Introduction to the Finite Difference Method: Filling and Draining a Cylindrical Tube

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Consider filling a cylinder with water.

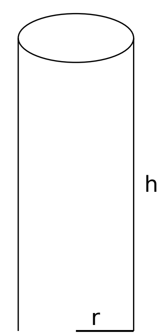


Figure 1: Cylinder with dimensions. r is the radius and h is the height.

The water flows in at a constant rate of 5 cm^3/s. The inflow rate (Q) can be written as the change in volume (V) over the change in time (t) (the Δ symbol represents change):

Inflow Rate

$$Q = \frac{\Delta V}{\Delta t} = 5\ cm^3/s \tag{1}$$

1 Filling the Tube Calculations and Equations

1.1 Conceptual Physics Approach

So, at this inflow rate, after 10 seconds there will be 50 cm^3 added to the cylinder.

$$\begin{aligned} V &= 5\ cm^3/s \cdot 10\ s \\ &= 50\ cm^3 \end{aligned}$$

In terms of the equation, the change in volume is equal to the inflow rate (Q) times the time period (t).

$$V = Q \cdot t \tag{2}$$

How high will the water have risen in the cylinder in those 10 seconds when 50 cm^3 of water was added? Well, we know that the volume of a cylinder is given by the equation:

$$V = \pi r^2 h \tag{3}$$

So, if we know the volume and the radius of the cylinder (r) we can solve this equation for the height (h):

Divide both sides by πr^2:

$$\frac{V}{\pi r^2} = \frac{\pi r^2 h}{\pi r^2} \tag{4}$$

To get:

$$\frac{V}{\pi r^2} = h \tag{5}$$

Which can be rewritten as:

$$h = \frac{V}{\pi r^2} \tag{6}$$

or:

$$h = \frac{1}{\pi r^2} V \tag{7}$$

Thus, for our given problem where the radius is 2.25 cm, and the volume of water added is 50 cm^3:

$$h = \frac{1}{\pi \cdot 2.25^2} \cdot 50 = 3.1 \text{ cm} \tag{8}$$

Now we can substitute for volume using Equation 2 to get:

$$h = \frac{1}{\pi r^2}\, Q \cdot t \tag{9}$$

Now, lets rewrite this equation so we just consider what happens over a small time period (call it a *time step* denoted by Δt). It's the change from moment to moment and results in a small change in height (Δh). So our final equation becomes:

$$\Delta h = \frac{1}{\pi r^2}\, Q \cdot \Delta t \tag{10}$$

Which we rearrange a little to get:

$$\boxed{\Delta h = \frac{\Delta t}{\pi r^2}\, Q} \tag{11}$$

Having calculated the change in the height of the water in the cylinder in a given time step (Δt), for each timestep we calculate the new height of water (h_{new}) as the old height plus the change:

$$\boxed{h_{new} = h_{old} + \Delta h} \tag{12}$$

We can now use these two equation to create a computer model that gives the height of water in the column over time.

1.2 Using Calculus to Find the Discrete Equations

Same problem–filling a cylinder–but using calculus to end up with the same equations in the end.

Start with the equation for the volume of a cylinder:

$$V = \pi r^2 h \tag{13}$$

There are two variables that change with time as the cylinder fills, the volume (V) and the height (h) since the radius (r) does not change. So, if we differentiate this equation with respect to time (implicit differentiation), we get:

$$\frac{dV}{dt} = \pi r^2 \frac{dh}{dt} \tag{14}$$

Solving for $\frac{dh}{dt}$ gives the **height change equation**:

$$\frac{dh}{dt} = \frac{1}{\pi r^2} \frac{dV}{dt} \tag{15}$$

The expression $\frac{dh}{dt}$ represents the instantaneous change in height with time: the rate at which height changes at any instant. To write a program to solve this equation we'll **discretize** the expression by using $\frac{\Delta h}{\Delta t}$:

$$\frac{\Delta h}{\Delta t} \approx \frac{dh}{dt} \tag{16}$$

The Δ means that we're taking the difference between two discrete value of h, so:

$$\Delta h = h_2 - h_1 \tag{17}$$

Since this is the rate of change over time it can be easier to think of the change in height as the difference between the new height and the old height over the short (Δt) time period.

$$\Delta h = h_{new} - h_{old} \tag{18}$$

So now we rewrite our height change equation (Eqn. 15) as:

$$\frac{\Delta h}{\Delta t} = \frac{1}{\pi r^2}\frac{dV}{dt} \tag{19}$$

which we can solve for the change in height (Δh):

$$\Delta h = \frac{\Delta t}{\pi r^2}\frac{dV}{dt} \tag{20}$$

Since the inflow rate (Q) is the change in volume over time, and it remains constant for our model, we can say:

Change in Height Equation

$$\boxed{\Delta h = \frac{\Delta t}{\pi r^2}\, Q} \tag{21}$$

Which is the same equation (Eqn. 11) we found when we took the conceptual approach in the previous section.

Now we substitute in the discrete change for Δh (Eqn. 18) to get:

$$h_{new} - h_{old} = \frac{\Delta t}{\pi r^2}\, Q \tag{22}$$

Which we can solve for the new height:

$$h_{new} = h_{old} + \frac{\Delta t}{\pi r^2}\, Q \tag{23}$$

Which is the same as:

Height Update Equation

$$\boxed{h_{new} = h_{old} + \Delta h} \tag{24}$$

We can use the **Change in Height** (Eqn. 21) and **Height Update** (Eqn. 18) equations to create a computer model of the height of the water in the cylinder as it fills it up.

2 Code

The following example code that solves this water-filling problem uses the ezGraph class (https://github.com/lurbano/ezGraph) which requires matplotlib and numpy. However, the code in the following section avoids the use of most imported modules, but does not graph.

2.0.1 Code with Graphical Output

water-filling-fd.py

```python
import numpy as np
import time
from ezGraph import *

# Finite Difference Model

# PARAMETERS
dt = 1.
nsteps = 20

r = 2.25        # radius (cm)
Q = 5           # Volume inflow rate: (cubic cm / s)
h = 0           # Initial height (cm)

# GRAPH
graph = ezGraph(xmax=30, ymin=0, ymax=10, xLabel="Time (s)", yLabel="Height (cm)")
graph.add(0, h)                 # add initial values


# TIME LOOP
for t in range(1, nsteps):
    modelTime = t * dt

    dh = Q * dt / (np.pi * r**2)    # find the change in height
    h = h + dh                      # update height

    print(modelTime, h)
    graph.add(modelTime, h)
```

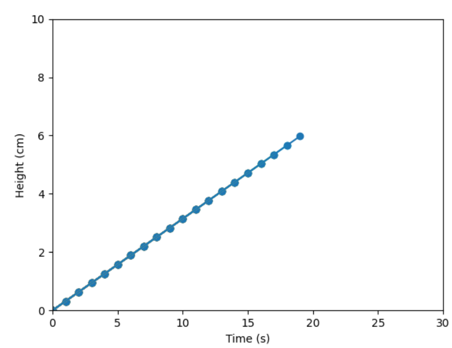


Figure 2: Model output: Graph of height of water in the column over time when filling the cylinder.

```
    graph.wait(0.1)

# DRAW GRAPH
graph.keepOpen()
```

2.1 Code without Graphical Output

A stripped down version of the code with no graph and no external modules except "math".

water-filling-fd-noGraph.py

```
import math
# Finite Difference Model

# PARAMETERS
dt = 1.
nsteps = 20

r = 2.25      # radius (cm)
Q = 5         # Volume inflow rate: (cubic cm / s)
h = 0         # Initial height (cm)

print(0, h)       # print initial values

# TIME LOOP
for t in range(1, nsteps):
    modelTime = t * dt

    dh = Q * dt / (math.pi * r**2)     # find the change in height
    h = h + dh                         # update height

    print(modelTime, h)
```

Which should produce a table of time and height output:

```
0 0
1.0 0.3143801345025093 5
2.0 0.6287602690050187
3.0 0.943140403507528
4.0 1.2575205380100374
5.0 1.5719006725125468
```

```
6.0 1.8862808070150563
7.0 2.2006609415175657
8.0 2.515041076020075
9.0 2.8294212105225847
10.0 3.143801345025094
11.0 3.4581814795276036
12.0 3.772561614030113
13.0 4.086941748532622
14.0 4.4013218830351315
15.0 4.715702017537641
16.0 5.03008215204015
17.0 5.34446228654266
18.0 5.658842421045169
19.0 5.973222555547679
```

3 Analytical Solutions using Calculus

The analytical solution to this problem will help confirm the accuracy of our model.+

3.1 Filling

As we saw in the section on using calculus (Section 1.2), we can start with the equation for the volume of a cylinder:

$$V = \pi r^2 h \tag{25}$$

And differentiate with respect to time to get:

$$\frac{dV}{dt} = \pi r^2 \frac{dh}{dt} \tag{26}$$

Assuming a constant inflow rate ($\frac{dV}{dt} = Q$):

$$Q = \pi r^2 \frac{dh}{dt} \tag{27}$$

And solve for $\frac{dh}{dt}$:

$$\frac{dh}{dt} = \frac{Q}{\pi r^2} \tag{28}$$

This we can separate:

$$dh = \frac{Q}{\pi r^2} dt \tag{29}$$

and integrate:

$$\int dh = \int \frac{Q}{\pi r^2} dt \tag{30}$$

to get:

$$h = \frac{Q}{\pi r^2} t + c \tag{31}$$

When $t = 0$, c can be shown to be the initial height ($c = h_i$) so:

$$h = \frac{Q}{\pi r^2} t + h_i \tag{32}$$

And since $\frac{Q}{\pi r^2}$ is constant, we can see that this term is the slope in a linear equation of the form.

$$y = mx + b \tag{33}$$

So the linear pattern produced by the filling model is correct (Figure 2).

4 Draining

4.1 Draining: Analytical Solution using Calculus

Experiments (which you may have done) show that if you're draining a cylinder by gravity the outflow rate of water is linearly proportional to the height of water in the tube.

$$\frac{dV}{dt} \propto h \tag{34}$$

Converting the proportionality statement to an equation requires us to introduce a constant (k):

$$\frac{dV}{dt} = kh \tag{35}$$

So in draining, the outflow rate ($\frac{dV}{dt}$) is not constant, it slows down as the height of water in the tube decreases.

Now, lets substitute the equation for the volume of a cylinder:

$$V = \pi r^2 h \tag{36}$$

into the draining equation (Eq. 51) to get:

$$\frac{d[\pi r^2 h]}{dt} = kh \tag{37}$$

we can extract π and r^2 from the differential because they are constant:

$$\pi r^2 \frac{dh}{dt} = kh \tag{38}$$

separating the variables gives:

$$\pi r^2 \frac{dh}{h} = k \cdot dt \tag{39}$$

and rearranging:

$$\frac{dh}{h} = \frac{k \cdot dt}{\pi r^2} \tag{40}$$

$$\frac{dh}{h} = \frac{k}{\pi r^2} dt \tag{41}$$

To simplify a little, lets consolidate the constants on the left hand side into one therm K:

$$K = \frac{k}{\pi r^2} \tag{42}$$

so:

$$\frac{dh}{h} = K \cdot dt \tag{43}$$

which we can integrate (remember K is a constant):

$$\int \frac{dh}{h} = K \int dt \tag{44}$$

$$\ln h = K \cdot t + c \tag{45}$$

we can solve for h by raising both sides by e to cancel the ln:

$$e^{\ln h} = e^{Kt+c} \tag{46}$$

$$h = e^{Kt+c} \tag{47}$$

Because of math, we can pull the constant out to get:

$$h = Ce^{Kt} \tag{48}$$

Where the constant is the initial value of the height (h_0):

$$h = h_0 \cdot e^{Kt} \tag{49}$$

This is an exponential function. If K is less than 1 ($K < 1$) then this is a decay curve.

4.2 Numerical Solution

For draining, the outflow rate (change in volume over time, $\frac{dV}{dt}$) is not constant. The outflow rate is proportional to the height of water in the tube, since the higher the water level the greater pressure at the bottom of the tube and the faster the outflow rate.

$$\frac{dV}{dt} \propto h \tag{50}$$

Converting the proportionality statement to an equation requires us to introduce a constant (k):

$$\frac{dV}{dt} = k \cdot h \tag{51}$$

So, let's substitute this relationship into the height change equation (Eq. 11):

$$\Delta h = \frac{\frac{dV}{dt} \cdot \Delta t}{\pi r^2}$$

to get:

$$\boxed{\Delta h = \frac{k \cdot h \cdot \Delta t}{\pi r^2}} \tag{52}$$

so, in our code we just need to change this line and set up a few different constants.

4.3 Code: Draining

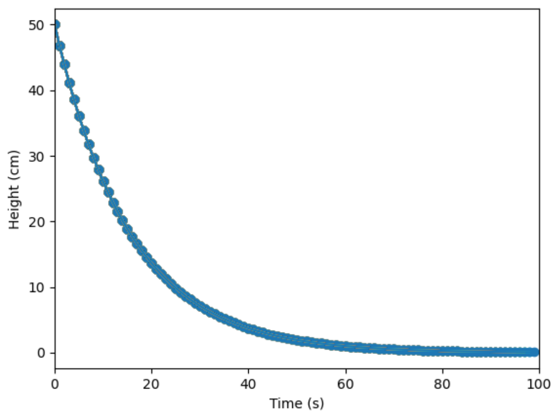
This program is based off the filling code. We're using an initial height of 50 cm ($h_i = 50$), and set the constant $K = 1$.

water-draining-fd.py

```
import numpy as np
import time
from ezGraph import *

# Finite Difference Model

# PARAMETERS
dt = 1
nsteps = 100

r = 2.25     # radius (cm)
Q = 0        # Volume inflow rate (dV/dt): (cubic cm / s)
h = 50        # Initial height (cm)
k = 1.0      # outflow rate constant

# GRAPH
graph = ezGraph(xmax=100,
xLabel="Time (s)", yLabel="Height (cm)")
graph.add(0, h)               # add initial values


# TIME LOOP
for t in range(1, nsteps):
modelTime = t * dt

# Filling
dh = Q * dt / (np.pi * r**2)    # find the change in height
h = h + dh                      # update height

# Draining
dVdt = -k * h
dh = dVdt * dt / (np.pi * r**2)
h = h + dh

print(modelTime, h)
graph.add(modelTime, h)
graph.wait(0.1)

# DRAW GRAPH
graph.keepOpen()
```

The output from the model (Fig. 3) looks like an exponential

Figure 3: Model output: Graph of height of water in the column over time when draining the cylinder.



decay curve, which is what we find from the analytical solution (Eqn. 49).