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MATH 5000
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(Aa) There is no set of uncountably many nonintersecting circles in the plane with distinct centers.

Proof. Given a set of nonintersecting circles with distinct centers. There is a set of nonintersecting open sets around the centers of the circles. Pick a rational out of each such set, and you have a 1-1 map from circles to $\mathbb{Q}$. $\square$

(Ab) There is no set of uncountably many figure-8s in the plane.

Proof. Similar argument as **(Aa)**, using the crossing at the middle of the figure-8 instead of the center. $\square$

(B) ...

(C) The Sheffer Stroke ($p|q$ iff $\neg p \wedge \neg q$) is a complete logical connective.

Proof. $\neg p$ iff $p|p$. $p \wedge q$ iff $(\neg p)|(\neg q)$. $\square$

(D) Logical equivalence together with negation is not a complete set of logical connectives.

Proof. I will show that the expression $p \Rightarrow q$ is not representable by proving that every truth table in p and q with these connectives has an even number of "true" values. By induction on ϕ:

Trivial for $\phi = p$, $\phi = q$.

For $\phi = \neg\psi$, then the number of true values in ϕ is four minus the number of true values in ψ, still even.

For $\phi = \psi \Leftrightarrow \chi$, it suffices to show that the number of true values in ϕ is even when the number of true values in ψ and χ is exactly two (it is trivial for zero and four). Wlg assume that ψ's truth table reads $TTFF$ (we can assume negated p and q as appropriate). Then for each truth table of χ:

- $\chi = TTFF$ then $\phi = TTTT$.
- $\chi = TFTF$ then $\phi = TFFT$.
- $\chi = TFFT$ then $\phi = TFTF$.
- $\chi = FTTF$ then $\phi = FTFT$.
- $\chi = FTFT$ then $\phi = FTTF$.
- $\chi = FFTT$ then $\phi = FFFF$.

All of which have even numbers of "true" values. $\square$

Ea

$$\begin{aligned}
&\forall x \forall y \forall z\ (x+y)+z = x+(y+z) \\
\wedge &\forall x \forall y \forall z\ (xy)z = x(yz) \\
\wedge &\forall x \forall y\ x+y = y+x \\
\wedge &\forall x \forall y\ xy = yx \\
\wedge &\forall x \forall y \forall z\ a(b+c) = (ab)+(ac) \\
\wedge &\exists x \exists y\ (x \neq y \\
&\qquad \wedge \forall z\ x+z = z \\
&\qquad \wedge \forall z\ yz = z \\
&\qquad \wedge \forall z \exists w\ z+w = x \\
&\qquad \wedge \forall z\ (z \neq x \Rightarrow \exists w\ zw = y))
\end{aligned}$$

Eb I'll use symbols other than x, y, z, w as variables for readability.

Write $(x|y)$ as shorthand for $\exists z\ xz = y$. Write $P(x)$ as shorthand for $\forall y\ ((y|x) \Rightarrow (y = 1 \vee y = x))$ (which must only be used when the variable 1 is in scope). Write $x \leq y$ as shorthand for $\exists z\ x+z = y$.

$$\begin{aligned}
&\exists 1 (\forall x\ 1x = x \\
&\quad \wedge \forall x \exists y\ (x \leq y \wedge P(y) \wedge P((y+1)+1)))
\end{aligned}$$