Numerical Optimization

Shuai Lu 170742

Homework 7

Exercise 7.1 (a) [3 points] Consider the constrained optimization problem

$$\begin{gathered} x^2 + y^3 \to \min \\ y \geq 1 \end{gathered}$$

Write down its Lagrage function. Formulate the first order necessary condition for the local minimum. Find the minimizer. Find the critical cone at the minimizer. Check the second order sufficient condition at the minimizer. Is the constraint active or inactive at the minimizer?

Solution:

The Lagrange function $L(x, y, \lambda) = x^2 + y^3 - \lambda(y-1)$. The first order necessary condition are:

$$\nabla_x L(x^*, y^*, \lambda^*) = 2x^* = 0$$

$$\nabla_y L(x^*, y^*, \lambda^*) = 3{y^*}^2 - \lambda^* = 0$$

$$y^* - 1 \geq 0$$

$$\lambda^* > 0$$

$$\lambda^*(y^* - 1) = 0$$

So $x^* = 0, y^* = \sqrt{\frac{\lambda^*}{3}} \geq 1$. It is obvious that $x^* = 0, y^* = 1$ is the minimizer and $\lambda^* = 3$.
Since $c = y^* - 1 = 0$, the constraint is active.
$\nabla c = (0,1)^T$ and $\lambda^* > 0$, so the critical cone is $\mathcal{C}(x^*, \lambda^*) = \left\{(w_1, w_2)^T \mid (0,1)(w_2, w_2)^T = 0\right\} = \left\{(w_1, 0)^T\right\}$.
Since $\nabla^2_{xx}\ \mathrm{L}(x^*, \lambda^*) = \begin{bmatrix} 2 & 0 \\ 0 & 6 \end{bmatrix}$, the second necessary condition $w^T \nabla^2_{xx}\ \mathrm{L}(x^*, \lambda^*)\, w = \begin{bmatrix} w_1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 6 \end{bmatrix} \begin{bmatrix} w_1 \\ 0 \end{bmatrix} = 2w_1^2 \geq 0$ is satisfied.

(b) [1 point] The minimizer of the problem in Ex. 6.1 is evidently $(x, y) = (0, 0)$. Is the constrait active or inactive at it? What does this imply for the Lagrange multiplier?

Solution:

Since $c(x^*) = 1 - x - y = 1 > 0$, the constraint is inactive. According to the complementarity condition $\lambda^* c(x^*) = 0$, we have $\lambda^* = 0$.

Exercise 7.2 [4 points] Consider the quadratic programming problem

$$\begin{aligned} & x_1^2 - x_1 x_2 + x_2^2 + x_1 + x_2 \to \min \\ & s.t. \quad x_1 + x_2 = 1 \end{aligned}$$

Write its KKT system. Solve it by the Schur-complement method.

Solution:

The standard form of this problem is:

$$\frac{1}{2}\mathbf{x}^T \mathbf{G} \mathbf{x} + \mathbf{x}^T \mathbf{c} \to min$$

$$\mathbf{A}\mathbf{x} = \mathbf{b}$$

where

$$\mathbf{G}=\left[\begin{array}{cc} 2 & -1 \\ -1 & 2 \end{array}\right], \mathbf{x}=\left[\begin{array}{c} x_1 \\ x_2 \end{array}\right], \mathbf{c}=\left[\begin{array}{c} 1 \\ 1 \end{array}\right], \mathbf{A}=\left[\begin{array}{cc} 1 & 1 \end{array}\right], \mathbf{b}=1$$

The KKT matrix is:

$$\left(\begin{array}{cc} \mathbf{G} & \mathbf{A}^{\mathrm{T}} \\ \mathbf{A} & 0 \end{array}\right)\left(\begin{array}{c} -\mathbf{p} \\ \lambda^* \end{array}\right)=\left(\begin{array}{c} \mathbf{g} \\ \mathbf{h} \end{array}\right)$$

where

$$\mathbf{h}=\mathbf{A}\mathbf{x}-\mathbf{b}, \mathbf{g}=\mathbf{c}+\mathbf{G}\mathbf{x}, \mathbf{p}=\mathbf{x}^*-\mathbf{x}$$

Since $\mathbf{A}$ has full row rank and the kernal basis matrix $\mathbf{Z}$ can be $\left[\begin{array}{c} 1 \\ -1 \end{array}\right]$, the reduced-Hessian matrix $\mathbf{Z^T G Z}=6$ is positive definite. So there is a unique solution $\mathbf{x}^*$. Using the Schur-complement method, the decomposition of the KKT matrix can be writen as:

$$\left(\begin{array}{cc} \mathbf{G} & \mathbf{A}^{\mathrm{T}} \\ \mathbf{A} & 0 \end{array}\right)\left(\begin{array}{c} -\mathbf{p} \\ \lambda^* \end{array}\right)=\left(\begin{array}{cc} \mathbf{I} & 0 \\ \mathbf{A}\mathbf{G}^{-1} & \mathbf{I} \end{array}\right)\left(\begin{array}{cc} \mathbf{G} & \mathbf{A}^{\mathrm{T}} \\ 0 & -\mathbf{A}\mathbf{G}^{-1}\mathbf{A}^{\mathrm{T}} \end{array}\right)\left(\begin{array}{c} -\mathbf{p} \\ \lambda^* \end{array}\right)=\left(\begin{array}{c} \mathbf{g} \\ \mathbf{h} \end{array}\right)$$

Let the estimate $\mathbf{x}=\left[\begin{array}{c} 0 \\ 0 \end{array}\right]$, then the equation becomes:

$$\left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{array}\right]\left[\begin{array}{ccc} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 0 & 0 & -2 \end{array}\right]\left(\begin{array}{c} -\mathbf{p} \\ \lambda^* \end{array}\right)=\left[\begin{array}{c} 1 \\ 1 \\ -1 \end{array}\right]$$

Then

$$\left[\begin{array}{ccc} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 0 & 0 & -2 \end{array}\right]\left(\begin{array}{c} -\mathbf{p} \\ \lambda^* \end{array}\right)=\left[\begin{array}{c} 1 \\ 1 \\ -3 \end{array}\right]$$

Finally, we get

$$\left(\begin{array}{c} -\mathbf{p} \\ \lambda^* \end{array}\right)=\left[\begin{array}{c} -\frac{1}{2} \\ -\frac{1}{2} \\ \frac{3}{2} \end{array}\right]$$

So

$$\mathbf{x}^*=\mathbf{p}+\mathbf{x}=\left[\begin{array}{c} \frac{1}{2} \\ \frac{1}{2} \end{array}\right]$$