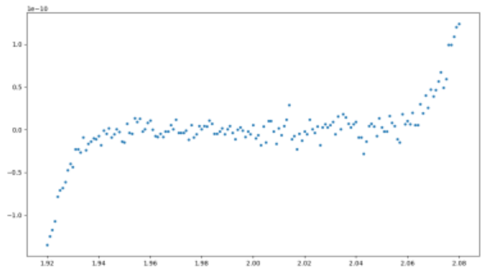
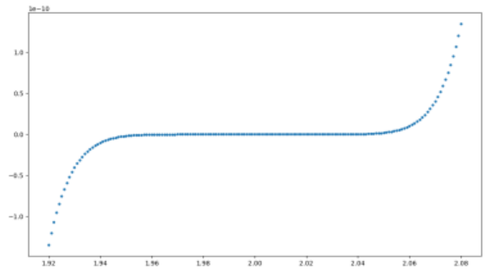
HW 1

Tuesday, September 5, 2023 7:59 PM

1) See code in repo! Comments answer questions.





2.i) Evaluate $\sqrt{x+1} - 1$ for $x \approx 0$ to avoid cancellation

For $x \approx 0$, $\sqrt{x+1} \approx 1$, so $\sqrt{x+1} - 1 = 0$

$$\sqrt{x+1} - 1 \cdot \frac{\sqrt{x+1}+1}{\sqrt{x+1}+1}$$

$$\frac{x+1-1}{\sqrt{x+1}+1}$$

$$\boxed{\frac{x}{\sqrt{x+1}+1}}$$

No more subtraction: $\sqrt{x+1}+1 \neq 0$

2.ii) Evaluate $\sin(x) - \sin(y)$ for $x \approx y$ to avoid cancellation

$$\sin(x) - \sin(y)$$

Let $x = a+b$, $y = a-b \Rightarrow x+y = 2a \Rightarrow a = \frac{x+y}{2}$

$$\sin(a+b) - \sin(a-b)$$

$x-y = 2b \Rightarrow b = \frac{x-y}{2}$

$\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b)$

$\sin(a-b) = \sin(a)\cos(b) - \cos(a)\sin(b)$

$\sin(a)\cos(b) + \cos(a)\sin(b) - \sin(a)\cos(b) + \cos(a)\sin(b)$

$2\cos(a)\sin(b)$

$$\boxed{2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)}$$

2.iii) Evaluate $\frac{1-\cos(x)}{\sin(x)}$ for $x \approx 0$ to avoid cancellation

For $x \approx 0$, $\cos(x) \approx 1$, so $1-\cos(x) = 0$, so $\frac{1-\cos(x)}{\sin(x)} = 0$

$$\frac{1-\cos(x)}{\sin(x)} \cdot \frac{1+\cos(x)}{1+\cos(x)}$$

$$\frac{1-\cos^2(x)}{\sin(x)+\sin(x)\cos(x)}$$

$\cos^2(x) = 1 - \sin^2(x) \quad -\cos^2(x) = \sin^2(x) - 1$

$$\frac{1+\sin^2(x)-1}{\sin(x)+\sin(x)\cos(x)}$$

$$\boxed{\frac{\sin(x)}{1+\cos(x)}} \qquad 1+\cos(x) \neq 0 \text{ for } x \approx 0$$

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3) $f(x) = (1+x+x^3)\cos(x)$

Find Taylor polynomial $P_2(x)$ centered at $x_0 = 0$

$$P_2(x) = \sum_{n=0}^{2} \frac{f^{(n)}(0)}{n!} x^n$$

$$= f(0) + f'(0)x + \frac{f''(0)}{2}x^2$$

$f(0) = (1+0+0)(1) = 1$

$$= f(0) + f'(0)x + \frac{f''(0)}{2}x^2$$

$$f(0) = (1+0+0)(1) = 1$$

$$f'(x) = (1+3x^2)\cos(x) - (1+x+x^3)\sin(x)$$

$$f'(0) = (1)(1) - (1)(0) = 1$$

$$f''(x) = 6x\cos(x) - (1+3x^2)\sin(x) - (1+3x^2)\sin(x) - (1+x+x^3)\cos(x)$$

$$= -(x^3 - 5x + 1)\cos(x) - 2(3x^2+1)\sin(x)$$

$$f''(0) = -(1)(1) - 0 = -1$$

$$P_2(x) = 1 + x - \tfrac{1}{2}x^2$$

3a) $P_2(0.5) = 1 + 0.5 - \frac{1}{2}(0.5)^2 = 1.375$

$$f(0.5) = (1 + 0.5 + 0.5^3)\cos(0.5) \approx 1.426$$

Remainder term: $R_n(x) = \dfrac{f^{(n+1)}(z(x))}{(n+1)!}(x-a)^{n+1}$

$R_2(x) = \dfrac{f'''(z(x))}{3!}x^3$ $z(x)$ between x and $a=0$

$$f'''(x) = (x^3 - 17x + 1)\sin(x) + (3 - 9x^2)\cos(x)$$

$$R_2(x) = \frac{((z(x))^3 - 17z(x) + 1)\sin(z(x)) + (3 - 9(z(x))^2)\cos(z(x))}{6}x^3$$

$$0 < z(x) < 0.5$$

$$R_2(0.5) = \frac{((z(x))^3 - 17z(x) + 1)\sin(z(x)) + (3 - 9(z(x))^2)\cos(z(x))}{48}$$

For $0 < z(x) < 0.5$, $-\frac{54}{\ }\leq ((z(x))^3 - 17z(x) + 1) \leq 1$

For $0 < z(x) < 0.5$, $-\frac{54}{8} < \left((z(x))^3 - 17z(x) + 1\right) < 1$,

$0 < \sin(z(x)) < 1/2$

$3/4 < (3 - 9(z(x))^2) < 3$,

$4/5 < \cos(z(x)) < 1$

using Desmos!

Lower $R_2(0.5)$: $\frac{0 + (3/4)(4/5)}{48} = \frac{1}{80}$

Upper $R_2(0.5)$: $\frac{(1)(1/2) + (3)(1)}{48} = \frac{7}{96}$

$$\boxed{0.0125 = 1/80 \leq |R_2(0.5)| \leq 7/96 \approx 0.0729}$$

Actual error: $|f(0.5) - P_2(0.5)|$

$|1.426 - 1.375| = \boxed{0.051}$

We note that $0.051 \in (0.0125, 0.0729)$ ✓

3b) $f(x) = P_2(x) + R_2(x)$

$f(x) - P_2(x) = R_2(x)$

$|f(x) - P_2(x)| = |R_2(x)|$

As computed in 3a):

$$R_2(x) = \frac{((z(x))^3 - 17z(x) + 1)\sin(z(x)) + (3 - 9(z(x))^2)\cos(z(x))}{6} x^3$$

6

$$|R_2(x)| = \frac{\left|((z(x))^3 - 17z(x)+1)\sin(z(x)) + (3-9(z(x))^2\cos(z(x))\right|}{6} x^3$$

3c) $$\int_0^1 (1+x+x^3)\cos(x)\,dx \approx \int_0^1 1+x-\frac{1}{2}x^2\,dx$$

$$= \left[x+\frac{1}{2}x^2-\frac{1}{6}x^3\right]_0^1$$

$$= 1+\frac{1}{2}-\frac{1}{6} = \boxed{4/3}$$

3d) $$\int_0^1 (1+x+x^3)\cos(x)\,dx = \int_0^1 1+x-\frac{1}{2}x^2\,dx + \int_0^1 R_2(x)\,dx$$

For $0 < x < 1$, $-15 < (x^3-17x+1) < 1$,

$0 < \sin(x) < 9/10$,

$-6 < 3-9x^2 < 3$,

$1/2 < \cos(x) < 1$

$$\frac{(0)+(-6)(1/2)}{6}x^3 < R_2(x) < \frac{(1)(9/10)+(3)(1)}{6}x^3$$

$$-\frac{1}{2}x^3 < R_2(x) < \frac{13}{20}x^3$$

$$\int_0^1 -\frac{1}{2}x^2\,dx < \int_0^1 R_2(x)\,dx < \int_0^1 \frac{13}{20}x^3$$

$$\boxed{-\frac{1}{8} < \int_0^1 R_2(x)\,dx < \frac{13}{80}}$$

4a) $x^2 - 56x + 1$ Precision: $\sqrt{2} = 1.414 \pm 0.0005$

Exact: $r_1 = \frac{-(-56) + \sqrt{3136 - 4(1)(1)}}{2(1)} = \frac{56 + \sqrt{3132}}{2}$

$$= \frac{56 + 6\sqrt{87}}{2}$$

$$= 28 + 3\sqrt{87}$$

$r_2 = 28 - 3\sqrt{87}$

With 3 sigfigs:

$$r_1 = \frac{56 + \sqrt{3132}}{2} \approx \frac{56 + 55.964}{2} = 55.982$$

$$r_2 \approx \frac{56 - 55.964}{2} = 0.018$$

r_1 error: $\frac{|28 + 3\sqrt{87} - 55.982|}{28 + 3\sqrt{87}} \approx \boxed{2.45 \cdot 10^{-6}}$

r_2 error: $\frac{|28 - 3\sqrt{87} - 0.018|}{28 - 3\sqrt{87}} \approx \boxed{0.008}$

$$\overline{} \over 28-3\sqrt{87}$$ $\approx \boxed{0.008}$

↑ bad root!

4b) $$r_2 = \frac{-b-\sqrt{b^2-4ac}}{2a} \cdot \frac{-b+\sqrt{b^2-4ac}}{-b+\sqrt{b^2-4ac}} = \frac{-2c}{b-\sqrt{b^2-4ac}}$$

$$r_2 = \frac{-2(1)}{(-56)-\sqrt{(-56)^2-4(1)(1)}} = \frac{-2}{-56-\sqrt{3132}}$$

$$= \frac{-2}{-56-55.964} = \boxed{\frac{2}{111.964}} \approx 0.01786$$

New relative error: $$\frac{\left|28-3\sqrt{87}-\frac{2}{111.964}\right|}{28-3\sqrt{87}} \approx \boxed{2.450\cdot 10^{-6}}$$

Much better!

5a) $y = x_1 - x_2 \quad \tilde{x}_1 = x_1 + \Delta x_1 \quad \tilde{x}_2 = x_2 + \Delta x_2$

$$x_1 = \tilde{x}_1 - \Delta x_1 \quad x_2 = \tilde{x}_2 - \Delta x_2$$

$$y = (\tilde{x}_1 - \Delta x_1) - (\tilde{x}_2 - \Delta x_2)$$

$$y = \tilde{x}_1 - \Delta x_1 - \tilde{x}_2 + \Delta x_2$$

$$\tilde{y} = y + (\Delta x_1 - \Delta x_2)$$

$$\tilde{y} = \tilde{x}_1 - \Delta x_1 - \tilde{x}_2 + \Delta x_2 + \Delta x_1 - \Delta x_2 = \tilde{x}_1 - \tilde{x}_2$$

$$|\Delta y| = |\Delta x_1 - \Delta x_2|$$

Using triangle inequality:

$$|\Delta y| = |\Delta x_1 - \Delta x_2| \leq |\Delta x_1| - |\Delta x_2|$$

$$\boxed{\frac{|\Delta y|}{y} \leq \frac{|\Delta x_1| - |\Delta x_2|}{x_1 - x_2}}$$

5b) $\cos(\delta + x) - \cos(x)$

Let $\delta + x = A + B$ and $x = A - B$

$\Rightarrow A = \frac{\delta}{2} + x, \quad B = \frac{\delta}{2}$

$\cos(A+B) - \cos(A-B)$

Using $\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B)$,

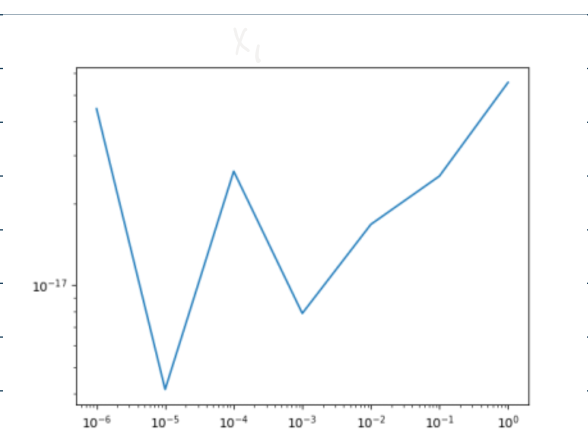
$\cos(A-B) = \cos(A)\cos(B) + \sin(A)\sin(B)$

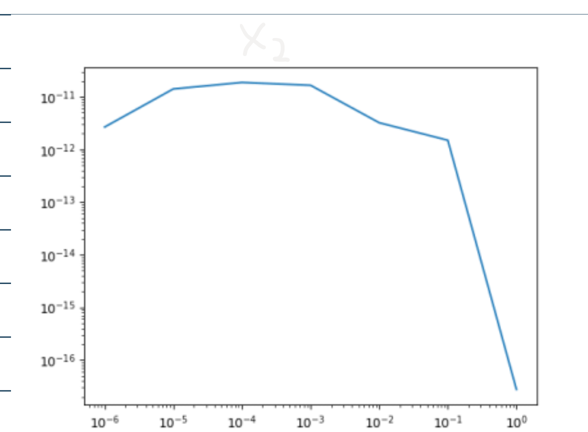
$\cos(A)\cos(B) - \sin(A)\sin(B) - \cos(A)\cos(B) - \sin(A)\sin(B)$

$-2\sin(A)\sin(B)$

$-2\sin\left(\frac{\delta}{2} + x\right)\sin\left(\frac{\delta}{2}\right)$

Using $x = \pi$ and $x = 10^6$:





5c) $f(x+\delta) - f(x) = \delta f'(x) + \frac{\delta^2}{2!} f''(z) \qquad z \in [x, x+\delta]$

$[\cos(x)]' = -\sin(x) \qquad [\cos(x)]'' = -\cos(x)$

$$\cos(x+\delta) - \cos(x) = -\delta \sin(x) - \frac{\delta^2}{2}\cos(z)$$

$$-1 \le \cos(z) \le 1$$

$$-1 \le -\cos(z) \le 1$$

$$-\frac{\delta^2}{2} \le -\frac{\delta^2}{2}\cos(z) \le \frac{\delta^2}{2}$$

$$\left| -\frac{\delta^2}{2}\cos(z) \right| \le \frac{\delta^2}{2}$$

Choose $z = x$