Test Two

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Prob 1:

proof: a. $E[M(\frac{1}{3})] = M(\frac{1}{3}) = 1.5 \times \frac{1}{3} - 0.25 \times (1 - e^{-6\times\frac{1}{3}})$

$= 0.5 - 0.25 \cdot (1 - e^{-2}) = 0.25 + 0.25 \cdot e^{-2}$

≈ 0.2838

$\approx \boxed{0.284}$

b. By elementary renewal theorem, we have:

$\frac{1}{\mu} = \lim_{t\to\infty} \frac{M(t)}{t} = \lim_{t\to\infty} \frac{1.5t - 0.25(1-e^{-6t})}{t} = \lim_{t\to\infty} 1.5 - \frac{0.25(1-e^{-6t})}{t} = 1.5$

Then By Blackwell's Theorem, we have:

Long run expected number of renewals per shift $:= \lim_{t\to\infty}(M(t+\frac{1}{3}) - M(t))$

$= \frac{1}{3} \cdot \frac{1}{\mu} = 1.5 \times \frac{1}{3} = \boxed{0.5}$

c. (i) $24E[V(\frac{1}{3})] = [\mu(M(\frac{1}{3})+1) - \frac{1}{3}] \cdot 24$

$= [\frac{1}{1.5}(1.5\times\frac{1}{3} - 0.25\times(1-e^{-6\cdot\frac{1}{3}})+1) - \frac{1}{3}] \cdot 24$

$= [\frac{2}{3}(0.25 + 1 + 0.25 \cdot e^{-2}) - \frac{1}{3}] \cdot 24$

$= [\frac{1}{2} + \frac{1}{6}e^{-2}] \cdot 24 = 12 + 4e^{-2}$

≈ 12.5413

$\boxed{\approx 12.541 \text{ hours}}$

(ii) $4 \times e^{-2} \times 60 \approx 32.48 \text{min} \approx 32 \text{ min}$

$\Rightarrow$ 8:00 am + 12.541 hours $\approx$ $\boxed{20:32 \text{ p.m.}}$

d.

$$\lim_{t\to\infty} P(M(t+\tfrac{1}{3})-M(t)\geq 1)$$

$$= 1-\lim_{t\to\infty} P(M(t+\tfrac{1}{3})-M(t)=0)$$

$$= \lim_{t\to\infty} P(V(t) \leq \tfrac{1}{3})$$

$$= 1-\lim_{t\to\infty} P(V(t) > \tfrac{1}{3})$$

$$= 1-\frac{1}{\mu}\int_{\frac{1}{3}}^{\infty}(1-F(x))\,dx$$

$$= 1-1.5\int_{\frac{1}{3}}^{\infty} e^{-3x}+3xe^{-3x}\,dx$$

$$= 1-1.5\left[\int_{\frac{1}{3}}^{\infty} d\,\frac{-2e^{-3x}}{3}-xe^{-3x}\right]$$

$$= 1-\frac{3}{2}\cdot e^{-1}$$

$$\approx 0.4481$$

$$\approx \boxed{0.448}$$

prob 2:

proof: a. $\{X(t): t \geq 0\}$ is regenerative process

Reasons: Assume passagen: $S_n := \sum_{k=1}^{n} X_k$, $X_k \sim EXP(\lambda)$, $\{X_n\} \to \{S_n\} \to \{N(t)\}$ poisson renewal process

train: $T_m := \sum_{k=1}^{m} Y_k$, $Y_k \sim F$, $\{Y_n\} \to \{T_m\} \to \{Z(t)\}$ renewal process

(i) $\{T_m\}$ is stopping time

(ii) $\{X(t): t \geq 0\}$ at a given renewal point is a probablistic replicate of $\{X(t): t \geq 0\}$

$\Rightarrow \{X(t): t \geq 0\}$ is regenerative process

b. $P\{X(t) = k\} = P\{X(t)=k, t<T_1\} + P\{X(t)=k, t>T_1\}$

when $t<T_1$, $P\{X(t)=k \mid t<T_1\} = P\{N(t)=k\} = \dfrac{(\lambda t)^k e^{-\lambda t}}{k!}$

$P\{X(t)=k, t<T_1\} = P\{X(t)=k \mid t<T_1\} \cdot P\{t<T_1\} = \dfrac{(\lambda t)^k e^{-\lambda t}}{k!} \cdot [1-F(t)]$

when $t>T_1$. $P\{X(t)=k, t>T_1\} \overset{\text{renewal}}{\underset{\text{process}}{=}} \int_0^t F(ds) \cdot P\{X(t-s)=k\}$

$\Rightarrow P\{X(t)=k\} = \dfrac{(\lambda t)^k e^{-\lambda t}}{k!} [1-F(t)] + \int_0^t F(ds) \cdot P\{X(t-s)=k\}$

By renewal equation $\Rightarrow P\{X(t)=k\} = \dfrac{(\lambda t)^k e^{-\lambda t}}{k!} [1-F(t)] + \int_0^t M(ds) \cdot \dfrac{(\lambda(t-s))^k e^{-\lambda(t-s)}}{k!} [1-F(t-s)]$

Here $M := \sum_{n=1}^{\infty} F_n$, $F_n = \underbrace{F * F * \cdots * F}_{n}$

c. $\lim_{t\to\infty} P\{X(t)=k\} \overset{\text{由 b}}{=} \underbrace{\lim_{t\to\infty} \dfrac{(\lambda t)^k e^{-\lambda t}}{k!} \cdot [1-F(t)]}_{0} + \lim_{t\to\infty} \int_0^t M(ds) \cdot \dfrac{(\lambda(t-s))^k}{k!} e^{-\lambda(t-s)} [1-F(t-s)]$

$\{T_m\}$ recurrent, $\lim_{t\to\infty} P\{X(t)=k\} = \dfrac{1}{\mu} \int_0^{\infty} \dfrac{(\lambda t)^k e^{-\lambda t}}{k!} (1-F(t)) dt$, Here $\mu = E[T_1]$