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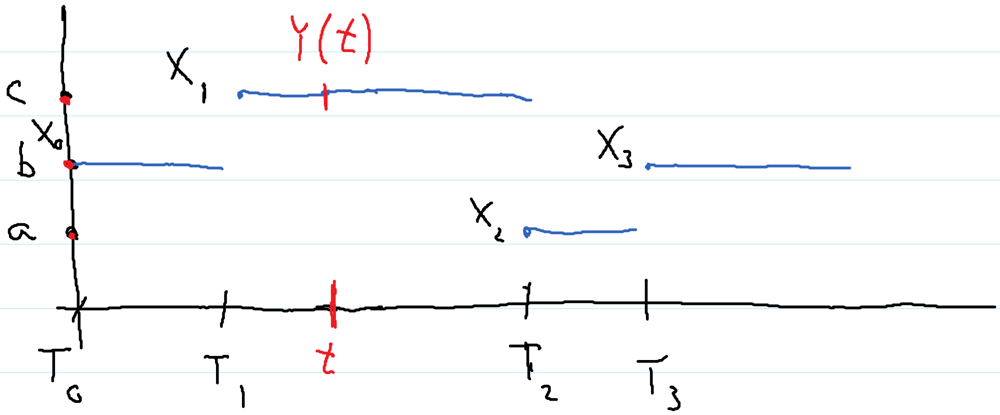

$\{X_n, T_n\}$ abbreviation for $\{(X_n, T_n) : n = 0, 1, \cdots\}$

$\{Y(t)\}$

Example → Markov processes are MRP

Consider Markov process with generator $\begin{bmatrix} -3 & 3 & 0 \\ 2 & -5 & 3 \\ 0 & 2 & -2 \end{bmatrix}$

Fixed t

$Q(t)$ represent the matrix whose elements are $Q(i,j,t)$

$$Q(t) = \begin{bmatrix} 0 & 1-e^{-3t} & 0 \\ 0.4(1-e^{-5t}) & 0 & 0.6(1-e^{-5t}) \\ 0 & 1-e^{-2t} & 0 \end{bmatrix}$$

$Q(i,j,t)$ = prob. of going from state i to state j within t time units.

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Fix (i,j) and consider the function $t \to Q(i,j,t)$

Let $P(i,j) = \lim_{t \to \infty} Q(i,j,t) \leq 1$

$P(i,j) \geq 0 \quad \forall\, i,j$ and $\sum_j P(i,j) = 1 \Rightarrow P$ is a Markov matrix

$\Rightarrow \{X_n\}$ is a Markov chain

$$P\{T_1 \leq t \mid X_0 = i, X_1 = j, X_2 = k\} = \frac{Q(i,j,t)}{P(i,j)} \text{ if } P(i,j) > 0$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$P\{T_1 \leq t, X_1 = j, X_2 = k \mid X_0 = i\} = P\{X_1 = j, T_1 \leq t \mid X_0 = i\} P(\cancel{j}, k)$$

$$P\{X_1 = j, X_2 = k \mid X_0 = i\} = P(i,j)\, P(\cancel{j}, k)$$

$$P\{T_1 - T_0 \le t_1,\ T_2 - T_1 \le t_2,\ T_3 - T_2 \le t_3 \mid X_0 = i_0,\ X_1 = i_1,\ X_2 = i_2\}$$

$$= G(i_0, i_1, t_1)\, G(i_1, i_2, t_2)\, G(i_2, i_3, t_3)$$

where $G(i,j,t) = \dfrac{Q(i,j,t)}{P(i,j)}$ with $\dfrac{0}{0} = 1$

$\Longrightarrow$ increments $T_1 - T_0,\ T_2 - T_1,\ T_3 - T_2, \cdots$ are conditionally independent given the Markov chain $\{X_n\}$

and the increments $T_{n+1} - T_n$, only depends on $\underline{X_n \text{ and } X_{n+1}}$

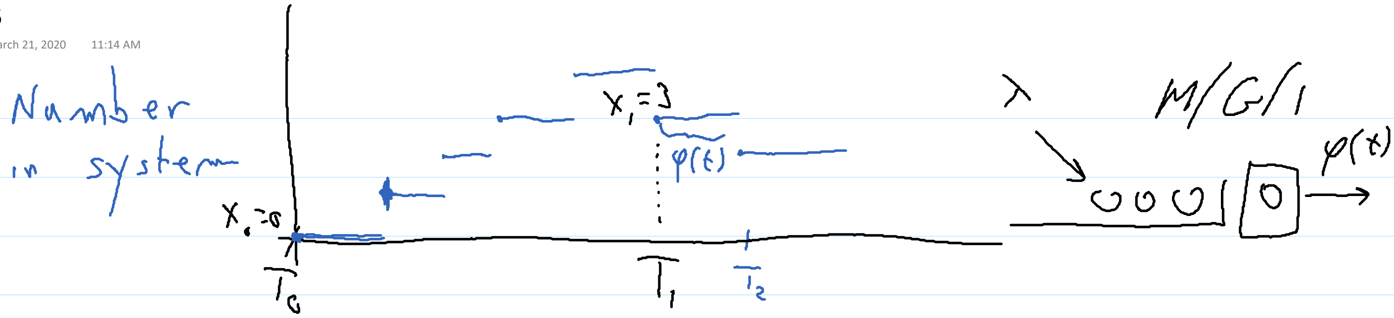
Corollary: If E has only one state, $\{T_n\}$ forms a renewal process

Counter Type I: $E = \{0, 1\} = \{\text{unlocked}, \text{locked}\}$

$Q(0,1,t) \rightarrow$ probability of going from unlocked to locked in less than or equal to t time units

$Q(1,0,t) \rightarrow$ prob. of going from locked to unlocked in $\leq t$ time units

$$\tilde{Q}(t) = \begin{bmatrix} 0 & 1-e^{-\lambda t} \\ \psi(t) & 0 \end{bmatrix}$$



$Q(i,j,t) \rightarrow$ a departure has just occurred and there are i customers in the system. $Q(i,j,t)$ is the probability that the next departure will occur $\le t$ time units and immediately after that departure, there will be j customers in the system.

$Q(3,0,t) = 0$ and $Q(3,1,t) = 0$

$Q(3,2,t) = \int_{[0,t]} \varphi(ds)\, e^{-\lambda s}$ ← No arrivals before service

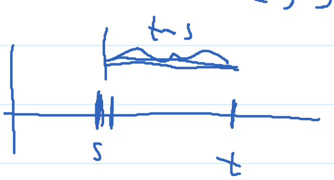
$Q(3,3,t) = \int_{[0,t]} \varphi(ds)\, e^{-\lambda s}(\lambda s)$ ← exactly one arrival during service time

$Q(3,4,t) = \int_{[0,t]} \varphi(ds)\, \frac{e^{-\lambda s}(\lambda s)^2}{2!}$ ← exactly two arrivals during service time

$$q_n(t) = \int_{[0,t]} \varphi(ds)\, \frac{e^{-\lambda s}(\lambda s)^n}{n!}$$

$$Q(t) = \begin{array}{c|ccccc} 0 & p_0(t) & p_1(t) & p_2(t) & p_3(t) & \cdots \\ 1 & q_0(t) & q_1(t) & q_2(t) & q_3(t) & \cdots \\ 2 & 0 & q_0(t) & q_1(t) & q_2(t) & \cdots \\ 3 & 0 & 0 & q_0(t) & q_1(t) & \cdots \\ 4 & 0 & 0 & 0 & q_0(t) & \cdots \end{array}$$

$$Q(0,0,t) = \int_{[0,t]} \lambda e^{-\lambda s} q_0(t-s)\, ds$$



$$p_n(t) = \int_{[0,t]} \lambda e^{-\lambda s} q_n(t-s)\, ds$$

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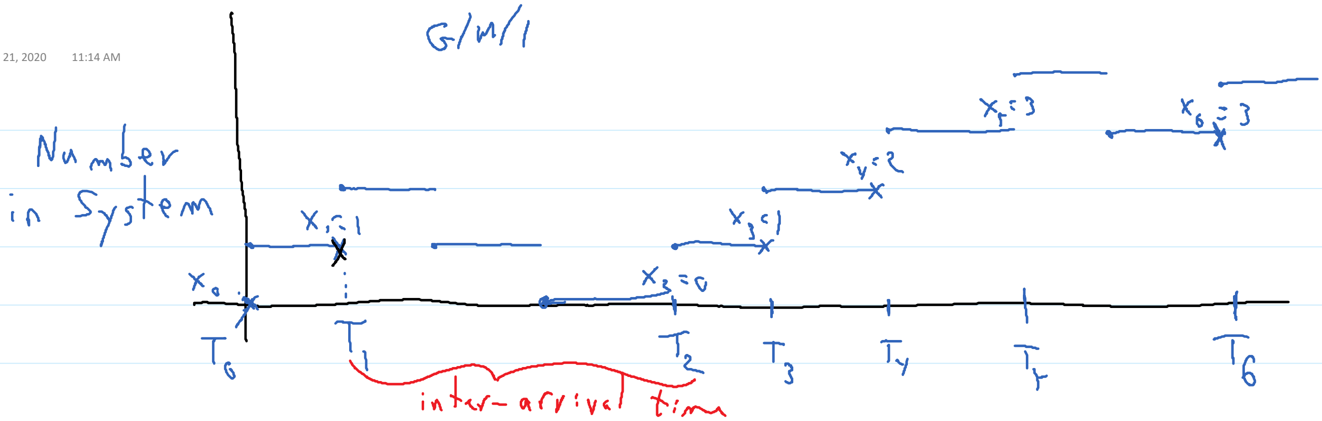

$Q(i,j,t)$ = arrival just occurred and i are in system, not including the arriving customer. $Q(i,j,t)$ is prob. next arrival occurs before t and j in system at arrival time, not counting the arriving customer.

$$Q(t) = \begin{bmatrix} r_0(t) & q_0(t) & & & \\ r_1(t) & q_1(t) & q_0(t) & & 0 \\ r_2(t) & q_2(t) & q_1(t) & q_0(t) & \\ r_3(t) & q_3(t) & q_2(t) & q_1(t) & q_0(t) \end{bmatrix}$$

$$q_n(t) = \int_{[0,t]} \varphi(ds) \boxed{\frac{e^{-\mu s}(\mu s)^n}{n!}}$$ ← n departure during an inter-arrival time

$$r_n(t) = \varphi(t) - \sum_{k=0}^{n} q_k(t)$$