Stabilization schemes of Unfitted FEM/DG methods

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December 31, 2025

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Introduction

Model problem

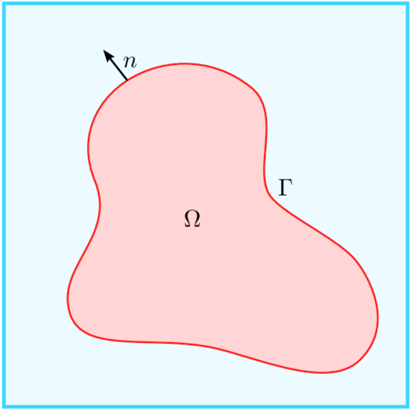


Figure: Computational domain for the boundary value problem.

We consider a model Poisson problem posed on an embedded domain:

$$
\begin{aligned}
-\Delta u &= f \qquad \text{in } \Omega, \\
u &= g \qquad \text{on } \Gamma.
\end{aligned}
$$

Objectives:

- Discretize the problem on an unfitted background mesh.
- Investigate stability and conditioning issues induced by cut cells.

Nitsche's method

We discretize the unfitted problem using Nitsche's method: find $u_h \in V_h$ such that

$$a_h(u_h, v) = l_h(v) \quad \text{for all } v \in V_h,$$

with the following bilinear forms:

- **FEM:**

$$a_h(v, w) = (\nabla v, \nabla w)_\Omega - (n \cdot \nabla v, w)_\Gamma - (v, n \cdot \nabla w)_\Gamma + \beta h^{-1}(v, w)_\Gamma.$$

- **DG:**

$$\begin{aligned} a_h(v, w) =& (\nabla v, \nabla w)_\Omega - (n \cdot \nabla v, w)_\Gamma - (v, n \cdot \nabla w)_\Gamma + \beta h^{-1}(v, w)_\Gamma \\ & - (\{n \cdot \nabla v\}, [w])_{\mathcal{E}_h} - ([v], \{n \cdot \nabla w\})_{\mathcal{E}_h} + \beta h^{-1}([v], [w])_{\mathcal{E}_h}. \end{aligned}$$

$\Rightarrow$ On unfitted meshes, these formulations may suffer from instability due to the terms defined on Γ.

Issues in the proof of coercivity

A key step in the stability analysis is to control the boundary term

$$(n \cdot \nabla v, w)_\Gamma,$$

which typically relies on a trace inequality of the form

$$\|\nabla v \cdot n\|_e \leq C_t |e|^{1/2} |T|^{-1/2} \|\nabla v\|_T, \quad \forall v \in \mathbb{P}_k(T),\ \forall e \subset \partial T.$$

On *shape-regular* fitted meshes ($\frac{h_T}{\rho_T} \leq C$), this yields the standard estimate

$$\|\nabla v \cdot n\|_e \leq \widetilde{C}_t h_T^{-1/2} \|\nabla v\|_T,$$

with constants independent of the mesh size.

Issues in the proof of coercivity

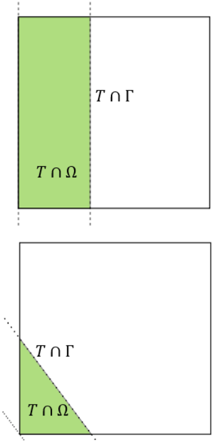


Figure: Small cut elements.

Key difficulty: For a cut element T, the ratio

$$\frac{|e|}{|T|} \to \frac{|T \cap \Gamma|_{d-1}}{|T \cap \Omega|_d}$$

is no longer uniformly bounded by Ch_T^{-1}.

This is because

- $|T \cap \Omega|_d$ can be arbitrarily small,
- while $|T \cap \Gamma|_{d-1} \sim h_T$.

$\Rightarrow$ The constant in the trace inequality strongly depends on the cut configuration, leading to loss of coercivity.

Issues about the condition number

Consider the **mass matrix** $M_{ij} = \int_\Omega \phi_i \phi_j \, dx$.

Using the Rayleigh quotient,

$$\lambda_{\min}(M) = \min_{v \neq 0} \frac{v^T M v}{v^T v}.$$

Let $v_h = \phi_i$ be a basis function supported on a cut cell T with $\eta_T = |T \cap \Omega|/|T| \ll 1$. Then

$$v^T M v = \|\phi_i\|^2_{L^2(T\cap\Omega)} \sim \eta_T h^d, \qquad v^T v = 1.$$

Hence $\lambda_{\min}(M) \lesssim \eta_{\min} h^d$, while $\lambda_{\max}(M) \sim h^d$, and therefore

$$\kappa(M) \sim \eta_{\min}^{-1}.$$

As a result, the discrete problem becomes ill-conditioned when $\eta_{\min} \to 0$.

Stabilization strategies for unfitted methods

- **Underlying issue**
  - Small cut elements may have arbitrarily small physical measure
  - Loss of discrete coercivity and deterioration of the condition number

- **Stabilization approaches**
  - **Ghost penalty:** introduce penalty terms on inter-element faces to recover uniform stability by weakly coupling basis functions across cut elements
  - **Agglomeration:** modify the discrete space by constraining outer degrees of freedom through extrapolation from interior ones, effectively forming larger aggregates

Ghost penalty type

Basic ideas

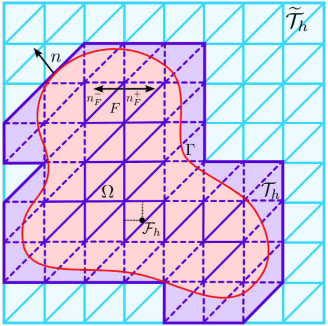


Figure: Background mesh and active mesh.

- $\tilde{\mathcal{T}}_h$: a quasi-uniform **background mesh** consisting of d-dimensional shape-regular simplices $\{T\}$ covering $\bar{\Omega}$.
- $\mathcal{T}_h$: the active mesh consisting of those elements $T \in \tilde{\mathcal{T}}_h$ which intersect the interior of Ω.
- $\mathcal{F}_h$: the set of interior faces in the active mesh.
- V_h: the discrete function space defined as the broken polynomial space of order k on the active mesh, $V_h := \mathbb{P}_k(\mathcal{T}_h) := \bigoplus_{T \in \mathcal{T}_h} \mathbb{P}_k(T)$.

Motivation: trace inequalities on cut elements

- Recall that the following trace inequality may fail on small cut elements, since the physical subdomain $T \cap \Omega$ can have arbitrarily small measure:

$$\|\nabla v \cdot n\|_{e\cap\Omega} \leq \widetilde{C}_t h_T^{-1/2} \|\nabla v\|_{T\cap\Omega}.$$

- Remedy: extend the estimate to the full background element T.

$$\|\nabla v \cdot n\|_{e\cap\Omega} \leq \|\nabla v \cdot n\|_{e} \leq C_1 h_T^{-1/2} \|\nabla v\|_{T}.$$

$$\|\nabla v \cdot n\|_{\Gamma\cap\Omega} \leq C_2 h_T^{-1/2} \|\nabla v\|_{T}.$$

Motivation

Assumption EP1

The ghost penalty g_h extends the H^1 semi-norm from the physical domain Ω to the entire active mesh $\mathcal{T}_h$ in the sense that

$$\|\nabla v\|_{\mathcal{T}_h}^2 \leq C_g(\|\nabla v\|_\Omega^2 + g_h(v, v))$$

holds for $v \in V_h$, with the hidden constants depending only on the dimension d, the polynomial order k and the shape-regularity of $\mathcal{T}_h$.

As a consequence of standard trace inequalities on the active mesh and Assumption EP1, it follows that for all $v \in V_h$,

$$\|h^{1/2}\nabla v \cdot n\|_\Gamma^2 + \|h^{1/2}\nabla v \cdot n\|_{\mathcal{F}_h \cap \Omega}^2 \leq C_g(C_1 + C_2)(\|\nabla v\|_\Omega^2 + g_h(v, v)).$$

Numerical scheme of cutDG

Under the assumption EP1, we can define the cutDG scheme as follows: find $u_h \in V_h$ such that

$$A_h(u_h, v) := a_h(u_h, v) + g_h(u_h, v) = l_h(v).$$

And we can define the following discrete norms:

$$\begin{aligned} |||v|||_{a_h}^2 &= \|\nabla v\|_\Omega^2 + \|h^{-1/2}[v]\|_{\mathcal{F}_h \cap \Omega}^2, \\ |v|_{g_h}^2 &= g_h(v, v), \\ |||v|||_{A_h}^2 &= |||v\,|||_{a_h}^2 + |v|_{g_h}^2. \end{aligned}$$

The following stability result holds.

Theorem

The discrete form A_h is coercive and stable with respect to the discrete energy norm $|||\cdot|||_{A_h}$.

A priori error analysis

A key subtlety of unfitted methods is the mismatch between the continuous and discrete domains:

- The exact solution u is defined only on the physical domain Ω.
- The discrete space V_h is defined on the active mesh $\mathcal{T}_h$.

As a consequence, the error

$$e = u - u_h \in H^2(\mathcal{T}_h) + V_h$$

is not confined to Ω only, and standard interpolation arguments cannot be applied directly.

We therefore introduce the following strengthened norm:

$$|||v|||_{a_h,*}^2 = |||v|||_{a_h}^2 + \|h^{1/2}\{\nabla v \cdot n\}\|_{\mathcal{F}_h \cap \Omega}^2 + \|h^{1/2} \nabla v \cdot n\|_{\Gamma}^2.$$

Key idea: extend the exact solution to the active mesh and construct a suitable unfitted approximation operator.

Unfitted approximation operator π_h^e

To construct a suitable approximation on the active mesh, we proceed in two steps.

- **Extension operator.** There exists a bounded extension

$$(\cdot)^e : H^s(\Omega) \to H^s(\Omega^e), \qquad \|u^e\|_{s,\Omega^e} \lesssim \|u\|_{s,\Omega},$$

  where $\Omega_h^e = \bigcup_{T\in\mathcal{T}_h} T \subset \Omega^e$.

- **Standard L^2 projection.** Let $\pi_h : L^2(\mathcal{T}_h) \to V_h$ be the elementwise L^2-orthogonal projection, which satisfies the local approximation property

$$|v - \pi_h v|_{T,r} \lesssim h_T^{s-r}|v|_{s,T}, \quad 0 \le r \le s, \ \forall v \in H^s(T).$$

Combining the two operators, we define the **unfitted projection** $\pi_h^e : H^s(\Omega) \to V_h$ by setting

$$\boxed{\pi_h^e u := \pi_h u^e.}$$

Approximation properties of π_h^e

Combining the local approximation properties of π_h with the stability of the extension operator $(\cdot)^e$, $\pi_{h,e}$ satisfies the global error estimates:

$$
\begin{aligned}
\|v - \pi_h^e v\|_{\mathcal{T}_h,r} &\lesssim h^{s-r}\|v\|_{s,\Omega}, && 0 \leq r \leq s, \\
\|v - \pi_h^e v\|_{\mathcal{F}_h,r} &\lesssim h^{s-r-1/2}\|v\|_{s,\Omega}, && 0 \leq r \leq s - 1/2, \\
\|v - \pi_h^e v\|_{\Gamma,r} &\lesssim h^{s-r-1/2}\|v\|_{s,\Omega}, && 0 \leq r \leq s - 1/2.
\end{aligned}
$$

As a direct consequence, we can easily estimate the approximation error in the $\|\cdot\|_{a_h,*}$-norm as follows.

Corollary

Let $u \in H^s(\Omega)$ and assume that $V_h = P^k(\mathcal{T}_h)$. Then for $r = \min\{s, k+1\}$, the approximation error of π_h^e satisfies

$$\|u - \pi_h^e u\|_{a_h,*} \lesssim h^{r-1}\|u\|_{r,\Omega}.$$

Consistency

The addition of the stabilization term g_h leads to a **perturbed Galerkin orthogonality**.

Lemma

Let u be the exact solution and u_h the discrete solution. Then

$$a_h(u - u_h, v) = g_h(u_h, v), \quad \forall v \in V_h.$$

As a consequence, the error analysis hinges on controlling the additional consistency error introduced by g_h.

Assumption EP2

For $v \in H^s(\Omega)$ and $r = \min\{s, k+1\}$, the semi-norm $|\cdot|_{g_h}$ satisfies the estimate

$$|\pi_h^e v|_{g_h} \lesssim h^{r-1} \|v\|_{r,\Omega}.$$

A priori error estimates

With the above ingredients, we can derive the final a priori error bounds.

Theorem

Let $u \in H^s(\Omega)$ with $s \geq 2$ be the exact solution, and let $u_h \in \mathbb{P}_k(\mathcal{T}_h)$ be the discrete solution. Then, for $r = \min\{s, k+1\}$, it holds that

$$|||u - u_h|||_{a_h,*} \lesssim h^{r-1} \|u\|_{r,\Omega},$$

and

$$\|u - u_h\|_{\Omega} \lesssim h^{r} \|u\|_{r,\Omega}.$$

Condition number estimates: main ingredients

A bound for the condition number $\kappa(A)$ is obtained by combining three standard ingredients:

1. **Norm equivalence (algebraic vs. functional):**

$$h^{d/2}\|V\|_{\mathbb{R}^N} \lesssim \|v_h\|_{L^2(\mathcal{T}_h)} \lesssim h^{d/2}\|V\|_{\mathbb{R}^N},$$

   valid for quasi-uniform meshes, which allows switching between coefficient vectors and FE functions.

2. **Discrete Poincaré-type inequality:** controls the L^2 norm by the discrete energy norm

$$\|v_h\|_{L^2(\mathcal{T}_h)} \lesssim |||v_h|||_{A_h} .$$

3. **Inverse inequality:** bounds the discrete energy norm in terms of the L^2 norm

$$|||v_h|||_{A_h} \lesssim h^{-1}\|v_h\|_{L^2(\mathcal{T}_h)}.$$

Condition number estimates: role of ghost penalty

To ensure the required discrete inequalities uniformly with respect to cut configurations, the ghost penalty must satisfy the following assumptions.

Assumption EP3

The ghost penalty extends control from the physical domain to the active mesh:

$$\|v\|_{L^2(\mathcal{T}_h)}^2 \lesssim \|v\|_{L^2(\Omega)}^2 + |v|_{g_h}^2, \qquad \forall v \in V_h.$$

Assumption EP4

The ghost penalty semi-norm is bounded by the L^2 norm:

$$|v|_{g_h} \lesssim h^{-1}\|v\|_{L^2(\mathcal{T}_h)}, \qquad \forall v \in V_h.$$

These assumptions restore the discrete Poincaré and inverse inequalities uniformly with respect to small cut cells.

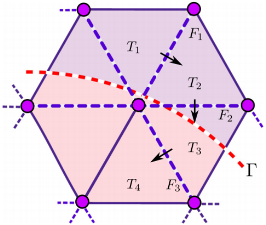
Face-based ghost penalties: motivation

The key idea of face-based ghost penalties is that **local L^2 control on one element can be propagated to neighboring elements** by penalizing jumps of normal derivatives across interior faces.

Lemma

Let $T_1, T_2 \in \mathcal{T}_h$ be two elements sharing a common face F. Then for $v_h \in V_h$ the inequality

$$\|v\|_{T_1}^2 \lesssim \|v\|_{T_2}^2 + \sum_{0 \leq j \leq k} h^{2j+1} \left([\partial_n^j v], [\partial_n^j v] \right)_F.$$



Face-based ghost penalties: definition

We define the set of **ghost penalty faces**

$$\mathcal{F}_h^g = \{F \in \mathcal{F}_h : T^+ \cap \Gamma \neq \emptyset \ \vee \ T^- \cap \Gamma \neq \emptyset\},$$

i.e., interior faces whose neighboring elements are cut by the boundary Γ.

On these faces, we introduce the face-based ghost penalty

$$g_h^1(v, w) := \sum_{j=0}^{k} \sum_{F \in \mathcal{F}_h^g} \gamma_j \, h_F^{2j-1} \big([\partial_n^j v], \, [\partial_n^j w]\big)_F.$$

- The penalty g_h^1 provides control across cut elements and can be shown to satisfy **Assumptions EP1 - EP4**.
- For piecewise linear elements $v \in \mathbb{P}_1(\mathcal{T}_h)$, penalizing the full gradient jump does not satisfy **Assumptions EP3**:

$$\widetilde{g}_h^1(v, w) = \sum_{F \in \mathcal{F}_h^g} h_F \big([\nabla v], [\nabla w]\big)_F.$$

Experiment 1: Convergence rate studies

The analytical reference solution given by

$$u(x, y) = \cos(2\pi x)\cos(2\pi y)\sin(2\pi x)\sin(2\pi y).$$

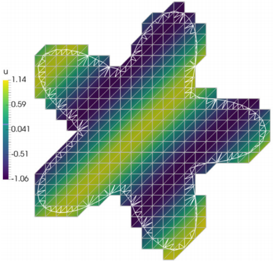


Figure: Solution plot for the two-dimensional convergence study.

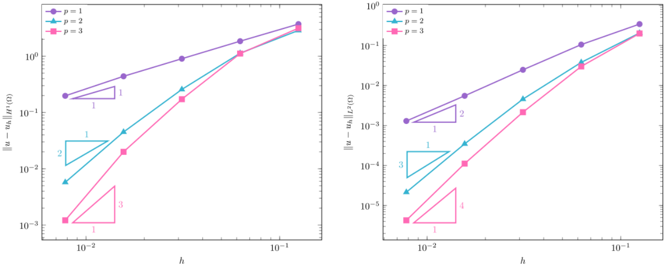


Figure: Convergence rates for the two-dimensional test case using different approximation orders.

Experiment 1: Convergence rate studies

For the three-dimensional case, the analytic reference solution is chosen as

$$u(x, y, z) = e^{x+y+z} \cos(x + y + z) \sin(x + y + z).$$

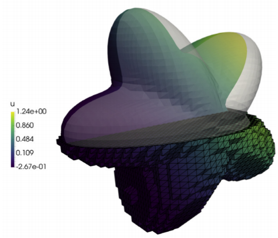


Figure: Solution plot for the three-dimensional convergence study.

N_k	$\|e_k^1\|_{H_1(\Omega)}$	EOC	$\|e_k^1\|_{L^2(\Omega)}$	EOC
6	$4.53 \cdot 10^{-1}$	–	$6.33 \cdot 10^{-2}$	–
12	$2.46 \cdot 10^{-1}$	0.88	$1.88 \cdot 10^{-2}$	1.75
24	$1.26 \cdot 10^{-1}$	0.97	$5.04 \cdot 10^{-3}$	1.90
48	$6.35 \cdot 10^{-2}$	0.99	$1.28 \cdot 10^{-3}$	1.98
96	$3.18 \cdot 10^{-2}$	1.00	$3.14 \cdot 10^{-4}$	2.03

Figure: Convergence rates for three-dimensional test case using $\mathbb{P}_1(\mathcal{T}_h)$.

Experiment 2: Condition number studies

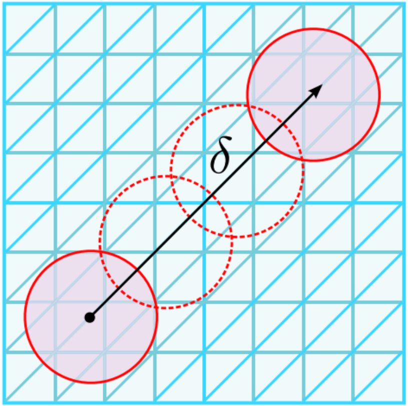


Figure: Experimental set-up.

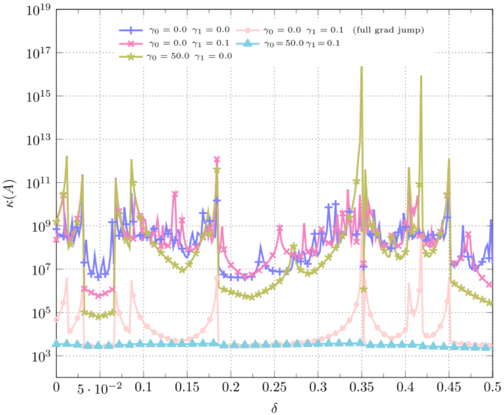


Figure: Condition number analysis with and without ghost penalty stabilization.

Experiment 2: Condition number studies

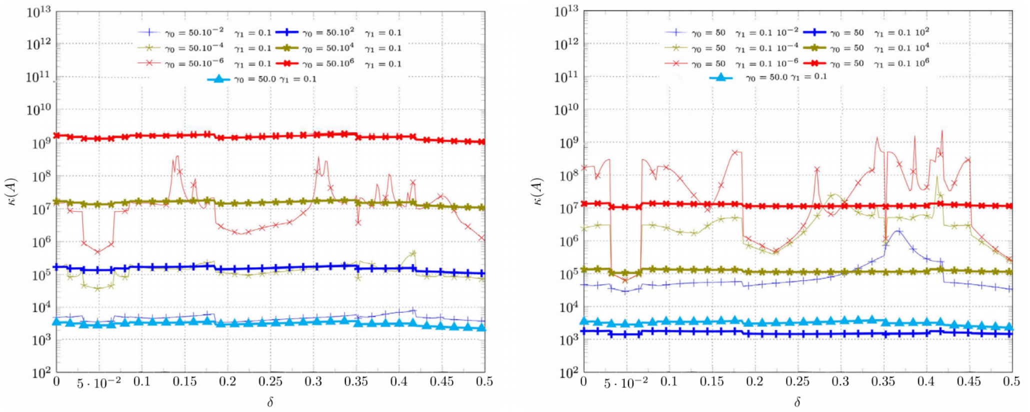
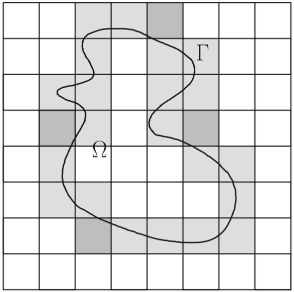


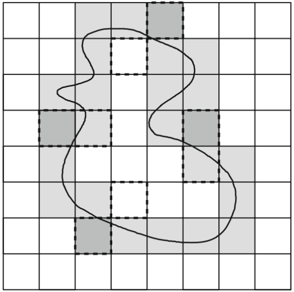
Figure: Condition numbers computed for varying ghost penalty parameter γ_0 and γ_1.

Agglomeration type

Basic ideas for aggregated DG

Goal: Eliminate the adverse effects of arbitrarily small cut elements by merging them into geometrically well-behaved aggregates.





- $\mathcal{K}_{\text{cube}}$: a structured background mesh with uniform mesh size h.
- $\mathcal{K} = \{K \cap \Omega : K \in \mathcal{K}_{\text{cube}},\ K \cap \Omega \neq \emptyset\}$, the mesh induced on the physical domain Ω.

Large and small elements

Let S_δ denote a cube with side length δh, $0 < \delta \leq 1$.

- An element $K \in \mathcal{K}$ is called **large** if there exists a subcube $S_\delta \subset K$.
- Otherwise, K is called **small**.

This classification induces the decomposition

$$\mathcal{K}_{\text{large}} = \{K \in \mathcal{K} : \exists S_\delta \subset K\},$$
$$\mathcal{K}_{\text{small}} = \mathcal{K} \setminus \mathcal{K}_{\text{large}}.$$

Remark: Uncut elements are always large elements; hence, small elements only appear near the boundary Γ.

Extended mesh

For each small element $K \in \mathcal{K}_{\text{small}}$, we associate one neighboring large element $\mathcal{N}(K) \in \mathcal{K}_{\text{large}}$.

- Each small element is merged with its associated large neighbor to form an **extended element** $K \cup \mathcal{N}(K)$.
- Large elements that are not used in any merging remain unchanged.

The resulting extended mesh is denoted by $\mathcal{K}_{\text{ext}}$ as follows:

$$\mathcal{K}_{\text{ext}} = \mathcal{M} \cup \{K \cup \mathcal{N}(K) : K \in \mathcal{K}_{small}\},$$

where

$$\mathcal{M} = \mathcal{K}_{large} \backslash \{\mathcal{N}(K) : K \in \mathcal{K}_{small}\}.$$

Discrete spaces on $\mathcal{K}_{\text{ext}}$

We define the space of discontinuous piecewise polynomials on the extended mesh $\mathcal{K}_{\text{ext}}$ by

$$\mathcal{V}_h = \bigoplus_{K \in \mathcal{K}_{\text{ext}}} \mathbb{P}_k(K),$$

where $\mathbb{P}_k(K)$ denotes the space of tensor-product polynomials of degree k defined on an (extended) element K.

Remark:

- Since DG method does not impose inter-element continuity, the discrete space can be defined on the extended mesh in a natural way.
- Agglomeration restores stability and a uniformly bounded condition number by eliminating arbitrarily small cut elements and yielding aggregates of uniformly bounded size, on which standard inverse and Poincaré inequalities hold uniformly.

Restored inverse estimates on extended elements

Lemma

There exist constants $C_1, C_2 > 0$, depending only on the aggregation parameter δ, such that for all $K \in \mathcal{K}_{ext}$ and all $v \in \mathbb{P}_k(K)$,

$$\|v\|_{\partial K} \leq C_1 h^{-1/2} \|v\|_K, \qquad \|\nabla v\|_K \leq C_2 h^{-1} \|v\|_K.$$

Proof strategy:

- Embed each extended element into a uniformly bounded macro cube.
- Transfer the estimates to reference domains.
- Use finite dimensional norm equivalence and scale back.

Restored inverse estimates: geometric embedding

- Since extended elements may be geometrically irregular, we embed each $K \in \mathcal{K}_{\text{ext}}$ into a macro cube

$$S_\Delta = \{x \in \mathbb{R}^d : x = \alpha(y - y_c) + y_c,\ y \in S_\delta\},$$

  where α is chosen such that $K \subset S_\Delta$.
- The size of S_Δ is uniformly bounded and depends only on δ.
- This allows us to work on reference cubes and avoid degeneration due to small cut elements.

Restored inverse estimates

- Using the Lipschitz regularity of Γ, we first estimate

$$h\|v\|^2_{K\cap\Gamma} \lesssim h^d\|v\|^2_{L^\infty(K\cap\Gamma)} \lesssim h^d\|v\|^2_{L^\infty(K)} \lesssim h^d\|v\|^2_{L^\infty(S_\Delta)}.$$

- Mapping to the reference macro cube $\widehat{S}_\Delta$ yields

$$h^d\|v\|^2_{L^\infty(S_\Delta)} = h^d\|\hat{v}\|^2_{L^\infty(\widehat{S}_\Delta)} \lesssim h^d\|\hat{v}\|^2_{L^2(\widehat{S}_\delta)},$$

  where the inequality follows from finite dimensional norm equivalence.
- Mapping back and using $S_\delta \subset K$, we obtain

$$h^d\|\widehat{v}\|^2_{L^2(\widehat{S}_\delta)} = \|v\|^2_{L^2(S_\delta)} \leq \|v\|^2_K.$$

- The gradient estimate follows by the same argument, combined with scaling of derivatives.

Key ideas behind aggregation stability

- **The role of S_δ:** S_δ is not only used to classify elements as large or small, but also provides a *uniform lower bound on the measure* of all admissible subregions. This ensures the validity of the estimate

$$\|v\|_{L^\infty(\widehat{S}_\Delta)} \lesssim \|v\|_{L^2(\widehat{S}_\delta)},$$

  since vanishing on $\widehat{S}_\delta$ implies vanishing on $\widehat{S}_\Delta$ for finite-dimensional polynomial spaces.

- **Why aggregation restores stability:** Aggregation does not modify the discrete space, but guarantees that each extended element contains a uniformly non-degenerate subregion. This restores uniform inverse and trace inequalities via finite-dimensional norm equivalence, with constants independent of the cut configuration.

From Aggregated DG to Aggregated FEM

Aggregated DG

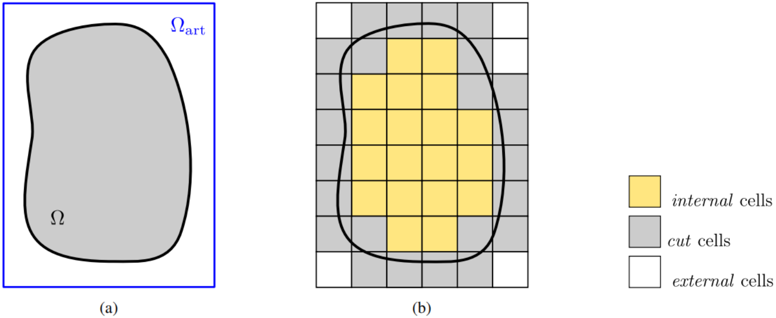
- Stability is recovered by merging small cut cells into larger aggregates.
- The resulting spaces are defined on non-standard elements.
- Naturally suited for DG/FVM, but not for conforming FEM.

Aggregated FEM

- Preserve the stability of aggregation within a conforming FE framework.
- Keep the background mesh unchanged.
- Control instability by constraining DOFs on cut cells using interior cells.

Key idea: Aggregated FEM acts as an extension from interior DOFs to the active mesh.

Basic Ideas of Cell Aggregation



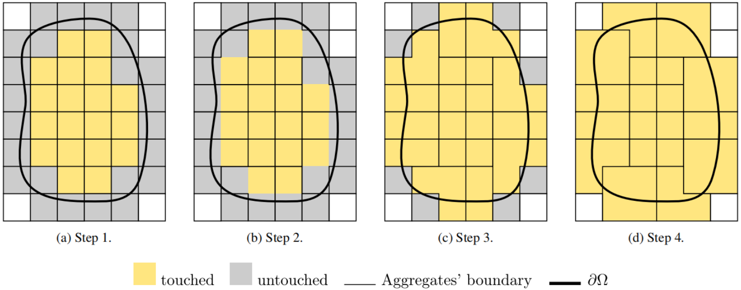
- $\mathcal{T}_h^{in}$: set of interior cells
- $\mathcal{T}_h^{cut}$: set of cut cells
- $\mathcal{T}_h^{act} := \mathcal{T}_h^{in} \cup \mathcal{T}_h^{cut}$: set of active cells

Construction of $\mathcal{T}_h^{agg}$

Cell Aggregation Scheme

1. Mark all interior cells as *touched* and all cut cells as *untouched*.
2. For each untouched cut cell, if there exists at least one touched cell sharing a facet F such that $F \cap \Omega \neq \emptyset$, aggregate the cut cell to the aggregate containing the closest interior cell (measured by the Euclidean distance between cell barycenters).
3. If multiple touched cells satisfy the above condition, one is selected arbitrarily, e.g., the one with the smallest global index.
4. Mark all newly aggregated cells as touched.
5. Repeat Steps 2–4 until all cells are aggregated.

Properties of the Aggregation



(a) Step 1. (b) Step 2. (c) Step 3. (d) Step 4.

Remarks:

- Each aggregate contains exactly one interior cell, referred to as the *root cell* of the aggregate.
- Every cut cell belongs to one and only one aggregate.
- For $\forall K \in \mathcal{T}_h^{act}$, its root cell is defined as the interior cell of the aggregate to which K belongs.
- A one-to-one correspondence between aggregates $A_K \in \mathcal{T}_h^{agg}$ and interior root cells $K \in \mathcal{T}_h^{in}$ exists. Consequently, the same index can be used to denote an aggregate and its root cell.

Aggregated Lagrangian FE spaces

Goal: Construct a FE space by constraining ill-posed DOFs using cell aggregates.

Step 1: FE spaces on active and interior cells

- *Active FE space* (defined on all active cells):

$$V_h^{\text{act}} \doteq \left\{ v \in \mathcal{C}^0(\Omega_{\text{act}}) : v|_K \in V(K), \ \forall K \in \mathcal{T}_h^{\text{act}} \right\}.$$

- *Interior FE space* (defined only on interior cells):

$$V_h^{\text{in}} \doteq \left\{ v \in \mathcal{C}^0(\Omega_{\text{in}}) : v|_K \in V(K), \ \forall K \in \mathcal{T}_h^{\text{in}} \right\}.$$

Step 2: Classification of FE nodes

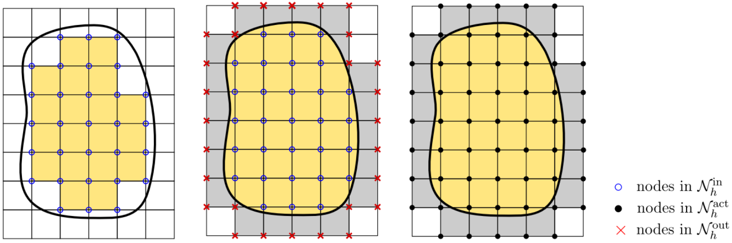
Let $\mathcal{N}_h^{\text{act}}$ and $\mathcal{N}_h^{\text{in}}$ denote the sets of FE nodes associated with V_h^{act} and V_h^{in}, respectively. We define the set of *outer nodes* as

$$\mathcal{N}_h^{\text{out}} \doteq \mathcal{N}_h^{\text{act}} \setminus \mathcal{N}_h^{\text{in}}.$$

Why outer nodes are problematic

- **Outer nodes:** supported only on cut cells with arbitrarily small intersection with Ω, which leads to loss of uniform inverse inequalities and ill-conditioned system matrices.
- **Interior nodes:** well posed but do not define a FE space on the whole physical domain Ω.

Key idea: stabilize the space by constraining outer-node DOFs using information from interior (root) cells.



(a) V_h^{in}. (b) V_h^{agg}. (c) V_h^{act}.

Mapping DOFs to Root Cells

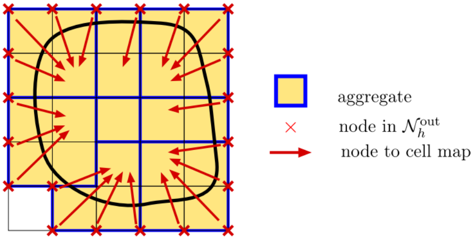


Figure: Mapping from outer nodes to interior (root) cells.

We can define a map from DOFs (nodes) to root cells as follows:

1. For each node, define its *owner VEF* (vertex, edge, or face) as the lowest-dimensional VEF that contains it.
2. For each VEF, select one of the active cells touching it as its *cell owner*.
3. Each active cell belongs to exactly one aggregate, which has a unique root (interior) cell.
4. Therefore, each DOF is mapped to the root cell of the aggregate containing its cell owner.

Extension Operator: Local Construction

- **Nodal Vector Spaces:**

$$V_h^{\text{in}} \leftrightarrow \{\phi^b : b \in \mathcal{N}_h^{\text{in}}\} \leftrightarrow \underline{\mathbf{u}}^{\text{in}} \in \mathbb{R}^{|\mathcal{N}_h^{\text{in}}|},$$
$$V_h^{\text{act}} \leftrightarrow \{\phi^b : b \in \mathcal{N}_h^{\text{act}}\} \leftrightarrow \underline{\mathbf{u}}^{\text{act}} \in \mathbb{R}^{|\mathcal{N}_h^{\text{act}}|}.$$

- **Structure of Active Dof Vector:** The active nodes are partitioned into interior and outer nodes:

$$\underline{\mathbf{u}}^{\text{act}} = \begin{bmatrix} \underline{\mathbf{u}}^{\text{in}} \\ \underline{\mathbf{u}}^{\text{out}} \end{bmatrix}, \quad \text{where } \underline{\mathbf{u}}^{\text{out}} \in \mathbb{R}^{|\mathcal{N}_h^{\text{out}}|}$$

- For each $b \in \mathcal{N}_h^{\text{out}}$, let $K(b)$ be its associated interior cell.

Constraint

Given $u_h \in V_h^{\text{in}}$ and the corresponding nodal values $\underline{\mathbf{u}}^{\text{in}}$, we compute the outer nodal values as follows:

$$\underline{\mathbf{u}}_b^{\text{out}} = \sum_{a \in \mathcal{N}(K(b))} \phi^a(x_b)\underline{\mathbf{u}}_a^{\text{in}}, \quad \text{for } b \in \mathcal{N}_h^{\text{out}}.$$

Construction of Aggregated Space V_h^{agg}

1. Global Matrix Representation

- The extension operator can be written compactly as: $\underline{\mathbf{u}}^{\text{out}} = \mathbf{C}\underline{\mathbf{u}}^{\text{in}}$, where $\mathbf{C}$ is the global matrix of constraints.
- We define the **Global Extension Matrix E** : $\mathbb{R}^{|\mathcal{N}_h^{\text{in}}|} \to \mathbb{R}^{|\mathcal{N}_h^{\text{act}}|}$ as:

$$\mathbf{E}\underline{\mathbf{u}}^{\text{in}} = \begin{bmatrix} \mathbf{I} \\ \mathbf{C} \end{bmatrix} \underline{\mathbf{u}}^{\text{in}} = \left[\underline{\mathbf{u}}^{\text{in}}, \mathbf{C}\underline{\mathbf{u}}^{\text{in}}\right]^T$$

2. The Extension Operator and Space

- Let $\mathcal{E} : V_h^{\text{in}} \to V_h^{\text{act}}$ be the operator such that $\mathcal{E}(u_h)$ has nodal values $\mathbf{E}\underline{\mathbf{u}}^{\text{in}}$.
- The **Aggregated FE Space** is defined as the range of $\mathcal{E}$:

$$V_h^{\text{agg}} \doteq \text{range}(\mathcal{E}(V_h^{\text{in}})) \subset V_h^{\text{act}}$$

- Let $\mathcal{C}(a)$ represents the set of outer nodes in $\mathcal{N}_h^{\text{out}}$ that are constrained by a, then the basis for V_h^{agg}:

$$\mathcal{E}(\phi^a) = \phi^a + \sum_{b \in \mathcal{C}(a)} C_{ba}\phi^b, \quad \text{for } a \in \mathcal{N}_h^{\text{in}}.$$

Construction of the aggregated basis $V_h^{\mathbf{agg}}$

We start from the standard finite element representation on the active mesh:

$$u(x) = \sum_{i \in \mathcal{N}_h^{\text{in}}} u_i \phi^i + \sum_{i \in \mathcal{N}_h^{\text{out}}} u_i \phi^i.$$

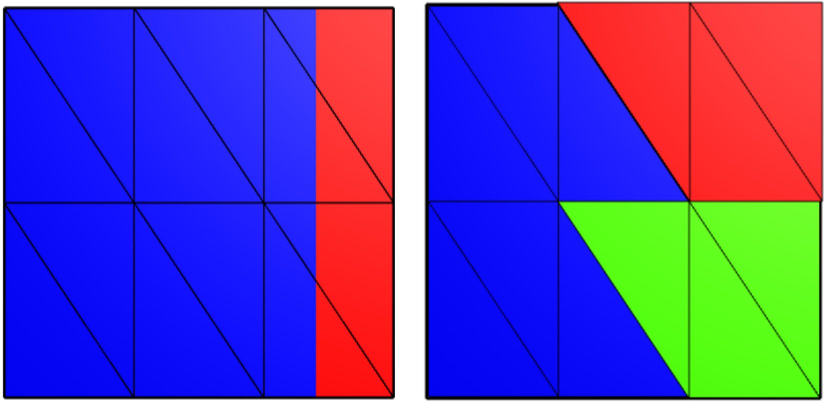
In the aggregated space V_h^{agg}, the degrees of freedom associated with outer nodes are constrained by interior ones:

$$u_i = \sum_{j \in \mathcal{N}_h^{\text{in}}} C_{ij} u_j, \quad \forall i \in \mathcal{N}_h^{\text{out}}.$$

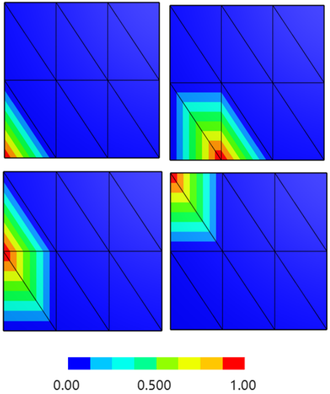
Substituting the constraints into the expansion, we obtain

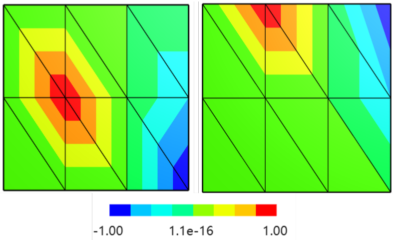
$$\begin{aligned} u(x) &= \sum_{i \in \mathcal{N}_h^{\text{in}}} u_i \phi^i + \sum_{i \in \mathcal{N}_h^{\text{out}}} \sum_{j \in \mathcal{N}_h^{\text{in}}} C_{ij} u_j \phi^i \\ &= \sum_{j \in \mathcal{N}_h^{\text{in}}} u_j \left(\phi^j + \sum_{i \in \mathcal{N}_h^{\text{out}}} C_{ij} \phi^i \right). \end{aligned}$$

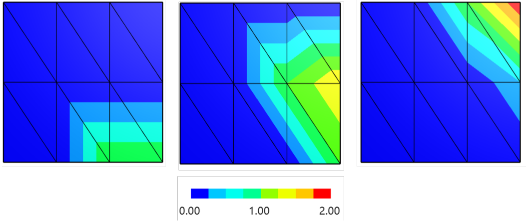
An example



An example







Stability of the Coordinate Vector Extension

Cell-wise extension operator: For each aggregate A_K with root cell $K \in \mathcal{T}_h^{\text{in}}$, we define

$$\mathbf{E}_{A_K}\underline{\mathbf{u}} = [\underline{\mathbf{u}}_K, \mathbf{C}_{\hat{A}_K}\underline{\mathbf{u}}_K],$$

where $\mathbf{C}_{\hat{A}_K}$ is the cell-wise constraint matrix such that

$$\mathbf{C}\underline{\mathbf{u}} \cdot \mathbf{C}\underline{\mathbf{u}} = \sum_{K \in \mathcal{T}_h^{\text{in}}} \mathbf{C}_{\hat{A}_K}\underline{\mathbf{u}}_K \cdot \mathbf{C}_{\hat{A}_K}\underline{\mathbf{u}}_K.$$

Lemma (Stability of the extension)

The cell-wise and global extension operators satisfy

$$1 \leq \|\mathbf{E}_{A_K}\|_2^2 \leq 1 + \|\mathbf{C}_{\hat{A}_K}\|_2^2,$$

and

$$1 \leq \|\mathbf{E}\|_2^2 \leq 1 + \|\mathbf{C}\|_2^2 \leq C_{\mathrm{e}},$$

where C_{e} is a constant independent of the cut configuration and mesh size.

Stability of the Coordinate Vector Extension

Proof idea (sketch):

- By construction, the extension operator has the form

$$\|\mathbf{E}\underline{\mathbf{u}}\|_2^2 = \|\underline{\mathbf{u}}\|_2^2 + \|\mathbf{C}\underline{\mathbf{u}}\|_2^2, \qquad \|\mathbf{E}_{A_K}\underline{\mathbf{u}}\|_2^2 = \|\underline{\mathbf{u}}\|_2^2 + \|\mathbf{C}_{\hat{A}_K}\underline{\mathbf{u}}\|_2^2.$$

  This proves the first result.
- Since constraints are defined *aggregate-wise*, we have

$$\|\mathbf{C}\underline{\mathbf{u}}\|_2^2 = \sum_{A_K \in \mathcal{T}_h^{\mathrm{agg}}} \|\mathbf{C}_{\hat{A}_K}\underline{\mathbf{u}}_K\|_2^2 \leq \sum_{A_K \in \mathcal{T}_h^{\mathrm{agg}}} \|\mathbf{C}_{\hat{A}_K}\|_2^2 \, \|\underline{\mathbf{u}}_K\|_2^2 \, .$$

- Each constraint matrix $\mathbf{C}_{\hat{A}_K}$ involves only a uniformly bounded number of DOFs, determined by the maximum aggregate size.
- Therefore,

$$\|\mathbf{C}\underline{\mathbf{u}}\|_2^2 \leq n_{\mathsf{cell}} \sup_{A_K \in \mathcal{T}_h^{\mathsf{agg}}} \|\mathbf{C}_{\hat{A}_K}\|_2^2 \|\underline{\mathbf{u}}\|_2^2,$$

  with the constant $C_{\mathrm{e}} = n_{\mathsf{cell}} \sup_{A_K \in \mathcal{T}_h^{\mathrm{agg}}} \|\mathbf{C}_{\hat{A}_K}\|_2^2$ independent of the cut configuration.

Continuity of the extension operator

Goal. Prove the L^2-stability of the extension operator

$$\mathcal{E} : V_h^{\text{in}} \to V_h^{\text{agg}}.$$

Key ingredients:

- Norm equivalence on interior cells.
- Stability of the coordinate vector extension $\mathbf{E}$.

For any interior cell $K \in \mathcal{T}_h^{\text{in}}$, standard FE theory yields

$$\lambda^- h_K^d \|\underline{\mathbf{u}}_K\|_2^2 \leq \|u_h\|_{L^2(K)}^2 \leq \lambda^+ h_K^d \|\underline{\mathbf{u}}_K\|_2^2.$$

Since interior cells are contained in the active domain,

$$\|\mathcal{E}(u_h)\|_{L^2(\Omega_{\text{act}})}^2 \geq \|u_h\|_{L^2(\Omega_{\text{in}})}^2 = \sum_{K \in \mathcal{T}_h^{\text{in}}} \|u_h\|_{L^2(K)}^2 \geq \sum_{K \in \mathcal{T}_h^{\text{in}}} \lambda^- h_K^d \|\underline{\mathbf{u}}_K\|_2^2 \gtrsim h^d \|\underline{\mathbf{u}}\|_2^2.$$

Continuity of the extension operator

On the other hand, using the norm equivalence on active cells, we have

$$\|\mathcal{E}(u_h)\|^2_{L^2(\Omega_{\text{act}})} = \sum_{K \in \mathcal{T}_h^{\text{act}}} \|\mathcal{E}(u_h)\|^2_{L^2(K)} \leq \sum_{K \in \mathcal{T}_h^{\text{act}}} \lambda^+ h_K^d \|\underline{\mathbf{u}}_K\|_2^2.$$

By the stability of the coordinate extension $\mathbf{E}$,

$$\sum_{K \in \mathcal{T}_h^{\text{act}}} h_K^d \|\underline{\mathbf{u}}_K\|_2^2 \lesssim h^d \|\mathbf{E}\underline{\mathbf{u}}\|_2^2 \lesssim h^d \|\underline{\mathbf{u}}\|_2^2.$$

Combining both estimates, we obtain the norm equivalence

$$\|\mathcal{E}(u_h)\|^2_{L^2(\Omega_{\text{act}})} \sim h^d \|\underline{\mathbf{u}}\|_2^2, \quad \forall u_h \in V_h^{\text{in}}.$$

Corollary

The extension operator is L^2-stable:

$$\|\mathcal{E}(u_h)\|_{L^2(\Omega_{act})} \lesssim \|u_h\|_{L^2(\Omega_{in})}, \quad \forall u_h \in V_h^{in}.$$

Restored inverse inequality

For the active FE space V_h^{act}, the standard inverse inequality holds:

$$\|\nabla v_h\|_{L^2(\Omega_{\text{act}})} \lesssim h^{-1}\|v_h\|_{L^2(\Omega_{\text{act}})}, \quad \forall v_h \in V_h^{\text{act}}.$$

Since $\mathcal{E}(u_h) \in V_h^{\text{act}}$ and $\Omega_{\text{in}} \subset \Omega_{\text{act}}$, we obtain

$$\|\nabla \mathcal{E}(u_h)\|_{L^2(\Omega_{\text{act}})} \lesssim h^{-1}\|\mathcal{E}(u_h)\|_{L^2(\Omega_{\text{act}})} \lesssim h^{-1}\|u_h\|_{L^2(\Omega_{\text{in}})}.$$

Restored inverse inequality:

$$\boxed{\|\nabla \mathcal{E}(u_h)\|_{L^2(\Omega_{\text{act}})} \lesssim h^{-1}\|u_h\|_{L^2(\Omega_{\text{in}})}, \quad \forall u_h \in V_h^{\text{in}}.}$$

Restored trace inequality

Lemma (Restored trace inequality)

For any $u_h \in V_h^{agg}$ and any $K \in \mathcal{T}_h^{act}$, it holds

$$\|\boldsymbol{n} \cdot \nabla u_h\|_{L^2(\Gamma_D \cap K)} \leq C_\partial h_K^{-1/2} \|\nabla u_h\|_{L^2(\Omega_K)},$$

where C_∂ is independent of the cut configuration and the mesh size.

Proof idea.

- Reduce the trace bound to an L^∞ estimate on a cut cell.
- Control the L^∞ norm via the extension of gradient DOFs.
- Transfer the estimate to neighboring interior cells.

Proof of the restored trace inequality

We only need to consider the case where K is a cut cell. For simplicity, we denote $\boldsymbol{\xi}_h := \nabla u_h$ for $u_h \in V_h^{\mathrm{agg}}$. On a cut cell K, let $\underline{\boldsymbol{\xi}}_K$ denote the coordinate vector of $\boldsymbol{\xi}_h$ with respect to the local basis.
We first estimate the boundary L^2-norm:

$$\|\boldsymbol{\xi}_h\|_{L^2(\Gamma_D \cap K)}^2 = \int_{\Gamma_D \cap K} |\boldsymbol{\xi}_h|^2 \leq |\Gamma_D \cap K| \, \|\boldsymbol{\xi}_h\|_{L^\infty(\Gamma_D \cap K)}^2 \leq |\Gamma_D \cap K| \, \|\boldsymbol{\xi}_h\|_{L^\infty(K)}^2.$$

Using norm equivalence in finite-dimensional polynomial spaces together with standard scaling arguments, we obtain

$$\|\boldsymbol{\xi}_h\|_{L^\infty(K)} \simeq \|\boldsymbol{\xi}_h\|_{L^\infty(\hat{K})} \lesssim \|\underline{\boldsymbol{\xi}}_K\|_2.$$

To bound $\underline{\boldsymbol{\xi}}_K$, we introduce the set of constraining interior cells $K_1, \ldots, K_{m_K}$ $(m_K \geq 1)$, i.e., interior cells that constrain at least one degree of freedom of the cut cell K.

Proof of the restored trace inequality

The coordinate vector $\underline{\xi}_K$ can be written as an extension of the gradient degrees of freedom on the constraining interior cells:

$$\underline{\xi}_K = \mathbf{D}_K \left[\underline{\xi}_{K_1}, \dots, \underline{\xi}_{K_{m_K}}\right]^T$$

The matrix $\mathbf{D}_K$ is uniformly bounded, by an argument analogous to that used for the matrix $\mathbf{C}$. Consequently,

$$\|\underline{\xi}_K\|_2 \lesssim \sum_{i=1}^{m_K} \|\underline{\xi}_{K_i}\|_2 \lesssim \sum_{i=1}^{m_K} \|\xi_h\|_{L^2(\hat{K}_i)} \lesssim |K|^{-1/2} \sum_{i=1}^{m_K} \|\xi_h\|_{L^2(K_i)}.$$

Here we used standard scaling arguments and the fact that $|K| \lesssim |K_i| \lesssim |K|$. Combining the above estimates and using the scaling relation

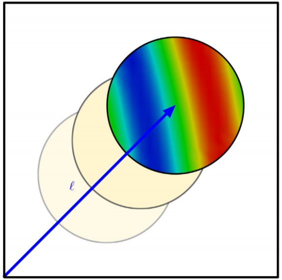
$$|\Gamma_D \cap K|^{1/2} \, |K|^{-1/2} \lesssim h_K^{-1/2},$$

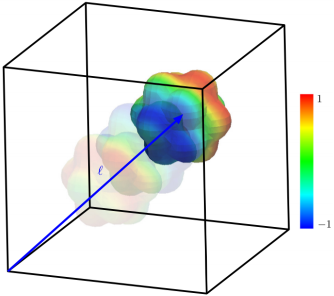
the proof of the lemma is complete.

Experiment: Moving domain test

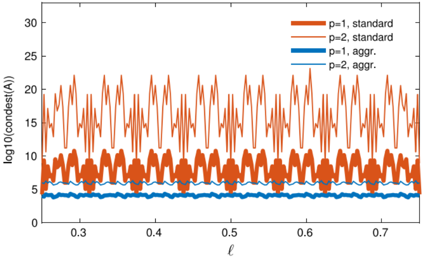
Test robustness with respect to geometric cut configurations:

- A domain moves along a diagonal of the computational box.
- The relative position between the boundary and the background mesh changes continuously. Therefore, arbitrarily small cut cells may appear.
- Fixed background mesh with $h = 2^{-5}$.
- Domain position controlled by a single parameter ℓ (i.e., the distance between the center of the body and a selected vertex of the box).

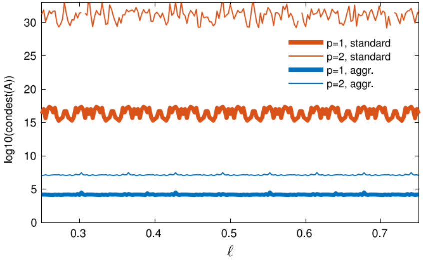




Experiment: Moving domain test



(a) 2D case.

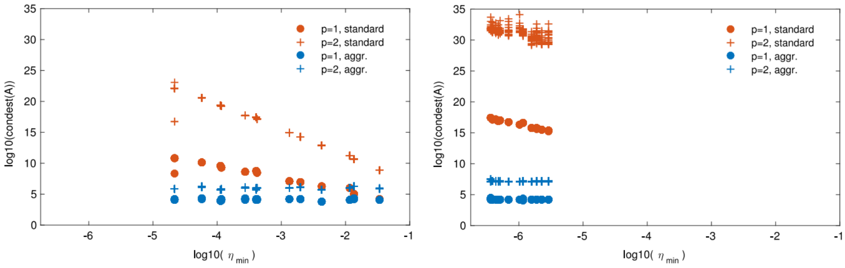


(b) 3D case.

Figure: Condition number vs. domain position.

- Standard unfitted FE: condition number is highly sensitive to the domain position.
- Aggregation-based FE: condition number is nearly independent of the position.
- Severe deterioration for standard FE when increasing order or dimension (2D $\to$ 3D).

Experiment: Moving domain test



(a) 2D case. (b) 3D case.

Figure: Condition number vs. smallest volume fraction $\eta_{\min} = \min_{K \in \mathcal{T}_h^{\text{act}}} \eta_K$.

- Standard unfitted FE: condition number scales inversely with $\eta_{\min}$.
- Aggregation-based FE: condition number is independent of $\eta_{\min}$.

Conclusions

Comparison of stabilization strategies for unfitted FEM

Aspect	**Ghost penalty methods**	**Aggregation-based FEM**
Stabilization level	Variational formulation	Discrete FE space
Main idea	Add penalty terms on cut facets	Eliminate problematic DOFs
Handling of small cut cells	Penalize jumps to recover stability	Avoid small supports by construction
Conditioning / errors	Sensitive to the choice of penalty parameter	Robust
High-order discretizations	Difficult (high-order normal derivatives)	Naturally supported
Implementation effort	Low	High (aggregation + constraints)

Locking phenomenon in ghost penalty methods

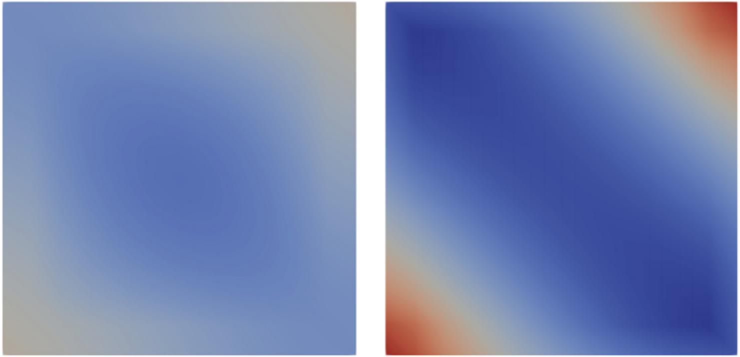


Figure: F-GP with large penalty parameter ($\gamma = 10^4$) vs. AggFEM.

Observation: As the penalty parameter increases, the ghost penalty formulation exhibits locking, leading to an overly stiff solution, while AggFEM remains unaffected.

Sensitivity to the penalty parameter

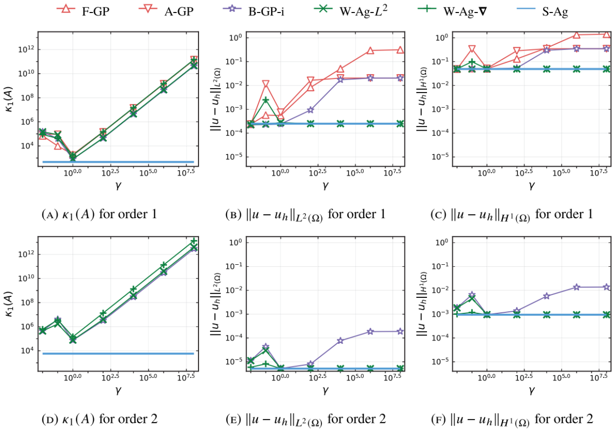


Figure: Condition number and error norms vs. penalty parameter γ.

- Ghost penalty methods show a strong sensitivity to the choice of γ, affecting both conditioning and accuracy.
- Aggregation-based FEM avoids penalty parameters and exhibits robust behavior.

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