APPM 4600 — Midterm 1 Review Sheet

Here is a list of important concepts to review from the material for Midterm 1.

- **Floating Point arithmetic:**
  (a) What is the floating point representation of a number? (significant digits / mantissa, base and exponent)
  (b) Define the following terms: Overflow (OFL), underflow (UFL), machine epsilon ϵ_M, absolute and relative rounding errors.
  (c) On the machine, what is the maximum number of (correct) significant digits in the answer to $2^e - 2^{e-1}$ in double precision? How is this related to the machine epsilon?
  (d) Why is double precision associated with 16 significant digits? Like wise why is single precision associated with 8 significant digits?
  (e) What arithmetic operations can result in a loss of precision? Why causes the loss of precision? List a few notable examples from class, homework, flipped day exercises, etc.

- **Algorithms, pseudocodes and stability:**
  (a) What is an algorithm? How is an algorithm different from a pseudocode?
  (b) What does it mean for an algorithm to be stable? Unstable? Conditionally stable?
  (c) Suppose that you have developed an algorithm for a mathematical problem, and you realize that it is not stable. What are some general strategies to (try to) modify the algorithm so that it is stable?
  (d) Define: Big O and little o notation.

- **Conditioning of a problem**

- **Nonlinear equations and root finding methods (1D):**
  (a) What is the relationship between solving a nonlinear equation and root finding?
  (b) Suppose that you want to use the **Bisection method:** to find a root of $f(x)$ on $[a, b]$. What assumptions on $f(x)$ need to be satisfied? If these assumptions are met, does bisection always converge?
  (c) What are the different options for stopping criteria of iterative methods? What are the differences between the options?
  (d) Define: order of convergence α, linear convergence and convergence rate.
  (e) Suppose the sequence $\{x_n\}_{n=1}^{\infty}$ converges to x. How can we determine the order of convergence by looking at the iterates x_n and the errors at each iterations e_n. What does the log-plot of e_n look like for linear and super-linear convergence?
  (f) What is a fixed point of a function $g(x)$? Under what conditions are we guaranteed that a fixed point exists in $[a, b]$?
  (g) What is the fixed point iteration? Under what conditions is convergence of the **Fixed point iteration** guaranteed? When it does converge, when does it converge linearly? Quadratically?

(h) How can you apply the fixed point iteration to a root finding problem? How does the choice of function used in the fixed point iteration $g(x)$, given the goal of finding x such that $f(x) = 0$, affect the convergence of this approach?

(i) Derive a step in **Newton's method**. What is the technique based on? Under what assumptions does Newton converge quadratically? What does it mean for an initial guess for Newton's method to be in the basin of convergence for the algorithm?

(j) How does Newton perform near a repeated root? How can we fix this slower order of convergence? What are other possible issues with the convergence of Newton's method?

(k) What is the idea behind **Secant method**? Under what assumptions does it converge super-linearly; i.e/ convergence order is greater than 1?

(l) How does the secant method perform when compared to the Newton method? What is the advantages and disadvantages of using the secant method?

- **Systems of Nonlinear equations and root finding methods (nD):**

(a) Consider the fixed point iteration applied to $F : R^n \to R^n$. How would you apply this to a root finding problem? Under what conditions on its Jacobian G is F a contractive map?

(b) Derive Newton's method for systems of nonlinear equations. Under what assumptions does this method converge quadratically? How does the Newton's method for nonlinear systems of equations differ from Newton's method for scalar equations?

Note: Below are some practice problems. This list of problems is not exhaustive. You should not expect the exam to look exactly like this. These are simply some problems that you should be able to solve using the material from class. Making sure you understand all the concepts and algorithms introduced in class, homeworks and flipped days is the best way to prepare you for the exam. Be sure to understand why, when and how an algorithm works in addition to the limitations.

1. In evaluating the function $f(x) = \frac{\sin x^2}{1-\cos x}$ using finite precision arithmetic, the computed result may be inaccurate when x is near zero. Explain why this is the case, and propose an algorithm for evaluating this expression that avoids the issue.

2. Let $f(x) = \sqrt{1+x} - \sqrt{1-x}$ and $g(x) = 2x/(\sqrt{1+x} + \sqrt{1-x})$. Show that $f(x) = g(x)$ for all x such that $|x| \leq 1$. If you are using finite precision arithmetic, which expression is better to use when $x \approx 0$? Explain.

3. In the context of root finding for scalar equations, state one advantage of using:

   (a) Newton's method over the bisection method.

   (b) The bisection method over Newton's method.

   (c) The secant method over Newton's method.

4. Let $f(x)$ be a continuous function with a root r in the interval $[1, 2]$. Suppose that the bisection method is applied to find the root with starting guess $x_0 = 1.5$ (assume that $f(1)f(2) < 0$). What is the maximum number of steps needed to ensure that the error is less than 10^{-4}?

5. Recall problem 4 of homework 2 on applying the bisection method to $f(x) = (x-5)^9$ and its expanded form. Indicate at which step it is key to have an accurate evaluation of $f(x)$ for this method to converge to the root to high accuracy.

6. Consider the following fixed point iterations $x_{n+1} = g(x_n)$ and corresponding fixed point $x*$. Compute $g'(x)$ and use this to determine whether the method converges (for $x0$ close to the root), and if so, what is the order of convergence α. If it converges linearly, determine the convergence rate λ.

   (a) $x_{n+1} = \frac{2x_n}{x_n^2-1}$ and $x* = 0$.

   (b) $x_{n+1} = x_n - 0.5x_n \cos(x_n)$ and $x* = 0$.

   (c) $x_{n+1} = \frac{1}{1+4x_n^2}$ and $x* = 1/2$.

7. Simplicio and Salviati are asked to solve for the root closest to 0 of the following polynomial using Newton's method.

$$p(x) = 80 - 116x + 60x^2 - 13x^3 + x^4 = 0.$$

(a) Simplicio implements Newton's method and comes up with the following sequence of iterations and proclaims that the iteration converges quadratically to the root $x = 2$. Is that correct? What kind of convergence is exhibited in the second column of the table?

iteration	Approx by Simplicio	Approx by Salviati
1	0.6896551724E+00	0.1379310345E+01
2	0.1174998031E+01	0.1894775372E+01
3	0.1503982576E+01	0.1995765052E+01
4	0.1715901111E+01	0.1999992554E+01
5	0.1844400978E+01	0.2000000000E+01
6	0.1917745107E+01	0.2000000000E+01
7	0.1957555056E+01	
8	0.1978415265E+01	
9	0.1989112319E+01	
10	0.1994531690E+01	

Meanwhile, Salviati observes Simplicio's implementation, and after computing its derivative, he obtains:

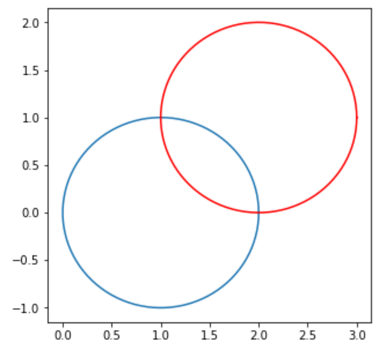
$$p'(x) = 4x^3 - 39x^2 + 120x - 116$$

After evaluating $p'(2)$, he realizes that he can apply Newton to a different function in order to obtain a higher order of convergence. The resulting iterates are included in the third column of the table.

(b) Based on the table only, indicate how we can tell whether Salviati's method is linearly or super-linearly convergent.

(c) How, and more importantly, why did Salviati have to modify Newton's method in order to obtain faster convergence to the root?

8. Consider the problem of finding the intersections between two circles of radius 1, with centers at $(1, 0)$ and $(2, 1)$, respectively.



From this picture, we can clearly see two roots: $(1, 1)$ and $(2, 0)$. We can write this as a system of two nonlinear equations for coordinates (x, y):

$$F_1(x) = (x-1)^2 + y^2 - 1 = 0$$
$$F_2(x) = (x-2)^2 + (y-1)^2 - 1 = 0$$

Where $F(x) = [F_1(x) \quad F_2(x)]$. We find the Jacobian of $F(x)$:

$$J_F(x) = \begin{bmatrix} 2(x-1) & 2y \\ 2(x-2) & 2(y-1) \end{bmatrix}$$

(a) We use the Newton method starting at initial point $(x0, y0) = (1, 0)$. We immediately get a complaint from our software that reads 'LinAlgError: Singular Matrix'. Compute the Jacobian at this initial point and explain why we got this message.

(b) After running the Newton method for a good amount of initial points $(x0, y0)$, we notice a pattern. Iterations starting above the line $y = x - 1$ converge to $(1, 1)$ and iterations starting below it converge to $(2, 0)$. If we start exactly at this line, we get the same error message we got in (a). Plug in $y = x - 1$ in the formula for the Jacobian and explain why that might be the case.