Luke Palmer
MATH 4730
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(11a) Given a non-empty subset S of $\mathbb{Z}$, consider the disjoint subsets $S^+ = \{s \in S : s >= 0\}$ and $S^- = \{-s \in S : s < 0\}$ covering S, and observe that each is order-isomorphic to a subset of ω. If S^+ is nonempty, then the least element of S is $\min S^+$, otherwise the least element is $-\min S^-$. $\square$

(11b) $E(3) = 3$. $E(-1) = \omega$. $E(-2) = \omega + 1$. $\operatorname{rng} E = \omega + \omega$.

(14) Given $a, b \in A$ such that $a < b$, show $F(a) \subset F(b)$. Notice that $a \in F(a)$ but $b \notin F(a)$, so $F(a) \neq F(b)$. If $F(a) = \emptyset$, then $F(a) \subseteq F(b)$. So assume that $F(a) \neq \emptyset$ and $a' \in F(a)$. $a' \leq a < b$, so $a' \in F(b)$. $\square$

(16) The following three cases follow from Theorem 7K about well-orderings:

- $\alpha^+ = \operatorname{seg} b$ for some $b \in \beta$: Then $\alpha^+ \in \beta$ and therefore $\alpha^+ \in \beta^+$.
- $\alpha^+ = \beta$: Then $\alpha^+ \in \beta \cup \{\beta\} = \beta^+$.
- $\beta = \operatorname{seg} a$ for some $a \in \alpha^+$: Then $\beta \in \alpha^+$, so $\beta \in \alpha \subseteq \beta$ or $\beta = \alpha \in \beta$, a contradiction. $\square$

(18) If $\bigcup S = \alpha^+$ then for every element s of S, $s \subseteq \alpha^+$. Since s is an ordinal, either $s \in \alpha^+$ or $s = \alpha^+$. Also, $\bigcup S = \alpha^+ \in S$, so α^+ is the greatest element. This proves part *(i)* by contrapositive, and its negation trivially implies part *(ii)*. $\square$

(20) Suppose that S were infinite. We can construct an infinitely ascending sequence of R by induction. However, this sequence will be an infinitely descending sequence of R^{-1}, contradicting its being a well-ordering. $\square$