Prelim1:

Q1. $f_{X}(\underline{x};\theta) = \prod_{i=1}^{n} 1_{\{-\theta \le X_i \le \theta\}} = \prod_{i=1}^{n} 1_{\{|X_i| \le \theta\}}$

$= 1_{\{\max_i |X_i| \le \theta\}}$

(1) & (2). By the Factorization thm, $\max_{1\le i\le n} |X_i|$ is suff.

also, $\max_{1\le i\le n} |X_i|$ is the MLE.

Since $|X_i| \overset{d}{=} Y_i$, where $\{Y_i\} \sim$ iid Unif $[0,\theta]$.

and $Y_{(n)}$ is complete for $\theta$. (shown in class).

then $\max_{1\le i\le n} |X_i|$ is also complete for $\theta$.

(3). $f_X(\underline{x};\theta) = \prod_{i=1}^{n} 1_{\{-\theta \le X_i \le \theta\}}$.

$= 1_{\{X_{(1)} \ge -\theta\}} 1_{\{X_{(n)} \le \theta\}}$.

$\Rightarrow$ $(X_{(1)}, X_{(n)})$ is also suff.

By (2)., $\max_{1\le i\le n} |X_i|$ is complete & suff $\Rightarrow$ $\max_{1\le i\le n} |X_i|$ is min. suff.

$\Rightarrow$ $(X_{(1)}, X_{(n)})$ is NOT min suff. since it is 2-dim.

(4). Check $E_\theta(X_{(n)} - X_{(1)})$. , if $X_{(n)} - X_{(1)}$ ancillary $\Rightarrow$ $E_\theta(X_{(n)} - X_{(1)})$ not depend on $\theta$.

Note that $X_i \overset{d}{=} Z_i - \theta$, $\{Z_i\} \sim$ iid Unif $[0, 2\theta]$.

$\Rightarrow$ $E_\theta(X_{(n)} - X_{(1)}) = E_\theta(Z_{(n)} - Z_{(1)}) = \frac{n}{n+1} \cdot 2\theta - \frac{1}{n+1} 2\theta = \frac{n-1}{n+1} \cdot 2\theta$,

as $E_\theta(Z_{(n)}) = \frac{n}{n+1} 2\theta$. (pdf shown in class).

$E_\theta(Z_{(1)}) = \int_0^{2\theta} (1 - \frac{z}{2\theta})^n dz = \frac{2\theta}{n+1}$.

$\Rightarrow$ $X_{(n)} - X_{(1)}$ is not ancillary.

(Another way: Unif $[-\theta, \theta]$ is a scale family. but $X_{(n)} - X_{(1)}$ measures location).

$Q2$: (1). $L(\lambda^2; \underline{x}) = \prod_{i=1}^{n} ((x_i)!)^{-1} e^{-i\lambda^2} (i\lambda^2)^{x_i}$.

$\Rightarrow \log L(\lambda^2; \underline{x}) = -\sum_{i=1}^{n} i \cdot \lambda^2 + \sum_{i=1}^{n} x_i \log(i\lambda^2) - \sum_{i=1}^{n} \log(x_i!)$.

$= -\lambda^2 \sum_{i=1}^{n} i + \log(\lambda^2) \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} i x_i - \sum_{i=1}^{n} \log(x_i!)$

$\Rightarrow \frac{\partial}{\partial \lambda^2} \log L(\lambda^2; \underline{x}) = -\sum_{i=1}^{n} i + \sum_{i=1}^{n} x_i \cdot \frac{1}{\lambda^2} \overset{\text{set}}{=} 0.$

$\Rightarrow \hat{\eta} = \frac{\sum_i x_i}{\sum_i i} = \frac{2 \sum x_i}{n(n+1)}$ is the unique solution.

and $\frac{\partial^2}{\partial (\lambda^2)^2} \log L(\lambda^2; \underline{x}) = -\frac{1}{\lambda^4} \sum_{i=1}^{n} x_i < 0$. for $\forall \lambda$.

$\Rightarrow \hat{\eta}$ is the MLE for $\eta \equiv \lambda^2$.

$\Rightarrow$ By the invariance property of MLE: $\hat{\lambda}_{MLE} = \left(\frac{2 \sum_{i=1}^{n} x_i}{n(n+1)}\right)^{1/2}$.

Note that $\mathbb{E}_{\lambda^2}(\hat{\eta}) = \frac{2}{n(n+1)} \sum_{i=1}^{n} i \cdot \lambda^2 = \lambda^2$.

$f(x) := x^2$ is a convex fcn in $x$.

$\Rightarrow$ By Jensen's ineq: $\underset{\underset{\lambda^2}{\parallel}}{\mathbb{E}_\lambda(f(\hat{\lambda}_{MLE}))} \geq f(\mathbb{E}_\lambda(\hat{\lambda}_{MLE}))$

$\Rightarrow \mathbb{E}_\lambda(\hat{\lambda}_{MLE}) \leq \lambda$. ✓

(2). $\mathbb{P}_{\lambda,\mu}(Z > z, W = 0.) = \mathbb{P}_{\lambda,\mu}(X > Y > z)$

$= \int_z^\infty \int_y^\infty \lambda e^{-\lambda x} dx \cdot \mu e^{-\mu y} dy.$

$= \frac{\mu}{\lambda + \mu} e^{-(\lambda+\mu) z}$.

Similarly, $\mathbb{P}_{\lambda,\mu}(Z > z, W = 1) = \frac{\lambda}{\lambda+\mu} e^{-(\lambda+\mu) z}$.

Note that $\mathbb{P}_{\lambda,\mu}(Z > z) = e^{-(\lambda+\mu) z}$. $z \geq 0$.

and $\mathbb{P}_{\lambda,\mu}(W = 0) = \frac{\mu}{\lambda+\mu}$.

$\Rightarrow Z \perp\!\!\!\perp W$.

$\therefore L(\lambda, \mu; \underset{\sim}{z}, \underset{\sim}{w})$.

$$= \prod_{i=1}^{n} (\lambda+\mu) e^{-(\lambda+\mu)z_i} \prod_{i=1}^{n} \left(\frac{\lambda}{\lambda+\mu}\right)^{w_i} \left(\frac{\mu}{\lambda+\mu}\right)^{1-w_i}$$

$\Rightarrow \log L(\lambda, \mu; \underset{\sim}{z}, \underset{\sim}{w})$.

$$= -(\lambda+\mu) \sum_{i=1}^{n} z_i + \log \lambda \sum_{i=1}^{n} w_i + \log \mu \left(n - \sum_{i=1}^{n} w_i\right)$$

$$\Rightarrow \frac{\partial}{\partial \lambda} \log L = -\sum_{i=1}^{n} z_i + \frac{1}{\lambda} \sum_{i=1}^{n} w_i \overset{\text{set}}{=} 0 \Rightarrow \hat{\lambda} = \frac{\bar{w}}{\bar{z}}.$$

$$\frac{\partial}{\partial \mu} \log L = -\sum_{i=1}^{n} z_i + \frac{1}{\mu} \left(n - \sum_{i=1}^{n} w_i\right) \overset{\text{set}}{=} 0 \Rightarrow \hat{\mu} = \frac{1-\bar{w}}{\bar{z}}.$$

Check: $\frac{\partial^2}{\partial \lambda^2} \log L = -\frac{1}{\lambda^2} n \cdot \bar{w} < 0$. $\quad \frac{\partial^2}{\partial \mu^2} \log L = -\frac{1}{\mu^2} n(1-\bar{w}) < 0$.

$\therefore (\hat{\lambda}, \hat{\mu})$ are MLE for $(\lambda, \mu)$.

Q3:

$$\left.\begin{aligned} f(\underset{\sim}{\theta}; \underset{\sim}{\alpha}) &\propto \prod_{i=1}^{k} \theta_i^{\alpha_i - 1} \\ p(\underline{x} \mid \underline{\theta}) &\propto \prod_{i=1}^{k} \theta_i^{x_i} \end{aligned}\right\} \Rightarrow p(\underline{\theta} \mid \underline{x}) \propto \prod_{i=1}^{k} \theta_i^{x_i + \alpha_i - 1}.$$

$\Rightarrow (\underline{\theta} \mid \underline{X}) \sim \text{Dirichlet}(X_1+\alpha_1, \cdots, X_k+\alpha_k)$.

By the given hint,

$(\theta_j \mid \underline{X}) \sim \text{Beta}\left(x_j+\alpha_j, \sum_{i \neq j} (x_i + \alpha_i)\right)$

$$\Rightarrow \mathbb{E}(\theta_j \mid \underline{x}) = \frac{x_j+\alpha_j}{\sum_{i=1}^{k} (x_i+\alpha_i)} = \frac{x_j+\alpha_j}{n + \sum_{i=1}^{k} \alpha_i}$$

is the Bayes estimator.

With data plugged in:

$$\mathbb{E}(\theta_1 \mid \underline{X}) = \frac{256+5}{1000+20} = 0.2559.$$