Name:______________________________________

Full name on front, one name back

Test One

Closed Book, Closed Notes

You may use your calculator

1. Consider a five state Markov chain. For each state a through e below, some specific condition is given. ***Circle*** the appropriate state classification from among the three choices given. Note, evaluate each part independently from the other parts; in other words, when considering the condition given in part (b), the condition given in part (a) is not relevant.

Remember that $R(i, j) = E[N^j \mid X_0=i]$ and $F(i, j) = P\{T^j < \infty \mid X_0=i\}$.

state	condition	choice		
a	$R(a, a) = \infty$	recurrent	transient	not enough info to classify
b	$F(b, b) = 1$	recurrent	transient	not enough info to classify
c	$F(c, c) = 0.8$	recurrent	transient	not enough info to classify
d	$R(d, d) = 123.3$	recurrent	transient	not enough info to classify
e	$F(e, c) = 1.0$	recurrent	transient	not enough info to classify

2. There are two common types of failure to a critical electronic element of some machinery: either component A or component B may fail. If either component fails, the machinery goes down. Component A fails according to a Poisson process with mean rate 1.1 failures *per shift*. (The company operates 24/7 using eight-hour shifts.) Component B fails according to a Poisson process with a mean rate of 2.7 failures *per day*. Upon the failure the electronic element is immediately replaced, and for this problem, assume the time to replace the element is negligible.

a. What is the average time, in hours, between failures?

Rate/day = 1.183 + 2.7 = 6/day. Time in hours = 24/6 = 4 hr (rate and time are reciprocals)

b. What is the probability that there will be exactly four failures of the machine within a given day?

$\text{Exp}(-\lambda t)\,(\lambda t)^4 / 4! = e^{-6}\, 6^4 / 24 = 0.1339$

c. It is now noon and the most recent failure occurred two hours ago. What is the probability that the next failure will occur between 1PM and 5PM today?

Next occurrence is exponentially distributed and since exponential is memoryless, the time of the most recent failure is not relevant.

$P\{1 < T < 5\} = \text{Exp}(-6*1/24) - \text{Exp}(-6*5/24) = 0.4923$

3. Calls to a fire station arrive according to a non-stationary Poisson process. The arrival rate of calls from midnight to 6AM is 0.1 per hour; from 6AM to 11AM it is 0.2 per hour; from 11AM to 1PM it is 0.4 per hour; from 1PM to 6PM it is 0.3 per hour; and from 6PM to midnight it is 0.2 per hour.

a. It is now 8PM. What is the probability that there will be no calls tomorrow? (In other words, the probability that no calls arrive between midnight tonight and midnight tomorrow night.)

Remember memoryless property so what happens between now and midnight is not relevant.

Rate/day = 6*0.1 + 5*0.2 + 2*0.4 + 5*0.3 + 6*0.2 = 5.1/day

P{zero arrivals during a one-day period} = exp(-5.1) = 0.0061

b. It is now 8PM. What is the probability that there will be no calls within the next four hours?

P{zero arrivals during a four-hour period} = exp(-0.2*4) = 0.4493

4. Consider the following Markov matrix representing a Markov chain with state space {a,b,c,d,e,f} and profit function given by f = (10, 20, 30, 40, 50, 60).

$$P = \begin{array}{c|cccccc} a & 0.5 & 0.0 & 0.5 & 0.0 & 0.0 & 0.0 \\ b & 0.8 & 0.2 & 0.0 & 0.0 & 0.0 & 0.0 \\ c & 0.0 & 1.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ d & 0.0 & 0.0 & 0.0 & 0.8 & 0.2 & 0.0 \\ e & 0.0 & 0.0 & 0.0 & 0.6 & 0.4 & 0.0 \\ f & 0.0 & 0.4 & 0.4 & 0.0 & 0.0 & 0.2 \end{array}$$

Round your answers to be accurate to four digits to the right of the decimal.

Note that the following sets are irreducible: {a,b,c}, and {d, e}. Find the following:

a. $\lim_{n\to\infty} P\{X_n = a \mid X_0 = a\}$. Solve the following three equations and three unknowns

$0.5\pi_a + 0.8\pi_b = \pi_a$, $0.2\pi_b + \pi_c = \pi_b$, $\pi_a + \pi_b + \pi_c = 1$

First eq.: $\pi_a = 1.6\pi_b$, Second eq: $\pi_c = 0.8\pi_b$, Third eq: $1.6\pi_b + \pi_b + 0.8\pi_b = 1$

Therefore, $\pi_b = 1/3.4$ and $\pi_a = 1.6/3.4 =$ 0.4706, and $\pi_c = 0.8/3.4$

b. $\lim_{n\to\infty} P\{X_n = a \mid X_0 = b\} =$ 0.4706, since a and b both in same irreducible sets.

c. $\lim_{n\to\infty} P\{X_n = a \mid X_0 = e\} =$ 0, since a and e in different irreducible sets.

d. $\lim_{n\to\infty} P\{X_n = a \mid X_0 = f\} = 0.5*0.4706 =$ 0.2353, since F(f,a) = 0.5).

e. $\lim_{n\to\infty} E[f(X_n) \mid X_0 = a] = 10*(1.6/3.4) + 20*(1/3.4) + 30*(0.8/3.4) =$ 17.6471

5. Let {N(t); t>=0} be a renewal process with renewal function $m(t) = \sum_{n=1}^{\infty} F_n(t)$ for t>=0. (The parameter t is in terms of hours.) Furthermore, assume inter-renewal times are distributed according to an Erlang distribution with a mean of 30 minutes and standard deviation 15 minutes; in other words, F is from the sum of 4 exponentials, each with mean 7.5 minutes.

Let {Y(t); t>=0} be a related stochastic process such that

$$P\{Y(t) = k\} = e^{-4t} + \int_{s\in[0,t]} F(ds)P\{Y(t-s) = k\}$$

Or, if you prefer the old fashioned way to write it:

$$P\{Y(t) = k\} = e^{-4t} + \int_{s\in[0,t]} P\{Y(t-s) = k\}\, dF(s)$$

Give the numerical value for $\lim_{t\to\infty} P\{Y(t) = k\}$.

Solution is $P\{Y(t) = k\} = e^{-4t} + \int_{s\in[0,t]} m(ds)\, e^{-4(t-s)}$,

By the Key Renewal Theorem: $\lim_{t\to\infty} P\{Y(t) = k\} = (1/0.5)\int_0^{\infty} e^{-4t}\, dt = 2/4 =$ 0.5