Markov Renewal Processes – Basics

Definition: The stochastic process $\{(X_n,T_n)\}$ is called a *Markov renewal process* with state space E if

$$P\{X_{n+1} = j, T_{n+1} - T_n \le t \mid X_0, \ldots, X_n; T_0, \ldots, T_n\} = P\{ X_{n+1} = j, T_{n+1} - T_n \le t \mid X_n\}$$

for all n=0, 1, …, $j \in E$, and $t \ge 0$.

We will always assume (1) the process is time homogeneous and (2) E is discrete.

Definition: The family of probabilities $Q = \{Q(i,j,t): i,j \in E, \text{ and } t\ge 0\}$ is called a *semi-Markov kernel* and is defined by

$$P\{ X_{n+1} = j, T_{n+1} - T_n \le t \mid X_n = i\} = Q(i, j, t).$$

Definition: The process $\{Y(t)\}$ defined by

$$Y(t) = X_n \text{ for } T_n \le t < T_{n+1} \text{ and } Y(t) = \Delta \text{ if } t \ge \sup \{T_n : n=0, 1, \ldots \},$$

is called a *semi-Markov process*, where Δ is a state not in E. For most of our examples, there will be no Δ.

Two properties of a Markov renewal process:

1. $\{Xn\}$ is a Markov chain.
2. $P\{ T_{n+1} - T_n \le t \mid X_0, X_1, \ldots ; T_0, \ldots , T_n\} = P\{ T_{n+1} - T_n \le t \mid X_n, X_{n+1}\}$

The Markov renewal convolution operator

$$Q*f(i,t) = \sum_{k\in E} \int_{[0,t]} Q(i,k,ds)f(k,t-s)$$

$$Q^{n+1}(i, j, t) = \sum_{k\in E} \int_{[0,t]} Q^n(i,k,ds)Q(k,j,t-s) \qquad \text{for fixed } j$$

$$R(i, j, t) = \sum_{n=0}^{\infty} Q^n (i,j,t) \qquad \text{this also implies} \qquad R*Q(i, j, t) = R(i, j, t) - I(i, j)\, I_{[0,\infty]}(t)$$

Let $\nu P= \nu$ and $\mu(i) = E_i[T_1]$, then the rate between visits to state i is $\eta(i) = \nu(i) / \nu\bullet\mu$

Markov renewal type equation: find collection of functions $\{t\to h(i,t): \text{for } i\in E\}$ such that

$$h(i, t) = g(i, t) + Q*h(i, t) \qquad \text{for } i\in E \text{ and } t \ge 0.$$

Solution: $h(i, t) = R*g(i, t)$ for $i\in E$ and $t \ge 0$ and

$$\lim_{t\to\infty} R*g(i, t) = (1/\nu\bullet\mu) \sum_{k\in E} \nu(k)\int_0^{\infty} g(k,t)dt .$$

Examples

1. A Markov process forms a Markov renewal process. Or, one could also say that a semi-Markov process is a generalization of a Markov process.

2. *Counters of Type I*: Arrivals to a particle counter form a Poisson process with rate λ. An arriving particle which finds the counter free gets registered and locks it for a random duration with distribution function ψ. Arrivals during a locked period have no effect. Define State 0 to be the state when the counter is unlocked and let State 1 be when the counter is locked. Let $T_0=0$, T_1, T_2, etc. be the successive instants of changes in the state of the counter and let X_n be the state immediately after T_n. Then $\{(X_n, T_n)\}$ is a Markov renewal process with state space E= {0, 1}. The semi-Markov process $\{Y(t)\}$ associated with $\{(X_n, T_n)\}$ represents the state of the counter at time t. The semi-Markov kernel for this process is relatively simple.

3. *M/G/1 Queueing System*. An M/G/1 system represents a single-server queueing system with a Poisson arrival process with rate λ and independent service times with the common distribution φ. Let $T_0=0$, T_1, T_2, etc. be the successive instants of departures, and let X_n be the number of customers left behind by the n^{th} departure. Then $\{(X_n, T_n)\}$ is a Markov renewal process with state space E= {0, 1, … }. The semi-Markov kernel for this process is more complex than the previous example, but we do have $Q(i,j,t) = 0$ for $i - j \geq 2$.

   Consider example 4 after example 3 has been discussed

4. *G/M/1 Queueing System*. A G/M/1 system represents a single-server queueing system with the arrival process being a renewal process with φ being the distribution of inter-arrival times and independent service times governed by an exponential distribution with mean rate μ. Let $T_0=0$, T_1, T_2, etc. be the successive instants of arrivals, and let X_n be the number of customers just **before** the n^{th} arrival. Then $\{(X_n, T_n)\}$ is a Markov renewal process with state space E= {0, 1, … }. The semi-Markov kernel for this process is more complex than the previous example, but we do have $Q(i,j,t) = 0$ for $j - i \geq 2$.

5. *System Availability* (from April 13 lecture) Consider a piece of equipment with a finite number of components; suppose that the failure of any one component is a failure for the equipment itself. Let $T_0=0$, T_1, T_2, … be the times of successive failures, and let X_n be the type of component causing the n^{th} failure. The time $T_{n+1} - T_n$ between two failures is the sum of the repair time of the component which failed at T_n and a failure-free interval following the repair. We suppose all components have exponential lifetimes, with the component j having the parameter $\lambda(j)$; and suppose that the repair time of the component j has distribution $t \to \varphi(j, t)$. Under these assumptions, $\{X_n, T_n\}$ is a Markov renewal process. We need to give the semi-Markov kernel and derive some probability expressions. For the probabilities, we will define Y(t) to denote the component that caused the last failure before time t and let W(t)=1 if the equipment is working at time t and W(t)=0 if a component is under repair at time t. Or goal will be to obtain an expression for $P_i\{Y(t)=j, W(t)=0\}$ and $\lim_{t\to 0} P_i\{Y(t)=j, W(t)=0\}$.