**STAT 611 Homework 7 Solutions**

1. (a) Suppose that $\theta_2 > \theta_1 > 0$. Then the likelihood ratio and its derivative are

$$\frac{f(x|\theta_2)}{f(x|\theta_1)} = \frac{\theta_2(\theta_1^2 + x^2)}{\theta_1(\theta_2^2 + x^2)}$$

and

$$\frac{d}{dx} = \frac{f(x|\theta_2)}{f(x|\theta_1)} = \frac{\theta_2(\theta_2^2 - \theta_1^2)}{(\theta_2^2 + x^2)^2}x$$

Since the sign of the derivative is the same as the sign of $x$ which changes sign, the ratio is not monotone. Hence, the MLR does not exist.

(b) It can be shown that $|X|$ is sufficient. The PDF of $T = |X|$ is

$$f(t|\theta) = \frac{2\theta}{\pi(\theta^2 + t^2)}$$

for $t > 0$. Differentiating we find that the sign of the derivative is the same as the sign of $y$ which is positive. Hence, the distribution of $|X|$ has an MLR.

2. (a) The marginal distribution of $Y_1$ is

$$f(y_1|\theta) = n(1 - (y_1 - \theta))^{n-1}$$

for $\theta < y_1 < \theta + 1$. The joint pdf of $(Y_1, Y_n)$ is

$$f(y_1, y_n|\theta) = n(n-1)(y_n - y_1)^{n-2}$$

for $\theta < y_1 < y_n < \theta + 1$. Under $H_0$, $P(Y_n \geq 1) = 0$ so

$$\alpha = P(Y_1 \geq k|\theta = 0) = (1-k)^n$$

Hence, for a size $\alpha$ test, take $k = 1 - \alpha^{1/n}$.

(b) For $\theta \leq k - 1$, $\beta(0) = 0$ and for $k - 1 < \theta \leq 0$,

$$\beta(\theta) = \int_k^{\theta+1} n(1 - (y_1 - \theta))^{n-1} dy_1 = (1 - k + \theta)^n$$

For $0 < \theta \leq k$,

$$\begin{aligned}\beta(\theta) &= \int_k^{\theta+1} n(1 - (y_1 - \theta))^{n-1} dy_1 + \int_\theta^k \int_1^{\theta+1} n(n-1)(y_n - y_1)^{n-2} dy_n dy_1 \\ &= \alpha + 1 - (1-\theta)^n\end{aligned}$$

For $k < \theta$, $\beta(\theta) = 1$.

(c) $(Y_1, Y_n)$ are sufficient statistics. Using Corollary 8.3.13 and the joint pdf, we can attempt to find the UMP test. For $\theta \in (0, 1)$, the ratio of the pdfs is

$$\frac{f(y_1, y_n|\theta)}{f(y_1, y_n|0)} = \begin{cases} 0 & 0 < y_1 \leq \theta, y_1 < y_n < 1 \\ 1 & \theta < y_1 < y_n < 1 \\ \infty & 1 \leq y_n < \theta + 1, \theta < y_1 < y_n \end{cases}$$

For $1 \le \theta$, the ratio of the pdfs is

$$\frac{f(y_1, y_n|\theta)}{f(y_1, y_n|0)} = \begin{cases} 0 & y_1 < y_n < 1 \\ \infty & \theta < y_1 < y_n < \theta + 1 \end{cases}$$

For $0 < \theta < k$, use $k' = 1$. The given test always rejects if $f(y_1, y_n|\theta)/f(y_1, y_n|0) > 1$ and always accepts if this ratio is less than 1. For $\theta \ge k$, use $k' = 0$. The given test always rejects if the ratio of pdfs is greater than 0 and always accepts if it is less than 0. Thus, the given test is UMP.

(d) From the power function in (b), $\beta(\theta) = 1$ for all $\theta \ge k = 1 - \alpha^{1/n}$. Thus, these conditions are satisfied for any $n$.

3. Let $R_1$ denote the rejection region, that is, $R_1$ is the set of all sequences $\mathbf{X}$ such that its likelihood ratio is greater than or equal to $\gamma_1$. Similarly, let $R_0$ be the acceptance region or the set of all sequences $\mathbf{X}$ whose likelihood ratio is less than or equal to $\gamma_0$. The power is

$$\begin{aligned} \beta = P(\text{reject } H_0|H_1) &= \int_{R_1} p(x|p_1)dx \\ &= \int_{R_1} \frac{p(x|p_1)}{p(x|p_0)} p(x|p_0)dx \\ &\ge \gamma_1 \int_{R_1} p(x|p_0)dx \\ &= \gamma_1 \times P(\text{reject } H_0|H_0) \\ &= \gamma_1 \alpha \end{aligned}$$

Hence, $\gamma_1 \le \beta/\alpha$. Using a similar derivation,

$$\begin{aligned} 1 - \beta = P(\text{accept } H_0|H_1) &= \int_{R_0} p(x|p_1)dx \\ &= \int_{R_0} \frac{p(x|p_1)}{p(x|p_0)} p(x|p_0)dx \\ &\le \gamma_0 \int_{R_0} p(x|p_0)dx \\ &= \gamma_0 \times P(\text{accept } H_0|H_0) \\ &= \gamma_0(1 - \alpha) \end{aligned}$$

so $(1 - \beta)/(1 - \alpha) \le \gamma_0$.