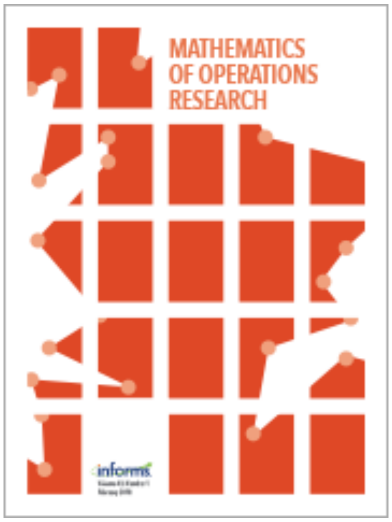

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Stationary Almost Markov Perfect Equilibria in Discounted Stochastic Games

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In this paper, we study discounted stochastic games with Borel state and compact action spaces depending on the state variable. The primitives of our model satisfy standard continuity and measurability conditions. The transition probability is a convex combination of finitely many probability measures depending on states, and it is dominated by some finite measure on the state space. The coefficients of the combination depend on both states and action profiles. This class of models contains stochastic games with Borel state spaces and finite, state-dependent action sets. Our main result establishes the existence of subgame perfect equilibria, which are stationary in the sense that the equilibrium strategy for each player is determined by a single function of the current and previous states of the game. This dependence is called almost Markov. Our result enhances both the theorem of Mertens and Parthasarathy established in 1991 for games with finite, state-independent action sets, where the equilibrium strategies were also depended on the calendar time, and their result on stationary equilibria proved under an additional condition that the transition probabilities are atomless. A counterexample given very recently by Levy shows that stationary Markov perfect equilibria may not exist in the class of games considered in this paper. The presented results in this work are illustrated by the Cournot dynamic games, which were already considered in the literature under much stronger assumptions.

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1. Introduction. Discounted stochastic games were introduced by Shapley [32], who studied zero-sum games with finite state and action spaces. Since then, they have received much attention from both mathematicians and economists, especially in the general sum case. The growing attraction of stochastic games in modelling of economic behaviour has made an equilibrium existence problem a major issue and active area of research. There is a long tradition in the game theory literature of dealing with equilibrium existence problems in discounted stochastic games on general state spaces.

A major contribution belongs to Mertens and Parthasarathy [23, 24], who established the existence of a subgame perfect equilibrium but with strategies depending on the entire history of the game (see also Maitra and Sudderth [20] and Solan [33] for alternative proofs and some extensions). The view that equilibrium strategies should only depend on payoff-relevant data was emphasised by Maskin and Tirole [21], who motivated the development of the concept of Markov perfect equilibrium. Such equilibrium strategies are then Markovian; i.e., they may only depend on the current state of the game. Another desired feature of equilibrium strategies is stationarity, i.e., independence of the calendar time. The main reason for searching for stationary equilibria is their simplicity and easy implementation as well as satisfying the aforementioned property highlighted by Maskin and Tirole [21]. It is worthwhile to recall that Himmelberg et al. [13] discovered the first class of games having a stationary Nash equilibrium. They assumed that the payoff and transition probability functions must meet certain restrictive separability conditions. The second class was pointed out by Parthasarathy and Sinha [30], who considered models with finite action spaces of the players and state independent and atomless transition probabilities. Later, Nowak and Raghavan [29] proved the existence of correlated equilibria in stationary Markov strategies. Similar results were reported by Duffie et al. [6] and Harris et al. [11]. Recently, Duggan [7], dropping the correlation scheme, showed that so-called "noisy stochastic games" also possess stationary Markov perfect equilibria. Finally, Nowak [27] pointed out another class of games for which the aforementioned type of equilibria exists. Such a class consists of models in which the transition probability is a convex combination of finitely many probability measures on the state space. It turns out that most of the discussed classes of discounted stochastic games can be described in a unified manner. Namely, He and Sun [12] formulated certain general condition called "decomposable coarser transition kernel" and showed that under this requirement a discounted stochastic game with the transition probability dominated by some measure on the state space possesses a stationary Markov perfect equilibrium. In addition, they illustrated that games studied in Duggan [7], Nowak [27], and Nowak and Raghavan [29] indeed satisfy their assumption.

The existence results of stationary Markov perfect equilibria has proved elusive. It has been lately confirmed by Levy and McLennan [19], who provided an example of a discounted stochastic game having no stationary Markov perfect equilibrium when the state space is an interval in the real line, the action sets are finite and state independent, and the transitions are dominated by a probability measure. This measure may be atomless.[1] However, the counterexample presented in Levy and McLennan [19] does not satisfy the "decomposable coarser transition kernel" condition. Consequently, the class of games studied by He and Sun [12] does not include the models with finite action spaces even if the transitions are atomless.

The ongoing research emphasises the value of finding equilibria in the class of strategies that are less complex than those considered in Mertens and Parthasarathy [23, 24]. For instance, Mertens and Parthasarathy [23] examined stochastic games with finite, state-independent action sets and general state space and showed that in the atomless case the equilibrium strategies may depend on the current and previous states. However, their proof technique cannot be applied to stochastic games with infinite sets of actions. Recently, Barelli and Duggan [2], with the aid of a result of Mertens [22] and under absolute continuity assumption, showed the existence of stationary semi-Markov perfect equilibria. Namely, the equilibrium strategies in Barelli and Duggan [2] may depend on the current state, the previous state, and the action profile chosen by the players in the previous state.

In this paper, we discover another class of games for which stationary almost Markov perfect equilibria exist. Stationarity means that an equilibrium strategy for each player is determined by the same function depending on the current and previous state of the game. Since this dependence on the two consecutive states occurs, this equilibrium is not called Markov but "almost Markov." We presume that the transition probability is a convex combination of finitely many measures depending on the state variable. The coefficients in this combination, on the other hand, may depend, as in Nowak [27], on the state and actions of the players and are continuous with respect to the actions. In contrast to Barelli and Duggan [2] and Mertens and Parthasarathy [23], we allow the transition probability to possess atoms. The transition structure assumed in this paper is the same as Levy and McLennan [19] and Mertens and Parthasarathy [23] imposed for games with finite action spaces. Such a transition structure with infinite action sets is natural in a number of economic applications. We also show how our main results can (after some minor modification) be applied to some dynamic Cournot games.

The paper is organised as follows. In §2, we describe our basic model and assumptions. In §3, we present our main theorems with some comments on the existing results in the literature. Finally, §4 is devoted to applications of certain modifications of our main results to dynamic games in economics.

2. The model. By $\mathbb{N}$ we denote the set of all positive integers and by $\mathbb{R}$ the set of all real numbers. Let Y be a compact metric space. By $P(Y)$ we denote the space of all Borel probability measures on Y endowed with the weak topology. Then $P(Y)$ becomes a compact metrisable subset of a linear topological Hausdorff space; see Proposition 7.22 in Bertsekas and Shreve [3]. Recall that this topology can be characterised in terms of convergent sequences; see Proposition 7.21 in Bertsekas and Shreve [3]. Namely, a sequence $(p_n)_{n\in\mathbb{N}}$ converges to some $p\in P(Y)$ in the weak topology if and only if

$$\lim_{n\to\infty}\int_Y u(y)p_n(dy)=\int_Y u(y)p(dy)$$

for any continuous function $u\colon Y\mapsto\mathbb{R}$.

We consider an m-person nonzero-sum discounted stochastic game G for which

(i) $(S,\mathscr{B})$ is a nonempty Borel *state space* with its Borel σ-algebra $\mathscr{B}$.

(ii) X_i is a nonempty compact metric *space of actions* for player $i\in I:=\{1,\dots,m\}$.

(iii) $A_i(s)\subset X_i$ is a *set of actions* available to player $i\in I$ in state $s\in S$. The correspondence $A_i\colon S\rightrightarrows X_i$ is lower measurable and compact valued. Define

$$X:=X_1\times\cdots\times X_m \qquad\text{and}\qquad A(s)=A_1(s)\times\cdots\times A_m(s),\quad s\in S.$$

(iv) $r_i\colon S\times X\mapsto\mathbb{R}$ is a product measurable *reward* (or *payoff*) *function* for player $i\in I$. It is assumed that r_i is a Carathéodory function; i.e., $r_i(s,\cdot)$ is continuous on X for every $s\in S$ and $r_i(\cdot,a)$ is measurable on S for each $a\in X$. Moreover,

$$L:=\max_{i\in I}\sup_{s\in S}\max_{a\in A_i(s)}|r_i(s,a)|<\infty.$$

[1] The transition probability used in Levy and McLennan [19] has an atom, but their example may be easily modified to work in the atomless case.

(v) q: $S \times X \times \mathcal{B} \mapsto [0, 1]$ is a *transition probability*. We assume that there exist l Carathéodory functions $g_1, \ldots, g_l$: $S \times X \mapsto [0, 1]$ such that $\sum_{j=1}^{l} g_j(s, a) = 1$ for every $(s, a) \in S \times X$ and transition probabilities $q_1, \ldots, q_l$: $S \times \mathcal{B} \mapsto [0, 1]$ such that

$$q(\cdot \mid s, a) = \sum_{j=1}^{l} g_j(s, a) q_j(\cdot \mid s), \quad (s, a) \in S \times X. \tag{1}$$

Moreover, each q_j is dominated by a probability measure μ defined on $(S, \mathcal{B})$.

(vi) $\beta \in (0, 1)$ is a *discount factor*.

The above components are used to describe a discrete-time dynamic game in which each period begins with a state $s \in S$, and after observing s, the players simultaneously choose their actions from their available sets $A_i(s)$ and obtain rewards $r_i(s, a)$. A new state s' is realised from the distribution $q(\cdot \mid s, a)$, and a new period begins with rewards discounted by β.

A *strategy* for a player is a sequence of Borel measurable mappings, where each mapping associates with the given history a probability distribution on the available set of actions. The set of strategies for player $i \in I$ is denoted by Π_i and its generic element by π_i. Let F_i be the set of all Borel measurable mappings f_i: $S \times S \mapsto P(X_i)$ such that $f_i(s^-, s) \in P(A_i(s))$ for each $s^-, s \in S$. A *stationary almost Markov strategy* for player $i \in I$ is a constant sequence $(\sigma_{it})_{t \in \mathbb{N}}$ where $\sigma_{it} = f_i$ for some $f_i \in F_i$ and for all $t \in \mathbb{N}$. If s_t is a state of the game on its t-stage with $t \geq 2$, then player i chooses an action using the mixed strategy $f_i(s_{t-1}, s_t)$. The mixed strategy used at an initial state s_1 is $f_i(s_1, s_1)$. A stationary almost Markov strategy for player $i \in I$ is identified with a Borel measurable mapping $f_i \in F_i$. We would like to indicate that F_i is a special class of stationary semi-Markov strategies considered in Barelli and Duggan [2], where their dependence on the current state, previous state, and the actions selected by all players in the previous state is assumed. Furthermore, a stationary almost Markov strategy f_i for player $i \in I$ is called *Markov,* if $f_i(s^-, s) = f_i(s)$; that is, f_i is independent of the previous state $s^- \in S$.

For any strategy profile $\pi = (\pi_1, \ldots, \pi_m) \in \Pi_1 \times \cdots \times \Pi_m$, an initial state $s \in S$, and $t \in \mathbb{N}$, by $r_i^{(t)}(s, \pi)$ we denote the expected reward for player $i \in I$ in the t-th period of the game. The *expected discounted payoff* or *reward function* for player $i \in I$ is

$$J_i(s, \pi) = \sum_{t=1}^{\infty} \beta^{t-1} r_i^{(t)}(s, \pi).$$

Let $y = (y_1, \ldots, y_m)$, where y_j belongs to some set Z_j $(j \in I)$ and assume that $z_i \in Z_i$. Using standard notation we shall write (z_i, y_{-i}) to denote y with y_i replaced by z_i.

A profile of strategies $\pi^* \in \Pi_1 \times \cdots \times \Pi_m$ is called a *Nash equilibrium* if

$$J_i(s, \pi^*) \geq J_i(s, (\pi_i, \pi^*_{-i})) \quad \text{for all } s \in S, \ \ \pi_i \in \Pi_i, \ \text{ and } \ i \in I.$$

A *stationary almost Markov perfect equilibrium* (*SAMPE*) is a Nash equilibrium that belongs to the class of strategy profiles $F := F_1 \times \cdots \times F_m$.

Henceforth, we shall refer to the game $\tilde{G}$ with the state space $S \times S$, action spaces X_i, and available action spaces $\tilde{A}_i(s^-, s) = A_i(s)$ for all $(s^-, s) \in S \times S$ and $i \in I$. The reward function for player $i \in I$ in game $\tilde{G}$, denoted by $\tilde{r}_i$, is defined as follows:

$$\tilde{r}_i((s^-, s), a) := r_i(s, a) \quad \text{for all } (s^-, s) \in S \times S \ \text{ and } \ a \in A(s).$$

The transition probability $\tilde{q}$ in game $\tilde{G}$ is of the following form:

$$\tilde{q}(C_1 \times C_2 \mid (s^-, s), a) := \delta_s(C_1) q(C_2 \mid s, a)$$

for all $(s^-, s) \in S \times S$, $a \in A(s)$, and C_1, $C_2 \in \mathcal{B}$. Here, δ_s denotes the Dirac measure concentrated at $s \in S$.

Stationary Markov strategies are defined in game $\tilde{G}$ in an obvious manner. Note that a stationary almost Markov strategy $f_i \in F_i$ of player $i \in I$ in game G is stationary Markov in game $\tilde{G}$. The discounted payoff for player $i \in I$, when a strategy profile $f \in F$ is used, is denoted by $\tilde{J}_i((s^-, s), f)$. Note that if s is an initial state in game G, then for each $f \in F$, we have

$$J_i(s, f) = \tilde{J}_i((s^-, s), f) \quad \text{with } s^- = s. \tag{2}$$

3. Main Results. Our main result in this paper is as follows.

THEOREM 1. *Every game G satisfying assumptions* (i)–(vi) *has an SAMPE.*

COROLLARY 1. *Consider a game where the set $X = X_1 \times \cdots \times X_m$ is finite, assumptions* (i)–(iv) *and* (vi) *are satisfied, and the transition probability q is Borel measurable*; *then the game has an SAMPE.*

PROOF. Because of Theorem 1 we have to show that the game meets condition (v). Let $l \in \mathbb{N}$ be such that $X = \{a^1, \ldots, a^l\}$. Now for $j = 1, \ldots, l$ define

$$g_j(s,a) := \begin{cases} 1, & \text{if } a \in A(s),\ \ a = a^j \\ 0, & \text{otherwise,} \end{cases} \qquad q_j(\cdot \mid s) := \begin{cases} q(\cdot \mid s,a), & \text{if } a \in A(s),\ \ a = a^j \\ \mu(\cdot), & \text{otherwise.} \end{cases}$$

Then we can see that $q(\cdot \mid s,a) = \sum_{j=1}^{l} g_j(s,a) q_j(\cdot \mid s)$ and the conclusion follows from Theorem 1. □

REMARK 1. The above corollary extends the result in Mertens and Parthasarathy [23], where it is additionally assumed that $A_i(s) = X_i$ for all $s \in S$, $i \in I$ and that μ is atomless; see Comment on p. 147 in Mertens and Parthasarathy [23] or Theorem VII.1.8 on p. 398 in Mertens et al. [25]. If μ admits some atoms, then they obtained a subgame perfect equilibrium in which the strategy of player $i \in I$ is of the form $(f_{i1}, f_{i2}, \ldots)$ with $f_{it} \in F_i$ for each $t \in \mathbb{N}$. Thus, the equilibrium strategy of player $i \in I$ is stage dependent.

REMARK 2. A related result to Theorem 1 is given in Barelli and Duggan [2]. The assumption imposed on the transition probability in Barelli and Duggan [2] is weaker, but an equilibrium is shown to exist in a larger class of stationary semi-Markov strategies, where the players take into account the current state, previous state, and the actions chosen by the players in the previous state.

REMARK 3. Recently, Levy [18] provided an example of a deterministic game that does not possess a stationary Markov perfect equilibrium. Later, Levy and McLennan [19] showed that a stochastic game need not have a stationary Markov perfect equilibrium either. In their model, each set X_i is finite and $A_i(s) = X_i$ for every $i \in I$, $s \in S$. Moreover, the transition law is a convex combination of a probability measure (depending the current state) and the Dirac measure. Such a model satisfies the absolute continuity condition. Hence, this example shows that we cannot expect to obtain an equilibrium in stationary Markov strategies even in games with finite action spaces. Therefore, Corollary 1 is meaningful.

Let $B(S)$ be the space of all bounded Borel measurable real-valued functions on S and $B^m(S) := B(S) \times \cdots \times B(S)$ (m times). With any $s \in S$ and $v = (v_1, \ldots, v_m) \in B^m(S)$, we associate the one-shot game $\Gamma_v(s)$ in which the payoff function for player $i \in I$ is

$$U_i(v_i, s, a) := (1-\beta) r_i(s,a) + \beta \int_S v_i(s') q(ds' \mid s, a), \qquad a \in A(s). \tag{3}$$

Under our assumptions, $a \mapsto U_i(v_i, s, a)$ is continuous on $A(s)$ for every $v_i \in B(S)$, $s \in S$, $i \in I$. Let $N_v(s)$ be the set of all Nash equilibria in the game $\Gamma_v(s)$. By $P_v(s)$ $[coP_v(s)]$ we denote the set of payoff vectors [convex combinations of payoff vectors] corresponding to all equilibria in $N_v(s)$. By Lemma 6 in Nowak and Raghavan [29], the correspondence $coP_v\colon S \rightrightarrows \mathbb{R}^m$ is lower measurable, and by the Kuratowski and Ryll-Nardzewski [17] theorem, it admits a Borel measurable selector.

The *outline* of the proof of Theorem 1 is as follows. First, we show that there exists a bounded Borel measurable function $w^*\colon S \mapsto \mathbb{R}^m$ such that $w^*(s) \in coP_{w^*}(s)$ on the nonatomic part of μ and $w^*(s) \in P_{w^*}(s)$ for every atom s of μ. This result is the content of Theorem 2 and is obtained by applying a generalisation of the Kakutani fixed point theorem due to Glicksberg [10]. More precisely, its proof is based on combining the methods used to study correlated equilibria in general state space games and Nash equilibria in games with countable state space. In the proof of Theorem 1, we apply the Mertens [22] measurable choice theorem. We show that there exists a bounded Borel measurable mapping $v^*\colon S \times S \mapsto \mathbb{R}^m$ such that

$$\int_S w^*(s') q_j(ds' \mid s) = \int_S v^*(s,s') q_j(ds' \mid s), \qquad j = 1, \ldots, m.$$

Moreover, we have that $v^*(s,s') \in P_{v^*}(s')$ for all states s and s'. In this place we exploit the particular form of the transition probability defined in (1) and the fact that every q_j is independent of players' actions. Furthermore, making use of a measurable implicit function theorem, we claim that $v^*(s,s')$ is the vector of equilibrium payoffs corresponding to some stationary almost Markov strategy profile.

We now give a refinement of a fixed point result stated in Nowak and Raghavan [29] that plays a crucial role in the proof of Theorem 1. Its proof contains some new ingredients.

THEOREM 2. *Assume that* (i)–(vi) *hold. Let D be the countable set of atoms of μ and $C := S\backslash D$. There exists some $w^* \in B^m(S)$ such that*

$$w^*(s) \in coP_{w^*}(s) \quad \text{for all } s \in C \qquad \text{and} \qquad w^*(s) \in P_{w^*}(s) \quad \text{for all } s \in D.$$

PROOF. By Φ_i we denote the set of all mappings ϕ_i: $D \mapsto P(X_i)$ such that $\phi_i(s) \in P(A_i(s))$ for each $s \in D$, $i \in I$. We point out that Φ_i can be recognised as a compact convex subset of a linear topological Hausdorff space. A sequence $(\phi_i^n)_{n\in\mathbb{N}}$ converges to $\phi_i \in \Phi_i$ if and only if $\phi_i^n(s) \to \phi_i(s)$ (as $n \to \infty$) in the weak topology on $P(A_i(s))$ for each $s \in D$. Put $\Phi^m := \Phi_1 \times \cdots \times \Phi_m$ and endow Φ^m with the product topology. Then Φ^m is also a compact convex metrisable subset of a linear topological Hausdorff space.

Let V be the space of all μ-equivalence classes of Borel measurable functions h: $S \mapsto \mathbb{R}$ such that $|h(s)| \le L$ μ-a.e. It is well known that because of the Banach-Alaoglu Theorem, V is a compact convex metrisable subset of $L_\infty(\mu) := L_\infty(S, \mathcal{B}, \mu)$ when endowed with the weak-star topology $\sigma(L_\infty(S, \mathcal{B}, \mu), L_1(S, \mathcal{B}, \mu))$; see Theorem 6.21 in Aliprantis and Border [1]. Recall that a sequence $(\delta^n)_{n\in\mathbb{N}}$ converges in the weak-star topology to $\delta \in L_\infty(\mu)$ if and only if

$$\lim_{n\to\infty} \int_S \delta^n(y)\eta(y)\mu(dy) = \int_S \delta(y)\eta(y)\mu(dy) \quad \text{for every } \eta \in L_1(S, \mathcal{B}, \mu).$$

Put $V^m := V \times \cdots \times V$ (m times) and endow V^m with the product topology. Then V^m is a compact convex metrisable subset of a linear topological Hausdorff space. The game $\Gamma_v(s)$ with payoff functions (3) can also be defined for any $v = (v_1, \ldots, v_m) \in V^m$.

Let $\phi = (\phi_1, \ldots, \phi_m) \in \Phi^m$. For any atom $s \in D$, we define $U_i(v_i, s, \phi(s))$ as the integral of the function $a \mapsto U_i(v_i, s, a)$ on $A(s)$ with respect to the product probability measure $\phi_1(s) \otimes \cdots \otimes \phi_m(s)$. If $s \in D$ and $\nu \in P(A_i(s))$, then

$$U_i(v_i, s, (\nu, \phi_{-i}(s))) = U_i(v_i, s, \hat{\phi}(s))$$

where $\hat{\phi} = (\hat{\phi}_1, \ldots, \hat{\phi}_m)$ with $\hat{\phi}_i(s) = \nu$ and $\hat{\phi}_j(s) = \phi_i(s)$ for $j \ne i$.

Let $v = (v_1, \ldots, v_m) \in V^m$, $\phi = (\phi_1, \ldots, \phi_m) \in \Phi^m$ and $s \in D$. For every player $i \in I$, define

$$p_i(v_i, s, \phi_{-i}(s)) := \max_{\nu\in P(A_i(s))} U_i(v_i, s, (\nu, \phi_{-i}(s))).$$

Then we put

$$p(v, s, \phi(s)) := (p_1(v_1, s, \phi_{-1}(s)), \ldots, p_m(v_m, s, \phi_{-m}(s))), \quad s \in D.$$

For any $v \in V^m$ and $\phi \in \Phi^m$, we define the set $M(v, \phi) \subset V^m$ in the following way. A Borel measurable function $\tilde{w}$: $S \mapsto \mathbb{R}^m$ represents a μ-equivalent class in $M(v, \phi)$ if and only if $\tilde{w}(s) = p(v, s, \phi(s))$ for all $s \in D$ and there exists a Borel measurable selector w of the correspondence coP_v: $C \rightrightarrows \mathbb{R}^m$ such that $\tilde{w} = w$ μ-a.e. on C.

Let $R_i(v_i, \phi_{-i})$ be the set of all $\tilde{\phi}_i \in \Phi_i$ such that

$$p_i(v_i, s, (\tilde{\phi}_i(s), \phi_{-i}(s))) := \max_{\nu\in P(A_i(s))} U_i(v_i, s, (\nu, \phi_{-i}(s)))$$

for all $s \in D$. Under our continuity assumptions, the set $R_i(v_i, \phi_{-i})$ is compact in Φ_i. It is also convex. Define

$$R(v, \phi) := R_1(v_1, \phi_{-1}) \times \cdots \times R_m(v_m, \phi_{-m}).$$

Then $R(v, \phi)$ is a compact convex subset of Φ^m.

Finally, define the compact convex valued correspondence

$$\Psi(v, \phi) := M(v, \phi) \times R(v, \phi) \subset V^m \times \Phi^m.$$

We now show that Ψ has a closed graph. Assume that $v = (v_1, \ldots, v_m) \in V^m$, $\phi = (\phi_1, \ldots, \phi_m) \in \Phi^m$, and

$$v^n = (v_1^n, \ldots, v_m^n) \to v \qquad \text{and} \qquad \phi^n = (\phi_1^n, \ldots, \phi_m^n) \to \phi \quad \text{as } n \to \infty.$$

Suppose that $w^n = (w_1^n, \ldots, w_m^n) \in M(v^n, \phi^n)$, $\varphi^n = (\varphi_1^n, \ldots, \varphi_m^n) \in R(v^n, \phi^n)$ for each $n \in \mathbb{N}$ and

$$w^n \to w = (w_1, \ldots, w_m) \in V^m, \qquad \varphi^n \to \varphi = (\varphi_1, \ldots, \varphi_m) \in \Phi^m \quad \text{as } n \to \infty.$$

We must show that $(w, \varphi) \in \Psi(v, \phi)$. Observe that since $q_j(\cdot \mid s) \ll \mu$ for all $s \in S$ and $j = 1, \ldots, l$, we have

$$\lim_{n\to\infty} \int_S v_i^n(s') q_j(ds' \mid s) = \int_S v_i(s') q_j(ds' \mid s), \quad i \in I.$$

Hence by (1), it follows that

$$\lim_{n\to\infty} \max_{a\in A(s)} |U_i(v_i^n, s, a) - U_i(v_i, s, a)| = 0, \quad i \in I, \quad s \in S. \tag{4}$$

From the convergence of w^n to w in V^m, it follows that $w_i^n(s) \to w_i(s)$ (as $n \to \infty$) for every atom $s \in D$ and for each $i \in I$. This, $\varphi^n \to \varphi$, $\phi^n \to \phi$, and (4) imply that

$$w_i(s) = p(v_i, s, (\varphi_i(s), \phi_{-i}(s))) = \max_{\nu \in P(A_i(s))} U_i(v_i, s, (\nu, \phi_{-i}(s))) \tag{5}$$

for all $s \in D$, $i \in I$.

By Mazur's theorem (see Dunford and Schwartz [8]), there exists a sequence $(\tilde{w}^k)_{k\in\mathbb{N}}$ of convex combinations of the functions w^n that converges pointwise to w on some Borel set $S_1 \subset S$ such that $\mu(S_1) = 1$. Note that $D \subset S_1$. Since $w^n(s) \to w(s)$ as $n \to \infty$ for each $s \in D$, we have that

$$\lim_{k\to\infty} \tilde{w}^k(s) = \lim_{n\to\infty} w^n(s) = w(s)$$

for all $s \in D$. On the other hand, as in Nowak and Raghavan [29][2] we claim that for each $s \in S_1 \cap C$

$$w(s) = \lim_{k\to\infty} \tilde{w}^k(s) \in coP_v(s).$$

By (5) and that $w(s) \in coP_v(s)$ μ-a.e. on C, it follows that $w \in M(v, \phi)$. From (5) we also conclude that $\varphi \in R(v, \phi)$. Thus, we obtain that $(w, \varphi) \in \Psi(v, \phi)$. By the generalised version of Kakutani's fixed point theorem due to Glicksberg [10], we infer that there exists $(v^0, \phi^*) \in \Psi(v^0, \phi^*)$. Hence, there exists a Borel set $C\backslash S_0$ such that $\mu(S_0) = 0$ and

$$v^0(s) \in coP_{v^0}(s) \quad \text{for all } s \in C\backslash S_0. \tag{6}$$

Moreover, $v^0(s) = p(v^0, s, \phi^*(s))$ for each $s \in D$; i.e.,

$$v_i^0(s) = p(v_i^0, s, \phi^*(s)) = \max_{\nu\in P(A_i(s))} U_i(v_i^0, s, (\nu, \phi^*_{-i}(s))), \quad s \in D, \quad i \in I. \tag{7}$$

Clearly, (7) means that $v^0(s) \in P_{v^0}(s)$ for each $s \in D$. By the Kuratowski and Ryll-Nardzewski [17] theorem, there exists a Borel measurable function $\hat{v}^0$ such that $\hat{v}^0(s) \in coP_{v^0}(s)$ for all $s \in S_0$. Define $w^*(s) := v^0(s)$ for each $s \in (C\backslash S_0) \cup D$ and $w^*(s) := \hat{v}^0(s)$ for each $s \in S_0$. Since $\mu(S_0) = 0$, we have $q(S_0 \mid s, a) = 0$ for all $s \in S$, $a \in A(s)$. Therefore, $U_i(w_i^*, s, a) = U_i(v_i^0, s, a)$ for all $s \in S$, $a \in A(s)$, and $i \in I$. Consequently, we have $P_{w^*}(s) = P_{v^0}(s)$ (and hence $coP_{w^*}(s) = coP_{v^0}(s)$) for all $s \in S$. The result now follows from the definition of w^*, (6), and (7). $\square$

Remark 4. Theorem 2 can be used to construct a stationary correlated equilibrium with public signals in game G as in Nowak and Raghavan [29]. An additional feature of this equilibrium is that the players may choose actions independently of each other in every state (atom) $s \in D$. A correlation using signals is important only in states $s \in C$. This property is not shown in Nowak and Raghavan [29]. We would like to point out that Theorem 2 remains true for games with transition probabilities, dominated by some probability measure μ, that are norm continuous in actions of the players. The particular form of transition probabilities given in (1) need not be then exploited. Within such a framework, equality (4) can be proved in the same manner as in Nowak and Raghavan [29].

Let $B(S \times S)$ be the space of all bounded Borel measurable real-valued functions on $S \times S$. For any $s \in S$ and $v_1, \ldots, v_m \in B(S \times S)$, we define

$$U_i(v_i, s, a) := (1-\beta) r_i(s, a) + \beta \int_S v_i(s, s')\, q(ds' \mid s, a), \quad a \in A(s).$$

Let $f = (f_1, \ldots, f_m) \in F$. For any $(s^-, s) \in S \times S$, we define $U_i(v_i, s, f(s^-, s))$ as the integral of $U_i(v_i, s, a)$ with respect to the product probability measure $f_1(s^-, s) \otimes \cdots \otimes f_m(s^-, s)$ on $A(s)$. The definition of $U_i(v_i, s, (\nu, f_{-i}(s^-, s)))$ for any $\nu \in P(\tilde{A}_i(s^-, s))$ and $f \in F$ is standard.

[2] A more detailed discussion of such a conclusion is provided in the proof of equilibrium theorem for semi-Markov games on pp. 32–35 in Jaśkiewicz and Nowak [15].

Proof of Theorem 1. By Theorem 2, there exists $w^* = (w_1^*, \ldots, w_m^*) \in B^m(S)$ such that

$$w^*(s) \in coP_{w^*}(s) \quad \text{for all } s \in C \qquad \text{and} \qquad w^*(s) \in P_{w^*}(s) \quad \text{for all } s \in D.$$

Moreover, from the proof of Theorem 2 we know that there exists $\phi^* = (\phi_1^*, \ldots, \phi_m^*) \in \Phi^m$ such that

$$w_i^*(s) = U_i(w_i^*, s, \phi^*(s)) = \max_{\nu \in P(A_i(s))} U_i(w_i^*, s, (\nu, \phi_{-i}^*(s))) \tag{8}$$

for all $s \in D$, $i \in I$. Define the l-dimensional stochastic kernel

$$K(\cdot \mid s) := (q_1(\cdot \mid s), \ldots, q_l(\cdot \mid s)) \quad \text{for } s \in S.$$

Put

$$\bar{w}^*(s) := \int_C w^*(s')K(ds' \mid s) = \left(\int_C w^*(s')q_1(ds' \mid s), \ldots, \int_C w^*(s')q_l(ds' \mid s)\right).$$

Recall that by Lemma 6 in Nowak and Raghavan [29], the correspondences $P_{w^*}\colon S \rightrightarrows \mathbb{R}^m$ and $coP_{w^*}(s)\colon S \rightrightarrows \mathbb{R}^m$ are lower measurable. Let Δ be the graph of the mapping $s \mapsto \int_C P_{w^*}(s')K(ds' \mid s)$; i.e.,

$$\Delta := \left\{(s, y) \in S \times \mathbb{R}^{ml}\colon s \in S,\ y \in \int_C P_{w^*}(s')K(ds' \mid s)\right\}.$$

From part 3 of Theorem on p. 108 in Mertens [22], it follows that there exists a Borel measurable mapping $\gamma\colon \Delta \times S \mapsto \mathbb{R}^m$ such that $\gamma(s, y, s') \in P_{w^*}(s')$ for all $(s, y) \in \Delta$, $s' \in C$, and

$$y = \int_C \gamma(s, y, s')K(ds' \mid s), \quad s \in S. \tag{9}$$

Observe that every measure $q_i(\cdot \mid s)$ is atomless on C. Therefore, from Lyapunov's theorem, it follows that $\int_C P_{w^*}(s')K(ds' \mid s) = \int_C coP_{w^*}(s')K(ds' \mid s)$; see Corollary 18.1.10 in Klein and Thompson [16]. Hence,

$$\bar{w}^*(s) \in \int_C P_{w^*}(s')K(ds' \mid s) \quad \text{for each } s \in S.$$

Put

$$v^*(s, s') := \begin{cases} \gamma(s, \bar{w}^*(s), s'), & \text{if } s \in S,\ s' \in C, \\ w^*(s'), & \text{if } s \in S,\ s' \in D. \end{cases}$$

Clearly, $v^* \in B(S \times S)$. By (9), we have that

$$\bar{w}^*(s) = \int_C v^*(s, s')K(ds' \mid s) = \int_C w^*(s')K(ds' \mid s) \quad \text{for every } s \in S.$$

This fact implies that

$$\int_C v^*(s, s')q_j(ds' \mid s) = \int_C w^*(s')q_j(ds' \mid s) \tag{10}$$

for every $s \in S$, and $j = 1, \ldots, l$. From the definition of v^*, it immediately follows that

$$\int_D v^*(s, s')q_j(ds' \mid s) = \int_D w^*(s')q_j(ds' \mid s) \tag{11}$$

for every $s \in S$, and $j = 1, \ldots, l$. By (10), (11), and (1), we have that

$$\int_S v^*(s, s')q(ds' \mid s, a) = \int_S w^*(s')q(ds' \mid s, a) \tag{12}$$

for every $a \in A(s)$, $s \in S$. Write $v^* = (v_1^*, \ldots, v_m^*)$. Note that for any pair $(s^-, s) \in S \times S$ we have $v^*(s^-, s) \in P_{w^*}(s)$. By Filippov's implicit function theorem (see Theorem 18.17 in Aliprantis and Border [1] or Lemma 4 in Nowak and Raghavan [29]), there exists a profile $g^* = (g_1^*, \ldots, g_m^*)$ where $g_i^*\colon S \times C \mapsto P(X_i)$ is Borel measurable, $g_i^*(s^-, s) \in P(A_i(s))$ for all $i \in I$ and $(s^-, s) \in S \times C$, and

$$v_i^*(s^-, s) = U_i(w_i^*, s, g^*(s^-, s)) = \max_{\nu \in P(\tilde{A}_i(s^-, s))} U_i(w_i^*, s, (\nu, g_{-i}^*(s^-, s))). \tag{13}$$

By (12), it follows that $U_i(w_i^*, s, a) = U_i(v_i^*, s, a)$ for each $a \in A(s)$, $s \in S$ and $i \in I$. Thus from (13), we conclude that

$$v_i^*(s^-, s) = U_i(v_i^*, s, g^*(s^-, s)) = \max_{\nu \in P(\tilde{A}_i(s^-, s))} U_i(v_i^*, s, (\nu, g_{-i}^*(s^-, s))) \tag{14}$$

for every $s^- \in S$, $s \in C$, and $i \in I$. Let $f^*(s^-, s) = g^*(s^-, s)$ for $(s^-, s) \in S \times C$ and $f^*(s^-, s) = \phi^*(s)$ for $(s^-, s) \in S \times D$. By (8) and (14), we obtain

$$v_i^*(s^-, s) = U_i(v_i^*, s, f^*(s^-, s)) = \max_{\nu \in P(\tilde{A}_i(s^-, s))} U_i(v_i^*, s, (\nu, f_{-i}^*(s^-, s)))$$

for each $(s^-, s) \in S \times S$ and $i \in I$. Thus, we have obtained the Bellman equation for each player $i \in I$ in the game $\tilde{G}$ with $\tilde{r}_i$ replaced by $(1-\beta)\tilde{r}_i$. By standard dynamic programming arguments (see Blackwell [4]), these equations imply that

$$(1-\beta)\tilde{J}_i((s^-, s), f^*) = \max_{f_i \in F_i}(1-\beta)\tilde{J}_i((s^-, s), (f_i, f_{-i}^*)) \tag{15}$$

for every $(s^-, s) \in S \times S$, $i \in I$. Dividing both sides of (15) by $(1-\beta)$, setting $s^- = s$, and making use of (2), we obtain that

$$J_i(s, f^*) = \max_{f_i \in F_i} J_i(s, (f_i, f_{-i}^*)) \quad \text{for every } s \in S, \ \ i \in I.$$

These equations and standard dynamic programming arguments (see Blackwell [4] or Bertsekas and Shreve [3]) imply that $f^* \in F$ is a Nash equilibrium in the class of all strategies of the players. □

Remark 5. In certain cases, the game $\Gamma_v(s)$ has a nonempty compact set $N_v^o(s)$ of pure Nash equilibria. Then replacing in the above proofs of Theorems 1 and 2 $P_v(s)$ by the set $P_v^o(s)$ of payoff vectors corresponding to all pure equilibria in $N_v^o(s)$ and $coP_v(s)$ by $coP_v^o(s)$, one can obtain a pure *SAMPE* in the considered game. This situation, for instance, concerns two classes of games, namely concave and supermodular, discussed in the following section.

4. Applications to economic models.

This section is devoted to the presentation of some examples of Cournot games that satisfy our assumptions and for which a pure *SAMPE* exists. The first example concerns a "game with observable production parameters" related to the models considered on p. 98 in Horst [14] and Example 1 in Nowak [28], whereas the others refer to a "quantity competition with complementary goods and learning-by-doing" model (see p. 197 in Curtat [5]). In most of the games discussed in the literature, assumption (v) is formulated with the help of q_i independent of $s \in S$, which imposes an extra restriction. Here, we relax this condition and allow the transition probabilities to depend on the current state of the game. Moreover, we also notice that a strict diagonal dominance condition used by Curtat [5] can be dropped.

4.1. Concave games.

Example 1 (Dynamic Cournot Oligopoly). Let $S = [0, \bar{s}]$ and let a state $s \in S$ represent a realisation of a random *demand shock* that is modified at each stage of the game. Every player $i \in I$ (oligopolist) sets a quantity $a_i \in A_i(s) = [0, 1]$. If $P(s, \sum_{j=1}^m a_j)$ is the inverse demand function, $c_i(s, a_i)$ is the cost function for player i, then

$$r_i(s, a) := a_i P\left(s, \sum_{j=1}^m a_j\right) - c_i(s, a_i), \quad a = (a_1, \ldots, a_m).$$

A typical example of the inverse demand function is

$$P\left(s, \sum_{j=1}^m a_j\right) = s\left(m - \sum_{j=1}^m a_j\right).$$

Usually, the function $a_i \mapsto a_i P(s, \sum_{j=1}^m a_j)$ is concave for each $i \in I$. Moreover, we assume that

$$q(\cdot \mid s, a) = (1-\bar{a})q_1(\cdot \mid s) + \bar{a}q_2(\cdot \mid s), \quad \bar{a} := \frac{1}{m}\sum_{j=1}^m a_j,$$

where $q_1(\cdot \mid s)$ and $q_2(\cdot \mid s)$ are for all $s \in S$ absolutely continuous with respect to some probability measure μ on $(S, \mathcal{B})$. A similar example was also discussed on p. 97 in Escobar [9], but with a countable state space, fixed costs, and a different meaning of the transition probabilities q_i. Here we observe that if s is the state at some period of the game and the total sale $m\bar{a} = 0$, then the next demand level s' is drawn from $q_1(\cdot \mid s)$. On the other hand, if $m\bar{a} = m$, then s' follows $q_2(\cdot \mid s)$. In order to provide further interpretation of q we note that

$$q(\cdot \mid s, a) = q_1(\cdot \mid s) + \bar{a}(q_2(\cdot \mid s) - q_1(\cdot \mid s)). \tag{16}$$

Now let

$$E_q(s, a) := \int_S s' q(ds' \mid s, a), \qquad E_{q_j}(s) := \int_S s' q_j(ds' \mid s), \quad j = 1, 2$$

be the mean values of the distributions $q(\cdot \mid s, a)$ and $q_j(\cdot \mid s)$, respectively. By (16), we have

$$E_q(s, a) := E_{q_1}(s) + \bar{a}(E_{q_2}(s) - E_{q_1}(s)).$$

Assume that $E_{q_1}(s) \geq s \geq E_{q_2}(s)$. This condition is consistent with our interpretation of q_1 and q_2 and implies that

$$E_{q_2}(s) - E_{q_1}(s) \leq 0.$$

Thus, the expectation of the next demand shock $E_q(s, a)$ decreases if the total sale $m\bar{a}$ increases in the current state $s \in S$.

Observe that the game $\Gamma_v(s)$ is concave, if $r_i(s, \cdot, a_{-i})$ is concave on $A_i(s)$ for all $i \in I$. This requirement is met, for instance, if $a_i \mapsto a_i P(s, \sum_{j=1}^m a_j)$ is concave and the cost function $c_i(s, \cdot)$ is convex. In general, the game $\Gamma_v(s)$ may have many pure Nash equilibria. From Remark 5, we conclude that the dynamic game has a pure *SAMPE*. Finally, we wish to stress that this game is neither supermodular in the sense of Curtat [5] nor belongs to the classes of models examined in Escobar [9], Horst [14], and Nowak [28], where stationary Markov perfect equilibria are discussed.

Example 2 (Cournot Competition with Substituting Goods in Differentiated Markets). This model is to some extent inspired by a dynamic game with complementary goods studied in Curtat [5], although related static games were already discussed in Spence [34] and Vives [36]. There are m firms in the market, and firm $i \in I$ produces a quantity $a_i \in A_i(s) = X_i = [0, 1]$ of a differentiated product. The inverse demand function is given by a twice differentiable function $P_i(a)$, where $a = (a_1, \ldots, a_m)$. The goods are *substitutes*; i.e., $\partial P_i(a)/\partial a_j < 0$ for all $i, j \in I$, (see Spence [34]). In other words, consumption of one good will decrease consumption of the others. Further, we assume that $S = [0, 1]^m$, where i-th coordinate $s_i \in [0, 1]$ is a measure of the cumulative experience of firm $i \in I$. We equip both S and $X = [0, 1]^m$ with the usual component-wise ordering. Then S and X are complete lattices. If $c_i(s_i)$ is the marginal cost for firm $i \in I$, then

$$r_i(s, a) := a_i[P_i(a) - c_i(s_i)], \qquad a = (a_1, \ldots, a_m), \quad s = (s_1, \ldots, s_m) \in S. \tag{17}$$

The transition probability of the next state (experience vector) is of the following form

$$q(\cdot \mid s, a) = h\left(\sum_{j=1}^m (s_j + a_j)\right) q_2(\cdot \mid s) + \left(1 - h\left(\sum_{j=1}^m (s_j + a_j)\right)\right) q_1(\cdot \mid s), \tag{18}$$

where

$$h\left(\sum_{j=1}^m (s_j + a_j)\right) = \frac{\sum_{j=1}^m s_j + \sum_{j=1}^m a_j}{2m} \tag{19}$$

and $q_1(\cdot \mid s)$ and $q_2(\cdot \mid s)$ are for each $s \in S$ absolutely continuous with respect to some probability measure μ on $(S, \mathscr{B})$. Recall that in Curtat [5] it is assumed that q_1 and q_2 are independent of $s \in S$ and also that q_2 stochastically dominates q_1. Then the underlying Markov process governed by q captures the ideas of learning-by-doing and spillover (see p. 197 in Curtat [5]). Recall that $q_2(\cdot \mid s)$ stochastically dominates $q_1(\cdot \mid s)$ for every $s \in S$ if and only if for any nondecreasing function b: $S \mapsto \mathbb{R}$, it holds

$$\int_S b(s') q_2(s' \mid s) \geq \int_S b(s') q_1(s' \mid s) \tag{20}$$

for every $s \in S$; see Topkis [35], pp. 159–161. In particular, if $b(s') = \sum_{j=1}^m s'_j$, then $\int_S b(s') q_i(s' \mid s)$ is the expected value of the total experience of all firms with respect to the transition probability $q_i(\cdot \mid s)$. Then inequality (20) has a natural interpretation. Observe that the transition probability in (18) with the function h given in (19) can be seen in the following way: if $h(\sum_{j=1}^m (s_j + a_j)) > 1/2$ for some vectors $s \in S$ and $a \in X$, i.e., the firms' cumulative experience and the total production are large enough, then it is more likely that the next state is drawn according to $q_2(\cdot \mid s)$.

It is easy to see that our dummy game $\Gamma_v(s)$ is concave if $r_i(s, \cdot, a_{-i})$ is concave on $[0, 1]$. Clearly, it is satisfied if for each $i \in I$, we have

$$2\frac{\partial P_i(a)}{\partial a_i} + \frac{\partial^2 P_i(a)}{\partial a_i^2} a_i < 0.$$

If the goods are substitutes, this condition holds when $\partial^2 P_i(a)/\partial a_i^2 \leq 0$ for all $i \in I$.

From Remark 5, it follows that a pure *SAMPE* exists. It is obvious that this game is neither supermodular in the sense of Curtat [5] nor supermodular in the sense of Milgrom and Roberts [26] or Topkis [35]. The game $\Gamma_v(s)$ may have multiple pure Nash equilibria.

4.2. Supermodular games. In this subsection, we provide two examples of supermodular games for which a pure *SAMPE* exists. We wish to draw the reader's attention to the terminology of a "supermodular game." By a "supermodular game" we mean a game satisfying conditions introduced by Milgrom and Roberts [26] or Topkis [35]. Curtat [5], on the other hand, by a "supermodular game" understands a model meeting additional requirements to those considered in Milgrom and Roberts [26] and Topkis [35]. Our subsequent illustrations refer to Example 2, but with products that are *complements*. This means that the state space and action spaces for firms are the same as in Example 2. Furthermore, we assume that the transition probability is defined in (18) and $q_1(\cdot \mid s)$ and $q_2(\cdot \mid s)$ are for all $s \in S$ absolutely continuous with respect to some probability measure μ on $(S, \mathcal{B})$. The payoff function for every firm is given in (17).

Example 3 (Cournot Oligopoly with Complementary Goods in Differentiated Markets I). Let h be given as in (19). Suppose that the payoff function in the dummy game $\Gamma_v(s)$ satisfies the following condition:[3]

$$\frac{\partial^2 U_i(v_i, s, a)}{\partial a_i \partial a_j} \geq 0 \quad \text{for } j \neq i.$$

Then by Theorem 4 in Milgrom and Roberts [26] the game $\Gamma_v(s)$ is supermodular. Note that within our framework, it is sufficient to prove that for $r_i(s, a)$, defined in (17), it holds $\partial^2 r_i(s, a)/\partial a_i \partial a_j \geq 0$, $j \neq i$. But

$$\frac{\partial^2 r_i(s, a)}{\partial a_i \partial a_j} = a_i \frac{\partial^2 P_i(a)}{\partial a_i \partial a_j} + \frac{\partial P_i(a)}{\partial a_j}, \quad j \neq i$$

and they are likely to be nonnegative, if the goods are complements; i.e., $\partial P_i(a)/\partial a_j \geq 0$ for $j \neq i$ (see Vives [36]). By Theorem 5 in Milgrom and Roberts [26], the game $\Gamma_v(s)$ has a pure Nash equilibrium. Hence, again by Remark 5, the dynamic game possesses a pure *SAMPE*.

Remark 6. The game described in Example 3 is also studied by Curtat [5], but with additional restrictive assumptions; namely, q_1 and q_2 are independent of $s \in S$. Then such transition probability q has so-called increasing differences in (s, a); see Curtat [5] or Topkis [35]. If q_1 or q_2 depends on $s \in S$, then the increasing differences property of q does not hold. The other assumption made in Curtat [5] is that the payoff functions $r_i(s, a)$ are increasing in a_{-i} and, more importantly, satisfy the so-called strong diagonal dominance condition for each $s \in S$. For details the reader is referred to Curtat [5] and Rosen [31]. This additional condition is imposed in order to have a unique pure Nash equilibrium in every auxiliary game $\Gamma_v(s)$ under consideration. The advantage is that Curtat [5] can directly work with Lipschitz continuous strategies for the players and find an equilibrium in that class using Schauder's fixed point theorem. Without the strict diagonal dominance condition, $\Gamma_v(s)$ may have many pure Nash equilibria.

Remark 7. Although the function h in Examples 2 and 3 is affine, it can be replaced by a convex function. However, in this case we need to impose a new assumption on q_1 and q_2. Let $V_d \subset V$ be the space of all μ-a.e. nondecreasing functions and let $V_d^m = V_d \times \cdots \times V_d$ (m times). In the next example, we shall assume the following:

(D0) For any $v_i \in V_d$ and $i \in I$, the function

$$s \mapsto \theta_i(v_i, s) := \int_S v_i(s') q_2(ds' \mid s) - \int_S v_i(s') q_1(ds' \mid s)$$

is nondecreasing.

[3] For our models of interest we assume that we deal with smooth supermodular games. Therefore, the condition of increasing differences is formulated it terms of derivatives.

Let $(0, y]$ be an interval in S. From Corollary 3.9.1(a) in Topkis [35], condition (D0) is equivalent to the fact that the function

$$s \mapsto q_2((0, y] \mid s) - q_1((0, y] \mid s) \tag{21}$$

is nonincreasing for every $y \in S$. The monotonicity assumption of the mapping defined in (21) holds, if, for instance, q_2 is stochastically increasing and q_1 is stochastically decreasing in $s \in S$.

EXAMPLE 4 (COURNOT OLIGOPOLY WITH COMPLEMENTARY GOODS IN DIFFERENTIATED MARKETS II). Assume (D0) and

(D1) the function h: $[0, 2m] \mapsto [0, 1]$ is convex, increasing, and twice continuously differentiable.
(D2) $q_2(\cdot \mid s)$ stochastically dominates $q_1(\cdot \mid s)$ for every $s \in S$.
(D3) c_i is continuously differentiable and decreasing on $[0, 1]$ for every $i \in I$.
(D4) $\partial^2 r_i(s, a)/\partial a_i \partial a_j \geq 0$ for $j \neq i$, $i \in I$.
(D5) $\partial r_i(s, a)/\partial a_i \geq 0$ for $i \in I$.

Fix $v \in V_d^m$. Let us recall that the payoff function for firm $i \in I$ is now of the form

$$U_i(v_i, s, a) = (1-\beta) r_i(s, a) + \beta \left[\int_S v_i(s') q_1(ds' \mid s) + h\left(\sum_{k=1}^{m} (a_k + s_k) \right) \theta_i(v_i, s) \right], \tag{22}$$

where $s \in S$, $a \in A(s)$, and $r_i(s, a)$ defined in (17). Note that from Theorem 4 in Milgrom and Roberts [26], it follows that $\Gamma_v(s)$ is supermodular. Indeed, under our conditions (D1), (D2), and (D4), for any $i \in I$ and $j \neq i$, it follows that

$$\frac{\partial^2 U_i(v_i, s, a)}{\partial a_i \partial a_j} = (1-\beta) \frac{\partial^2 r_i(s, a)}{\partial a_i \partial a_j} + \beta h'' \left(\sum_{k=1}^{m} (a_k + s_k) \right) \theta_i(v_i, s) \geq 0.$$

Moreover, by (17) and (22), for any $i \in I$, we conclude from our assumptions (D0)–(D3) that

$$\frac{\partial U_i(v_i, s, a)}{\partial a_i} = (1-\beta) \left(P_i(a) - c_i(s_i) + a_i \frac{\partial P_i(a)}{\partial a_i} \right) + \beta h' \left(\sum_{k=1}^{m} (a_k + s_k) \right) \theta_i(v_i, s)$$

is nondecreasing in $s \in S$. Hence, we conclude that $U_i(v_i, s, a)$ has increasing differences in (s, a_i). From Theorem 6 in Milgrom and Roberts [26], it follows that the game with payoff functions defined in (22) has a pure Nash equilibrium that is a nondecreasing function of $s \in S$. Note that since the goods are complementary and (D1), (D5) hold, the payoff function for each firm $i \in I$ corresponding to any nondecreasing equilibrium is also nondecreasing in $s \in S$. For any $v \in V_d^m$, let M_v^d be the space of all μ-a.e. nondecreasing selections of the correspondence coP_v^o: $S \rightrightarrows \mathbb{R}^m$. Clearly, M_v^d is a nonempty, convex, and compact subset of M_v. Now in the proofs of Theorems 1 and 2, it suffices to replace the space V^m by V_d^m, consider the set of all pure Nash equilibria $N_v^o(s)$ in any game $\Gamma_v(s)$ and replace M_v by M_v^d. The correspondence $v \mapsto M_v^d$ is upper semicontinuous. The adapted proofs allow us to conclude that the dynamic game in Example 4 possesses a pure *SAMPE*.

REMARK 8. We would like to mention that instead of (18), we may deal with the following transition probabilities

$$q(\cdot \mid s, a) = \frac{\sum_{j=1}^{m} (\tau_j(s_j) + \xi_j(a_j))}{2m} q_2(\cdot \mid s) + \left(1 - \frac{\sum_{j=1}^{m} (\tau_j(s_j) + \xi_j(a_j))}{2m} \right) q_1(\cdot \mid s),$$

where τ_j, ξ_j: $[0, 1] \mapsto [0, 1]$ are twice continuously differentiable increasing functions for each $j \in I$. Then under our conditions for every $j \neq i$

$$\frac{\partial^2 U_i(v_i, s, a)}{\partial a_i \partial a_j} = (1-\beta) \frac{\partial^2 r_i(s, a)}{\partial a_i \partial a_j} \geq 0,$$

and hence the game $\Gamma_v(s)$ is supermodular for each fixed state $s \in S$ and $N_v^o(s) \neq \varnothing$. Moreover, for every $i \in I$, we have

$$\frac{\partial U_i(v_i, s, a)}{\partial a_i} = (1-\beta) \left(P_i(a) - c_i(s_i) + a_i \frac{\partial P_i(a)}{\partial a_i} \right) + \frac{\beta \xi_i'(a_i)}{2m} \theta_i(v_i, s).$$

If $\theta_i(v_i, s)$ is nondecreasing in $s \in S$ and the cost function c_i is decreasing in $s_i \in [0, 1]$, then this partial derivative is nondecreasing in $s \in S$ as well. Hence, the function $U_i(v_i, s, a)$ has increasing differences in (s, a_i).

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