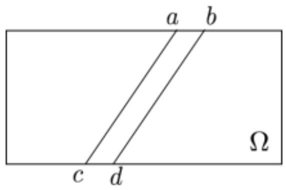
Fast Solvers for Large Systems of Equations

Shuai Lu 170742

Homework 4

Exercise 4.1 Consider the unstructured, triangular mesh for the domain Ω shown in the figure below. The corners of the rectangle are at points $(-1,-0.5),(-1,0.5),(1,0.5),(1,-0.5)$. Furthermore, $a=(0.3,0.5), b=(0.45,0.5), c=(-0.3,-0.5), d=(-0.15,-0.5)$. (Cf. stripe.ugx.) The stripe $abdc$ and the 2 d rest of the domain belong to subsets "Stripe" and "Bulk", respectively. Besides that, each boundary segment of Ω is placed in a separate subsets ("LeftBnd", "RightBnd" for the left and the right ones, "Wall" for the upper and the bottom ones).



(a) (1 point) Consider the diffusion equation

$$\nabla \cdot (d \nabla u) = 0, \quad x \in \Omega$$

Impose Dirichlet boundary conditions

$$u|_{x=-1} = 1, \quad u|_{x=1} = 0$$

on the left and right boundaries, respectively, and Neumann- 0 boundary conditions

$$u_y|_{y=-0.5} = 0, \quad u_y|_{y=0.5} = 0$$

on the other parts of the boundary. Explain, how these conditions can be interpreted physically. What impact do they have if applied to a biological diffusion problem like the drug penetration into the skin?

Solution:
Dirichlet boundary conditions mean the diffusion density on the left equals to 1 and that on the right equals to 0. Neumann boundary conditions mean the gradient of diffusion density along y direction on the upper and lower boundary equal to 0. The greater the drug density the larger the value of Dirichlet boundary condition on the left boundary.

(c) (2 points) Perform simulations of the diffusion described by this problem using the script $diffusion\_with\_stripe.lua$. Set the diffusion coefficient $d=1$ in $\Omega \backslash abdc$. Try to different values $d=0.0001$ and $d=0.1$ in the stripe $abdc$ (used the option -stripeD). Visualize your results with ParaView. Compare your two solutions and explain your observations physically by adressing both, the different diffusion coefficients and the boundary conditions.

Solution:
The results are shown in Fig.1 and 2. A low diffusion coefficient stripe will block the drug from left to right. A high diffusion coefficient stripe will allow more drug transform from left to right.

Exercise 4.2* (a) (2 points) Perform the simulation from Exercise. 4.1(c) with $d=0.0001$ in the stripe. Compute the solution on 3 different refinement levels $(3,4$ and $5)$ of the mesh (use the option -numRefs). As linear solver for the discrete system apply the linear iteration preconditioned by the geometric multigrid method (gmg) with the ILU smoother. For every refinement level, report the number of the degrees of freedom, the average and the last convergence rate of the iterative method, as well as number of iterations needed.

Solution:
The results are shown in Fig.3.

Figure 1: Stripe diffusion = 0.0001

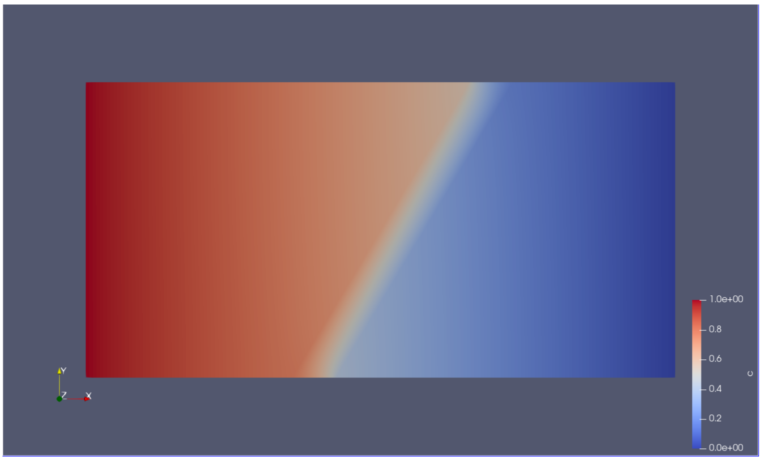


Figure 2: Stripe diffusion = 0.1

	degrees of freedom	average convergence rate	last convergence rate	number of iterations
refinement level=3	1697	2.50E-02	5.89E-02	4
refinement level=4	6593	2.11E-02	3.52E-02	4
refinement level=5	25985	2.23E-02	3.88E-02	4

Figure 3: The solution on refinement level = 3, 4, 5 (gmg)

(b) (4 points) Perform the same numerical experiments but use the linear iteration preconditioned by the ILU method. For every refinement level i, report the number of the degrees of freedom, the average and the last convergence rates ($\rho_{h,ave,i}$ and $\rho_{h,i}$) of the iterative method, as well as number of iterations needed. Compare your results with the ones for the gmg. Explain your observations with the properties you learned about the gmg. Estimate the order of the asymptotic convergence rate w.r.t. the mesh size h by comparing $\eta_{h,i} = 1 - \rho_{h,i}$ for different refinement levels. What does this mean for the amount of the computational work (cf. Exercise 1.4).

Solution:
The results are shown in Fig.4. The convergence rates become worse with the increase of refinement levels when using ILU method. The convergence rates is independent of refinement levels and keep on a small value when using gmg method. The comparison of $\eta_{h,i}$ is shown in Fig.5. The order of asymptotic convergence rate w.r.t. the mesh size h are 0 and 2 for gmg method and ILU method respectively. The amount of the computational work of gmg method and ILU method are $O\left(h^{-2}\right)$ and $O\left(h^{-4}\right)$ respectively.

	degrees of freedom	average convergence rate	last convergence rate	number of iterations
refinement level=3	1697	0.9945184	0.9961163	2514
refinement level=4	6593	0.9984803	0.9990336	9084
refinement level=5	25985	0.9995716	0.9997591	32240

Figure 4: The solution on refinement level = 3, 4, 5 (ILU)

	degrees of freedom	average convergence rate	last convergence rate	number of iterations	1-rho	2^alpha
refinement level=3	1697	0.02497848	0.05891133	4	0.94108867	
refinement level=4	6593	0.02107885	0.0352273	4	0.9647727	0.975451181
refinement level=5	25985	0.02230583	0.03883919	4	0.96116081	1.003757842
						So alpha=0
	degrees of freedom	average convergence rate	last convergence rate	number of iterations		
refinement level=3	1697	0.9945184	0.9961163	2514	0.0038837	
refinement level=4	6593	0.9984803	0.9990336	9084	0.0009664	4.018729305
refinement level=5	25985	0.9995716	0.9997591	32240	0.0002409	4.01162308
						So alpha=2

Figure 5: The comparison of convergence rates(Linear iteration)

Exercise 4.3* (3 points) Perform the same experiments as in Ex. 4.2(a, b) but with the cg method instead of the linear iteration. Report the average convergence rate $\rho_{h,ave,i}$ of the iterative method, as well as number of iterations needed. For the case of the plain ILU preconditioner, compute $\eta_{h,ave,i} = 1 - \rho_{h,ave,i}$ for the refinement levels and estimate the order of the averaged convergence rate w.r.t. the mesh size h.

Solution:
The results are shown in Fig.6. For the case of the plain ILU preconditioner, the order of the averaged convergence rate w.r.t. the mesh size h is 1.

CG iteration					
gmg	degrees of freedom	average convergence rate	number of iterations	1-rho	2^alpha
refinement level=3	1697	1.979081e-02	4	0.98020919	
refinement level=4	6593	1.376070e-02	4	0.9862393	0.993885754
refinement level=5	25985	1.266355e-02	4	0.98733645	0.998888778
					So alpha=0
ILU	degrees of freedom	average convergence rate	number of iterations		
refinement level=3	1697	8.185143e-01	69	0.1814857	
refinement level=4	6593	9.026617e-01	136	0.0973383	1.86448397
refinement level=5	25985	9.492833e-01	266	0.0507167	1.919255393
					So alpha=1

Figure 6: The comparison of convergence rates(CG iteration)

Exercise 4.4* (4 points) Consider the boundary value problem given on the unit square by

$$\Delta u(x, y) = f(x, y) = 400[\sin(20x) + \sin(20y)], \quad (x, y) \in \Omega = (0, 1) \times (0, 1)$$
$$u(x, y)|_{\delta\Omega} = \sin(20x) + \sin(20y)$$

For the finite volume method, it can be proven that in the Euclidean norm, the numerical solution u_h converges to the analytical solution u with the convergence order $q = 2$, i.e.

$$\|u - u_h\|_2 = O\left(h^2\right)$$

Use the script *poisson_error.lua* to measure the order q numerically: Compute the solution u_{h_i} for 6 successive mesh refinement levels, $i = 4, \ldots, 9$, with mesh sizes $h_i = 1/2^{i+1}$. (Use option -numRefs.) The numerical order is given by

$$q_{\text{num},i} := \log \frac{\|u - u_{h_i}\|_2}{\left\|u - u_{h_{i+1}}\right\|_2} / \log \frac{h_i}{h_{i+1}}$$

Report the error $\|u - u_{h_i}\|_2$ for each level i of refinement as well as the measured numerical order of the convergence.

Solution:
The results are shown in Fig.7.

	L2 error	average convergence rate	q
refinement level=4	0.024245789008882	1.525714e-02	2.03060129200871
refinement level=5	0.005934230664526	2.311703e-02	2.00776412220429
refinement level=6	0.00147559507964	2.548942e-02	2.00194949988233
refinement level=7	0.000368400617213	0.02405918	2.00048626541541
refinement level=8	9.20691168548E-05	0.02075345	2.00011597862362
refinement level=9	2.30154289231E-05	0.01991294	

Figure 7: The L2 error and the numerical order of the convergence