Identification and Estimation of Dynamic Games when Players' Beliefs Are Not in Equilibrium

ONLINE APPENDIX

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A. BOUNDS APPROACH IN DYNAMIC GAMES

The purpose of this part of the appendix is to explain why Aradillas-Lopez and Tamer's bounds approach, while useful for identification and estimation of static binary choice games, has very limited applicability to dynamic games. Aradillas-Lopez and Tamer (2008) (we use the abbreviation ALT from now on) consider a static, two-player, binary-choice game of incomplete information. The model they consider can be seen as a specific case of our framework. To see this, consider the final period of the game T in our model. For the sake of notational simplicity, we omit here the vector of state variables $\mathbf{X}$ as an argument of payoff and belief functions. At the last period T, the decision problem facing the players is equivalent to that of a static game. At period T there is no future and the difference between the conditional choice value functions is simply the difference between the conditional choice current profits. For the binary choice game with two players, the Best Response Probability Function (BRP) function is:

$$P_{iT}(1) = \Lambda\left(\; B_{iT}^{(T)}(0) \; [\pi_{iT}(1,0) - \pi_{iT}(0,0)] + B_{iT}^{(T)}(1) \; [\pi_{iT}(1,1) - \pi_{iT}(0,1)] \; \right) \tag{A.1}$$

ALT assume that players' payoffs are submodular in players' decisions (Y_i,Y_j), i.e., for every value of the state variables $\mathbf{X}$, we have that $[\pi_{it}(1,0) - \pi_{it}(0,0)] > [\pi_{it}(1,1) - \pi_{it}(0,1)]$. Under this restriction, they derive informative bounds around players' conditional choice probabilities when players are *level-k rational*, and show that the bounds become tighter as k increases. For instance, without further restrictions on beliefs (i.e., rationality of level 1), player i's conditional choice probability $P_{iT}(1)$ takes its largest possible value when $B_{iT}^{(T)}(1) = 0$, and it takes its smallest possible value when beliefs are $B_{iT}^{(T)}(0) = 1$. This result yields informative bounds on the period T choice probabilities of player i:

$$\Lambda\left(\pi_{iT}(1,1) - \pi_{iT}(0,1)\right) \leq P_{iT}(1) \leq \Lambda\left(\pi_{iT}(1,0) - \pi_{iT}(0,0)\right) \tag{A.2}$$

These bounds on conditional choice probabilities can be used to "set-identify" the structural parameters in players' preferences.

In their setup, the monotonicity of players' payoffs in the decisions of other players implies monotonicity of players' BRP functions in the beliefs about other players actions. This type of monotonicity is very convenient in their approach, not only from the perspective of identification, but also because it yields a very simple approach to calculate upper and lower bounds on conditional choice probabilities. In particular, the maximum and minimum possible values of the CCPs are reached when the belief probability is equal to 0 or 1, respectively. Unfortunately, this property does not extend to dynamic games, even the simpler ones. We now discuss this issue.

Consider the two-players, binary-choice, dynamic game at some period t smaller than T. To obtain bounds on players' choice probabilities analogous to the ones obtained at the last period, we need to find, for every value of the state variables $\mathbf{X}$, the value of beliefs $\mathbf{B}^{(t)}$ that generate the smallest (and the largest) values of the best response probability $\Lambda(v_{it}^{\mathbf{B}^{(t)}}(1, \mathbf{X}) - v_{it}^{\mathbf{B}^{(t)}}(0, \mathbf{X}))$. That is, we need to minimize (or maximize) this best response probability with respect to the vector of beliefs $\{B_{it}^{(t)}, B_{it+1}^{(t)}, ..., B_{iT}^{(t)}\}$. Without making further assumptions, this best response function is not monotonic in beliefs at every possible state. In fact, this monotonicity is only achieved under very strong conditions not only on the payoff function but also on the transition probability of the state variables and on belief functions themselves.

Therefore, in a dynamic game, to find the largest and smallest value of a best response (and ultimately the bounds on choice probabilities) at periods $t < T$, one needs to explicitly solve a non-trivial optimization problem. In fact, the maximization (minimization) of the BRP function with respect to beliefs is a extremely complex task. The main reason is that the best response probability evaluated at a value of the state variables depends on beliefs at every period in the future and at every possible value of the state variables in the future. Therefore, to find bounds on best responses we must solve an optimization problem with a dimension equal to the number of values in the space of state variables times the number of future periods. This is because, in general, the maximization (or minimization) of a best response with respect to beliefs does not have a time-recursive structure except under very special assumptions (see Aguirregabiria, 2008). For instance, though $B_{iT}^{(T)}(1|\mathbf{x}) = 0$ maximizes the best response at the last period T, in general the maximization of a best response at period $T-1$ is not achieved setting $B_{iT}^{(T-1)}(1|\mathbf{x}) = 0$ for any value $\mathbf{x}$. More generally, the beliefs from period t to T that provide the maximum (minimum) value of the best response at period t are not equal to the beliefs from period t to T that provide the maximum (minimum) value of the best response at $t-1$. So at each point in time we need to re-optimize with respect to beliefs about strategies at every period in the future. That is, while the optimization of expected and discounted payoffs has the well-known time-recursive structure, the maximization (or minimization) of the value of BRP functions does not.

B. INTEGRATED VALUE FUNCTION AND CONTINUATION VALUES

Our proofs of Propositions 1 - 3 apply the concepts of the *integrated value function* and the *continuation value function.* The *integrated value function* is defined as $\bar{V}_{it}^{\mathbf{B}(t)}(\mathbf{X}_t) \equiv \int V_{it}^{\mathbf{B}(t)}(\mathbf{X}_t, \varepsilon_{it})$ $dG_{it}(\varepsilon_{it})$ (see Rust, 1994). Applying this definition to the Bellman equation, we obtained the *integrated Bellman equation:*

$$
\begin{aligned}
\bar{V}_{it}^{\mathbf{B}(t)}(\mathbf{x}_t) &= \int \max_{y_i \in \mathcal{Y}} \left\{ v_{it}^{\mathbf{B}(t)}(y_i, \mathbf{x}_t) + \varepsilon_{it}(y_i) \right\} dG_{it}(\varepsilon_{it}) \\
&= \int \max_{y_i \in \mathcal{Y}} \left\{ \pi_{it}^{\mathbf{B}(t)}(y_i, \mathbf{x}_t) + \beta \sum_{\mathbf{x}_{t+1}} \bar{V}_{it+1}^{\mathbf{B}(t)}(\mathbf{x}_{t+1}) \; f_{it}^{\mathbf{B}}(\mathbf{x}_{t+1}|y_i, \mathbf{x}_t) + \varepsilon_{it}(y_i) \right\} dG_{it}(\varepsilon_{it})
\end{aligned} \tag{B.1}
$$

where it is understood that the function $\bar{V}_{it+1}^{\mathbf{B}(t)}$ depends on current beliefs about future events, that is, $B_{it+1}^{(t)}, B_{it+2}^{(t)}$... and so on. The expected payoff function $\pi_{it}^{\mathbf{B}}$ by contrast depends only on contemporaneous beliefs $B_{it}^{(t)}$. If $\{\varepsilon_{it}(0),\, \varepsilon_{it}(1),\, ...,\, \varepsilon_{it}(A-1)\}$ are i.i.d. extreme value type 1, the integrated Bellman equation has the following closed-form expression:

$$\bar{V}_{it}^{\mathbf{B}(t)}(\mathbf{x}_t) \quad = \quad \gamma + \ln\left(\sum_{y_i \in \mathcal{Y}} \exp\left\{\pi_{it}^{\mathbf{B}}(y_i, \mathbf{x}_t) + \beta \sum_{\mathbf{X}_{t+1}} \bar{V}_{it+1}^{\mathbf{B}(t)}(\mathbf{x}_{t+1})\ f_{it}^{\mathbf{B}}(\mathbf{x}_{t+1}|y_i, \mathbf{x}_t)\right\}\right) \tag{B.2}$$

where γ is Euler's constant. In the case of a finite horizon model with beliefs that do not vary over time (i.e., $B_{it+s}^{(t)} = B_{it+s}^{(t')}$ for every pair of periods t and t'), with knowledge of payoffs and beliefs, we could use this formula to obtain the integrated value function by backwards induction, starting at the last period T.

The *continuation value function* provides the expected and discounted value of *future* payoffs given current choices of all the players and beliefs of player i about future decisions. It is defined as:

$$c_{it}^{\mathbf{B}}(\mathbf{y}_t, \mathbf{x}_t) \equiv \beta \sum_{\mathbf{x}_{t+1}} \bar{V}_{it+1}^{\mathbf{B}(t)}(\mathbf{x}_{t+1})\ f_t(\mathbf{x}_{t+1}|\mathbf{y}_t, \mathbf{x}_t) \tag{B.3}$$

Note that continuation values $c_{it}^{\mathbf{B}}$ depend on beliefs for decisions at periods $t+1$ and later, but not on beliefs for decisions at period t. By definition, the relationship between the conditional choice value function $v_{it}^{\mathbf{B}(t)}$ and the continuation value function $c_{it}^{\mathbf{B}(t)}$ is the following:

$$v_{it}^{\mathbf{B}(t)}(y_i, \mathbf{x}) \quad = \quad \sum_{\mathbf{y}_{-i} \in \mathcal{Y}^{N-1}} \left[\pi_{it}(y_i, \mathbf{y}_{-i}, \mathbf{x}) + c_{it}^{\mathbf{B}}(y_i, \mathbf{y}_{-i}, \mathbf{x})\right]\ B_{it}^{(t)}(\mathbf{y}_{-i}|\mathbf{x}) \tag{B.4}$$

Finally, we define two other objects that will be useful in what follows. First, the continuation value differences:

$$\widetilde{c}_{it}^{\mathbf{B}}(\mathbf{y}_{-i}, \mathbf{x}) \equiv c_{it}^{\mathbf{B}}(1, \mathbf{y}_{-i}, \mathbf{x}) - c_{it}^{\mathbf{B}}(0, \mathbf{y}_{-i}, \mathbf{x}) \tag{B.5}$$

and the sum of current payoffs and the continuation value differences:

$$g_{it}^{\mathbf{B}}(y_i, \mathbf{y}_{-i}, \mathbf{x}) \equiv \pi_{it}(y_i, \mathbf{y}_{-i}, \mathbf{x}) + \widetilde{c}_{it}^{\mathbf{B}}(\mathbf{y}_{-i}, \mathbf{x}) \tag{B.6}$$

C. PROOF OF PROPOSITION 1

The proof has two parts. First, we show that given CCPs of player i only, it is possible to identify a function that depends on beliefs of players but not on payoffs. Second, under the assumption of equilibrium beliefs, the identified function of beliefs can be also identified using only CCPs of player j. Therefore, we have identified the same object using two different sources of data. If the hypothesis of equilibrium beliefs is correct, the two approaches should give us the same result, but if beliefs are biased the two approaches provide different results. This can be used to construct a test statistic.

There are $N = 2$ players, i and j, the vector of state variables $\mathbf{X}$ is $(S_i, S_j, \mathbf{W})$, and players' actions are y_i and y_j. Under Assumption ID-3(iv), the transition of the state variables has the form $f_t(\mathbf{X}_{t+1}|\mathbf{Y}_t, \mathbf{W}_t)$ and we have that continuation values $c_{it}^{\mathbf{B}}(\mathbf{Y}_t, \mathbf{X}_t)$ do not depend on $\mathbf{S}_t$. Therefore, the restrictions of the model can be written as:

$$q_{it}(y_i, \mathbf{x}) \quad = \mathbf{B}_{it}^{(t)}(\mathbf{x})' \; \mathbf{g}_{it}^{\mathbf{B}}(y_i, s_i, \mathbf{w}) \tag{C.1}$$

where $\mathbf{B}_{it}^{(t)}(\mathbf{x})$ and $\mathbf{g}_{it}^{\mathbf{B}}(y_i, s_i, \mathbf{w})$ are $A \times 1$ vectors. For notational simplicity and without loss of generality, we omit $\mathbf{W}$ for the rest of this proof.

Let s_j^0 be an arbitrary value in the set $\mathcal{S}$. And let $\mathcal{S}^{(a)}$ and $\mathcal{S}^{(b)}$ be two different subsets included in the set $\mathcal{S} - \{s_j^0\}$ such that they satisfy two conditions: (1) each of these sets has $A-1$ elements; and (2) $\mathcal{S}^{(a)}$ and $\mathcal{S}^{(b)}$ have at least one element that is different. Since $|\mathcal{S}| \geq A + 1$, it is always possible to construct two subsets that satisfy these conditions. Given one of these subsets, say $\mathcal{S}^{(a)}$, we can construct the following system of $A - 1$ equations:

$$\Delta \mathbf{q}_{it}^{(a)}(y_i, s_i) \quad = \Delta \mathbf{B}_{it}^{(a)}(S_i) \; \widetilde{\mathbf{g}}_{it}(y_i, s_i) \tag{C.2}$$

where: $\Delta \mathbf{q}_{it}^{(a)}(y_i, s_i)$ is an $(A-1) \times 1$ vector with elements $\{q_{it}(y_i, s_i, s_j) - q_{it}(y_i, s_i, s_j^0) : \text{for } s_j \in \mathcal{S}^{(a)}\}$; $\Delta \mathbf{B}_{it}^{(a)}(s_i)$ is a $(A-1) \times (A-1)$ matrix with elements $\{B_{it}^{(t)}(y_j, s_i, s_j) - B_{it}^{(t)}(y_j, s_i, s_j^0) : \text{for } y_j \in \mathcal{Y} - \{0\}$ and $s_j \in \mathcal{S}^{(a)}\}$; and $\widetilde{\mathbf{g}}_{it}(y_i, s_i)$ is a $(A-1) \times 1$ vector with elements $\{g_{it}^{\mathbf{B}}(y_i, y_j, s_i) - g_{it}^{\mathbf{B}}(y_i, 0, s_i) : y_j \in \mathcal{Y}\}$. Using the other subset, $\mathcal{S}^{(b)}$, we can construct a similar system of $A - 1$ equations. Given that matrices $\Delta \mathbf{B}_{it}^{(a)}(s_i)$ and $\Delta \mathbf{B}_{it}^{(b)}(s_i)$ are non-singular, we can use these systems to obtain two different solutions for $\widetilde{\mathbf{g}}_{it}(y_i, s_i)$:

$$\begin{aligned} \widetilde{\mathbf{g}}_{it}(y_i, s_i) \quad &= \quad \left[\Delta \mathbf{B}_{it}^{(a)}(s_i)\right]^{-1} \Delta \mathbf{q}_{it}^{(a)}(y_i, s_i) \\ &= \quad \left[\Delta \mathbf{B}_{it}^{(b)}(s_i)\right]^{-1} \Delta \mathbf{q}_{it}^{(b)}(y_i, s_i) \end{aligned} \tag{C.3}$$

For given s_i, we have these two solutions of $\widetilde{\mathbf{g}}_{it}(y_i, s_i)$ for every value of y_i in the set $\mathcal{Y} - \{0\}$. Putting these $A-1$ solutions in matrix form, we have:

$$\left[\Delta\mathbf{B}_{it}^{(a)}(s_i)\right]^{-1} \Delta\mathbf{Q}_{it}^{(a)}(s_i) \quad = \quad \left[\Delta\mathbf{B}_{it}^{(b)}(s_i)\right]^{-1} \Delta\mathbf{Q}_{it}^{(b)}(s_i) \tag{C.4}$$

where $\Delta\mathbf{Q}_{it}^{(a)}(s_i)$ and $\Delta\mathbf{Q}_{it}^{(b)}(s_i)$ are $(A-1)\times(A-1)$ matrices with columns $\Delta\mathbf{q}_{it}^{(a)}(y_i, s_i)$ and $\Delta\mathbf{q}_{it}^{(b)}(y_i, s_i)$, respectively. Given that $\Delta\mathbf{Q}_{it}^{(a)}(s_i)$ is an invertible matrix, we can rearrange the previous system in the following way:

$$\Delta\mathbf{B}_{it}^{(a)}(s_i)\ \left[\Delta\mathbf{B}_{it}^{(b)}(s_i)\right]^{-1} \quad = \quad \Delta\mathbf{Q}_{it}^{(a)}(s_i)\ \left[\Delta\mathbf{Q}_{it}^{(b)}(s_i)\right]^{-1} \tag{C.5}$$

This expression shows that we can identify the $(A-1)\times(A-1)$ matrix $\Delta\mathbf{B}_{it}^{(a)}(s_i)\ \left[\Delta\mathbf{B}_{it}^{(b)}(s_i)\right]^{-1}$ that depends only on beliefs, using only the CCPs of player i. That is, we can identify $(A-1)\times(A-1)$ objects or functions of beliefs.

Under the assumption of unbiased beliefs, we can use the CCPs of the other player, j, to identify matrix $\Delta\mathbf{B}_{it}^{(a)}(s_i)\ \left[\Delta\mathbf{B}_{it}^{(b)}(s_i)\right]^{-1}$:

$$\Delta\mathbf{B}_{it}^{(a)}(s_i)\ \left[\Delta\mathbf{B}_{it}^{(b)}(s_i)\right]^{-1} \quad = \quad \Delta\mathbf{P}_{jt}^{(a)}(s_i)\ \left[\Delta\mathbf{P}_{jt}^{(b)}(s_i)\right]^{-1} \tag{C.6}$$

where $\Delta\mathbf{P}_{jt}^{(a)}(s_i)$ is $(A-1)\times(A-1)$ matrix with elements $\{P_{jt}(y_j, s_i, s_j) - P_{jt}(y_j, s_i, s_j^0)$: for $y_j \in \mathcal{Y}-\{0\}$ and $s_j \in \mathcal{S}^{(a)}\}$, and $\Delta\mathbf{P}_{jt}^{(b)}(s_i)$ has a similar definition. Therefore, under the assumption of unbiased beliefs by player i the CCPs of player i and player j should satisfy the following $(A-1)^2$ restrictions:

$$\Delta\mathbf{Q}_{it}^{(a)}(s_i)\ \left[\Delta\mathbf{Q}_{it}^{(b)}(s_i)\right]^{-1} - \Delta\mathbf{P}_{jt}^{(a)}(s_i)\ \left[\Delta\mathbf{P}_{jt}^{(b)}(s_i)\right]^{-1} \quad = \quad \mathbf{0} \tag{C.7}$$

These restrictions are testable. ■

D. PROOF OF PROPOSITION 2

Proposition 2. Part (2.2). Identification of values. The restrictions of the model that come from best response behavior of player i can be represented using the following equation. For any $(y_i, \mathbf{x}) \in \mathcal{Y} \times \mathcal{X}$,

$$q_{it}(y_i, \mathbf{x}) = \mathbf{B}_{it}^{(t)}(\mathbf{x})'\ \left[\boldsymbol{\pi}_{it}(y_i, \mathbf{x}) + \widetilde{\mathbf{c}}_{it}^{\mathbf{B}}(y_i, \mathbf{x})\right] \tag{D.1}$$

where $\mathbf{B}_{it}^{(t)}(\mathbf{x})$, $\boldsymbol{\pi}_{it}(y_i, \mathbf{x})$, and $\widetilde{\mathbf{c}}_{it}^{\mathbf{B}}(y_i, \mathbf{x})$ are vectors with dimension $A^{N-1} \times 1$ containing beliefs, payoffs, and continuation values, respectively, for every possible value of $\mathbf{y}_{-i}$ in the set $\mathcal{Y}^{N-1}$. Recalling our definition: $\mathbf{g}_{it}^{\mathbf{B}}(y_i, \mathbf{x}) \equiv \boldsymbol{\pi}_{it}(y_i, \mathbf{x}) + \widetilde{\mathbf{c}}_{it}^{\mathbf{B}}(y_i, \mathbf{x})$, we can re-write equation D.1 as $q_{it}(y_i, \mathbf{x}) = \mathbf{B}_{it}(\mathbf{x})' \ \mathbf{g}_{it}^{\mathbf{B}}(y_i, \mathbf{x})$, and $\mathbf{g}_{it}^{\mathbf{B}}(y_i, \mathbf{x})$ is also a vector with dimension $A^{N-1} \times 1$.

Let $\mathcal{S}_{-i}^{(R)}$ be the set $[\mathcal{S}^{(R)}]^{N-1}$. By assumption ID-4, for any $\mathbf{x}$ such that $\mathbf{s}_{-i} \in \mathcal{S}_{-i}^{(R)}$ we have that $B_{it}^{(t)}(\mathbf{y}_{-i}|\mathbf{x}) = P_{-it}(\mathbf{y}_{-i}|\mathbf{x})$ and $P_{-it}(\mathbf{y}_{-i}|\mathbf{x})$ is known to the researcher. Consider the system of equations formed by equation (D.1) at a fixed value of $(y_i, s_i, \mathbf{w})$ and for every value of $\mathbf{s}_{-i}$ in $\mathcal{S}_{-i}^{(R)}$. This is a system of R^{N-1} equations, and we can represent this system in vector form using the following expression:

$$\mathbf{q}_{it}^{(R)}(y_i, s_i) = \mathbf{P}_{-it}^{(R)}(s_i) \ \mathbf{g}_{it}^{\mathbf{B}}(y_i, s_i) \tag{D.2}$$

where $\mathbf{P}_{-it}^{(R)}(s_i)$ is the $R^{N-1} \times A^{N-1}$ matrix $\{P_{-it}(\mathbf{y}_{-i}|s_i, \mathbf{s}_{-i}) : \mathbf{y_{-i}} \in \mathcal{Y}^{\mathbf{N-1}}, \ \mathbf{s}_{-i} \in \mathcal{S}_{-i}^{(R)}\}$. Under conditions (i)-(ii) in Proposition 2, matrix $\mathbf{P}_{-it}^{(R)}(s_i)'\mathbf{P}_{-it}^{(R)}(s_i)$ is non-singular and therefore we can solve for vector $\mathbf{g}_{it}^{\mathbf{B}}(y_i, s_i)$ in the previous system of equations:

$$\mathbf{g}_{it}^{\mathbf{B}}(y_i, s_i) = \left[\mathbf{P}_{-it}^{(R)}(s_i)' \ \mathbf{P}_{-it}^{(R)}(s_i)\right]^{-1} \mathbf{P}_{-it}^{(R)}(s_i)' \ \mathbf{q}_{it}^{(R)}(y_i, s_i) \tag{D.3}$$

This expression shows that, given continuation values at period t, the vector of payoffs $\mathbf{g}_{it}^{\mathbf{B}}(y_i, s_i)$ is identified, i.e., part (b) of Proposition 2.

Proposition 2. Part (2.1). Identification of beliefs. Now, we show the identification of the beliefs function for states outside the subset $\mathcal{S}_{-i}^{(R)}$. Again, we start with the system equations implied by the best response restrictions, but now we take into account that the vector $\mathbf{g}_{it}^{\mathbf{B}}(y_i, s_i)$ is identified and then look at the identification of beliefs at states $\mathbf{x}$ with $\mathbf{s}_{-i}$ outside the subset $\mathcal{S}_{-i}^{(R)}$. We stack equation (C.1) for every value of $y_i \in \mathcal{Y} - \{0\}$ to obtain a system of equations. Note that $\mathbf{B}_{it}^{(t)}(\mathbf{x})$ is a vector of A^{N-1} probabilities, one element for each value of $\mathbf{y}_{-i}$ in $\mathcal{Y}^{N-1}$. The probabilities in this vector should sum to one, and therefore, $\mathbf{B}_{it}^{(t)}(\mathbf{x})$ satisfies the restriction $\mathbf{1}'\mathbf{B}_{it}(\mathbf{x}) = 1$, where $\mathbf{1}$ is a vector of ones. Therefore, we have the following system of A equations:

$$\mathbf{q}_{it}(\mathbf{x}) = \widetilde{\mathbf{V}}_{it}(\mathbf{x}) \ \mathbf{B}_{it}^{(t)}(\mathbf{x}) \tag{D.4}$$

$\mathbf{q}_{it}(\mathbf{x})$ is an $A \times 1$ vector with elements $\{q_{it}(1, \mathbf{x}),\ ...,\ q_{it}(A-1, \mathbf{x})\}$ at rows 1 to $A-1$, and a 1 at the last row. And $\widetilde{\mathbf{V}}_{it}(\mathbf{x})$ is an $A \times A^{N-1}$ matrix where rows 1 to $A-1$ are $\mathbf{g}_{it}^{\mathbf{B}}(1, s_i, \mathbf{w})'$, ..., $\mathbf{g}_{it}^{\mathbf{B}}(A-1, s_i, \mathbf{w})'$, and the last row of the matrix is a row of ones. Since the values $g_{it}^{\mathbf{B}}(y_i, \mathbf{y}_{-i}, s_i, \mathbf{w})$ are identified from Proposition 2.2, we have that matrix $\widetilde{\mathbf{V}}_{it}(\mathbf{x})$ is identified. We now prove that condition (ii) implies that matrix $\widetilde{\mathbf{V}}_{it}(\mathbf{x})$ is non-singular. Our proof of part (b) implies that:

$$\widetilde{\mathbf{V}}_{it}(\mathbf{x}) = \mathbf{Q}_{it}^{(R)}(s_i)\ \mathbf{P}_{-it}^{(R)}(s_i)\ \left[\mathbf{P}_{-it}^{(R)}(s_i)'\ \mathbf{P}_{-it}^{(R)}(s_i)\right]^{-1} \tag{D.5}$$

and $\mathbf{Q}_{it}^{(R)}(s_i)$ is the $A \times R$ matrix with $\mathbf{q}_{it}^{(R)}(y_i, s_i)'$ at the first $A-1$ rows, and ones at the last row. By Assumption ID-4, $\mathbf{P}_{-it}^{(R)}(s_i)$ is full column rank, and then a sufficient condition for $\widetilde{\mathbf{V}}_{it}(\mathbf{x})$ to be non-singular matrix is that $\mathbf{Q}_{it}^{(R)}(s_i)$ has rank A, which is a condition in Proposition 2. Therefore, the vector of beliefs $\mathbf{B}_{it}^{(t)}(\mathbf{x})$ is identified as:

$$\mathbf{B}_{it}^{(t)}(\mathbf{x}) = \left[\widetilde{\mathbf{V}}_{it}(\mathbf{x})\right]^{-1} \mathbf{q}_{it}(\mathbf{x}) \tag{D.6}$$

Proposition 2. Part (2.3): Identification of payoff differences. Consider the function $g_{it}^{\mathbf{B}}(y_i, \mathbf{y}_{-i}, s_i)$ at two different values of s_i, say $s^{(a)}$ and $s^{(b)}$, i.e., $g_{it}^{\mathbf{B}}(y_i, \mathbf{y}_{-i}, s^{(a)}) = \pi_{it}(y_i, \mathbf{y}_{-i}, s^{(a)}) + \widetilde{c}_{it}^{\mathbf{B}}(\mathbf{y}_{-i})$ and $g_{it}^{\mathbf{B}}(y_i, \mathbf{y}_{-i}, s^{(b)}) = \pi_{it}(y_i, \mathbf{y}_{-i}, s^{(b)}) + \widetilde{c}_{it}^{\mathbf{B}}(\mathbf{y}_{-i})$. By assumption ID-3(iv), variable s_i enters $g_{it}^{\mathbf{B}}(y_i, \mathbf{y}_{-i}, s_i)$ through the current payoff. Therefore, the difference: $g_{it}^{\mathbf{B}}(y_i, \mathbf{y}_{-i}, s^{(a)}) - g_{it}^{\mathbf{B}}(y_i, \mathbf{y}_{-i}, s^{(b)})$ gives the payoff difference:

$$g_{it}^{\mathbf{B}}(y_i, \mathbf{y}_{-i}, s^{(a)}) - g_{it}^{\mathbf{B}}(y_i, \mathbf{y}_{-i}, s^{(b)}) = \pi_{it}(y_i, \mathbf{y}_{-i}, s^{(a)}) - \pi_{it}(y_i, \mathbf{y}_{-i}, s^{(b)}) \qquad \blacksquare \tag{D.7}$$

E. ASYMPTOTIC DISTRIBUTION OF TWO-STEP ESTIMATORS

The derivation of the asymptotic distribution of our two-step estimators of payoffs and beliefs is an application of properties of two-step semiparametric estimators as shown in Newey (1994), Andrews (1994), and McFadden and Newey (1994). In fact, given our maintained assumption that the space of state variables $\mathcal{X}$ is discrete and finite, all the structural functions in our model live in a finite dimensional Euclidean space. Therefore, we do not need to apply stochastic equicontinuity results, as in Newey (1994) and Andrews (1994), to show root-M consistency and asymptotic normality of these estimators. Here we apply results in Newey (1984) who provides a methods of moments interpretation of sequential estimators.

We begin by establishing the consistency and asymptotic normality of our estimator of CCPs. The estimator of the CCP $P_{it}(y|\mathbf{x})$ is based on the moment condition:

$$\mathbb{E}\left[1\{\mathbf{X}_{mt}=\mathbf{x}\}\ \ (1\{Y_{imt}=y\}-P_{it}(y|\mathbf{x}))\right]=0 \tag{E.1}$$

In vector form, we have the system, $\mathbb{E}\left[f_{\mathbf{x}}(\mathbf{X}_{mt},Y_{imt},\mathbf{P}_{it,\mathbf{x}})\right]\equiv\mathbb{E}\left[1\{\mathbf{X}_{mt}=\mathbf{x}\}\ \ (\mathbf{1}_{Y_{imt}}-\mathbf{P}_{it,\mathbf{x}})\right]=\mathbf{0}$, where $\mathbf{1}_{Y_{imt}}$ and $\mathbf{P}_{it,\mathbf{x}}$ are the $(A-1)\times 1$ vectors $\mathbf{1}_{Y_{imt}}\equiv\{1\{Y_{imt}=y\}:y=1,2,...,A-1\}$ and $\mathbf{P}_{it,\mathbf{x}}\equiv\{P_{it}(y|\mathbf{x}):y=1,2,...,A-1\}$, respectively. The corresponding sample moment condition that defines the estimator $\widehat{\mathbf{P}}_{it,\mathbf{x}}$ is: $\sum_{m=1}^{M}f_{\mathbf{x}}(\mathbf{X}_{mt},Y_{imt},\widehat{\mathbf{P}}_{it,\mathbf{x}})=\mathbf{0}$. For notational simplicity, for the rest of this Appendix we omit the player and time subindexes (i,t) from variables, parameters, and functions. As the observations are *i.i.d.* across markets, this estimator satisfies the standard regularity conditions for consistency and asymptotic normality of the Method of Moments estimator, such that as M goes to infinity, we have that $\widehat{\mathbf{P}}_{\mathbf{x}}\rightarrow_p\mathbf{P}_{\mathbf{x}}$, and

$$\sqrt{M}\left(\widehat{\mathbf{P}}_{\mathbf{x}}-\mathbf{P}_{\mathbf{x}}\right)\rightarrow_d N\left(0\ ,\ \mathbf{F}_{\mathbf{x}}^{-1}\ \boldsymbol{\Omega}_{ff}\ \mathbf{F}_{\mathbf{x}}^{-1\prime}\right) \tag{E.2}$$

where $\mathbf{F}_{\mathbf{x}}\equiv\mathbb{E}\left[\partial f_{\mathbf{x}}(\mathbf{X},Y,\mathbf{P}_{\mathbf{x}})/\partial\mathbf{P}_{\mathbf{x}}'\right]$ and $\boldsymbol{\Omega}_{ff}\equiv\mathbb{E}\left[f_{\mathbf{x}}(\mathbf{X},Y,\mathbf{P}_{\mathbf{x}})\ f_{\mathbf{x}}(\mathbf{X},Y,\mathbf{P}_{\mathbf{x}})'\right]$.

We now establish the consistency and asymptotic normality of the estimator of the function $g_{it}^{\mathbf{B}}(y_i,\mathbf{y}_{-i},\mathbf{X})$. This implies the distribution of the estimator of payoffs $\pi_{it}^{\mathbf{B}}(y_i,\mathbf{y}_{-i},\mathbf{X})$ is also consistent and asymptotically normal as, by Proposition 3, payoffs are a deterministic, linear combination of $g_{it}^{\mathbf{B}}(y_i,\mathbf{y}_{-i},\mathbf{X})$. The population restrictions that our estimator of payoff must satisfy at a given value of $(y_i,S_i,\mathbf{W})$ are given by:

$$\mathbf{q}_i^{(R)}(y_i,S_i,\mathbf{W})-\mathbf{P}^{(R)}(S_i,\mathbf{W})\ \mathbf{g}_i^{\mathbf{B}}(y_i,S_i,\mathbf{W})=0 \tag{E.3}$$

Or in vector form, for any value of y_i (and omitting the player subindex i), $h_{S,\mathbf{W}}(\mathbf{P}_{\mathbf{x}},\boldsymbol{\pi}_{S,\mathbf{W}})\equiv\mathbf{q}^{(R)}(.,S,\mathbf{W})-\mathbf{P}^{(R)}(.,S,\mathbf{W})\ \boldsymbol{\pi}_{S,\mathbf{W}}(.,S,\mathbf{W})=\mathbf{0}$. In the just-identified nonparametric model, the estimator $\widehat{\mathbf{g}}_{S,\mathbf{W}}$ of the vector of payoffs $\mathbf{g}_{S,\mathbf{W}}$ is the value that solves the system of equations $h_{S,\mathbf{W}}(\widehat{\mathbf{P}}_{\mathbf{x}},\widehat{\boldsymbol{\pi}}_{S,\mathbf{W}})=\mathbf{0}$. Under the conditions of Proposition 2, the mapping $h_{S,\mathbf{W}}(\mathbf{P}_{\mathbf{x}},\boldsymbol{\pi}_{S,\mathbf{W}})$ satisfies the regularity conditions to apply Slutsky's Theorem and the Continuous Mapping Theorem (or Mann-Wald Theorem) such that as M goes to infinity, we have that $\widehat{\mathbf{g}}_{S,\mathbf{W}} \to_p \mathbf{g}_{S,\mathbf{W}}$, and $\sqrt{M}\left(\widehat{\mathbf{g}}_{S,\mathbf{W}} - \mathbf{g}_{S,\mathbf{W}}\right) \to_d N\left(0\ ,\ \mathbf{V}_{\mathbf{g}_{S,\mathbf{W}}}\right)$, where applying Newey (1984)

$$\mathbf{V}_{\mathbf{g}_{S,\mathbf{W}}} = \mathbf{H}_{\mathbf{g}}^{-1}\left(\ \mathbf{\Omega}_{hh} + \mathbf{H}_{\mathbf{P}}[\mathbf{F}_{\mathbf{x}}^{-1}\ \mathbf{\Omega}_{ff}\ \mathbf{F}_{\mathbf{x}}^{-1\prime}]\ \mathbf{H}_{\mathbf{P}}' - \mathbf{H}_{\mathbf{P}}[\mathbf{F}_{\mathbf{x}}^{-1}\ \mathbf{\Omega}_{f,h}\ +\ \mathbf{\Omega}_{h,f}\mathbf{F}_{\mathbf{x}}^{-1\prime}]\mathbf{H}_{\mathbf{P}}'\ \right)\ \mathbf{H}_{\mathbf{f}}^{-1\prime} \tag{E.4}$$

with $\mathbf{H}_{\mathbf{g}} \equiv \partial h_{S,\mathbf{W}}(\mathbf{P}_{\mathbf{x}}, \mathbf{g}_{S,\mathbf{W}})/\partial \mathbf{g}_{S,\mathbf{W}}'$, $\mathbf{H}_{\mathbf{P}} \equiv \partial h_{S,\mathbf{W}}(\mathbf{P}_{\mathbf{x}}, \mathbf{g}_{S,\mathbf{W}})/\partial \mathbf{P}_{\mathbf{x}}'$, $\mathbf{\Omega}_{hh} \equiv \mathbb{E}[h_{S,\mathbf{W}}(\mathbf{P}_{\mathbf{x}}, \mathbf{g}_{S,\mathbf{W}})\ h_{S,\mathbf{W}}(\mathbf{P}_{\mathbf{x}}, \mathbf{g}_{S,\mathbf{W}})']$, $\mathbf{\Omega}_{fh} \equiv \mathbb{E}[f_{\mathbf{x}}(\mathbf{X}, Y, \mathbf{P}_{\mathbf{x}})\ h_{S,\mathbf{W}}(\mathbf{P}_{\mathbf{x}}, \mathbf{g}_{S,\mathbf{W}})']$, and $\mathbf{\Omega}_{hf} \equiv \mathbb{E}[h_{S,\mathbf{W}}(\mathbf{P}_{\mathbf{x}}, \mathbf{g}_{S,\mathbf{W}})\ f_{\mathbf{x}}(\mathbf{X}, Y, \mathbf{P}_{\mathbf{x}})']$.

Our estimator of beliefs takes as given the estimates of CCPs and payoffs. Specifically, for a given vector of the state variables $\mathbf{x}$, beliefs are given by the system

$$\ell_{\mathbf{x}}\left(\mathbf{P}_{\mathbf{x}}, \mathbf{B}_{\mathbf{x}}\right) \equiv \widetilde{V}_i(\mathbf{X})\ \mathbf{B}_i(\mathbf{X}) - \mathbf{q}_i(\mathbf{X}) = \mathbf{0} \tag{E.5}$$

The estimator $\widehat{\mathbf{B}}_{\mathbf{x}}$ of the vector of beliefs $\mathbf{B}_{\mathbf{x}}$ is the value that solves the system of equations $\ell_{\mathbf{x}}\left(\widehat{\mathbf{P}}_{\mathbf{x}}, \widehat{\mathbf{B}}_{\mathbf{x}}\right) = 0$. Under the conditions of Proposition 2, the mapping $\ell_{\mathbf{x}}\left(\mathbf{P}_{\mathbf{x}}, \mathbf{B}_{\mathbf{x}}\right)$ satisfies the regularity conditions to apply Slutsky's Theorem and the Continuous Mapping Theorem such that as M goes to infinity, we have that $\widehat{\mathbf{B}}_{\mathbf{x}} \to_p \mathbf{B}_{\mathbf{x}}$, and $\sqrt{M}\left(\widehat{\mathbf{B}}_{\mathbf{x}} - \mathbf{B}_{\mathbf{x}}\right) \to_d N\left(0\ ,\ \mathbf{V}_{\mathbf{B}_{\mathbf{x}}}\right)$, where applying Newey (1984),

$$\mathbf{V}_{\mathbf{B}_{\mathbf{x}}} = \mathbf{L}_{\mathbf{B}}^{-1}\left(\ \mathbf{\Omega}_{\ell\ell} + \mathbf{L}_{\mathbf{P}}[\mathbf{F}_{\mathbf{x}}^{-1}\ \mathbf{\Omega}_{ff}\ \mathbf{F}_{\mathbf{x}}^{-1\prime}]\ \mathbf{L}_{\mathbf{P}}' - \mathbf{L}_{\mathbf{P}}[\mathbf{F}_{\mathbf{x}}^{-1}\ \mathbf{\Omega}_{f,\ell}\ +\ \mathbf{\Omega}_{\ell,f}\mathbf{F}_{\mathbf{x}}^{-1\prime}]\mathbf{L}_{\mathbf{P}}'\ \right)\ \mathbf{L}_{\mathbf{B}}^{-1\prime} \tag{E.6}$$

with $\mathbf{L}_{\mathbf{B}} \equiv \partial \ell_{\mathbf{x}}\left(\mathbf{P}_{\mathbf{x}}, \mathbf{B}_{\mathbf{x}}\right)/\partial \mathbf{B}_{\mathbf{x}}'$, $\mathbf{L}_{\mathbf{P}} \equiv \partial \ell_{\mathbf{x}}\left(\mathbf{P}_{\mathbf{x}}, \mathbf{B}_{\mathbf{x}}\right)/\partial \mathbf{P}_{\mathbf{x}}'$, $\mathbf{\Omega}_{\ell\ell} \equiv \mathbb{E}[\ell_{\mathbf{x}}\left(\mathbf{P}_{\mathbf{x}}, \mathbf{B}_{\mathbf{x}}\right)\ \ell_{\mathbf{x}}\left(\mathbf{P}_{\mathbf{x}}, \mathbf{B}_{\mathbf{x}}\right)']$, $\mathbf{\Omega}_{f,\ell} \equiv \mathbb{E}[f_{\mathbf{x}}(\mathbf{X}, Y, \mathbf{P}_{\mathbf{x}})\ \ell_{\mathbf{x}}\left(\mathbf{P}_{\mathbf{x}}, \mathbf{B}_{\mathbf{x}}\right)']$, and $\mathbf{\Omega}_{\ell,f} \equiv \mathbb{E}[\ell_{\mathbf{x}}\left(\mathbf{P}_{\mathbf{x}}, \mathbf{B}_{\mathbf{x}}\right)\ f_{\mathbf{x}}(\mathbf{X}, Y, \mathbf{P}_{\mathbf{x}})']$. ■

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