HW9

Thursday, November 2, 2023 9:40 PM

1. Find the least squares solution to the overdetermined linear system

$$\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} u \\ v \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}_{\vec{b}}$$

$$A\hat{x} = \vec{b}$$

$$A^T A \hat{x} = A^T \vec{b}$$

$$\hat{x} = (A^T A)^{-1} A^T \vec{b}$$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}, \quad A^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$A^T A = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}}_{2\times 3} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}}_{3\times 2} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

$$\left[\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1/2 \end{array}\right] \quad (A^T A)^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}$$

$$(A^T A)^{-1} A^T = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}}_{2\times 2} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}}_{2\times 3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 1/2 \end{bmatrix}$$

$$\hat{x} = (A^T A)^{-1} A^T \vec{b} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 1/2 \end{bmatrix}}_{2\times 3} \underbrace{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}_{3\times 1} = \boxed{\begin{bmatrix} 1 \\ 1/2 \end{bmatrix}}$$

2. Find the vector $\boldsymbol{x}$ that minimizes the quantity $E^2 = b_1^2 + 4b_2^2 + 25b_3^2 + 9b_4^2$, when it holds that

$$\begin{bmatrix} 1 & 3 \\ 6 & -1 \\ 4 & 0 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

Hint: When we solve a linear system of equations $A\boldsymbol{x} = \boldsymbol{b}$, multiplication from the left with a nonsingular matrix will leave the solution unchanged. This is **not** the case when finding the least squares solution to an overdetermined system. Exploit this and multiply the system above with a suitable diagonal matrix, so that the problems becomes a regular least squares problem (for which we can apply the normal equation approach.)

E^2 minimized when $b_1 = b_2 = b_3 = b_4 = 0$

$$\underbrace{\begin{bmatrix} 1 & 3 \\ 6 & -1 \\ 4 & 0 \\ 2 & 7 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}}_{\vec{b}}$$

$A\hat{x} = \vec{b}$

$A^T A \hat{x} = A^T \vec{b}$

$\hat{x} = (A^T A)^{-1} A^T \vec{b}$

$$A^TA = \begin{bmatrix} 1 & 6 & 4 & 2 \\ 3 & -1 & 0 & 7 \end{bmatrix}_{2\times 4} \begin{bmatrix} 1 & 3 \\ 6 & -1 \\ 4 & 0 \\ 2 & 7 \end{bmatrix}_{4\times 2} = \begin{bmatrix} 57 & 11 \\ 11 & 59 \end{bmatrix}$$

$$(A^TA)^{-1} = \begin{bmatrix} 59/3242 & -11/3242 \\ -11/3242 & 57/3242 \end{bmatrix}$$

$$(A^TA)^{-1}A^T = \begin{bmatrix} 13/1621 & 365/3242 & 118/1621 & 41/3242 \\ 80/1621 & -123/3242 & -22/1621 & 377/3242 \end{bmatrix}$$

$$\hat{x} = (A^TA)^{-1}A^T\vec{b} = \boxed{\begin{bmatrix} 814/1621 \\ 645/1621 \end{bmatrix}}$$

3. If a function is identically zero over an interval, all its derivatives must also be identically zero over the same interval. Based on this observation:

(a) Prove that $\{1, x, x^2, \ldots, x^n\}$ are linearly independent.

Suppose $\{1, x, x^2, \cdots, x^n\}$ are linearly dependent. Then, there exists $\{a_0, a_1, a_2, \cdots, a_n\}$ s.t.

$$a_0(1) + a_1x + a_2x^2 + \cdots + a_nx^n \equiv 0$$

with at least one $a_i \neq 0$. The above equation is identically equal to 0 and so has infinite roots. However, the above equation is an n-degree polynomial, which only has n roots.

Therefore, $\{1, x, \cdots, x^n\}$ must be linearly independent.

(b) Show that the function set

$$\{1, \cos(x), \cos(2x), \ldots, \cos(nx), \sin(x), \ldots, \sin(nx)\}$$

is linearly independent (also over any interval).

The inner product for functions on an interval $[a,b]$ is:

$$\langle f(x), g(x) \rangle = \int_a^b f(x)\, g(x)\, dx$$

On the interval $[-\pi, \pi]$:

$$\int_{-\pi}^{\pi} \sin(nx)\sin(mx)\, dx = \begin{cases} 0 : & n \neq m \\ \pi : & n = m \end{cases}$$

$$\int_{-\pi}^{\pi} \cos(nx)\sin(mx)\, dx = \begin{cases} 0 : & n \neq m \\ \pi : & n = m \neq 0 \\ 2\pi : & n = m = 0 \end{cases}$$

$$\int_{-\pi}^{\pi} \cos(nx)\sin(mx) = 0 \quad \forall n, \forall m$$

The set is orthogonal and therefore linearly independent.

4. Prove the three-term recursion formula for orthogonal polynomials:

$$\phi_k(x) = (x - b_k)\phi_{k-1}(x) - c_k\phi_{k-2}(x)$$

where

$$b_k = \frac{< x\phi_{k-1}, \phi_{k-1} >}{< \phi_{k-1}, \phi_{k-1} >} \quad c_k = \frac{< x\phi_{k-1}, \phi_{k-2} >}{< \phi_{k-2}, \phi_{k-2} >}$$

Hint: Since $\phi_k(x)$ is a polynomial of degree k and of the form $\phi_k = x^k + \{\text{lower order terms}\}$, we can clearly select b_k and c_k so that the right hand side (RHS) of (1) matches $\phi_k(x)$ for powers x^k , x^{k-1} and x^{k-2} . We have no obvious reason to expect that the two sides will match the other lower order terms. Hence, we would expect to need to include a lot more terms in the RHS to get the two sides to become equal:

$$\phi_k(x) = (x - b_k)\phi_{k-1}(x) - c_k\phi_{k-2}(x) - \{a_{k-3}\phi_{k-3}(x) + a_{k-4}\phi_{k-4}(x) + \cdots + a_0\phi_0(x)\} \quad (1)$$

We now need to show that all these a's are in fact are zero. To show that $a_j = 0$, $j \leq k-3$, we form the scalar product of (1) with $\phi_j(x)$ for $j = 0, \ldots, k-1$. You need to show that everything in (1) apart from $a_j < \phi_j, \phi_j >$ then vanishes, thereby showing that $a_j = 0$, $j \leq k-3$. After that, it remains to determine the values of b_k and c_k . These coefficients follow by again forming suitable scalar products.

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5. One of the many formulas for computing the Chebychev polynomials $T_n(x)$ is

$$T_n(x) = \frac{1}{2}\left(z^n + \frac{1}{z^n}\right), \quad (2)$$

where z is implicitly defined through x via $x = \frac{1}{2}\left(z + \frac{1}{z}\right)$. Confirm that the formula (2) indeed generates the same polynomials as the standard definition of the Chebychev polynomials. **Hint:** One way would be to verify that it produces the correct result for T_0 and T_1 and that it satisfies the 3 term recursion.

$$2x = z + z^{-1}$$

$$2xz = z^2 + 1$$

$$z^2 - 2xz + 1 = 0$$

$$z^2 - (2x)z = -1$$

$$z^2 - (2x)z + x^2 = x^2 - 1$$

$$(z - x)^2 = x^2 - 1$$

$$z - x = \pm\sqrt{x^2-1}$$

$$z = x \pm \sqrt{x^2-1}$$

$$T_n(x) = \frac{1}{2}\left(\left(x+\sqrt{x^2-1}\right)^n + \frac{1}{\left(x+\sqrt{x^2-1}\right)^n}\right)$$

$$T_0(x) = \frac{1}{2}(1+1) = 1 \checkmark$$

$$T_1(x) = \frac{1}{2}\left(x+\sqrt{x^2-1} + \frac{1}{x+\sqrt{x^2-1}}\right)$$

$$= \frac{1}{2}\left(x+\sqrt{x^2-1} + \frac{x-\sqrt{x^2-1}}{x^2-x^2+1}\right)$$

$$= \frac{1}{2}\left(2x\right) = x \checkmark$$

$$T_2(x) = \frac{1}{2}\left(\left(x+\sqrt{x^2-1}\right)^2 + \frac{1}{\left(x+\sqrt{x^2-1}\right)^2}\right)$$

$$= \frac{1}{2}\left(\left(x+\sqrt{x^2-1}\right)^2 + \frac{\left(x-\sqrt{x^2-1}\right)^2}{\left(x+\sqrt{x^3-1}\right)^2\left(x-\sqrt{x^2-1}\right)^2}\right)$$

$$= \frac{1}{2}\left(\left(x+\sqrt{x^2-1}\right)^2 + \left(x-\sqrt{x^2-1}\right)^2\right)$$

$$= \frac{1}{2}\left(x^2+2x\sqrt{x^2-1}+x^2-1+x^2-2x\sqrt{x^2-1}+x^2-1\right)$$

$$= \frac{1}{2}\left(x^2 + 2x\sqrt{x^2-1} + x^2 - 1 + x^2 - 2x\sqrt{x^2-1} + x^2 - 1 \right)$$

$$= \frac{1}{2}\left(4x^2 - 2 \right) = 2x^2 - 1 = 2x(x) - 1 = 2x\, T_1(x) - T_0(x)$$