In[•]:=
```
(*-------------------------------------------*)
(*1. Parameters and scalar potential*)
(*-------------------------------------------*)ClearAll["Global`*"];

R0 = SetPrecision[2.0, 80];
w = SetPrecision[0.2, 80];
A = SetPrecision[0.01, 80];

Phi[r_] := Module[{rr = SetPrecision[r, 80]}, -A (1 - R0 / rr) Exp[-(rr - R0)^2 / w^2]]

(*-------------------------------------------*)
(*2. BD scalar field equation residual*)
(*phi''+2 phi'/r=0*)
(*-------------------------------------------*)

BDResidual[r_] := Module[{rr, dp, ddp}, rr = SetPrecision[r, 80];
  dp = SetPrecision[D[Phi[x], x] /. x → rr, 80];
  ddp = SetPrecision[D[Phi[x], {x, 2}] /. x → rr, 80];
  SetPrecision[ddp + (2 / rr) dp, 80]]

(*-------------------------------------------*)
(*3. Evaluate at throat and far outside*)
(*-------------------------------------------*)

Print["Φ(R0)        = ", Phi[R0]];
Print["Φ'(R0)       = ", BDResidual[R0] /. (_ + _ / R0) → N[D[Phi[x], x] /. x → R0]];
Print["Φ''(R0)      = ", N[D[Phi[x], {x, 2}] /. x → R0]];

Print["BD residual at r = R0:   ", BDResidual[R0]];
Print["BD residual at r = 2R0:  ", BDResidual[2 R0]];
Print["BD residual at r = 5R0:  ", BDResidual[5 R0]];
Print["BD residual at r = 10R0: ", BDResidual[10 R0]];

Print["Limit r→∞ BD residual:   ", Limit[BDResidual[r], r → Infinity]];

(*-------------------------------------------*)
(*4. Plot BD-residual*)
(*-------------------------------------------*)

Plot[Evaluate[BDResidual[r]], {r, R0, 10 R0}, WorkingPrecision → 60, PlotRange → All,
 AxesLabel → {"r", "BD residual"}, PlotLabel → "Brans-Dicke residual (high precision)"]
```

$\Phi(R0) = 0. \times 10^{-82}$

$\Phi'(R0) = 0$

$\Phi''(R0) = 0.005$

BD residual at r = R0: 0

BD residual at r = 2R0:

$-1.832137418190281680794316499415026089769824601412259712235526754150660064350481 0 \times 10^{-42}$

BD residual at r = 5R0:

$-1.720408511077324623733590736339164592153604603040262887286741456371677252220086 5 \times 10^{-692}$

BD residual at r = 10R0:

$-1.194933342802428726561847929674986219106374171878081430836491637653584920564824 1 \times 10^{-3514}$

Limit r→∞ BD residual: 0

Out[•]=