Game 0713

lusun

June 2021

1 Purpose

In section 2, it shows the results of collaborate game under both complete case and partial observed case.

In section 3, it shows the basic idea of extend model. Namely, for the market entry and exist model, different agents share different belief. The problem is how to establish a two-dimension linear interpolation (need to be figure out in the future).

2 Toy Model for Comparison (Collaborate Game)

2.1 Definition

- s : hidden state $s_t \in \{0, 1\}$
- a : action $a_t = (a_t^1, a_t^2) \in \{0, 1\} \times \{0, 1\}$
- π : policy $\pi_1(a_t^1|x_t), \pi_2(a_t^2|x_t)$
- β : discount factor
- x : belief vector $x_{t+1} = \lambda(z_{t+1}, x_t, a_t) = P(s_{t+1}|x_t, a_t, z_{t+1})$
- z : observation state $z_t \in \{0, 1\}$ (for s_t)
- T : time length (length of each sample path) $T = 150$
- N : number of sample path $N = 5000$
- N_{sub} : number of sample path with the same initial belief $N_{sub} = 100$
- $\bar{V}$: belief value function $\bar{V}_1(x), \bar{V}_2(x)$
- Q : belief and action value function $Q_1(x, a_1), Q_2(x, a_2)$
- r : reward $r(s_t, a_t)$
  $\theta_1 = (2, -1)$,
  $r(s, a) = \begin{cases} \theta_1 = 2, & s = a_1 = a_2 \\ -1, & otherwise \end{cases}$

$s=0$	$a_1=0$	$a_1=1$
$a_2=0$	2	-1
$a_2=1$	-1	-1

$s=1$	$a_1=0$	$a_1=1$
$a_2=0$	-1	-1
$a_2=1$	-1	2

Table 1: Reward

- $P(s',z'|s,a)$:dynamic $P(s_{t+1},z_{t+1}|s_t,a_t) = P(s_{t+1}|s_t,a_t)*P(z_{t+1}|s_{t+1},a_t)$
- $P(s'|s,a)$:transition $P(s_{t+1}|s_t,a_t)$

  $\theta_2 = (0.9, 0.5, 0.5, 0.5, 0.5, 0.5, 0.5, 0.1)$

  $P(s'|s,a) = \begin{cases} \theta_2, & s_{t+1}=0 \\ 1-\theta_2, & s_{t+1}=1 \end{cases}$

$a_t=(0,0)$	$s_t=0$	$s_t=1$
$s_{t+1}=0$	0.9	0.5
$s_{t+1}=1$	0.1	0.5
$a_t=(0,1)$	$s_t=0$	$s_t=1$
$s_{t+1}=0$	0.5	0.5
$s_{t+1}=1$	0.5	0.5
$a_t=(1,0)$	$s_t=0$	$s_t=1$
$s_{t+1}=0$	0.5	0.5
$s_{t+1}=1$	0.5	0.5
$a_t=(1,1)$	$s_t=0$	$s_t=1$
$s_{t+1}=0$	0.5	0.1
$s_{t+1}=1$	0.5	0.9

- $P(z'|s',a)$: observation $P(z_{t+1}|s_{t+1},a_t)$

  $\theta_3 = (0.9, 0.2, 0.8, 0.2, 0.8, 0.2, 0.8, 0.1)$

  $P(z'|s',a) = \begin{cases} \theta_3, & z_{t+1}=0 \\ 1-\theta_3, & z_{t+1}=1 \end{cases}$

$a_t=(0,0)$	$s_{t+1}=0$	$s_{t+1}=1$
$z_{t+1}=0$	0.9	0.2
$z_{t+1}=1$	0.1	0.8
$a_t=(0,1)$	$s_{t+1}=0$	$s_{t+1}=1$
$z_{t+1}=0$	0.8	0.2
$z_{t+1}=1$	0.2	0.8
$a_t=(1,0)$	$s_{t+1}=0$	$s_{t+1}=1$
$z_{t+1}=0$	0.8	0.2
$z_{t+1}=1$	0.2	0.8
$a_t=(1,1)$	$s_{t+1}=0$	$s_{t+1}=1$
$z_{t+1}=0$	0.8	0.1
$z_{t+1}=1$	0.2	0.9

2.2 Formula

2.2.1 POMDP

- Sigma : $\sigma(z_{t+1}, x_t, a_t) := \sum_{s_{t+1}} \sum_{s_t} P(s_{t+1}, z_{t+1}|s_t, a_t) x_t(s_t)$
- Belief : $x_{t+1} = \lambda(z_{t+1}, x_t, a_t) := \frac{\sum_{s_t} P(s_{t+1}, z_{t+1}|s_t, a_t) x_t(s_t)}{\sigma(z_{t+1}, x_t, a_t)}$
- Bellman Equation : $i \in \{1, 2\}$
  $\bar{V}_i(x) = \gamma + \log(\sum_{a_i} \exp(Q_i(x, a_i)))$

$$Q_i(x, a_i) = \sum_{a_{-i}} \pi_{-i}(a_{-i}|x) \left(\sum_{s_t} x(s_t) r(s_t, a_i, a_{-i})) + \beta \sum_z \bar{V}_i(\lambda(z, x, a_i, a_{-i})) \sigma(z, x, a_i, a_{-i}) \right) \tag{1}$$

- Bayes-Nash Equilibrium:

$$\pi_i(a_i|x) = \frac{\exp Q_i(x, a_i)}{\sum_a \exp Q_i(x, a)}, i \in \{1, 2\} \tag{2}$$

2.2.2 Complete

- $P(z_{t+1}|z_t, a_t)$: dynamic matrix (observation state in POMDP as the true state in complete case)
- Bellman Equation : $i \in \{1, 2\}$
  $\bar{V}_i(z) = \gamma + \log(\sum_{a_i} \exp(Q_i(z, a_i)))$

$$Q_i(z, a_i) = \sum_{a_{-i}} \pi_{-i}(a_{-i}|z) \left(r(z, a_i, a_{-i}) + \beta \sum_{z'} \bar{V}_i(z') P(z'|z, a_i, a_{-i}) \right) \tag{3}$$

- Bayes-Nash Equilibrium:

$$\pi_i(a_i|z) = \frac{\exp Q_i(z, a_i)}{\sum_a \exp Q_i(z, a)}, i \in \{1, 2\} \tag{4}$$

2.3 ALG for Data Generation

- Random generate s_0 with probability $(\frac{1}{2}, \frac{1}{2})$.
  Initial $x_0 = \{0/49, 1/49, ..., 49/49\} \times N_{sub}$.
  Generate z_0 for s_0 based on x_0, namely, $z_0 \in \{0, 1\} \sim (x_0, 1 - x_0)$.

- Generate Perfect Markov Equilibrium, get $\pi_i(a_i|x), i = 1, 2$
  (By Value Iteration Method)
  Initial $\pi_i^{(0)} = \frac{1}{2}, \forall x, a_i$
  For $m = 0, ..., 1000$:
  1. Generate $Q_{-i}^{(m+1)}$ with $\pi_i^{(m)}$ by equation 1 and value iteration method
  2. Generate $\pi_{-i}^{(m+1)}$ with $Q_{-i}^{(m+1)}$ by equation 2
  3. Generate $Q_i^{(m+1)}$ with $\pi_{-i}^{(m+1)}$ by equation 1 and value iteration method
  4. Update $\pi_i^{(m+1)}$ with $Q_i^{(m+1)}$ by equation 2
- For $t = 0, ...T$:
  1. Generate $a_{t,i}$ with probability $\pi_i(a_{t,i}|x_t), i = 1, 2$
  2. Generate s_{t+1}, z_{t+1} with probability $P(s_{t+1}, z_{t+1}|s_t, a_t)$
  3. Update x_{t+1} by $\lambda(z_{t+1}, x_t, a_t)$

2.4 ALG for Recover Process (MLE)

2.4.1 POMDP

$$\text{Log-like1}: \ L1(\theta_1; \theta_2, \theta_3, x_0, a, z) := \sum_{n=1}^{N} \sum_{t=0}^{T} \log \pi_{\theta_1}(a_{nt,1}|x_{nt}) + \log \pi_{\theta_1}(a_{nt,2}|x_{nt})$$
$$\text{Log-like2}: \ L2(\theta_2, \theta_3; x_0, a, z) := \sum_{n=1}^{N} \sum_{t=0}^{T-1} \log \sigma_{\theta_2,\theta_3}(z_{n,t+1}, x_{nt}, a_{nt})$$

- Given x_0, z, a, generate $x_{t=1,...,T}$ by $\lambda_{\theta_2,\theta_3}$
- $(\theta_2^{est}, \theta_3^{est}) = \arg\max_{\theta_2,\theta_3} L2(\theta_2, \theta_3; x_0, a, z)$
- Given $x_0, z, a, \theta_2^{est}, \theta_3^{est}$, generate $x_{t=1,...,T}$ and function $\pi_{\theta_1}(a_i|x), i = 1, 2$ by Perfect Markov Equilibrium.
- $\theta_1^{est} = \arg\max_{\theta_1} L1(\theta_1; \theta_2^{est}, \theta_3^{est}, x_0, a, z)$

2.4.2 Complete

$$\text{Log-like1}: \ L1(\theta_1; \theta_2, a, z) := \sum_{n=1}^{N} \sum_{t=0}^{T} \log \pi_{\theta_1}(a_{nt,1}|z_{nt}) + \log \pi_{\theta_1}(a_{nt,2}|z_{nt})$$
$$\text{Log-like2}: \ L2(\theta_2; a, z) := \sum_{n=1}^{N} \sum_{t=0}^{T-1} \log P_{\theta_2}(z_{n,t+1}|z_{nt}, a_{nt})$$

- Given z, a, $\theta_2^{est} = \arg\max_{\theta_2} L2(\theta_2; a, z)$

- Given z, a, θ_2^{est}, generate function $\pi_{\theta_1}(a_i|z), i = 1, 2$ by Perfect Markov Equilibrium.

- $\theta_1^{est} = \arg\max_{\theta_1} L1(\theta_1; \theta_2^{est}, a, z)$

2.5 Estimation Results

2.5.1 POMDP

	θ_2							
True value	0.9	0.5	0.5	0.5	0.5	0.5	0.5	0.1
Initial	0.72	0.4	0.4	0.4	0.4	0.4	0.4	0.08
Estimate	0.89959	0.49736	0.49154	0.47387	0.50219	0.50407	0.52450	0.10071

Table 2: Estimation for Transition Matrix

	θ_3							
True value	0.9	0.2	0.8	0.2	0.8	0.2	0.8	0.1
Initial	0.72	0.16	0.64	0.16	0.64	0.16	0.64	0.08
Estimate	0.90033	0.16949	0.79959	0.19520	0.81242	0.18113	0.84671	0.09699

Table 3: Estimation for Observation Matrix

	θ_1	
True value	2	-1
Initial	1.7	-0.85
Estimate	1.90595	-1.05773

Table 4: Estimation for Reward

2.5.2 Complete

	θ_2							
Initial	0.72	0.4	0.4	0.4	0.4	0.4	0.4	0.08
Estimate	0.80159	0.49642	0.49749	0.32732	0.67041	0.49910	0.50625	0.19658

Table 5: Estimation for Transition Matrix

	θ_1	
True value	2	-1
Initial	1.7	-0.85
Estimate	1.59095	-0.79547

Table 6: Estimation for Reward

3 Market Entry and Exit Model (Extend)

3.1 Definition

- s : hidden state (condition) $s_t \in \{0,1\}$ (good, bad)
- a : action $a_t = (a_t^1, a_t^2) \in \{0,1\} \times \{0,1\}$(exit,enter)
- π : policy $\pi_1(a_t^1|z_t, a_{t-1}, x_t), \pi_2(a_t^2|z_t, a_{t-1}, x_t)$
- β : discount factor 0.95
- x : belief vector $x_t = (x_t^1, x_t^2)$, $x_t^i(s_t^c) = P^i(s_t^c|\zeta_{t-1}), i = 1, 2$
- z : observation state $z_t \in \{0,1,2\}$ market size (for s_t)
- T : time length (length of each sample path) $T = 150$
- N : number of sample path $N = 3000$
- N_{sub} : total number of replications for each initial belief x_0 $N_{sub} = 30$
- M: total number of discrete intervals for belief x_0 $M = 100$
- $\bar{V}$: belief value function $\bar{V}_1(z_t, a_{t-1}, x_t), \bar{V}_2(z_t, a_{t-1}, x_t)$
- Q : belief and action value function $Q_1(z_t, a_{t-1}, x_t, a_t^1), Q_2(z_t, a_{t-1}, x_t, a_t^2)$
- $P(z', s'|z, s, a)$: dynamic $P(z_{t+1}, s_{t+1}|z_t, s_t, a_t) = P(s_{t+1}|s_t, a_t) {*} P^1(z_{t+1}|z_t, s_{t+1})$ (agent 1 is an expert, true observation matrix is the same as $P^1(z_{t+1}|z_t, s_{t+1})$, agent 2 is not an expert, observatio matrix $P^2(z_{t+1}|z_t, s_{t+1})$ is different from $P^1(z_{t+1}|z_t, s_{t+1})$)
- $P(s'|s, a)$:transition $P(s_{t+1}|s_t, a_t)$

  $\theta_2 = (0.9, 0.3, 0.8, 0.85)$

$$P(s'|s,a) = \begin{cases} \theta_2(1) = 0.9, & s = s' = a1 = a2 \\ 1 - \theta_2(1) = 0.1, & s' \neq s = a1 = a2 \\ \theta_2(2) = 0.3, & s \neq s' = a1 = a2 \\ 1 - \theta_2(2) = 0.7, & s = s' \neq a1 = a2 \\ \theta_2(3) = 0.8 & a1 = 0, a2 = 1, s = s' \\ 1 - \theta_2(3) = 0.2 & a1 = 0, a2 = 1, s \neq s' \\ \theta_2(4) = 0.85 & a1 = 1, a2 = 0, s = s' \\ \theta_2(4) = 0.15 & a1 = 1, a2 = 0, s \neq s' \end{cases}$$

$a_t = (0,0)$	$s_{t+1} = 0$	$s_{t+1} = 1$
$s_t = 0$	0.9	0.1
$s_t = 1$	0.3	0.7

$a_t = (0,1)$	$s_{t+1} = 0$	$s_{t+1} = 1$
$s_t = 0$	0.8	0.2
$s_t = 1$	0.2	0.8

$a_t = (1,0)$	$s_{t+1} = 0$	$s_{t+1} = 1$
$s_t = 0$	0.85	0.15
$s_t = 1$	0.15	0.85

$a_t = (1,1)$	$s_{t+1} = 0$	$s_{t+1} = 1$
$s_t = 0$	0.7	0.3
$s_t = 1$	0.1	0.9

- $P(z'|z,s')$: observation $P^1(z_{t+1}|z_t, s_{t+1}), P^2(z_{t+1}|z_t, s_{t+1})$

  $\theta_3 = (0.1, 0.4, 0.2, 0.3)$

$$P^1(z'|z, s'=0) = \left\{ \begin{array}{cc} \theta_3(1) = 0.1, & |z - z'| = 1 \\ 1 - 2\theta_3(1) = 0.8, & z = z' = 1 \\ 1 - \theta_3(1) = 0.9 & z = z' \neq 1 \\ 0 & otherwise \end{array} \right.$$

$$P^1(z'|z, s'=1) = \left\{ \begin{array}{cc} \theta_3(1) = 0.4, & |z - z'| = 1 \\ 1 - 2\theta_3(1) = 0.2, & z = z' = 1 \\ 1 - \theta_3(1) = 0.6 & z = z' \neq 1 \\ 0 & otherwise \end{array} \right.$$

$$P^2(z'|z, s'=0) = \left\{ \begin{array}{cc} \theta_3(2) = 0.2, & |z - z'| = 1 \\ 1 - 2\theta_3(2) = 0.6, & z = z' = 1 \\ 1 - \theta_3(2) = 0.8 & z = z' \neq 1 \\ 0 & otherwise \end{array} \right.$$

$$P^2(z'|z, s'=1) = \left\{ \begin{array}{cc} \theta_3(3) = 0.3, & |z - z'| = 1 \\ 1 - 2\theta_3(3) = 0.4, & z = z' = 1 \\ 1 - \theta_3(3) = 0.7 & z = z' \neq 1 \\ 0 & otherwise \end{array} \right.$$

$s' = 0$	$z' = 0$	$z' = 1$	$z' = 2$	$s' = 1$	$z' = 0$	$z' = 1$	$z' = 2$
$z = 0$	0.9	0.1	0	$z = 0$	0.6	0.4	0
$z = 1$	0.1	0.8	0.1	$z = 1$	0.4	0.2	0.4
$z = 2$	0	0.1	0.9	$z = 2$	0	0.4	0.6

Table 7: Agent 1

$s' = 0$	$z' = 0$	$z' = 1$	$z' = 2$	$s' = 1$	$z' = 0$	$z' = 1$	$z' = 2$
$z = 0$	0.8	0.2	0	$z = 0$	0.7	0.3	0
$z = 1$	0.2	0.6	0.2	$z = 1$	0.3	0.4	0.3
$z = 2$	0	0.2	0.8	$z = 2$	0	0.3	0.7

Table 8: Agent 2

- r : reward $r(z_t, a_{t-1}, s_t, a_t)$

  $\theta_1 = (\theta_{RS,good}, \theta_{RS,bad}, \theta_{FC,1}, \theta_{FC,2}, \theta_{EC}) = (1.5, 1, -1.9, -1.8, 1)$,

$$r_i(z_t, a_{t-1}, s_t, a_t) = \begin{cases} z_t \frac{\theta_{RS,good}}{(2+a_t^{-i})^2} - \theta_{FC,i} - \theta_{EC}(1 - a_{t-1}^i), & a_t^i = 1, s_t = good \\ z_t \frac{\theta_{RS,bad}}{(2+a_t^{-i})^2} - \theta_{FC,i} - \theta_{EC}(1 - a_{t-1}^i), & a_t^i = 1, s_t = bad \\ 0, & a_t^i = 0 \end{cases}$$

$s=0$	$a_{t-1,1}=0$	$a_{t-1,1}=1$	$s=1$	$a_{t-1,1}=0$	$a_{t-1,1}=1$
$a_{t,2}=0$	$3z_t/8+0.9$	$3z_t/8+1.9$	$a_{t,2}=0$	$1z_t/4+0.9$	$1z_t/4+1.9$
$a_{t,2}=1$	$z_t/6+0.9$	$z_t/6+1.9$	$a_{t,2}=1$	$1z_t/9+0.9$	$1z_t/9+1.9$

Table 9: Reward of agent 1 with $a_{t,1}=1$

$s=0$	$a_{t-1,2}=0$	$a_{t-1,2}=1$	$s=1$	$a_{t-1,2}=0$	$a_{t-1,2}=1$
$a_{t,1}=0$	$3z_t/8+0.8$	$3z_t/8+1.8$	$a_{t,1}=0$	$1z_t/4+0.8$	$1z_t/4+1.8$
$a_{t,1}=1$	$z_t/6+0.8$	$z_t/6+1.8$	$a_{t,1}=1$	$1z_t/9+0.8$	$1z_t/9+1.8$

Table 10: Reward of agent 2 with $a_{t,2}=1$

3.2 Formula

- Sigma : $\sigma(z_{t+1}, z_t, x_t^i, a_t) := \sum_{s_{t+1}} \sum_{s_t} P^i(z_{t+1}, s_{t+1}|z_t, s_t, a_t)x_t^i(s_t)$
  $= \sum_{s_{t+1}} \sum_{s_t} P(s_{t+1}|s_t, a_t)P^i(z_{t+1}|z_t, s_{t+1})x_t^i(s_t), i = 1, 2$

- Belief : $x_{t+1}^i = \lambda(z_{t+1}, z_t, x_t^i, a_t) := \frac{\sum_{s_t} P^i(z_{t+1}, s_{t+1}|z_t, s_t, a_t)x_t^i(s_t)}{\sigma(z_{t+1}, z_t, x_t^i, a_t)}$
  $= \frac{\sum_{s_t} P(s_{t+1}|s_t, a_t)P^i(z_{t+1}|z_t, s_{t+1})x_t^i(s_t)}{\sum_{s_{t+1}} \sum_{s_t} P(s_{t+1}|s_t, a_t)P^i(z_{t+1}|z_t, s_{t+1})x_t^i(s_t)}$

- Bellman Equation : $i \in \{1, 2\}$
  $\bar{V}_i(z_t, a_{t-1}, x_t) = \gamma + \log(\sum_{a_t^i} \exp(Q_i(z_t, a_{t-1}, x_t, a_t^i)))$

$$\begin{aligned} Q_i(z_t, a_{t-1}, (x_t^1, x_t^2), a_t^i) &= \sum_{a_t^{-i}} \pi_{-i}(a_t^{-i}|z_t, a_{t-1}, x_t)\Big(\sum_{s_t} x^i(s_t) r_i(z_t, a_{t-1}, s_t, a_t) \\ &+ \beta \sum_{z'} \sigma(z', z_t, x_t^i, a_t)\bar{V}_i(z', a_t, \{\lambda(z', z_t, x_t^j, a_t)\}_{j=1}^2)\Big) \end{aligned} \tag{5}$$

- Bayes-Nash Equilibrium:

$$\pi_i(a_t^i|z_t, a_{t-1}, x_t) = \frac{\exp Q_i(z_t, a_{t-1}, x_t, a_t^i)}{\sum_a \exp Q_i(z_t, a_{t-1}, x_t, a)}, i \in \{1, 2\} \tag{6}$$

3.3 ALG for Data Generation

- Random generate:
  $s_0 \in \{0,1\}$ with prob $(1/2, 1/2)$,
  $x_0 = \{0, 1/99, 2/99, ...1\} \times N_{sub}(30)$,
  $z_0 \in \{0,1,2\}$ with prob $(1/3, 1/3, 1/3)$,
  $a_0 = (1,1)$.
- Generate Perfect Markov Equilibrium, get $\pi_i(a_i'|z', a, x'), i = 1,2$
  (By Value Iteration Method)
  Initial $\pi_i^{(0)} = \frac{1}{2}, \forall x, a_i$
  For $m = 0, ..., 1000$:
  1. Generate $Q_{-i}^{(m+1)}$ with $\pi_i^{(m)}$ by equation 5 and value iteration method
  2. Generate $\pi_{-i}^{(m+1)}$ with $Q_{-i}^{(m+1)}$ by equation 6
  3. Generate $Q_i^{(m+1)}$ with $\pi_{-i}^{(m+1)}$ by equation 5 and value iteration method
  4. Update $\pi_i^{(m+1)}$ with $Q_i^{(m+1)}$ by equation 6
- For $t = 1, ...T$:
  1. Generate s_t, z_t with probability $P(s_t|s_{t-1}, a_{t-1})$, $P^1(z_t|z_{t-1}, s_t)$
  2. Update x_t^1, x_t^2 by $\lambda(z_t, z_{t-1}, x_{t-1}^1, a_{t-1}), \lambda(z_t, z_{t-1}, x_{t-1}^2, a_{t-1})$
  3. Generate $a_{t,i}$ with ccp $\pi_i(a_{t,i}|z_t, a_{t-1}, x_t)$

3.4 ALG for Recover Process (MLE)

Log-like1 : $L1(\theta_1; \theta_2, \theta_3, x_0, a, z) := \sum_{n=1}^{N} \sum_{t=1}^{T} \log \pi_{1_{\theta_1}}(a_{nt,1}|z_{nt}, a_{n(t-1)}, x_{nt}) + \log \pi_{2_{\theta_1}}(a_{nt,2}|z_{nt}, a_{n(t-1)}, x_{nt})$

Log-like2 : $L2(\theta_2, \theta_3; x_0, a, z) := \sum_{n=1}^{N} \sum_{t=1}^{T} \sum_{i=1}^{2} \log \sigma_{\theta_2,\theta_3}(z_{n,t}, z_{n,t-1}, x_{n,t-1}^i, a_{n,t-1})$

- Given x_0, a, z, generate $x_{t=1,...,T}$ by $\lambda_{\theta_2,\theta_3}$
- $(\theta_2^{est}, \theta_3^{est}) = \arg\max_{\theta_2,\theta_3} L2(\theta_2, \theta_3; x_0, a, z)$
- Given $x_0, z, a, \theta_2^{est}, \theta_3^{est}$, generate new $x_{t=1,...,T}$ and function $\pi_{i_{\theta_1}}(a_i'|z', a, x',), i = 1, 2$ by Perfect Markov Equilibrium.
- $\theta_1^{est} = \arg\max_{\theta_1} L1(\theta_1; \theta_2^{est}, \theta_3^{est}, x_0, a, z)$

3.5 Real Data Description

Store location of McDonalds(MD) and Burger King(BK) database is originally from Toivanen and Waterson(2005) which can't be found online. The available database is from Aguirregabiria and Magesan(2019) which includes 422 local markets (sample paths=422) data in U.K. and 5 years (time length=5) data from 1991 to 1995. The database also includes 27 different characteristic data. Only the following 17 terms are listed which are used in the model of Aguirregabiria and Magesan(2019).

1. district_code: Distinguish different sample path. Different market has a different code from 1 to 422
2. year: Distinguish different 5 year, 1991, 1992, 1993, 1994, 1995. (Since the original data is from McDonalds' annul report, only data in years is available not in months.)
3. mcd_stock: Number of MD stores
4. mcd_entry: Action of MD (exit:0, entry: 1,2,3. Here 2 or 3 doesn't often occur resulting in that we can equal them to be just 1)
5. bk_stock: Number of BK stores
6. bk_entry: Action of BK (exit:0, entry: 1,2,3. Here 2 or 3 doesn't often occur resulting in that we can equal them to be just 1)
7. district_area: Each market's area
8. population: Each market's population (in thousand)
9. pop_0514: Percentage of each market's people aging form 5 to 14
10. pop_1529: Percentage of each market's people aging form 15 to 29
11. pop_4559: Percentage of each market's people aging form 45 to 59
12. pop_6064: Percentage of each market's people aging form 60 to 64
13. pop_6574: Percentage of each market's people aging form 65 to 74
14. avg_rent: Average weekly rent per dwelling for each market
15. ctax: Council tax
16. ue: unemployment benefits
17. gdp_pc: GDP per capita for each market

The way to handle market size:

In Aguirregabiria and Mira (2007), Market Size = Population, changed with time

In Aguirregabiria and Magesan (2019), Market Size = Market Character $\times \vec{\gamma}$, doesn't change with time.

$$\text{Here, Market Character} = \begin{bmatrix} \texttt{Population} \\ \texttt{Population/area} \\ \texttt{percentage of population in age group 15-29} \\ \texttt{GDP per capita} \\ \texttt{Unemployment benefits/population} \\ \texttt{average rent} \\ \texttt{council tax} \end{bmatrix}$$

$\vec{\gamma} = [1, r_1, r_2, r_3, r_4, r_5, r_6]$ (I guess $r_1, ..., r_6$ are generated through regression method. It's not show clearly in their code and paper)

I think maybe we can use *Population/area* the population density to define our market size, namely, divided *Population/area* into 3 classes: low, medium, high and labeled as $\{0, 1, 2\}$.

3.6 Real Data Modeling

1.Definition

- agent 1: MD, agent 2: BK
- s : hidden state (condition) $s_t \in \{0, 1\}$ (good, bad)
- a : action $a_t = (a_t^1, a_t^2) \in \{0, 1\} \times \{0, 1\}$(exit,enter) when $a_t >= 2$, put $a_t = 1$
- π : policy $\pi_1(a_t^1 | z_t, a_{t-1}, x_t), \pi_2(a_t^2 | z_t, a_{t-1}, x_t)$
- β : discount factor 0.95
- x : belief vector $x_{t+1} = \lambda(z_{t+1}, z_t, x_t) = P(s_{t+1} | z_{t+1}, z_t, x_t)$
- z : observation state $z_t \in \{0, 1, 2\}$ market size (for s_t) direct use population density (0: low market size, 1: medium market size, 2: high market size)

2.Results

- market size is divided according to population within each sample path

	$s_{t+1} = 0$	$s_{t+1} = 1$
$s_t = 0$	0.92156	0.07844
$s_t = 1$	0.297521	0.702479

- market size is divided according to population/area among all the sample path

$s_{t+1}=0$	$z_{t+1}=0$	$z_{t+1}=1$	$z_{t+1}=2$
$z_t=0$	0.40475	0.49858	0.09667
$z_t=1$	0.00744	0.22770	0.76486
$z_t=2$	0.16510	0.02723	0.80767
$s_{t+1}=1$	$z_{t+1}=0$	$z_{t+1}=1$	$z_{t+1}=2$
$z_t=0$	0.68532	0.04433	0.27035
$z_t=1$	0.76340	0.23657	0.00003
$z_t=2$	0.03244	0.51718	0.45038

Parameter	θ_1				
	$\theta_{RS,good}$	$\theta_{RS,bad}$	$\theta_{FC,1}$	$\theta_{FC,2}$	θ_{EC}
Estimate	0.00000	0.00000	-0.00000	-0.00000	4.07429

Table 11: Estimation for Reward

	$s_{t+1}=0$	$s_{t+1}=1$
$s_t=0$	0.01830	0.98170
$s_t=1$	0.00001	0.99999

$s_{t+1}=0$	$z_{t+1}=0$	$z_{t+1}=1$	$z_{t+1}=2$
$z_t=0$	0.84524	0.15473	0.00003
$z_t=1$	0.00001	0.00001	0.99998
$z_t=2$	0.00001	0.000001	0.99998
$s_{t+1}=1$	$z_{t+1}=0$	$z_{t+1}=1$	$z_{t+1}=2$
$z_t=0$	0.99997	0.00001	0.00002
$z_t=1$	0.00952	0.99045	0.00003
$z_t=2$	0.00001	0.00001	0.99998

Parameter	θ_1				
	$\theta_{RS,good}$	$\theta_{RS,bad}$	$\theta_{FC,1}$	$\theta_{FC,2}$	θ_{EC}
Estimate	0.00000	0.00000	-0.00000	-0.00000	4.07428

Table 12: Estimation for Reward