HW10

Thursday, November 9, 2023 10:42 AM

1. For the function $f(x) = \sin(x)$. Determine the Padé approximations of degree 6 with
   (a) Both the numerator and denominator are cubic
   (b) The numerator is quadratic and the denominator is a fourth degree polynomial.
   (c) The numerator is a fourth degree polynomial and the denominator is quadratic.

   Compare the accuracy of these approximations with the sixth order Maclaurin polynomial by ploting the error over the interval $[0, 5]$.

a)

$$f(x) = \sin(x) \approx r(x) = \frac{p(x)}{q(x)} = \frac{p_0 + p_1 x + p_2 x^2 + p_3 x^3}{1 + q_1 x + q_2 x^2 + q_3 x^3}$$

$$\sin(x) \approx x - \frac{1}{6}x^3 + \frac{1}{120}x^5$$

$$\left(x - \frac{1}{6}x^3 + \frac{1}{120}x^5\right)\left(1 + q_1 x + q_2 x^2 + q_3 x^3\right) = p_0 + p_1 x + p_2 x^2 + p_3 x^3$$

$$0 + 1x + q_1 x^2 + \left(q_2 - \frac{1}{6}\right)x^3 + \left(q_3 - \frac{1}{6}q_1\right)x^4 + \left(\frac{1}{120} - \frac{1}{6}q_2\right)x^5$$

$$+\left(\frac{1}{120}q_1 - \frac{1}{6}q_3\right)x^6 + \frac{1}{120}q_2 x^7 + \frac{1}{120}q_3 x^8 = p_0 + p_1 x + p_2 x^2 + p_3 x^3$$

$p_0 = 0$, $p_1 = 1$, $p_2 = q_1$, $p_3 = q_2 - 1/6$, $p_4 = 0 = q_3 - 1/6\, q_1$,

$p_5 = 0 = 1/120 - 1/6\, q_2$, $p_6 = 0 = 1/120\, q_1 - 1/6\, q_3$, $p_7 = 0 = 1/120\, q_2$,

$p_8 = 0 = 1/120\, q_3$

$\Rightarrow p_0 = 0$, $p_1 = 1$, $p_2 = 0$, $p_3 = -7/60$, $q_1 = 0$, $q_2 = 1/20$, $q_3 = 0$

$$\boxed{\sin(x) \approx P_3^3(x) = \frac{x - \frac{7}{60}x^3}{1 + \frac{1}{20}x^2}}$$

b) $$\sin(x) \approx P_4^2(x) = \frac{p_0 + p_1 x + p_2 x^2}{1 + q_1 x + q_2 x^2 + q_3 x^3 + q_4 x^4}$$

$$\left(x - \frac{1}{6}x^3 + \frac{1}{120}x^5\right)\left(1 + q_1 x + q_2 x^2 + q_3 x^3 + q_4 x^4\right) = p_0 + p_1 x + p_2 x^2$$

$$0 + 1x + q_1 x^2 + \left(q_2 - \frac{1}{6}\right)x^3 + \left(q_3 - \frac{1}{6}q_1\right)x^4 + \left(\frac{1}{120} - \frac{1}{6}q_2 + q_4\right)x^5$$

$$+ \left(\frac{1}{120}q_1 - \frac{1}{6}q_3\right)x^6 + \left(\frac{1}{120}q_2 - \frac{1}{6}q_4\right)x^7 + \frac{1}{120}q_3 x^8 + \frac{1}{120}q_4 x^9$$

$$= p_0 + p_1 x + p_2 x^2$$

$p_0 = 0$, $p_1 = 1$, $p_2 = q_1$, $p_3 = 0 = q_2 - 1/6$, $p_4 = 0 = q_3 - 1/6\, q_1$,

$p_5 = 0 = 1/120 - 1/6\, q_2 + q_4$, $p_6 = 0 = 1/120\, q_1 - 1/6\, q_3$, $p_7 = 0 = 1/120\, q_2 - 1/6\, q_4$,

$p_8 = 0 = 1/120\, q_3$, $p_9 = 0 = 1/120\, q_4$

$\Rightarrow p_0 = 0$, $p_1 = 1$, $p_2 = 0$, $q_1 = 0$, $q_2 = 1/6$, $q_3 = 0$, $q_4 = 7/360$

$$\sin(x) \approx P_4^2(x) = \boxed{\frac{x}{\frac{7}{360}x^4 + \frac{1}{6}x^2 + 1}}$$

c) $$\sin(x) \approx \frac{p_0 + p_1 x + p_2 x^2 + p_3 x^3 + p_4 x^4}{1 + q_1 x + q_2 x^2}$$

$$\sin(x) \approx \frac{\cdots}{1+q_1x+q_2x^2}$$

$$\left(x - \frac{1}{6}x^3 + \frac{1}{120}x^5\right)\left(1+q_1x+q_2x^2\right) = p_0 + p_1x + p_2x^2 + p_3x^3 + p_4x^4$$

$$0 + 1x + q_1x^2 + \left(q_2 - \frac{1}{6}\right)x^3 - \frac{1}{6}q_1x^4 + \left(\frac{1}{120} - \frac{1}{6}q_2\right)x^5$$

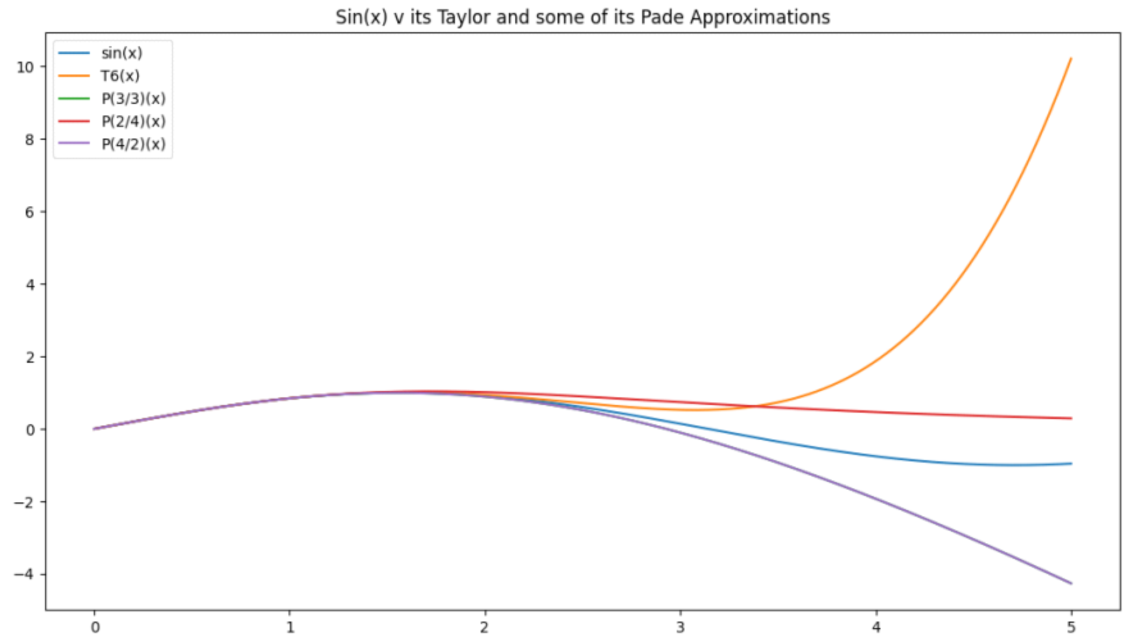
$$+ \frac{1}{120}q_1x^6 + \frac{1}{120}q_2x^7 = p_0 + p_1x + p_2x^2 + p_3x^4 + p_4x^4$$

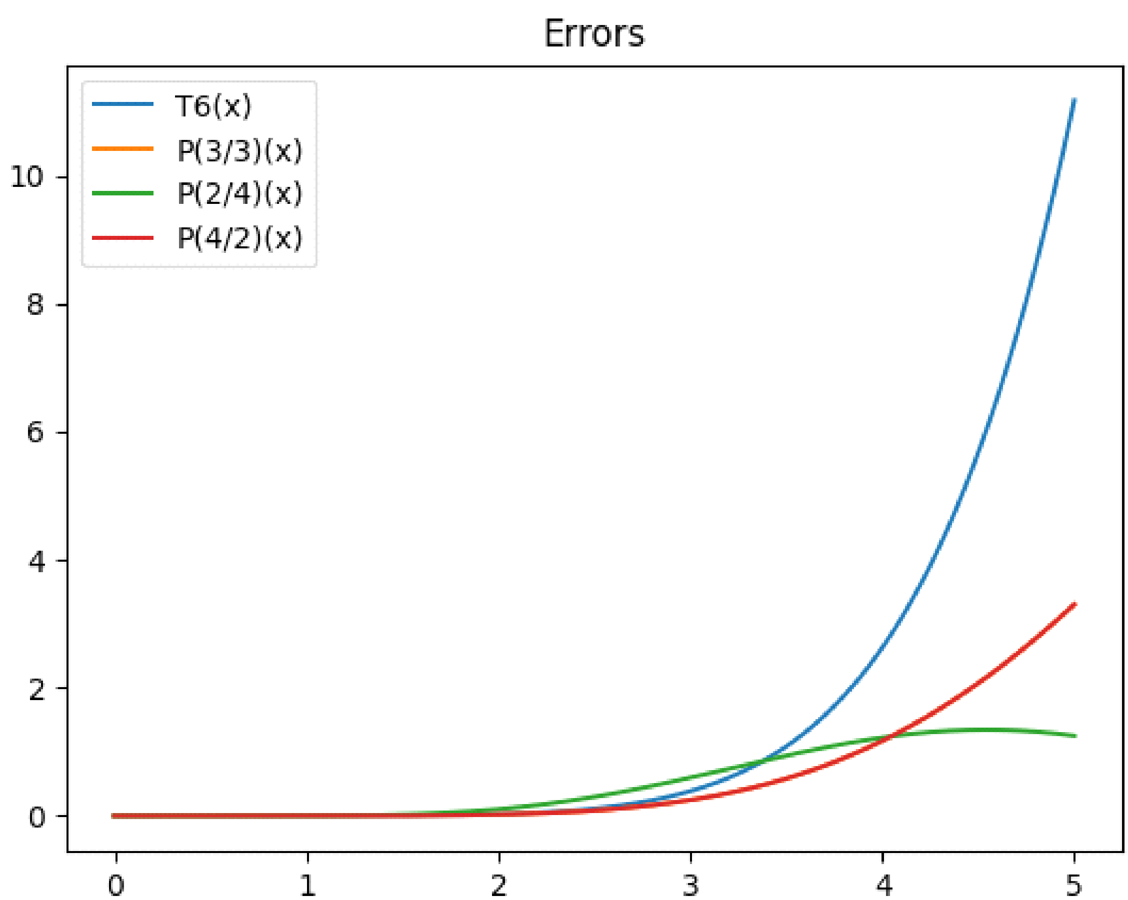
$p_0 = 0,\ p_1 = 1,\ p_2 = q_1,\ p_3 = q_2 - 1/6,\ p_4 = -1/6\, q_1,$

$p_5 = 0 = 1/120 - 1/6\, q_2,\ p_6 = 0 = 1/120\, q_1,\ p_7 = 0 = 1/120\, q_2$

$p_0 = 0,\ p_1 = 1,\ p_2 = 0,\ p_3 = 7/60,\ p_4 = 0,\ q_1 = 0,\ q_2 = 1/20$

$$\sin(x) \approx P_2^4(x) = P_3^3(x) = \boxed{\frac{x - \frac{7}{60}x^3}{1 + \frac{1}{20}x^2}}$$


Sin(x) v its Taylor and some of its Pade Approximations
sin(x)
T6(x)
P(3/3)(x)
P(2/4)(x)
P(4/2)(x)
10
8
6
4
2
0
−2
−4
0
1
2
3
4
5


Errors
T6(x)
P(3/3)(x)
P(2/4)(x)
P(4/2)(x)
10
8
6
4
2
0
0
1
2
3
4
5

2. Find the constants x_0, x_1 and c_1 so that the quadrature formula

$$\int_0^1 f(x)dx = \frac{1}{2}f(x_0) + c_1 f(x_1)$$

has the highest possible degree of precision.

Degree 0: $f(x) = x^0 = 1$

$$\int_0^1 1\,dx = 1 = \frac{1}{2}(1) + c_1(1)$$

$$1 = 1/2 + c_1 \Rightarrow c_1 = 1/2$$

Degree 1: $f(x) = x^1 = x$

$$\int_0^1 x\,dx = \left(\frac{1}{2}x^2\right)\Big|_{x=0}^{x=1} = \frac{1}{2}x_0 + \frac{1}{2}x_1$$

$$\frac{1}{2} = \frac{1}{2}\left(x_0 + x_1\right)$$

$$1 = x_0 + x_1 \Rightarrow x_1 = 1 - x_0$$

Degree 2: $f(x) = x^2$

$$\int_0^1 x^2\,dx = \left(\frac{1}{3}x^3\right)\Big|_{x=0}^{x=1} = \frac{1}{2}x_0^2 + \frac{1}{2}x_1^2$$

$$\frac{1}{3} = \frac{1}{2}x_0^2 + \frac{1}{2}\left(1 - x_0\right)^2$$

$$\frac{1}{3} = \frac{1}{2}x_0^2 + \frac{1}{2}\left(1 - 2x_0 + x_0^2\right)$$

$$\frac{1}{3} = \frac{1}{2}x_0^2 + \frac{1}{2} - x_0 + \frac{1}{2}x_0^2$$

$$x_0^2 - x_0 + \frac{1}{6} = 0$$

$$x_0^2 - x_0 = -\frac{1}{6}$$

$$x_0^2 - x_0 + \frac{1}{4} = -\frac{1}{6} + \frac{1}{4}$$

$$\left(x_0 - \frac{1}{2}\right)^2 = \frac{1}{12}$$

$$x_0 - \frac{1}{2} = \pm\sqrt{1/12}$$

$$x_0 = \frac{1}{2} \pm \frac{1}{\sqrt{12}} \qquad 2\sqrt{3}$$

$$x_0 = \frac{1}{2} \pm \frac{1}{2\sqrt{3}}$$

$$= \frac{\sqrt{3}}{2\sqrt{3}} \pm \frac{1}{2\sqrt{3}} = \frac{\sqrt{3}-1}{2\sqrt{3}}, \quad \frac{\sqrt{3}+1}{2\sqrt{3}}$$

$$= \frac{1}{2} - \frac{\sqrt{3}}{6} \quad \frac{1}{2} + \frac{\sqrt{3}}{6}$$

$$\Rightarrow x_1 = \frac{1}{2} + \frac{\sqrt{3}}{6}, \quad \frac{1}{2} - \frac{\sqrt{3}}{6}$$

$$\boxed{(x_0, x_1) = \left(\frac{1}{2} - \frac{\sqrt{3}}{6}, \ \frac{1}{2} + \frac{\sqrt{3}}{6}\right)}$$

$$\boxed{(x_0, x_1) = \left(\frac{1}{2} - \frac{\sqrt{3}}{6}, \ \frac{1}{2} + \frac{\sqrt{3}}{6}\right)}$$

3. (a) Write a code to approximate $\int_{-5}^{5} \frac{1}{1+s^2} ds$ using a composite Trapezoidal rule. To do this, partition the interval $[-5, 5]$ into equally spaced points $t_0, t_1, \ldots, t_n$.
Write another code to approximate $\int_{-5}^{5} \frac{1}{1+s^2} ds$ using a composite Simpson's rule. To do this, partition the interval $[-5, 5]$ into equally spaced points $t_0, t_1, \ldots, t_n$ where $n = 2k$ is even. The even indexed points should be the endpoints of your subintervals.
You may combine the two into one code that selects the desired method if you wish.
Turn in a listing of your code(s).

See hw10.py

b) Use the error estimates derived in class to choose n so that

$$\left| \int_{-5}^{5} \frac{1}{1+s^2} ds - T_n \right| < 10^{-4} \quad \text{and} \quad \left| \int_{-5}^{5} \frac{1}{1+s^2} ds - S_n \right| < 10^{-4},$$

where T_n is the result of the composite Trapezoidal rule and where S_n is the result of the composite Simpson's rule. Be sure to explain your reasoning for choosing n in both cases (these n values will be different in the two cases).

Trapezoidal error:

$$|E_T(f(x))| = \left| -\frac{b-a}{12} \cdot \left(\frac{b-a}{n}\right)^2 f''(\xi) \right| = \frac{250}{3n^2} |f''(\xi)|$$

$$|f''(x)| = \left| \frac{6x^2 - 2}{(x^2+1)^3} \right|, \quad \text{max is 2 at } x = 0$$

$$|E_T| \le \frac{500}{3n^2} < 10^{-4} \Rightarrow n > 1000 \cdot \sqrt{\frac{5}{3}} \approx 1{,}290$$

$$\boxed{n \geq 1{,}291}$$

Simpson's Error:

$$\left|E_S(f(x))\right| = \left|-\frac{b-a}{180}h^4 \cdot f^{(4)}(\xi)\right| = \frac{5000}{9n^4} \cdot \left|f^{(4)}(\xi)\right|$$

Max of $|f^{(4)}(\xi)|$ is 24 at $x=0$.

$$|E_S| \leq \frac{40{,}000}{3n^4} < 10^{-4} \Rightarrow n > \frac{100\sqrt{2}}{\sqrt[4]{3}} \approx 107$$

$$\boxed{n \geq 108}$$

c) Run your code with the predicted values of n and compare your computed values S_n and T_n with that of SCIPY's **quad** routine on the same problem. Run the built in quadrature twice, once with the default tolerance of 10^{-6} and another time with the set tolerance of 10^{-4}. Report the number of function evaluations required in both cases and compare these to the number of function values your codes (both S_n and T_n) required to meet the tolerance

Turn in your codes and the results of this test.

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Problem 3)
Trapezoidal, n = 1,291:
2.746801385962377

Simpsons, n = 108:
2.746801528748204

scipy.integrate.quad, tol = 1 x 10^-6:
2.7468015338900327
Iterations needed: 147

scipy.integrate.quad, tol = 1 x 10^-4:
2.746801533909586
Iterations needed: 63
```