N-P Lemma (2nd Ver): (Randomized Test).

$H_0: \theta = \theta_0$ vs $H_1: \theta = \theta_1$.

Test fcn:
$$\phi(X) = \begin{cases} 1 & \text{if } f(X;\theta_1) > k f(X;\theta_0) \\ \gamma & \text{if } f(X;\theta_1) = k f(X;\theta_0) \\ 0 & \text{if } f(X;\theta_1) < k f(X;\theta_0). \end{cases}$$

$\gamma \in [0,1]$. chosen s.t. $\mathbb{E}_{\theta_0}(\phi(X)) = \alpha$

then $\phi(X)$ is a UMP test of size α.

Ex: $\{X_i\} \sim$ iid Bern(θ).

$H_0: \theta = \theta_0$. $H_1: \theta = \theta_1$. $0 < \theta_0 < \theta_1 < 1$.

LR: $\lambda(\underline{X}) = \left(\frac{\theta_1}{\theta_0}\right)^{\sum X_i} \left(\frac{1-\theta_1}{1-\theta_0}\right)^{n - \sum X_i}$ increase in $\sum X_i$

UMP test:
$$\phi(\sum X_i) = \begin{cases} 1 & \text{if } \sum X_i > k' \\ \gamma & \text{if } \sum X_i = k' \\ 0 & \text{if } \sum X_i < k'. \end{cases}$$

and $\alpha = \mathbb{P}_{\theta_0}(\sum X_i > k') + \gamma \mathbb{P}_{\theta_0}(\sum X_i = k')$
$$= \sum_{j=k'+1}^{n} \binom{n}{j} \theta_0^j (1-\theta_0)^{n-j} + \gamma \binom{n}{k'} \theta_0^{k'} (1-\theta_0)^{n-k'}.$$

Unless $\sum_{j=k'+1}^{n} \binom{n}{j} \theta_0^j (1-\theta_0)^{n-j} = \alpha$,

$\phi(\sum X_i)$ is a randomized test !!

Rmk: ① The UMP test can be determined uniquely except on the set $\{\underline{x} : f(\underline{x};\theta_1) = k' f(\underline{x};\theta_0)\} =: B$

② If $\mathbb{P}(B) = 0$. then $\exists$ a unique, non-randomized UMP test.

Test between 2 distn: one obs. X.

$H_0: X \sim N(0,1)$. $\quad H_1: X \sim$ Double Expo. pdf: $\frac{1}{4} e^{-\frac{1}{2}|x|}$.

$$\mathbb{P}\left(\frac{1}{4} e^{-\frac{1}{2}|X|} = \frac{k}{\sqrt{2\pi}} e^{-\frac{1}{2}X^2} \right) = 0.$$

$\Rightarrow \exists$ a unique non-randomized UMP test.

Test fcn:

$\phi(X) = 1$ iff $\frac{1}{4} e^{-\frac{1}{2}|x|} > \frac{k}{\sqrt{2\pi}} e^{-\frac{1}{2}X^2}$

$\Leftrightarrow |X|^2 - |X| > \underbrace{C}_{\text{constant}}$.

$\Leftrightarrow |X| > t$ or $|X| < 1-t$ for some $t > \frac{1}{2}$.

Assume $\alpha < \frac{1}{3}$. determine t from:

$$\alpha = \mathbb{P}_{H_0}(|X|>t) + \mathbb{P}_{H_0}(|X|<1-t)$$

If $t \le 1$ $\Rightarrow \mathbb{P}_{H_0}(|X|>t) \geq \mathbb{P}_{H_0}(|X|>1) = 0.3374 > \alpha$.

So $t > 1$. and $\alpha = \mathbb{P}_{H_0}(|X|>t) = 2(1-\Phi(t))$.

$\Rightarrow t = \Phi^{-1}(1-\frac{1}{2}\alpha)$.

and $\phi(X) = \mathbb{1}\{|X| > \Phi^{-1}(1-\frac{1}{2}\alpha)\}$.

Rmk: - Reject H_0 if $|X|$ is large.

- The probability of getting a large $|X|$ is higher under H_1.

- Power fcn: $\mathbb{P}_{H_1}(|X|>t) = e^{-\frac{1}{2}t}$.