HW2

Thursday, September 14, 2023 4:54 PM

1. (a) Show that $(1+x)^n = 1 + nx + o(x)$ as $x \to 0$.
   (b) Show that $x \sin\sqrt{x} = O(x^{3/2})$ as $x \to 0$.
   (c) Show that $e^{-t} = o(\frac{1}{t^2})$ as $t \to \infty$.
   (d) Show that $\int_0^{\varepsilon} e^{-x^2} dx = O(\varepsilon)$ as $\varepsilon \to 0$.

a) $(1+x)^n = 1 + nx + o(x)$ as $x \to 0$

$$(1+x)^n = \frac{1}{0!}x^0 + \frac{n}{1!}x^1 + \underbrace{\frac{n(n-1)}{2!}x^2 + \cdots + nx^{n-1} + x^n}_{g(x)}$$

Need to show f(x) grows slower than g(x) near $x=0$

f(x) dominated by nx term, so let $h(x) = nx$

$$\lim_{x\to 0} \left| \frac{g(x)}{h(x)} \right|$$

$$\lim_{x\to 0} \left| \frac{\frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots + nx^{n-1} + x^n}{nx} \right|$$

$$\lim_{x\to 0} \left| \frac{\frac{n(n-1)}{2!}x + \frac{n(n-1)(n-2)}{3!}x^2 + \cdots + nx^{n-2} + x^{n-1}}{n} \right| = 0 \checkmark$$

Therefore, $(1+x)^n = 1 + nx + o(x)$ as $x \to 0$

b) $x\sin(\sqrt{x}) = O(x^{3/2})$ as $x \to 0$

$$\sin(x) = \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\sin(\sqrt{x}) = \sum_{n=1}^{\infty} \frac{(-1)^n (x^{1/2})^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+\frac{1}{2}}}{(2n+1)!} \quad \text{for } x \geq 0$$

$$= \sqrt{x} \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{(2n+1)!}$$

$$x\sin(\sqrt{x}) = x^{3/2} \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{(2n+1)!} \quad x \geq 0$$

Let $f(x) = x\sin(\sqrt{x}) = x^{3/2} \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{(2n+1)!}$ and $g(x) = x^{3/2}$

Then $f(x) = O(g(x))$, i.e. $x\sin(\sqrt{x}) = O(x^{3/2})$ if:

$$\lim_{x \to 0^+} \left| \frac{f(x)}{g(x)} \right| \leq k, \quad k > 0, \quad x \geq 0$$

$$\lim_{x \to 0^+} \left| \frac{\cancel{x^{3/2}} \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{(2n+1)!}}{\cancel{x^{3/2}}} \right|$$

Note that $x^n = (x^{1/2})^{2n} = x^{-1/2}(x^{1/2})^{2n+1} = \frac{1}{\sqrt{x}} (\sqrt{x})^{2n+1}$

$$\lim_{x \to 0^+} \left| \sum_{n=0}^{\infty} \frac{(-1)^n (\sqrt{x})^{2n+1}}{(2n+1)!} \cdot \frac{1}{\sqrt{x}} \right|$$

$$\lim_{x\to 0^+} \frac{\sin(\sqrt{x})}{\sqrt{x}}$$

let $u = \sqrt{x}$

$$\lim_{x\to 0^+} \frac{\sin(u)}{u} = 1 \checkmark$$

Therefore, $x\sin(\sqrt{x}) = O(x^{3/2})$

c) $e^{-t} = o\left(\frac{1}{t^2}\right)$ as $t \to \infty$

Let $f(t) = e^{-t}$, $g(t) = \frac{1}{t^2}$

Need: $\lim_{t\to\infty} \left|\frac{f(t)}{g(t)}\right| = 0$

$$\lim_{t\to\infty} \left|\frac{e^{-t}}{1/t^2}\right| = \frac{0}{0}$$

$$\lim_{t\to\infty} \left|\frac{t^2}{e^t}\right| = \frac{\infty}{\infty}$$

L'H

$$\lim_{t\to\infty} \left|\frac{2t}{e^t}\right| = \frac{\infty}{\infty}$$

L'H

$$\lim_{t\to\infty}\left|\frac{2}{e^t}\right| = 0 \checkmark$$

Therefore, $e^{-t} = o(1/t^2)$

d) $\displaystyle\int_0^{\varepsilon} e^{-x^2}\,dx = O(\varepsilon)$ as $\varepsilon \to 0$

Let $f(\varepsilon) = \displaystyle\int_0^{\varepsilon} e^{-x^2}\,dx$, $g(\varepsilon) = \varepsilon$

Need:
$$\lim_{\varepsilon\to 0}\left|\frac{\int_0^{\varepsilon} e^{-x^2}\,dx}{\varepsilon}\right| = 0$$

$$e^{-x^2} = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{n!}$$

$$= \frac{1}{0!} - \frac{x^2}{1!} + \frac{x^4}{2!} - \cdots$$

$$\int_0^{\varepsilon} e^{-x^2}\,dx = \int_0^{\varepsilon}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{n!}\,dx$$

$$= \left[\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)\,n!}\right]_0^{\varepsilon}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n \varepsilon^{2n+1}}{(2n+1)\,n!} = \frac{\varepsilon}{(1)0!} - \frac{\varepsilon^3}{(3)1!} + \frac{\varepsilon^5}{(5)2!} - \cdots$$

$$\lim_{\varepsilon \to 0} \left| \frac{f(\varepsilon)}{g(\varepsilon)} \right|$$

$$\lim_{\varepsilon \to 0} \left| \frac{\varepsilon - \frac{\varepsilon^3}{3} + \frac{\varepsilon^5}{10} - \cdots}{\varepsilon} \right|$$

$$\lim_{\varepsilon \to 0} \left| 1 - \frac{\varepsilon^2}{3} + \frac{\varepsilon^4}{10} - \cdots \right| = 1 \checkmark$$

Therefore, $\int_0^{\varepsilon} e^{-x^2}\,dx = O(\varepsilon)$

2. Consider solving $\mathbf{A}\mathbf{x} = \mathbf{b}$ where $\mathbf{A} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1+10^{-10} & 1-10^{-10} \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. The exact solution is $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and the inverse of $\mathbf{A}$ is $\begin{bmatrix} 1-10^{10} & 10^{10} \\ 1+10^{10} & -10^{10} \end{bmatrix}$. In this problem we will investigate a perturbation in $\mathbf{b}$ of $\begin{bmatrix} \Delta b_1 \\ \Delta b_2 \end{bmatrix}$ and the numerical effects of the condition number.

(a) Find an exact formula for the change in the solution between the exact problem and the perturbed problem Δx.

(b) What is the condition number of $\mathbf{A}$?

(c) Let Δb_1 and Δb_2 be of magnitude 10^{-5}; not necessarily the same value. What is the relative error in the solution? What is the relationship between the relative error, the condition number, and the perturbation. Is the behavior different if the perturbations are the same? Which is more realistic: same value of perturbation or different value of perturbation?

a) $A\vec{x} = \vec{b} \Rightarrow \vec{x} = A^{-1}\vec{b}$

$A\hat{x} = \vec{b} + \Delta\vec{b}$

$$\hat{x} = A^{-1}(\vec{b} + \Delta\vec{b})$$

$$\hat{x} = A^{-1}\vec{b} + A^{-1}\Delta\vec{b}$$

$$\hat{x} = \vec{x} + \underbrace{A^{-1}\Delta\vec{b}}_{\Delta\vec{x}}$$

$$\Delta\vec{x} = A^{-1}\Delta\vec{b}$$

b) Relative condition number: $\kappa(A) = \|A\|_2 \|A^{-1}\|_2 = \frac{\sigma_1}{\sigma_n}$

See hw2.py in repo

$$\boxed{\kappa(A) = 19999973849.225224}$$

c) Relative error: $< \kappa(A) \cdot \|\Delta\vec{x}\|$

$\Delta b_1, \Delta b_2 \propto 10^{-5}$ let $\Delta b_1, \Delta b_2 = 5 \cdot 10^{-5}$

$$\Delta\vec{x} = A^{-1}\Delta\vec{b}$$

From hw2.py:

$\|\Delta\vec{x}\|_2 =$ 7.071068147724373e-05

relative error $< \kappa(A)\,\|\Delta\vec{x}\|$

From hw2.py: $<$ 1414211.780405769

Not sure what relationship is between relative error condition

Not sure what relationship is between relative error, condition number, and perturbation :/

3)

3. Let $f(x) = e^x - 1$

(a) What is the relative condition number $\kappa(f(x))$? Are there any values of x for which this is ill-conditioned?

(b) Consider computing $f(x)$ via the following algorithm:

```
1: y = math.e^x
2: return y -1
```

Is this algorithm stable? Justify your answer

(c) Let x have the value $9.999999995000000 \times 10^{-10}$, in which case $f(x)$ is equal to 10^{-9} up to 16 decimal places. How many correct digits does the algorithm listed above give you? Is this expected?

(d) Find a polynomial approximation of $f(x)$ that is accurate to 16 digits for $x = 9.999999995000000 \times 10^{-10}$. Hint: use Taylor series, and remember that 16 digits of accuracy is a relative error, not an absolute one.

(e) Verify that your answer from part (d) is correct.

(f) [Optional] How many digits of precision do you have if you do a simpler Taylor series?

(g) [Fact; no work required] Matlab provides `expm1` and Python provides `numpy.expm1` which are special-purpose algorithms to compute $e^x - 1$ for $x \approx 0$. You could compare your Taylor series approximation with `expm1`.

a) $f(x) = e^x - 1$

$$K(f(x)) = \frac{|f'(c)|\,|x|}{|f(x)|} \qquad f'(x) = \frac{\delta f}{\delta x}$$

$$f'(x) = e^x \qquad f'(c) = e^c, \qquad x < c < x + \Delta x$$

$$K(f(x)) = \frac{|e^c|\,|x|}{|e^x - 1|} = \boxed{e^c \frac{|x|}{|e^x - 1|}}$$

b) $\lim_{x \to \infty} K(f(x)) \qquad \lim_{x \to 0} K(f(x))$

$$\lim_{x \to \infty} e^c \frac{|x|}{|e^x - 1|} \qquad \lim_{x \to 0} e^c \frac{|x|}{|e^x - 1|} \qquad |x| = \begin{cases} x : x > 0 \\ -x : x < 0 \end{cases}$$

$$\lim_{x\to\infty} e^c \frac{|x|}{|e^x-1|} \qquad \lim_{x\to 0} e^c \frac{|x|}{|e^x-1|} \qquad |x| = \begin{cases} x : x>0 \\ -x : x<0 \\ 0 : x=0 \end{cases}$$

$$e^c \lim_{x\to\infty} \frac{x}{e^x-1} \qquad e^c \lim_{x\to 0} \frac{|x|}{|e^x-1|} \qquad |e^x-1| = \begin{cases} e^x-1 : x>0 \\ 1-e^x : x<0 \\ 0 : x=0 \end{cases}$$

L'H

$$e^c \lim_{x\to\infty} \frac{1}{e^x} = e^c$$

$$e^c \lim_{x\to 0^+} \frac{x}{e^x-1} \qquad e^c \lim_{x\to 0^-} \frac{-x}{1-e^x}$$

L'H   L'H

$$e^c \lim_{x\to 0^+} \frac{1}{e^x} = e^c \qquad e^c \lim_{x\to 0^-} \frac{-1}{-e^x} = e^c$$

$$\lim_{x\to 0} e^c \frac{|x|}{|e^x-1|} = e^c$$

The algorithm $f(x)$ is stable around $x=0$, the limit as $K(f(x))$ approaches 0 is e^c. For c near $x=0$, e^c is close to 1, and this low condition number indicates high stability.

However, $f(x)$ is not stable for larger values of x, as the limit as $K(f(x))$ approaches ∞ is again e^c. For c near large values of x, e^c will also be very large, and this high condition number indicates high instability.

c) See hw2.py

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$f(\underbrace{9.999999995000000\cdot 10^{-10}}_{x_1}) =$ 1.000000082740371e-09

$= 0.\underbrace{00000000}_{8}\underbrace{1}_{1}\underbrace{0000000}_{7}827\cdots$

Expected: $f(x_1) = 1\cdot 10^{-9}$

$= 0.\underbrace{000000001}_{9}$

Correct digits = 16, expected as input also had 16 digits of precision, and we know $f(x)$ is stable for x close to 0.

d) $e^x = \sum_{k=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

$e^x - 1 = x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = f(x)$

Let $g_n(x) = x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}$

Relative error: $\frac{|\text{actual} - \text{observed}|}{\text{actual}}$

$$\frac{|f(x) - g_n(x)|}{f(x)} \ll 1\cdot 10^{-16}$$

$x_1 = 9.999999995000000\cdot 10^{-10}$

$x_1 = 9.999999995000000 \cdot 10^{-10}$

$$\frac{|f(x_1) - g_n(x_1)|}{f(x_1)} \leq 1 \cdot 10^{-16}$$

$$\frac{|(e^{x_1} - 1) - g_n(x_1)|}{e^{x_1} - 1} \leq 1 \cdot 10^{-16}$$

From hw2.py, only 2 terms are needed in order for the relative error to be $\leq 1 \cdot 10^{-16}$ at x_1

$$\boxed{g_2(x) = x + \frac{x^2}{2!}}$$

e) view hw2.py

4) see hw2.py