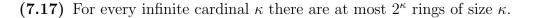
Luke Palmer MATH 5000 2006-10-18



*Proof.* If R is a set of size  $\kappa$ , then the number of functions from  $R \times R \mapsto R$  is  $\kappa^{\kappa \cdot \kappa} = 2^{\kappa}$ . That is the number of possible + and  $\cdot$  functions, regardless of whether they satisfy the ring definition, so there cannot possibly be more rings than that.

(7.22) For each  $f: \omega \mapsto \omega$  let  $X_f = \{f \upharpoonright m : m \in \omega\}$ . Then each  $X_f$  has size  $\aleph_0$ , and if  $f \neq g$  then  $X_f \cap X_g$  is finite.

*Proof.* It is obvious that  $X_f$  has size at most  $\aleph_0$  by its definition. Also, each  $m^+ \in \omega$  will give a different function, since  $\langle m, y \rangle$  (for some y) must be in  $f \upharpoonright m^+$  but not in  $f \upharpoonright m$ .

Given  $f, g : \omega \mapsto \omega$ ,  $f \neq g$ . Then there must be a least  $x \in \omega$  for which  $f(x) \neq g(x)$ . We have that  $\{f \upharpoonright m : m \in x\} = \{g \upharpoonright m : m \in x\}$ , however,  $f \upharpoonright m \neq g \upharpoonright m$  for any m > x, so the size of  $X_f \cap X_g$  is x.  $\square$ 

(7.23) There exists a family  $\langle A_{\alpha} : \alpha < 2^{\aleph_0} \rangle$ , with  $A_{\alpha} \subseteq \mathscr{P}(\omega)$  and  $A_{\alpha}$  infinite, such that for any distinct  $\alpha, \beta < 2^{\aleph_0}$ ,  $A_{\alpha} \cap A_{\beta}$  is finite. As a consequence, every even integer greater than 2 is the sum of two primes.

*Proof.* Trivial.  $\Box$ 

(7.24) If (A, <) and  $(B, \prec)$  are countable linear orderings, neither of which has a least or greatest element and both dense, then A is isomorphic to B.

*Proof.* First, well-order A and B (i.e. take the isomorphisms between them and  $\omega$ ). By recursion there is a function  $g:\omega\mapsto (A\times B)$ , where g(x) is defined as follows:

Let  $h = g \upharpoonright x$ . It will be clear that this is a function later.

If x is even, then pick the least  $a \in A - \text{dom } h$  by our well-ordering of A. If a is less than all elements of dom h then pick the least  $b \in B - \text{rng } h$ 

 $\prec$  all elements of rng h, which can be done since our orderings have no endpoints. Similarly if a is greater than all elements of dom h. If neither is the case, then there are  $a', a'' \in \text{dom } h$  such that a' < a < a'', so pick the least b with  $h(a') \prec b \prec h(a'')$ , which can be done because our orderings are dense. Let  $g(x) = \langle a, b \rangle$ .

If x is odd, pick a  $b \in B - \operatorname{rng} h$  and a corresponding a analogous to above. Let  $g(x) = \langle a, b \rangle$ .

The h used is a function because each time we picked an a outside of the domain of the previous h. And by the same argument, rng g is a function. Because of how we picked the elements of g, g is 1-1 and an order isomorphism. dom g = A because if it weren't, then there would be a least a that we didn't pick, meaning that A's well-order type wasn't  $\omega$ , which we assumed away. Similarly rng g = B.

**Lemma.** The Banach-Tarski-Cantor-Schröder-Bernstein theorem. If  $A \subseteq B \subseteq C$  and  $A \equiv C$ , then  $A \equiv B$ .

*Proof.* Let  $f: C \xrightarrow{1-1 \text{ onto}} A$  be the function "witnessing"  $A \equiv C$ .  $B \subseteq C$ , so  $f[B-A] \subseteq A$ . Define  $S = \bigcup_{i \in \omega} f^i[B-A]$  (including  $f^0$ , the identity). We have that  $f[S] \subseteq A$ , since each operand is a subset of B and thus C.

Note that  $B-S=A-S\subseteq A$ , because  $f^0[B-A]=B-A\subseteq S$ . Convince yourself with a Venn diagram.

Define q as follows.

$$g(x) = \begin{cases} f(x) & x \in S \\ x & x \in B - S \end{cases}$$

If f splits C into n pieces and performs rigid motions to get to A, our g splits B into n+1 pieces and performs rigid motions to get to A (identity is certainly a rigid motion). It is 1-1 because both f and the identity are 1-1. It is onto because  $f[S] \cup (B-S) = (A \cap S) \cup (A-S) = A$ .