Topology 2 Notes

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1 March 6, 2015

NOTICE: Reminder, there is an open book exam on the 11th, next week on Wednesday. There will be four multipart questions during the 55 minute exam.

Back to the two-torus, we identify the sides of $I \times I$ Let $Y = S^1 \times S^1$ and $Y^1 = A \cup B$. We showed last time that $H_q(Y, Y^1) \cong H_1(E^2, \dot{E}^2) = \mathbb{Z}$ for $q = 2$ and 0 otherwise. Recall from the long exact sequence in homology, we have

$$H_2(Y^1) \cong 0 \xrightarrow{i_*} H_2(Y) \xrightarrow{j_*} H_2(Y, Y^1) \xrightarrow{\partial_2} H_1(Y^1) \cong \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{i_*} H_1(Y) \xrightarrow{j_*} H_1(Y, Y^1) = 0$$

then we get the sequence

$$0 \to H_2(Y) \xrightarrow{j_*} H_2(Y, Y^1) \cong \mathbb{Z} \xrightarrow{\partial_2} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{i_*} H_1(Y) \xrightarrow{j_*} 0$$

Then we have $\text{im}(j_*) \cong H_2(Y) \cong \ker(\partial_*)? = H_2(Y, Y^1) \cong \mathbb{Z}$.

We make a guess before diving into the mess, $H_2(Y) \cong \mathbb{Z}$. So $j_*(n) = dn$ for $|d| > 1$? If $\text{im}(j_*) \cong \ker i_*$. But then $\mathbb{Z}/|d|\mathbb{Z}$ would be a subgroup of $H_1(Y^1)$, which is a contradition since every free abelian group is torsion free.

So if $H_2(Y, Y^1) \neq 0$, then j_* is onto $H_2(Y, Y^1) \cong \mathbb{Z}$. We want to show ∂_* is the zero map for $H_2(Y, Y^1) \cong \mathbb{Z} \to H_1(Y^1) \cong \mathbb{Z} \oplus \mathbb{Z}$.

Notice that a mpa $\mathbb{Z} \xrightarrow{M} \mathbb{Z} \oplus \mathbb{Z}$ for $\ker(M) = |d|\mathbb{Z}$. Then by algebra

$$\begin{tikzcd}
\mathbb{Z} \arrow[r, "M"] \arrow[d, "\pi"] & \mathbb{Z} \oplus \mathbb{Z} \\
\mathbb{Z}/\ker(M) \arrow[ur, dotted, "\tilde{M}"] &
\end{tikzcd}$$

Recall part of diagram from page 202

$$\begin{tikzcd}
H_2(E^2, \dot{E}^2) \arrow[r, "\cong\tilde{\partial}_*"] \arrow[d, "\cong f_*"] & H_1(\dot{E}^2) \arrow[d, "\cong f_*"] \\
H_2(Y, Y^1) \arrow[r, "\cong\partial_*"] & H_1(Y^1)
\end{tikzcd}$$

Looking at the graph of the boundary, the generator for $H_1(\dot{E}^2)$ is $T_{e_1} + T_{e_2} + T_{e_3} + T_{e_4}$ where $T_{e_1}(x) = (x, 0)$, $T_{e_2}(x) = (1, x)$, $T_{e_3}(x) = (1 - x, 1)$, $T_{e_4}(x) = (0, 1 - x)$,

LOOK AT IMAGE AND FILL IN DETAILS

therefore f_* of the generators of $H_1(\dot{E}^2)$ is $f_*(T_{e_1} + T_{e_2} + T_{e_3} + T_{e_4}) = f_*(T_{e_1}) + f_*(T_{e_2}) + f_*(T_{e_3}) + f_*(T_{e_4}) = 0$.

Therefore $\partial_* = (f_*) \circ \tilde{\partial_*} f_*^{-1} = 0$. Then we have the exact sequence

$$0 \xrightarrow{\partial_*} H_1(Y^1) \cong \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{i_*} H_1(Y) \to 0$$

hence i_* is an isomorphism. Therefore we have the following computation for Y, the torus

$$H_q(Y) = 0q \geq 2\mathbb{Z}q = 2\mathbb{Z} \oplus \mathbb{Z}q = 1\mathbb{Z}q = 0$$

Now we condsider the homology of the real projective plane $\mathbb{RP}^2$. Let $g : E^2 \to E^2/\sim$ relation on the boundary. Let $X = g(E^2)$ and $X^1 = g(\dot{E}^2) = S^1$. Remark, we can go through the same proof from the torus involving deformation retracts, excision, and the 5-lemma: to prove $H_1(E^2, \dot{E}^2) \xrightarrow{f_* \cong} H_q(X, X^1)$ go through as before to show

$$H_q(E^2, \dot{E}^2) \xrightarrow{g_* \cong} H_q(X, X^1) = \mathbb{Z}q = 2 0 \text{otherwise}$$

Now look back at the long exact sequence

$$H_3(X, X^1) \longrightarrow H_2(X^1) \longrightarrow H_2(X) \longrightarrow H_2(X, X^1) \cong \mathbb{Z} H_1(X^1) \xrightarrow{\partial_*} H_1(X) \xrightarrow{j_*} H_1(X, X^1) \cong 0$$

Then we look at the generators

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the ∂_* is multiplication by 2!

2 March 9, 2015 (Monday)

We have an exam on Wednesday. This covers chapter II sections 1-6 and chapter III sections 1-3. Overview of material

(1) Definition of singular n-cubes: $C_n(X) = Q_n(X)/D_n(X)$ where $T : I^n \to X$ is degenerate if for one of the coordinates, T is invariant.

(2) Boundary map $\partial_n(T) = \sum_{j=1}^{n} [A_j T - B_j T]$. Note that $\partial_{n-1} \circ \partial n = 0$. We have n-cycles $Z_n(X) = \ker(\partial_n)$ and n-boundaries $B_n(X) = \operatorname{im}(\partial_{n+1})$.

(3) We get the homology groups $H_n(X) = Z_n(X)/B_n(X)$ and we have the augmentation map $\varepsilon : C_n(X) \to \mathbb{Z}$ where $\tilde{Z}_0 = \ker(\varepsilon_*)$. We have $B_0(X) \subset Z_n(X)$ so we get the reduced homology $\tilde{H}_0(X) = \tilde{Z}_0(X)/B_0(X) \subset H_0(X)$.

(4) We get homomophisms in homoloy from continuous maps $f : X \to Y$. We obtain $f_* : H_n(X) \to H_n(Y)$ for $n \geq 0$ and $f_* : \tilde{H}_0(X) \to \tilde{H}_0(Y)$. If $f, g : X \to Y$ are homotopic then there is a $F : X \times I \to I$ such that $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$. Then $f_* = g_*$ on $H_n(X) \to H_n(Y)$ and $\tilde{H}_0(X) \to \tilde{H}_0(Y)$.

(5) We say spaces X and Y are homotopic if there exists continuous functions $f : X \to Y$ and $f : Y \to X$ such that $f \circ g \simeq id_X$ and $f \circ g \simeq id_Y$. If X and Y are homotopy equivalent then $f_* : H_n(X) \xrightarrow{\cong} H_n(Y)$ for each $n \geq 0$.

(6) Let $A \subseteq X$ then A is said to be a deformation retraction of X if there exists $r : X \to A$ such that r is homotopic to the identity map on X, id_X. If A is a deformation retract of X then $i_* : H_n(A) \xrightarrow{\cong} H_n(X)$ and $r_* : H_n(X) \xrightarrow{\cong} H_n(A)$.

(7) Homology of a pair (X, A). We have $C_n(X, A) = C_n(X)/C_n(A)$ and we have $\partial_n : C_n(X, A) \to C_{n-1}(X, A)$ where $\partial_{n-1} \circ \partial_n = 0$ for all $n \geq 1$. We have the relative n-boundaries $B_n(X, A) = \mathrm{im}\partial_{n+1} \subset Z_n(X, A)$. We have the relative homology group $H_n(X, A) = Z_n(X, A)/B_n(X, A)$.

(8) Using the short exact sequence of chain complexes

$$0 \longrightarrow C_\bullet(A) \xrightarrow{i} C_\bullet(X) \xrightarrow{j} C_\bullet(X, A) \longrightarrow 0$$

We get the long exact sequence:

ADD SNAKING LONG EXACT SEQUENCE

(9) Let $f : (X, A) \to (Y, B)$; that is, $f : X \to Y$ where $f(A) \subseteq B$. We can apply the five lemma to get

$$\begin{tikzcd}
H_n(A) \arrow[r] \arrow[d, "f_*"] & H_n(X) \arrow[r] \arrow[d, "f_*"] & H_n(X, A) \arrow[r] \arrow[d, "f_*"] & H_{n-1}(A) \arrow[d, "f_*"] \\
H_n(B) \arrow[r] & H_n(Y) \arrow[r] & H_n(Y, B) \arrow[r] & H_{n-1}(B)
\end{tikzcd}$$

(10) We then got the excision theorem: Let (X, A) be a pair of topological spaces and suppose $W \subseteq A$ such that $\bar{W} \subset \mathrm{Int}(A)$. Then $i_* : H_n(X \sim W, A \sim W) \to H_n(X, A)$ for $n \geq 0$ is an isomorphism.

(11) Once we got this theorem, we were able to deduce the homology groups of the n-sphere. Then $S^n \subseteq E^{n+1}$, with $S^n = \{(x_1, \ldots, x_{n+1} : \sum_{j=1}^{n+1} (x_i)^2 = 1\}$. For $n \geq 1$ this gives

$$H_1(S^n) = \begin{cases} \mathbb{Z} & \text{if } q \in \{0, n\} \\ 0 & \text{otherwise} \end{cases}$$

(12) From proposition 2.4, we get that $H_q(E_n, S^{n-1}) = \mathbb{Z}$ for $q = n$ and 0 otherwise.

(13) If $f : S^n \to S^n$ is a continuous map, define $deg(f)$ as the integer k such that $f_*([1]_{\tilde{H}_n(S^n))}) = k \cdot [1]_{\tilde{H}_n(S^n)}$

(14) Suspension: If $f : S^n \to S^n$ is continuous, then we get the suspension $\Sigma f : S^{n+1} \to S^{n+1}$, we have $def(f) = deg(\Sigma f)$.

(15) Finite regular graphs (X, X^0) where $X^0 = \{v_i\}_{i=1}^n$ and $X^1 = \{e_i\}_{j=1}^k$ we have $f_i : [0, 1] \to \bar{e}_i \subset X^1$ where $\bar{e}_i = \{v\} \cup e_i \cup \{v'\}$ where $v \neq v'$. Then

$$H_q(X, X^0) = \begin{cases} \mathbb{Z}^k & \text{for } q = 1 \\ 0 & \text{otherwise} \end{cases}$$

(16) Theorem 3.4: If (X, X^0) be a finite regular graph with P path components, n vertices, and k edges.

$$H_q(X) = \begin{cases} \mathbb{Z}^{p-(n-k)} = \mathbb{Z}^{p-\chi(X,X^0)} & \text{if } q = 1 \\ \mathbb{Z}^p & \text{if } q = 0 \\ 0 & \text{otherwise} \end{cases}$$

Note this implies $\tilde{H}_0(X) = \mathbb{Z}^{p-1}$. We write S^1 as a graph.

3 Meyer-Vietoris Exact Sequence

Imagine a torus, look at pieces A and B which have nonempty intersection that looks like a disjoint union of cylinders, and each subset looks like a cylinder. All pieces deformation retract to circles.

Therefore, we know the homology of A, B, and $A \cap B$. The inclusion maps

$$\begin{aligned} i &: A \cap B \hookrightarrow A & j &: A \cap B \hookrightarrow B \\ k &: A \hookrightarrow A \cup B & l &: B \hookrightarrow A \cup B \end{aligned}$$

We have the induced morphisms in homology

$$\begin{aligned} i_* &: H_n(A \cap B) \hookrightarrow H_n(A) & j_* &: H_n(A \cap B) \hookrightarrow H_n(B) \\ k_* &: H_n(A) \hookrightarrow H_n(A \cup B) & l_* &: H_n(B) \hookrightarrow H_n(A \cup B) \end{aligned}$$

So consider maps

$$\begin{aligned} \phi_* := (i_*, j_*) : H_n(A \cap B) \to H_n(A) \oplus H_n(B) \text{ for all } n \geq 0 \\ \psi_* := k_* - l_* : H_n(A) \oplus H_n(B) \to H_n(A \cup B) \end{aligned}$$

where $\psi_*([S], [T]) = k_*([S]) - l_*([T])$

Theorem: Let A and B be subspaces of a topological space X such that $X = \text{Int}(A) \cup \text{Int}(B)$. Then there exists a natural homomorphism $\Delta_* : H_n(X) \to H_{n-1}(A \cap B)$ for all $n \geq 1$ giving the following long exact sequence in homology:

$$\cdots \longrightarrow H_n(A \cap B)$$

Note that in degree 0 the sequence works in reduced homology.

Question: What do we mean by Δ_* being natural?
Given (X, A, B) and (X', A', B') and a continuous map $f : X \to X'$ such that $f(A) \subseteq A'$ and $f(B) \subset B'$. Then the following diagram commutes:

$$\begin{array}{ccccccc} H_n(A \cap B) & \xrightarrow{\phi_*} & H_n(A) \oplus H_n(B) & \xrightarrow{\psi_*} & H_n(X) & \xrightarrow{\Delta_*} & H_{n-1}(A \cap B) \\ \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_* \times f_*} & & & & \downarrow \\ H_n(A' \cap B') & \xrightarrow{\phi'_*} & H_n(A') \oplus H_n(B') & \xrightarrow{\Delta'_*} & H_n(X') & \longrightarrow & H_{n-1}(A' \cap B') \end{array}$$

Proof. Let $\mathcal{U} = \{A, B\}$ be a "special" open cover. From the book, we have an isomorphism $\sigma_* : H_n(X,\mathcal{U}) \xrightarrow{\cong} H_n(X)$, the restricted chain $C_n(X,\mathcal{U})$ can be written as $C_n(X,\mathcal{U}) = C_n(A) + C_n(B) \subset C_n(X)$. Recall that we have maps on chains

$$\begin{aligned} i_\sharp &: C_n(A\cap B) \to C_n(A) & j_\sharp &: C_n(A\cap B) \to C_n(A) \\ k'_\sharp &: C_n(A) \to C_n(X,\mathcal{U}) \hookrightarrow C_n(X) & k'_\sharp &: C_n(B) \to C_n(X,\mathcal{U}) \hookrightarrow C_n(X) \end{aligned}$$

Then there exists a homomorphism

$$\begin{aligned} \Phi : C_n(A\cap B) &\xrightarrow{i_\sharp \times j_\sharp} C_n(A) \oplus C_n(B) \\ \Psi : C_n(A) \oplus C_n(B) &\xrightarrow{k'_* - l'_*} \end{aligned}$$

Now consider the following complexes

$$\begin{aligned} \mathcal{C} &= (C_\bullet = C_\bullet(A\cap B), \partial_\bullet) \\ \mathcal{D} &= (D_\bullet = C_\bullet(A) \oplus C_\bullet(B), \partial'_\bullet = (\partial_\bullet, \partial_\bullet)) \\ \mathcal{E} &= (E_\bullet = C_n(X,\mathcal{U}), \partial_\bullet) \end{aligned}$$

You can check that the following diagram has exact rows and commutes

$$\begin{tikzcd} 0 \arrow[r] & C_n(A\cap B) \arrow[r, "\Phi_n"] \arrow[d, "\partial_n"] & C_n(A) \oplus C_n(B) \arrow[r, "\Psi_n"] \arrow[d, "\partial'_n"] & C_n(X,\mathcal{U}) \arrow[r] \arrow[d, "\partial_n"] & 0 \\ 0 \arrow[r] & C_n(A\cap B) \arrow[r, "\Phi_{n-1}"] & C_n(A) \oplus C_n(B) \arrow[r, "\Psi_{n-1}"] & C_n(X,\mathcal{U}) \arrow[r] & 0 \end{tikzcd}$$

We have $\mathrm{Im}(\Phi_n) = \ker \Psi_n$. Therefor we have the short exact sequence of chain complexes

$$0 \to \mathcal{C} \xrightarrow{\Phi} \mathcal{D} \xrightarrow{\Psi} \mathcal{E} \to 0$$

giving us a long exact sequence with boundary maps $\Delta_n : H_n(\mathcal{E}) \to H_{n-1}(\mathcal{C})$. Because σ_* is an isomorphism, we have the natural map. $\square$

4 March 16, 2015 (Monday)

Suppose A and B are subsets of X such that $\mathrm{int}(A) \cup \mathrm{int}(B) = X$. Let $\phi_* : (i_*, j_*) : H_n(A\cap B) \to H_n(A) \oplus H_n(B)$ and $\psi_* = k_* - k_* : H_n(A) \oplus H_n(B) \to H_n(X)$. The for all $n \geq 1$ there is a natural isomorphism $\Delta_* : H_n(X) \to H_{n-1}(A\cap B)$ giving us the short exact sequence

$$0 \to C_n(A\cap B) \to C_n(A) \oplus C_n(B) \to C_n(X) \to 0$$

Giving us a long exact sequence

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Moreover, if $A \cap B \neq \varnothing$ we can use reduced homology in dimension 0.

Examples:

(1) Homology of the torus $X = S^1 \times S^1$. We take two sleeves A and B which each deformation retract to a cirlce, and their intersection deformation retracts of a disjoint union of two circles. Drawing out the long exact sequence, we get the part

$$0 \xrightarrow{\psi_*} H_2(X) \xrightarrow{\Delta_*} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\phi_*} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\psi_*} H_1(X) \xrightarrow{\Delta_*} \tilde{H}_0(A \cap B) \cong \mathbb{Z} \to 0$$

Recall that Δ_* is one to one, so its image is the kernel of ϕ_*. Likewise, the image of $\phi_* = \ker(\psi_*)$, and $\Delta_* : H_1(X) \to \tilde{H}_0(A \cap B)$ is onto. We must choose generators on our topological space, so $H_1(A \cap B) = \mathbb{Z}(\text{gen1}) \oplus \mathbb{Z}(\text{gen2})$. Then $\phi_*(m, n) = (m - n, m - n) \in H_1(A) \oplus H_1(B)$. Hence the kernel of ϕ_* is the image of Δ_*, so it's isomorphic to $\mathbb{Z}$. Then, the image of ϕ_* is the diagonal, which is the kernel of ψ_*. Now, we look at the short exact sequence

$$0 \to H_1(A) \oplus H_1(B)/\text{im}\phi_* = \ker\psi_* \xrightarrow{\psi_*} H_1(X) \xrightarrow{\Delta_*} \mathbb{Z} \to 0$$

Notice $\{(m, n) : m, n \in \mathbb{Z}\}/\{(l, l) : l \in \mathbb{Z}\}$, which is $\mathbb{Z}$. (observe this quotient is just $\mathbb{Z}^2/\mathbb{Z}$). Now we can look at

$$0 \to \mathbb{Z} \to H_1(X) to \mathbb{Z} \to 0$$

Because the term on the right is free, the sequence splits.

(2) We can use induction to find the homology groups of the spheres using overlapping overhemispheres.

Jordan-Brauer Separation Theory: (Related to the Jordan curve theorem) Let $\gamma : [0, 1] \to \mathbb{R}^2$ be a simple closed curve so that $\gamma(0) = \gamma(1)$. Let $\mathcal{C} = \{\gamma(t) : 0 \leq t \leq 1\}$. Then $\mathbb{R}^2 \sim \mathcal{C}$. In homological terms, we only need to show $H_0(\mathbb{R}^2 \sim \mathcal{C}) \cong \mathbb{Z}^2$ or $\tilde{H}_0(\mathbb{R}^2 \sim \mathcal{C}) \cong \mathbb{Z}$. Consider $\mathbb{R}^2 \sim \mathcal{C}$ on S^2. Let $(0, 0)$ be the inside of the curve. Put $S62$ on "top of" $(0, 0)$. Using the inverse of stereographic projection, we may take the curve onto S^2. Call it $\tilde{\mathcal{C}}$. We can show that $\tilde{H}_0(S^2 \sim \tilde{\mathcal{C}}) \cong \mathbb{Z}$.

Theorem(Jordan-Brauer Separation Theorem) Let $A \subset S^n$ with $n \geq 1$. and suppose A is homeomorphic to S^k for $0 \leq k \leq n - 1$, then

$$\tilde{H}_q(S^n - A) = \begin{cases} \mathbb{Z} & q = n - k - 1 \\ 0 & q \neq n - k - 1 \end{cases}$$

For $n = 2, k = 1$, we have the original theorem.

Lemma 6.1 Let Y be a subset of S^n that is homeomorphic to intervals I^k for $0 \leq k \leq n$. Then the reduced homology of $\tilde{H}_q(S^n \sim Y) = 0$ for $q \in \mathbb{N} \cup \{0\}$.

5 March 18, 2015

Jordan Brauer Separation Theorem – Continued

Lemma 6.1 Let $Y \subset S^n$ that is homeomorphic to a k-cube I^k for $0 \leq k \leq n$. Then $\tilde{H}_i(S^n \sim Y) = 0$ for each $i \in \mathbb{N} \cup \{0\}$. True for $Y = h(I^0) =$ single point. We have $\tilde{H}_i(S^n \sim \{pt\}) \cong \tilde{H}_i(\mathbb{R}^n) \cong 0$ for each $n \geq 0$. Assume now that if $f_j : I^j \to S^n$ are homeomorphisms onto their image $0 \leq j \leq k-1 < n$, then $\tilde{H}_i(S^n \sim f_j(I^j)) = 0$ for each $i \in \mathbb{N} \cup \{0\}$. Now let $h : I^k \to S^n$ be a homeomorphism onto its image. We have to show $\tilde{H}_i(S^n \sim h(I^k))$ for each $i \in \mathbb{N} \cup \{0\}$. Let $Y_0 = h([0, \frac{1}{2}] \times I^{k=1})$, $Y_1 = h([\frac{1}{2}, 1] \times I^{k-1})$, so $Y = Y_0 \cup Y_1 = h(I_k)$. Note $Y_0 \cap Y_1 = h(\{\frac{1}{2}\} \times I^{k-1})$ which is in the homeomorphic image of $I^k - 1$. Not $\tilde{H}_i(S^n \sim Y_0 \cap Y_1) = 0$ for each $i \geq 0$, by induction hypothesis. We have $Y_0, Y_1, Y, Y_0 \cap Y_1$ all closed and let $A = S^n \sim Y_0$ and $B = S^n \sim Y_1$, which are open. Note $A \cup B = S^n \sim (Y_0 \cap Y_1)$ and $A \cap B = S^n \sim (Y_0 \cup Y_1) = S^n \sim Y$. We use the Meyer-Vietoris sequence

$$\tilde{H}_k(A \cup B) \xrightarrow{\Delta_*} \tilde{H}_i(A \cap B) \xrightarrow{\phi_*} \tilde{H}_i(A) \oplus \tilde{H}_i(B) \xrightarrow{\psi_*} \tilde{H}_i(A \cup B)$$

We have $\tilde{H}_{i+1}(A \cup B) = 0$ an $\tilde{H}_i(A \cup B) = 0$, therefore ϕ_* is an isomorphism. Suppose by way of contradiction that $\exists [s] \in \tilde{H}_i(A \cap B)$ such that $[s] \neq 0$. Then $\phi_*([s]) = ((i_0)_*([s]), (i_1)_*([s])) \neq 0$ in $\tilde{H}_i(A) \oplus \tilde{H}_i(B)$.

Now fix n, by induction on k. Then $\tilde{H}_i(A)$

$Y_{0,0} \cup Y_{0,1} = Y_0 \sim$ k-cube so $S^n \sim Y_{0,0}$ and $S^n \sim Y_{0,1}$ are open an ... Look at page 63-65 for the proof.

Applications?:

6 March 20, 2015 (Friday)

Proposition: Let $A \subset S^n$ with A homeomorphic to S^{n-1}. Then write $S^n \sim A \cong X_1 \coprod X_2$ where X_1 and X_2 are two path-components of $S^n \sim A$. Then $\partial X_1 = \partial X_2$.

We are interested in finding the relationship between the fundamental group of X and it's first homology group.

Definition: Let G be a discrete group. The commutator subgroup $[G, G]$ is the smallest subgroup of G that contains $\{aba^{-1}b^{-1} : a, b \in G\}$.

Theorem (group theory): Let G be a discrete group then $[G, G]$ is a normal subgroup and $G/[G, G]$ is abelian. Also, if $\phi : G \to A$ is a group homomorphism into an abelian group A, then $[G, G] \subset \ker\phi$.

Proof. Let $aba^{-1}b^{-1}$ and consider $g(aba^{-1}b^{-1})g^{-1}$ for some $g \in G$. This equals $gag^{-1}gbg^{-1}ga^{-1}g^{-1}b^{-1}g^{-1}$. This simplifies to $a'b'(a')^{-1}(b')^{-1}$ for $a' = gag^{-1}$ and $b' = gbg^{-1}$. Therefore $g[G, G]g^{-1} \subset [G, G]$, hence is normal. We know $G/[G, G]$ is abelian since $aba^{-1}b^{-1} \mapsto 1$ under the quotient map.

Similarly, if $\phi : G \to A$ is a homomorphism with A abelian, we have $\phi(aba^{-1}b^{-1}) = A$, hence $[G, G] \subset \ker\phi$. $\square$

Let X be a path connected space and fix some $\{x_0\} \subset X$. We want to show there is a homomorphism $h_X : \pi_1(X, x_0) \to H_1(X)$ and moreover $\pi_1(X, x_0)/[\pi_1(X, x_0), \pi_1(X, x_0)]$.

We now define $h_X : \pi_1(X, x_0) \to H_1(X)$. Let $\alpha \in \pi_1(X, x_0)$. Suppose $f : I \to X$ is a loop based at x_0 with $[f] = \alpha$. Since $\partial f = -(A_1 f - B_1 f) = f(1) - f(0) = 0$, $f \in Z_1(X)$. Define $h_X(\alpha) = [f]$ where f is a loop representing α. Suppose f and g are homotopic maps, hence both representing $\alpha \in \pi_1(X, x_0)$, we want $h_X(f) = h_X(g)$. $F : I \times I \to X$ such that $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$. We have $F \in C_2(X)$ since $F : I^2 \to X$. We check that $g = f + \partial F$. Then $[g] = [f + \partial F] = [f]$ in $H_1(X)$. Then h_X is well-define. Next we show that this map is infact a homomorphism. $h_X(f_1 \cdot f_2) = h_X(f_1) + h_X(f_2)$. Recall that

$$f_1 \cdot f_2 = \begin{cases} f_1(2t) & 0 \le t \le \frac{1}{2} \\ f_2(2t-1) & \frac{1}{2} \le t \le 1 \end{cases}$$

After break, we'll show $[f_1 \cdot f_2]_{H_1(X)} = [f_1]_{H_1(X)} + [f_2]_{H_1(X)}$

7 March 30, 2015 (Monday)

Today we study the relationship between $H_1(X)$ and $\pi_1(X, x_0)$ for a path connected space X.

Theorem: Fix $x_0 \in X$ in a path connected space X. Then, there is agroup homomorphism

$$h_x : \pi_1(X, x_0) \to H_1(X)$$

Moreover, h_X is surjective and its kernel is

$$[\pi_1(X, x_0), \pi_1(X, x_0)]$$

Proof. Recall for $\alpha \in \pi_1(X, x_0)$ is represented by $f_\alpha : [0, 1] \to X$ with $f_\alpha(0) = f_\alpha(1) = x_0$. Then $h_X(\alpha) = [f_\alpha]$ where $f_\alpha \in Z_1(X)$ and $\partial f_\alpha = 0$. Suppose $f, g : [0, 1] \to X$ with $f(0) = f(1) = g(0) = g(1) = x_0$ that both represent $\alpha \in \pi_1(X, x_0)$. Then f, g are homotopic as loops in X. So there exists $F : [0, 1] \times [0, 1] \to X$ with $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$ for each $x \in X$. Also, $F(0, y) = x_0 = F(1, y)$ for each $y \in [0, 1]$. We view $F \in Q_2(X)/D_2(X) = C_2(X)$. Then $\partial F : [0, 1] \to X$ is given by $\partial F(x) = -[F(1, x) - F(0, x)] + [F(x, 1) - F(x, 0)] = -[T_{x_0} - T_{x_0}] + g(x) - f(x)$. Therefore $\partial F(x) + f(x) = g(x)$. Notice that the homology classes are equal; that is, $[f] = [g] \in H_1(X)$. We now show h_X preserves group multiplication. Let $\alpha, \beta \in \pi_1(X, x_0)$ with $\alpha = [f]$ and $\beta = [g]$. We have $f : [0, 1] \to X$ and $g : [0, 1] \to X$ with $f(0) = f(1) = x_0 = g(0) = g(1)$. Then $\alpha \cdot \beta = [g \star f]$. We set

$$g \star f(x) = \begin{cases} f(2x) & 0 \le x \le \frac{1}{2} \\ g(2x-1) & \frac{1}{2} \le x \le 1 \end{cases}$$

We want to show $h_X([g \star f]) = h_X([g]) + h_X([f])$. We define $T : [0,1] \times [0,1] \to X$ with $T \in Q_2(X)$ by

$$T(x_1, x_2) = \begin{cases} f(x_1 + 2x_2) & x_1 + 2x_2 \leq 1 \\ g(\frac{x_1+2x_2-1}{x_1+1}) & 1 \leq x_1 + 2x_2 \end{cases}$$

Then we have

$$\begin{aligned} \partial T(x) &= -(T(1,x) - T(0,x)) + T(x,1) - T(x,0) \\ &= -g(\frac{1+2x-1}{1+1}) + \begin{cases} f(2x) & 0 \leq x \leq \frac{1}{2} \\ g(2x-1) & \frac{1}{2} \leq x \leq 1 \end{cases} \\ &+ g(\frac{x+2-1}{x+1}) - g(x) \\ \partial T(x) &= -g(x) + g \star f(x) + g(1) - f(x) \end{aligned}$$

Notice $g(1) \in D_1(X)$ hence $f + g + \partial T = g \star f$ in $H_1(X)$, so $[f] + [g] = [g \star f]$ therefore $h_X([g] \star [f]) = h_X([g]) + h_X([f])$. Fill in the gaps that h_X is a homomorphism is surjective into $H_1(X)$ and $\ker(h_X) = [\pi_1(X, x_0), \pi_1(X, x_0)]$. You can also check that $h_X : \pi_1(X, x_0) \to H_1(X)$ is natural in the following sense: Let (X, x_0) and (Y, y_0) be path connected spaces with base points x_0 and y_0 respectively. Let $\phi : X \to Y$ be a basepoint preserving map; that is $\phi(x_0) = y_0$. Then there exists a homomorphism $\phi_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ given by $\phi_*(f_\alpha) = \phi \circ f_\alpha$ such that the following diagram commutes:

$$\begin{array}{ccc} \pi_1(X, x_0) & \xrightarrow{\phi_*} & \pi_1(Y, y_0) \\ \downarrow{\scriptstyle h_X} & & \downarrow{\scriptstyle h_Y} \\ H_1(X) & \xrightarrow{\phi_*} & H_1(Y) \end{array}$$

□

Adjoining cells to spaces: Let X, Y be topological spaces and $A \subset Y$ a subspace. Let $f : A \to X$ by a continuous map. Define an equivalence relation on $Y \coprod X$ by $x \sim f(x)$ for any $x \in A$. We call this space X^*. This is also notated $Y \cup_f X$ in other books.

The process we described is called attaching Y to X.

Observe that Y may not always be embedded into $Y \cup_f X$. Consider $Y = [0,1]$, $A \subset Y$ by $A = \{0,1\}$, and $X = S^2 \subset \mathbb{R}^3$. Define $f : A \to S^2$ given by $f(0) = f(1) = (0,0,1)$. Now suppose X is fixed and let $Y = \coprod_{i=1}^k E_i^n$ for $n, k \in \mathbb{N}^+$ with $E_i^n = \{(x_i) \in \mathbb{R}^n : \sum x_i^2 \leq 1\}$. Let $A = \coprod_{i=1}^n S_i^{n-1}$ where each $S_i^{n-1} \subset E_i^n$ where each $S_i^{n-1} \subset E_i^n$ with the boundary $\dot{E}_i^n = S_i^{n-1}$. The question is, how do we compute the (relative) homology groups of $Y \cup_f X$?

8 April 1, 2015 (Wednesday)

Recall $X = \coprod_{i=1}^{k} E_i^n$ and $Y = \coprod_{i=1}^{n} S_i^{n-1}$. We have $S^{n-1} \subseteq E^n$. Let X be a compact Hausdorff space. The map $f : A \to X$ defined as the restriction of a continuous function $f : S^{n-1} \to X$. We let $X^* = Y \cup_f X$ and $\pi : Y \coprod X \to X^* = X \coprod Y / \sim$. Note that π is one to one on X into X^* therefore X is a subspace X^*, but π may not be one to one on Y.

Lemma: (EXERCISE) Let (X, A) and (Y, B) be pairs of spaces where $B \subseteq A \subseteq X$ and $Y \subseteq X$. Suppose that B is a deformation retract of A, and Y is a deformation retract of X. Then the inclusion map $i : (Y, B) \hookrightarrow (X, A)$ induces an isomorphism $i_* : H_q(Y, B) \to H_q(X, A)$ in relative homology.

Proof. Trivial. This proof comes from two long exact sequences in homology and the five-lemma. □

Theorem: Let X^* and X be as described from above. Then

$$H_q(X^*, X) = \begin{cases} 0 & q \neq n \\ \mathbb{Z}^k & q = n \end{cases}$$

Proof. For each i we have a map $\pi_i = \pi|_{E_n^{(i)}} : (E_i^n, S_i^{n-1}) \to (X^*, X)$, thus we obtain $(\pi|_{E_n^{(i)}})_* : H_q(E_i^n, S_i^{n-1}) \to H_q(X^*, X)$. We thus obtain $H_q(X^*, X) = \bigoplus_{i=1}^{k} (\pi_i)_*(H_q(E_i^n, S_i^{n-1}))$.

Let $D_i^n = \{x \in \mathbb{R}^n : \|x\| \leq \frac{1}{2}\} \subseteq E_i^n$. Note $S_i^{n-1} \subseteq E_i^n - D_i^n$. Let $\mathcal{D}_i = \prod_i (D_i^n) \subset X^*$. Let $a_i \in \mathcal{D}_i = \pi_i(0)$ with $0 \in D_i^n \subseteq E_i^n$ with $a_i \in \mathcal{D}_i$ Let $\mathcal{D} = \cup_{i=1}^{k} \mathcal{D}_i \subset X^*$. Let $A^0 = \cup_{i=1}^{k} \subset \mathcal{D} \subseteq X^*$. For each i with $1 \leq i \leq k$. We have $(\mathcal{D}_i, \mathcal{D}_i - \{a_i\}) \subseteq (X^*, X' = X^* - A^0)$, then $(\mathcal{D}, \mathcal{D} - A^0) \subset (X^*, X' = X^* - A^0)$. So by the second problem in the last problem, we have

$$\begin{aligned} H_q(\mathcal{D}, \mathcal{D} - A^0) &= \bigoplus_{i=1}^{k} H_q(\mathcal{D}_i, (\mathcal{D} - A^0) \cap \mathcal{D}_i) \\ &= \bigoplus_{i=1}^{k} H_q(\mathcal{D}_i, \mathcal{D}_i - \{a_i\}) \end{aligned}$$

We now note that $X = \pi(X)$ is a deformation retract of $X^* - A^0$. Since

$$\begin{aligned} X^* - A^0 &= X^* - \bigcup_{i=1}^{k} \{a_i\} \\ &= \pi(\coprod_{i=1}^{k} E_i^n \coprod X) - \pi(\coprod \{0\}_i \coprod \varnothing) \\ &= \pi(\coprod_{i=1}^{k} (E_i^n - \{0\}) \coprod X) \end{aligned}$$

which deforms to

$$\pi(\coprod_{i=1}^{k} S_i^{n-1} \coprod X) = X$$

So by the next lemma, the inclusion $i : (X^*, \pi(X)) \to (X^*, X^* - A^0)$ induces an isomorphism $(i)_* : H_q(X^*, X) \xrightarrow{\cong} H_q(X^*, X^* - A^0)$. Consider the inclusion $\Phi : (\mathcal{D}, \mathcal{D} - A^0) \to (X^*, X^* - A^0)$. We claim $\Phi_* : H_q(\mathcal{D}, \mathcal{D} - A^0) \xrightarrow{\cong} H_q(X^*, X^* - A^0)$.

Let $W = X^* - \mathcal{D}$ which is open since $\mathcal{D}$ is closed. Then

$$\begin{aligned}
\bar{W} &= X^* - \mathrm{Int}(\mathcal{D}) \\
&= X^* - \pi(\bigcup_{i=1}^{k} \mathrm{Int}(D_i^n)) \\
&= X^* - \pi(\bigcup_{i=1}^{k} \{x \in \mathbb{R}^n : \|x\| < \frac{1}{2}\}) \qquad \subseteq \mathrm{Int}(X^* - A^0) = X^* - A^0
\end{aligned}$$

By the excision theorem $\Phi_* : H_q(X^* - W, (X^* - A^0) - W) \to H_q(X^*, X^* - A^0)$ is an isomorphism for each $q \in \mathbb{N}$. That is $\Phi_* : H_q(\mathcal{D}, \mathcal{D} - A^0) \to H_q(X^*, X^* - A^0)$ is an isomorphism for each $q \in \mathbb{N}$.

Using these facts, $(i_*)^{-1} \circ \Phi_* : H_q(\mathcal{D}, \mathcal{D} - A^0) \to H_q(X^*, X)$ is an isomorphism for each $q \in \mathbb{N}$. To finish the theorem, note

$$\begin{aligned}
H_q(\mathcal{D}, \mathcal{D} - A^0) &= \bigoplus_{i=1}^{k} H_q(D_i, D_i - \{0\}) \\
&= \bigoplus_{i=1}^{k} H_q(E_i^n, S_i^{n-1}) \\
&= \begin{cases} 0 & q \neq n \\ \mathbb{Z}^k & q = n \end{cases}
\end{aligned}$$

□

Corollary: Let $X \to X^*$ where $X^* = U \bigcup_f X$ as in the previous theorem. Then $i_* : H_q(X) \to H_q(X^*)$ is an isomorphism except for possibly $q \in \{n, n-1\}$. For those values of q we have the long exact sequence

$$0 \to H_n(X) \to H_n(X^*) \to H_n(X^*, X) \to H_{n-1}(X) \to H_{n-1}(X^*) \to 0$$

Proof. For $q > n$ we have $H_{q+1}(X^*, X) = 0$. Look at the photo for the rest of the details. □

9 3 March, 2015 (Friday)

We give $\mathbb{RP}^n$ the quotient topology from S^n.

Theorem: $\mathbb{RP}^n$ is homeomorphic to $E^n_1 \bigcup_f \mathbb{RP}^{n-1}$ for

$$f : \dot{E}^n_1 = S^{n-1} \to\to\to\to\to\to\to \mathbb{RP}^{n-1} = S^{n-1}/\sim$$

Proof. Recall $S^n \subseteq \mathbb{R}^{n+1}$ is the set $E^N_+ \bigcup E^n_- \bigcup S^{n-1}$. When attaching cells, we view $E^n = \{(x_1, \ldots, x_n) : \sum_{i=1}^n x_i^2 =\leq 1\}$ and S^{n-1} is the boundary of E_n. Observe that E^n_+ is homeomorphic to E^n by

$$(x_1, \ldots, x_n, x_{n+1}) \mapsto \left(\frac{x_1}{\sqrt{1 - x_{n+1}^2}}, \ldots, \frac{x_1}{\sqrt{1 - x_{n+1}^2}} \right)$$

This map takes the image of S^{n-1} in E^n_+ to S^{n-1}. Then, consider the diagram

$$\begin{tikzcd}
 & S^{n-1} \arrow[dl] \arrow[d] \arrow[r, "\pi_{n-1}"] & \mathbb{RP}^{n-1} \arrow[d, "\iota"] \\
E^n_+ \arrow[r, hook] & S^n \arrow[r, "\pi_n"] & \mathbb{RP}^n
\end{tikzcd}$$

We have a map $E^n_+ \coprod \mathbb{RP}^{n-1} \to \mathbb{RP}^n$ by the diagram: if $x \in E^n_+$, $\pi_n(x) \in \mathbb{RP}^n$ and if $l \in \mathbb{RP}^{n-1}$ with $\iota(l) \in \mathbb{RP}^n$. Now we chack that is $y \in S^{n-1}$, $\iota \circ \pi_{n-1}(y) = \pi_n(y) \in \mathbb{RP}^n$. Observer this map is surjective and if it's continuous, and only not 1-1 on S^{n-1} from the diagram. Since $\mathbb{RP}^n$ is compact Hausdorff, $\mathbb{RP}^n \cong E^n_+ \bigcup_f \mathbb{RP}^{n-1}$. $\square$

Definition: A **finite CW-complex** (or a **cell-complex**) is a topological space X with a filtration

$$\varnothing = X^{(-1)} \subseteq X^{(0)} \subseteq X^{(1)} \subseteq \cdots \subseteq X^{(n)} = X$$

where you get from the $(j-1)$-**skeleton** X^{j-1} to the j-skeleton X^j by attaching a finite number of j-cells

$$\coprod_{i=1}^{k_j} E_i^j$$

along the boundaries

$$\coprod_{i=1}^{k_j} S_i^{j-1}$$

via continuous maps $f_i : S_i^{j-1} \to X^{(j-1)}$ such that

$$X^{(j)} = \left[\coprod_{i=1}^{k_j} E_i^j \right] \bigcup_f X^{(j-1)}$$

Examples:

(1) We get S^n from S^{n-1} by attaching the lower and upper hemispheres of S^n along S^{n-1}.

(2) $\mathbb{RP}^n$ is obtained from the last theorem.

(3)

10 April 6, 2015 (Monday)

Before talking about homology with coefficients, we need tensor products.

Tensor Products: Observe that the category of abelian groups is equivalent to the category of $\mathbb{Z}$-modules. For $\mathbb{Z}$-modules A, B, we have $A \times B = \{(a,b) | a \in A \text{ and } b \in B\}$. Let $F(A \times B)$ be the free abelian group generated by the pairs (a,b). This is the collection of finite sums of the generators, or more generally

$$\{\sum m_i(a_i, b_i) | m_i \in \mathbb{Z} \text{ and all but finitely many } m_i = 0\}$$

Let $R(A \times B)$ be the subgroup of $F(A \times B)$ generated by elements of the form

$$\begin{array}{rr} (a_1 + a_2, b) + (-1)(a_1, b) + (-1)(a_2, b) & (a, b_1 + b_2) + (-1)(a, b_1) + (-1)(a, b_2) \\ (ma, b) + (-1)m(a, b) & (a, mb) + (-1)m(a, b) \end{array}$$

for $a, a_1, a_2 \in A$, $b, b_1, b_2 \in B$, and $m \in \mathbb{Z}$. We define

$$A \otimes B = A \otimes_{\mathbb{Z}} B = F(A \times B)/R(A \times B)$$

We omit $\mathbb{Z}$ from $\otimes$ since we only look at $\mathbb{Z}$-modules, but in general, it's a wise idea to specify the ring your modules are over. The **simple tensors** are the elements under the image of this quotient map. They are of the form

$$\pi(1 \cdot (a,b)) = a \otimes b$$

They have the following nice properties:

$$\begin{aligned} (a_1 + a_2) \otimes b &= a_1 \otimes b + a_2 \otimes b \\ a \otimes (b_1 + b_2) &= a \otimes b_1 + a \otimes b_2 \\ m(a \otimes b) &= (ma) \otimes b = a \otimes (mb) \end{aligned}$$

Proposition: Tensor products satisfy the following universal mapping property:

$$\begin{tikzcd} A \times B \arrow[r, "\phi"] \arrow[d, "\otimes"'] & \mathbb{Z} \\ A \otimes B \arrow[ur, "\hat{\phi}"'] & \end{tikzcd}$$

That is; for any bilinear map $\phi : A \times B \to \mathbb{Z}$ factors uniquely through $A \otimes B$. This means that there exists a unique map $\hat{\phi} : A \otimes B \to \mathbb{Z}$ such that $\phi = \hat{\phi} \circ \otimes$.

Definition: Given a chain complex $\mathcal{C} = (C_\bullet, \partial_\bullet)$, we define the **chain complex with coefficients** as

$$\mathcal{C} \otimes G = (C_\bullet \otimes G, \partial'_\bullet = \partial_\bullet \otimes \mathbb{1}_G)$$

Clearly, we need to check that this is in fact a chain complex, but we also need to understand what the tensor product of morphisms actually means! We have the map $\partial_n \times \mathbb{1}_G : F(C_n \times G) \to F(C_{n-1} \times G)$ is defined by

$$(\partial_n \times \mathbb{1}_G) \left(m \sum_{i=1}^{N} (a_i, g_i) \right) = \sum_{i=1}^{N} (\partial_n(a_i), g_i)$$

Now, we must quotient by the induced relations $R(C_n \times G)$. I'll give one of them, but leave the rest as an exercise!

$$\begin{aligned} &(\partial_n \times \mathbb{1}_G)[(c_1 + c_2, g) - (c_1, g) - (c_2, g)] \\ &= (\partial_n(c_1 + c_2), g) + (-1)(\partial_n(c_1), g) + (-1)(\partial_n(c_2), g) \\ &= (\partial_n(c_1) + \partial_n(c_2), g) + (-1)(\partial_n(c_1), g) + (-1)(\partial_n(c_2), g) \end{aligned}$$

Consider the diagram

$$\{C_n \otimes G \xrightarrow{\partial'_n} C_{n-1} \otimes G\} = \{C_n \otimes G \xrightarrow{\partial_n \otimes \mathbb{1}_G} C_{n-1} \otimes G\}$$

We have the maps $\partial'_n(c \otimes g) = \partial_n(c) \otimes g$. Since

$$0 \otimes g = 0(1 \otimes g) = 1 \otimes 0 = 0$$

and

$$(\partial'_{n-1} \circ \partial'_n)(c \otimes g) = (\partial_{n-1} \circ \partial_n)(c) \otimes g = 0 \otimes g$$

for each $c \otimes g \in C_n \otimes G$, the "complex" with coefficients in G is indeed a complex! Now we can define the homology by

$$H_n(C_\bullet \otimes G) = \frac{Z_n(C_\bullet \otimes G)}{B_n(C_\bullet \otimes G)}$$

The exact and boundary elements are given the obvious definition; that is, the abstract definition.

Question: When does $H_n(C_\bullet \otimes G) = H_n(C_\bullet) \otimes G$. For $\mathbb{Z}$, this is trivial since for any $\mathbb{Z}$-module M, $M \otimes \mathbb{Z} = M$ (note this holds true over any ring as well). We answer this question using the universal coefficient theorem.

11 April 8, 2015 (Wednesday)

Let (X, A) be a pair of topological spaces, G and abelian group, and

$$0 \to C_\bullet \to D_\bullet \to E_\bullet \to 0$$

a short exact sequence of chain complexes. We form the complexes of local coefficients. Observe that in general $- \otimes_{\mathbb{Z}} G$ is only right exact; that is,

$$C_\bullet \otimes G \to D_\bullet \otimes G \to E_\bullet \otimes G \to 0$$

is exact. We can form $H_n(C_\bullet \otimes G) = H_n(A; G)$, $H_n(D_\bullet \otimes G) = H_n(X; G)$, and $H_n(E_\bullet \otimes G) = H_n(X, A; G)$, but it's not clear if a long exact sequence exists.

Claim: However, if the original short exact sequence is split, then the complex with local coefficients is exact. It becomes clear since the complex is isomorphic to

$$0 \to C_\bullet \to C_\bullet \oplus E_\bullet \to E_\bullet \to 0$$

and because $(C_\bullet \oplus E_\bullet) \otimes G = (C_\bullet \otimes G) \oplus (E_\bullet \otimes G)$, the claim becomes obvious. In this case, we can get a long exact sequence! Since $C_n(X, A)$ is still a free abelian group, we can get the desired long exact sequence

$$\cdots \xrightarrow{\delta_{k+1}} H_k(A; G) \to H_k(X; G) \to H_k(X, A; G) \xrightarrow{\delta_k} H_{k-1}(A; G) \to \cdots$$

from homological methods.

Definition Let A be an abelian group. a free resolution for A is a short exact sequence

$$0 \to R \xrightarrow{i} F \xrightarrow{\pi} A \to 0$$

where $R \subset F$ and F is a free abelian group.

Factoid: Any abelian group A has a free resolution. Notice that for the free abelian group generated from A, $F(A)$, has elements of the form

$$\sum_{i=1}^{N} m_i [a_i] \text{ for } m_i \in \mathbb{Z} \text{ and } a_i \in A$$

Consider the homomorphism $\pi : F(a) \to A$ given by

$$\pi(\sum_{i=1}^{N} m_i [a_i]) = \sum_{i=1}^{N} m_i a_i$$

The kernel of this map is a subgroup of $F(A)$. Then, we have the free resolution

$$0 \to R = \ker(\pi) \xrightarrow{\iota} F(A) \xrightarrow{\pi} A \to 0$$

In general, given any free resolution, $0 \to R \to F \to A \to 0$, and for G an abelian group, $- \otimes_{\mathbb{Z}} G$ is still right-exact on this sequence, but we have a tool which measures the failure for left exactness. This is called the *Tor* functor. We define it as

$$Tor^1_{\mathbb{Z}}(A; G) := \ker(R \xrightarrow{\iota \otimes \mathbb{1}_G} F)$$

Remark: $Tor(A; G)$ is independent of the free resolution.

Theorem:(Universal Coefficient Theorem for Homology) For a chain complex $C_\bullet$, there is a short exact sequence

$$0 \to H_n(X) \otimes G \to H_n(X; G) \to Tor(H_{n-1}(X); G) \to 0$$

12 April 10, 2015

Theorem: (6.2/ Universal Coefficient Theorem) Let $(C_\bullet, \partial_\bullet)$ be a chain complex of free abelian groups with corresponding homology groups $\{H_k(C_\bullet)\}_{k\in\mathbb{N}_+}$. Then, there exists a collection of short exact sequences

$$0 \to H_n(X) \otimes G \to H_n(X;G) \to Tor(H_{n-1}(X);G) \to 0$$

Corollary: Let G be an abelian group, and (X, A) a pair of topological spaces, then we have the long exact sequence with coefficients in G:

$$\cdots \to H_{k+1}(X, A; G) \xrightarrow{\delta_{k+1}} H_k(A; G) \to H_k(X; G) \to H_k(X, A; G) \xrightarrow{\delta_k} H_{k-1}(A; G) \to \cdots$$

These groups are given by

$$H_n(X;G) \cong H_n(X) \otimes G \oplus Tor(H_n(X), G)$$
$$H_n(X, A; G) \cong H_n(X, A; G) \oplus Tor(H_n(X, A), G)$$

Observe that the splitting is noncanonical:

Proposition: Let $f : (X, A) \to (Y, B)$ be a map of pairs, with corresponding homology maps $f_* : H_n(X) \to H_n(Y)$ and $f_* : H_n(X) \to H_n(Y)$. The splitting just mentioned in noncanonical.

Proof. Consider a free resolution of G

$$0 \to G_1 \xrightarrow{j} G_0 \xrightarrow{\tau} G \to 0$$

Dedekind? proved that this implies G_0 is a free abelian group and G_1 a subgroup of a free abelian group which is also free. Since the groups in $C_\bullet$ are free, tensoring our chain complex with the short exact sequence in the hypothesis gives

$$\begin{array}{ccccccc}
0 \to C_n \otimes G_1 & \xrightarrow{Id_{C_\bullet}\otimes j} & C_n \otimes G_0 & \xrightarrow{Id_{C_\bullet}\otimes \tau} & C_n \otimes G & \longrightarrow & 0 \\
\Big\downarrow{\scriptstyle \partial_n \otimes Id_{G_1}} & & \Big\downarrow{\scriptstyle \partial_n \otimes Id_{G_0}} & & \Big\downarrow{\scriptstyle \partial_n \otimes Id_G} & & \\
0 \to C_{n-1} \otimes G_1 & \xrightarrow{Id_{C_\bullet}\otimes j} & C_{n-1} \otimes G_0 & \xrightarrow{Id_{C_\bullet}\otimes \tau} & C_{n-1} \otimes G & \longrightarrow & 0
\end{array}$$

This commuting diagram gives a short exact sequence of chain complexes giving us a long exact sequence of homology groups for the chain complexes $\{C_\bullet \otimes G_1, \partial_n\}$, $\{C_\bullet \otimes G_0, \partial'_n\}$, $\{C_\bullet \otimes G_1 \partial_n\}$, we have the corresponding long exact sequence

$$\cdots \xrightarrow{\delta_{q+1}} H_q(C_\bullet \otimes G_1) \xrightarrow{J_*} H_q(C_\bullet \otimes G_0) \xrightarrow{T_*} H_q(C_\bullet \otimes G) \xrightarrow{\delta_q} \cdots$$

$\square$

Remark: If F is a free abelian gruop, and $\{C_\bullet, \partial_\bullet\}$ is a chain complex, then tensoring this chain complex yields a complex $\{C_\bullet \otimes F, \partial'_\bullet\}$ with the property $H_n(C_\bullet \otimes G) = H_n(C_\bullet) \otimes G$. Notice that this implies that our long exact sequence found in the previous proposition is a long exact sequence of free abelian groups isomorphic to

$$\cdots \xrightarrow{\delta_{q+1}} H_q(C_\bullet) \otimes G_1 \xrightarrow{J_*} H_q(C_\bullet) \otimes G_0 \xrightarrow{T_*} H_q(C_\bullet) \otimes G \xrightarrow{\delta_q} \cdots$$

We construct this map

$$H_q(C_\bullet) \otimes G \xrightarrow{\iota_*} H_q(C_\bullet \otimes G)$$

by sending $[f] \otimes g \mapsto [f \otimes g]$ for f a homology class and g an element of the group.

13 April 13, 2015 (Monday)

More universal coefficient theorem.

Example We have already seen

$$H_n(\mathbb{RP}^2) = \begin{cases} 0 & n \geq 2 \\ \mathbb{Z}_2 & n = 1 \\ \mathbb{Z} & n = 0 \end{cases}$$

Given $G = \mathbb{Z}/2 = \mathbb{Z}/2\mathbb{Z}$,

$$H_n(\mathbb{RP}^2; \mathbb{Z}/2) = (H_n(\mathbb{RP}^2) \otimes \mathbb{Z}/2) \oplus Tor(H_{n-1}(\mathbb{RP}^2), \mathbb{Z}/2)$$

Then

$$H_n(\mathbb{RP}^2; \mathbb{Z}/2) = \begin{cases} (0 \otimes \mathbb{Z}/2) \oplus Tor(\mathbb{Z}/2, \mathbb{Z}/2) & n = 2 \\ (\mathbb{Z}/2 \otimes \mathbb{Z}/2) \oplus Tor(\mathbb{Z}, \mathbb{Z}/2) & n = 1 \\ \mathbb{Z} \otimes \mathbb{Z}/2 \oplus Tor(\{0\}, \mathbb{Z}/2) & n = 0 \end{cases}$$

so

$$H_n(\mathbb{RP}^2; \mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2 & n \in \{0, 1, 2\} \\ 0 & n \geq 3 \end{cases}$$

Exercise: Show $Tor(G, A) = 0$ for any free abelian group A.

Let G be an abelian group and G_0, G_1 abelian free groups, and

$$0 \to G_1 \to G_0 \to G \to 0$$

a free resolution. A is a free abelian group, so we write $A = \bigoplus_{\alpha \in \Sigma} [\mathbb{Z}]_\alpha$, so for $i \in \{0, 1\}$

$$G_i \otimes A = G_i \otimes \left(\bigoplus_{\alpha \in \Sigma} [\mathbb{Z}]_\alpha \right) = \bigoplus_{\alpha \in \Sigma} [G_i]_\alpha$$

Since $(- \otimes A)$ is a right exact functor,

$$\bigoplus_{\alpha \in \Sigma} G_1 \to \bigoplus_{\alpha \in \Sigma} G_0 \to \bigoplus_{\alpha \in \Sigma} G \to 0$$

is an exact sequence. Then,

$$Tor_1(G, A) = \ker \left\{ \bigoplus_{\alpha \in \Sigma} G_1 \to \bigoplus_{\alpha \in \Sigma} G_0 \right\}$$

Hence $Tor_1(G, A) = 0$.

For the trivial topological space $X = \{*\}$ and G an abelian group,

$$\begin{aligned} H_n(X, G) &= \begin{cases} (H_n(X) \otimes G) \oplus Tor(H_{n-1}(*), G) & n \geq 1 \\ (\mathbb{Z} \otimes G) \oplus Tor(0, G) & n = 0 \end{cases} \\ &= \begin{cases} 0 & n \geq 1 \\ G & n = 0 \end{cases} \end{aligned}$$

Cohomology! Given a chain complex $(C_\bullet, \partial_\bullet)$ and an abelian group G, we form a cochain complex $(C^\bullet, \delta^\bullet)$ as follows: Given a fixed n, let $f_1, f_2 \in Hom(C_n, G)$. Define $f_1 + f_2$ as

$$(f_1 + f_2)(l) = f_1(l) + f_2(l)$$

for $l \in C_n$. It is a quick exercise to show that the additive structure gives an abelian group structure on $Hom(C_n, G)$; denote it by C^n. Consider the following diagram:

$$\begin{tikzcd} \cdots \arrow[r, "\partial_{n+2}"] & C_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[dr] & C_n \arrow[r, "\partial_n"] \arrow[d] & C_{n-1} \arrow[r, "\partial_{n-1}"] \arrow[dl, "\psi"] & \cdots \\ & & G & & \end{tikzcd}$$

Observe that composition of ψ with ∂_n gives a map $C_n \to G$. It's a quick exercise to show this defines a group homomorphism $\delta^{n-1} : C^{n-1} \to C^n$. Because $(\delta^n \circ \delta^{n-1})(\psi) = \psi \circ (\partial_n \circ \partial_{n+1}) = 0$, the induced maps are differentials. Therefore, we can rightly give $(C^\bullet, \delta^\bullet)$ the name of a **cochain complex**. Notice that the indices go up in index; ie, we have the diagram

$$\cdots \xrightarrow{\delta^{n-2}} C^{n-1} \xrightarrow{\delta^{n-1}} C^n \xrightarrow{\delta^n} C^{n+1} \xrightarrow{\delta^{n+1}} \cdots$$

Definition: Given a cochain complex $(C^\bullet, \delta^\bullet)$, we define the **cohomology** of it as

$$H^n(C^\bullet) = \frac{\ker\{\delta^n : C^n \to C^{n+1}\}}{\mathrm{im}\{\delta^{n-1} : C^{n-1} \to C^n\}} = \frac{Z^n}{B^n}$$

We call Z^n and B^n the **n-cocycles** and **n-coboundaries**, respecitvely.

14 April 15, Wednesday

Given a topological space X with a subspace A, and G an abelian group, we have the singular chain complexes $(C_\bullet(X), \partial_\bullet)$, $(C_\bullet(A), \partial_\bullet)$, and $(C_\bullet(X, A), \partial_\bullet)$. Dualizing these complexes with $\operatorname{Hom}_{\mathbb{Z}}(-, G)$ gives the cochain complexes $(C^\bullet(A; G), \delta^\bullet)$, $(C^\bullet(X; G), \delta^\bullet)$, and $(C^\bullet(X, A; G), \delta^\bullet)$ where

$$\begin{aligned} C^n(A; G) &= \operatorname{Hom}(C_n(A), G) \\ C^n(X; G) &= \operatorname{Hom}(C_n(X), G) \\ C^n(X, A; G) &= \operatorname{Hom}(C_n(X, A), G) \end{aligned}$$

We have the corresponding cohomology groups

$$\begin{aligned} H^n(C^\bullet(A; G)) &= \frac{Z^n(A; G)}{B^n(A; G)} \\ H^n(C^\bullet(X; G)) &= \frac{Z^n(X; G)}{B^n(X; G)} \\ H^n(C^\bullet(X, A; G)) &= \frac{Z^n(X, A; G)}{B^n(X, A; G)} \end{aligned}$$

Given three cochain complexes $(C^\bullet, \delta^\bullet)$, $(D^\bullet, \delta^\bullet)$, and $(E^\bullet, \delta^\bullet)$, and morphisms $f^\bullet : C^\bullet \to D^\bullet$ and $g^\bullet : D^\bullet \to E^\bullet$ such that the following diagram of exact sequences commutes

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & {\scriptstyle\delta^{k+1}}\big\uparrow & & {\scriptstyle\delta^{k+1}}\big\uparrow & & {\scriptstyle\delta^{k+1}}\big\uparrow & & \\
0 & \longrightarrow & C^{k+1} & \xrightarrow{f^{k+1}} & D^{k+1} & \xrightarrow{g^{k+1}} & E^{k+1} & \longrightarrow & 0 \\
 & & {\scriptstyle\delta^{k}}\big\uparrow & & {\scriptstyle\delta^{k}}\big\uparrow & & {\scriptstyle\delta^{k}}\big\uparrow & & \\
0 & \longrightarrow & C^{k} & \xrightarrow{f^{k}} & D^{k} & \xrightarrow{g^{k}} & E^{k} & \longrightarrow & 0 \\
 & & {\scriptstyle\delta^{k-1}}\big\uparrow & & {\scriptstyle\delta^{k-1}}\big\uparrow & & {\scriptstyle\delta^{k-1}}\big\uparrow & & \\
0 & \longrightarrow & C^{k-1} & \xrightarrow{f^{k-1}} & D^{k-1} & \xrightarrow{g^{k-1}} & E^{k-1} & \longrightarrow & 0 \\
 & & {\scriptstyle\delta^{k-2}}\big\uparrow & & {\scriptstyle\delta^{k-2}}\big\uparrow & & {\scriptstyle\delta^{k-2}}\big\uparrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

we can apply the same methodology as before to get the long exact sequence

$$\begin{tikzcd}
0 \arrow[r] & H^0(E^\bullet) \arrow[r] & H^0(D^\bullet) \arrow[r] & H^0(C^\bullet) \arrow[dll, "\Delta^0"] \\
& H^1(E^\bullet) \arrow[r] & H^1(D^\bullet) \arrow[r] & H^1(C^\bullet) \arrow[dll, "\Delta^1"] \\
& H^2(E^\bullet) \arrow[r] & H^2(D^\bullet) \arrow[r] & H^2(C^\bullet) \arrow[dll, "\Delta^2"] \\
& \cdots & &
\end{tikzcd}$$

of cohomology groups. Hence we have the long exact sequence

$$\begin{tikzcd}
0 \arrow[r] & H^0(C^\bullet(X,A;G)) \arrow[r] & H^0(C^\bullet(X;G)) \arrow[r] & H^0(C^\bullet(A;G)) \arrow[dll, "\Delta^0"] \\
& H^1(C^\bullet(X,A;G)) \arrow[r] & H^1(C^\bullet(X;G)) \arrow[r] & H^1(C^\bullet(A;G)) \arrow[dll, "\Delta^1"] \\
& H^2(C^\bullet(X,A;G)) \arrow[r] & H^2(C^\bullet(X;G)) \arrow[r] & H^2(C^\bullet(A;G)) \arrow[dll, "\Delta^2"] \\
& \cdots & &
\end{tikzcd}$$

on relative pairs. These details are left for the reader!

15 April 17, Friday

Proposition: For a fixed abelian group G, and a split short exact sequence

$$0 \to F \xrightarrow{i} H \xrightarrow{\pi} K \to 0$$

applying the functor $\mathrm{Hom}(-, G)$ preserves split exactness; that is, the following sequence is exact

$$0 \to \mathrm{Hom}(K, G) \to \mathrm{Hom}(H, G) \to \mathrm{Hom}(F, G) \to 0$$

Corollary: For each $n \geq 0$, and for any abelian group G, and any pair of topological spaces (X, A), the following sequence is exact:

$$0 \to \mathrm{Hom}(C_n(X, A), G) \to \mathrm{Hom}(C_n(X), G) \to \mathrm{Hom}(C_n(A), G) \to 0$$

This implies we have a split exact sequence of cochain complexes

$$0 \to C^\bullet(X, A; G) \to C^\bullet(X; G) \to C^\bullet(A; G) \to 0$$

Proposition: Note if $f : (X, A) \to (Y, B)$ is a map of pairs, we obtain cochain maps

$$C^\bullet(A; G) \to C^\bullet(B; G)$$
$$C^\bullet(X; G) \to C^\bullet(Y; G)$$
$$C^\bullet(X, A; G) \to C^\bullet(Y, A; G)$$

for a fixed abelian group G. Then we have the following diagram useful for the five lemma

$$\begin{array}{ccccccc}
H^{k-1}(C^\bullet(B;G)) & \xrightarrow{\Delta^{k-1}} & H^k(C^\bullet(Y,B;G) & \longrightarrow & H^k(C^\bullet(Y;G)) & \xrightarrow{\Delta^k} & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
H^{k-1}(C^\bullet(A;G)) & \xrightarrow{\Delta^{k-1}} & H^k(C^\bullet(X,A;G) & \longrightarrow & H^k(C^\bullet(X;G)) & \xrightarrow{\Delta^k} & \cdots
\end{array}$$

$$\begin{array}{ccccc}
\cdots & \xrightarrow{\Delta^k} & H^k(C^\bullet(B;G)) & \longrightarrow & H^{k-1}(C^\bullet(Y,B;G)) \\
& & \downarrow & & \downarrow \\
\cdots & \xrightarrow{\Delta^k} & H^k(C^\bullet(A;G)) & \longrightarrow & H^{k-1}(C^\bullet(X,A;G))
\end{array}$$

Definition: Now we want to capture homotopic spaces under cochain morphisms. Given a map of cochain complexes $f^\bullet : C^\bullet \to D^\bullet$, we define a **cochain homotopy** of cochain complexes by the following commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r, "\delta^{k-2}"] & C^{k-1} \arrow[r, "\delta^{k-1}"] \arrow[d, "f^{k-1}"'] \arrow[dl, "s^{k-1}"'] & C^k \arrow[r, "\delta^k"] \arrow[d, "f^k"'] \arrow[dl, "s^{k-1}"'] & C^{k+1} \arrow[r, "\delta^k"] \arrow[d, "f^{k+1}"'] \arrow[dl, "s^{k+1}"'] & \cdots \arrow[dl, "s^{k+1}"'] \\
\cdots \arrow[r, "\delta^{k-2}"] & D^{k-1} \arrow[r, "\delta^{k-1}"] & D^k \arrow[r, "\delta^k"] & D^{k+1} \arrow[r, "\delta^{k+1}"] & \cdots
\end{tikzcd}$$

with the condition

$$f^k = \delta^{k-1} \circ s^{k-1} + s^{k+1} \circ \delta^k$$

Definition: We say cochain maps $f^\bullet, g^\bullet : C^\bullet \to D^\bullet$ are homotopic if for each k

$$f^k - g^k = \delta^{k-1} \circ s^{k-1} + s^{k+1} \circ \delta^k$$