$$P_i\{A\} = P\{A \mid X_0 = i\}$$

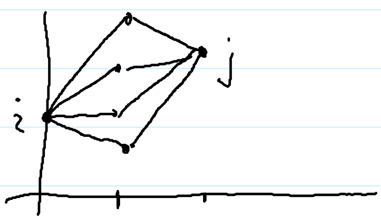
Markov renewal process

$$P_i\{T_1 \le t \mid X_1, X_2, \cdots\} = P_i\{T_1 \le t \mid X_1\}$$

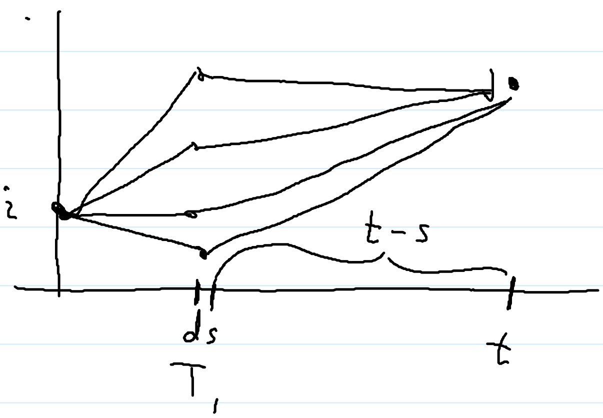
$$P_i\{X_{n+1} = j \mid X_n = k\} = P_k\{X_1 = j\}$$

$$P_i\{X_{n+1} = j,\ T_{n+1} - T_n \le t \mid X_n = k\} = P_k\{X_1 = j,\ T_1 \le t\} = Q(k,j,t)$$

$$P_i\{X_2 = j\} = P^2(i,j)$$
$$= \sum_k P(i,k)P(k,j)$$



$$P_i\{X_2 = j,\ T_2 \le t\} = \sum_k \int_{[0,t]} Q(i,k,ds)\,Q(k,j,t-s)$$



Let $t \to f(i,t)$ with $f(i,t)=0$ if $t<0$

Define convolution operator for Markov renewal processes

$$Q * f(i,t) = \sum_{k \in E} \int_{[0,t]} Q(i,k,ds)\, f(k,t-s)$$

Define $$Q^2(i,j,t) = \sum_k \int_{[0,t]} Q(i,k,ds)\, Q(k,j,t-s)$$

$$Q^{n+1}(i,j,t) = \sum_k \int_{[0,t]} Q^n(i,k,ds)\, Q(k,j,t-s)$$

$$P_i\{X_n = j, T_n \le t\} = Q^n(i,j,t)$$

$$\longrightarrow Q^0(i,j,t) = I(i,j)\, I_{[0,\infty)}(t)$$

Fix state j

Let $\{S_n^j : n=0,1,\cdots\}$ be successive visits to state j

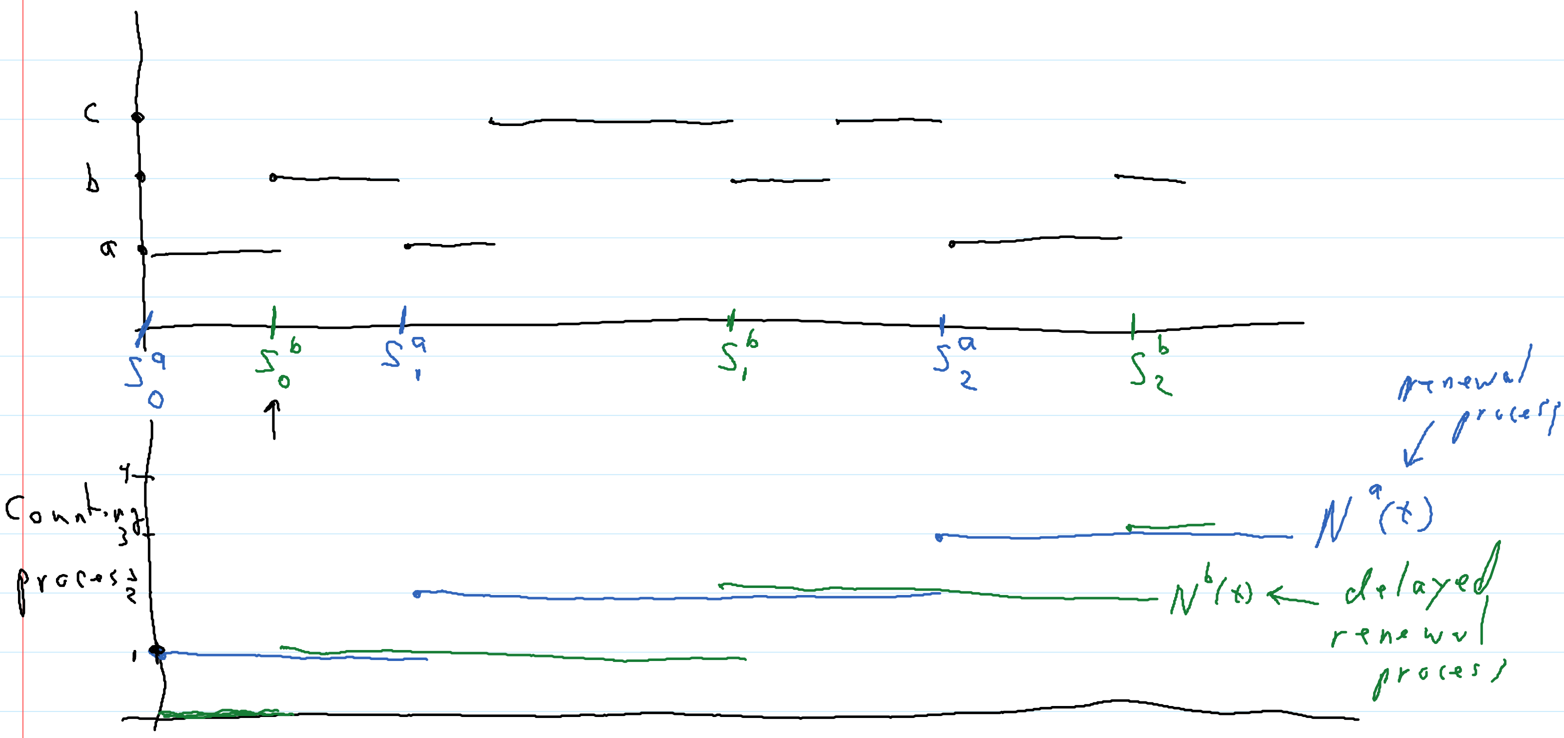
Let $N^j(t) = \sum_{n=0}^{\infty} I_{[0,t]}(S_n^j) = \sum_{n=0}^{\infty} I(X_n, j) I_{[0,t]}(T_n)$

Finally, let $R(i,j,t) = E_i[N^j(t)] = E_i\left[\sum_{n=0}^{\infty} I(X_n,j) I_{[0,t]}(T_n)\right]$

$\Rightarrow R(i,j,t) = \sum_{n=0}^{\infty} P_i\{X_n = j, T_n \le t\} = \sum_{n=0}^{\infty} Q^n(i,j,t)$

For fixed i and j, the function $t \to R(i,j,t)$ is called a Markov renewal function.

The collection of functions $R = \{R(i,j,\cdot) \text{ for } i,j \in E\}$ is called a Markov renewal kernel


c
b
a
S_0^a
S_0^b
S_1^a
S_1^b
S_2^a
S_2^b
Counting process
renewal process
$N^a(t)$
$N^b(t)$
delayed renewal process

If $i \neq j$, let $F(i,j,t) = P_i\{S_0^j \leq t\}$

otherwise let $F(j,j,t) = P_j\{S_1^j \leq t\}$

$F(i,j,t) \rightarrow$ For $i \neq j$, starting in state i, probability of reaching state j before time t.

$F(j,j,t) \rightarrow$ starting in state j, probability of returning to state j before time t.

$F(j,j,t) \longrightarrow$ inter-renewal

$$F_n(j,j,t) = \int_{[0,t]} F_{n-1}(j,j,ds)\, F(j,j,t-s)$$

$$m(t) = \sum_{n=1}^{\infty} F_n(j,j,t) \quad , \text{ let } F_0(j,j,t) = I_{[0,\infty)}(t)$$

$$R(j,j,t) = \sum_{n=0}^{\infty} F_n(j,j,t) = 1 + m(t) \quad \text{for } t \geq 0$$

Renewal equation

$$h(t) = g(t) + \int_{[0,t]} F(j,j,ds)\, h(t-s)$$

sol $\longrightarrow$ $h(t) = g(t) + \int_{[0,t]} m(ds)\, g(t-s)$ (1)

$\boxed{\text{OR}}$

sol $\longrightarrow$ $h(t) = \int_{[0,t]} R(j,j,ds)\, g(t-s)$ (2)

$\varphi(t) = I_{[0,\infty)}(t)$ what is $\int_{[0,t]} \varphi(ds)\, g(t-s) = g(t)$

$$R(i,i,t) = \sum_{n=0}^{\infty} F_n(i,i,t)$$

for $i \neq j \implies R(i,j,t) = \int_{[0,t]} F(i,j,ds)\,R(j,j,t-s)$

If $E = \{a\}$ and $Q(a,a,t) = 1 - e^{-5t}$ for $t \geq 0$
write out $R(a,a,t)$

$$R(a,a,t) = \begin{cases} 0 & \text{for } t < 0 \\ 1 + 5t & \text{for } t \geq 0 \end{cases}$$

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$$R(i,j,t) = \sum_{n=0}^{\infty} Q^n(i,j,t)$$

$$R = \sum_{n=0}^{\infty} Q^n$$

$$R * Q = \sum_{n=1}^{\infty} Q^n = R - I$$

$$\boxed{R * Q = R - I}$$

Markov renewal type equation

Let fixed, let $t \to g(i,t)$ and $t \to h(i,t)$ be non-negative functions with support on $[0,\infty)$

$$\longrightarrow \quad Q * h(i,t) = \sum_k \int_{[0,t]} Q(i,k,ds)\, h(k,t-s)$$

$$\longrightarrow \quad R * g(i,t) = \sum_k \int_{[0,t]} R(i,k,ds)\, g(k,t-s)$$

Find the collection of functions $t \to h(i,t)$ that satisfy

$$h(i,t) = g(i,t) + Q * h(i,t)$$

for $t \geq 0$ and $i \in E$.

Markov renewal type equation

Solution: $$h(i,t) = \sum_k \int_{[0,t]} R(i,k,ds)\, g(k,t-s)$$

$$h(i,t) = R * g(i,t)$$

for $t \geq 0$ and $i \in E$.

If E is finite, the solution is unique.

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$$Q(t) = \begin{array}{c} a \\ b \end{array} \begin{bmatrix} 0.6\left(1-e^{-5t}\right) & 0.4\left(1-e^{-3t}\right) \\ 0.5\left(1-e^{-2t}\right) & 0.5\left(1-e^{-4t}\right) \end{bmatrix}$$