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MATH 4730: Set Theory
Final Exam

(1) If κ, λ, and μ are cardinals, then $\kappa^{\lambda+\mu} = \kappa^\lambda \cdot \kappa^\mu$.

Proof. Given $\bar{\bar{K}} = \kappa$, $\bar{\bar{L}} = \lambda$, $\bar{\bar{M}} = \mu$, and $L \cap M = \emptyset$. Given a function $f : L \cup M \mapsto K$. Construct $g : L \mapsto K = f \restriction L$ and $h : M \mapsto K = f \restriction M$. So $\kappa^{\lambda+\mu} \le \kappa^\lambda \cdot \kappa^\mu$.

Given a pair of functions $g : L \mapsto K$ and $h : M \mapsto K$. Construct $f : L \cup M \mapsto K = g \cup h$. g and h have disjoint domains, so f is a function. Therefore $\kappa^\lambda \cdot \kappa^\mu \le \kappa^{\lambda+\mu}$, and therefore they are equal. □

(2) $2^{\aleph_0} \ne \aleph_{\omega_1 \cdot 49 + \omega}$

Proof. Let $\beta = \omega_1 \cdot 49 + \omega$. β is a limit ordinal, so $\operatorname{cf} \aleph_\beta = \operatorname{cf} \beta = \aleph_0$ (by Enderton's Theorem 9N). However, by König's theorem, $\aleph_0 < \operatorname{cf} 2^{\aleph_0}$. □

(3) If A is a set well-ordered by $(<)$ and $f : A \mapsto A$ satisfies $x < y \Rightarrow f(x) < f(y)$, then for all $x \in A$, $x \le f(x)$.

Proof. If not, then there is a least x such that $f(x) < x$. Construct by recursion $R : \omega \mapsto A$ where $R(0) = x$; $R(a+1) = f(R(a))$. $R(1) < R(0)$, and given $R(n+1) < R(n)$, $f(R(n+1)) < f(R(n))$, so $R(n+2) < R(n+1)$. Therefore R is an infinitely descending sequence in A, a contradiction. □

(4) There exists a sequence of sets $A_0 \supseteq A_1 \supseteq \cdots \supseteq A_\alpha \supseteq \cdots$ $(\alpha < \omega_1)$ with $\bigcap_\alpha A_\alpha = \emptyset$, and all A_α uncountable.

Proof. Namely, let $A_\alpha = \omega_1 - \alpha$. Every $\alpha < \omega_1$ is countable, so subtracting it from uncountable ω_1 leaves an uncountable set. Also, any ordinal $\alpha \notin A_{\alpha+1}$, so the intersection is empty. □

(5) For λ an infinite cardinal, $2^\lambda = \lambda^\lambda$.

Proof. By the Schröder-Bernstein theorem, $2^\lambda \le \lambda^\lambda \le (2^\lambda)^\lambda \le 2^{\lambda\cdot\lambda} \le 2^\lambda$. $\square$

(6) The set of hereditarily finite sets is equal to V_ω.

Proof. It suffices to show that $x \in V_\omega$ iff $TC(x)$ is finite.

($\Rightarrow$) Of course, x has finite rank iff $x \in V_\omega$. By induction on $\operatorname{rnk} x$. If $\operatorname{rnk} x = 0$ then $x = TC(x) = \emptyset$. Suppose all sets of rank α or less have finite transitive closure. Given $\operatorname{rnk} A = \alpha + 1$. $TC(A) = \{A\} \cup \bigcup\limits_{a\in A} TC(a)$. Every element of A must have rank α or less, and A itself must be finite because it has finite rank, so $TC(A)$ must be finite.

($\Leftarrow$) If $TC(x)$ is finite, then it has finite rank (for if it didn't, then it would have to contain an infinite set, and $TC(x)$ would have to be infinite). So every element of $TC(x)$ has a smaller rank, and is thus an element of V_ω. $x \in TC(x)$, so $x \in V_\omega$. $\square$

(7) Given a set P of $\aleph_0$-many people. There is an infinite subset of that set such that all people have met each other or all people have not met each other.

Proof. Let $M \cup \tilde{M} = P^{(2)}$, where $P^{(2)}$ is the set of 2-element subsets of P. M represents the set of pairs of people who have met, and $\tilde{M}$ represents the set of pairs of people who have not. Then by Ramsey's theorem, there exists a $P' \subseteq P$ where $\bar{\bar{P}}' = \aleph_0$ and either $P'^{(2)} \subseteq M$ or $P'^{(2)} \subseteq \tilde{M}$. $\square$

(8ai) There is a countable subset of $[0, 1]$ isomorphic to ω^2 under the usual $(<)$ of the real line.

Proof. Define $v : \mathbb{R} \times \mathbb{R} \times \omega \mapsto \mathbb{R}$ by $v(a, b, n) = b - 2^{-n}(b - a)$. Now define $w : \omega^2 \mapsto \mathbb{R}$ by $w(\omega \cdot c + d) = v(1 - 2^{-c}, 1 - 2^{-(c+1)}, d)$. The set $W = \operatorname{rng} w$ is a subset of $[0, 1]$ and isomorphic to ω^2. Since the domain of w is a set of ordinals, if w is monotonically increasing then W is well-ordered. Notice that $v(a, b, n) \in [a, b)$ for any n and it is monotonically increasing in n. Then it is easy to see that w is monotonically increasing. $\square$

(8aii) There is no $X \subseteq [0,1]$ with X well-ordered by the usual $(<)$ of the real line and X uncountable.

Proof. Assume that there is. Let $f : \bar{\bar{X}} \mapsto X$ be the function that well-orders X (for example, $f(0) =$ the smallest element in X, etc.). Let $q : \mathbb{R} \times \mathbb{R} \mapsto \mathbb{Q}$ be a function that selects a rational between its two arguments. Let $g : \bar{\bar{X}} \xrightarrow{1-1} \mathbb{Q}$, by induction: $g(0) = q(f(0), f(1))$; $g(\alpha + 1) = q(f(\alpha + 1), f(\alpha + 2))$; $g(\lambda) = q(f(\lambda), f(\lambda + 1))$. We have just shown that $\bar{\bar{X}} \leq \bar{\bar{\mathbb{Q}}}$, contradicting that X is uncountable. $\square$

(8b)

Lemma. *Given an uncountable $Z \subseteq [a,b]$, there exists an $r \in (a,b)$ such that both $Z \cap [a,r]$ and $Z \cap [r,b]$ are uncountable.*

Proof. Suppose not. Then for every $r \in [a,b]$, either $[a,r] \cap Z$ is countable or $[r,a] \cap Z$ is countable. Let $L = \sup \{r \in Z | Z \cap [a,r] \text{ countable}\}$, $R = \sup \{r \in Z | Z \cap [r,b] \text{ countable}\}$. Clearly $L < R$, because if $R \leq L$, then $\frac{1}{2}(R+L)$ would have a countable number of elements below it and above it, and Z itself would be countable. But then $\frac{1}{2}(R+L)$ has an uncountable number of things on both sides of it, contradicting our assumption that it didn't. $\square$

Theorem. *If $Y \subseteq [0,1]$, closed, and uncountable, then $\bar{\bar{Y}} = 2^{\aleph_0}$.*

Proof. First, choose a well-ordering of the reals $(\prec)$. Construct a function $f :\ {}^{<\omega}2 \mapsto \mathcal{P}(\mathbb{R})$ by the following: $f(\langle\rangle) = Y$. Given some finite sequence S, pick the least r (according to $(\prec)$) so that both $f(S) \cap [0,r]$ and $f(S) \cap [r,1]$ are uncountable. Let $f(S : \langle 0 \rangle) = f(S) \cap [0,r]$ and $f(S : \langle 1 \rangle) = f(S) \cap [r,1]$ (where the notation $S : \langle n \rangle$ means the sequence S with n tacked on the end). Now define $g :\ {}^{\omega}2 \mapsto \mathcal{P}(\mathbb{R})$ by: $g(S) = \bigcap_{s \in \omega} f(S \restriction s)$ for S an infinite sequence.

Notice that $g(S)$ for any sequence is nonempty, because it is made up of descending sequences of nonempty closed sets. Also $g(S) \cap g(T) = \emptyset$ when $S \neq T$ by the construction. Pick a choice function $h : \mathcal{P}(\mathbb{R}) \mapsto \mathbb{R}$. $h \circ g$ is 1-1, so $2^{\aleph_0} \leq \bar{\bar{Y}} \leq 2^{\aleph_0}$. $\square$

(9) $\aleph_\omega^{\aleph_0} > \aleph_\omega$.

Proof. Let $\overline{\overline{I}} = \aleph_0$, and let A_i for $i \in I$ be a bunch of disjoint sets each with cardinality $\aleph_\omega$. By a theorem due to König, $\aleph_\omega^{\aleph_0} = \text{card} \bigotimes_i A_i > \text{card} \bigcup_i A_i = \aleph_\omega \cdot \aleph_\omega = \aleph_\omega$. $\square$