Sankhyā : The Indian Journal of Statistics
1982, Volume 44, Series A, Pt. 1, pp. 114–127.

EXISTENCE OF EQUILIBRIUM STATIONARY STRATEGIES IN DISCOUNTED STOCHASTIC GAMES*

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SUMMARY. In this paper we show that the existence of p-equilibrium stationary strategies imply the existence of equilibrium stationary strategies when the state space is uncountable under the condition that the reward functions and transition probabilities are continuous. Using this and a theorem of Himmelberg *et al* we are able to give a set of sufficient conditions for the existence of an equilibrium pair. This answers partially a question raised by Ward Whitt recently.

1. Introduction

A stochastic game is determined by six objects : S, A, B, q, r_1, r_2. Here S is the unit interval [0, 1], the set of states of the system; $A = \{1, 2, ..., k\}$ and $B = \{1, 2, ..., l\}$ are finite sets denoting the available actions or alternatives to players I and II respectively; $r_1(s, a, b)$ and $r_2(s, a, b)$ are the (immediate) rewards to I and II respectively when s is the state and a and b are the actions chosen by I and II respectively. As a consequence of the actions chosen by the players two things happen : players I and II receive $r_1(s, a, b)$ and $r_2(s, a, b)$ and the system moves to a new state s' according to $q(\cdot \mid s, a, b)$. Then the whole process is repeated from the new state s'. The game is played over the infinite future. Both I and II want to maximise their accumulated income. The problem then is to find good strategies for the two players.

A strategy Π for I is a sequence $(\Pi_1, \Pi_2, ..., \Pi_n, ...)$ where Π_n specifies the action to be chosen on the n-th day depending on the past history h_n. Here Π_n is a probability distribution on A given the past history, and $\Pi_n(E \mid h_n)$ is measurable in h_n for every fixed Borel set E. A strategy Π is called stationary if there is a Borel map $f : S \to P_A$ (= class of all probability distribution on A) such that $\Pi_n = f$ for all n. Analogously strategies and stationary strategies are defined for player II.

*Paper presented at the Conference on Mathematical Statistics and Probability, Indian Statistical Institute, New Delhi, December 1980 in honour of C. R. Rao on the occasion of his sixtieth birthday.

Let β be a fixed number with $0 \leqslant \beta < 1$. A pair (Π, Γ) of strategies for I and II associates with each initial state s an n-th day expected reward $r_i^{(n)}$ for player I and a total expected discounted reward for player i

$$I_i(\Pi, \Gamma)(s) = \sum_{n=1}^{\infty} \beta^{n-1} r_i^{(n)}(\Pi, \Gamma)(s).$$

Call a pair (Π^*, r^*) an equilibrium pair in the sense of Nash if

$$I_1(\Pi^*, \Gamma^*)(s) \geqslant I_1(\Pi, \Gamma^*)(s)$$

and

$$I_2(\Pi^*, \Gamma^*)(s) \geqslant I_2(\Pi^*, \Gamma)(s) \quad \text{for all } \Pi, \Gamma \text{ and } s.$$

Let p be a probability distribution on $S = [0, 1]$. Call a pair Π^*, Γ^* a p-equilibrium pair if there exists a Borel set E with $P(E) = 1$ and for every $s \epsilon E$ and

$$I_1(\Pi^*, \Gamma^*)(s) \geqslant I_1(\Pi, \Gamma^*)(s)$$

and

$$I_2(\Pi^*, \Gamma^*)(s) \geqslant I_2(\Pi^*, \Gamma)(s) \quad \text{for all } \Pi \text{ and } \Gamma.$$

In other words they will be an equilibrium pair on a set of p-measure one. We are now ready to state our main results.

Theorem 1: *Let* $S = [0, 1]$, $A = \{1, 2, \ldots, k\}$, $B = \{1, 2, \ldots, l\}$ *and* $\beta \epsilon (0, 1)$. *Let* $r_1(s, a, b)$ *and* $r_2(s, a, b)$ *be continuous over* $S \times A \times B$ *and let* $q(\cdot \mid s, a, b)$ *be strongly continuous over* $S \times A \times B$. *Let* p *be a probability distribution over* $[0, 1]$ *with* $q(\cdot \mid s, a, b)$ *absolutely continuous with respect to* p. *If there exists a* p*-equilibrium stationary pair for the discounted stochastic game then there exists an equilibrium pair.*

Theorem 2: *Let* $S = [0, 1]$, $A = \{1, 2, \ldots, k\}$, $B = \{1, 2, \ldots, l\}$ *and* $\beta \epsilon (0, 1)$. *Let* $r_i(s, a, b) = l_i(s, a) + m_i(s, b)$ *for* $i = 1, 2, \ldots,$ *where* l_i *and* m_i *are continuous functions in* s. *Let* $q(\cdot \mid s, a, b) = [q'(\cdot \mid s, a) + q''(\cdot \mid s, b)] | 2$ *where* q' *and* q'' *are probability measures which are strongly continuous in* s *for each* a, b. *More over suppose* q' *and* q'' *are absolutely continuous with respect to Lebesgue measure. Then the discounted stochastic game has a pair* (f^*, g^*) *of equilibrium stationary strategies and further* $I_1(f^*, g^*)(s)$ *and* $I_2(f^*, g^*)(s)$ *are Borel measurable in* s.

Before proving these results, we would like to mention results known in this direction. When $r_1 = -r_2$, Theorem 1 as well as Theorem 2 is true without any assumption. We need only the assumption that r_1 and $q(\cdot \mid s, a, b)$

are measurable in s for every a, b. This is proved in Parthasarathy (1973). In Himmelberg, Parathasarathy, Raghavan and Vleck (1976), notion of p-equilibrium pair is introduced. Theorem 2 is a strengthening of the result in Himmelberg, Parthasarathy, Raghavan and Vleck (1976) of course with a stronger hypothesis namely that the functions r_1, r_2 and q are assumed to be continuous over $S \times A \times B$. Very few modifications in the proof of Theorem 1 of Himmelberg, Parthasarathy, Raghavan and Vleck (1976) are required but to simplify the readers' task we will retrace the main steps and do some of them in detail. This will be done in Section 3. Theorem 2 is a partial answer to a question raised in remark 4, page 47 in Whitt (1980). There, ϵ-equilibrium stationary pair is shown to exist when r_1, r_2 and q are uniformly continuous by the approximation procedure. In Rieder (1979) existence of equilibrium pair (not necessarily stationary) is proved when r_1, r_2 are continuous and q is strongly continuous in the sense that whenever $(s_n, a_n, b_n) \to s_0, a_0, b_0)$, $\int v(\cdot)dq\ (\cdot \mid s_n, a_n, b_n)$ tends to $\int v(\cdot)dq(\cdot \mid s_0, a_0, b_0)$ for every bounded measurable function v on S.

Organization of the present paper is as follows. Section 2 contains a selection lemma and a proof of Theorem 1. Section 3 gives a detailed proof of Theorem 2. Section 4 contains a generalization, few remarks; and open problems.

2. Selection lemma and a proof of theorem

Proof of Theorem 1 can be explained as follows. We know that there exists a p-equilibrium stationary pair (f_0, g_0) for the two players—that is, (f_0, g_0) will be an equilibrium pair for some stochastic game (S_1, A, B, r_1, r_2), where $p(S_1) = 1$. Since q is absolutely continuous with respect to p, $q(S_1 \mid s, a, b) = 1$ for every s, a, b. Hence it follows that if the initial state $s \notin S_1$, the state of the system from the second day onwards will be in S_1. Exploiting these facts we will construct a pair of stationary equilibrium strategies (f_1, g_1) for the game $(S_1', A, B, q, r_1, r_2)$ where S_1' is the complement of S_1 with respect to $[0, 1]$. Now one can easily define a pair of equilibrium stationary strategies with the help of (f_0, g_0) and (f_1, g_1) for the entire game. We need the following selection lemma for the proof of Theorem 1.

Selection Lemma : *Let S' be an arbitrary Borel subset of $S = [0, 1]$. Let P_A and P_B be the space of all probability distributions on A and B respectively. Let $h_1(s, \mu, \lambda)$ and $h_2(s, \mu, \lambda)$ be two continuous functions on $S' \times P_A \times P_B$ with the further property that both h_1 and h_2 are bilinear functions on $P_A \times P_B$ for each fixed s. That is for each fixed s and μ, $h_i(s, \mu, \lambda)$ is linear* *in λ and for each fixed s and λ, $h_i(s, \mu, \lambda)$ is linear in μ for $i = 1, 2$. Then there exist two Borel functions f_1 and g_1 such that*

$$\left.\begin{aligned} h_1(s, f_1(s), g_1(s)) &\geqslant h_1(s, \mu, g_1(s)) \\ h_2(s, f_1(s), g_1(s)) &\geqslant h_2(s, f_1(s), \lambda) \end{aligned}\right\}$$

for all $s \in S'$, $\mu \in P_A$, $\lambda \in P_B$ where $f_1 : S' \to P_A$ and $g_1 : S' \to P_B$.

Proof: Define a set valued map $\psi : S' \to P_A \times P_B$ as follows. For each s, $\psi(s)$ = the set of all Nash equilibrium pair corresponding to the non-cooperative game whose pay-offs are given by $h_1(s, \mu, \lambda)$ for $i = 1, 2$. In fact

$$\psi(s) = \{(\mu', \lambda') : h_1(s, \mu', \lambda') \geqslant h_1(s, \mu, \lambda')$$

and

$$h_2(s, \mu', \lambda') \geqslant h_2(s, \mu', \lambda) \text{ for all } \mu \in P_A, \ \lambda \in P_B\}.$$

From Nash (1951), it follows that $\psi(s) \neq \phi$. Furthermore one can check that the mapping $s \to \psi(s)$ is upper-semi-continuous. Invoking the fundamental selection theorem (Kuratowski and Ryll-Nardzewski, 1965) we have measurable function $k(s) \in \psi(s)$ for each $s \in S'$, where $k : S' \to P_A \times P_B$. Write $k(s) = (f_1(s), g_1(s))$. Since $k(s)$ is measurable, f_1 and g_1 are measurable and they satisfy the conclusion of the lemma. This terminates the proof of the selection lemma.

Proof of Theorem 1 : From hypothesis, we have a pair of (f_0, g_0) of p-equilibrium stationary strategies. That is (f_0, g_0) will be a pair of equilibrium stationary strategies for some stochastic game (S_1, A, B, q, r_1, r_2) where $p(S_1) = 1 = q(S_1 \mid s, a, b)$ for each s, a, b. For each $s \in S_1$, define $u_0(s) = I_1(f_0, g_0)(s)$ and $v_0(s) = I_2(f_0, g_0)(s)$ and further these u_0 and v_0 will satisfy the following functional equations, (see Theorem 6f of Blackwell, 1965)

$$u_0(s) = \max_{\mu \in P_A} [r_1(s, \mu, g_0(s)) + \beta \int u_0(\cdot) dq(\cdot \mid s, \mu, g_0(s))]$$

and

$$v_0(s) = \max_{\lambda \in P_B} [r_2(s, f_0(s), \lambda) + \beta \int v_0(\cdot) dq(\cdot \mid s, f_0(s), \lambda)].$$

Note that f_0 and g_0 are measurable stationary strategies defined on S_1.

We will now extend (f_0, g_0) to the whole of $S = [0, 1]$ in such a way that they still form an equilibrium pair for the whole game.

Let S_1' denote the complement of S_1 with respect to $[0, 1]$. Define h_1 and h_2 as follows :

$$h_1(s, a, b) = r_1(s, a, b) + \beta \int u_0(\cdot) dq(\cdot \mid s, a, b)$$

and

$$h_2(s, a, b) = r_2(s, a, b) + \beta \int v_0(\cdot) dq(\cdot \mid s, a, b).$$

Since A and B are finite sets, $r_i(s, a, b)$ for $i = 1, 2$, are continuous on S for each $(a, b) \epsilon A \times B$ and $q(\cdot \mid s, a, b)$ is strongly continuous on S for each $(a, b) \epsilon A \times B$, it follows that $h_1(s, \mu, \lambda)$ and $h_2(s, \mu, \lambda)$ are continuous over $S \times P_A \times P_B$ where $h_1(s, \mu, \lambda) = r_1(s, \mu, \lambda) + \beta \int u_0(\cdot) dq(\cdot \mid s, \mu, \lambda)$ and $h_2(s, \mu, \lambda) = r_2(s, \mu, \lambda) + \beta \int v_0(\cdot) dq(\cdot \mid s, \mu, \lambda)$. Clearly h_1 and h_2 satisfy the conditions of the selection lemma, on $S_1' \times P_A \times P_B$. Consequently we can find two Borel functions f_1 and g_1 such that

$$h_1(s, f_1(s), g_1(s)) \geqslant h_1(s, \mu, g_1(s))$$

and

$$h_2(s, f_1(s), g_1(s)) \geqslant h_2(s, f_1(s), \lambda)$$

for all $s \epsilon S_1'$, $\mu \epsilon P_A$, $\lambda \epsilon P_B$ where $f_1 : S_1' \to P_A$ and $g_1 : S_1' \to P_B$.

We will now define a pair of stationary strategies (f^*, g^*) and show that they form an equilibrium pair for the original stochastic game (S, A, B, q, r_1, r_2). Define f^* and g^* as follows :

$$f^*(s) = \begin{cases} f_0(s) & \text{if } s \epsilon S_1 \\ f_1(s) & \text{if } s \epsilon S_1' \end{cases} \quad \text{and } g^*(s) = \begin{cases} g_0(s) & \text{if } s \epsilon S_1 \\ g_1(s) & \text{if } s \epsilon S_1'. \end{cases}$$

Also define $u^*(s)$ and $v^*(s)$ as follows :

$$u^*(s) = \begin{cases} u_0(s) & \text{if } s \epsilon S_1 \\ h_1(s, f_1(s), g_1(s)) & \text{if } s \epsilon S_1' \end{cases} \quad \text{and } v^*(s) = \begin{cases} v_0(s) & \text{if } s \epsilon S_1 \\ h_2(s, f_1(s), g_1(s)) & \text{if } s \epsilon S_1'. \end{cases}$$

Observe that u^* and v^* satisfy the following functional equations, for every $s \epsilon [0, 1]$.

$$\max_{\mu \epsilon P_A} \; [r_1(s, \mu, g^*(s)) + \beta \int u^*(\cdot) dq(\cdot \mid s, \mu, g^*(s))] = u^*(s)$$

and

$$\max_{\lambda \epsilon P_B} \; [r_2(s, f^*(s), \lambda) + \beta \int v^*(\cdot) dq(\cdot \mid s, f^*(s), \lambda)] = v^*(s).$$

For every stationary g, define the operator T_g from $M(S)$ = space of Borel functions on S to $M(S)$ as follows. Let $u \in M(S)$. Then

$$T_g u(s) = \max_{\mu \epsilon P_A} [r_1(s, \mu, g(s) + \beta \int u(.) dq(\cdot \mid s, \mu, g(s))].$$

Clearly $T_g u \in M(S)$.

One important point to note here is that,

$$\int u^*(\cdot) dq(. \mid s, \mu, g^*(s)) = \int u_0(\cdot) dq(\cdot \mid s, \mu, g^*(s))$$

and

$$\int v^*(\cdot) dq(\cdot \mid s, f^*(s), \lambda) = \int v_0(\cdot) dq(\cdot \mid s, f^*(s), \lambda)$$

since $q(S_1 \mid s, a, b) = 1$ for every s, a, b. Since T_{g^*} is a contraction operator from $M(S)$ to $M(S)$, u^* is the unique fixed point of the operator T_{g^*} and hence u^* is a Borel measurable function on S. Similarly one can show that v^* is also Borel measurable. If we fix g^* (or f^*) it becomes a problem in dynamic programming and from Theorem 6f of Blackwell (1969) we have that

$$u^*(s) = \max_{\Pi} I_1(\Pi, g^*)(s) \quad \text{for all } s \in S$$

and

$$v^*(s) = \max_{\Gamma} I_2(f^*, \Gamma)(s) \quad \text{for all } s \in S.$$

The equalities asserted above have in them maxima taken over plans in the dynamic programming problem and they are still true even if we allow strategies of the game problem. This can be done as in Theorem 3.1 in Maitra and Parthasarathy (1970). Also it is easy to verify that $u^*(s) = I_1(f^*, g^*)(s)$ and $v^*(s) = I_2(f^*, g^*)(s)$. Thus we have shown that (f^*, g^*) forms a pair of equilibrium stationary strategies and the return functions u^* and v^* and Borel measurable functions. This terminates the proof of Theorem 1.

One can generalize Theorem 1 as follows. Proof of the following is the same as that of Theorem 1 and hence omitted.

Theorem 3 : *Let S, A and B compact metric spaces. Let $r_1(s, a, b)$ and $r_2(s, a, b)$ be jointly continuous on $S \times A \times B$ and $q(. \mid s, a, b)$ be strongly continuous on $S \times A \times B$. Let p be a probability distribution on S with $q(. \mid s, a, b)$ absolutely continuous with respect to p. If there exists a p-equilibrium pair then there exists an equilibrium pair.*

Remark 1 : Theorem 1 remains true even if S is not a compact set provided A and B are finite sets, while in Theorem 3, S compact is essential to conclude $r(s, \mu, \lambda)$ is continuous on $S \times P \times P$ given $r(s, a, b)$ is continuous on $S \times A \times B$ (see Lemma 2.1 in Maitra and Parthasarathy (1970).

Remark 2 : We are not able to determine whether there always exists a p-equilibrium stationary strategies under the conditions of Theorem 1. We know from Fink (1964) and Takahasi (1964) and Theorem 5.1 in Parthasarathy (1973) that it is true when S is either finite or countable.

3. Existence of p-equilibrium pair

As already remarked, proof of Theorem 2 requires very few changes in the proof of Theorem 1 of Himmelberg, Parthasarathy, Raghavan and Vleck (1976) but to make the paper self-contained we will retrace the main steps and do some of them in detail. The real problem is to suitably topologize the space of stationary strategies so that it becomes compact and metrizable consequently we can use sequential arguments and fixed point theorems of Kakutani and Glicksberg (1952). To achieve this we follow closely Warga (1967).

Let $\mathcal{B}$ a Banach space of real-valued functions on $S \times A$ where $S = [0, 1]$ and A has k-elements. An element $\phi \in \mathcal{B}$ if $\phi(s, a)$ is measurable in s for each $a \in A$ and there exists an integrable (with respect to Lebesgue measure) scalar function $\psi(s)$ such that $|\phi(s, a)| \leqslant \psi(s)$ for every s, a. We define the norm in $\mathcal{B}$ as follows :

$$|\phi| = \int_0^1 \max_a \ |\phi(s, a)| ds.$$

Let f be a stationary strategy, that is a measurable function from $S \to P_A$ where P_A as usual denote the space of probability distributions on A. We will write $f(s) = (f(s, 1), f(s, 2), \ldots, f(s, k)$ where $f(s, a)$ denote the probability with which action a is chosen in state s. We will identify f with the bounded linear functional Λ_f in $\mathcal{B}^* =$ dual of $\mathcal{B}$ that maps ϕ into $\Lambda_f(\phi)$ where

$$\Lambda_f(\phi) = \int_0^1 (\sum_a f(s, a)\phi(s, a))ds.$$

In this way we can identify $M_A =$ space of stationary strategies as a subset of $\mathcal{B}^*$. We will say $f_1 = f_2$ if they coincide almost everywhere with respect to the Lebesgue measure. We will equip $\mathcal{B}^*$ with the weak-star topology. Specifically we shall say that a sequence $\Lambda_1, \Lambda_2, \ldots,$ in $\mathcal{B}^*$ converges to Λ if $\Lambda_n(\phi) \to \Lambda(\phi)$ for every $\phi \in \mathcal{B}$. We will now prove the following :

Proposition : *M_A is compact and metrizable in the weak-star topology (considered as a subset of $\mathcal{B}^*$).*

Proof: If we show that M_A is a closed subset of the unit sphere in $\mathcal{B}^*$, we are through. For every $\phi \in \mathcal{B}$

$$|\Lambda_f(\phi)| = \left| \int_0^1 \sum_a f(s, a)\phi(s, a) \right|$$

$$\leqslant \int_0^1 \max_a |\phi(s, a)| ds = |\phi|$$

which shows that $\mathcal{B}^*$ norm of Λ_f is less than or equal to one and consequently every sequence in M_A has a subsequence converging to some point in $\mathcal{B}^*$.

Let $\Lambda \in \mathcal{B}^*$ be a limit of a sequence $\Lambda_{f_n} \in M_A$. Clearly $\Lambda(\phi) \geqslant 0$ whenever $\phi \geqslant 0$ and further $\Lambda(\phi) = L(E)$ (= Lebesgue measure of E) if $\phi(s, a) = 1$ on $E \times A$ and $\phi(s, a) = 0$ on $S - E \times A$ for every measurable set $E \subset S$. It is well-known that (see Lemma 4.3, pp. 633 in Warga 1967), there exists a measurable mapping f' from S to the class of regular signed Borel measures on A such that $\Lambda(\phi) = \int_0^1 (\sum_a f'(s, a)\phi(s, a))ds$. Consequently it follows that $f'(s) \in P_A$ for almost all s. Hence $\Lambda = \Lambda_{f'}$ for some $f' \in M_A$. This terminates the proof of the proposition.

Similarly one can show that M_B is also compact and metrizable in the weak-star topology. Observe both M_A and M_B are convex. Proof of Theorem 2 will follow from the following two lemmas and Theorem 1.

Lemma 1 : *Let $\{f_n\} \in M_A$ and $\{g_n\} \in M_B$ with $f_n \to f_0 \in M_A$ and $g_n \to g_0 \in M_B$ in the weak-star topology. Let $r(s, a, b) = l(s, a) + m(s, b)$ for every a, b where l and m are bounded measurable functions in s. Let*

$$\phi_n(s) = r(s, f_n(s), g_n(s)) = \sum_b \sum_a r(s, a, b) f_n(s, a) g_n(s, b)$$

and

$$\phi_0(s) = r(s, f_0(s), g_0(s) = \sum_b \sum_a r(s, a, b) f_0(s, a) g_0(s, b).$$

Then $\phi_n(s) \to \phi(s)$ in the weak-star topology that is

$$\int_0^1 h(s)\phi_n(s) ds \to \int_0^1 h(s)\phi_0(s)\, ds$$

for every integrable function $h(s)$.

Proof: Let h be any integrable function. Then

$$\int_0^1 h(s)\phi_n(s)ds = \int_0^1 h(s)(\Sigma\Sigma(l(s, a)+m(s, b))f_n(s, a)g_n(s, b)ds$$

$$= \int_0^1 h(s) \sum_b \sum_a l(s, a)f_n(s, a)g_n(s, b)ds$$

$$+ \int_0^1 h(s) \sum_a \sum_b m(s, b)f_n(s, a)g_n(s, b)ds$$

$$= \int_0^1 \sum_a (h(s)l(s, a))f_n(s, a)ds + \int_0^1 \sum_b (h(s)m(s, b))g_n(s, b)ds$$

$$\to \int_0^1 \sum_a h(s)l(s, a)f_0(s, a)ds + \int_0^1 \sum_b h(s)m(s, b)g_0(s, b)ds$$

$$= \int_0^1 h(s)\phi_0(s)ds.$$

This terminates the proof of the lemma.

Lemma 1 is crucial to the proof of the next lemma.

Lemma 2 : *Let* $\tau : M_A \times M_B \to 2^{M_A \times M_B}$ (*= all nonempty weak-star closed convex subsets of* $(M_A \times M_B)$ *where*

$$\tau(f, g) = \{(f', g') : u_g(s)$$
$$= r_1(s, f'(s), g(s)) + \beta \int u_g(\cdot)dq(\cdot \mid s, f'(s), g(s)) \text{ a.e.}$$

and

$$v_f(s) = r_2(s, f(s), g'(s)) + \beta \int v_f(\cdot)dq(\cdot \mid s, f(s), g'(s)) \text{ a.e.}\}.$$

Then the map τ is upper-semi-continuous and further there exists $(f, g_0) \in \tau(f_0, g_0)$ for some $f_0 \in M_A$, $g_0 \in M_B$.

First we will explain the set valued mapping clearly and then give the proof of Lemma 2. As in the proof of Theorem 1, u_g is the fixed point of the operator T_g and v_f is the fixed point of the corresponding operator given by f_0, with one difference. Now the domain and the range of the operator T_g is the space L_∞ of essentially bounded (with respect to the Lebesgue measure) measurable functions on S. This is the reason that the term a.e. is occurring in the definition of $\tau(f, g)$.

Proof of Lemma 2 : We will first show that the map τ is upper semi-continuous. Since M_A and M_B are compact metric spaces we will resort to sequential arguments. Let $f_n \to f_0$, $g_n \to g_0$, $(f_n^*, g_n^*) \epsilon \tau(f_n, g_n)$ with $f_n^* \to f^*$, $g_n^* \to g^*$. We will show that $(f^*, g^*) \epsilon \tau(f_0, g_0)$. We will write u_n for u_{g_n} and v_n for v_{f_n}. Then

$$u_n(s) = r_1(s, f_n^*(s), g_n(s)) + \beta \int u_n(\cdot) dq(\cdot \mid s, f_n^*(s), g_n(s)) \text{ a.e.}$$

and

$$v_n(s) = r_2(s, f_n(s), g_n^*(s)) + \beta \int v_n(\cdot) dq(\cdot \mid s, f_n(s), g_n^*(s)) \text{ a.e.}$$

Since the functions u_n are uniformly bounded integrable functions. (This follows from the fact that r is bounded and $\beta \epsilon (0, 1)$ it has a convergent subsequence, (see pp. 294 of Dunford and Schwartz, 1957). We will assume without loss of generality that $u_n \to u_0$ in the weak-star sense. (In this connection see also problem 6 in pp. 339, Dunford and Schwartz, 1957). From Lemma 1, it follows that $r_1(s, f_n^*(s), g_n(s))$ tends to $r_1(s, f^*(s), g_0(s))$ in the weak-star sense. We will now show

$$\int_0^1 u_n(\cdot) dq(\cdot \mid s, f_n^*(s), g_n(s)) \to \int_0^1 u_0(\cdot) dq(\cdot \mid s, f^*(s), g_0(s))$$

in the weak-star sense.

Note that

$$\int_0^1 u_n(\cdot) dq(\cdot \mid s, a, b) \to \int_0^1 u_0(\cdot) dq(\cdot \mid s, a, b)$$

for each s, a, b, since q is absolutely continuous with respect to the Lebesgue measure and $u_n \to u_0$ in the weak-star sense. Set

$$\phi_n(s, a, b) = \int_0^1 u_0(\cdot) dq(\cdot \mid s, a, b)$$

and

$$\phi_0(s, a, b) = \int_0^1 u_0(\cdot) dq(\cdot \mid s, a, b).$$

Let h be any integrable,

$$\left| \int_0^1 h(s) \sum_b \sum_a \phi_n(s, a, b) f_n^*(s, a) g_n(s, b)\, ds \right.$$

$$\left. - \int_0^1 h(s) \sum_b \sum_a \phi_0(s, a, b) f^*(s, a) g_0(s, b)\, ds \right|$$

$$\leqslant \left| \int_0^1 \sum_b \sum_a h(s)(\phi_n(s, a, b) - \phi_0(s, a, b)) f_n^*(s, a) g_n(s, b)\, ds \right|$$

$$+ \left| \int_0^1 \sum_b \sum_a h(s)\phi_0(s, a, b)(f_n^*(s, a) g_n(s, b) - f^*(s, a) g_0(s, b))\, ds \right|$$

$$\leqslant \sum_b \sum_a \int_0^1 |h(s)|\, |\phi_n(s, a, b) - \phi_0(s, a, b)|\, ds$$

$$+ \left| \int_0^1 \sum_b \sum_a h(s)\phi_0(s, a, b)(f_n^*(s, a) g_n(s, b) - f^*(s, a) g_0(s, b))\, ds \right|$$

$$\to 0 \text{ as } n \to \infty.$$

Here we use the fact that $|f_n^*(s, a) g_n(s, b)| \leqslant 1$. Above the first expression goes to zero by the Lebesgue's Bounded Convergence Theorem and the second expression goes to zero by Lemma 1. Hence it follows that the two limits of u_n must coincide a.e. That is

$$u_0(s) = r_1(s, f^*(s), g_0(s)) + \beta \int u_0(\cdot) dq(\cdot \mid s, f^*(s), g_0(s)) \text{ a.e.}$$

Similarly

$$v_0(s) = r_2(s, f_0(s), g^*(s)) + \beta \int v_0(\cdot) dq(\cdot \mid s, f_0(s), g^*(s)) \text{ a.e.}$$

Now it is not difficult to check that u_0 and v_0 are the fixed points associated with T_{g_0} and the operator corresponding to f_0. This proves that the map is upper-semi-continuous. Now apply Kakutani-Glicksberg (1952), fixed point theorem to get an $(f_0, g_0) \in b(f_0, g_0)$. This completes the proof of Lemma 2.

Proof of Theorem 2 : We will write u_0 and v_0 for u_{g_0} and v_{f_0} where $(f_0, g_0) \in \tau(f_0, g_0)$ as given by lemma. Once again using (Theorem 6f in Blackwell, 1965 and Theorem 3.1 in Maitra and Parthasarathy, 1970) we can conclude that (f_0, g_0) is a pair of p-equilibrium pair. Now use Theorem 1 to show that (f_0, g_0) is an equilibrium pair. This completes the proof of Theorem 2.

Remark 3 : Theorem 2 remains true, if instead of assuming q' and q'' are absolutely continuous with respect to the Lebesgue measure if we assume q' and q'' are absolutely continuous with respect to any probability distribution p on $S = [0, 1]$. As already remarked S need not be a compact set—it can be any measurable subset of the real line.

4. Further generalization, remarks and open problems

In this section we will generalize Theorem 2, make a few remarks and mention some open problems.

Let u_1 and u_2 be any two bounded measurable functions on S. Let A and B are finite sets. Set

$$h_1(s, a, b) = r_1(s, a, b) + \beta \int u_1(\cdot) dq(\cdot \mid s, a, b)$$

and

$$h_2(s, a, b) = r_2(s, a, b) + \beta \int u_2(\cdot) dq(\cdot \mid s, a, b).$$

Let

$$\psi(s) = \{(\mu', \lambda') : h_1(s, \mu', \lambda') \geqslant h_1(s, \mu, \lambda')$$

and

$$h_2(s, \mu', \lambda') \geqslant h_2(s, \mu', \lambda) \text{ for all } \mu \text{ and } \lambda\}.$$

This is the set valued function as defined in the selection lemma. Graph of $\psi = \{(s, \mu', \lambda') : (\mu', \lambda') \in \psi(s)\}$. We suppose Graph of ψ is a Borel subset of $S \times \boldsymbol{P_A} \times \boldsymbol{P_B}$ for every u_i and u_2. We will call this condition as (C). Now we are ready to state a generalization of Theorem 2.

Theorem 4 : *Let S be a Borel subset of a complete separable metric space, $A = \{1, 2, \ldots, k\}$, $B = \{1, 2, \ldots, l\}$ and $\beta \in (0,1)$. Let $r_i(s, a, b) = l_i(s, a) + m_i(s,b)$ for $i = 1, 2$ where l_i and m_i are bounded measurable functions in s. Let $q(\cdot \mid s, a, b) = [q'(\cdot \mid s, a) + q'(\cdot \mid s, b)]/2$ where $q'(\cdot \mid s, a)$ and $q''(\cdot \mid s, b)$ are measurable in s. Let p be any probability distribution over S. Suppose q' and q'' are absolutely continuous with respect to p. Further suppose condition (C) is satisfied. Then the discounted stochastic game has a pair of equilibrium stationary strategies.*

The proof of Theorem 4 is the same as the proof of Theorem 2. However, one should note that selection lemma holds good under condition (C)—this can be seen from Corollary 1 in Brown and Purves (1973) or Theorem 3 in Himmelberg, Parthasarathy and Vleck (1976).

We would like to make a few remarks now, *Remark* 4 : When S is countable, clearly Lemma 1 holds good without any separability assumption on r and q, since in that case we can always have a p-distribution on S which has non-zero mass on every point of S. Further more in that case weak-star convergence and strong convergence of sequences in $L_1(S, p)$ are the same (see Corollary 13, pp. 295 of Dunford and Schwartz, 1957).

Remark 5 : Trivially condition (C) is satisfied whenever S is countable. When S, A, B are compact metric and further if $r_1(s, \mu, \lambda)$ and $\int u(\cdot)dq(\cdot | s, \mu, \lambda)$ are continuous on $S \times P_A \times P_B$ for every measurable u, then also condition (C) is satisfied.

Open Problems : (1) We are not able to determine whether Theorem 2 remains true without the separability condition on r_i and q. We cannot imitate the proof given in the present paper completely for Lemma 1 may fail to hold good without the separability assumptions.

(2) Let $S = [0, 1]$, $A = \{1, 2, ..., k\}$, $B = \{1, 2, ..., l\}$ and $\beta \epsilon (0, 1)$. Let r_1 and r_2 be bounded measurable functions (or continuous functions) even $S \times A \times B$. Does there exist an equilibrium stationary pair for the discounted stochastic game ?

Problem 2 is harder compared to problem 1. We strongly believe that problem 1 has an affirmative answer.

Acknowledgement. The author would like to thank the referee and Professor Rolf van Dawen for some useful comments.

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Paper received : January, 1981.
Revised : July, 1981.