A LAW OF LARGE NUMBERS FOR THE MAXIMUM IN A STATIONARY GAUSSIAN SEQUENCE[1]

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1. Introduction. Let X_1, X_2, $\cdots$ be sequence of random variables which are unbounded above, and let

$$Z_n = \max(X_1, \cdots, X_n).$$

The law of large numbers (LLN) is said to hold for the sequence $\{Z_n\}$ if there exists a sequence of constants $\{A_n\}$ such that

$$Z_n - A_n \to 0 \qquad \text{in probability.} \tag{1}$$

The necessary and sufficient conditions for the LLN for Z_n in the case where $\{X_n\}$ is a sequence of mutually independent random variables with a common d.f. $F(x)$ were found by B. V. Gnedenko [2]. In particular, he mentioned that the standard normal distribution satisfies the conditions and that (1) holds with

$$A_n = (2 \log n)^{\frac{1}{2}}. \tag{2}$$

The main result of this paper is that if $\{X_n : n \geqq 1\}$ is a stationary Gaussian process with

$$EX_i = 0, \qquad EX_i^2 = 1, \qquad EX_1X_i = r_i,$$

then Z_n satisfies (1) with A_n given by (2), under the condition that $nr_n \to 0$.

Lemma 1 furnishes a condition for a stationary process under which the maximum behaves (in probability) almost as if the underlying random variables were mutually independent. Lemma 2 generalizes a result of G. S. Watson [3] on the tail of a bivariate normal d.f. The results of Lemma 2 are used to show that the given stationary Gaussian process satisfies the conditions of Lemma 1.

2. Gnedenko's conditions. It has been shown by Gnedenko [2] that (1) holds if and only if for every $\epsilon > 0$,

$$\begin{aligned} &\lim_{n\to\infty} n(1 - F(A_n + \epsilon)) = 0 \\ &\lim_{n\to\infty} n(1 - F(A_n - \epsilon)) = \infty. \end{aligned} \tag{3}$$

This can be seen from the fact that (1) holds if and only if for every $\epsilon > 0$,

$$\begin{aligned} 1 &= \lim_{n\to\infty} P\{A_n - \epsilon < Z_n \leqq A_n + \epsilon\} \\ &= \lim_{n\to\infty} F^n(A_n + \epsilon) - \lim_{n\to\infty} F^n(A_n - \epsilon), \end{aligned} \tag{4}$$

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and the equivalence of (3) and (4) follows from the logarithmic expansion of the terms in the last part of (4).

3. Preliminaries.

LEMMA 1. *Let* $\{X_n : n \geqq 1\}$ *be a stationary sequence, with the marginal* d.f. $F(x)$, *which satisfies* (3) *for some sequence* $\{A_n\}$ *and for every* $\epsilon > 0$; *let* $Z_n = \max(X_1, \cdots, X_n)$. *If for every* $\epsilon > 0$,

$$\lim_{n\to\infty} \frac{2}{n^2} \sum_{j=2}^{n} (n - j + 1) \frac{P\{X_1 > A_n - \epsilon, X_j > A_n - \epsilon\}}{P^2\{X_1 > A_n - \epsilon\}} = 1, \tag{5}$$

then (1) *holds.*

PROOF. From the relations

$$P\{Z_n > A_n + \epsilon\} = P\left(\bigcup_{i=1}^{n} \{X_i > A_n + \epsilon\}\right)$$
$$\leqq \sum_{i=1}^{n} P\{X_i > A_n + \epsilon\} = n(1 - F(A_n + \epsilon))$$

and from (3), it follows that

$$P\{Z_n > A_n + \epsilon\} \to 0.$$

The proof of the lemma will be completed by showing that

$$P\{Z_n \leqq A_n - \epsilon\} \to 0. \tag{6}$$

Let $I(H)$ denote the indicator function of the event H. It follows from (3), (5), and the stationarity of the sequence $\{X_n\}$ that

$$\frac{E\left(\sum_{i=1}^{n} I[X_i > A_n - \epsilon]\right)^2}{n^2(1 - F(A_n - \epsilon))^2} = \frac{1}{n(1 - F(A_n - \epsilon))}$$
$$+ \frac{\sum_{i \neq j} P\{X_i > A_n - \epsilon, X_j > A_n - \epsilon\}}{n^2(1 - F(A_n - \epsilon))^2} = \frac{1}{n(1 - F(A_n - \epsilon))}$$
$$+ \frac{2}{n^2} \sum_{j=2}^{n} (n - j + 1) \frac{P\{X_1 > A_n - \epsilon, X_j > A_n - \epsilon\}}{P^2\{X_1 > A_n - \epsilon\}} \to 1;$$

hence,

$$\text{l.i.m.} \frac{\sum_{i=1}^{n} I[X_i > A_n - \epsilon]}{n(1 - F(A_n - \epsilon))} = 1,$$

and from (3),

$$\sum_{i=1}^{n} I[X_i > A_n - \epsilon] \to \infty \qquad \text{in probability.}$$

By the application of the elementary inequality

$$1 - x \leqq e^{-x}, \qquad x \geqq 0,$$

and the bounded convergence theorem, one may now conclude that

$$P\{Z_n \leqq A_n - \epsilon\} = E \prod_{i=1}^{n} (1 - I[X_i > A_n - \epsilon])$$

$$\leqq E\left\{\exp\left[-\sum_{i=1}^{n} I[X_i > A_n - \epsilon]\right]\right\} \to 0;$$

therefore, (6) is verified.

LEMMA 2. *If X and Y have a bivariate normal distribution with expectations* 0, *unit variances, and correlation coefficient r, then*

$$\lim_{c\to\infty} \frac{P\{X > c, Y > c\}}{[2\pi(1 - r)^{\frac{1}{2}}c^2]^{-1} \exp\left\{-\dfrac{c^2}{1 + r}\right\} (1 + r)^{\frac{3}{2}}} = 1$$

uniformly for all r such that $|r| \leqq \delta$, for any δ, $0 < \delta < 1$.

PROOF.

$$P\{X > c, Y > c\}$$

$$= \frac{1}{2\pi(1 - r^2)^{\frac{1}{2}}} \int_c^\infty \int_c^\infty \exp\left\{-\frac{1}{2(1 - r^2)} (x^2 - 2rxy + y^2)\right\} dxdy.$$

After the change of variables

$$x = [w(1 + r)/c] + c; \qquad y = [z(1 + r)/c] + c,$$

the integral becomes

$$\frac{(1 + r)^{\frac{3}{2}} \exp\{-c^2/(1 + r)\}}{2\pi(1 - r)^{\frac{1}{2}}c^2}$$

$$\cdot \int_0^\infty \int_0^\infty \exp\left\{-\frac{1 + r}{2(1 - r)c^2} (w^2 - 2rwz + z^2)\right\} e^{-w-z}\, dwdz.$$

The first exponent in the integrand is never positive; hence, as $c \to \infty$, it follows from the bounded convergence theorem that

$$\left|\int_0^\infty \int_0^\infty \exp\left\{-\frac{1 + r}{2(1 - r)c^2} (w^2 - 2rwz + z^2)\right\} e^{-w-z}\, dwdz - 1\right|$$

$$\leqq \int_0^\infty \int_0^\infty \left(1 - \exp\left[-\frac{1 + \delta}{2(1 - \delta)c^2} (w^2 + 2\delta wz + z^2)\right]\right) e^{-w-z}\, dwdz \to 0,$$

where the convergence is independent of r.

4. The main result.

THEOREM. *Let $\{X_n\}$ be a stationary Gaussian process such that*

$$EX_i = 0, \qquad EX_i^2 = 1, \qquad i = 1, 2, \cdots,$$
$$EX_1X_i = r_i, \qquad i = 2, 3, \cdots.$$

If

$$\lim_{n\to\infty} nr_n = 0, \tag{7}$$

then $Z_n - (2 \log n)^{\frac{1}{2}} \to 0$ in probability.

REMARK. The theorem requires that the covariance sequence tend to zero faster than n^{-1}. This holds, e.g., for the Markov process where $r_n = r^{n-1}$ for some r such that $0 < r < 1$.

PROOF. Condition (7) and the stationarity of the sequence imply that $|r_n| < 1$ for all n; therefore, condition (7) also implies the existence of a δ, $0 < \delta < 1$, such that $|r_n| \leqq \delta$ for all n. To prove the theorem, it will be shown that (5) holds for A_n given by (2).

From Lemma 2 and the well-known asymptotic expression for the tail of the univariate normal d.f.

$$P\{X > c\} \sim (2\pi)^{-\frac{1}{2}}c^{-1} \exp\left(-\tfrac{1}{2}c^2\right),$$

it follows that the expression corresponding to the left side of (5) is asymptotic to

$$\left(\sum_{j=2}^{[\log n]} + \sum_{j=[\log n]+1}^{n}\right)\frac{2}{n^2}(n-j+1) \exp\left[(2\log n - 2\epsilon(2\log n)^{\frac{1}{2}} + \epsilon^2)\frac{r_j}{1+r_j}\right]\frac{(1+r_j)^{\frac{1}{2}}}{(1-r_j)^{\frac{1}{2}}}, \tag{8}$$

since the convergence in Lemma 2 is uniform in r.

The first sum in (8) tends to zero; since

$$r/(1+r) \qquad \text{and} \qquad (1+r)^{\frac{1}{2}}(1-r)^{-\frac{1}{2}}$$

are increasing functions of r, the first sum is bounded above by

$$\frac{2}{n^2}\exp\left\{\epsilon^2\frac{\delta}{1+\delta}\right\}\frac{(1+\delta)^{\frac{1}{2}}}{(1-\delta)^{\frac{1}{2}}}\cdot n^{2\delta/(1+\delta)}\sum_{j=2}^{[\log n]}(n-j+1),$$

which tends to zero.

The second sum in (8) converges to 1. Since $r_n \to 0$, the factors

$$(1+r_j)^{\frac{1}{2}}(1-r_j)^{-\frac{1}{2}}$$

and $r_j/(1+r_j)$ are uniformly close to 1 and 0, respectively, for sufficiently large j; furthermore, for $j > [\log n]$, from (7),

$$|r_j/(1+r_j)|\, 2\log n \sim 2|r_j| \log n \leqq 2|r_j|j \to 0.$$

The entire second sum in (8) is therefore asymptotic to

$$\frac{2}{n^2} \sum_{j=[\log n]+1}^{n} (n - j + 1) \to 1.$$

5. Concluding remarks. The referee has pointed out that the assumption of stationarity in the theorem is not critically used; condition (7) may be replaced by

$$\lim_{n\to\infty} nEX_iX_{i+n} = 0 \qquad \text{for all } i,$$

and the proofs go through without difficulty after a suitable modification of Lemma 1. The referee's suggestions were also helpful in the elimination of unnecessary calculations in an earlier version of the paper.

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