Numerical Optimization

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Homework 6

Exercise 6.1 [1 point] Consider the constrained optimization problem

$$\begin{aligned} x^2 + y^2 &\to \min \\ x + y &\leq 1 \end{aligned}$$

Write down its Lagrage function. Formulate the first order necessary condition for the local minimum.

Solution:

The Lagrange function $L(x, \lambda) = x^T A x + \lambda \left(b^T x - c\right)$
where

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, c = 1$$

So $L(x_1, x_2, \lambda) = x_1^2 + x_2^2 + \lambda(x_1 + x_2 - 1)$.
The first order necessary condition is

$$\nabla_x L(x_1, x_2, \lambda) = \begin{pmatrix} \nabla_{x_1} L(x_1, x_2, \lambda) \\ \nabla_{x_2} L(x_1, x_2, \lambda) \end{pmatrix} = \begin{pmatrix} 2x_1 + \lambda \\ 2x_2 + \lambda \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Exercise 6.2 [6 points] Modify your code of the Newton's method from Ex.4.3 (by implementing the corresponding ∇f and B_k) to get the Gauss-Newton method for the non-linear least squares problem

$$f(x, y) := \frac{1}{2}\left[(r_1 - 10)^2 + (r_2 - 6)^2 + (r_3 - 11)^2\right] \to \min$$

where

$$\begin{aligned} r_1(x, y) &:= 3x + 4y + \tfrac{1}{8}xy \\ r_2(x, y) &:= 2x + 2y + \tfrac{1}{10}xy \\ r_3(x, y) &:= 4x + 3y + \tfrac{1}{6}x^2 \end{aligned}$$

Test this program for $\mathbf{x}_0 = (x_0, y_0) = (0, 0)$ as the initial guess. Take the same parameters for the line search and the termination criteria ex in Ex.4.3. In every iteration, as well as for the initial guess, print (x_k, y_k), $f(x_k, y_k)$ and $\|\nabla f(x_k, y_k)\,|$.

Solution:

Since

$$\nabla f(\mathbf{x}) = \sum_{i=1}^{n} r_i(\mathbf{x}) \nabla r_i(\mathbf{x}) = \mathbf{J}(\mathbf{x})^T \mathbf{r}(\mathbf{x})$$

$$\nabla^2 f(\mathbf{x}) \approx \mathbf{J}(\mathbf{x})^T \mathbf{J}(\mathbf{x})$$

$$\mathbf{J}(\mathbf{x}) = \frac{1}{\sqrt{2}} \begin{bmatrix} 3 + \frac{1}{8}y & 4 + \frac{1}{8}x \\ 2 + \frac{1}{10}y & 2 + \frac{1}{10}x \\ 4 + \frac{1}{3}x & 3 \end{bmatrix}$$

$$\mathbf{r}(\mathbf{x}) = \frac{1}{\sqrt{2}} \begin{bmatrix} 3x + 4y + \frac{1}{8}xy - 10 \\ 2x + 2y + \frac{1}{10}xy - 6 \\ 4x + 3y + \frac{1}{6}x^2 - 11 \end{bmatrix}$$

So

$$\nabla f(\mathbf{x}) = \frac{1}{2} \begin{bmatrix} 3 + \frac{1}{8}y & 2 + \frac{1}{10}y & 4 + \frac{1}{3}x \\ 4 + \frac{1}{8}x & 2 + \frac{1}{10}x & 3 \end{bmatrix} \begin{bmatrix} 3x + 4y + \frac{1}{8}xy - 10 \\ 2x + 2y + \frac{1}{10}xy - 6 \\ 4x + 3y + \frac{1}{6}x^2 - 11 \end{bmatrix}$$

$$\nabla^2 f(\mathbf{x}) \approx \frac{1}{2}\left[\begin{array}{lll} 3+\frac{1}{8}y & 2+\frac{1}{10}y & 4+\frac{1}{3}x \\ 4+\frac{1}{8}x & 2+\frac{1}{10}x & 3 \end{array}\right]\left[\begin{array}{ll} 3+\frac{1}{8}y & 4+\frac{1}{8}x \\ 2+\frac{1}{10}y & 2+\frac{1}{10}x \\ 4+\frac{1}{3}x & 3 \end{array}\right]$$

The output of the program is shown in Figure 1.

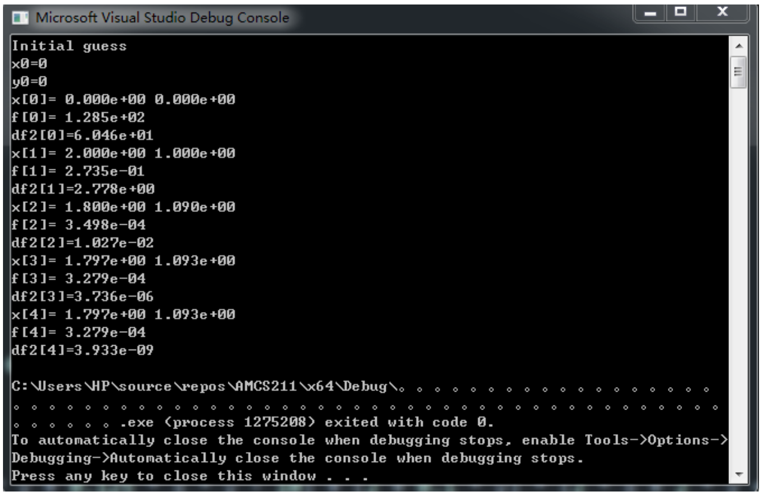

Figure 1: Gauss-Newton method

Exercise 6.3 Consider the linear programming optimization problem.

$$\begin{array}{ll} 4x+2y & \rightarrow \max \\ x+y & \leq 5, \\ 2x+\frac{1}{2}y & \leq 8, \\ x,y & \geq 0 \end{array}$$

(a) [2 points] Plot the feasible set of this problem graphically on the xy-plain. Basing on this drawing, find the solution.

Solution:

The feasible set has been drawn in Figure 2. The vertices of the polygon are $(0,0)$, $(4,0)$, $(0,5)$, $(\frac{11}{3},\frac{4}{3})$. The corresponding values are $f(0,0)=0$, $f(4,0)=16$, $f(0,5)=10$, $f(\frac{11}{3},\frac{4}{3})=\frac{52}{3}$. So the maximum is $f(\frac{11}{3},\frac{4}{3})=\frac{52}{3}$.

(b) [1 point] Transform this problem to the standard form $\mathbf{c}^T\mathbf{x} \rightarrow \min$ s.t. $\mathbf{A}\mathbf{x}=\mathbf{b}, \mathbf{x} \geq 0$ (by introducing slack variables). For this form, write down the dual problem.

Solution:

The standard form of this problem:

$$\mathbf{c}^T\mathbf{x} = [-4\ -2]\left[\begin{array}{c} x \\ y \end{array}\right] \rightarrow \min$$

$$\mathbf{A}\mathbf{x} = \left[\begin{array}{cc} 1 & 1 \\ 2 & \frac{1}{2} \end{array}\right]\left[\begin{array}{c} x \\ y \end{array}\right] = \mathbf{b} = \left[\begin{array}{c} 5 \\ 8 \end{array}\right], x \geq 0, y \geq 0$$

Introducing slack variables z_1, z_2, the dual problem is

$$\min\{z_1+z_2 \mid \mathbf{A}\mathbf{x}+\mathbf{z}, x, z_1, z_2 \geq 0\}$$

Exercise 6.4 [6 points] Solve the linear programming problem

$$\begin{array}{c} \min_{x_1,x_2,x_3,x_4} -4x_1-2x_2 \\ s.t. \quad x_1+x_2+x_3=5 \\ 2x_1+\frac{1}{2}x_2+x_4=8 \\ x_1,x_2,x_3,x_4 \geq 0 \end{array}$$

Figure 2: The feasible set

by the simplex method.

Solution:

The simplex tables are shown in Figure 3. So the minimum value is $-\frac{52}{3}$ when $x_1 = \frac{11}{3}$ and $x_2 = \frac{4}{3}$.

Tableau1			4	2	0	0
Base	Cb	P0	P1	P2	P3	P4
P3	0	5	1	1	1	0
P4	0	8	2	1/2	0	1
Z		0	-4	-2	0	0
Tableau2			4	2	0	0
Base	Cb	P0	P1	P2	P3	P4
P3	0	1	0	3/4	1	-1/2
P1	4	4	1	1/4	0	1/2
Z		16	0	-1	0	2
Tableau3			4	2	0	0
Base	Cb	P0	P1	P2	P3	P4
P2	2	4/3	0	1	4/3	-2/3
P1	4	11/3	1	0	-1/3	2/3
Z		52/3	0	0	4/3	4/3

Figure 3: The simplex tables