STAT 611 Homework 1 Solutions

(1) The joint pdf of $(X_1, \ldots, X_n)$ is

$$f(x_1, x_2, \ldots, x_n|\theta) = \prod_{i=1}^{n} e^{i\theta - x_i} \mathbb{1}_{x_i \in (i\theta, \infty)}$$

Notice that $x_i > i\theta$ for all i if, and only if, $\min_i(x_i/i) > \theta$. Hence, the condition that $\mathbf{1}_{x_i \in (t\theta, \infty)}$ for each i is the same as $\mathbf{1}_{T(\mathbf{x}) \in (\theta, \infty)}$ so

$$f(x_1, x_2, \ldots, x_n|\theta) = \underbrace{e^{in\theta} \mathbb{1}_{T(\mathbf{x}) \in (\theta, \infty)}}_{g(T(\mathbf{x}))|\theta} \cdot \overbrace{e^{-\sum_i x_i}}^{h(\mathbf{x})}$$

Take $g(T(\mathbf{x})|\theta) = e^{in\theta} \mathbb{1}_{T(\mathbf{x}) \in (\theta, \infty)}$ and $h(\mathbf{x}) = e^{-\sum_i x_i}$. By the Factorization Theorem, $T(\mathbf{x}) = \min_i(x_i/i)$ is a sufficient statistic for θ.

(2) First, define the statistics

$$x_{min} = \min\{x_1, \ldots, x_n\} \quad \text{and} \quad x_{max} = \max\{x_1, \ldots, x_n\}$$

and

$$y_{min} = \min\{y_1, \ldots, y_n\} \qquad y_{max} = \max\{y_1, \ldots, y_n\}$$

Now put $\mathbf{x} = (x_1, \ldots, x_n)$, $\mathbf{y} = (y_1, \ldots, y_n)$, and $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_4)$. Write the joint pdf as

$$f(\mathbf{x}, \mathbf{y}|\boldsymbol{\theta}) = \prod_{i=1}^{n} \frac{1}{(\theta_3 - \theta_1)(\theta_4 - \theta_2)} \mathbb{1}_{(\theta_1, \theta_3)}(x_i) \mathbb{1}_{(\theta_2, \theta_4)}(y_i)$$

We can rewrite the previous expression in terms of the statistics defined above as

$$\begin{aligned} f(\mathbf{x}, \mathbf{y}|\boldsymbol{\theta}) = &\left(\frac{1}{(\theta_3 - \theta_1)(\theta_4 - \theta_2)}\right)^n \\ &\times \mathbb{1}_{(\theta_1, \infty)}(x_{min}) \mathbb{1}_{(-\infty, \theta_3)}(x_{max}) \mathbb{1}_{(\theta_2, \infty)}(y_{min}) \mathbb{1}_{(-\infty, \theta_4)}(y_{max}) \end{aligned}$$

If we take $g(T(\mathbf{x}, \mathbf{y})|\boldsymbol{\theta})$ as the entirety of the previous expression and $h(\mathbf{x}, \mathbf{y}) = 1$, then by the Factorization Theorem, $(x_{min}, x_{max}, y_{min}, y_{max})$ is a sufficient statistic for $\boldsymbol{\theta}$.

(3) (a) The pdf of Y_1 is

$$f(y; \lambda, c) = \lambda e^{\lambda y} \mathbb{1}_{y<c} + e^{-\lambda c} \mathbb{1}_{y \geq c}$$

(b) Begin with the joint pdf of $\mathbf{y}$:

$$\begin{aligned} f(y_1, \ldots, y_n|\lambda) &= \prod_{i=1}^{n} \left[\lambda e^{-\lambda y_i} \mathbb{1}_{y_i<c} + e^{-\lambda c} \mathbb{1}_{y_i \geq c}\right] \\ &= \exp\left(\sum_{i=1}^{n} \log\left(\lambda e^{-\lambda y_i} \mathbb{1}_{y_i<c} + e^{-\lambda c} \mathbb{1}_{y_i \geq c}\right)\right) \end{aligned}$$

Now rewrite the previous expression as (convince yourself that these two expressions are equivalent!):

$$= \exp\left(\sum_{i=1}^{n} \log\left(\left[\lambda e^{-\lambda y_i}\right]^{\mathbb{1}_{y_i<c}} \times \left[e^{-\lambda c}\right]^{\mathbb{1}_{y_i \geq c}}\right)\right)$$

Now use properties of the logarithm and distribute the summation to obtain

$$= \exp\left(\sum_{i=1}^{n} \mathbb{1}_{y_i<c}(\log \lambda - \lambda y_i) - \lambda c \sum_{i=1}^{n} \mathbb{1}_{y_i \geq c}\right)$$

Since $\mathbb{1}_{y_i<c} = 1 - \mathbb{1}_{y_i \geq c}$, we can rewrite this as

$$= \exp\left(\sum_{i=1}^{n} \log \lambda - \lambda \sum_{i=1}^{n} y_i - \sum_{i=1}^{n} (\log \lambda - \lambda y_i + \lambda c) \mathbb{1}_{y_i \geq c}\right)$$

Notice that $y_i \geq c$ (or $\mathbb{1}_{y_i \geq c} = 1$) implies $y_i = c$ so

$$= \exp\left(\sum_{i=1}^{n} \log \lambda - \lambda \sum_{i=1}^{n} y_i - \log \lambda \sum_{i=1}^{n} \mathbb{1}_{y_i \geq c}\right)$$

Hence, this exponential family is curved.

(c) The sufficient statistics are

$$T(\mathbf{y}) = \sum_{i=1}^{n} y_i$$

and

$$T(\mathbf{y}) = \sum_{i=1}^{n} \mathbb{1}_{y_i \geq c}$$