HW3

Tuesday, September 19, 2023 7:08 PM

1. Consider the equation $2x - 1 = \sin x$.

   (a) Find a closed interval $[a, b]$ on which the equation has a root r, and use the Intermediate Value Theorem to prove that r exists.

   (b) Prove that r from (a) is the only root of the equation (on all of $\mathbb{R}$).

   (c) Use the bisestion code from class (or your own) to approximate r to eight correct decimal places. Include the calling script, the resulting final approximation, and the total number of iterations used.

1a) $2x - 1 = \sin(x)$

$f(x) = 2x - \sin(x) - 1$

$f(\pi) = 2\pi - 1 > 0$

$f(-\pi) = -2\pi - 1 < 0$

Since f is continuous, $f(-\pi) < 0$ and $f(\pi) > 0$, $\exists r \in [-\pi, \pi]$ s.t. $f(r) = 0$ by the IVT.

1b)

Suppose that there are 2 roots, r_1 and r_2, s.t.:

$f(r_1) = 0 = 2r_1 - 1 - \sin(r_1)$ and

$f(r_2) = 0 = 2r_2 - 1 - \sin(r_2)$

We must have then that $f(r_1) - f(r_2) = 0$, which implies that

$$\frac{f(r_1) - f(r_2)}{r_1 - r_2} = 0$$

However, since f is continuous on $[r_1, r_2]$ $\forall r_1, \forall r_2 \in \mathbb{R}$ and differentiable on (r_1, r_2) $\forall r_1, \forall r_2 \in \mathbb{R}$, by the MVT, $\exists c \in \mathbb{R}$ such that:

$$f'(c) = \frac{f(r_1) - f(r_2)}{r_1 - r_2} = 0$$

However, for $f(x) = 2x - 1 - \sin(x)$,

$$f'(x) = 2 - \cos(x) = 0$$

$$2 = \cos(x) \quad \times$$

There is no c s.t. $f'(c) = 0$. Therefore, there can only be one root for $f(x) = 2x - 1 - \sin(x)$ for $x \in \mathbb{R}$.

c) See hw3.py

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Problem 1c)
Iterations ran: 30, Current approximation: 0.8878622125563768
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2. The function $f(x) = (x-5)^9$ has a root (with multiplicity 9) at $x = 5$ and is monotonically increasing (decreasing) for $x > 5$ ($x < 5$) and should thus be a suitable candidate for your function above. Use `a=4.82` and `b=5.2` and `tol = 1e-4` and use `bisection` with:

(a) $f(x) = (x-5)^9$.

(b) The expanded expanded version of $(x-5)^9$, that is, $f(x) = x^9 - 45x^8 + \ldots - 1953125$.

(c) Explain what is happening.

2a) See hw3.py.

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Problem 2a)
Iterations ran: 12, Current approximation: 5.000026855468751
```

2b) See hw3.py

```
Problem 2b)
Iterations ran: 12, Current approximation: 5.1118188476562505
```

2c) The same number of iterations occur to make the interval smaller than the given tolerance. However, the expanded form of the function is numerically unstable due to the large amounts of subtraction when compared to the factored form.

3. (a) Use a theorem from class (Theorem 2.1 from text) to find an upper bound on the number of iterations in the bissection needed to approximate the solution of $x^3 + x - 4 = 0$ lying in the interval $[1, 4]$ with an accuracy of 10^{-3}.

(b) Find an approximation of the root using the bisection code from class to this degree of accuracy. How does the number of iterations compare with the upper bound you found in part (a)?

3a) Let $f(x) = x^3 + x - 4 = 0$

Since $f \in C[1,4]$ and $f(1)f(4) = (-2)(64) < 0$, the sequence $\{p_n\}_{n=1}^{\infty}$ generated by the bisection method approximating a zero, p, of f has an accuracy of:

$$|p_n - p| \leq \frac{3}{2^n} \qquad n \geq 1$$

For accuracy of 10^{-3}:

$$10^{-3} \leq \frac{3}{2^n}$$

$$2^n \leq 3 \cdot 10^3$$

$2^n \leq 3 \cdot 10^3$

$n \leq \log_2(3 \cdot 10^3) \approx 11.55$

$\boxed{\text{11 iterations needed}}$

3b)

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Problem 3b)
Iterations ran: 10, Current approximation: 1.37939453125
```

1 less iteration was needed

4. **Definition 1** *Suppose $\{p_n\}_{n=0}^{\infty}$ is a sequence that converges to p with $p_n \neq p$ for all n. If there exists postive constants λ and α such that*

$$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|^{\alpha}} = \lambda$$

then $\{p_n\}_{n=1}^{\infty}$ converges to p with an order α and asympotic error constant λ. If $\alpha = 1$ and $\lambda < 1$ then the sequence converges linearly. *If $\alpha = 2$, the sequence is* quadratically *convergent.*

Which of the following iterations will converge to the indicated fixed point x_* (provided x_0 is sufficiently close to x_*)? If it does converge, give the order of convergence; for linear convergence, give the rate of linear convergence.

(a) (10 points) $x_{n+1} = -16 + 6x_n + \frac{12}{x_n}$, $x_* = 2$

(b) (10 points) $x_{n+1} = \frac{2}{3}x_n + \frac{1}{x_n^2}$, $x_* = 3^{1/3}$

(c) (10 points) $x_{n+1} = \frac{12}{1+x_n}$, $x_* = 3$

4a) $X_{n+1} = g(X_n) = -16 + 6X_n + \frac{12}{X_n}$, $X^* = 2$

$g'(X_n) = 6 - \frac{12}{X_n^2}$

$|g'(2)| = \left|6 - \frac{12}{4}\right| = |6-3| = 3 > 1$

$$\boxed{\text{Diverges, as derivative near fixed point is greater than one}}$$

4b) $x_{n+1} = g(x_n) = \frac{2}{3}x_n + \frac{1}{x_n^2}, \quad x^* = 3^{1/3}$

$$g'(x_n) = \frac{2}{3} - \frac{2}{x_n^3}$$

$$g'(3^{1/3}) = 0$$

$$g''(x_n) = \frac{6}{x_n^4}$$

$$g''(3^{1/3}) = \frac{6}{3^{4/3}} = \frac{2}{3^{1/3}} = \frac{2}{\sqrt[3]{3}}$$

Taylor Series of $g(x_n)$ centered at $x^* = 3^{1/3}$

$$\cancelto{x_{n+1}}{g(x_n)} = \cancelto{3^{1/3}}{g(3^{1/3})} + \cancelto{0}{g'(3^{1/3})}(x_n - 3^{1/3}) + \frac{g''(\xi)}{2!}(x_n - 3^{1/3})^2$$

$$\xi \in [x_n, 3^{1/3}]$$

$$x_{n+1} - 3^{1/3} = \frac{g''(\xi)}{2!}(x_n - 3^{1/3})^2$$

$$\frac{x_{n+1} - 3^{1/3}}{(x_n - 3^{1/3})^2} = \frac{g''(\xi)}{2!}$$

$$\left|\frac{x_{n+1}-3^{1/3}}{(x_n-3^{1/3})^2}\right| = \left|\frac{g''(\xi)}{2!}\right|$$

$$\lim_{n\to\infty}\frac{|x_{n+1}-3^{1/3}|}{(x_n-3^{1/3})^2} = \lim_{n\to\infty}\frac{|g''(\xi)|}{2!}$$

Since $x_{n+1}=g(x_n)$ converges, as $n\to\infty$, $x_n\to x^*=3^{1/3}$.

For $\xi\in[x_n, 3^{1/3}]$, as $x_n\to 3^{1/3}$, $\xi\in[3^{1/3},3^{1/3}] \Rightarrow \xi = 3^{1/3}$

$$\lim_{n\to\infty}\frac{|x_{n+1}-3^{1/3}|}{(x_n-3^{1/3})^{\overset{\alpha}{2}}} = \lim_{n\to\infty}\frac{|g''(3^{1/3})|}{2!}$$

$$= \lim_{n\to\infty}\frac{2/\sqrt[3]{3}}{2!} = \frac{1}{\sqrt[3]{3}}$$

$\boxed{\text{Converges quadratically}}$

4c) $x_{n+1}=g(x_n)=\dfrac{12}{1+x_n}$, $x^*=3$

$$g'(x_n) = -\frac{12}{(1+x_n)^2}$$

$$|g'(3)| = \left|-\frac{12}{4^2}\right| = \left|-\frac{12}{16}\right| = \frac{3}{4} < 1$$

Taylor Series of $g(x_n)$ centered at $x^*=3$

$$\underbrace{g(x_n)}_{x_{n+1}} = \underbrace{g(3)}_{3} + g'(\xi)(x_n-3) \qquad \xi\in[x_n,3]$$

$$\frac{x_{n+1} - 3}{x_n - 3} = g'(\xi)$$

$$\left|\frac{x_{n+1} - 3}{x_n - 3}\right| = |g'(\xi)|$$

$$\lim_{n\to\infty} \frac{|x_{n+1} - 3|}{|x_n - 3|} = |g'(\xi)|$$

Since $x_{n+1} = g(x_n)$ converges, as $n \to \infty$, $x_n \to x^* = 3$.

For $\xi \in [x_n, 3]$, as $x_n \to 3$, $\xi \in [3,3] \Rightarrow \xi = 3$

$$\lim_{n\to\infty} \frac{|x_{n+1} - 3|}{|x_n - 3|} = |g'(3)| = \left|-\frac{3}{4}\right| = \frac{3}{4}$$

$\boxed{\text{Linear with convergence rate of } 3/4}$

5. All the roots of the scalar equation

$$x - 4\sin(2x) - 3 = 0,$$

are to be determined with at least 10 accurate digits[1].

(a) Plot $f(x) = x - 4\sin(2x) - 3$ (using your Python toolbox). All the zero crossings should be in the plot. How many are there?

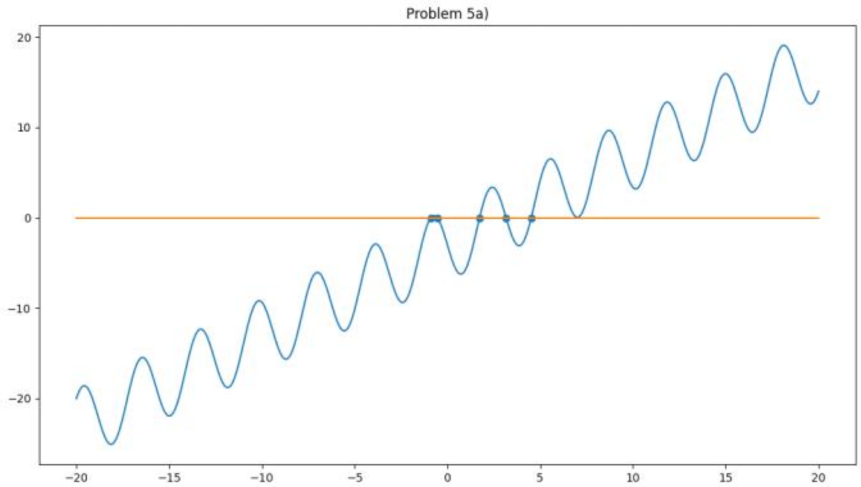
(b) Write a program or use the code from class to compute the roots using the fixed point iteration

$$x_{n+1} = -\sin(2x_n) + 5x_n/4 - 3/4.$$

Use a stopping criterium that gives an answer with ten correct digits. (*Hint: you may have to change the error used in determing the stopping criterion.*) Find, empirically which of the roots that can be found with the above iteration. Give a theoretical explanation.

DPM

5a) See hw3.py



There are 5 crossings

5b)

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Problem 5b)
For root near x = -0.898357
Iterations ran: 1000, Current approximation: -4.573077362541188e+96
ERROR: MAX ITERATIONS EXCEEDED

For root near x = -0.544442
Iterations ran: 18, Current approximation: -0.5444424006352793

For root near x = 1.73207
Iterations ran: 69, Current approximation: 3.1618264867077195

For root near x = 3.16183
Iterations ran: 71, Current approximation: 3.1618264867223598

Root at x~= 4.51779
Iterations ran: 1000, Current approximation: 1.740927240979749e+97
ERROR: MAX ITERATIONS EXCEEDED
```

Note that for $x_{n+1} = g(x_n) = -\sin(2x_n) + \frac{5x_n}{4} - \frac{3}{4}$,

$g'(x_n) = \frac{5}{4} - 2\cos(x_n)$

And, for values near our roots:

$g'(-0.89) \approx 1.66 > 1$, $g'(-0.54) \approx 0.3 < 1$, $g'(1.73) \approx 3.15 > 1$,

$g'(-0.89) \approx 1.66 > 1$, $g'(-0.54) \approx 0.3 < 1$, $g'(1.73) \approx 3.15 > 1$,

$g'(3.16) \approx -0.74 > -1$, $g'(4.51) \approx 3.08 > 1$

The roots that the algorithm can converge to satisfy $|g'(r)| < 1$, while the roots that don't converge don't satisfy that.