$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r} \quad \text{for } |r|<1$$

$$\sum_{n=1}^{\infty} n r^{n-1} = \frac{1}{(1-r)^2} \quad \text{for } |r|<1$$

$$\sum_{n=0}^{\infty} \frac{r^n}{n!} = e^r$$

Feb 5-7:59 AM

Renewa Theory

$\{X_n\}$ nonegative i.i.d. with $\mu = \int_{[0,\infty)} x\, F(dx)$

$$S_n = \sum_{k=1}^{n} X_n$$

$$N(t) = \sum_{n=1}^{\infty} I_{[0,t]}(S_n)$$

Renewal function: $m(t) = \sum_{n=1}^{\infty} F_n(t)$

$$m * F(t) = F * m(t) = m(t) - F(t)$$

Feb 5-8:06 AM

Renewal process is periodic with period d if $\{X_n\}$ take values in discrete set $\{0, d, 2d, \cdots\}$ and d is the largest such number; otherwise, if no such $d>0$ exists, then the renewal process is called aperiodic.

Renewal process is recurrent if $\lim_{t\to\infty} F(t) = 1$; otherwise, it is called transient.

Feb 5-8:09 AM

If $\{N(t)\}$ is recurrent aperiodic, then

1) $\lim_{t\to\infty} [m(t+s) - m(t)] = \frac{s}{\mu}$

(Blackwell's Thm)

2) $\lim_{t\to\infty} \frac{m(t)}{t} = \frac{1}{\mu}$

(Elementary renewal Thm)

Feb 5-8:13 AM

Renewal Eq:

Find the function $h(\cdot)$ such that

$$h(t) = g(t) + \int_{[0,t]} F(ds)\, h(t-s)$$

$$h = g + F * h \leftarrow \text{the renewal eq.}$$

Solution: $h(t) = g(t) + \int_{[0,t]} m(ds)\, g(t-s)$

$$\boxed{h = g + m * g}$$

Feb 5-8:15 AM

$$h = g + F * h = g + m * g$$

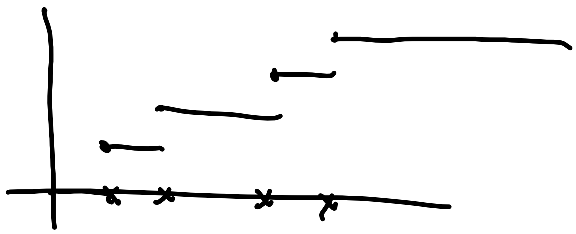
let $h = g + m * g$

$$F * h = F * g + F * m * g = F * g + (m - F) * g$$

$$= \cancel{F * g} + m * g - \cancel{F * g}$$

Feb 5-8:19 AM

$\{N(t)\}$ is transient



let $N = \lim_{t\to\infty} N(t)$

$P\{X_1 \le t\} = F(t)$

$P\{X_1 < \infty\} = \lim_{t\to\infty} F(t) = F(\infty)$

$F(\infty) < 1$

$P\{N = 0\} = (1 - F(\infty))$

$P\{N = 1\} = F(\infty)(1 - F(\infty))$

$P\{N = 2\} = (F(\infty))^2 (1 - F(\infty))$

$\vdots$

$$E[N] = \frac{F(\infty)}{1 - F(\infty)}$$

$$\lim_{t\to\infty} m(t) = F(\infty)/(1 - F(\infty))$$

Feb 5-8:23 AM

Let $\{N(t)\}$ be a transient renewal process.

$(\lim_{t\to\infty} F(t) < 1)$

Define $L = \sup\{S_n : S_n < \infty\}$

$P\{L > t\}$

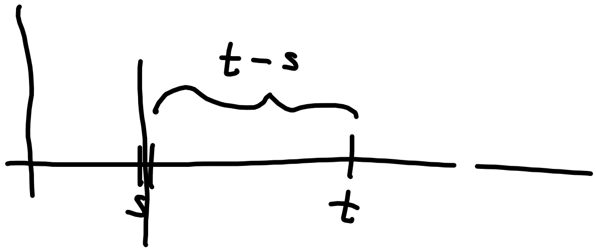
Case I: on the set $\{t < S_1 < \infty\}$ we have $P\{L > t \mid S_1\} = 1$

Case II: on the set $\{S_1 \le t\}$

$\rightarrow$ process is renewed at S_1 with new $\{N^*(t)\}$ with lifetime L^*

$L = S_1 + L^*$

Feb 5-8:37 AM



$$P\{L > t\} = 1\cdot\big(F(\infty) - F(t)\big) + \int_{[0,t]} F(ds)\, P\{L > t-s\}$$

Let $h(t) = P\{L > t\}$

$g(t) = F(\infty) - F(t)$ $\quad h = g + F * h \Rightarrow h = g + m * g$

$$P\{L > t\} = \big(F(\infty) - F(t)\big) + \int_{[0,t]} m(ds)\big[F(\infty) - F(t-s)\big]$$

$$= \big(F(\infty) - F(t)\big) + \int_{[0,t]} m(ds)\, F(\infty) - \int_{[0,t]} m(ds)\, F(t-s)$$

Feb 5-8:43 AM

$$P\{L > t\} = F(\infty) - F(t) + F(\infty)\int_{[0,t]} m(dh) - m * F$$

$$= F(\infty) - \cancel{F(t)} + F(\infty)\, m(t) - m(t) + \cancel{F(t)}$$

(I used fact $m * F = m - F$)

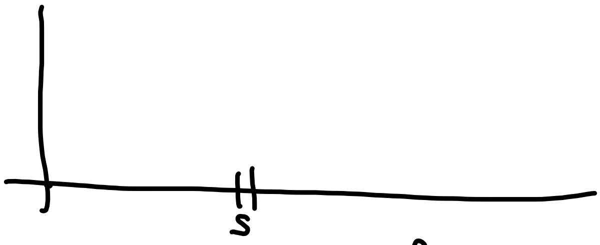
$$P\{L \le t\} = 1 - F(\infty) - F(\infty)\, m(t) + m(t)$$

$$= \big(1 - F(\infty)\big)\big(1 + m(t)\big)$$

$$E[L] = \int_0^\infty P\{L > t\}\, dt$$

Feb 5-8:51 AM

$E[L]$ on set $\{S_1 = \infty\}$ $E[L \mid S_1] = 0$



$$E[L] = 0 \cdot (1 - F(\infty)) + \int_{[0,\infty)} F(ds)\,[s + E[L]]$$

$$= \int_{[0,\infty)} s\,F(ds) + F(\infty)\,E[L]$$

$$E[L] = \left(\int_{[0,\infty)} s\,F(ds)\right) \Big/ \left(1 - F(\infty)\right)$$

Feb 5-8:56 AM

$$\int_{[0,\infty)} s\,F(ds) = \int_{s\in[0,\infty)} F(ds) \int_{u=0}^{s} du$$

$$= \int_{u=0}^{\infty} du \int_{s\in(u,\infty)} F(ds)$$

$$= \int_0^{\infty} [F(\infty) - F(u)]\,du$$

$$E[L] = \frac{\int_0^{\infty} [F(\infty) - F(u)]\,du}{1 - F(\infty)}$$

Feb 5-9:00 AM

Pedestrian Delay

Let $S_1, S_2, \dots$ be the successive instants at which vehicles cross a certain fixed on the highway.

Let $W_n = S_n - S_{n-1}$ be time between vehicles and let $\varphi(t) = P\{W_n \le t\}$.

Assume pedestrian needs τ between cars to cross.

Let $X_n = \begin{cases} W_n & \text{if } W_n \le \tau \\ +\infty & \text{otherwise} \end{cases}$

Feb 5-9:03 AM