{A} \models twin(a)$. If the twin prime conjecture were false in $(\omega, +, \cdot)$, then there would be some $b \in \omega$ where $(\omega, +, \cdot) \models \neg\exists x\, (b < x \wedge twin(x))$. But, of course, b can be written in the form $1 + 1 + \cdots + 1$ a finite number of times, so that statement could be written for (and would be true in) $\mathcal{A}$, too, contradicting the existence of a.

$(\Rightarrow)$ Suppose that $(\omega, +, \cdot) \models \forall x \exists y\, (x < y \wedge twin(y))$. This sentence is also true in $\mathcal{A}$, so pick some nonstandard $a \in A$. Therefore, there must exist $b \in A$ with $a < b$ (so b is nonstandard) where $\mathcal{A} \models twin(b)$. $\square$

(B) Given an infinite set of primes P. There is a countable model $\mathcal{B} = (B, +, \cdot)$ of $\mathrm{Th}\,(\omega, +, \cdot)$ and a $b \in B$ such that for every prime $p \in \omega$, $\mathcal{B} \models \bar{p}|b$ if and only if $p \in P$.

Proof. Let $a|b$ mean $\exists n\ n \cdot a = b$. Let $\mathcal{L} = (+, \cdot, c)$. Let $\Sigma = \mathrm{Th}\,(\omega, +, \cdot) \cup \{(\bar{p}|c) | p \in P\} \cup \{\neg(\bar{p}|c) | p \in \omega - P\}$.

Given some finite $F \subseteq \Sigma$, F will have some sentences from $\mathrm{Th}\,(\omega, +, \cdot)$ and some sentences saying certain primes do and do not divide c. Build a model of F by letting the universe be ω and c be the product of the primes which Σ says should divide c. So by the compactness theorem, Σ has a model $\mathcal{B}$, and $c^{\mathcal{B}}$ has our desired properties by construction. $\square$

(Di) $\{\varphi_0, \varphi_1, \ldots, \varphi_{n-1}\} \vdash \psi$ if and only if $\vdash (\varphi_0 \rightarrow (\varphi_1 \rightarrow \cdots (\varphi_{n-1} \rightarrow \psi)))$.

Proof. By induction on n. The atomic case $n = 0$ is trivial. $\{\varphi_0, \varphi_1, \ldots, \varphi_n\} \vdash \psi$ iff (by IH) $\{\varphi_0\} \vdash (\varphi_1 \to \cdots (\varphi_n \to \psi))$ iff (by the deduction theorem) $\vdash (\varphi_0 \to \cdots (\varphi_n \to \psi))$. $\square$

(Dii) If $\Sigma \cup \{\varphi\} \vdash \psi$ and $\Sigma \cup \{\neg\varphi\} \vdash \psi$, then $\Sigma \vdash \psi$.

Proof. By (Di) we have that $\Sigma \vdash (\varphi \to \psi)$ and that $\Sigma \vdash (\neg\varphi \to \psi)$, so ψ follows as a logical consequence (and can thus be proved by the completeness theorem). $\square$