Differential Geometry Notes

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1 Notice

These notes contain errors. Please put an issue on github or just fork the repository and make the changes yourself.

2 March 6, 2015

If we have $f : M \to N$ and $q \in N$. The claim was $f^{-1}(q) \subset M$ is a submanifold. We have to find charts. Let $p \in f^{-1}(q)$. By the rank theorem on charts, there are charts (U, x) around p and (V, y) around q such that $x(p) = 0$ and $y(q) = 0$, and $y \circ f \circ x^{-1}(v) = (v_1, \dots, v_n)$. for v in a neighborhood of origin in $\mathbb{R}^m$. For v in a neighborhood of origin in $\mathbb{R}^m$ the chart we are looking for, find $f^{-1}(q)$ is $(x_{n+1}, \dots, x_m)$. Then $f^{-1}(q) \cap U = (x_{n+1}, \dots, x_m)^{-1}(\mathbb{R}^{m-n})$

Theorem (Ehresmann): A proper surjective submersion is a fiber bundle with fiber

Orientation: Let M be a smooth connected m-manifold. and let $\Lambda^k T^*M \sim O_N := \{\omega_p \in \Lambda^k T^*M | p \in M \text{ and } \omega_p \neq 0\} \subset \Lambda^m T^*M$. These ω's are nonzero in some fiber. Then $\dim \Lambda^m T^*_p M = 1$, this is the space of determinants. This is a subspace. A manifold M is called **orientable** if $\Lambda^k T^*M \sim 0_M$ has exactly two connected components. An **orientation** of an orientable smooth manifold M is a choice of a component of $\Lambda^k T^*M \sim 0_M$.

The zero section of a vector bundle $p : V \to M$ is a $M \to v, p \mapsto 0_p$ where 0_M is the zero section of $\Lambda^k T^*M$ or better its image.

The tangent bundle recap: take TM of some smooth manifold M, and let (U, x) and (V, y) be smooth charts such that $U \cap V \neq \varnothing$. Then $x \circ x^{-1}|_{U \cap V} : x(U \cap V) \to y(U \cap V)$ are smooth transition maps of M. Then the induced map on trivializations $x(U \cap V) \times \mathbb{R}^m \to y(U \cap V) \times \mathbb{R}^m$ where $(p, v) \mapsto (y \circ x^{-1}(p), D(y \circ x^{-1})(p)v)$

Counterexample: Mobius band.

Example: Chiral molecules have a defined orientation. Similar, but not the same.

Theorem: Let M be a connected smooth m-manifold. Then the following are equivalent:

(1) M is orientable

(2) There is an atlas $\mathcal{A}$ of M such that the $\det(D(x\circ y^{-1})(y(p)))>0$ for each $(U,x),(V,y)\in\mathcal{A}$ and $p\in U\cap V$

(3) There is a nowhere vanishing m-form $\omega\in\Omega^m(M)$.

Proof: (1) $\Rightarrow$ (2) Let Λ be an orientation have have $\Lambda\cap\Lambda^m T_p^*M\sim 0_p$ is a component of $\Lambda^m T_p^*M\sim 0_p$ Define $\mathcal{A}$ be the set of all charts (U,x) of M such that $dx_1\wedge\cdots\wedge dx_m(p)\in\Lambda$ for all $p\in U$. Assume also that each U is connected. We need to compute the transition functions to check ....WHAT?..... Let (V,y) be a second chart from $\mathcal{A}$ such that $p\in U\cap V$ then

$$dx_1\wedge\cdots\wedge dx_m|_p=\det(\frac{\partial x_k}{\partial y_j})(p)dy_1\wedge\cdots\wedge dy_m|p$$

Since $\det(\frac{\partial x_k}{\partial y_j})(p)=\det(D(x\circ y^{-1})(y(p)))>0$. hence the transition functions are positive.

3 March 9, 2015

Theorem Let M be a connected manifold. The following are equivalent

(1) M is orientable

(2) There exists an atlas $\mathscr{A}$ of M such that the detereminant of $D(x\circ y^{-1})(y(p))>0$ for all $(U,x),(V,y)\in\mathscr{A}$ and $p\in U\cap V$.

(3) There is a nowhere vanishing $\omega\in\Omega^m(M)$ with $m=\dim(M)$.

Proof. (2) $\Rightarrow$ (3). Under the hypotheses of (2). Chose a smooth partition of unity $(\phi_i)_{i\in\mathbb{N}}$ of M subordinate to $\mathscr{A}$; that is, for each $i\in\mathbb{N}$ there is $(U_i,x^{(i)})\in\mathscr{A}$ such that $\text{supp}(\phi_i)\subset U$ is relatively compact; that is, it's closure is compact and contained in U. ($\sum\phi_i=1$, $(\text{supp}(\phi_i))$ is locally finite). Put $\omega:=\sum_{i\in\mathbb{N}}\phi_i\cdot dx_1^{(i)}\wedge\cdots\wedge dx_m^{(i)}$. If $p\in U^{(i)}\cap U^{(j)}$, then $dx_1^{(i)}\wedge\cdots\wedge dx_m^{(i)}=\lambda_{ij}dx_1^{(j)}\wedge\cdots\wedge dx_m^{(j)}$ where $\lambda_{ij}=\det(D(x^{(i)}\circ x^{(j)-1})(x^{(j)}(p))>0$. Then $\omega(p)=(\sum_{i\in\mathbb{N}}\phi_j(p)\lambda_{ji}(p))dx_1^{(j)}\wedge\cdots\wedge dx_m^{(m)}(p)$. Since each term is greater than 0.

(3) $\Rightarrow$ (1). Under the hypotheses of (3), there is a nowhere vanishing $\omega\in\Omega^m(M)$ such that $\omega(p)$ is nonzero for all $p\in M$. Then $\Lambda^m T^*M\sim 0_m$ is the union of $\Lambda^+:=\{\rho\in\lambda^m T^*M:\rho=\lambda\cdot\omega_{\pi(\rho)}\text{ for some }\lambda>0\}$ and $\Lambda^-:=\{\rho\in\lambda^m T^*M:\rho=\lambda\cdot\omega_{\pi(\rho)}\text{ for some }\lambda<0\}$. Notice $\Lambda^+\cap\Lambda^-=\varnothing$. Then show Λ^+ and Λ^- are path connected. Take (p,ρ) and (q,τ). We may connect ρ by a path to $\omega(p)$ where $\gamma(t)=(\lambda(1-t)+t)\omega(p)+\rho=\lambda\omega(p)$ where $\lambda>0$. Then, $\omega(p)$ may be connected by a path with $\omega(q)$ by taking $\tilde{\gamma}(t)$ a path where $\tilde{\gamma}(0)=p$ and $\tilde{\gamma}(1)=q$ and put $\gamma(t)=\omega(\tilde{\gamma}(t))$. We do this so we can integrate on M. ∎

We have orientation so we may properly integrate. Reminder: $D\subset\mathbb{R}^n$ is open and bounded, $\phi:D\to\tilde{D}\subset\mathbb{R}^n$ is a diffeomorphism, and $f:\phi(D)\to\mathbb{R}$ a continuous function, then

$$\int_{\phi(A)}f=\int_A f\circ\phi|\det(D\phi)|\text{ (transformation formula)}$$

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Assume $\omega \in \Omega^n(\tilde{D})$. Then $\omega = f dx_1 \wedge \cdots \wedge dx_n$ for some $f \in C^\infty(\tilde{D})$. Put $\tilde{A} = \phi(A)$, and define

$$\int_{\tilde(A)} \omega := \int_{\tilde{A}} f$$

condsider $\phi^*(\omega) \in \Omega^n(D)$. Then $\phi^*(f dx_1 \wedge \cdots \wedge dx_n) = (f \circ \phi) \cdot \det(D\phi) \cdot dx_1 \wedge \cdots \wedge dx_n$

Let M be an oriented manifold and $\mathscr{A}$ an oriented atlas. Choose a partition of unity $(\phi_i)_{i \in \mathbb{N}}$ subordinate to $\mathscr{A}$. For each $\omega \in \Omega_c^m(M)$, put

$$\int_M \omega = \sum_{i \in \mathbb{N}} \int_{\tilde{U}_i} \phi_i x^{-1*} \omega$$

Prove this is independent of atlas.

4 March 11, 2015

Notations:

(1) Denote $\mathbb{H}^n$ as the upper-half space which is the set $\{(x_1, \ldots, x_n) \in \mathbb{R}^n | x_1 \geq 0\}$.

(2) The interior of a manifold with boundary is denoted $M^\circ = M \sim \partial M$.

Definition: A **manifold with boundary** M is a topolgical space which is Hausdorff and second countable subject to the following conditions:

(1) A chart of M in $\mathbb{H}^n$ is a homeomorphism $x : U \subset M \to \hat{U} \subset \mathbb{H}^n$ where $U, \hat{U}$ are open.

(2) Two charts $(U, x), (V, y)$ of M in $\mathbb{H}^n$ are called C^∞-compatible charts if $x \circ y^{-1}|_{U \cap V} : y(U \cap V) \to x(U \cap V)$ is a C^∞-diffeomorphism.

(3) An atlas of M in $\mathbb{H}^n$ consists of a set of C^∞-compatible charts in $\mathbb{H}^n$ which cover M.

(4) A maximal atlas $\mathscr{A}$

Definition: Let M be a manifold with boundary. Define $\partial M \subset M$ as the set of points $p \in M$ such that there is a chart (U, x) around p with $x(p) = (0, x_2(p), \ldots, x_n(p))$ with $p \in x^{-1}(\{0\} \times \mathbb{R}^{n-1}$.

Observations:

(1) ∂M is a manifold of dimension $n-1$. It's atlas is given by charts $(U \cap M, \bar{x}|_{U \cap \partial M})$ where $(U, x) \in \mathscr{A}$ and $p \in V$, with $\bar{x}(p) = (x_2(p), \ldots, x_n(p))$ (from $(0, x_2(p), \ldots, x_n(p))$). This gives us transition functions $\bar{x} \circ \bar{y}^{-1} : \bar{y}(U \cap V \cap \partial M) \to \bar{x}(U \cap V \cap \partial M)$ is a diffeomorphism.

(2) The tangent spaces of the interior are obvious. On the boundary, using curves ends up being very technical. The space of derivations definition gives a more obvious definition of the tangent space on the boundary. So, $T_p M = \text{Der}(C_p^\infty, \mathbb{R})$ for $p \in \partial M$. Then the tangent space is spanned by $\{\frac{\partial}{\partial x_1}|_p, \ldots, \frac{\partial}{\partial x_n}|_p\}$.

(3) Orientation is defined in the same way. Notice that the boundary has an induced orientation. We get this from ...

Theorem: (Stokes) Given a compact oriented m-manifold M with boundary. Then for each $\omega \in \Omega^{m-1}(M)$ Then

$$\int_{\partial M} \omega|_{\partial M} = \int_M d\omega$$

where ∂M has the indeuced orientation.

Proof. Recall the fundamental theorem of calculus:

$$\begin{aligned}\int_0^a \frac{\partial}{\partial s} f(s, t_2, \ldots, t_m) ds &= f(A, t_2, \ldots, t_m) - f(0, t_2, \ldots, t_m) \\ &= \int_{\{A\}} f(t, t_2, \ldots, t_n) dt - \int_{\{0\}} f(t, t_2, \ldots t_n) dt\end{aligned}$$

Now, let $Q \subset \mathbb{H}^n$ be a cube; that is, $Q = [a_1, b_1] \times \cdots [a_n, b_n]$ with $a_1 \geq 0$ and $a_2, \ldots a_n \in \mathbb{R}$, $b_i > a_i$ for all $i \in \{1, \ldots, n\}$. Let $\omega \in \Omega^{m-1}(Q)$ with the support of ω compactly contained in Q. Locally, we may represent ω as $\sum_{i=1}^m \omega_i dx_1 \wedge \cdots \wedge \hat{dx_i} \wedge \cdots \wedge dx_m$ for $\omega_i \in C^\infty(Q)$. Then

$$\begin{aligned}\int_Q d\omega &= \sum_i (-1)^i \int_Q \frac{\partial \omega_i}{\partial x_i} dx_1 \wedge \cdots \wedge dx_m \\ &= \sum_i \int_{Q_i}\end{aligned}$$

$\square$

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We have $Q = [A_1, B_1] \times \cdots \times [A_n, B_n] \subset \mathbb{R}^n$, $\omega \in \Omega^{n-1}(M)$, and

$$\operatorname{supp}(\omega) \subset\subset \begin{cases} (A_1, B_1) \times \cdots \times (A_n, B_n) & A_1 > 0 \\ [0, B_1) \times (A_2, B_2) \times \cdots \times (A_n, B_n) \end{cases}$$

$\omega = \sum_i \omega_i dx_1 \wedge \cdots \wedge \hat{dx_i} \wedge \cdots \wedge dx_n$, $d\omega = \sum_i (-1)^i \frac{\partial \omega_i}{\partial x_i} dx_1 \wedge \cdots dx_n$, and

$$\begin{aligned}\int_M d\omega &= \sum_i \int_{Q_i} (\int_{A_i}^{B_i} \frac{\partial \omega_i}{\partial x_i} dx_i) \wedge dx_1 \wedge \cdots \wedge \hat{dx_i} \wedge \cdots \wedge dx_n \\ &= -\int_{Q_1} \omega_1 dx_2 \wedge \cdots \wedge dx_n\end{aligned}$$

where $Q_i = [A_1, B_1] \times \cdots \times [A_i \hat{,} B_i] \times \cdots \times [A_n, B_n]$, because

$$\int_{A_i}^{B_i} \frac{\partial \omega_i}{\partial x_i} dx_i = \omega_i(B_i) - \omega_i(A_i) = 0 - 0 = 0$$

For the boundary, we have

$$\int_{\partial Q} \omega = \int_{Q_1} \omega = \int_{Q_1} -\omega_1 d\tilde{x}_2 \wedge \cdots \wedge d\tilde{x}_n$$

We want an outward pointing orientation. Notice $\frac{\partial}{\partial x_1}$ points invward to M (with respect to Q) but we want to orient Q, resp M such that

$$-\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n}$$

Notice $\partial Q_1 \cup \cup_{l \geq 2} Q_l$

Now we choose an oriented atlas $\mathscr{A}$ of M, after passing to a finite atlas, we can assume that $\tilde{U} \subset \mathbb{R}^n$ for $(U, x) \in \mathscr{A}$ has form $[A_1, B_1] \times \cdots \times [A_n, B_n] \subset \mathbb{H}^n$ and $x : U \to \tilde{U} \subset \mathbb{H}^n$ and such that $\mathscr{A}$ is countable. Choose a smooth partition of unity. Choose a smooth partition of unity $(\phi_{(U,x)})_{(U,x) \in \mathscr{A}}$ subordinate to $\mathscr{A}$ where $\text{supp}(\phi_{(U,x)}) \subset\subset U \Rightarrow x_*(\phi_{(U,x)}\omega \subset\subset \tilde{U} \subset \mathbb{H}^n)$ and

$$\begin{aligned}
\int_M d\omega &= \sum_{(U,x) \in \mathscr{A}} \int_{\tilde{U}} d(x_* \phi_{(U,x)} \omega) \\
&= \sum_{(U,x) \in \mathscr{A}} \int_{\partial \tilde{U}} x_*(\phi_{(U,x)} \omega) \\
&= \sum_{(U,x) \in \mathscr{A}} \int_{\partial \tilde{U}} x_*(\phi_{(U,x)} \omega|_{\partial M}) \\
&= \int_{\partial M} \omega
\end{aligned}$$

Corollary: If M is a closed manifold (compact and no boundary) then

$$\int_M d\omega = 0$$

For $\omega \in \Omega^{\dim M}(M)$

Integration of Vector Fields Let $\xi : M \to TM$ be a C^∞ vector field on a smooth manifold M. A curve $\gamma : I \to M$, $I = (a, b) \subset \mathbb{R}$ is called an integral curve of M if $\xi(\gamma(t)) \in T_{\gamma(t)}M$ is equal to $\dot{\gamma}(t)$ for all $t \in I$.

Question:

6 March 16, 2015

Integral Curves: Let $\xi : M \to TM$ be a smooth vector field on a manifold M. By an integral curve of ξ, one understands a smooth map $\gamma : I \to M$, with $I \subset \mathbb{R}$ an open interval, such that

$$\dot{\gamma}(t) = \xi(\gamma(t)) \text{ for all } t \in I$$

Observations:

(1) For each $p \in M$, there exists an open interval $I \subset \mathbb{R}$ containing the origin 0, and a smooth integral curve $\gamma : I \to M$ of ξ such that $\gamma(0) = p$.

Proof. Choose coordinates (U, x) of M around p, and then consider the following ordinary differential equation:

$$\dot{c}(t) = F(c(t)) \qquad\qquad c(0) = x(p)$$

where $F := (pr_2 \circ Tc \circ \xi \circ x^{-1}) : \hat{U} \to \mathbb{R}^n$. By existence and uniqueness (Picard-Lindelof theorem), there exists $c : (-\varepsilon, \varepsilon) \to \hat{U}$ such that the initial value problem is satisfied. We put $\gamma := x^{-1} \circ c : (-\varepsilon, \varepsilon) \to M$, then $\gamma(0) = p$ and $\dot{\gamma}(t) = (Tx)^{-1}(c(t), \dot{c}(t)) = Tx^{-1}(c(t), F(c(t))) = \xi(x^{-1}(c(t))) = \xi(\gamma(t))$. (Note that this holds true in Banach manifolds) $\square$

(2) If γ_1, γ_2 are integral curves of ξ with $\gamma_1(0) = \gamma_2(0) = p$, then $\gamma_1|_{I_1 \cap I_2} = \gamma_2|_{I_1 \cap I_2}$ (Note $I_1 \cap I_2$ is nonempty since they both implicitly contain 0)

Proof. Let $K = \{t \in I_1 \cap I_2 : \gamma_1(t) = \gamma_2(t)\}$. We have $K = (\gamma_1, \gamma_2)^{-1}(\Delta_M)$. (note $(\gamma_1, \gamma_2) : I_1 \cap I_2 \to M \times M$). By continuity of γ_1 and γ_2 and M being Hausdorff, $I_1 \cap I_2$ is an open interval around the origin, hence connected. Let $t \in K$. Consider $\tilde{\gamma_1} : I_1 - t \to M$ and $\tilde{\gamma_2} : I_2 + t \to M$ where $\tilde{\gamma_i}(s) = \gamma_i(s + t)$. So $\tilde{\gamma_1}(0) = \gamma_1(t) = \gamma_2(t) = \tilde{\gamma_2}(0)$, so $\dot{\tilde{\gamma_i}}(s) = \dot{\gamma_i}(s + t) = \xi(\gamma_i(x + t)) = \xi(\tilde{\gamma_i}(s))$. By local uniqueness of the initial value problem, there exists an ε such that $\tilde{\gamma_1}(s) = \tilde{\gamma_2}(s)$ for $s \in (-\varepsilon, \varepsilon)$. Hence γ_1 and γ_2 agree on an ε-neighborhood of t. $\square$

(3) For each $p \in M$, let $I_p = (t_p^-, f_p^+)$ with $t_p^- < t_p^+$ and $t_p^-, t_p^+ \in \mathbb{R} \cup \{\pm\infty\}$. There of all intervals I such that there exists an integral curve $\gamma : I \to M$ of ξ with $\gamma(0) = p$. Define $\gamma_p : I_p \to M$ by $t \mapsto \gamma(t)$, where $t \in I$ with $\gamma : I \to M$ (If M is compact , the $I_p = \mathbb{R}$, a counterexample is the plane with a point removed and having a constant vector field oriented upwards). Now put $\mathcal{D} = \cup_{p \in M} I_p \times \{p\} \subset \mathbb{R} \times M$, and $\phi : \mathcal{D} \to M$, $(t, p) \mapsto \gamma_p(t)$. Then ϕ is called the flow of the vector field ξ. It has the following nice properties:

(a) $\mathcal{D} \subset \mathbb{R} \times M$ is open.

(b) The domain $\phi_t \circ \phi_s \subset$ domain ϕ_{t+s} where $\phi_t : M \to M$ where $p \mapsto \phi(t, p)$

(c) $\phi_{t+s}(p) = \phi_t \circ \phi_s(p)$ for $p \in \mathrm{dom}(\phi_t \circ \phi_s)$.

(d) ϕ_d

Proof. $\square$

7 18 March, 2015 (Wednesday)

Banach Fixed Point Theorem: If you have a complete metric space with a Lipschitz contraction, then the space has a unique fixed point.

Proposition: Let J be an open interval containing 0, U an open set of a banach space $\mathbb{E}$, and $x_0 \in \mathbb{E}$. Let $a \in (0, 1)$ such that the closed ball $\bar{B}_{3a} \subset U$.

Assume that $f : J \times U \to \mathbb{E}$ be a bounded continuous map, bounded by constant $L \geq 1$, and satisfying on U uniformly with respect to J a Lipschitz condition with Lipschitz constant $K \geq 1$. Then $\|f(t,x) - f(t,y)\| \leq K\|x-y\|$ for all $t \in J$ and $x, y \in U$. If $b < \frac{a}{LK}$, then for each $x \in \bar{B}_a(x_0)$ there exists a unique flow $\phi : J_b \times B_a(x) \to U$; that is, $\frac{d}{dt}\phi(t,x) = f(t, \phi(t,x))$ and $\phi(0,x) = x$. Letting $I_b = [-b, b]$, and let x be fixed in $\bar{B}_a(x_0)$. Let M be a set of continuous maps

$$a : I_b \to \bar{B}_{2a}(x_0)$$

We have that M is a complete metric space with distance given by the sup-norm.

$$S : M \to M \qquad\qquad s\alpha(t) = x + \int_0^t f(u, \alpha(u))du$$

Choose S fulfills Lipschitz-condition with Lipschitz-constant $L_x < 1$ which implies there exists a unique fixed point by the Banach Fixed Point Theorem. Call this $\phi_x \in M$ with $s\phi_x = \phi_x$. By the fundamental theorem of calculus, we have $\phi_x(t) = x + \int_0^t f(u, \phi_x(u))du$ is differentiable; that is, $\dot{\phi_x}(t) = f(t, \phi_x(t))$ with $\phi_x(0) = x$. If f is C^k for $k \in \mathbb{N}^* \cup \{+\infty\}$, then ϕ is C^k. Look at Lang Differentiable Manifolds for the full proof.

Last lecture we had $\phi : \mathcal{D} \to M$ by $(t,p) \mapsto \gamma_p(t)$. Then ϕ has the following properties:

(1) $\mathcal{D} \subset \mathbb{R} \times M$ is open

(2) $\mathrm{dom}(\phi_s \circ \phi_t) \subset \mathrm{dom}(\phi_{s+t})$ where $\phi_t : \mathcal{D} \cap \{t\} \times M = \mathcal{D}_y = \mathrm{dom}(\phi_t)$ by $p \mapsto \phi(t,p)$

(3) We also have $\phi_{t+s} = \phi_t \circ \phi_s$ for $p \in \mathrm{dom}(\phi_t \circ \phi_s)$

(4) $\phi_t : \mathcal{D}_t \to \mathcal{D}_{-t}$ is a diffeomorphism with inverse ϕ_{-t}

Proof. (a) Local flow theorem from Lang

(b) Let $s \in (t_-(p), t_+(p))$ Then $f \mapsto \gamma_p(s+t)$ is an integral curve of $\mathcal{G}$ and has maximal domain $(t_-(p) - s, t_+(p) - s) = (t_-(\gamma_p(s)), t_+(\gamma_p(s)))$. Since $\gamma_p(s+0) = \gamma_p(s)$. Now $p \in \mathrm{dom}(\phi_t \circ \phi_s) \Rightarrow p \in \mathrm{dom}(\phi_s) \Rightarrow s \in (t_-(p), t_+(p))$ and $t \in (t_-(\gamma_p(s)), t_+(\gamma_p(s)) \Rightarrow t + s \in (t_-(p), t_+(p))$.

□

8 March 20, 2015 (Friday)

Lie Derivatives: We want to take derivatives of vector fields $\xi : M \to TM$ which gives a tangent map $T\xi : TM \to T(TM)$. Assume $W : M \to TM$ is a second vector field. We want to define a derivative of ξ with respect to W.

Lie Derivative: Looking at the flow of W, $\phi : \mathcal{D} \to M$ with

$$\mathcal{L}_W \xi(p) := \lim_{t \to 0} \frac{T\phi_{-t}(\xi_{\phi_t(p)}) - \xi_p}{t} = \frac{d}{dt} T\phi_{-t}(\xi_{\phi_t(p)})|_{t=0}$$

Notice that the limit exists in coordinates since all the functions are smooth. The map $\mathcal{L}_W$ is called the Lie derivative.

Observations:

(1) $\mathcal{L}_W f = W(f)$

(2) $\mathcal{L}_W \xi = [W, \xi]$

(3) $\mathcal{L}_W$ is tensorial in W only over $\mathbb{R}$, not $C^\infty(M)$.

(4) $\mathcal{L}_W : \Omega^\bullet(M) \to \Omega^\bullet(M)$ commutes with d.

(5) $\mathcal{L}_W(\omega \wedge \rho) = \mathcal{L}_W\omega \wedge \rho + \omega \wedge \mathcal{L}_W \rho$

(6) **Cartan's Magical Formula:** $\mathcal{L}_W\omega = i_W d\omega + di_W\omega$ for $\omega \in \Omega^k(M)$ where $i_W \in \Omega^{k-1}(M)$ is defined by $i_W\omega(Y_1, \ldots, Y_{k-1}) = \omega(W, Y_1, \ldots, Y_k)$ (useful for proving Poincare's lemma).

Proof. (1) $\mathcal{L}_W f(p) = \frac{d}{dt}(\phi_t^* f)(p) = \frac{d}{dt}(f \circ \phi_t(p))|_{t=0} = W(p) \cdot [f]_p$.
(2) We show that the bracket is a derivation to show that the bracket is still a vector field. **Exercise:** do this.
(3) Omitted
(4) We have $\mathcal{L}_W d\omega = \frac{d}{dt}\phi_t^*(d\omega)|_{t=0} = \frac{d}{dt}d(\phi_t^*\omega)|_{t=0} = d(\frac{d}{dt}\phi_t^*\omega)|_{t=0}$
(5) Same argument as (4)
(6) We prove this by induction on k. For $k = 0$, $\mathcal{L}_W f = Wf$ and $i_w df + di_W f = i_w df = Wf$. Assume this holds true for $k - 1$. $\square$

9 March 30, 2015 (Monday)

Proposition: $\mathcal{L}_X Y = [X, Y]$

Proof. For $f \in C^\infty(M)$ we have

$$\begin{aligned}
\mathcal{L}_X Y(f) &= (\lim_{t\to 0} \frac{TX_{-t}Y_{x_y(m)} - y_m}{t})(f) \\
&= \frac{d}{dt}|_{t=0}(TX_{-t}Y_{X_t(m)})(t) \\
&= \frac{d}{dt}|_{t=0}Y_{X_t(m)}(f \circ X_{-t})
\end{aligned}$$

For the auxillary function $H(t, u) = f(X_{-t}(Y_u(X_t(m))))$ with $(t, u) \in \mathbb{R}^2$, small enough. We have

$$Y_{X_t(m)}(f \circ X_{-t}) = \frac{\partial}{\partial r_2}|_{(t,0)}H(t, r_2)$$

Then we have $\mathcal{L}_X Y(f) = \frac{\partial^2}{\partial r_1 \partial r_2}|_{(0,0)}$. Consider another auxillary function

$K(t,u,s) = f(X_s(Y_u(X_t(m))))$ we have $H(t,u) = K(t,u,-t)$ Then

$$\begin{aligned}
\mathcal{L}_X Y(f) &= \frac{\partial^2 K}{\partial r_1 \partial r_2}|_{(0,0,0)} - \frac{\partial^2 K}{\partial r_2 \partial r_3}|_{(0,0,0)} \\
\frac{\partial K}{\partial r_2}|_{(t,0,0)} &= Y_{X_t(m)} f = (Yf)(X_t(m)) \\
\frac{\partial^2 K}{\partial r_1 \partial r_2}|_{(0,0,0)} &= X_m(Yf) \\
\frac{\partial K}{\partial r_3}|_{(0,0,0)} &= Xf(Y_u(m)) \\
\frac{\partial^2 K}{\partial r_1 \partial r_3}|_{(0,0,0)} &= Y_m(Xf)
\end{aligned}$$

□

Cartan's Magic Formula $\mathcal{L}_X \omega = i_X d\omega + d i_X \omega$ for $\omega \in \Omega^k(M)$.

Proof. This proof follows from induction. For $k = 0$

$$\mathcal{L}_X f = Xf = i_X df = i_X df + d i_X f$$

Now, for the induction step, take

$$\begin{aligned}
\mathcal{L}_X (df \wedge \omega) &= \mathcal{L}_X df \wedge \omega + df \wedge \mathcal{L}_X \omega \\
(i_X d + d i_X)(df \wedge \omega) &= -df \wedge i_X d\omega + d(i_X df \wedge \omega - df \wedge i_X \omega) \\
&= -df \wedge i_X d\omega + d i_X df \wedge \omega + (i_X df) \wedge d\omega + df \wedge d i_X \omega \\
&= \mathcal{L}_X df \wedge \omega \cdots \text{ look in Tu}
\end{aligned}$$

Exercise: Show $i_X(\rho \wedge \omega) = i_X \rho \wedge \omega + (-1)^{deg(\rho)} \rho \wedge i_X \omega$ □

10 April 1, 2015 (Wednesday)

Andy Given a smooth n-manifold M, a **Riemannian metric** g is a smooth symmetry covariant 2-tensor field on M that is positive definite at each point in M; that is, $g \in \Gamma(T*M \otimes T*M)$. Locally, we may express g as $g_{ij} dx^i \otimes dx^j$ for coordinates $(U, x^1, \ldots, x^n)$ where (g_{ij}) is a positive definite matrix of smooth functions.

A Kahler structure on a Riemannian manifold (M^n, g) is given by a 2-form ω and a field of endomorphisms J on the tangent bundle such that

Algebraic conditions:

(1) J is an almost complex structure; that is, $J^2 = -Id$ as an endomorphism on the tangent space

(2) $g(X,Y) = g(JX, JY)$ for each $X, Y \in \Gamma(TM)$

(3) $\omega(X,Y) = g(JX,Y)$

Analytic conditions:

(4) The 2-form ω is closed; ie, $d\omega = 0$
(5) J is integrable

Note that (1) and (5) are equivalent to having a holomorphic structure. If $N(X,Y) = 2([JX,JY] - [X,Y] - [JX,Y] - [X,JY]) = 0$ we have the holomorphic structure.

Locally, we may express ω as $ih_{\alpha\beta} dz_\alpha \wedge dz_{\bar\beta}$ where $h_{\alpha\beta} = h(\frac{\partial}{\partial z_\alpha}, \frac{\partial}{\partial z_{\bar\beta}})$ and h is hermitian. Also, $\frac{\partial^2 u}{\partial z_\alpha \partial z_{\bar\beta}}$ where u is the Kahler potential. As a side remark, the only solutions found to the Einstein vacuum equation $R_{\alpha\beta} = 0$ are Kahler manifolds.

A complex manifold is a smooth manifold of dimension $2n$ which admits a holomorphic atlas $\{U_i, \phi_i\}$ such that the transition functions ϕ_i are biholomorphic and map into $\mathbb{C}^n$. Remember that a functions $F = f + ig$ is holomorphic if it satisfies the Cauchy-Riemann equations

$$\frac{\partial f}{\partial x} = \frac{\partial g}{\partial y} \qquad\qquad \frac{\partial f}{\partial y} = -\frac{\partial g}{\partial x}$$

Exercise: Show that this is equivalent to the equation $\frac{\partial F}{\partial \bar z} = 0$

The canonical examples of a kahler manifolds are the complex projective plane, tori, $\mathbb{C}^n$, and Riemann surfaces. Note that every complex variety may be embedded in $\mathbb{CP}^n$.

Nicholas: A **Calabi-Yau manifold** is a compact Kahler manifold where the holonomy group is $SU(d)$ where d is the complex dimension.

Definition: Take $C^\infty(M,TM)$ as the space of vector fields on M. A bilinear map $\nabla : C^\infty(M,TM) \to C^\infty(M,TM)$ where $(X,Y) \mapsto \nabla_X Y$ is a connection if it satisfies

(1) $\nabla_{fX} Y = f\nabla_X Y$ for each $f \in C^\infty(M,TM)$
(2) $\nabla_X(fX) = X(f)Y + f\nabla_X Y$

Definition: A vector field X is parallel if $\nabla_Y X = 0$ for every $Y \in C^\infty(M,TM)$

Take $\gamma : [a,b] \to M$ be a smooth curve on M. A vector field X on $\gamma([a,b])$ is called a parallel transport of a vector $v \in T_{\gamma(a)M}$ if $\nabla_{\dot{\gamma(t)}} X = 0$ for each t and $X(a) = v$.

If X is a parallel transport of v and Y is a parallel transport of w (both along γ) Then $c_1X + c_2Y$ is the unique parallel transport of $c_1v + c_2w$ along γ. Let X^{e_i} be a parallel transport of e_i along γ. Taking $f_\gamma : T_{\gamma(a)}M \to T_{\gamma(b)}M$ by $v = i^i e_i \mapsto i^i X^{e_i}$.

Considering all loops in M based at $p \in M$. Taking α as a loop of M, the map $f_\alpha : T_{\gamma(a)} \to T_{\gamma(b)}M \in GL(n;\mathbb{R})$

11 April 6, 2015 (Monday)

Let V be a finite dimensional $\mathbb{R}$-vector space and $\lambda : V \times V \to \mathbb{R}$ a **symmetric bilinear form**; that is, λ satisfies the following properites:

(1) $\lambda(v + v', w) = \lambda(v,w) + \lambda(v',w)$
(2) $\lambda(av,w) = \lambda(v,aw) = a\lambda(v,w)$

(3) $\lambda(v,w) = \lambda(w,v)$

Moreover, we say λ is **nondegenerate** if $\lambda(v,w) = 0$ if $v = 0$ or $w = 0$.

Theorem:(Sylvester) If $\Lambda : V \times V \to \mathbb{R}$ is a symmetric bilinear form, then there is a basis $(b_i)_{i=1}^n$ of V such that λ has the matrix

Observation: λ is non-degenerate iff $ker(\lambda_i j) = 0$.

Definitions: The **signature** of λ is (n_+, n_-) where n_+ is the number of positive eigenvalues and n_- is the number of negative eigenvalues. If n_+ is the dimension of V, then λ is called **positive-definite**. Also, n_- is called the **index** of λ.

Definition: A **semi-riemannian** n-manifold is a manifold M together with a nondegenerate symmetric tensor $g \in \Gamma(T^*M \otimes T^*M)$ such that the index g_p at $p \in M$ is constant for any $p \in M$. If the index of g is 0, then (M,g) is called **riemannian**. Locally, for some chart (U,ϕ) with local coordinates, $x^1, \ldots, x^n$, we can express g as

$$g = g_{ij} dx^i \otimes dx^j$$

Sidenote: General relativity is the geometry of 4-dimensional semi-Riemannian manifolds with index 1. A semi-riemannian metric with index 1 is called a **Lorentz metric**.

Remark: There is no Lorentz metric on S^2. (Of topological nature)

Theorem: Every manifold admits a Riemannian metric

Proof. Let $\mathscr{A}$ be an atlas of M. For each $(U,x) \in \mathscr{A}$, put $g_U := x^*(\langle -,- \rangle)$ of the standard euclidean metric on $\mathbb{R}^n$. Choose a partition of unity subordinate to $\mathscr{A}$, (ϕ_V). Put

$$g(v,w) = \sum_{(U,\phi)\in\mathscr{A}} \phi_U(p) g_U(v,x) \text{ for } v,w \in T_pM$$

Notice that each point g_p is positive definite and symmetric. □

Observation: For a lorentz metric, it may cancel out on the partition of unity. Observe

Assume (M,g) is semi-riemannian metric. Let (U,x) be a chart and $\frac{\partial}{\partial x_i}$ a local frame of TM. Put $g_{ij}^{(U,x)} := g(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}) \in C^\infty(U)$. If (V,y) is another coordinate chart with $U \cap V \neq \varnothing$, we want to know how the local expression of g transforms.

$$\frac{\partial}{\partial y_j}|_p = \sum_{k=1}^n \frac{\partial(x_k \circ y^{-1})}{\partial y_j}(p) \frac{\partial}{\partial x_k}|p \text{ and}$$

$$g_{ij}^{(V,y)}(p) = \sum_{k,l=1}^n \frac{\partial(x_k \circ y^{-1})}{\partial y_j}(p) \cdot \frac{\partial(x_l \circ y^{-1})}{y_j}(p) g_{kl}^{(U,x)}(p)$$

Assume $N \hookrightarrow M$ is a submanifold, and that g is a semi-riemannian metric on M. Then, one can pull-back g to N to a get a symmetric 2-tensor $i^*g \in C^\infty(T^*M \otimes T^*M)$ with

$$i^*g(p)(v,w) = g(i(p))(Ti(y), Ti(w))$$

Observations:

(1) If g is positive definite, the i^*g is so as well.

(2) The pull-back of a semi-riemannian metric may not be semi-riemannian. The obstructions for this are topological, but

12 April 8, 2015 (Wednesday)

Exotic Spheres:(Milnor) There is a family of smooth 7-manifolds with are homeomorphic to $S^7 \subset \mathbb{R}^8$, but not diffeomorphic.

Example Consider

$$\begin{array}{ccc} \tilde{\mathbb{R}} & x & \mathbb{R} \\ \Big\downarrow{\scriptstyle \psi} & \Big\downarrow & \Big\downarrow{\scriptstyle Id} \\ \mathbb{R} & x^3 & \mathbb{R} \end{array}$$

Observe that these manifolds do not have the same smooth structure, but are diffeomorphic by $\tilde{\mathbb{R}} \xrightarrow{x^3} \mathbb{R}$.

(1) We want M to be homeomorphic to S^n

(2) Construct M_k^7 by sphere bundles $E \to S^4$

(3) Prove that $M_k^7 \cong S^n$ as a homeomorphism.

(4) (Black Magic) Construct an invariant $\lambda(M_k^7) \neq \lambda(S^7)$.

First $p \in M$ is a point, and $f : M \to \mathbb{R}$ is a morse function if the Hessian matrix of the critical points is non-singular. Recall that the critical points are the $p \in M$ such that $dH_p = 0$. The Hessian matrix can be represented as the matrix

$$\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)_{i,j}$$

Theorem: If M is a compact n-manifold with f a morse function with 2 critical points, then M is homeomorphic to S^n.

Theorem: Let $f \in C^\infty(M)$, $M^r = f^{-1}(-\infty, r)$, $a < b \in \mathbb{R}$. If $f^{-1}([a, b])$ is compact with no critical points, then M^a is diffeomorphic to M^b.

Proof. Let g be a Riemannian metric $g(X, Y) = \langle X, Y\rangle$. Let $\text{grad}(f) \in \mathcal{X}(M)$ with $\langle \text{grad}(f), Y\rangle = \tilde{X}(f)$. Observe $X = \phi\text{grad}(f)$ for $\phi \in C^\infty(f^{-1}[[a, b])$ with

$$\phi = \frac{1}{\|\text{grad}(f)\|^2}$$

is a vector field of compact support. Defines a flow ϕ_t with $X(p) = \frac{d}{dt}\phi_t(p)$; consider $f(\phi_t(q))$ as a function of t. If $\phi_t(q) \in f^{-1}[a, b]$, then $\frac{d}{dt} f(\phi(q))) = \langle \frac{d\phi_t(q)}{dt}, \text{grad}(f)\rangle = X(f) = \phi\|\text{grad}(f)\|^2 = 1$. This implies that $f(\phi_t(q)) = f(q) + t$. If $f(q) \leq a$, then $f(\phi_{b-a}(q)) = f(q) + b - a \leq b$. $\square$

(2) For constructing M_k^7, consider a sphere bundle $S^2 \hookrightarrow M_k^7 \to S^4$. Observe that $S^4 = U^+ \cup U^-$ for $U^+ = S^4 \sim N$ and $U^- = S^4 \sim S$ and each of these sets are homeomorphic to $\mathbb{R}^4$. Decompose M_k^7 are the union of the preimage of these sets, and denote them V^+ and V^- respectively, these are homeomorphic to $\mathbb{R}^4 \times S^3$. Define a map $V^+ \to V-$ by

$$(u; v) \mapsto \left(\frac{u}{\|u\|^2}; \frac{u^i v u^j}{\|u\|} \right) = (u'; v')$$

with $u \in \mathbb{H}$ and $v \in S^3 \subset \mathbb{H}$. We define a morse function $f(u; v)$ by

$$\frac{\mathrm{Re}(v)}{(1 + \|u\|^2)^{1/2}} = \frac{\mathrm{Re}(u'')}{(1 + \|u''\|^2)^{1/2}}$$

where $u'' = u'(v')^{-1}$.

13 April 10, 2015

Morse Theory: Studies smooth functions on a manifold to better understand the underlying topological structure.

Let $f : M \to \mathbb{R}$ be a smooth function. Then the points $p \in M$ such that the differential of f is the 0 map are called **critical points**. In local coordinates, this may be expressed as

$$\frac{\partial f}{\partial x_i}(p) = 0$$

Definition: The Hessian matrix is the matrix

$$H_f \left[\frac{\partial^2}{\partial x_i \partial x_j} \right] = \left(\frac{\partial^2 f}{\partial x_i \partial x_j} \right)_{i,j}$$

Definition: A critical point is nondegenerate at p of f is the Hessian matrix is nonsingular.

Proposition: The nondegeneracy of a point is independent of the chart used.

Definition: A smooth function $f \in C^\infty(M)$ is called a **Morse function** if all its critical points are nondegenerate.

Lemma:(Morse Lemma) For a smooth m-manifold M, a point b is a nondegenerate critical point of a smooth function f, there exists a chart $(x_1, \dots, x_m)$ such that $x_i(b) = 0$ and

$$f = -x_1^2 - x_2^2 - \cdots - x_\alpha^2 + x_{\alpha+1}^2 + \ldots + x_m^2 + f(b)$$

Corollary: Nondegenerate critical points are isolated (there exists a neighborhood of b such that b is the only critical point in this neighborhood)

Corollary: A Morse function on a compact m-manifold M has only finitely many critical points.

Definition: Two functions f, g on a smooth m-manifold M are called (C^2, ε)-close if the following three properties hold:

(1) $|f(p) - g(p)| < \varepsilon$

(2) $|\frac{\partial f}{\partial x_i}(p) - \frac{\partial g}{\partial x_i}(p)| < \varepsilon$

(3) $|\frac{\partial^2 f}{\partial x_i \partial x_j}(p) - \frac{\partial^2 g}{\partial x_i \partial x_j}(p)| < \varepsilon$

Theorem: Let $g : M \to \mathbb{R}$ be a smooth function. Then there exists a Morse function f such that f and g are (C^2, ε)-close.

14 13 April, 2015 (Monday)

Definition: The **Minkowski Metric** over $\mathbb{R}^4$ is the metric g such that for any vectors $v, w \in \mathbb{R}^4$, $g(v, w) = -v_1 w_1 + v_2 w_2 + v_3 w_3 + v_4 w_4$.

Definition: Recall that a local diffeomorphism

Definition: A local diffeomorphism $\phi : M \to N$ between semi-riemannian manifolds (M, g_M) and (N, g_N) is a local isometry if for all $p \in M$ and $v, w \in T_pM$,

$$g_M(v, w) = g_N(T_p\phi(v), T_p\phi(w))$$

Observation: For each semi-riemannian manifold (M, g), the set of isometries form a group, denoted by $Isom(M, g)$.

Exercise: Check that $Isom(M, g)$ is a group.

Examples:

(1) Maps from $\mathbb{R}^n, g_{euc})$ to itself of the form $f : \mathbb{R}^n \to \mathbb{R}^n$, by $v \mapsto Av + b$, where $A \in O(n, \mathbb{R})$ and $v, b \in \mathbb{R}^n$. This space of maps are called the **Euclidean transformations**. We denote this by $Trans_{euc}(\mathbb{R}^n)$. Notice that compositions of such transformations are an orthogonal transformation.

Theorem: $Trans_{euc}(\mathbb{R}^n) = Isom(\mathbb{R}^n, g_{euc})$ This is highly nontrivial to prove

(2) Maps $f : (\mathbb{R}^n, g_{Min}) \to (\mathbb{R}^n, g_{Min})$ of the form $f(v) = Av + b$ for $A \in O(n, 1) = \{A \in GL(n+1, \mathbb{R})) : g_{Min}(Av, Aw) = g_{Min}(v, w)\}$. The set of all transformations is a group called the poincare group. This is the isometry group.

(3) The set of isometries of the sphere S^n is $O(n+1)$.

Covariant Derivatives: Let $\eta : M \to TM$ be a vector field. It's exterior derivative is a map $T\eta : TM \to TTM$. If $\xi \in T_pM$, then $T\eta\xi \in T_{\eta(p)}TM \neq TM$; this is a problem!

Definition: By a covariant derivative (or connection) on a manifold M is a map $\nabla : \mathfrak{X}^\infty(M) \to \Omega^1(M) \otimes_{C^\infty(M)} \mathfrak{X}^\infty(M)$ such that the following holds true:

$$\nabla_\xi(f\eta) = df \otimes \eta + f\nabla_\xi(\eta)$$

This implies the following properties:

15 April 15, 2015 (Wednesday)