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A Comparative Theoretical and Computational Study on Robust Counterpart Optimization: I. Robust Linear Optimization and Robust Mixed Integer Linear Optimization

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ABSTRACT: Robust counterpart optimization techniques for linear optimization and mixed integer linear optimization problems are studied in this paper. Different uncertainty sets, including those studied in literature (i.e., interval set; combined interval and ellipsoidal set; combined interval and polyhedral set) and new ones (i.e., adjustable box; pure ellipsoidal; pure polyhedral; combined interval, ellipsoidal, and polyhedral set) are studied in this work and their geometric relationship is discussed. For uncertainty in the left-hand side, right-hand side, and objective function of the optimization problems, robust counterpart optimization formulations induced by those different uncertainty sets are derived. Numerical studies are performed to compare the solutions of the robust counterpart optimization models and applications in refinery production planning and batch process scheduling problem are presented.

1. INTRODUCTION

In many optimization applications, the problem data is assumed to be known with certainty. However, that is seldom the case in practice. Very often, the realistic data are subject to uncertainty due to their random nature, measurement errors, or other reasons. Since the solution of an optimization problem often exhibits high sensitivity to the data perturbations as illustrated by Ben-Tal and Nemirovski,[1] ignoring the data uncertainty could lead to solutions which are suboptimal or even infeasible for practical applications.

Robust optimization belongs to an important methodology for dealing with optimization problems with data uncertainty. In the first stage of this type of method, a deterministic data set is defined within the uncertain space, and in the second stage the best solution which is feasible for any realization of the data uncertainty in the given set is obtained. The corresponding second stage optimization problem is also called *robust counterpart optimization* problem. One major motivation for studying robust optimization is that in many applications the data set is an appropriate notion of parameter uncertainty, e.g., for applications in which infeasibility cannot be accepted at all (e.g., design of engineering structures like bridges considered in Ben-Tal and Nemirovski[2,3]), and for those cases that the parameter uncertainty is not stochastic, or if no distributional information is available.

One of the earliest papers on robust counterpart optimization is related to the work of Soyster,[4] who considered simple perturbations in the data and aimed at finding a reformulation of the original linear programming problem such that the resulting solution would be feasible under all possible perturbations. This approach, however, is the most conservative one since it ensures feasibility against all potential realizations. Thus, it is highly desirable to provide a mechanism to allow trade-off between robustness and performance. To address the issue of overconservatism in worst-case models, Ben-Tal, Nemirovski and co-workers[1,5−7], and El-Ghaoui and co-workers[8,9] independently proposed the ellipsoidal-set-based robust counterpart formulation for dealing with parameter uncertainty within linear and quadratic programming problems. El-Ghaoui and Lebret[8] studied the robust solutions to the uncertain least-squares problems, and El-Ghaoui et al.[9] studied uncertain semidefinite problems. Ben-Tal and Nemirovski[6,7] showed that when the uncertainty sets for a linear constraint are ellipsoids, the robust formulation turns out to be a conic quadratic problem. Ben-Tal et al.[5] considered LP problems where some of the decision variables must be determined before the realization of uncertain data, while the other decision variables can be set after the realization.

The robust optimization formulation introduced for linear programming problems with uncertain linear coefficients was extended by Lin et al.[10] and Janak et al.[11] to mixed integer linear optimization (MILP) problems under uncertainty. They developed the theory of the robust optimization framework for general mixed-integer linear programming problems and considered both bounded and several known probability distributions. The robust optimization framework is later extended by Verderame and Floudas[12] who studied both continuous (general, bounded, uniform, normal) and discrete (general, binomial, Poisson) uncertainty distributions and applied the framework to operational planning problems. The work was further compared with the conditional-value risk based method in Verderame and Floudas.[13] For a recent review on planning and scheduling under uncertainty, the reader is directed to Verderame et al.[14] and for process scheduling under uncertainty to Li and Ierapetritou.[15]

Bertsimas and Sim[16] considered robust linear programming with coefficient uncertainty using an uncertainty set with budgets. In this robust counterpart optimization formulation, a budget parameter is introduced to control the degree of conservatism of the solution. As it will be shown in this paper, this type of robust formulation is based on a combined interval and polyhedral uncertainty set. Bertsimas and co-workers[17] extended and applied a robust optimization framework in the fields of linear and discrete programming. Bertsimas et al.[18] characterized

Received: January 21, 2011
Accepted: July 27, 2011
Revised: July 1, 2011
Published: July 27, 2011

the robust counterpart of a linear programming problem with an uncertainty set described by an arbitrary norm. The ideas of the robust optimization approach in Bertsimas and Sim[16] have also been extended to conic optimization problems in Bertsimas and Sim,[19] and also used by Bertsimas and Thiele[20] to address inventory control problems to minimize total costs.

Kouvelis and Yu[21] proposed a framework for robust discrete optimization, which seeks to find a solution that minimizes the worst case performance under a set of scenarios for the data. Chen and Lin[22] proposed an approximate algorithm to solve the robust design problem in a stochastic-flow network. Atamtürk and Zhang[23] described a two-stage robust optimization approach for solving network flow and design problems with uncertain demand. They generalized the approach to multicommodity network flow and design, and studied applications to lot-sizing and location-transportation problems. Atamtürk[24] introduced alternative formulations to robust mixed 0−1 programming with interval uncertainty objective coefficients. Averbakh[25] proposed a general approach for finding minmax regret solutions for a class of combinatorial problems with interval uncertain objective function coefficients, based on reducing the problem with uncertainty to a set of deterministic problems. Kasperski and Zielinski[26] considered a similar class of problems and presented a polynomial time approximation algorithm. Bertsimas and Sim[17] proposed an approach to address data uncertainty for discrete optimization and network flow problems. They presented an algorithm for the special case of the robust network flow where only the objective uncertainty exists and the problem is a mixed 0−1 problem, and solved the problem by considering a polynomial number of nominal minimum cost flow problems in a modified network.

Chen et al.[27] proposed an asymmetrical uncertainty set that generalizes the symmetric ones. Chen et al.[28] studied the relationship between different conditional value-at-risk (CVaR) bound-based approximations to individual chance constraints and different set-based robust optimization formulations and showed the equivalence between them. Fischetti and Monaci[29] developed a robustness framework denoted as a "light robustness" approach to cope with the issue of overly conservative solutions in robust optimization. They placed a hard upper bound on the objective value and then minimized the degree of infeasibility with a fixed uncertainty set.

As pointed out by Goh and Sim,[30] if the exact distribution of uncertainties is precisely known, optimal solutions to the robust problem would be overly and unnecessarily conservative. Conversely, if the assumed distribution of uncertainties is in fact different from the actual distribution, the optimal solution using a stochastic programming approach may perform poorly. So several recent works aim at bridging the gap between the conservatism of robust optimization and the specificity of stochastic programming, where optimal decisions are sought for the worst-case probability distributions within a family of possible distributions, defined by certain properties such as their support and moments. Specifically, El Ghaoui et al.[31] developed worst-case value-at-risk (VaR) bounds for a robust portfolio selection problem when only the bounds on the means and covariance matrix of the assets are known. Chen et al.[27] introduced directional deviations as an additional means to characterize a family of distributions that were applied by Chen and Sim[32] to a goal-driven optimization problem. Delage and Ye[33] studied distributionally robust stochastic programs where the mean and covariance of the primitive uncertainties are themselves subject to uncertainty. Ben-Tal et al.[34] proposed a framework for robust optimization that relaxes the standard notion of robustness by allowing the decision maker to vary the protection level in a smooth way across the uncertainty set.

In this paper, we present a systematic study of the set-induced robust counterpart optimization techniques for both linear optimization (LP) and mixed integer linear optimization (MILP) problems. The new contributions of the paper are as follows: we have proposed several novel uncertainty sets (i.e., adjustable box; pure ellipsoidal; pure polyhedral; combined interval, ellipsoidal, and polyhedral set) and derived their robust counterparts for both LP and MILP problems; for the first time in the literature, we have discussed the connection among six different uncertainty sets (including those studied in the literature, that is, interval set introduced by Soyster,[4] combined interval and ellipsoidal set introduced by Ben-Tal and Nemirovski,[1] combined interval and polyhedral set introduced by Bertsimas and Sim[16]) and the differences among their corresponding robust counterparts, from both the geometrical point of view and the computational studies.

The paper is organized as follows. In section 2, we introduce the set-induced robust counterpart optimization for general linear and mixed integer linear optimization problems. In section 3, we introduce six different uncertainty sets and discuss their relationship from a geometric point of view. In section 4, we present the detailed robust counterpart formulations under different uncertainty sets for linear constraints and the derivation procedures. In section 5, a numerical example and a refinery production planning example are studied and the different robust counterpart optimization models are compared. In section 6, robust counterparts for mixed integer linear optimization problems are derived. In section 7, a numerical example and an application in process scheduling problem are presented. Finally, conclusions are presented in section 8.

2. UNCERTAINTY SET-INDUCED ROBUST OPTIMIZATION

In set-induced robust optimization, the uncertain data are assumed to be varying in a given uncertainty set, and the aim is to choose the best solution among those "immunized" against data uncertainty, that is, candidate solutions that remain feasible for all realizations of the data from the uncertainty set.

2.1. Robust Linear Optimization. *Motivating Example 1.* Consider the following linear optimization problem:

$$\begin{aligned} \max\quad & 8x_1 + 12x_2 \\ \text{s.t.}\quad & \tilde{a}_{11}x_1 + \tilde{a}_{12}x_2 \le 140 \\ & \tilde{a}_{21}x_1 + \tilde{a}_{22}x_2 \le 72 \\ & x_1, x_2 \ge 0 \end{aligned}$$

Assume that the left-hand side (LHS) constraint coefficients $\tilde{a}_{11}$, $\tilde{a}_{12}$, $\tilde{a}_{21}$, $\tilde{a}_{22}$ are subject to uncertainty and they are defined as follows:

$$\tilde{a}_{11} = 10 + \xi_{11}, \quad \tilde{a}_{12} = 20 + 2\xi_{12},$$
$$\tilde{a}_{21} = 6 + 0.6\xi_{21}, \quad \tilde{a}_{22} = 8 + 0.8\xi_{22}$$

where ξ_{11}, ξ_{12}, ξ_{21}, ξ_{22} are independent random variables. The random variables are distributed in the range $[-1, 1]$ (i.e., the constraint coefficients $\tilde{a}_{11}$, $\tilde{a}_{12}$, $\tilde{a}_{21}$, $\tilde{a}_{22}$ have maximum 10% perturbation around their nominal values 10, 20, 6, 8, respectively). Under the set-induced robust optimization framework, finding a robust solution for the above example means to find the best possible candidate solution such that the feasibility of the constraints is maintained no matter what value the random variables realize within a certain set that belongs to the uncertain space defined by $\xi_{ij} \in [-1,1]$.

In general, consider the following linear optimization problem

$$\begin{aligned} \max \quad & cx \\ \text{s.t.} \quad & \sum_j \tilde{a}_{ij} x_j \le \tilde{b}_i \quad \forall\, i \end{aligned} \tag{2.1}$$

where $\tilde{a}_{ij}$ and $\tilde{b}_i$ represent the true value of the parameters which are subject to uncertainty. Assume that the uncertainty affecting each constraint is independent of each other and consider the ith constraint of the above linear optimization problem where both the LHS constraint coefficients and RHS parameters are subject to uncertainty. Define the uncertainty as follows:

$$\tilde{a}_{ij} = a_{ij} + \xi_{ij}\hat{a}_{ij} \quad \forall\, j \in J_i \tag{2.2a}$$

$$\tilde{b}_i = b_i + \xi_{i0}\hat{b}_i \tag{2.2b}$$

where a_{ij} and b_i represent the nominal value of the parameters; $\hat{a}_{ij}$ and $\hat{b}_i$ represent constant perturbation (which are positive); J_i represents the index subset that contains the variable indices whose corresponding coefficients are subject to uncertainty; and ξ_{i0} and ξ_{ij} $\forall i$, $\forall j \in J_i$ are random variables which are subject to uncertainty. With the above definition, the original ith constraint can be rewritten as

$$\sum_{j \notin J_i} a_{ij} x_j + \sum_{j \in J_i} \tilde{a}_{ij} x_j \le \tilde{b}_i \tag{2.3}$$

which can be further reformulated as follows:

$$\sum_j a_{ij} x_j + \left[-\xi_{i0}\hat{b}_i + \sum_{j \in J_i} \xi_{ij}\hat{a}_{ij} x_j \right] \le b_i \tag{2.4}$$

In the set induced robust optimization method, with a predefined uncertainty set U, the aim is to find solutions that remain feasible for any ξ in the given uncertainty set U so as to immunize against infeasibility, that is,

$$\sum_j a_{ij} x_j + [\max_{\xi \in U} \{ -\xi_{i0}\hat{b}_i + \sum_{j \in J_i} \xi_{ij}\hat{a}_{ij} x_j \}] \le b_i \tag{2.5}$$

Finally, replacing the original constraint in LP problem 2.1 with the corresponding robust counterpart constraints, the robust counterpart of the original LP problem is obtained:

$$\begin{aligned} \max \quad & cx \\ \text{s.t.} \quad & \sum_j a_{ij} x_j + [\max_{\xi \in U} \{ -\xi_{i0}\hat{b}_i + \sum_{j \in J_i} \xi_{ij}\hat{a}_{ij} x_j \}] \le b_i \quad \forall\, i \end{aligned} \tag{2.6}$$

Motivating Example 1 (Continued). Applying the robust counterpart formulation 2.6 to the two constraints of the motivating example 1, their corresponding robust counterpart constraints become

$$10x_1 + 20x_2 + \max_{(\xi_{11}, \xi_{12}) \in U_1} \{\xi_{11} x_1 + 2\xi_{12} x_2\} \le 140$$

$$6x_1 + 8x_2 + \max_{(\xi_{21}, \xi_{22}) \in U_2} \{0.6\xi_{21} x_1 + 0.8\xi_{22} x_2\} \le 72$$

where U_1 and U_2 are predefined uncertainty sets for (ξ_{11}, ξ_{12}) and (ξ_{21}, ξ_{22}), respectively.

2.2. Robust Mixed Integer Linear Optimization. *Motivating Example 2.* Consider the following mixed integer linear optimization problem:

$$\begin{aligned} \max \quad & 3x_1 + 2x_2 - 10y_1 - 5y_2 \\ \text{s.t.} \quad & x_1 + x_2 \le 20 \\ & x_1 + 2x_2 \le 12 \\ & a_{31}x_1 + b_{31}y_1 \le 0 \\ & a_{42}x_2 + b_{42}y_2 \le 0 \\ & x_1 - x_2 \le -4 \\ & 0 \le x_1, x_2 \le 10, \ y_1, y_2 \in \{0, 1\}. \end{aligned}$$

Assume that the left-hand side (LHS) constraint coefficients of the third and the fourth constraints are subject to uncertainty and they are defined as follows:

$$a_{31} = 1 + 0.1\xi_{31}, \ b_{31} = -20 + 2\xi_{33},$$
$$a_{42} = 1 + 0.1\xi_{42}, \ b_{42} = -20 + 2\xi_{44}$$

where ξ_{31}, ξ_{33}, ξ_{42}, ξ_{44} are independent uncertain parameters distributed in the range $[-1,1]$. The robust solution for the problem is among the candidate solutions that remain feasible for all realizations of the data from the uncertainty set. For example, if the uncertainty set is defined as the bounded box with range $[-1,1]$ on each dimension, then the corresponding robust counterpart optimization solution should ensure the feasibility of all the constraints for any possible values of the uncertain parameters and maximize the objective at the same time.

Generally, consider the following mixed integer linear optimization (MILP) problem

$$\begin{aligned} \max \quad & \sum_m c_m x_m + \sum_k d_k y_k \\ \text{s.t.} \quad & \sum_m \tilde{a}_{im} x_m + \sum_k \tilde{b}_{ik} y_k \le \tilde{p}_i \quad \forall\, i \end{aligned} \tag{2.7}$$

where x and y represent the continuous and integer variables, respectively, and $\tilde{a}_{im}$, $\tilde{b}_{ik}$, $\tilde{p}_i$ represent the true value of the parameters which are possibly subject to uncertainty. Considering the ith constraint of the above problem, we assume that the uncertain parameters in the ith constraint are defined as follows:

$$\tilde{a}_{im} = a_{im} + \xi_{im}\hat{a}_{im}, \quad \forall\, m \in M_i \tag{2.8a}$$

$$\tilde{b}_{ik} = b_{ik} + \xi_{ik}\hat{b}_{ik}, \quad \forall\, k \in K_i \tag{2.8b}$$

$$\tilde{p}_i = p_i + \xi_{i0}\hat{p}_i \tag{2.8c}$$

where M_i and K_i represent the subsets that contain the continuous and discrete variable indices whose corresponding coefficients are subject to uncertainty, respectively; a_{im}, b_{ik}, p_i represent the nominal value of the parameters; $\hat{a}_{im}$, $\hat{b}_{ik}$, $\tilde{p}_i$ represent positive constant perturbation; and ξ_{im}, ξ_{ik}, ξ_i are random variables which are subject to uncertainty. With the above definitions, the original ith constraint can be rewritten as follows:

$$\sum_{m \notin M_i} a_{im}x_m + \sum_{k \notin K_i} b_{ik}y_k + \sum_{m \in M_i} \tilde{a}_{im}x_m + \sum_{k \in K_i} \tilde{b}_{ik}y_k \leq \tilde{p}_i \tag{2.9}$$

which after grouping the uncertain part can be further rewritten as

$$\sum_m a_{im}x_m + \sum_k b_{ik}y_k + \{-\xi_{i0}\hat{p}_i + \sum_{m \in M_i} \xi_{im}\hat{a}_{im}x_m + \sum_{k \in K_i} \xi_{ik}\hat{b}_{ik}y_k\} \leq p_i \tag{2.10}$$

With a predefined uncertainty set U for the random variables $\xi = \{\xi_{i0}, \xi_{im}, \xi_{ik}\}$, the objective is to find solutions that remain feasible for any ξ in the set so as to immunize against infeasibility; that is,

$$\sum_m a_{im}x_m + \sum_k b_{ik}y_k + \max_{\xi \in U}\{-\xi_{i0}\hat{p}_i + \sum_{m \in M_i} \xi_{im}\hat{a}_{im}x_m + \sum_{k \in K_i} \xi_{ik}\hat{b}_{ik}y_k\} \leq p_i \tag{2.11}$$

Correspondingly, the robust counterpart of the original MILP problem is obtained by replacing the original ith constraint with its robust counterpart constraint 2.11:

$$\begin{aligned} \max \quad & \sum_m c_m x_m + \sum_k d_k y_k \\ \text{s.t.} \quad & \sum_m a_{im}x_m + \sum_k b_{ik}y_k \\ & + \max_{\xi \in U}\{-\xi_{i0}\hat{p}_i + \sum_{m \in M_i} \xi_{im}\hat{a}_{im}x_m + \sum_{k \in K_i} \xi_{ik}\hat{b}_{ik}y_k\} \leq p_i \quad \forall\, i \end{aligned} \tag{2.12}$$

Motivating Example 2 (Continued). With the application of the robust counterpart formulation 2.12 to the two constraints of motivating example 2 and the realization that there is no RHS uncertainty (i.e., $\hat{p}_i = 0$), their corresponding robust counterpart constraints become

$$x_1 - 20y_1 + \max_{(\xi_{31}, \xi_{33}) \in U_1} \{0.1\xi_{31}x_1 + 2\xi_{33}y_1\} \leq 0$$

$$x_2 - 20y_2 + \max_{(\xi_{42}, \xi_{44}) \in U_2} \{0.1\xi_{42}x_2 + 2\xi_{44}y_2\} \leq 0$$

where U_1 and U_2 are predefined uncertainty sets for (ξ_{31}, ξ_{33}) and (ξ_{42}, ξ_{44}), respectively.

The set induced robust counterpart formulations 2.6 and 2.12 depend on the selection of the uncertainty set U. In the subsequent sections, several different uncertainty sets are studied first and the corresponding robust counterpart optimization formulations are then derived.

3. UNCERTAINTY SETS

As stated in the previous section, the formulation of robust counterpart optimization models is connected with the selection of the uncertainty set. In the sequel, several different uncertainty sets are introduced. For the sake of simplicity, we eliminate the constraint index i in the random vector ξ.

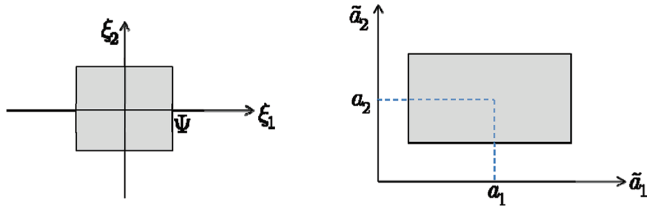


Figure 1. Illustration of box uncertainty set.

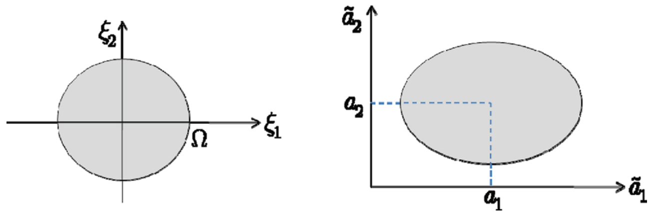


Figure 2. Illustration of ellipsoidal uncertainty set.

Definition 3.1 (Box Uncertainty Set). The box uncertainty set is described using the ∞-norm of the uncertain data vector as follows:

$$U_\infty = \{\xi \,|\, \|\xi\|_\infty \leq \Psi\} = \{\xi \,|\, |\xi_j| \leq \Psi, \ \forall\, j \in J_i\} \tag{3.1}$$

where Ψ is the adjustable parameter controlling the size of the uncertainty set.

Figure 1 illustrates the box uncertainty set for parameter $\tilde{a}_j$ defined by $\tilde{a}_j = a_j + \xi_j\hat{a}_j$, $j = 1{,}2$, where $\tilde{a}_j$ denotes the true value of the parameter, a_j denotes the nominal value of the parameter, ξ_j denotes the uncertainty and $\hat{a}_j$ represents a constant perturbation. If the uncertain parameters are known to be bounded in given intervals $\tilde{a}_{ij} \in [a_{ij} - \hat{a}_{ij}, a_{ij} + \hat{a}_{ij}]$ $\forall j \in J_i$, then the uncertainty can be represented by $\tilde{a}_{ij} = a_{ij} + \xi_j\hat{a}_{ij}$ and this results in the **interval uncertainty set**, which is a special case of box uncertainty set when $\Psi = 1$ (i.e., $U_\infty = \{\xi \,|\, |\xi_j| \leq 1, \forall j \in J_i\}$). Note that in this paper, we specifically use the "interval uncertainty set" to denote the box set with $\Psi = 1$, and use the "box uncertainty set" to represent a general adjustable bounded set.

Definition 3.2 (Ellipsoidal Uncertainty Set). The ellipsoidal uncertainty set is described using the 2-norm of the uncertain data vector as shown in Figure 2,

$$U_2 = \{\xi \,|\, \|\xi\|_2 \leq \Omega\} = \left\{\xi \,\middle|\, \sqrt{\sum_{j \in J_i} \xi_j^{\,2}} \leq \Omega \right\} \tag{3.2}$$

where Ω is the adjustable parameter controlling the size of the uncertainty set. Note that it is known from geometry that for bounded uncertainty $\xi_j \in [-1{,}1]$, when $\Omega \geq (|J_i|)^{1/2}$ (where $|J_i|$ is the cardinality of the set J_i), the entire uncertain space is covered by the ellipsoid uncertainty set.

Definition 3.3 (polyhedral Uncertainty Set). The polyhedral uncertainty set is described using the 1-norm of the uncertain data vector as shown in Figure 3,

$$U_1 = \{\xi \,|\, \|\xi\|_1 \leq \Gamma\} = \left\{\xi \,\middle|\, \sum_{j \in J_i} |\xi_j| \leq \Gamma \right\} \tag{3.3}$$

where Γ is the adjustable parameter controlling the size of the uncertainty set. Note that for bounded uncertainty $\xi_j \in [-1{,}1]$, when $\Gamma \geq |J_i|$, the overall uncertain space is covered by the polyhedral uncertainty set.

The above three uncertainty sets can be further combined to generate new uncertainty sets. Bounded uncertainty is a type of important uncertainty characteristic which is widely studied in practice. We will further introduce several uncertainty sets which are generated by combining the ellipsoid, or polyhedron, or both ellipsoid and polyhedral uncertainty set with the box uncertainty set.

Definition 3.4 ("Box+ellipsoidal" Uncertainty Set). This type of uncertainty set is the intersection between an ellipsoid and a box defined as follows,

$$U_{2\cap\infty} = \{\xi | \sum_{j\in J_i} \xi_j^2 \leq \Omega^2, \quad |\xi_j| \leq \Psi, \quad \forall j \in J_i\} \tag{3.4}$$

It is known from geometry that for an adjustable box defined by 3.1 and an adjustable ellipsoid defined by 3.2, in order to ensure that the intersection between them does not reduce to any one of them, the parameters should satisfy the following relationship:

$$\Psi \leq \Omega \leq \Psi\sqrt{|J_i|} \tag{3.5}$$

Remark 3.1. As $\Psi = 1$, the above set 3.4 defines the intersection between interval and ellipsoid, which is referred as "interval+ellipsoidal" uncertainty set in this paper. This type of uncertainty set is important for bounded uncertainty since it makes no sense to construct an uncertainty set exceeding the bounded uncertain space. For this kind of uncertainty set, when $\Omega = 1$, the ellipsoid is exactly inscribed by the box; when $\Omega = (|J_i|)^{1/2}$, the ellipsoid is circumscribed by the box (i.e., the intersection between the box and ellipsoid is exactly the box). Figure 4 illustrates the geometry of this uncertainty set for the case that the dimension of the uncertain parameter space is 2 (i.e., $|J_i| = 2$).

Definition 3.5 ("Box+Polyhedral" Uncertainty Set). This type of uncertainty set is the intersection between the polyhedral and the interval set defined with both 1-norm and infinite norm as follows,

$$U_{1\cap\infty} = \{\xi | \sum_{j\in J_i} |\xi_j| \leq \Gamma, \quad |\xi_j| \leq \Psi, \quad \forall j \in J_i\} \tag{3.6}$$

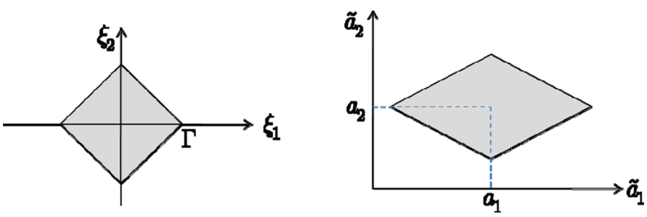


Figure 3. Illustration of polyhedral uncertainty set.

It is also known from geometry that for an adjustable box defined by 3.1 and an adjustable polyhedron defined by 3.3, the intersection between them does not reduce to any one of them if the parameters satisfy the following relationship:

$$\Psi \leq \Gamma \leq \Psi|J_i| \tag{3.7}$$

Remark 3.2. As $\Psi = 1$, the above set defines the intersection between the interval and polyhedral set, which is referred as "interval+polyhedral" uncertainty set. For this uncertainty set, when $\Gamma = 1$, the polyhedron is exactly inscribed by the box and the intersection between the polyhedron and the box is exactly the polyhedron; when $\Gamma = |J_i|$, the intersection between the polyhedron and the box is exactly the box, as shown in Figure 5.

Definition 3.6 ("Box+Ellipsoidal+Polyhedral" Uncertainty Set). This type of uncertainty set is the intersection between the ellipsoidal, polyhedral and box set defined as follows:

$$U_{1\cap 2\cap\infty} = \{\xi | \sum_{j\in J_i} |\xi_j| \leq \Gamma, \quad \sum_{j\in J_i} \xi_j^2 \leq \Omega^2, \quad |\xi_j| \leq \Psi, \quad \forall j \in J_i\} \tag{3.8}$$

For this type of uncertainty set, the intersection between polyhedron and ellipsoid is not reduced to any one of them if the adjustable parameters satisfy the following set of conditions:

$$\Psi \leq \Omega \leq \Psi\sqrt{|J_i|} \tag{3.9a}$$

$$\Omega \leq \Gamma \leq \Omega\sqrt{|J_i|} \tag{3.9b}$$

where the first equation is used to ensure that there is intersection between the ellipsoid and the box, the second equation is used to ensure that there is intersection between the ellipsoid and the polyhedron as shown in Figure 6.

Illustration 3.1. Assume $\tilde{a}_1 = 20 + 2\xi_1$, $\tilde{a}_2 = 10 + \xi_2$, $\xi_1, \xi_2 \in [-1, 1]$, then the corresponding ellipsoidal and polyhedral uncertainty sets for $\tilde{a}$ under different values of Ω and Γ can be illustrated as Figure 7 and Figure 8. From the illustration in Figure 7, it can be observed that when $\Gamma = \Omega$, the polyhedron is inscribed by the ellipsoid. On the other hand, it can be observed from Figure 8 that when $\Gamma = \Omega(|J_i|)^{1/2}$, the ellipsoid is inscribed by the polyhedron, which verifies the analysis in the previous definitions.

The different uncertainty sets are summarized in Table 1. Considering different types of uncertainty characteristics (bounded or unbounded), we also list the suggested range for the adjustable parameter of different uncertainty sets. Based on these definitions of the uncertainty sets, the corresponding robust counterpart optimization formulations for linear optimization problems are derived in the next section.

Remark 3.3

(1) All the parameter values should be non-negative.

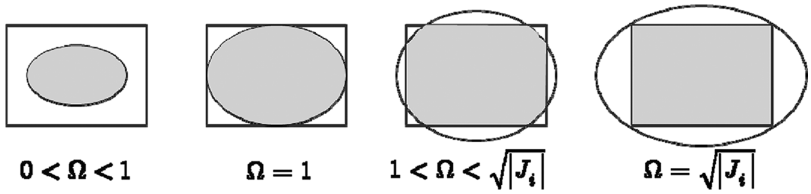


Figure 4. Illustration of the "interval+ellipsoidal" uncertainty set.

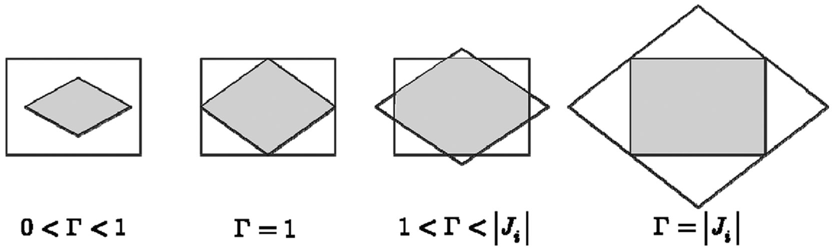


Figure 5. Illustration of combined interval and polyhedral uncertainty set.

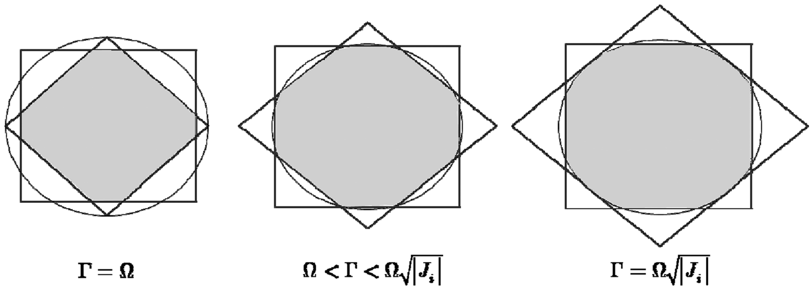


Figure 6. Illustration of combined interval, ellipsoidal and polyhedral uncertainty set.

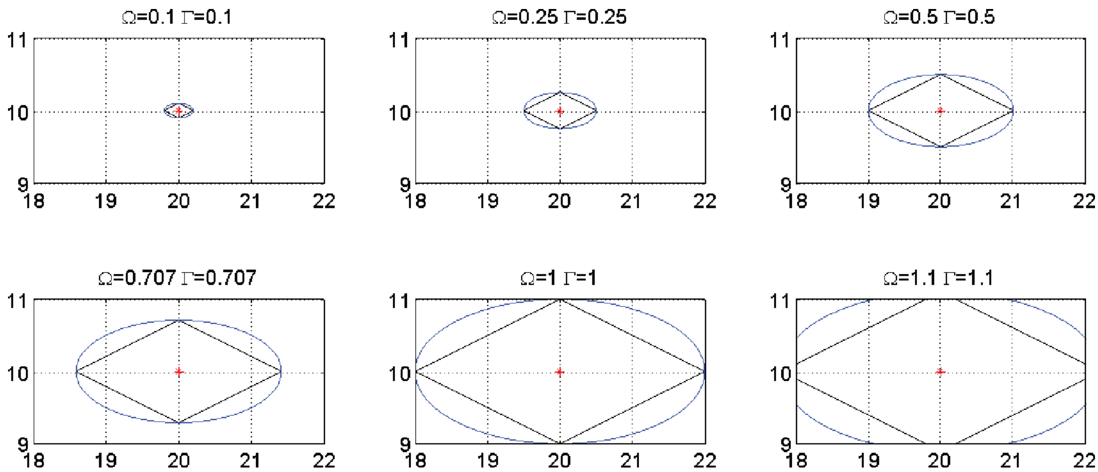


Figure 7. Illustration of the relationship between ellipsoidal and polyhedral uncertainty set ($\Gamma = \Omega$).

(2) The "interval+ellipsoidal", "interval+polyhedral", and "interval+ellipsoidal+polyhedral" uncertainty sets are not suggested for the unbounded uncertainty distribution since we do not want to restrict the set within a given interval.

(3) The suggested parameter range for bounded uncertainty is based on the following: when the adjustable parameter's value is equal to the upper bound given in the table, the bounded uncertain space is entirely covered by the corresponding uncertainty set. Thus, further increase of the value of the parameter could lead to a more conservative solution and will not improve the solution robustness.

(4) The suggested range for unbounded uncertainty is based on that we want to avoid that the intersection between different uncertainty sets is reduced to any one of them.

4. ROBUST COUNTERPART FORMULATIONS FOR LINEAR OPTIMIZATION PROBLEMS

To attain robust solutions, we look for solutions which are feasible for any realization of the uncertain data in a predefined uncertainty set. In the following subsections, we present the derivation procedure of the equivalent robust counterpart optimization models based on formulation 2.6. To eliminate the inner maximization problem in the *i*th constraint of 2.6, we first transform the inner maximization problem into its conic dual, and then incorporate the dual problem into the original constraint.

We will first derive the robust counterpart formulation for LHS only uncertainty of a linear optimization problem, then we will extend it to the RHS only uncertainty, and finally to the case of LHS and RHS uncertainty appearing simultaneously.

4.1. Left Hand Side (LHS) Uncertainty. When only LHS uncertainty is considered in the *i*th constraint of 2.1, the

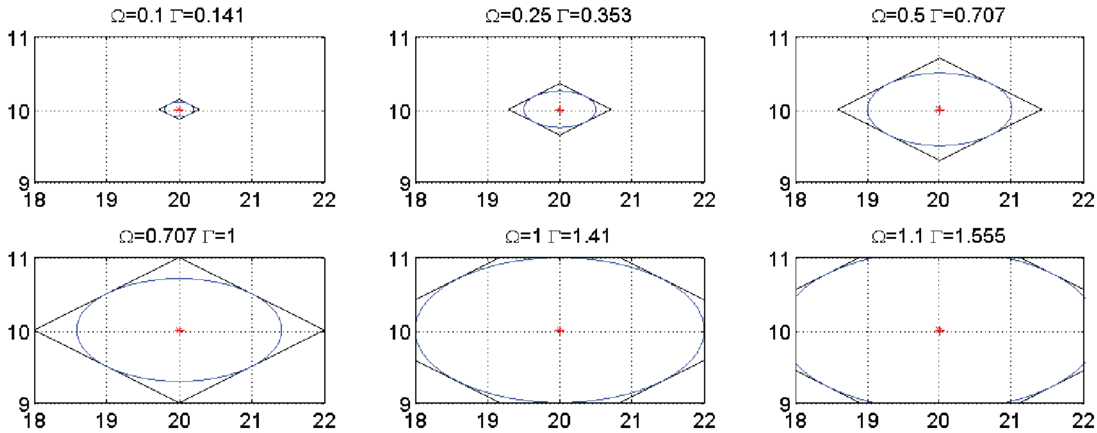
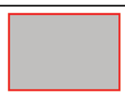
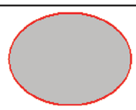
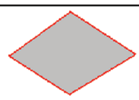
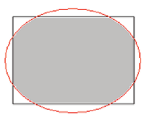
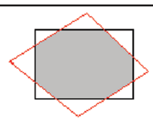
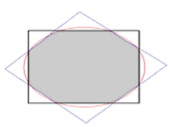


Figure 8. Illustration of the relationship between ellipsoidal and polyhedral uncertainty set ($\Gamma = \Omega\sqrt{2}$).

Table 1. Summary of the Uncertainty Set

Illustration	Type	Adjustable parameter	Suggested range for bounded uncertainty	Suggested range for unbounded uncertainty
	Box	Ψ	$\Psi \leq 1$	$\Psi < \infty$
	Ellipsoidal	Ω	$\Omega \leq \sqrt{\lvert J_i \rvert}$	$\Omega < \infty$
	Polyhedral	Γ	$\Gamma \leq \lvert J_i \rvert$	$\Gamma < \infty$
	Interval+Ellipsoidal	Ω	$\Omega \leq \sqrt{\lvert J_i \rvert}$	
	Box+Ellipsoidal	Ψ, Ω	$\Psi \leq 1, \Psi \leq \Omega \leq \Psi\sqrt{\lvert J_i \rvert}$	$\Psi \leq \Omega \leq \Psi\sqrt{\lvert J_i \rvert}$
	Interval+Polyhedral	Γ	$\Gamma \leq \lvert J_i \rvert$	
	Box+Polyhedral	Ψ, Γ	$\Psi \leq 1, \Psi \leq \Gamma \leq \Psi\lvert J_i \rvert$	$\Psi \leq \Gamma \leq \Psi\lvert J_i \rvert$
	Interval+Ellipsoidal +Polyhedral	Ω, Γ	$\Omega \leq \sqrt{\lvert J_i \rvert}, \Omega \leq \Gamma \leq \Omega\sqrt{\lvert J_i \rvert}$	
	Box+Ellipsoidal +Polyhedral	Ψ, Ω, Γ	$\Psi \leq 1, \Psi \leq \Omega \leq \Psi\sqrt{\lvert J_i \rvert}$ $\Omega \leq \Gamma \leq \Omega\sqrt{\lvert J_i \rvert}$	$\Psi \leq \Omega \leq \Psi\sqrt{\lvert J_i \rvert}$ $\Omega \leq \Gamma \leq \Omega\sqrt{\lvert J_i \rvert}$

corresponding robust counterpart constraint 2.5 for the *i*th constraint is reduced to

$$\sum_j a_{ij}x_j + \left[\max_{\xi \in U}\left\{\sum_{j \in J_i} \xi_{ij}\hat{a}_{ij}x_j\right\}\right] \leq b_i \qquad (4.1)$$

The robust counterpart is derived for different uncertainty sets introduced in section 3 as follows.

Property 4.1. If the set U is the box uncertainty set 3.1, then the corresponding robust counterpart constraint 4.1 is equivalent to the following constraint:

$$\sum_j a_{ij}x_j + \left[\Psi \sum_{j \in J_i} \hat{a}_{ij}|x_j|\right] \leq b_i \qquad (4.2)$$

Proof. For the box uncertainty set $U_\infty = \{\xi \mid |\xi_j| \leq \Psi, \forall j \in J_i\}$, we define $P_\infty = [I_{L\times L}; 0_{1\times L}]$, $p_\infty = [0_{L\times 1}; \Psi]$ and

$K_\infty = \{[\theta_{L\times 1}; t] \in R^{L+1} | \|\theta\|_\infty \le t\}$, where L is the cardinality of the uncertainty set (i.e., $L = |J_i|$). Then the inner maximization problem in 4.1 can be rewritten as

$$\max_{\xi}\{\sum_{j\in J_i} \xi_{ij}\hat{a}_{ij}x_j : P_\infty \xi + p_\infty \in K_\infty\}$$

Defining dual variable $y = [w_i; \tau_i] \in R^{L+1}$ and using the dual cone of K_∞: $K_\infty^* = \{[\theta_{L\times 1}; t] \in R^{L+1} | \|\theta\|_1 \le t\}$, the conic dual of the inner maximization problem can be formulated as

$$\min_{w,\tau}\left\{\Psi\tau_i : w_{ij} = \hat{a}_{ij}x_j \ \forall\ j,\ ||w_i||_1 \le \tau_i\right\}$$

Since the above problem is a minimization problem, it can be further rewritten as the following equivalent formulation by replacing τ_i with $\|w_i\|_1 = \Sigma_{j\in J_i}|w_{ii}|$,

$$\min_{w}\{\Psi \sum_{j\in J_i} |w_{ij}| : w_{ij} = \hat{a}_{ij}x_j \quad \forall\ j\}$$

Realizing that $\tilde{a}_{ij} \ge 0$, we can reformulate the conic dual of the inner maximization problem as follows:

$$\min_{w}\{\Psi \sum_{j\in J_i} |w_{ij}| : w_{ij} = \hat{a}_{ij}x_j\ \forall\ j\} = \Psi \sum_{j\in J_i} |\hat{a}_{ij}x_j| = \Psi \sum_{j\in J_i} \hat{a}_{ij}|x_j|$$

Replacing the original inner maximization problem with the above conic dual, then the following constraint is obtained:

$$\sum_j a_{ij}x_j + [\Psi \sum_{j\in J_i} \hat{a}_{ij}|x_j|] \le b_i$$

Remark 4.1. Constraint 4.2 contains absolute value terms $|x_j|$. If the variable is positive, the absolute value operator can be directly removed. Otherwise, it can be further equivalently transformed to the following constraints because their corresponding feasible sets are identical:

$$\begin{cases} \sum_j a_{ij}x_j + \Psi \sum_{j\in J_i} \hat{a}_{ij}u_j \le b_i \\ |x_j| \le u_j, j \in J_i \end{cases}$$

Thus, the absolute value term in 4.2 can be eliminated and the final equivalent robust formulation is obtained:

$$\begin{cases} \sum_j a_{ij}x_j + \Psi \sum_{j\in J_i} \hat{a}_{ij}u_j \le b_i \\ -u_j \le x_j \le u_j \end{cases} \tag{4.3}$$

Remark 4.2. When $\Psi = 1$ (i.e., the interval uncertainty set), the robust counterpart formulation is reduced to $\Sigma_j a_{ij}x_j + \Sigma_{j\in J_i}\hat{a}_{ij}|x_j| \le b_i$, which is exactly the robust counterpart formulation proposed by Soyster,[4] the so-called "worst case scenario" robust model for bounded uncertainty.

Motivating Example 1 (Continued). Considering the first constraint of motivating example 1,

$$(10 + \xi_{11})x_1 + (20 + 2\xi_{12})x_2 \le 140$$

and assuming that the uncertainty set related to ξ_{11},ξ_{12} is defined by 3.1, the corresponding robust counterpart for this constraint is

$$10x_1 + 20x_2 + \Psi(|x_1| + 2|x_2|) \le 140.$$

The first robust counterpart constraint with a different value of Ψ is illustrated in Figure 9a. It can be observed that as the parameter value Ψ increases (i.e., the size the uncertainty set increases), the feasible set of the resulting robust counterpart optimization problem contracts.

Similarly, for the second constraint of the example, the box uncertainty set induced robust counterpart is

$$6x_1 + 8x_2 + \Psi(0.6|x_1| + 0.8|x_2|) \le 72.$$

Notice that the robust counterpart formulation is constructed constraint by constraint and different parameter values can be applied for different constraints. The complete box uncertainty set induced robust counterpart formulation of this motivating example with different parameters Ψ_1 and Ψ_2 for the two constraints is

$$\begin{aligned} \max\quad & 8x_1 + 12x_2 \\ \text{s.t.}\quad & 10x_1 + 20x_2 + \Psi_1(|x_1| + 2|x_2|) \le 140 \\ & 6x_1 + 8x_2 + \Psi_2(0.6|x_1| + 0.8|x_2|) \le 72 \\ & x_1, x_2 \ge 0 \end{aligned}$$

which is equivalent to the following problem since the variables are positive:

$$\begin{aligned} \max\quad & 8x_1 + 12x_2 \\ \text{s.t.}\quad & 10x_1 + 20x_2 + \Psi_1(x_1 + 2x_2) \le 140 \\ & 6x_1 + 8x_2 + \Psi_2(0.6x_1 + 0.8x_2) \le 72 \\ & x_1, x_2 \ge 0 \end{aligned}$$

Property 4.2. If the set U is the ellipsoidal uncertainty set 3.2, then the corresponding robust counterpart constraint 4.1 is equivalent to the following constraint

$$\sum_j a_{ij}x_j + \left[\Omega\sqrt{\sum_{j\in J_i} \hat{a}_{ij}^2x_j^2}\right] \le b_i \tag{4.4}$$

Proof. Consider the ellipsoidal uncertainty set $U_2 = \{\xi | (\Sigma_{j\in J_i}\xi_j^2)^{1/2} \le \Omega\}$, we define $P_2 = [I_{L\times L}; 0_{1\times L}]$, $I = \mathrm{diag}\{1, ..., 1\}$, $p_2 = [0_{L\times 1}; \Omega]$ and $K_2 = \{[\theta_{L\times 1}; t] \in R^{L+1} | \|\theta\|_2 \le t\}$, then the inner maximization problem in 4.1 can be denoted as

$$\max_{\xi}\{\sum_{j\in J_i} \xi_{ij}\hat{a}_{ij}x_j : P_2\xi + p_2 \in K_2\}$$

Defining the dual variable $y = [z_i; \tau_i] \in R^{L+1}$ and using the dual cone $K_2^* = K_2$, the conic dual of the inner maximization problem is

$$\min\{\Omega\tau_i : z_i = \hat{a}_i x, ||z_i||_2 \le \tau_i\}$$

Since it is a minimization problem, we can make equivalent transformation of above problem by replacing τ_i with $\|z_i\|_2 = (\Sigma_{j\in J_i} z_{ij}^2)^{1/2}$ and get

$$\min_{z}\left\{\Omega\sqrt{\sum_{j\in J_i} z_{ij}^2} : z_i = \hat{a}_i x\right\} = \Omega\sqrt{\sum_{j\in J_i} \hat{a}_{ij}^2 x_j^2}$$

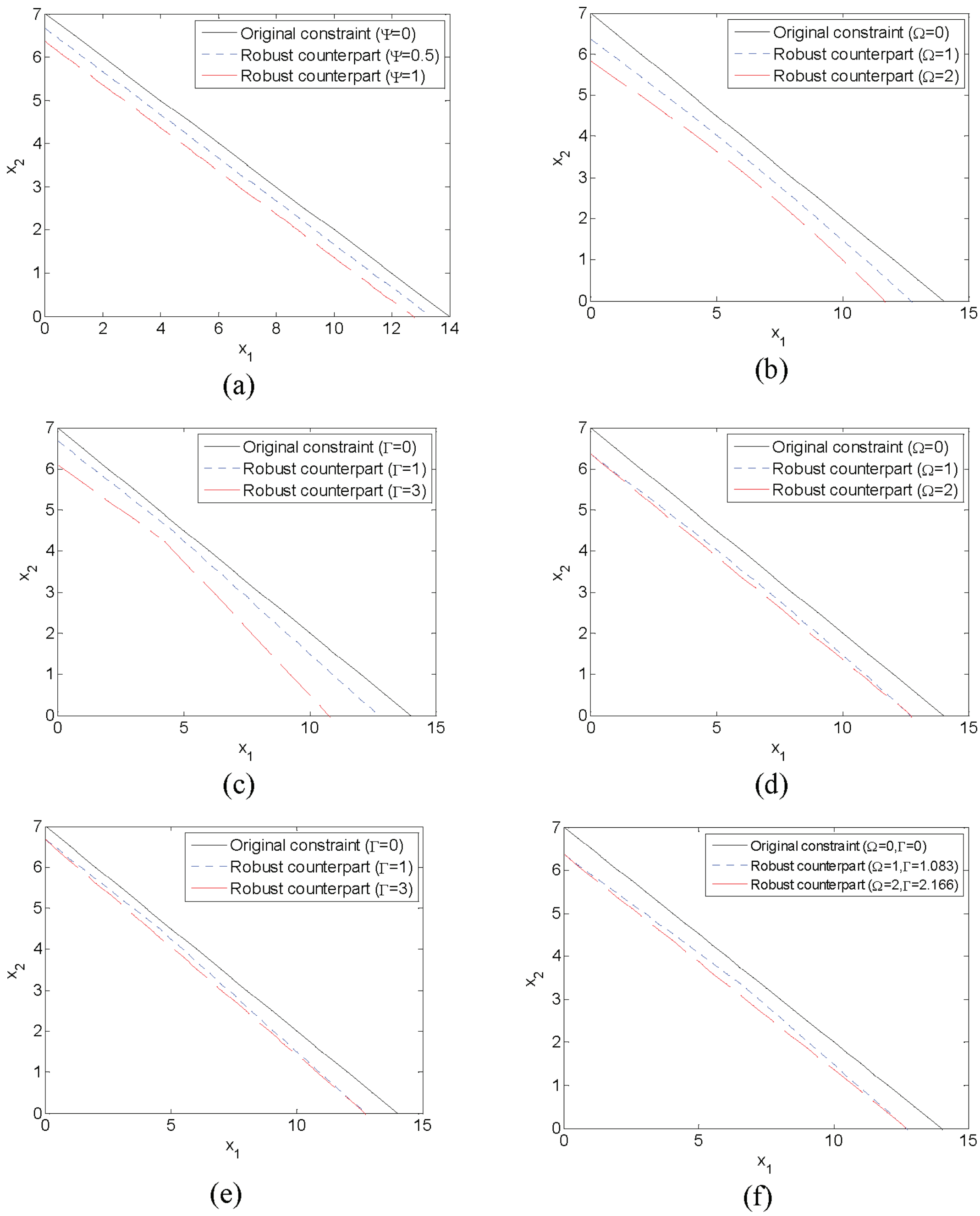


Figure 9. Illustration of the robust counterpart constraint. (a) Box uncertainty set; (b) ellipsoidal uncertainty set; (c) polyhedral uncertainty set; (d) "interval+ellipsoidal" uncertainty set; (e) "interval+polyhedral" uncertainty set; (f) "interval+ellipsoidal+polyhedral" uncertainty set.

After incorporating the above conic dual into the robust counterpart constraint, the following robust counterpart is obtained

$$\sum_j a_{ij}x_j + \left[\Omega\sqrt{\sum_{j\in J_i}\hat{a}_{ij}^2x_j^2}\right] \le b_i$$

Motivating Example 1 (Continued). The corresponding robust constraint for the first constraint of the motivating example 1 is

$$10x_1 + 20x_2 + \Omega\sqrt{x_1^{\,2} + 4x_2^{\,2}} \le 140$$

and its robust counterpart constraint with different value of Ω is illustrated in Figure 9b. It can be observed that as the parameter value Ω increases (i.e., the size the uncertainty set increases), the feasible set of the resulting robust counterpart optimization problem contracts.

Property 4.3. If the set U is the polyhedral uncertainty set 3.3, then the corresponding robust counterpart constraint 4.1 is equivalent to the following constraints

$$\begin{cases} \sum_j a_{ij}x_j + \Gamma p_i \le b_i \\ p_i \ge \hat{a}_{ij}|x_j|, \quad \forall j \in J_i \end{cases} \tag{4.5}$$

Proof. Consider the polyhedral uncertainty set $U_1 = \{\xi | \sum_{j \in J_i} |\xi_j| \leq \Gamma\}$, define $P_1 = [I_{L\times L}; 0_{1\times L}]$, $p_1 = [0_{L\times 1}; \Gamma]$, $K_1 = \{[\theta_{L\times 1}; t] \in R^{L+1} | \|\theta\|_1 \leq t\}$, then the set U_1 can be denoted as $U_1 = \{\xi \mid P_1\xi + p_1 \in K_1\}$ and the inner maximization problem in 4.1 can be denoted as

$$\max_{\xi} \{ \sum_{j \in J_i} \xi_{ij} \hat{a}_{ij} x_j : P_1 \xi + p_1 \in K_1 \}$$

Defining the dual variable $y = [z_i; \tau_i] \in R^{L+1}$ and based on the fact that the dual cone of K_1 is

$$K^*{}_1 = K_\infty = \{[\theta_{L\times 1}; t] \in R^{L+1} | ||\theta||_\infty \leq t\}$$

The conic dual of the inner optimization problem can be formulated as

$$\min_{z,\tau} \{\Gamma \tau_i : z_i = \hat{a}_i x, ||z_i||_\infty \leq \tau_i\}$$

which can be further rewritten as the following equivalent formulation by replacing τ_i with $\|z_i\|_\infty = \max_{j \in J_i} |z_i|$,

$$\min_z \{\Gamma \max_{j \in J_i} |z_{ij}| : z_i = \hat{a}_i x\} = \Gamma \max_{j \in J_i} |\hat{a}_{ij} x_j|$$

Since the above problem is a minimization problem, we can introduce an auxiliary variable p_i to replace $\max_{j \in J_i} |\hat{a}_{ij} x_j|$ and obtain the following equivalent description:

$$\min_z \{\Gamma \max_{j \in J_i} |z_{ij}| : z_i = \hat{a}_i x\} = \Gamma p_i, \quad p_i \geq \hat{a}_{ij} |x_j|, \quad \forall\, j \in J_i$$

Incorporating the above conic dual into the robust counterpart constraint, the following robust counterpart is obtained

$$\begin{cases} \sum_j a_{ij} x_j + \Gamma p_i \leq b_i \\ p_i \geq \hat{a}_{ij} |x_j|, \quad \forall j \in J_i \end{cases}$$

Remark 4.3. As shown in Remark 4.1, an equivalent robust formulation for 4.5 can be obtained by replacing the absolute value term $|x_j|$ with auxiliary variable u_j and constraint $-u_j \leq x_j \leq u_j$ as follows:

$$\begin{cases} \sum_j a_{ij} x_j + \Gamma p_i \leq b_i \\ p_i \geq \hat{a}_i u_j, \quad \forall j \in J_i \\ -u_j \leq x_j \leq u_j, \quad \forall j \in J_i \end{cases} \tag{4.6}$$

Motivating Example 1 (Continued). The corresponding robust counterpart constraint for the first constraint of the motivating example 1 is

$$\begin{cases} 10x_1 + 20x_2 + \Gamma p \leq 140 \\ p \geq |x_1|, \ p \geq 2|x_2| \end{cases}$$

The above robust counterpart for the first constraint with different value of Γ is illustrated in Figure 9c. It can be observed that as the parameter value Γ increases (i.e., the size the uncertainty set increases), the feasible set of the resulting robust counterpart optimization problem contracts.

Property 4.4. If the set U is the "box+ellipsoidal" uncertainty set 3.4, then the corresponding robust counterpart constraint 4.1 is equivalent to the following constraint

$$\sum_j a_{ij} x_j + \left[\Psi \sum_{j \in J_i} \hat{a}_{ij} |x_j - z_{ij}| + \Omega \sqrt{\sum_{j \in J_i} \hat{a}_{ij}^2 z_{ij}^2} \right] \leq b_i \tag{4.7}$$

Proof. The "box+ellipsoidal" uncertainty set $U_{2\cap\infty} = \{\xi | \sum_{j \in J_i} \xi_j^2 \leq \Omega^2, |\xi_j| \leq \Psi, \forall j \in J_i\}$ can be denoted using conic representation as follows,

$$U_{2\cap\infty} = \{\xi | P_2 \xi + p_2 \in K_2, P_\infty \xi + p_\infty \in K_\infty\}$$

where K_2 and K_∞ have the same definition as in the previous proof. Thus the inner maximization problem of 4.1 becomes

$$\max_{\xi} \{ \sum_{j \in J_i} \xi_{ij} \hat{a}_{ij} x_j : P_2 \xi + p_2 \in K_2, P_\infty \xi + p_\infty \in K_\infty \}$$

Defining the dual variable $y^1 = [w_i, \tau_1] \in R^{L+1}$, $y^2 = [z_i, \tau_2] \in R^{L+1}$ and using the dual cone $K_\infty^* = K_1$, $K_2^* = K_2$, the conic dual of the inner maximization problem can be formulated as follows:

$$\min\{\Psi \tau_1 + \Omega \tau_2 : w_i + z_i = \hat{a}_i x, ||w_i||_1 \leq \tau_1, ||z_i||_2 \leq \tau_2\}$$

After an equivalent transformation through replacing τ_1 and τ_2 with $\|w_i\|_1 = \sum_{j \in J_i} |w_{ij}|$ and $\|z_i\|_2 = (\sum_{j \in J_i} z_{ij}^2)^{1/2}$, respectively, we get

$$\min_{z,w} \left\{ \Psi \sum_{j \in J_i} |w_{ij}| + \Omega \sqrt{\sum_{j \in J_i} z_{ij}^2} : w_i + z_i = \hat{a}_i x \right\}$$

which is further equivalent to

$$\min_z \ \Psi \sum_{j \in J_i} |\hat{a}_{ij} x_j - z_{ij}| + \Omega \sqrt{\sum_{j \in J_i} z_{ij}^2}$$

Since z_{ij} are decision variables, we can replace z_{ij} with $\hat{a}_{ij}{}^{z_{ij}}$ and get an equivalent problem:

$$\min_z \ \Psi \sum_{j \in J_i} |\hat{a}_{ij} x_j - \hat{a}_{ij} z_{ij}| + \Omega \sqrt{\sum_{j \in J_i} \hat{a}_{ij}{}^2 z_{ij}{}^2}$$

Incorporation of the above conic dual into the robust counterpart constraint and removal of the minimization operator (it is a equivalent operation since the inner minimization is on the left-hand side of a "less or equal to" constraint) obtains the following robust counterpart:

$$\sum_j a_{ij} x_j + \left[\Psi \sum_{j \in J_i} \hat{a}_{ij} |x_j - z_{ij}| + \Omega \sqrt{\sum_{j \in J_i} \hat{a}_{ij}^2 z_{ij}^2} \right] \leq b_i$$

Remark 4.4. As shown in Remark 4.1, an equivalent robust formulation for 4.7 can be obtained by replacing the absolute value term $|x_j - z_{ij}|$ with auxiliary variable u_{ij} and constraint $-u_{ij} \leq x_j - z_{ij} \leq u_{ij}$ as follows:

$$\begin{cases} \sum_j a_{ij} x_j + \Psi \sum_{j \in J_i} \hat{a}_{ij} u_{ij} + \Omega \sqrt{\sum_{j \in J_i} \hat{a}_{ij}{}^2 z_{ij}{}^2} \leq b_i \\ -u_{ij} \leq x_j - z_{ij} \leq u_{ij} \end{cases} \tag{4.8}$$

Remark 4.5. When $\Psi = 1$ (i.e., the set U is defined as "interval+ellipsoidal" uncertainty set), the corresponding "interval+ellipsoidal" based robust counterpart optimization formulation takes the following form:

$$\begin{cases} \sum_j a_{ij}x_j + \sum_{j \in J_i} \hat{a}_{ij}u_{ij} + \Omega\sqrt{\sum_{j \in J_i} \hat{a}_{ij}{}^2 z_{ij}{}^2} \le b_i \\ -u_{ij} \le x_j - z_{ij} \le u_{ij} \end{cases} \tag{4.9}$$

which is exactly the robust counterpart formulation proposed by Ben-Tal and Nemirovski[1] (i.e., a special case of the combined adjustable box and adjustable ellipsoidal based robust counterpart).

Motivating Example 1 (Continued). The "interval+ellipsoidal" based robust constraint for the first constraint of the motivating example 1 is

$$\begin{cases} 10x_1 + 20x_2 + u_{11} + 2u_{12} + \Omega\sqrt{z_{11}^2 + 4z_{12}^2} \le 140 \\ -u_{11} \le x_1 - z_{11} \le u_{11}, \ -u_{12} \le x_2 - z_{12} \le u_{12} \end{cases}$$

The above constraint can be projected to the space spanned by the x_1, x_2 dimensions by fixing x_1 at different points and maximizing the corresponding x_2. The constraint can be illustrated as shown in Figure 9d. Comparing the robust counterpart constraint illustration Figure 9b and Figure 9d, it can be observed that for $\Omega = 1$, the two robust counterparts are the same, whereas for $\Omega = 2$, the "interval+ellipsoidal" based robust counterpart is less conservative because the resulting optimization feasible set is larger. This is consistent with the fact that as $\Omega \le 1$, the intersection between interval and ellipsoid is exactly the ellipsoid, but as $\Omega > 1$, the intersection between interval and ellipsoid is smaller than the ellipsoid itself.

Property 4.5. If the set U is the "box+polyhedral" uncertainty set 3.6, then the corresponding robust counterpart constraint 4.1 is equivalent to the following constraints

$$\begin{cases} \sum_j a_{ij}x_j + \Psi\sum_{j \in J_i} w_{ij} + \Gamma z_i \le b_i \\ z_i + w_{ij} \ge \hat{a}_{ij}|x_j| \quad \forall j \in J_i \\ z_i \ge 0, w_{ij} \ge 0 \end{cases} \tag{4.10}$$

Proof. The "box+polyhedral" uncertainty set $U_{1\cap\infty} = \{\xi | \sum_{j \in J_i} |\xi_j| \le \Gamma, |\xi_j| \le \Psi, \forall j \in J_i\}$ can be denoted as follows using conic representation:

$$U_{1\cap\infty} = \{\xi | P_1\xi + p_1 \in K_1, P_\infty\xi + p_\infty \in K_\infty\}$$

Defining the dual variable $y^1 = [w_i,\tau_1] \in R^{L\times 1}$, $y^2 = [v_i, \tau_2] \in R^{L\times 1}$ and using the dual cone $K_1^* = K_\infty$, $K_\infty^* = K_1$, the inner maximization problem is rewritten as

$$\max_\xi \left\{ \sum_{j \in J_i} \xi_{ij}\hat{a}_{ij}x_j : P_1\xi + p_1 \in K_1, P_\infty\xi + p_\infty \in K_\infty \right\}$$

The conic dual of the above problem can be formulated as follows

$$\min\{\Psi\tau_1 + \Gamma\tau_2 : w_i + v_i = \hat{a}_i x, ||w_i||_1 \le \tau_1, ||v_i||_\infty \le \tau_2\}$$

We can further get the following equivalent transformation through replacing τ_1 and τ_2 with $\|w\| = \sum_{j\in J_i}|w_{ij}|$ and $\|v\|_\infty = \max_{j\in J_i}|v_{ij}|$, respectively,

$$\min_{w,z}\{\Psi\sum_{j \in J_i}|w_{ij}| + \Gamma\max_{j \in J_i}|v_{ij}| : w_i + v_i = \hat{a}_i x\}$$

Since the above problem is a minimization problem, it can be equivalently transformed to the following problem:

$$\min_{w,z}\{\Psi\sum_{j \in J_i}|w_{ij}| + \Gamma z_i : z_i \ge |\hat{a}_{ij}x_j - w_{ij}|, \quad \forall j \in J_i\}$$

The above problem is further equivalent to the following problem since it is a minimization problem and optimal solution must be obtained with $|w_{ij}| \le |\hat{a}_{ij}x_j|$

$$\min_{w,z}\{\Psi\sum_{j \in J_i}|w_{ij}| + \Gamma z_i : z_i \ge |\hat{a}_{ij}x_j| - |w_{ij}|, \quad \forall j \in J_i, z_i \ge 0\}$$

which is further equivalent to the following problem by replacing $|w_{ij}|$ with w_{ij} and $w_{ij} \ge 0$

$$\min_{w,z}\{\Psi\sum_{j \in J_i} w_{ij} + \Gamma z_i : z_i \ge \hat{a}_{ij}|x_j| - w_{ij}, \ \forall j \in J_i, w_{ij} \ge 0, z_i \ge 0\}$$

After incorporation the above conic dual into the robust counterpart constraint, the following robust counterpart is obtained

$$\begin{cases} \sum_j a_{ij}x_j + \Psi\sum_{j \in J_i} w_{ij} + \Gamma z_i \le b_i \\ z_i + w_{ij} \ge \hat{a}_{ij}|x_j| \quad \forall j \in J_i \\ z_i \ge 0, w_{ij} \ge 0 \end{cases}$$

Remark 4.6. As pointed out in Remark 4.1, an equivalent robust formulation for 4.10 can be obtained by replacing the term $|x_j|$ with auxiliary variable u_j and constraint $-u_j \le x_j \le u_j$ as follows:

$$\begin{cases} \sum_j a_{ij}x_j + \Psi\sum_{j \in J_i} w_{ij} + \Gamma z_i \le b_i \\ z_i + w_{ij} \ge \hat{a}_{ij}u_j, \quad \forall j \in J_i \\ -u_j \le x_j \le u_j, \quad \forall j \in J_i \\ z_i \ge 0, w_{ij} \ge 0 \end{cases} \tag{4.11}$$

Remark 4.7. When $\Psi = 1$ (i.e., the set U is defined as the "interval+polyhedral" uncertainty set), the corresponding robust counterpart optimization formulation becomes

$$\begin{cases} \sum_j a_{ij}x_j + \sum_{j \in J_i} w_{ij} + \Gamma z_i \le b_i \\ z_i + w_{ij} \ge \hat{a}_{ij}u_j, \quad \forall j \in J_i \\ -u_j \le x_j \le u_j, \quad \forall j \in J_i \\ z_i \ge 0, w_{ij} \ge 0 \end{cases} \tag{4.12}$$

which is exactly the robust counterpart proposed by Bertsimas and Sim.[16]

Motivating Example 1 (Continued). The corresponding "interval+polyhedral" based robust counterpart for the first constraint of the motivating example 1 is

$$\begin{cases} 10x_1 + 20x_2 + w_1 + w_2 + \Gamma z \le 140 \\ z + w_1 \ge |x_1|, z + w_2 \ge 2|x_2| \\ z, w_1, w_2 \ge 0 \end{cases}$$

Figure 9e illustrates the projection of the above constraints to the x_1, x_2 dimensions. Comparing the robust counterpart constraint illustration Figure 9c and Figure 9e, it can be observed that for $\Gamma = 1$, the robust counterpart constraint is the same, whereas for $\Gamma = 3$, the "interval+polyhedral" based robust counterpart is less conservative. This is consistent with the fact that as $\Gamma \le 1$, the intersection between interval and polyhedron is exactly the polyhedron, but as $\Gamma > 1$, the intersection between interval and polyhedron is smaller than the polyhedron itself.

Property 4.6. If the set U is the "interval+ellipsoidal+polyhedral" uncertainty set 3.8, then the corresponding robust counterpart constraint 4.1 is equivalent to the following constraints

$$\begin{cases} \sum_j a_{ij}x_j + \left[\sum_{j \in J_i} |p_{ij}| + \Omega\sqrt{\sum_{j \in J_i} w_{ij}^2} + \Gamma z_i\right] \le b_i \\ z_i \ge |\hat{a}_{ij}x_j - p_{ij} - w_{ij}| \quad \forall j \in J_i \end{cases} \tag{4.13}$$

Proof. Consider the "interval+ellipsoidal+polyhedral" uncertainty set

$$U_{1 \cap 2 \cap \infty} = \{\xi | \sum_{j \in J_i} |\xi_j| \le \Gamma, \sum_{j \in J_i} \xi_j^2 \le \Omega^2, |\xi_j| \le 1, \ \forall j \in J_i\}$$

It can be denoted using conic representation as follows,

$$U_{1 \cap 2 \cap \infty} = \{\xi | P_1\xi + p_1 \in K_1, P_2\xi + p_2 \in K_2, P_\infty\xi + p_\infty \in K_\infty\}$$

Defining the dual variable $y^1 = [p_i, \tau_1]$, $y^2 = [w_i, \tau_2]$, $y^3 = [v_i, \tau_3]$ and using the dual cone $K_1^* = K_\infty$, $K_2^* = K_2$, $K_\infty^* = K_1$, the inner maximization problem can be written as

$$\max_{\xi}\{\sum_{j \in J_i} \xi_{ij}\hat{a}_{ij}x_j : P_1\xi + p_1 \in K_1, P_2\xi + p_2 \in K_2, P_\infty\xi + p_\infty \in K_\infty\}$$

The conic dual of the above problem can be formulated as follows

$$\min\{\tau_1 + \Omega\tau_2 + \Gamma\tau_3 : p_i + w_i + v_i = \hat{a}_i x, ||p_i||_1 \le \tau_1, ||w_i||_2 \le \tau_2, ||v_i||_\infty \le \tau_3\}$$

After equivalent transformation through replacing τ_1, τ_2, τ_3 with $||p_i||_1, ||w_i||_2, ||v_i||_\infty$ respectively, we get

$$\min_{p,w,v}\left\{\sum_{j \in J_i} |p_{ij}| + \Omega\sqrt{\sum_{j \in J_i} w_{ij}^2} + \Gamma \max_{j \in J_i} |v_{ij}| : p_i + w_i + v_i = \hat{a}_i x\right\}$$

Replacing $\max_{j \in J_i} |v_{ij}|$ with the auxiliary variable z_i, we get

$$\min_{p,w,z}\left\{\sum_{j \in J_i} |p_{ij}| + \Omega\sqrt{\sum_{j \in J_i} w_{ij}^2} + \Gamma z_i : z_i \ge |\hat{a}_i x_{ij} - p_{ij} - w_{ij}|, \ \forall j \in J_i\right\}$$

Table 2. Robust Counterpart Formulation for the *i*th Linear Constraint with LHS Uncertainty

Uncertainty set	Robust counterpart formulation
Box	$\sum_j a_{ij}x_j + \Psi\left[\sum_{j \in J_i} \hat{a}_{ij}|x_j|\right] \le b_i$
Ellipsoidal	$\sum_j a_{ij}x_j + \left[\Omega\sqrt{\sum_{j \in J_i} \hat{a}_{ij}^2 x_j^2}\right] \le b_i$
Polyhedral	$\begin{cases} \sum_j a_{ij}x_j + z_i\Gamma \le b_i \\ z_i \ge \hat{a}_{ij}|x_j| \quad \forall j \in J_i \end{cases}$
Interval+ Ellipsoidal	$\sum_j a_{ij}x_j + \sum_{j \in J_i} \hat{a}_{ij}|x_j - z_{ij}| + \Omega\sqrt{\sum_{j \in J_i} \hat{a}_{ij}^2 z_{ij}^2} \le b_i$
Interval+ Polyhedral	$\begin{cases} \sum_j a_{ij}x_j + \left[z_i\Gamma + \sum_{j \in J_i} p_{ij}\right] \le b_i \\ z_i + p_{ij} \ge \hat{a}_{ij}|x_j| \quad \forall j \in J_i \\ z_i \ge 0, p_{ij} \ge 0 \end{cases}$
Interval+ Ellipsoidal+ Polyhedral	$\begin{cases} \sum_j a_{ij}x_j + \left[z_i\Gamma + \sum_{j \in J_i} |p_{ij}| + \Omega\sqrt{\sum_{j \in J_i} w_{ij}^2}\right] \le b_i \\ z_i \ge |\hat{a}_{ij}x_j - p_{ij} - w_{ij}| \quad \forall j \in J_i \end{cases}$

Incorporate the above conic dual and removing the minimization operator. The following robust counterpart is then obtained

$$\begin{cases} \sum_j a_{ij}x_j + \left[\sum_{j \in J_i} |p_{ij}| + \Omega\sqrt{\sum_{j \in J_i} w_{ij}^2} + \Gamma z_i\right] \le b_i \\ z_i \ge |\hat{a}_{ij}x_j - p_{ij} - w_{ij}| \ \forall j \in J_i \end{cases}$$

Remark 4.8. An equivalent robust formulation for 4.13 can be obtained by replacing the term $|p_{ij}|$ with auxiliary variable v_{ij} and constraint $-v_{ij} \le p_{ij} \le v_{ij}$, replacing $|\hat{a}_{ij}x_j - p_{ij} - w_{ij}|$ with auxiliary variable u_{ij} and constraint $-u_{ij} \le \hat{a}_{ij}x_j - p_{ij} - w_{ij} \le u_{ij}$ as follows:

$$\begin{cases} \sum_j a_{ij}x_j + \sum_{j \in J_i} v_{ij} + \Omega\sqrt{\sum_{j \in J_i} w_{ij}^2} + \Gamma z_i \le b_i \\ -v_{ij} \le p_{ij} \le v_{ij}, \ \forall j \in J_i \\ -z_i \le \hat{a}_{ij}x_j - p_{ij} - w_{ij} \le z_i, \ \forall j \in J_i \end{cases} \tag{4.14}$$

Motivating Example 1 (Continued). The corresponding "interval+ellipsoidal+polyhedral" uncertainty set induced robust constraint for the first constraint of the motivating example 1 is:

$$\begin{cases} 10x_1 + 20x_2 + |p_1| + |p_2| + \Omega\sqrt{w_1{}^2 + w_2{}^2} + \Gamma z \le 140 \\ z \ge |x_1 - p_1 - w_1|, z \ge |2x_2 - p_2 - w_2| \end{cases}$$

The above constraints are also plotted on the x_1, x_2 dimensions as shown in Figure 9f. Comparing the robust counterpart constraint illustration Figure 9d and Figure 9f, it can be seen that for both $\Omega = 1$ and $\Omega = 2$, the "interval+ellipsoidal" set induced robust counterpart is more conservative than the combined interval, ellipsoidal and polyhedral set induced model because the resulting optimization feasible set is in general larger. This is consistent with the fact that when we further incorporate the polyhedral set

Table 3. Robust Counterpart for the *i*th Linear Constraint with RHS Uncertainty

uncertainty set	robust counterpart formulation
box	$\sum_j a_{ij}x_j + \Psi_i \le b_i$
ellipsoidal	$\sum_j a_{ij}x_j + \Omega_i \le b_i$
polyhedral	$\sum_j a_{ij}x_j + \Gamma_i \le b_i$
interval+ellipsoidal	$\sum_j a_{ij}x_j + \min(\Omega, 1)_i \le b_i$
interval+polyhedral	$\sum_j a_{ij}x_j + \min(\Gamma, 1)_i \le b_i$
interval+ellipsoidal+polyhedral	$\sum_j a_{ij}x_j + \min(\Omega, \Gamma, 1)_i \le b_i$

to construct the uncertainty set, the size of the resulting uncertainty set is actually decreased.

Finally, as a summary to the above derivation, we list the robust counterpart formulations for linear optimization problems with LHS uncertainty as shown in Table 2. Note that in Table 2, we list the "interval+ellipsoidal" based robust counterpart formulation but not "box+ellipsoidal" based model by realizing that the "interval+ellipsoidal" set is important for the bounded uncertainty distribution. Similarly, the "interval+polyhedral" and "interval+ellipsoidal+polyhedral" set induced robust counterpart optimization formulations are listed in the table.

Remark 4.9. For the sake of simplicity, only robust formulations with absolute value terms are listed in Table 2, and equivalent robust formulations after eliminating the absolute value terms can be found via eqs 4.3, 4.6, 4.9, 4.12, and 4.14. In the rest part of the paper, the absolute value term in the other robust counterpart formulations can be eliminated in a similar way.

4.2. Right Hand Side (RHS) Uncertainty. Consider the case that only RHS uncertainty exists in the ith constraint of 2.1 as follows

$$\sum_j a_{ij}x_j \le \tilde{b}_i \tag{4.15}$$

where $\tilde{b} = b_i + \xi_i\hat{b}_i$ and ξ_i is the random variable. Then the robust counterpart for the ith constraint 2.5 is reduced to

$$\sum_j a_{ij}x_j + [\max_{\xi \in U}\{-\xi_i\hat{b}\}] \le b_i \tag{4.16}$$

Property 4.7. For RHS only uncertainty of the ith constraint 4.15, the uncertainty set induced robust counterpart constraint 4.16 is equivalent to the following constraint

$$\sum_j a_{ij}x_j + \Delta\hat{b}_i \le b_i \tag{4.17}$$

where Δ is defined as Ψ, Ω, Γ, min(Ω,1), min(Γ,1), min(Ω, Γ, 1) for the box, ellipsoidal, polyhedral, "interval+ellipsoidal", "interval+polyhedral", and "interval+polyhedral+ellipsoidal" uncertainty sets, respectively.

Proof. Since the dimension of the uncertain space for RHS only uncertainty is 1 (i.e., $|J_i| = 1$), all the previously discussed different uncertainty sets are reduced to 1-dimensional interval sets which can be described as

$$U = \{\xi_i \,|\, |\xi_i| \le \Delta\} \tag{4.18}$$

where Δ is defined as Ψ, Ω, Γ, min(Ω,1), min(Γ,1), min(Ω,Γ,1) for the box, ellipsoidal, polyhedral, "interval+ellipsoidal", "interval+polyhedral", and "interval+polyhedral+ellipsoidal" uncertainty sets, respectively.

Table 4. Robust Counterpart Formulation for the *i*th Linear Constraint with LHS and RHS Uncertainty

Uncertainty set	Robust counterpart formulation
Box	$\sum_j a_{ij}x_j + \Psi\left[\sum_{j\in J_i}\hat{a}_{ij}\lvert x_j\rvert + \hat{b}_i\right] \le b_i$
Ellipsoidal	$\sum_j a_{ij}x_j + \left[\Omega\sqrt{\sum_{j\in J_i}\hat{a}_{ij}^2x_j^2 + \hat{b}_i^2}\right] \le b_i$
Polyhedral	$\begin{cases}\sum_j a_{ij}x_j + z_i\Gamma \le b_i \\ z_i \ge \hat{a}_{ij}\lvert x_j\rvert \quad \forall j \in J_i, \qquad z_i \ge \hat{b}_i\end{cases}$
Interval+ Ellipsoidal	$\sum_j a_{ij}x_j + \left[\sum_{j\in J_i}\hat{a}_{ij}\lvert x_j - z_{ij}\rvert + \hat{b}_i\lvert 1 + z_{i0}\rvert + \Omega\sqrt{\sum_{j\in J_i}\hat{a}_{ij}^2z_{ij}^2 + \hat{b}_i^2z_{i0}^2}\right] \le b_i$
Interval+ Polyhedral	$\begin{cases}\sum_j a_{ij}x_j + \left[z_i\Gamma + \sum_{j\in J_i}p_{ij} + p_{i0}\right] \le b_i \\ z_i + p_{ij} \ge \hat{a}_{ij}\lvert x_j\rvert \quad \forall j \in J_i, \quad z_i + p_{i0} \ge \hat{b}_i \\ z_i \ge 0, p_{ij} \ge 0, p_{i0} \ge 0\end{cases}$
Interval+ Ellipsoidal+ Polyhedral	$\begin{cases}\sum_j a_{ij}x_j + \left[z_i\Gamma + \sum_{j\in J_i}\lvert p_{ij}\rvert + \lvert p_{i0}\rvert + \Omega\sqrt{\sum_{j\in J_i}w_{ij}^2 + w_{i0}^2}\right] \le b_i \\ z_i \ge \lvert\hat{a}_{ij}x_j - p_{ij} - w_{ij}\rvert \quad \forall j \in J_i, \qquad z_i \ge \lvert\hat{b}_i + p_{i0} + w_{i0}\rvert\end{cases}$

Incorporating auxiliary variables x_0 and a constraint $x_0 =- 1$, the constraint 4.16 can be rewritten as

$$\sum_j a_{ij}x_j + [\max_{\xi \in U}\{\xi_i\hat{b}x_0\}] \le b_i$$

With the above reformulation and following the derivation process for box uncertainty set of LHS uncertainty, the corresponding robust counterpart formulation is obtained

$$\sum_j a_{ij}x_j + \Delta\hat{b}_i|x_0| \le b_i$$

Notice that $x_0 = -1$, so the above constraint is reduced to

$$\sum_j a_{ij}x_j + \Delta\hat{b}_i \le b_i$$

which is the robust counterpart for RHS only uncertainty for linear optimization problem.

Finally, the robust counterpart formulations for different uncertainty sets are summarized in Table 3. From this analysis, it is observed that for RHS only uncertainty of a linear constraint, there is no difference in defining different uncertainty sets since all of them reduce to a simple interval.

4.3. Simultaneous LHS and RHS Uncertainty. Let us consider the more general case where uncertainty appears on both the LHS and the RHS of the ith constraint:

$$\sum_{j \notin J_i} a_{ij}x_j + \sum_{j \in J_i} \tilde{a}_{ij}x_j \le \tilde{b}_i \tag{4.19}$$

Similarly, through incorporating auxiliary variable x_0 and a constraint $x_0 = -1$, moving the RHS to the LHS, the above constraint can be rewritten as

$$\sum_{j \notin J_i} a_{ij}x_j + \sum_{j \in J_i} \tilde{a}_{ij}x_j + \tilde{b}_ix_0 \le 0 \tag{4.20}$$

Thus, the robust counterpart formulations for simultaneous LHS and RHS uncertainty can be derived using the same procedure as shown in section 4.1, and they are summarized in Table 4. For a detailed derivation procedure, the reader is directed to Appendix A.

Remark 4.10. The objective max cx can be equivalently transformed as follows:

$$\begin{aligned} &\max\ z \\ &\text{s.t.} \quad z - cx \leq 0 \end{aligned} \tag{4.21}$$

Thus, the uncertainty in the objective coefficient $\tilde{c}$ can be treated as uncertainty in the following type of constraints

$$z - \tilde{c}x \leq 0 \tag{4.22}$$

Hence, the complete robust counterpart formulations for uncertainty in LHS, RHS, and objective function are obtained.

5. COMPUTATIONAL STUDIES FOR ROBUST LINEAR OPTIMIZATION

Example 5.1. Consider the following linear optimization problem

$$\begin{aligned} \max \quad & c_1x_1 + c_2x_2 \\ \text{s.t.} \quad & a_{11}x_1 + a_{12}x_2 \leq b_1 \\ & a_{21}x_1 + a_{22}x_2 \leq b_2 \\ & x_1, x_2 \geq 0 \end{aligned}$$

where

$$\begin{bmatrix} c_1 & c_2 \end{bmatrix} = \begin{bmatrix} 8 & 12 \end{bmatrix}, \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 10 & 20 \\ 6 & 8 \end{bmatrix}, \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 140 \\ 72 \end{bmatrix}$$

The uncertain version of the this LP problem can be described as the following problem:

$$\begin{aligned} \max \quad & \tilde{c}_1x_1 + \tilde{c}_2x_2 \\ \text{s.t.} \quad & \tilde{a}_{11}x_1 + \tilde{a}_{12}x_2 \leq \tilde{b}_1 \\ & \tilde{a}_{21}x_1 + \tilde{a}_{22}x_2 \leq \tilde{b}_2 \\ & x_1, x_2 \geq 0 \end{aligned}$$

where the possible uncertainty is related to the left-hand side (LHS) constraint coefficients $\tilde{a}_{11}$, $\tilde{a}_{12}$, $\tilde{a}_{21}$, $\tilde{a}_{22}$, the right-hand side (RHS) parameter $\tilde{b}_1$, $\tilde{b}_2$ and the objective (OBJ) coefficients $\tilde{c}_1$, $\tilde{c}_2$. Here we define the uncertainty as follows:

$$\tilde{c}_j = c_j + \hat{c}_j\xi_{i0}, \quad j = 1, 2$$

$$\tilde{a}_{ij} = a_{ij} + \hat{a}_{ij}\xi_{ij}, \quad i = 1, 2, \quad j = 1, 2$$

$$\tilde{b}_i = b_i + \hat{b}_i\xi_i, \quad i = 1, 2$$

where $\hat{a}_{ij} = 0.1a_{ij}$, $\hat{b}_i = 0.1b_i$, $\hat{c}_j = 0.1c_j$ represent constant perturbation around their nominal values and ξ_{10}, ξ_{20}, ξ_{11}, ξ_{12}, ξ_{21}, ξ_{22}, ξ_1, ξ_2 are independent random variables.

When we only consider the LHS uncertainty, the different uncertainty set induced robust counterparts can be formulated as shown in section 4.1. For example, the ellipsoidal uncertainty set based robust counterpart is

$$\begin{aligned} \max \quad & 8x_1 + 12x_2 \\ \text{s.t.} \quad & 10x_1 + 20x_2 + \Omega\sqrt{x_1{}^2 + 4x_2{}^2} \leq 140 \\ & 6x_1 + 8x_2 + \Omega\sqrt{0.36x_1{}^2 + 0.64x_2{}^2} \leq 72 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Note that the same uncertainty set parameter value Ω is applied for both constraints here. In the sequel, this will be similarly applied for the rest cases without further explanation. The solution of the different uncertainty set induced robust counterparts is shown in Figure 10a. Figure 10b illustrates the relationship among the "interval+ellipsoidal", "interval+polyhedral", and "interval+ellipsoidal+polyhedral" models (based on LHS+RHS uncertainty for both constraints).

Considering only the RHS uncertainty, the different uncertainty set induced robust counterparts can be formulated as shown in section 4.2. For example, the ellipsoidal uncertainty set based robust counterpart is as follows and the solution of the different robust counterparts are shown in Figure 11.

$$\begin{aligned} \max \quad & 8x_1 + 12x_2 \\ \text{s.t.} \quad & 10x_1 + 20x_2 + 14\Omega \leq 140 \\ & 6x_1 + 8x_2 + 7.2\Omega \leq 72 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Considering LHS and RHS uncertainty simultaneously, the ellipsoidal uncertainty set based robust counterpart is as follows and the solution is shown in Figure 12.

$$\begin{aligned} \max \quad & 8x_1 + 12x_2 \\ \text{s.t.} \quad & 10x_1 + 20x_2 + \Omega\sqrt{x_1^2 + 4x_2{}^2 + 196} \leq 140 \\ & 6x_1 + 8x_2 + \Omega\sqrt{0.36x_1^2 + 0.64x_2{}^2 + 51.84} \leq 72 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Considering LHS, RHS, and OBJ uncertainty simultaneously, we first equivalently transform the objective uncertainty into constraint uncertainty as eq 4.22 and then the different uncertainty set induced robust counterparts can be derived based on the simultaneous LHS and RHS uncertainty. For example, the ellipsoidal uncertainty set based robust counterpart is as follows and the solution for simultaneous LHS, RHS, and OBJ uncertainty is shown in Figure 13.

$$\begin{aligned} \max \quad & z \\ \text{s.t.} \quad & z - 8x_1 - 12x_2 + \Omega\sqrt{0.64x_1{}^2 + 1.44x_2{}^2} \leq 0 \\ & 10x_1 + 20x_2 + \Omega\sqrt{x_1{}^2 + 4x_2{}^2 + 196} \leq 140 \\ & 6x_1 + 8x_2 + \Omega\sqrt{0.36x_1{}^2 + 0.64x_2{}^2 + 51.84} \leq 72 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Based on the solution of the different cases of uncertainties, the following remarks can be made:

(1) It can be observed from Figures 10a, 12a, and 13a that for the ellipsoidal set based robust counterpart, when $\Omega \leq 1$, the ellipsoidal and "interval+ellipsoidal" has the same solution because the corresponding uncertainty sets are the same; when $\Omega \geq (|J_i|)^{1/2}$, the "interval+ellipsoidal" solution reaches the worst case and does not decrease anymore because the "interval+ellipsoidal" uncertainty

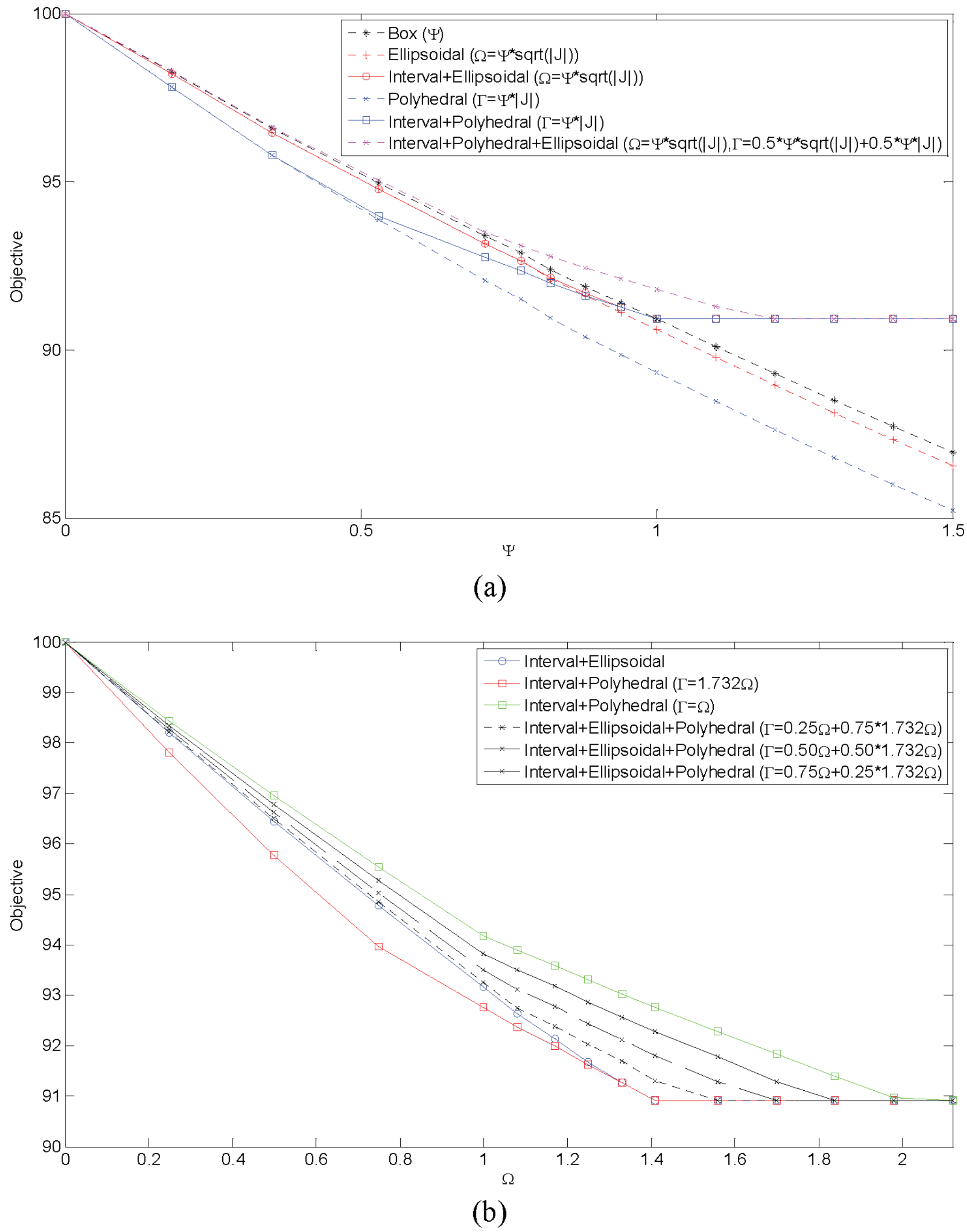


Figure 10. Only LHS uncertainty for both constraints ($|J_i| = 2$).

set is exactly the interval and does not change. For the polyhedral set induced robust counterpart, when $\Gamma \leq 1$, the polyhedral and "interval +polyhedral" set induced models have the same solution; when $\Gamma \geq |J_i|$, the "interval+polyhedral" solution reaches the worst case and does not decrease anymore. It can be concluded from those results that for bounded uncertainty, the uncertainty set should be combined with the interval to avoid conservative solutions.

(2) Comparing the "interval+ellipsoidal" and the "interval +polyhedral" set based model from Figures 10a, 12a, and 13a, when $\Gamma = \Omega(|J_i|)^{1/2}$, the "interval+polyhedral" set based solution is always worse than the "interval +ellipsoidal" based solution, which is verified by the fact that the "interval+polyhedral" uncertainty set is larger and completely covers the "interval+ellipsoidal" set; when $\Gamma = \Omega$, the "interval+polyhedral" set based solution is always better than the "interval+ellipsoidal" based solution because the "interval+polyhedral" uncertainty set is smaller and completely covered by the "interval+ellipsoidal" set.

(3) Comparing the "interval+ellipsoidal+polyhedral" set based model with others from Figures 10b, 12b, and 13b, for every Ω value, we adjust the value of Γ between Ω and $\Omega(|J_i|)^{1/2}$ and test three different values of Γ (as explained in section 3, only when $\Omega \leq \Gamma \leq \Omega(|J_i|)^{1/2}$, the intersection between the ellipsoidal and polyhedral set does not reduce to any one of them). It can be observed

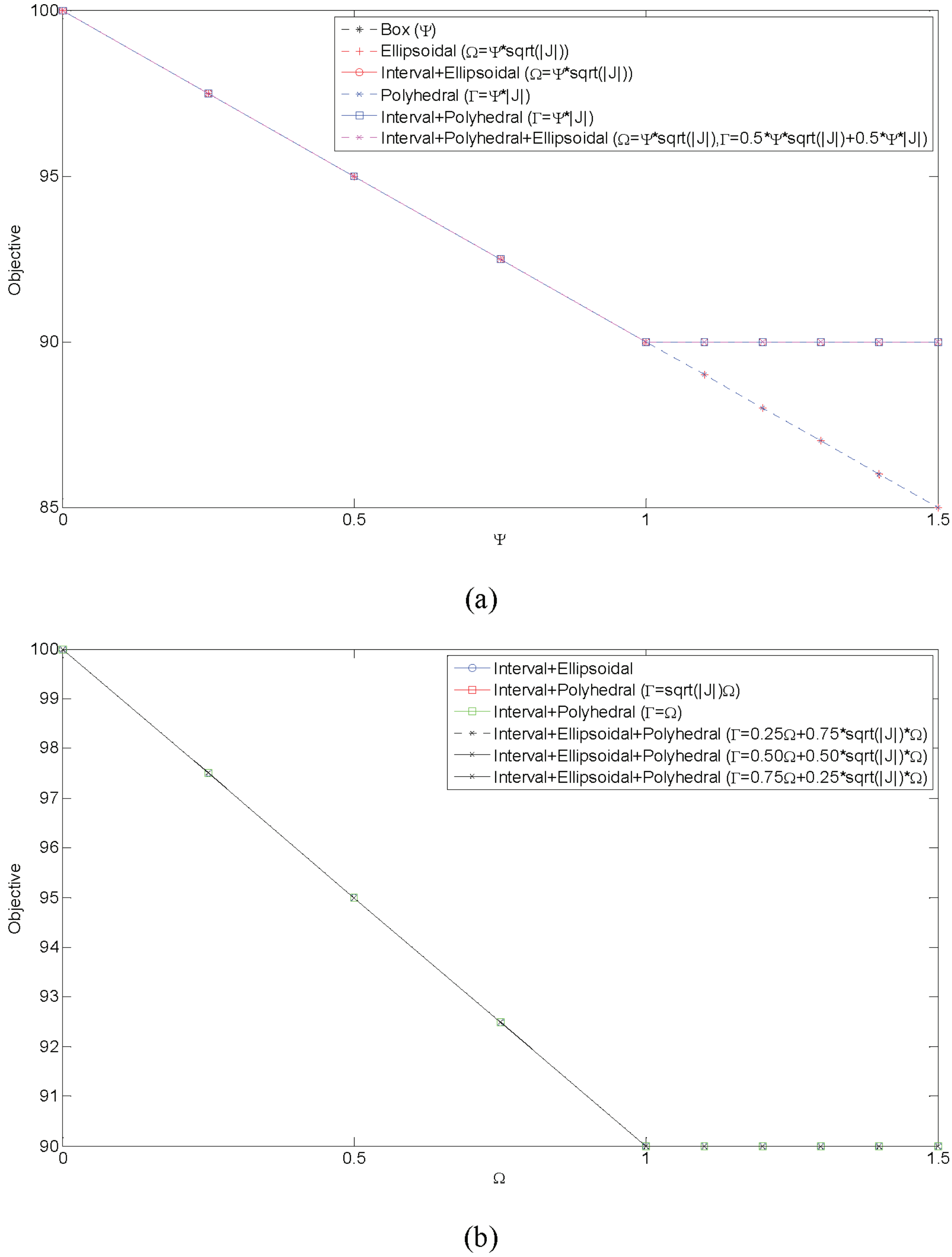


Figure 11. Only RHS uncertainty for both constraints ($|J_i| = 1$).

that as the value of Γ increases from Ω to $\Omega(|J_i|)^{1/2}$, the "interval+ellipsoidal+polyhedral" set based solution switches from the "interval+polyhedral" set based solution with $\Gamma = \Omega$ to the "interval+ellipsoidal" based solution with $\Gamma = \Omega(|J_i|)^{1/2}$, because the intersection between the ellipsoid and polyhedron is exactly changing from the polyhedron with $\Gamma = \Omega$ to the ellipsoid with parameter $\Gamma = \Omega(|J_i|)^{1/2}$.

(4) For RHS only uncertainty, which is a special case where the number of uncertain parameters for every constraint is 1, the solution is identical for ellipsoidal and polyhedral set induced models, and also for the "interval+ellipsoidal" and "interval+polyhedral" uncertainty sets as shown in Figure 11. Furthermore, as $\Omega \leq 1$ and $\Gamma \leq 1$, all the solutions are identical. This is consistent with the definition of the corresponding uncertainty set: as $\Omega = \Gamma \leq 1$; the four types of uncertainty sets are actually the same interval set.

Example 5.2 Refinery Production Planning Problem. Petroleum refinery production planning involves several types of uncertainty, such as prices and product demands. The refinery topology shown in Figure 14 and the operational planning model

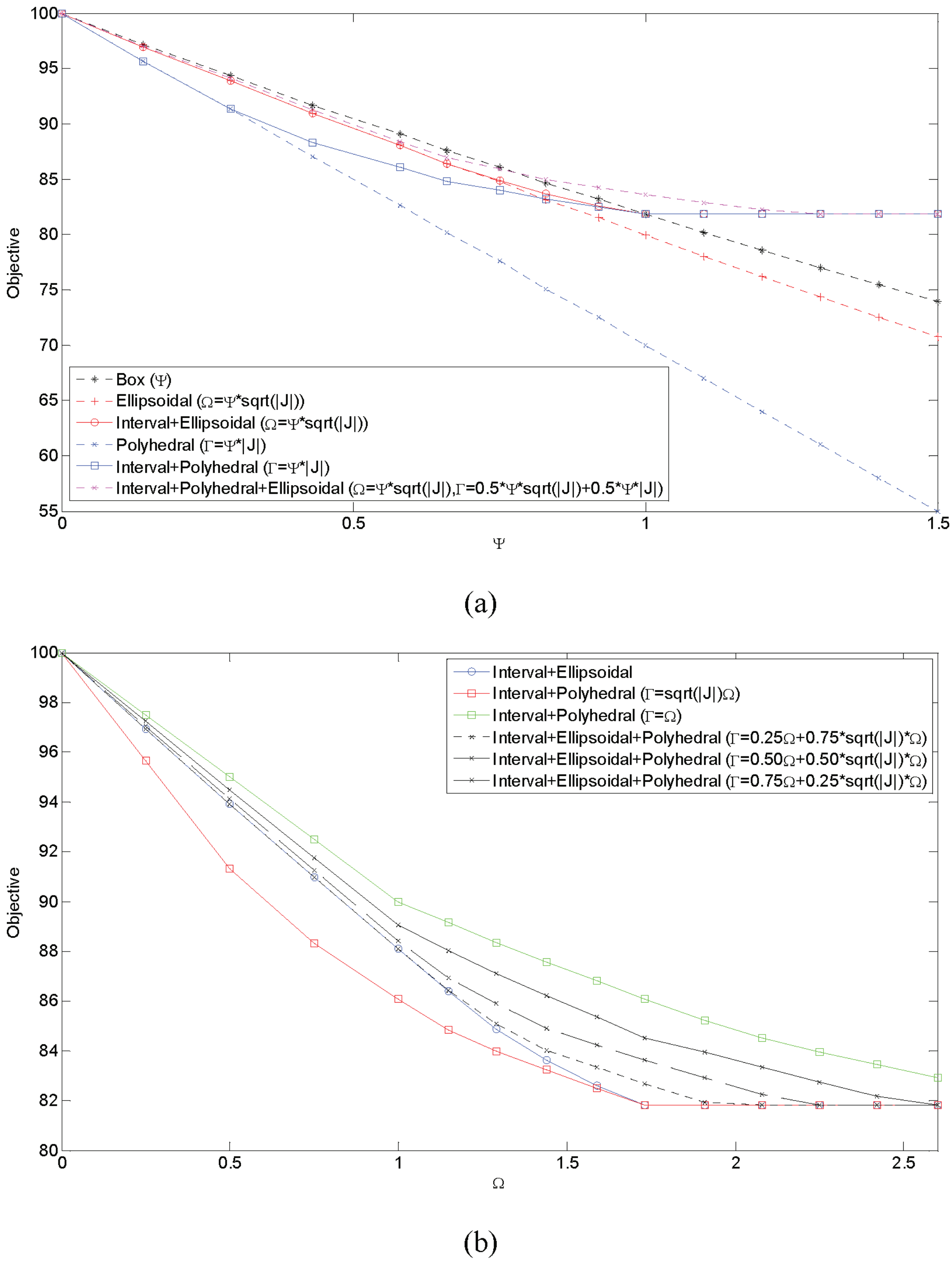


Figure 12. Simultaneous LHS and RHS uncertainty for both constraints ($|J_i| = 3$).

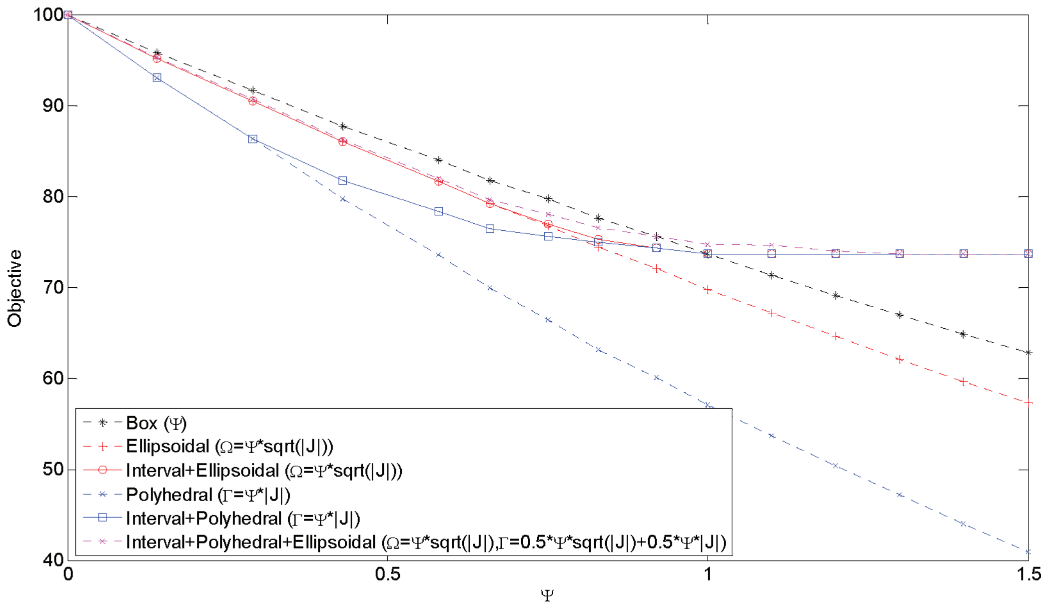
originally proposed by Alen[35] are used. Leiras et al.[36] illustrated the application of robust optimization framework which is based on the "interval+polyhedral" uncertainty set induced robust optimization methodology proposed by Bertsimas and Sim.[16]

In this example, the refinery includes three units: primary distillation unit (PDU), cracking, and blending. It processes crude oil (x_1) to produce gasoline (x_2), naphtha (x_3), jet fuel (x_4), heating oil (x_5), and fuel oil (x_6), where x_7— x_{20} are intermediary streams. The objective function maximizes the profit, which considers the crude oil cost and operating cost of the distillation and cracker units. Constraints include the production yield, fixed proportion blending, production balances, and production requirements. The deterministic model and the definitions of variables and parameters are shown as follows.

$$\max \sum_{t \in T} \sum_{j \in J^{\text{prod}}} p_{jt} x_{jt} - \sum_{t \in T} \sum_{j \in J^{\text{feed}}} c_{jt} x_{jt} \tag{5.1a}$$

s.t.

$$x_{jt} \le cap_{jt} \quad \forall\, j \in J^{\text{feed}},\ t \in T \tag{5.1b}$$



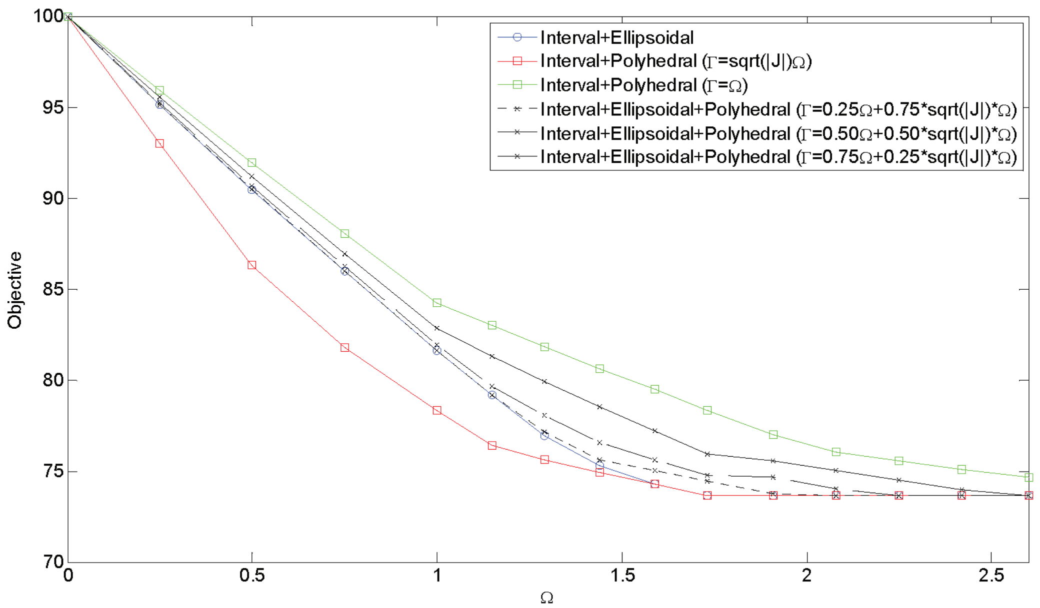


Figure 13. Simultaneous LHS, RHS and OBJ uncertainty ($|J_i| = 3$).

$$x_{jt} \leq \sum_i \eta_{ij} x_{it} \quad \forall\ i \in I,\ t \in T \tag{5.1c}$$

$$x_{it} \leq \sum_j \sigma_{ij} x_{jt} \quad \forall\ i \in I,\ t \in T \tag{5.1d}$$

$$x_{it} \leq \sum_j \alpha_{ij} x_{jt} \quad \forall\ i \in I,\ t \in T \tag{5.1e}$$

$$x_{jt} \leq \text{prod}_{jt} \quad \forall\ i \in I,\ t \in T \tag{5.1f}$$

$$x_{it} \geq 0 \quad \forall\ i \in I,\ t \in T \tag{5.1g}$$

where eq 5.1a represents the profit objective, eq 5.1b is plant capacity constraint, eq 5.1c is production yield constraint, eq 5.1d is fixed proportion blending constraints, eq 5.1e is production balance constraint, and eq 5.1f is the production demand constraint.

The uncertain parameters we focus on are the cost c_{jt}, the prices of products p_{jt}, the yields η_{ij}, and the demands prod_{jt}. We assume that those parameters are subject to bounded uncertainty and that there exists a maximum of 10% deviation of cost and

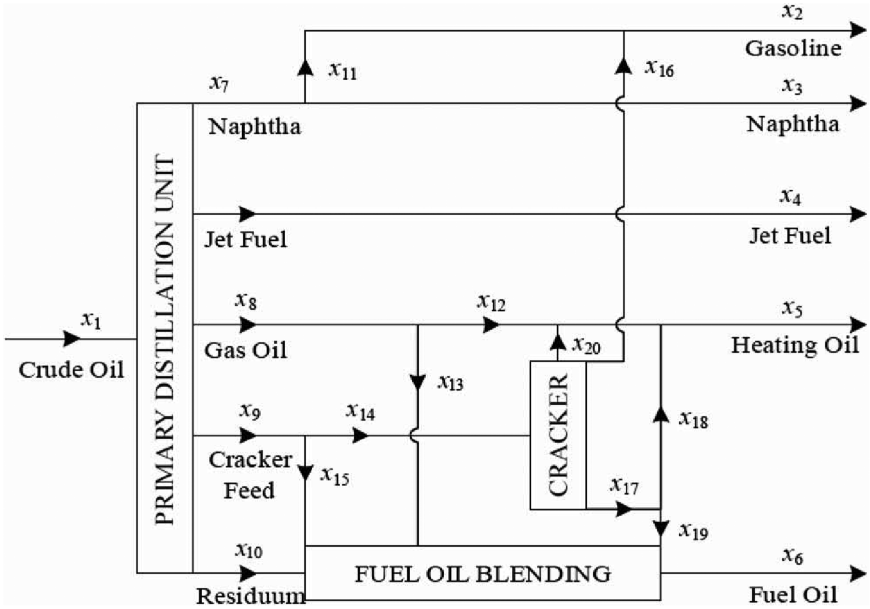


Figure 14. Refinery flowchart.

price coefficients, 5% of demand coefficient, and 1% of yield coefficient from their nominal values. It is also assumed that only the yields of products from the distillation unit are controlled. The cost c_{jt} and the prices of products p_{jt} appear in the same constraint and they are considered simultaneously. We applied the six different robust counterpart formulations for the three kinds of uncertainty separately, and all of them together subsequently. The solution of the nominal deterministic model is US $23,387.50/day. Considering the different types of uncertainty separately, the worst cases scenario results computed from box set induced robust counterpart optimization model with $\Psi = 1$ are listed in Table 5. From these results it can be observed that the cost and price uncertainty has the most significant effect on the overall profit since the objective value is much less than the objective value of pure yield uncertainty or pure demand uncertainty. In the sequel, we first analyze the different types of uncertainty separately, and then consider them simultaneously.

(1) Yield uncertainty. The set of yield constraints 5.1c contain the uncertain parameters. In each constraint, the number of the uncertain parameters (i.e., $|J_i|$) is 1. It belongs to the LHS case uncertainty. Applying the six different kinds of robust optimization formulations, the results are shown in Figure 15. It can be observed that the results of different formulations are the same, as the adjustable parameter is less than 1 because there is only 1 uncertain parameter in each individual constraint. When $\Psi, \Omega, \Gamma = 0$, the solutions are the same as in the deterministic model. When $\Psi = 1$, $\Omega = (|J_i|)^{1/2}$, and $\Gamma = |J_i|$, the results reach the worst case. When $\Omega > (|J_i|)^{1/2}$ and $\Gamma > |J_i|$, the results of "interval+ellipsoidal", "interval+polyhedral" and "interval+ellipsoidal+polyhedral" set induced models do not decrease anymore.

(2) Cost and price uncertainty. Since the uncertain parameters appear in the objective, we convert it into a constraint. The resulting problem has only LHS uncertainty, and the number of the uncertain parameters (i.e., $|J_i|$) is 7. The results are shown in Figure 16. When $\Psi, \Omega, \Gamma = 0$, the solutions are the same as in the deterministic model. When Ψ, Ω, Γ increase, the results of the box, ellipsoidal, and polyhedral uncertainty set induced models will decrease or even become infeasible. If the uncertainty set is combined with an "interval" set, the solution will finally reach the worst case value and will not decrease anymore. In this study, the following parameters are applied: for "interval+ellipsoidal" model, $\Omega = (|J_i|)^{1/2} = (7)^{1/2}$; for "interval+polyhedral" set induced model, $\Gamma = |J_i| = 7$; for "interval+ellipsoidal+polyhedral" set induced model, we take $\Omega = \Gamma/(|J_i|)^{1/2}$.

(3) Demand uncertainty. This belongs to the RHS uncertainty case and there is only one uncertain parameter in each constraint. The results are shown in Figure 17. From this figure, we can see that the results of different formulations are the same because there is only one uncertain parameter in each individual constraint. When $\Psi, \Omega, \Gamma = 0$, the solutions are the same as in the deterministic model. When $\Psi, \Omega, \Gamma = 1$, the results reach the worst case solution. When $\Omega > (|J_i|)^{1/2} = 1$, and $\Gamma > |J_i| = 1$, the results of the "interval+ellipsoidal", "interval+polyhedral" and interval+ellipsoidal+polyhedral" induced models do not decrease anymore.

(4) Simultaneous yield, price, cost, and demand uncertainty. Here we consider all uncertainties together. The x axis is Γ_{price} and we set $\Gamma_{\text{yield}} = \Gamma_{\text{demand}} = {}^{1}/_{7}\Gamma_{\text{price}}$ to plot the result using the same axis. The parameters are

Table 5. Objective Function Values for the Worst-Case Scenario Case

uncertain parameter	yield	demand	cost and price	cost, price, yield, and demand
objective value	22665.00	23134.97	7113.92	6569.14

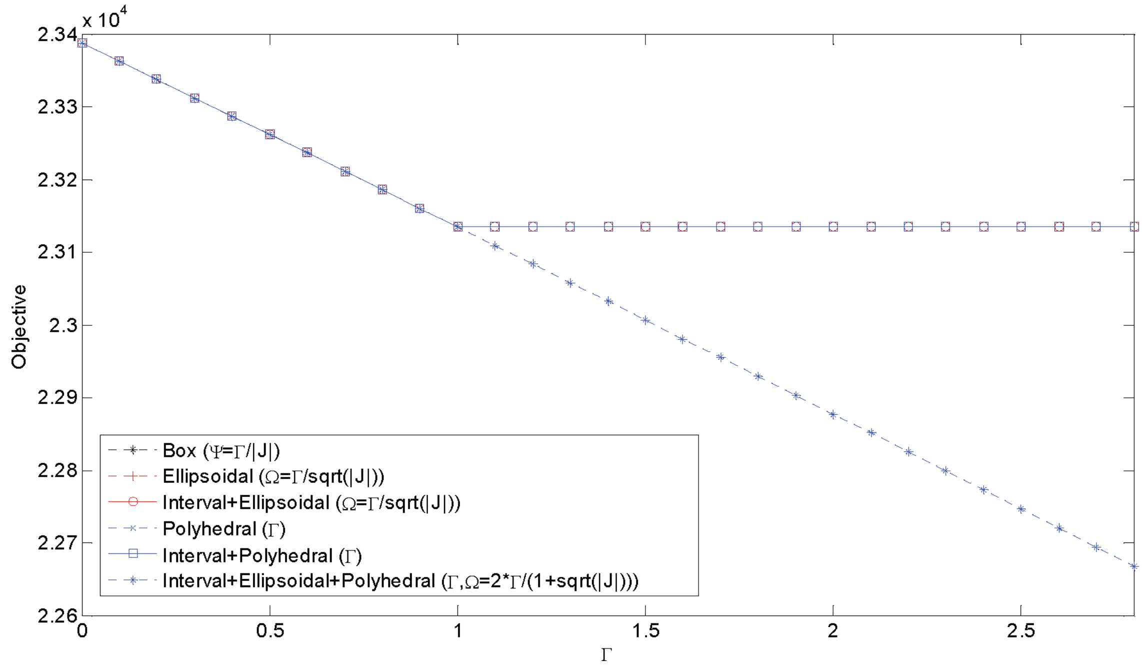


Figure 15. Solution for yield uncertainty.

as follows:

$$J_{\text{yield}} = 1,\ J_{\text{price}} = 7,\ J_{\text{demand}} = 1$$

$$\begin{aligned}\Psi_{\text{yield}} &= \Gamma_{\text{yield}}/|J_{\text{yield}}|,\\ \Psi_{\text{price}} &= \Gamma_{\text{price}}/|J_{\text{price}}|,\\ \Psi_{\text{demand}} &= \Gamma_{\text{demand}}/|J_{\text{demand}}|\end{aligned}$$

$$\begin{aligned}\Omega_{\text{yield}} &= \Gamma_{\text{yield}}/\sqrt{|J_{\text{yield}}|},\\ \Omega_{\text{price}} &= \Gamma_{\text{price}}/\sqrt{|J_{\text{price}}|},\\ \Omega_{\text{demand}} &= \Gamma_{\text{demand}}/\sqrt{|J_{\text{demand}}|}\end{aligned}$$

The results are shown as Figure 18. From this figure we can observe that when all the Ψ, Ω, Γ for yield, price, cost, and demand are 0, the results are equal to those of the deterministic model. When $\Gamma_{\text{price}} = 7$, $\Gamma_{\text{yield}} = \Gamma_{\text{demand}} = 1$, and $\Omega = (|J_i|)^{1/2}$, the results of the "interval+ellipsoidal", "interval+polyhedral", and "interval+ellipsoidal+polyhedral" set induced models reach the worst case and do not decrease anymore. At the same point, the "box" reaches the worst case also.

Finally, from this analysis, it can be concluded that for bounded uncertainty in the yield, demand, and price/cost data, the uncertainty set should be combined with the interval set so as to avoid too conservative or even infeasible solutions. On the other hand, all the different models have the flexibility to adjust the solution between the worst-case scenario and the deterministic solution, depending on the selection of the adjustable parameters for their corresponding uncertainty set. To perform a more rigorous comparison of the different models' conservatism, the evaluation of the probabilistic guarantees of constraint violation is necessary, and this will be the subject of a forthcoming publication.

6. ROBUST COUNTERPART FORMULATIONS FOR MIXED INTEGER LINEAR OPTIMIZATION PROBLEMS

In this section, different uncertainty set induced robust counterpart formulations are derived for a general mixed integer linear constraint. We first present the results for simultaneous constraint LHS and RHS uncertainty, and then extend the results to the case of objective function coefficients' uncertainty.

6.1. Uncertainty in LHS and RHS. For problem 2.7, introducing auxiliary variable x_0 and an additional constraint $x_0 = -1$, the original ith constraint's robust counterpart eq 2.11 can be rewritten as

$$p_i x_0 + \sum_m a_{im} x_m + \sum_k b_{ik} y_k + \max_{\xi \in U}\{\xi_{i0}\hat{p}_i x_0 + \sum_{m \in M_i} \xi_{im}\hat{a}_{im} x_m + \sum_{k \in K_i} \xi_{ik}\hat{b}_{ik} y_k\} \le 0 \tag{6.1}$$

With the following definition

$$\xi_i = [\xi_{i0}; \{\xi_{im}\}; \{\xi_{ik}\}] \tag{6.2a}$$

$$A_i = [p_i, \{a_{im}\}, \{b_{ik}\}] \tag{6.2b}$$

$$\hat{A}_i = [\hat{p}_i, \{\hat{a}_{im}\}, \{\hat{b}_{ik}\}] \tag{6.2c}$$

$$X = [x_0; \{x_m\}; \{y_k\}] \tag{6.2d}$$

$$j \in J_i = \{0\} \cup M_i \cup K_i \tag{6.2e}$$

the robust counterpart 6.1 can be rewritten as

$$\sum_j A_{ij} X_j + \max_{\xi_i \in U}\{\sum_{j \in J_i} \xi_{ij}\hat{A}_{ij} X_j\} \le 0 \tag{6.3}$$

To eliminate the inner maximization problem in 6.3, we first

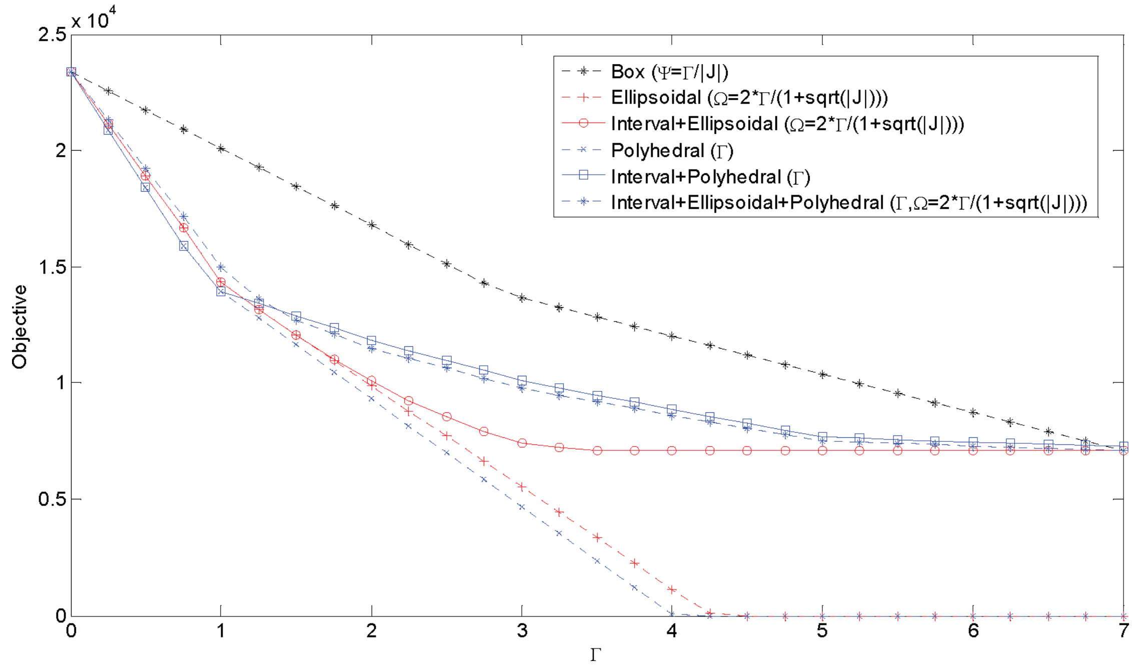


Figure 16. Solution for price and cost uncertainty.

transform the inner maximization problem into its conic dual, and then incorporate the dual problem into the original constraint. In the sequel, the robust counterpart formulations for the ith mixed integer linear constraint in 2.7 with simultaneous LHS and RHS uncertainty will be directly given. Detailed proofs of all properties can be found in Appendix B.

Property 6.1. If the set U is the box uncertainty set 3.1, then the corresponding robust counterpart constraint 6.3 becomes

$$\sum_m a_{im}x_m + \sum_k b_{ik}y_k + \Psi[\sum_{m \in M_i} \hat{a}_{im}|x_m| + \sum_{k \in K_i} \hat{b}_{ik}|y_k| + \hat{p}_i] \leq p_i \tag{6.4}$$

The proof is shown in Appendix B.

Remark 6.1. Notice that the absolute value operators in constraint 6.4 can be directly removed while the corresponding variable is positive. The robust formulation can be further equivalently transformed to the following constraints:

$$\begin{cases} \sum_m a_{im}x_m + \sum_k b_{ik}y_k + \Psi[\sum_{m \in M_i} \hat{a}_{im}u_m + \sum_{k \in K_i} \hat{b}_{ik}v_k + \hat{p}_i] \leq p_i \\ |x_m| \leq u_m \quad \forall m \in M_i \\ |y_k| \leq v_k \quad \forall\, k \in K_i \end{cases} \tag{6.5}$$

This constraint set can be further rewritten as the following form:

$$\begin{cases} \sum_m a_{im}x_m + \sum_k b_{ik}y_k + \Psi[\sum_{m \in M_i} \hat{a}_{im}u_m + \sum_{k \in K_i} \hat{b}_{ik}v_k + \hat{p}_i] \leq p_i \\ -u_m \leq x_m \leq u_m \quad \forall\, m \in M_i \\ -v_k \leq y_k \leq v_k \quad \forall\, k \in K_i \end{cases} \tag{6.6}$$

Motivating Example 2 (Continued). The robust counterpart for the original third constraint is as follows:

$$x_1 - 20y_1 + \Psi(0.1x_1 + 2y_1) \leq 0$$

The final complete robust counterpart optimization model is

$$\begin{aligned} \max \quad & 3x_1 + 2x_2 - 10y_1 - 5y_2 \\ \text{s.t.} \quad & x_1 + x_2 \leq 20 \\ & x_1 + 2x_2 \leq 12 \\ & x_1 - 20y_1 + \Psi_1(0.1x_1 + 2y_1) \leq 0 \\ & x_2 - 20y_2 + \Psi_2(0.1x_2 + 2y_2) \leq 0 \\ & x_1 - x_2 \leq -4 \\ & 0 \leq x_1, x_2 \leq 10, y_1, y_2 \in \{0, 1\} \end{aligned}$$

In the above formulation, different parameters Ψ_1, Ψ_2 are assigned to the two constraints. Note that the absolute value operator has been eliminated since the variables are all positive.

Property 6.2. If the set U is the ellipsoidal uncertainty set 3.2, then the corresponding robust counterpart constraint 6.3 becomes

$$\sum_m a_{im}x_m + \sum_m b_{ik}y_k + \Omega\sqrt{\sum_{m \in M_i} \hat{a}_{im}{}^2 x_m{}^2 + \sum_{k \in K_i} \hat{b}_{ik}{}^2 y_k{}^2 + \hat{p}_i{}^2} \leq p_i \tag{6.7}$$

The proof is shown in Appendix B).

Motivating Example 2 (Continued). The robust counterpart constraint for the third constraint of motivating example 2 is

$$x_1 - 20y_1 + \Omega_1\sqrt{0.01x_1{}^2 + 4y_1{}^2} \leq 0.$$

 dx.doi.org/10.1021/ie200150p |*Ind. Eng. Chem. Res.* 2011, 50, 10567–10603

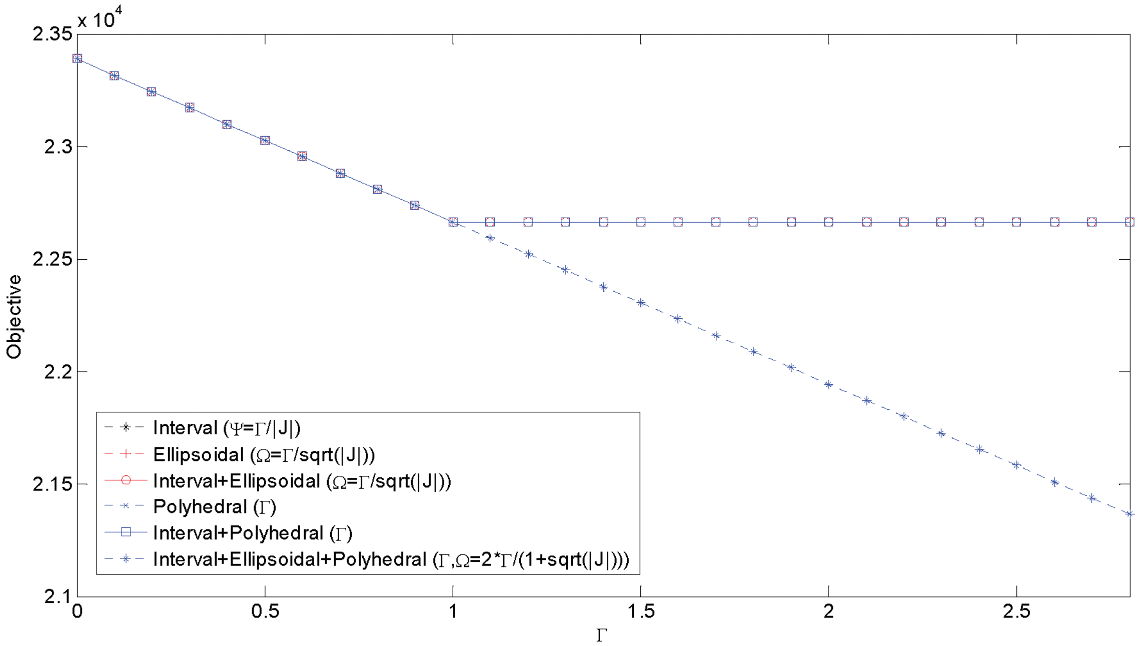


Figure 17. Solution for demand uncertainty.

Property 6.3. If the set U is defined as the polyhedral uncertainty set 3.3, then the corresponding robust counterpart constraint 6.3 becomes

$$\begin{cases} \sum_m a_{im}x_m + \sum_k b_{ik}y_k + z_i\Gamma \le p_i \\ z_i \ge \hat{a}_{im}|x_m| \quad \forall\ m \in M_i \\ z_i \ge \hat{b}_{ik}|y_k| \quad \forall\ k \in K_i \\ z_i \ge \hat{p}_i \end{cases} \tag{6.8}$$

The proof is shown in Appendix B.

. *Remark 6.2.* Similarly, as in Remark 6.1, the above robust formulation can be further transformed into the following equivalent constraint set after eliminating the absolute value operators:

$$\begin{cases} \sum_m a_{im}x_m + \sum_k b_{ik}y_k + z_i\Gamma \le p_i \\ z_i \ge \hat{a}_{im}u_m \quad \forall\ m \in M_i \\ z_i \ge \hat{b}_{ik}v_k \quad \forall\ k \in K_i \\ z_i \ge \hat{p}_i \\ -u_m \le x_m \le u_m \quad \forall\ m \in M_i \\ -v_k \le y_k \le v_k \quad \forall\ k \in K_i \end{cases} \tag{6.9}$$

Motivating Example 2 (Continued). The corresponding robust formulation for the third constraint of the motivating example is

$$x_1 - 20y_1 + z_1\Gamma_1 \le 0$$
$$z_1 \ge 0.1x_1,\ z_1 \ge 2y_1$$

Property 6.4. If the set U is the "interval+ellipsoidal" uncertainty set 3.4 with $\Psi = 1$, then the corresponding robust counterpart constraint 6.3 becomes

$$\sum_m a_{im}x_m + \sum_k b_{ik}y_k + \sum_{m \in M_i} \hat{a}_{im}|x_m - z_{im}| + \sum_{m \in K_i} \hat{b}_{ik}|y_k - z_{ik}| + \hat{p}_i|1 + z_{i0}| + \Omega\sqrt{\sum_{m \in M_i} \hat{a}_{im}^{\,2} z_{im}^{\,2} + \sum_{k \in K_i} \hat{b}_{ik}^{\,2} z_{ik}^{\,2} + \hat{p}_i^{\,2} z_{i0}^{\,2}} \le p_i \tag{6.10}$$

The proof is shown in Appendix B.

Remark 6.3. Constraint 6.10 can be rewritten as

$$\begin{cases} \sum_m a_{im}x_m + \sum_k b_{ik}y_k + \sum_{m \in M_i} \hat{a}_{im}u_{im} + \sum_{m \in K_i} \hat{b}_{ik}u_{ik} \\ \quad + \hat{p}_i u_{i0} + \Omega\sqrt{\sum_{m \in M_i} \hat{a}_{im}^{\,2} z_{im}^{\,2} + \sum_{k \in K_i} \hat{b}_{ik}^{\,2} z_{ik}^{\,2} + \hat{p}_i^{\,2} z_{i0}^{\,2}} \le p_i \\ u_{im} = |x_m - z_{im}| \quad \forall\ m \in M_i \\ u_{ik} = |y_k - z_{ik}| \quad \forall\ k \in K_i \\ u_{i0} = |1 + z_{i0}| \end{cases}$$

which can be further equivalently transformed to the following constraint sets as shown in Remark 6.1:

$$\begin{cases} \sum_m a_{im}x_m + \sum_k b_{ik}y_k + \sum_{m \in M_i} \hat{a}_{im}u_{im} + \sum_{m \in K_i} \hat{b}_{ik}u_{ik} \\ \quad + \hat{p}_i u_{i0} + \Omega\sqrt{\sum_{m \in M_i} \hat{a}_{im}^{\,2} z_{im}^{\,2} + \sum_{k \in K_i} \hat{b}_{ik}^{\,2} z_{ik}^{\,2} + \hat{p}_i^{\,2} z_{i0}^{\,2}} \le p_i \\ -u_{im} \le x_m - z_{im} \le u_{im} \quad \forall\ m \in M_i \\ -u_{ik} \le y_k - z_{ik} \le u_{ik} \quad \forall\ k \in K_i \\ -u_{i0} \le 1 + z_{i0} \le u_{i0} \end{cases} \tag{6.11}$$

Motivating Example 2 (Continued). The robust counterpart formulation for the third constraint is

$$\begin{aligned} & x_1 - 20y_1 + \Omega_1\sqrt{0.01z_{31}^2 + 4z_{33}^2} \le 0 \\ & -u_{31} \le x_1 - z_{31} \le u_{31} \\ & -u_{33} \le y_1 - z_{33} \le u_{33} \end{aligned}$$

Property 6.5. If the set U is defined as the "interval+polyhedral" uncertainty set 3.6 with $\Psi = 1$, then the corresponding robust counterpart constraint 6.3 is equivalent to the following constraint sets:

$$\begin{cases} \sum_m a_{im}x_m + \sum_k b_{ik}y_k + [z_i\Gamma_i + \sum_{m \in M_i} w_{im} + \sum_{k \in K_i} w_{ik} + w_{i0}] \le p_i \\ z_i + w_{im} \ge \hat{a}_{im}|x_m| \quad \forall\ m \in M_i \\ z_i + w_{ik} \ge \hat{b}_{ik}|y_k| \quad \forall\ k \in K_i \\ z_i + w_{i0} \ge \hat{p}_i \end{cases} \tag{6.12}$$

The proof is shown in Appendix B.

Remark 6.4. While the variables are positive, the absolute value operator can be directly removed. Otherwise, the robust formulation (6.12) can be rewritten as follows as shown in Remark 6.1:

$$\begin{cases} \sum_m a_{im}x_m + \sum_k b_{ik}y_k + [z_i\Gamma_i + \sum_{m \in M_i} w_{im} + \sum_{k \in K_i} w_{ik} + w_{i0}] \le p_i \\ z_i + w_{im} \ge \hat{a}_{im}u_m \quad \forall\ m \in M_i \\ z_i + w_{ik} \ge \hat{b}_{ik}v_k \quad \forall\ k \in K_i \\ z_i + w_{i0} \ge \hat{p}_i \\ -u_m \le x_m \le u_m \quad \forall\ m \in M_i \\ -v_k \le y_k \le v_k \quad \forall\ k \in K_i \end{cases} \tag{6.13}$$

Motivating Example 2 (Continued). Since all variables are positive, the robust counterpart for the third constraint becomes

$$\begin{aligned} & x_1 - 20y_1 + z_1\Gamma_1 + w_{31} + w_{33} \le 0 \\ & z_1 + w_{31} \ge 0.1x_1,\ z_1 + w_{33} \ge 2y_1 \end{aligned}$$

Property 6.6. If the set U is the "interval+ellipsoidal+polyhedral" uncertainty set 3.8 with $\Psi = 1$, then the corresponding robust counterpart constraint 6.3 becomes

$$\begin{cases} \sum_m a_{im}x_m + \sum_k b_{ik}y_k + z_i\Gamma + \sum_{m \in M_i} |q_{im}| + \sum_{k \in K_i} |q_{ik}| \\ \quad + |q_{i0}| + \Omega\sqrt{\sum_{m \in M_i} {w_{im}}^2 + \sum_{k \in K_i} {w_{ik}}^2 + {w_{i0}}^2} \le p_i \\ z_i \ge |\hat{a}_{im}x_m - q_{im} - w_{im}| \quad \forall m \in M_i \\ z_i \ge |\hat{b}_{ik}y_k - q_{ik} - w_{ik}| \quad \forall\ k \in K_i \\ z_i \ge |\hat{p}_i + q_{i0} + w_{i0}| \end{cases} \tag{6.14}$$

The proof is shown in Appendix B.

Remark 6.5. As in Remark 6.1, the robust counterpart can be equivalently rewritten as follows by introducing auxiliary variables and eliminating the absolute value operators:

$$\begin{cases} \sum_m a_{im}x_m + \sum_k b_{ik}y_k + z_i\Gamma + \sum_{m \in M_i} u_{im} + \sum_{k \in K_i} u_{ik} \\ \quad + u_{i0} + \Omega\sqrt{\sum_{m \in M_i} {w_{im}}^2 + \sum_{k \in K_i} {w_{ik}}^2 + {w_{i0}}^2} \le p_i \\ -u_{im} \le q_{im} \le u_{im} \quad \forall\ m \in M_i \\ -u_{ik} \le q_{ik} \le u_{ik} \quad \forall\ k \in K_i \\ -u_{i0} \le q_{i0} \le u_{i0} \\ -z_i \le \hat{a}_{im}x_m - q_{im} - w_{im} \le z_i \quad \forall\ m \in M_i \\ -z_i \le \hat{b}_{ik}y_k - q_{ik} - w_{ik} \le z_i \quad \forall\ k \in K_i \\ -z_i \le \hat{p}_i + q_{i0} + w_{i0} \le z_i \end{cases} \tag{6.15}$$

Motivating Example 2 (Continued). The robust counterpart formulation for the third constraint is

$$\begin{aligned} & x_1 - 20y_1 + z_1\Gamma_1 + u_{31} + u_{33} + \Omega_1\sqrt{{w_{31}}^2 + {w_{33}}^2} \le 0 \\ & -u_{31} \le q_{31} \le u_{31},\ -u_{33} \le q_{33} \le u_{33} \\ & -z_1 \le 0.1x_1 - q_{31} - w_{31} \le z_1,\ -z_1 \le 2y_1 - q_{33} - w_{33} \le z_1 \end{aligned}$$

The different uncertainty set induced robust counterpart formulations are summarized in Table 1. Finally, we point out that for the case of LHS only or RHS only uncertainty, the corresponding robust counterpart optimization formulations can be derived based on the above results of simultaneous LHS and RHS uncertainty.

For example, for LHS only uncertainty, we have $\hat{p}_i = 0$, then the box set induced robust counterpart 6.4 is reduced to

$$\sum_m a_{im}x_m + \sum_k b_{ik}y_k + \Psi[\sum_{m \in M_i} \hat{a}_{im}|x_m| + \sum_{k \in K_i} \hat{b}_{ik}|y_k|] \le p_i \tag{6.16}$$

Similarly, for RHS only uncertainty, $\hat{a}_{im} = 0$, $\hat{b}_{ik} = 0$, then the box set induced robust counterpart 6.4 is reduced to

$$\sum_m a_{im}x_m + \sum_k b_{ik}y_k + \Psi\hat{p}_i \le p_i \tag{6.17}$$

6.2. Objective Function Coefficients' Uncertainty. Considering the objective coefficients uncertainty in the mixed integer linear optimization problem 2.7:

$$\max \sum_m \tilde{c}_m x_m + \sum_k \tilde{d}_k y_k \tag{6.18}$$

To derive the corresponding robust counterpart formulation, the objective uncertainty is equivalently transformed into constraint LHS uncertainty as follows

$$\begin{aligned} \max \quad & z \\ \text{s.t.} \quad & z - \sum_m \tilde{c}_m x_m + \sum_k \tilde{d}_k y_k \le 0 \end{aligned} \tag{6.19}$$

Then the robust counterpart formulation can be applied on the resulting constraints which contain LHS only uncertainty. A summary of the robust counterpart formulation for the ith mixed integer linear constraint is shown in Table 6.

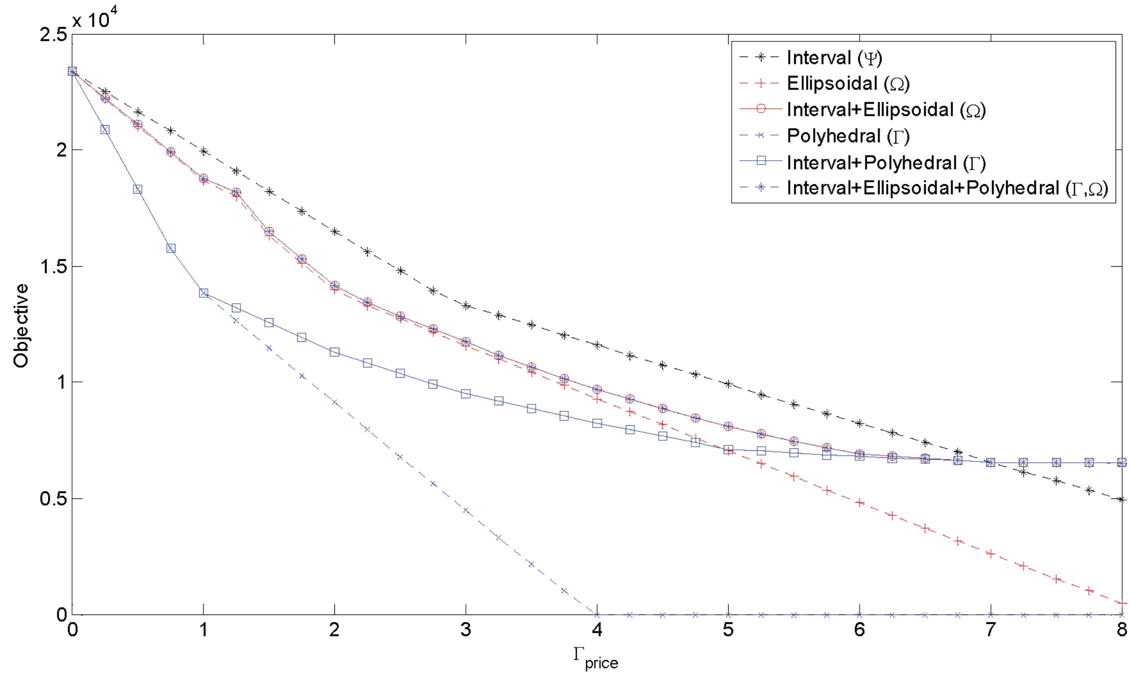


Figure 18. Solution for simultaneous yield, price, cost, and demand uncertainty.

Motivating Example 2 (Continued). To derive the robust counterpart for the objective function coefficients' uncertainty in motivating example 2, the original objective function is transformed into the following constraint first:

$$z - (3x_1 + 2x_2 - 10y_1 - 5y_2) \le 0$$

Then, the set induced robust counterpart constraint for the resulting new constraint can be formulated. For example, the box set induced robust formulation is

$$\begin{aligned} &z - (3x_1 + 2x_2 - 10y_1 - 5y_2) \\ &\quad + \Psi_0(0.3x_1 + 0.2x_2 + y_1 + 0.5y_2) \le 0 \end{aligned}$$

The ellipsoidal set induced robust counterpart constraint is

$$\begin{aligned} &z - (3x_1 + 2x_2 - 10y_1 - 5y_2) \\ &\quad + \Omega_0\sqrt{0.09{x_1}^2 + 0.04{x_2}^2 + {y_1}^2 + 0.25{y_2}^2} \le 0 \end{aligned}$$

The polyhedral set induced robust counterpart constraint is

$$\begin{cases} z - (3x_1 + 2x_2 - 10y_1 - 5y_2) + v_0\Gamma_0 \le 0 \\ v_0 \ge 0.3x_1,\ v_0 \ge 0.2x_2,\ v_0 \ge y_1,\ v_0 \ge 0.5y_2 \end{cases}$$

The "interval+ellipsoidal" set induced robust counterpart constraint is

$$\begin{cases} z - (3x_1 + 2x_2 - 10y_1 - 5y_2) + (0.3u_{01} + 0.2u_{02} + u_{03} \\ \quad + 0.5u_{04}) + \Omega_0\sqrt{0.09{z_{01}}^2 + 0.04{z_{02}}^2 + {z_{03}}^2 + 0.25{z_{04}}^2} \le 0 \\ -u_{01} \le x_1 - z_{01} \le u_{01},\ -u_{02} \le x_2 - z_{02} \le u_{02} \\ -u_{03} \le y_1 - z_{03} \le u_{03},\ -u_{04} \le y_2 - z_{04} \le u_{04} \end{cases}$$

7. COMPUTATIONAL STUDIES FOR ROBUST MIXED INTEGER LINEAR OPTIMIZATION

Example 7.1. Consider the following mixed 0−1 programming problem

$$\begin{aligned} \max\quad & 3x_1 + 2x_2 - 10y_1 - 5y_2 \\ \text{s.t.}\quad & x_1 + x_2 \le 20 \\ & x_1 + 2x_2 \le 12 \\ & x_1 - 20y_1 \le 0 \\ & x_2 - 20y_2 \le 0 \\ & x_1 - x_2 \le -4 \\ & 0 \le x_1,\ x_2 \le 10, y_1,\ y_2 \in \{0,1\} \end{aligned}$$

Let us assume that all the objective function coefficients, the LHS, and RHS of the constraints parameter are possibly subject to uncertainty. To find robust solutions of this problem, we first convert the objective uncertainty into LHS uncertainty as shown in section 6.2:

$$\begin{aligned} \max\quad & z \\ \text{s.t.}\quad & z - (3x_1 + 2x_2 - 10y_1 - 5y_2) \le 0 \\ & x_1 + x_2 \le 20 \\ & x_1 + 2x_2 \le 12 \\ & x_1 - 20y_1 \le 0 \\ & x_2 - 20y_2 \le 0 \\ & x_1 - x_2 \le -4 \\ & 0 \le x_1, x_2 \le 10,\ y_1, y_2 \in \{0,1\} \end{aligned}$$

The corresponding uncertain version of the above problem can be represented using the general form as follows

$$\begin{aligned} \max\quad & z \\ \text{s.t.}\quad & A_0 z + \tilde{A}x + \tilde{B}y \le \tilde{p} \\ & 0 \le x_1, x_2 \le 10,\ y_1, y_2 \in \{0,1\} \end{aligned}$$

Table 6. Summary on Robust Counterpart Formulation for the *i*th Mixed Integer Linear Constraint

Uncertainty Set	Robust Counterpart Formulation
Box	$\sum_m a_{im}x_m + \sum_k b_{ik}y_k + \Psi\left[\sum_{m\in M_i}\hat{a}_{im}\left\|x_m\right\| + \sum_{k\in K_i}\hat{b}_{ik}\left\|y_k\right\| + \hat{p}_i\right] \le p_i$
Ellipsoidal	$\sum_m a_{im}x_m + \sum_m b_{ik}y_k + \Omega\sqrt{\sum_{m\in M_i}\hat{a}_{im}^2x_m^2 + \sum_{k\in K_i}\hat{b}_{ik}^2y_k^2 + \hat{p}_i^2} \le p_i$
Polyhedral	$\begin{cases}\sum_m a_{im}x_m + \sum_k b_{ik}y_k + z_i\Gamma \le p_i \\ z_i \ge \hat{a}_{im}\left\|x_m\right\| & \forall m \in M_i \\ z_i \ge \hat{b}_{ik}\left\|y_k\right\| & \forall k \in K_i \\ z_i \ge \hat{p}_i\end{cases}$
Interval+ Ellipsoidal	$\sum_m a_{im}x_m + \sum_k b_{ik}y_k + \sum_{m\in M_i}\hat{a}_{im}\left\|x_m - z_{im}\right\| + \sum_{m\in K_i}\hat{b}_{ik}\left\|y_k - z_{ik}\right\| + \hat{p}_i\left\|1 + z_{i0}\right\| + \Omega\sqrt{\sum_{m\in M_i}\hat{a}_{im}^2z_{im}^2 + \sum_{k\in K_i}\hat{b}_{ik}^2z_{ik}^2 + \hat{p}_i^2z_{i0}^2} \le p_i$
Interval+ Polyhedral	$\begin{cases}\sum_m a_{im}x_m + \sum_k b_{ik}y_k + z_i\Gamma_i + \sum_{m\in M_i}w_{im} + \sum_{k\in K_i}w_{ik} + w_{i0} \le p_i \\ z_i + w_{im} \ge \hat{a}_{im}\left\|x_m\right\| & \forall m \in M_i \\ z_i + w_{ik} \ge \hat{b}_{ik}\left\|y_k\right\| & \forall k \in K_i \\ z_i + w_{i0} \ge \hat{p}_i\end{cases}$
Interval+ Ellipsoidal+ Polyhedral	$\begin{cases}\sum_m a_{im}x_m + \sum_k b_{ik}y_k + z_i\Gamma + \sum_{m\in M_i}\left\|q_{im}\right\| + \sum_{k\in K_i}\left\|q_{ik}\right\| + \left\|q_{i0}\right\| + \Omega\sqrt{\sum_{m\in M_i}w_{im}^2 + \sum_{k\in K_i}w_{ik}^2 + w_{i0}^2} \le p_i \\ z_i \ge \left\|\hat{a}_{im}x_m - q_{im} - w_{im}\right\| & \forall m \in M_i \\ z_i \ge \left\|\hat{b}_{ik}y_k - q_{ik} - w_{ik}\right\| & \forall k \in K_i \\ z_i \ge \left\|\hat{p}_i + q_{i0} + w_{i0}\right\|\end{cases}$

where $\tilde{A} = \{a_{im} + \hat{a}_{im}\xi_{im}\}$, $\tilde{B} = \{b_{ik} + \hat{b}_{ik}\xi_{ik}\}$, $\tilde{p} = \{p_i + \tilde{p}_i\xi_{i0}\}$, ξ_{im}, ξ_{ik}, ξ_{i0} are independent uncertain parameters, a_{im}, b_{ik}, and p_i are nominal data defined as follows

$$A_0 = \begin{bmatrix}1\\0\\0\\0\\0\\0\end{bmatrix}, \{a_{im}\} = \begin{bmatrix}-3 & -2\\1 & 1\\1 & 2\\1 & 0\\0 & 1\\1 & -1\end{bmatrix},$$

$$\{b_{ik}\} = \begin{bmatrix}10 & 5\\0 & 0\\0 & 0\\-20 & 0\\0 & -20\\0 & 0\end{bmatrix}, \{p_i\} = \begin{bmatrix}0\\20\\12\\0\\0\\-4\end{bmatrix}$$

Assuming 10% uncertainty level for the possible uncertainty (i.e., $\hat{a}_{im} = 0.1|a_{im}|$, $\hat{b}ik = 0.1|b_{ik}|$, $\hat{p}_i = 0.1|p_i|$), the robust counterpart model under different uncertainty sets can be formulated as shown in section 6. Note that for the constraints containing only continuous variables, their corresponding robust counterpart constraints can be formulated using the method presented in section 4.

In this example, several different uncertainty cases are studied, which include LHS only uncertainty, RHS only uncertainty, OBJ only uncertainty, simultaneous LHS, RHS, and OBJ uncertainty. Without giving a complete description of all the robust counterpart optimization models, we list several robust counterpart models using the box set induced robust counterpart formulation as follows:

(1) Considering LHS only uncertainty for all the constraints, the box set induced robust counterpart model is

$$\begin{aligned}\max\quad & 3x_1 + 2x_2 - 10y_1 - 5y_2\\ \text{s.t.}\quad & x_1 + x_2 + \Psi(0.1x_1 + 0.1x_2) \le 20\\ & x_1 + 2x_2 + \Psi(0.1x_1 + 0.2x_2) \le 12\\ & x_1 - 20y_1 + \Psi(0.1x_1 + 2y_1) \le 0\\ & x_2 - 20y_2 + \Psi(0.1x_2 + 2y_2) \le 0\\ & x_1 - x_2 + \Psi(0.1x_1 + 0.1x_2) \le -4\\ & 0 \le x_1, x_2 \le 10,\ y_1, y_2 \in \{0, 1\}\end{aligned}$$

Note that the same uncertainty set parameter Ψ is applied for all the constraints here. A similar setting will be applied for the rest of the models.

(2) Considering simultaneous LHS and RHS uncertainty, the box set induced robust counterpart model is

$$\begin{aligned}\max\quad & 3x_1 + 2x_2 - 10y_1 - 5y_2\\ \text{s.t.}\quad & x_1 + x_2 + \Psi(0.1x_1 + 0.1x_2 + 2) \le 20\\ & x_1 + 2x_2 + \Psi(0.1x_1 + 0.2x_2 + 1.2) \le 12\\ & x_1 - 20y_1 + \Psi(0.1x_1 + 2y_1) \le 0\\ & x_2 - 20y_2 + \Psi(0.1x_2 + 2y_2) \le 0\\ & x_1 - x_2 + \Psi(0.1x_1 + 0.1x_2 + 0.4) \le -4\\ & 0 \le x_1, x_2 \le 10,\ y_1, y_2 \in \{0, 1\}\end{aligned}$$

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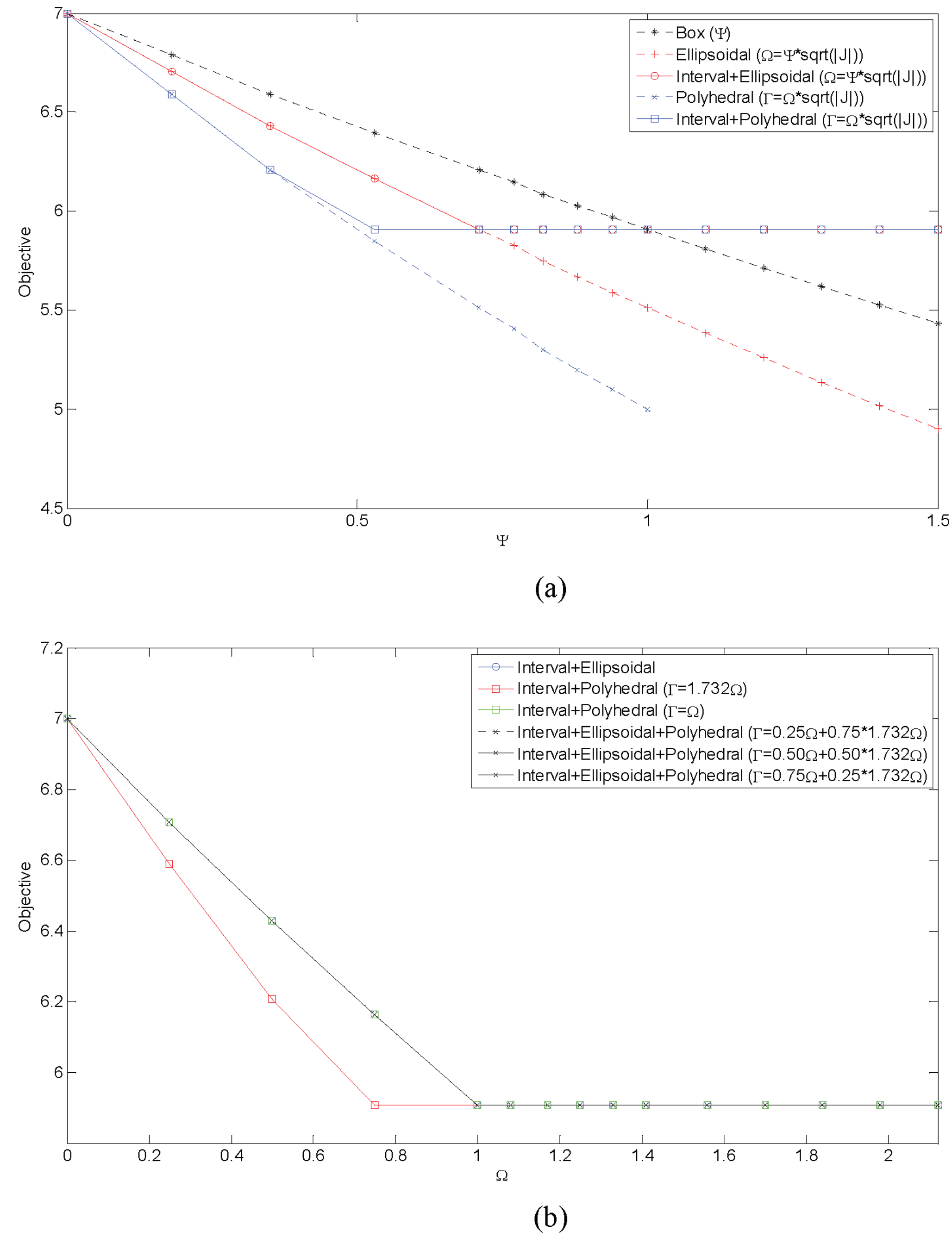


Figure 19. Only LHS uncertainty for all constraints ($|J_i| = 2$). Note: for polyhedral model, as the $\Gamma > \sqrt{2}$ model is infeasible.

(3) Considering simultaneous LHS, RHS and OBJ uncertainty, the robust counterpart model is

$$\begin{aligned}
&\max\ z\\
&\text{s.t.}\quad z - (3x_1 + 2x_2 - 10y_1 - 5y_2)\\
&\quad + \Psi(0.3x_1 + 0.2x_2 + y_1 + 0.5y_2) \le 0x_1\\
&\quad + x_2 + \Psi(0.1x_1 + 0.1x_2 + 2) \le 20x_1\\
&\quad + 2x_2 + \Psi(0.1x_1 + 0.2x_2 + 1.2) \le 12x_1 - 20y_1\\
&\quad + \Psi(0.1x_1 + 2y_1) \le 0x_2 - 20y_2\\
&\quad + \Psi(0.1x_2 + 2y_2) \le 0x_1 - x_2\\
&\quad + \Psi(0.1x_1 + 0.1x_2 + 0.4) \le -40 \le x_1,\\
&\quad x_2 \le 10,\ y_1, y_2 \in \{0, 1\}
\end{aligned}$$

On the basis of the solution of the robust formulations under different cases of uncertainties, the following remarks can be made:

(1) For RHS only uncertainty, which is a special case where the number of uncertain parameters for every constraint is 1, the solution is identical for ellipsoidal and polyhedral set induced models, and also for the “interval+ellipsoidal”, “interval+polyhedral” and “interval+ellipsoidal+polyhedral” uncertainty set induced models as shown in Figure 21. Furthermore, as $\Omega \le 1$ and $\Gamma \le 1$, all the solutions are identical because as $\Omega = \Gamma \le 1$, the different uncertainty sets are actually the same interval set.

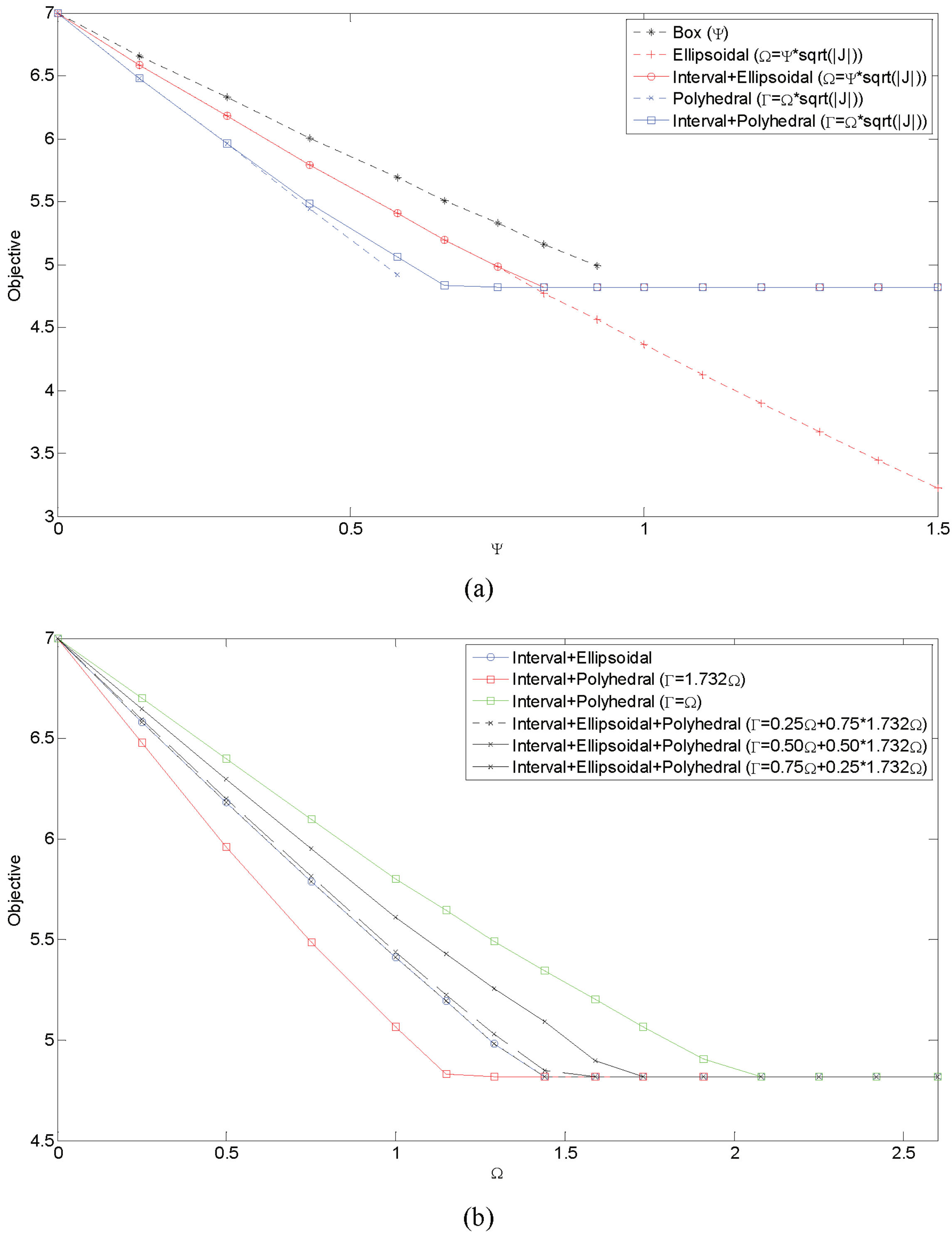


Figure 20. LHS+LHS uncertainty for all constraints ($|J_i| = 3$). Note: for polyhedral model and "box" model, infeasible for large Γ,Ψ.

(2) It can be observed from Figures 19a, 20a, 21a, 22a, and 23a that the ellipsoidal set based robust counterpart solution is equal or worse (even becomes infeasible with large Ω value) than the "interval+ellipsoidal" set based solution. Similarly, the polyhedral set based solution is equal or worse than the "interval +polyhedral" set based solution. This is because for the ellipsoidal set or polyhedral set, its combination with the interval set makes the resulting uncertainty set smaller, and thus less conservative. This suggests that for bounded uncertainty, the uncertainty set should be combined with interval to avoid conservative solutions.

(3) Comparing the "interval+ellipsoidal" and the "interval +polyhedral" induced model from Figures 19b, 20b, 22b, and 23b, when $\Gamma = \Omega(|J_i|)^{1/2}$, the "interval+polyhedral" based solution is always worse than the "interval+ellipsoidal" based solution, which is because the "interval+polyhedral"

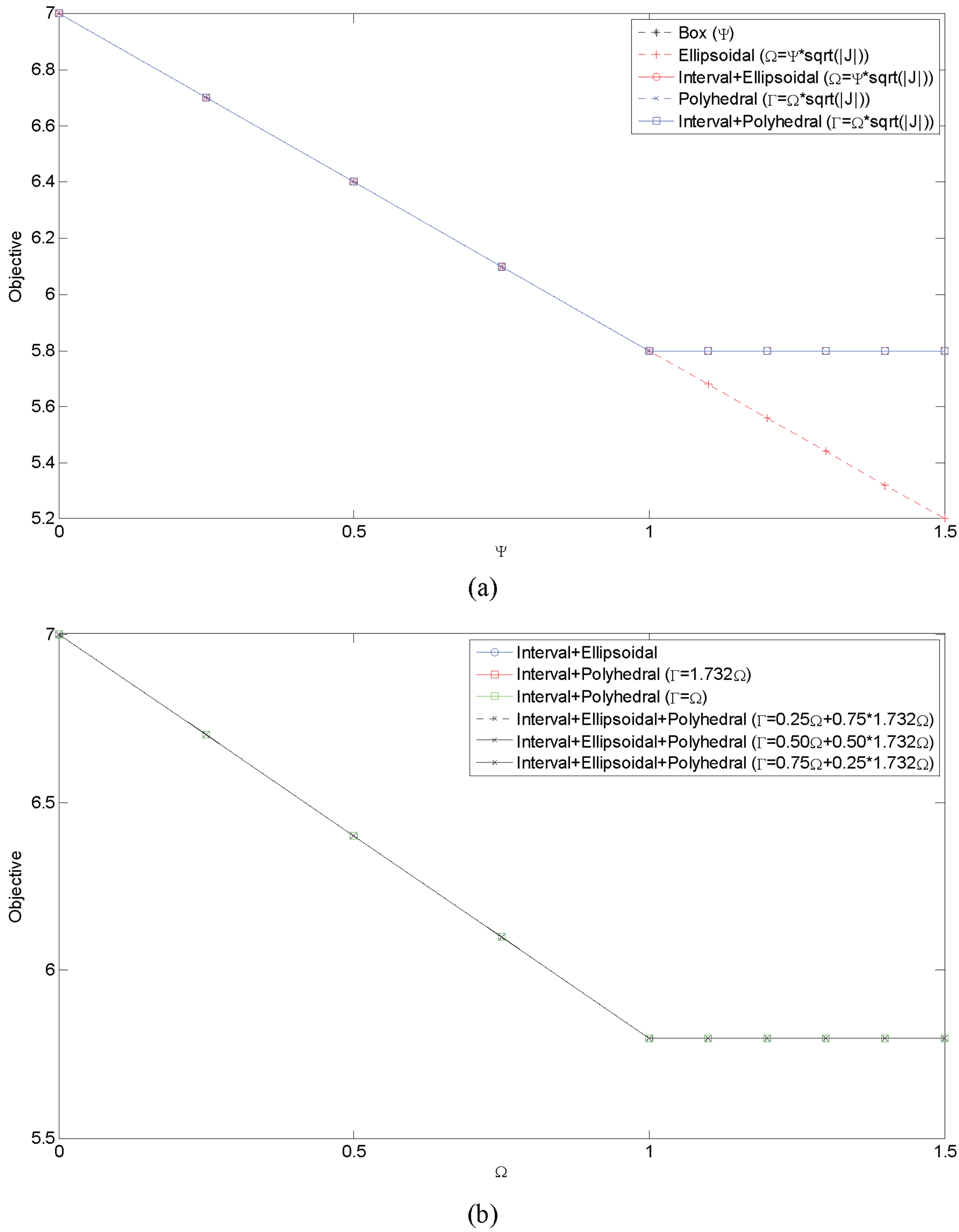


Figure 21. Only RHS uncertainty for all constraints ($|J_i| = 1$).

uncertainty set is larger and completely covers the "interval+ellipsoidal" set; when $\Gamma = \Omega$, the "interval+polyhedral" based solution is always better than the "interval+ellipsoidal" based solution because the "interval+polyhedral" uncertainty set is smaller and completely covered by the "interval+ellipsoidal" set.

(4) Comparing the "interval+ellipsoidal+polyhedral" set based model with others from Figures 19b, 20b, 22b, and 23b, it can be observed that as the Γ value increases from Ω to $\Omega(|J_i|)^{1/2}$, the "interval+ellipsoidal+polyhedral" based solution switches from the "interval+polyhedral" based solution with $\Gamma = \Omega$ to the "interval+ellipsoidal" based solution with $\Gamma = \Omega(|J_i|)^{1/2}$, because the intersection between ellipsoid and polyhedron is exactly changing from the polyhedral with $\Gamma = \Omega$ to the ellipsoid with parameter $\Gamma = \Omega(|J_i|)^{1/2}$.

Finally, from the above analysis, it can be concluded all the different models have the flexibility to adjust the solution between the worst-case scenario and the deterministic solution,

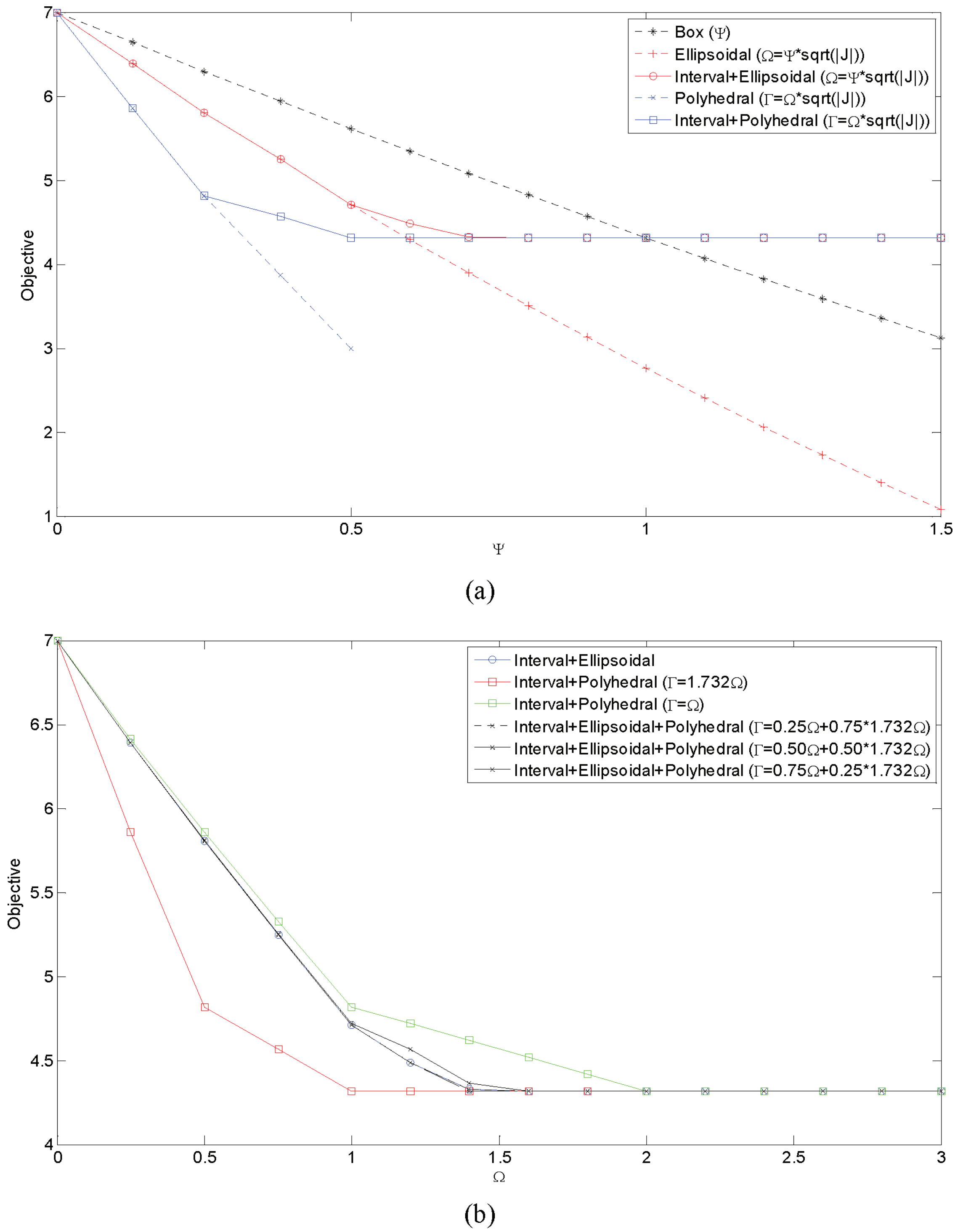
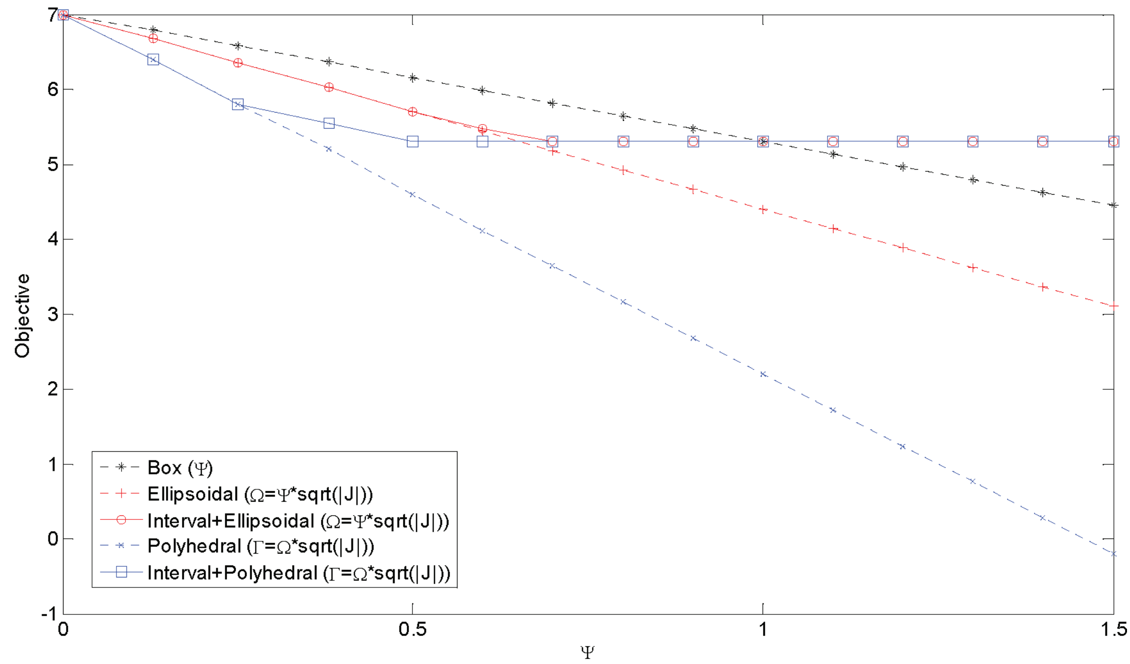


Figure 22. Only OBJ uncertainty ($|J_i| = 4$). Note: for polyhedral model; infeasible for large Γ.

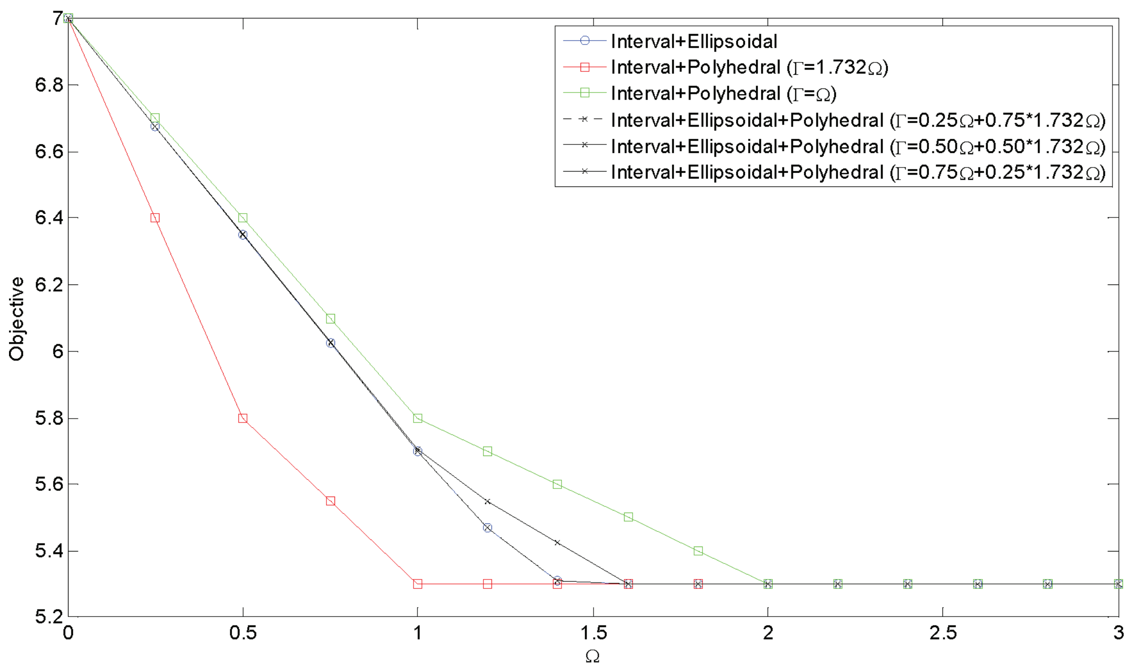
depending on the selection of the adjustable parameters for their corresponding uncertainty set. On the other hand, the degree of conservatism of the models differs, and some models even become infeasible with relatively large uncertainty set parameter values.

Example 7.2 Processing Scheduling Problem. This example involves the scheduling of a batch chemical process related to the production of two chemical products using three raw materials. The state-task-network (STN) representation of this example is shown in Figure 24. The deterministic MILP formulation (7.1) for the scheduling of this batch process is based on ref 37, and the detailed problem data can be found in ref 37.

Through this example, we study the different robust counterpart optimization formulations introduced in section 6 considering different types of uncertainty cases. The scheduling problem's MILP formulation is as follows:



(a)



(b)

Figure 23. Objective uncertainty ($|J_i| = 4$) and LHS uncertainty for the third and fourth constraints ($|J_i| = 2$).

$$\begin{aligned} \max \quad & \text{profit} \\ \text{s.t.} \quad & \text{profit} - \sum_{s \in s_p, n} \widetilde{\text{price}}_s d_{s,n} + \sum_{s \in s_r} \widetilde{\text{price}}_s (\text{STI}_s - \text{STF}_s) \leq 0 \end{aligned} \tag{7.1a}$$

$$\sum_{i \in I_j} wv_{i,j,n} \leq 1 \quad \forall\ i \in I \tag{7.1b}$$

$$st_{s,n} = st_{s,n-1} - d_{s,n} - \sum_{i \in I_s} \rho^C_{s,i} \sum_{j \in J_i} b_{i,j,n} + \sum_{i \in I_s} \rho^P_{s,i} \sum_{j \in J_i} b_{i,j,n-1} \quad \forall\ s \in S, \quad \forall\ n \in N \tag{7.1c}$$

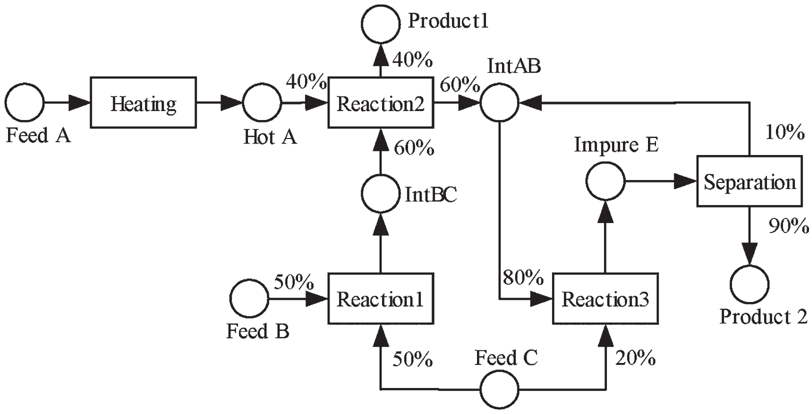


Figure 24. State Task Network (STN) representation of the batch chemical process.

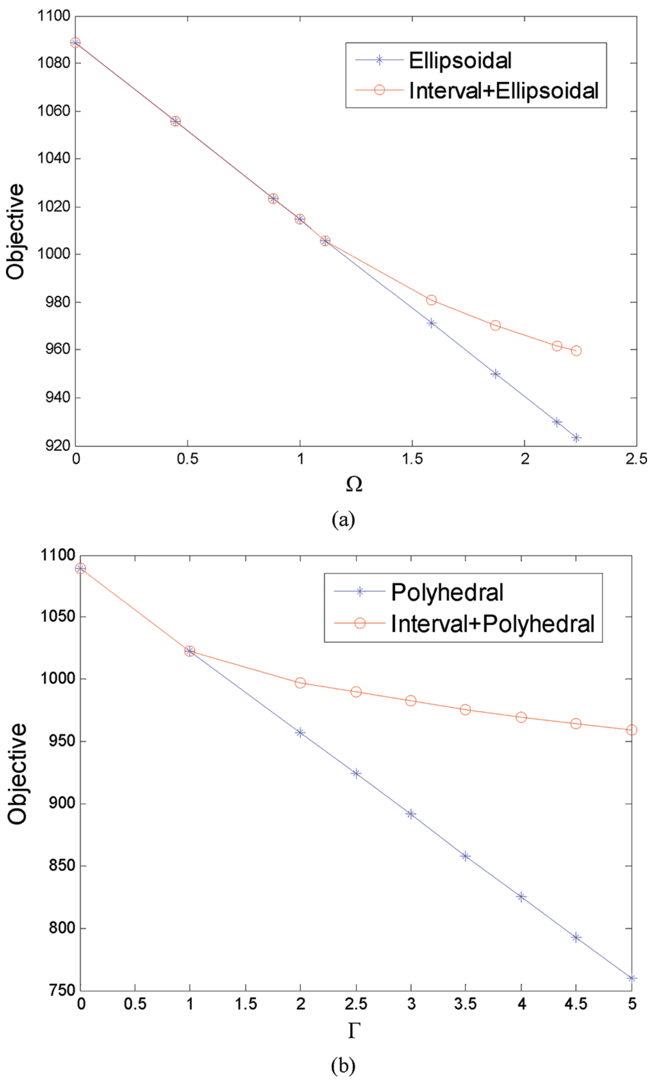


Figure 25. Price uncertainty ($|J_i| = 5$).

$$st_{s,n} \le st_s^{\max} \quad \forall\, s \in S, \quad \forall\, n \in N \tag{7.1d}$$

$$v_{i,j}^{\min} wv_{i,j,n} \le b_{i,j,n} \le v_{i,j}^{\max} wv_{i,j,n} \quad \forall\, i \in I, \quad \forall\, j \in J_i, \quad \forall\, n \in N \tag{7.1e}$$

$$\sum_n d_{s,n} \ge \tilde{r}_s \quad \forall\, s \in S \tag{7.1f}$$

$$Tf_{i,j,n} \ge Ts_{i,j,n} + \tilde{\alpha}_{i,j} wv_{i,j,n} + \tilde{\beta}_{i,j} b_{i,j,n} \quad \forall\, i \in I, \quad \forall\, j \in J_i, \quad \forall\, n \in N \tag{7.1g}$$

$$Ts_{i,j,n+1} \ge Tf_{i,j,n} - H(1 - wv_{i,j,n}) \quad \forall\, i \in I, \quad \forall\, j \in J_i, \quad \forall\, n \in N \tag{7.1h}$$

$$Ts_{i,j,n+1} \ge Tf_{i',j,n} - H(1 - wv_{i,'j,n}) \quad \forall\, i, i' \in I_j, \quad \forall\, j \in J, \quad \forall\, n \in N \tag{7.1i}$$

$$Ts_{i,j,n+1} \ge Tf_{i',j',n} - H(1 - wv_{i,'j',n}) \quad \forall\, i, i' \in I_j, i \ne i', \quad \forall\, j, j' \in J, \quad \forall\, n \in N \tag{7.1j}$$

$$Ts_{i,j,n+1} \ge Ts_{i,j,n} \quad \forall\, i \in I, \quad \forall\, j \in J_i, \quad \forall\, n \in N \tag{7.1k}$$

$$Tf_{i,j,n+1} \ge Tf_{i,j,n} \quad \forall\, i \in I, \quad \forall\, j \in J_i, \quad \forall\, n \in N \tag{7.1l}$$

$$Ts_{i,j,n} \le H \quad \forall\, j \in J_i, \quad \forall\, n \in N \tag{7.1m}$$

$$Tf_{i,j,n} \le H \quad \forall\, i \in I, \quad \forall\, j \in J_i, \quad \forall\, n \in N \tag{7.1n}$$

In the above formulation, the objective function 7.1a maximizes the profit; allocation constraints 7.1b state that only one of the tasks can be performed in each unit at an event point (n); constraints 7.1c represent the material balances for each state (s) expressing that at each event point (n) the amount is equal to that at event point ($n - 1$), adjusted by any amounts produced and consumed between event points ($n - 1$) and (n), and delivered to the market at event point (n); the storage and capacity limitations of production units are expressed by constraints 7.1d and 7.1e; constraints 7.1f are written to satisfy the demands of final products; and constraints 7.1g to 7.1n represent time limitations due to task duration and sequence requirements in the same or different production units.

In this example, uncertainties in material and product prices, processing times of tasks in different units, and product demands are studied. We assume bounded uncertainty and assign a

dx.doi.org/10.1021/ie200150p |Ind. Eng. Chem. Res. 2011, 50, 10567–10603

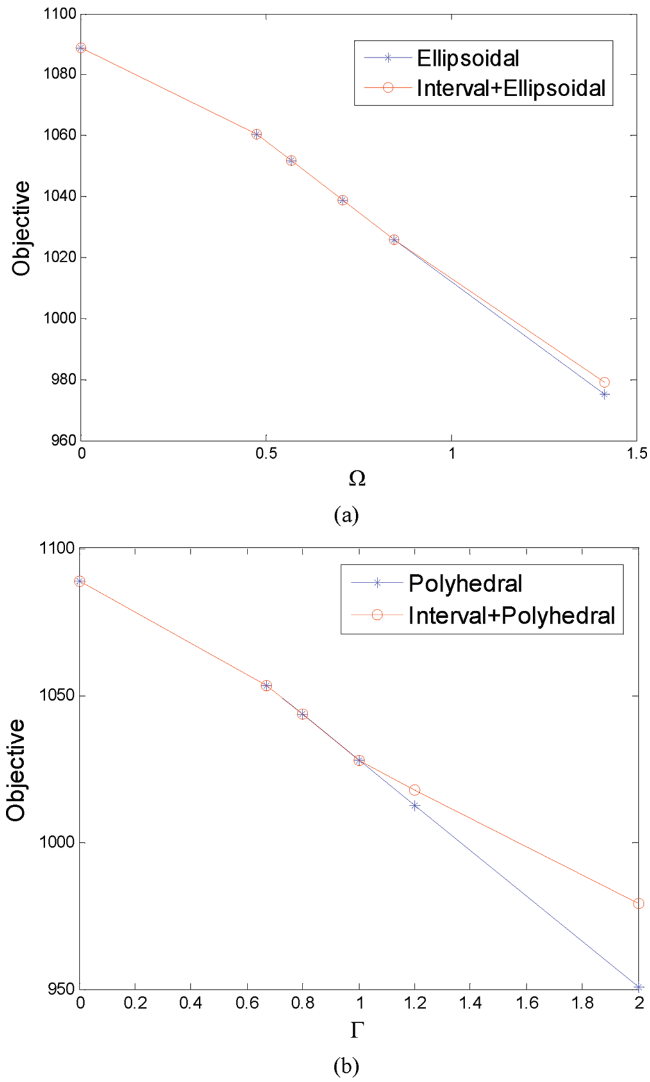


Figure 26. Processing time uncertainty ($|J_i| = 2$).

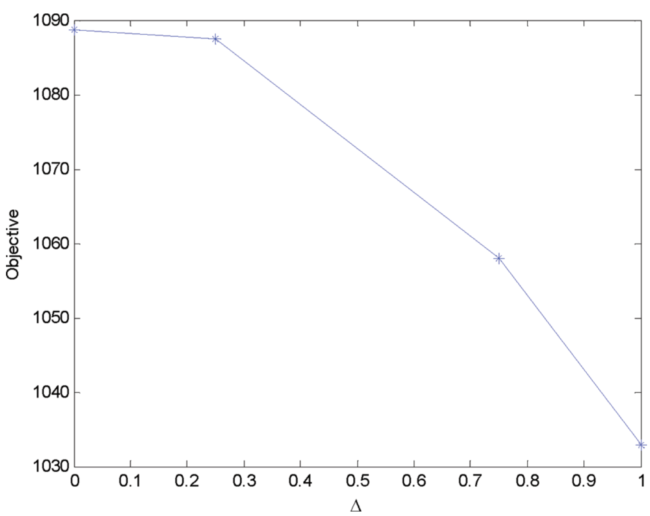


Figure 27. Demand uncertainty ($|J_i| = 1$).

maximum of 5% deviation of price data, 5% of processing times and 20% of demand data from their nominal values. *1. Price Uncertainty.* Considering only price uncertainty, then only constraint 7.1a is affected, where $\widetilde{\text{price}}_s$ are the uncertain parameters. For the process network in this example, there are three raw materials and two products, so the total number of uncertain parameters in the constraint is 5 (i.e., $|J_i| = 5$). We first study the ellipsoidal and polyhedral sets related robust formulations presented in section 6 and apply them on this constraint. The results are shown in Figure 25. From the results shown in Figure 25, it is seen that when $\Omega \le 1$ and $\Gamma \le 1$, (a) the ellipsoidal and the "interval+ellipsoidal" set based solutions are identical, and (b) the polyhedral and the "interval+polyhedral" set based solutions are identical. This is because the corresponding uncertainty sets are also identical. As $\Omega > 1$ and $\Gamma > 1$, the combined uncertainty sets based solutions are better because their uncertainty sets are smaller with the restriction of the bounded box comparing to the pure ellipsoidal and pure polyhedral set, whose corresponding solutions quickly deteriorate. The above analysis further verifies the earlier observation that for bounded uncertainty, a combination set is preferred to obtain less conservative solution. Finally, considering the "interval+ellispsoidal+polyhedral" set will only lead to solutions between the "interval+ellispoidal" and the "interval+polyhderal" cases and require a more complex model. Hence it is not suggested for the solution of robust scheduling problems.

Table 7. Worst-Case Scenario Solution

	deterministic	price uncertainty	processing time uncertainty	demand uncertainty
objective value	1088.75	959.56	974.95	1032.71

2. Processing Time Uncertainty. Here we consider only processing time uncertainty in constraints 7.1g, where ${}_{i,j}$ and ${}_{i,j}$ are uncertain parameters. Thus, every such constraint has two uncertain parameters (i.e., $|J_i| = 2$). We study the ellipsoidal and polyhedral sets related robust formulations presented in section 6 and apply them on this constraint. The results are shown in Figure 26. From the solution, same conclusions can be made as in the analysis for price uncertainty.

3. Demand Uncertainty. Considering only demand uncertainty, then constraints 7.1f are affected. For each one of these constraints, there is only uncertain parameter on the RHS of the constraint, and the uncertain parameter is the demand data $\tilde{r}_s$. Considering that the uncertainty is bounded, we only need to study the box set and those combined sets. Since the number of uncertain parameters is 1, for each constraint, the different uncertainty sets are reduced to 1-dimension interval set which can be described as

$$U = \{\xi_i || \xi_i| \le \Delta\} \tag{7.2}$$

where Δ is defined as Ψ, min(Ω,1), min(Γ,1), min(Ω,Γ,1) for the box, "interval+ellipsoidal", "interval+polyhedral", "interval +polyhedral+ellipsoidal" uncertainty set, respectively. Thus, the different uncertainty set induced robust counterpart formulations will be identical with same uncertainty set parameter value Δ. Here, we plot the result of their robust counterpart solution as shown in Figure 27.

Finally, we studied the worst-case scenario solution for the different uncertainty cases. The worst-case scenario solution means that the uncertainty set covers the whole uncertainty space. Among the different uncertainty sets to cover the whole bounded uncertain space, the box uncertainty set takes the smallest size, and here the box set with $\Psi = 1$ (i.e., interval set) is applied for the three types on uncertainty individually and the results are shown in Table 7. Comparing the result, we can conclude that with the given uncertainty characteristics, the price uncertainty has the largest effect on the final profit, whereas the demand uncertainty has the least effect on the final profit.

8. CONCLUSIONS

Set-induced robust counterpart optimization techniques are systematically studied in this paper. Several important uncertainty sets are studied, including those studied in the literature and also several new ones proposed in this work. New uncertainty sets such as the adjustable box, ellipsoidal, polyhedral, and "interval+ellipsoidal+polyhedral" set are introduced and their relationship with some well-known uncertainty sets presented in the literature is discussed. The relationships between those different uncertainty sets are extensively discussed, and useful insights are gained for their corresponding robust counterpart models. For uncertainty in the left-hand side, right-hand side, and objective function, the robust counterpart formulations induced by those different uncertainty sets for linear optimization problems and mixed integer linear optimization problems are derived. The different uncertainty set based robust counterpart formulations are also compared through numerical studies, a production planning and a process scheduling problem.

APPENDIX A

Derivation of the robust counterpart for a linear constraint under simultaneous LHS and RHS uncertainty.

Consider the ith linear constraint of problem 2.1 with simultaneous LHS and the RHS uncertainty:

$$\sum_{j \notin J_i} a_{ij}x_j + \sum_{j \in J_i} \tilde{a}_{ij}x_j \le \tilde{b}_i \tag{A.1}$$

where $\tilde{a}_{ij} = a_{ij} + \xi_{ij\hat{a}}ij\ \forall j \in J_i$, $\tilde{b}_i = b_i + \xi_{i0\hat{b}i}$. Incorporating auxiliary variable x_0 and an additional constraint $x_0 = -1$, the constraint can be rewritten as

$$b_i x_0 + \sum_j a_{ij}x_j + [\xi_{i0}\hat{b}_i x_0 + \sum_{j \in J_i} \xi_{ij}\hat{a}_{ij}x_j] \le 0 \tag{A.2}$$

With a given uncertainty set U for ξ_{i0} and ξ_{ij}, the corresponding set induced robust counterpart is

$$b_i x_0 + \sum_j a_{ij}x_j + [\max_{\xi \in U}\{\xi_{i0}\hat{b}_i x_0 + \sum_{j \in J_i} \xi_{ij}\hat{a}_{ij}x_j\}] \le 0 \tag{A.3}$$

With the following definition

$$\xi_i = [\xi_{i0}; \{\xi_{ij}\}] \tag{A.4a}$$

$$A_i = [b_i, \{a_{ij}\}] \tag{A.4b}$$

$$\hat{A}_i = [\hat{b}_i, \{\hat{a}_{ij}\}] \tag{A.4c}$$

$$X = [x_0; \{x_j\}] \tag{A.4d}$$

$$J_i' = J_i \cup \{0\} \tag{A.4e}$$

constraint A.3 can be rewritten as

$$\sum_j A_{ij}X_j + \max_{\xi \in U}\{\xi_i\hat{A}_iX\} \le 0 \tag{A.5}$$

Property A.1. The box uncertainty set 3.1 induced robust counterpart formulation A.5 is equivalent to

$$\sum_j a_{ij}x_j + \Psi\left[\sum_{j \in J_i'} \hat{a}_{ij}|x_j| + \hat{b}_i\right] \le b_i \tag{A.6}$$

Proof. Applying Property 3.1 on A.5, we obtain the following equivalent problem

$$\sum_j A_{ij}X_j + \left[\Psi \sum_{j \in J_i} \hat{A}_{ij}|X_j|\right] \le 0$$

Expanding the above constraints using the previously defined variables, the resulting robust counterpart formulation is

$$b_i x_0 + \sum_j a_{ij}x_j + \Psi[\sum_{j \in J_i} \hat{a}_{ij}|x_j| + \hat{b}_i|x_0|] \le 0$$

Notice that $x_0 = -1$, so the absolute value operation on them is automatically eliminated. The final robust counterpart formulation is

$$\sum_j a_{ij}x_j + \Psi[\sum_{j \in J_i} \hat{a}_{ij}|x_j| + \hat{b}_i] \le b_i$$

Property A.2. The ellipsoidal uncertainty set induced robust counterpart formulation A.5 is equivalent to

$$\sum_j a_{ij}x_j + \left[\Omega\sqrt{\sum_{j \in J_i} \hat{a}_{ij}^2 x_j^2 + \hat{b}_i^2}\right] \le b_i \tag{A.7}$$

Proof. Applying Property 3.2 on A.5, the ellipsoidal based uncertainty set induced robust counterpart is

$$\sum_j A_{ij}X_j + \left[\Omega\sqrt{\sum_{j \in J_i} \hat{A}_{ij}^2 X_j^2}\right] \le 0$$

Expanding the above constraints, the resulting robust counterpart formulation is

$$b_i x_0 + \sum_j a_{ij}x_j + \Omega\sqrt{\sum_{j \in J_i} \hat{a}_{ij}^2 x_j^2 + \hat{b}_i^2 x_0^2} \le 0$$

Notice that $x_0 = -1$, so the final robust counterpart formulation is

$$\sum_j a_{ij}x_j + \left[\Omega\sqrt{\sum_{j \in J_i} \hat{a}_{ij}^2 x_j^2 + \hat{b}_i^2}\right] \le b_i$$

Property A.3. The polyhedral uncertainty set induced robust counterpart formulation A.5 is equivalent to

$$\begin{cases} \sum_j a_{ij}x_j + z_i\Gamma \le b_i \\ z_i \ge \hat{a}_{ij}|x_j| \quad \forall j \in J_i,\ z_i \ge \hat{b}_i \end{cases} \tag{A.8}$$

Proof. Applying Property 3.3 on A.5, the ellipsoidal based uncertainty set induced robust counterpart is

$$\begin{cases} \sum_j A_{ij}X_j + \Gamma z_i \le 0 \\ z_i \ge \hat{A}_i|X_j|, \quad \forall j \in J_i \end{cases}$$

Expanding the above constraints, the resulting robust counterpart formulation is

$$\begin{cases} b_i x_0 + \sum_j a_{ij}x_j + z_i\Gamma \le 0 \\ z_i \ge \hat{a}_{ij}|x_j| \quad \forall\ j \in J_i; \quad z_i \ge \hat{b}_i x_0 \end{cases}$$

Notice that $x_0 = -1$, so the final robust counterpart formulation is

$$\begin{cases} \sum_j a_{ij}x_j + z_i\Gamma \le b_i \\ z_i \ge \hat{a}_{ij}|x_j| \quad \forall\ j \in J_i,\ z_i \ge \hat{b}_i \end{cases}$$

Property A.4. The "interval+ellipsoidal" uncertainty set induced robust counterpart formulation A.5 is equivalent to

$$\sum_j a_{ij}x_j + \left[\sum_{j\in J_i} \hat{a}_{ij}|x_j - z_{ij}| + \hat{b}_i|1 + z_{i0}| + \Omega\sqrt{\sum_{j\in J_i} \hat{a}_{ij}^{\,2} z_{ij}^{\,2} + \hat{b}_i^{\,2} z_{i0}^{\,2}}\right] \le b_i \tag{A.9}$$

Proof. Applying Property 3.4 on A.5, the "interval+ellipsoidal" based uncertainty set induced robust counterpart is

$$\sum_j A_{ij}X_j + \left[\sum_{j\in J_i} \hat{A}_{ij}|X_j - z_{ij}| + \Omega\sqrt{\sum_{j\in J_i} \hat{A}_{ij}^{\,2} z_{ij}^{\,2}}\right] \le 0$$

Expanding the above constraints, the resulting robust counterpart formulation is

$$b_i x_0 + \sum_j a_{ij}x_j + \left[\sum_{j\in J_i} \hat{a}_{ij}|x_j - z_{ij}| + \hat{b}_i|x_0 - z_{i0}| + \Omega\sqrt{\sum_{j\in M_i} \hat{a}_{ij}^{\,2} z_{ij}^{\,2} + \hat{b}_i^{\,2} z_{i0}^{\,2}}\right] \le 0.$$

Notice that $x_0 = -1$, the final robust counterpart formulation is

$$\sum_j a_{ij}x_j + \left[\sum_{j\in J_i} \hat{a}_{ij}|x_j - z_{ij}| + \hat{b}_i|1 + z_{i0}| + \Omega\sqrt{\sum_{j\in J_i} \hat{a}_{ij}^{\,2} z_{ij}^{\,2} + \hat{b}_i^{\,2} z_{i0}^{\,2}}\right] \le b_i$$

Property A.5. The "interval+polyhedral" uncertainty set induced robust counterpart formulation A.5 is equivalent to

$$\begin{cases} \sum_j a_{ij}x_j + \left[z_i\Gamma + \sum_{j\in J_i} p_{ij} + p_{i0}\right] \le b_i \\ z_i + p_{ij} \ge \hat{a}_{ij}|x_j| \quad \forall\ j \in J_i, \quad z_i + p_{i0} \ge \hat{b}_i \\ z_i \ge 0,\ p_{ij} \ge 0,\ p_{i0} \ge 0 \end{cases} \tag{A.10}$$

Proof. Applying Property 3.5 on A.5, the robust counterpart is

$$\begin{cases} \sum_j A_{ij}x_j + \sum_{j\in J_i} p_{ij} + \Gamma z_i \le 0 \\ z_i + p_{ij} \ge \hat{A}_{ij}|x_j| \quad \forall\ j \in J_i \\ z_i \ge 0,\ p_{ij} \ge 0 \end{cases}$$

Expanding the above constraints, the resulting robust counterpart formulation is

$$\begin{cases} b_i x_0 + \sum_j a_{ij}x_j + \left[\sum_{j\in J_i} p_{ij} + p_{i0} + z_i\Gamma\right] \le 0 \\ z_i + p_{ij} \ge \hat{a}_{ij}|x_j|, \quad \forall\ j \in J_i; \quad z_i + p_{i0} \ge \hat{b}_i|x_0| \\ z_i \ge 0, p_{ij} \ge 0,\ p_{i0} \ge 0 \end{cases}$$

Notice that $x_0 = -1$, so the final robust counterpart formulation is

$$\begin{cases} \sum_j a_{ij}x_j + \left[z_i\Gamma + \sum_{j\in J_i} p_{ij} + p_{i0}\right] \le b_i \\ z_i + p_{ij} \ge \hat{a}_{ij}|x_j| \quad \forall\ j \in J_i, \quad z_i + p_{i0} \ge \hat{b}_i \\ z_i \ge 0, p_{ij} \ge 0, p_{i0} \ge 0 \end{cases}$$

Property A.6. The "interval+polyhedral+ellipsoidal" uncertainty set induced robust counterpart formulation A.5 is equivalent to

$$\begin{cases} \sum_j a_{ij}x_j + \left[z_i\Gamma + \sum_{j\in J_i} |p_{ij}| + |p_{i0}| + \Omega\sqrt{\sum_{j\in J_i} w_{ij}^2 + w_{i0}^2}\right] \le b_i \\ z_i \ge |-\hat{b}_i - p_{i0} - w_{i0}| \quad \text{as} \quad z_i \ge |\hat{b}_i + p_{i0} + w_{i0}| \end{cases} \tag{A.11}$$

Proof. Applying Property 3.6 on A.5, the robust counterpart is

$$\begin{cases} \sum_j A_{ij}x_j + \left[\sum_{j\in J_i} |p_{ij}| + \Omega\sqrt{\sum_{j\in J_i} w_{ij}^2} + \Gamma z_i\right] \le 0 \\ z_i \ge |\hat{A}_{ij}x_j - p_{ij} - w_{ij}| \quad \forall\ j \in J_i \end{cases}$$

Expanding the above constraints, the resulting robust counterpart formulation is

$$\begin{cases} b_i x_0 + \sum_j a_{ij} x_j + \left[\sum_{j \in J_i} |p_{ij}| + |p_{i0}| + \Omega \sqrt{\sum_{j \in J_i} w_{ij}^2 + w_{i0}^2} + z_i \Gamma \right] \le 0 \\ z_i \ge |\hat{a}_{ij} x_j - p_{ij} - w_{ij}| \quad \forall\, j \in J_i; \quad z_i \ge |\hat{b}_i x_0 - p_{i0} - w_{i0}| \end{cases}$$

Notice that $x_0 = -1$, so the final robust counterpart formulation is

$$\begin{cases} \sum_j a_{ij} x_j + \left[z_i \Gamma + \sum_{j \in J_i} |p_{ij}| + |p_{i0}| + \Omega \sqrt{\sum_{j \in J_i} w_{ij}^2 + w_{i0}^2} \right] \le b_i \\ z_i \ge |\hat{a}_{ij} x_j - p_{ij} - w_{ij}| \quad \forall\, j \in J_i, \quad z_i \ge |\hat{b}_i + p_{i0} + w_{i0}| \end{cases}$$

APPENDIX B

Derivation of the robust counterpart for a mixed integer linear constraint under simultaneous LHS and RHS uncertainty

As presented in section 6.1, the robust counterpart formulation for the ith mixed integer linear constraint in problem 2.7 with simultaneous LHS and RHS uncertainty can be rewritten as eq 6.3; that is,

$$\sum_j A_{ij} X_j + \max_{\xi_i \in U} \left\{ \sum_{j \in J_i} \xi_{ij} \hat{A}_{ij} X_j \right\} \le 0$$

where A_i, $\hat{A}_i$, X, ξ_i, and J_i are defined in equations 6.2. In the following, proofs for properties 6.1−6.6 are presented.

B.1. Proof of Property 6.1. Notice that the derivation procedure in Section 4 for the robust linear counterpart constraint also applies for the mixed integer linear constraint since it applies for both continuous and integer variables. So, applying Property 4.1 on constraint 6.3, we can obtain the following equivalent problem

$$\sum_j A_{ij} X_j + \left[\Psi \sum_{j \in J_i} \hat{A}_{ij} |X_j| \right] \le 0$$

Expand the above constraints using the definition in equations 6.2 and the resulting robust counterpart formulation is

$$p_i x_0 + \sum_m a_{im} x_m + \sum_k b_{ik} y_k + \Psi \left[\sum_{m \in M_i} \hat{a}_{im} |x_m| + \sum_{k \in K_i} \hat{b}_{ik} |y_k| + \hat{p}_i |x_0| \right] \le 0$$

Notice that $x_0 = -1$, so the final robust counterpart formulation 6.4 is obtained:

$$\sum_m a_{im} x_m + \sum_k b_{ik} y_k + \Psi \left[\sum_{m \in M_i} \hat{a}_{im} |x_m| + \sum_{k \in K_i} \hat{b}_{ik} |y_k| + \hat{p}_i \right] \le p_i$$

B.2. Proof of Property 6.2. Applying Property 4.2 on 6.3, we obtain the following equivalent problem

$$\sum_j A_{ij} X_j + \left[\Omega \sqrt{\sum_{j \in J_i} \hat{A}_{ij}^2 X_j^2} \right] \le 0$$

Expand the above constraints using the definition in equations 6.2, and then the resulting robust counterpart formulation is

$$p_i x_0 + \sum_m a_{im} x_m + \sum_m b_{ik} y_k + \Omega \sqrt{\sum_{m \in M_i} \hat{a}_{im}^2 x_m^2 + \sum_{k \in K_i} \hat{b}_{ik}^2 y_k^2 + \hat{p}_i^2 x_0^2} \le 0$$

Notice that $x_0 = -1$, so the final robust counterpart formulation 6.7 is obtained:

$$\sum_m a_{im} x_m + \sum_m b_{ik} y_k + \Omega \sqrt{\sum_{m \in M_i} \hat{a}_{im}^2 x_m^2 + \sum_{k \in K_i} \hat{b}_{ik}^2 y_k^2 + \hat{p}_i^2} \le p_i$$

B.3. Proof of Property 6.3. Applying Property 4.3 on 6.3, we obtain the following equivalent problem

$$\begin{cases} \sum_j A_{ij} X_j + \Gamma z_i \le 0 \\ z_i \ge \hat{A}_{ij} |X_j|, \quad \forall j \in J_i \end{cases}$$

Expand the above constraints using the definition in equations 6.2, and the resulting robust counterpart formulation is

$$\begin{cases} p_i x_0 + \sum_m a_{im} x_m + \sum_k b_{ik} y_k + z_i \Gamma \le 0 \\ z_i \ge \hat{a}_{im} |x_m| \quad \forall m \in M_i \\ z_i \ge \hat{b}_{ik} |y_k| \quad \forall\, k \in K_i \\ z_i \ge \hat{p}_i |x_0| \end{cases}$$

Notice that $x_0 = -1$, so the final robust counterpart formulation 6.8 is obtained:

$$\begin{cases} \sum_m a_{im} x_m + \sum_k b_{ik} y_k + z_i \Gamma \le p_i \\ z_i \ge \hat{a}_{im} |x_m| \quad \forall\, m \in M_i \\ z_i \ge \hat{b}_{ik} |y_k| \quad \forall\, k \in K_i \\ z_i \ge \hat{p}_i \end{cases}$$

B.4. Proof of Property 6.4. Applying Property 4.4 on 6.3, we obtain the following equivalent problem

$$\sum_j A_{ij} X_j + \left[\sum_{j \in J_i} \hat{A}_{ij} |X_j - z_{ij}| + \Omega \sqrt{\sum_{j \in J_i} \hat{A}_{ij}^2 z_{ij}^2} \right] \le 0$$

Expand the above constraints using the definition in equations 6.2, and the resulting robust counterpart formulation is obtained:

$$p_i x_0 + \sum_m a_{im} x_m + \sum_k b_{ik} y_k + \sum_{m \in M_i} \hat{a}_{im} |x_m - z_{im}| + \sum_{m \in K_i} \hat{b}_{ik} |y_k - z_{ik}| + \hat{p}_i |x_0 - z_{i0}| + \Omega \sqrt{\sum_{m \in M_i} \hat{a}_{im}{}^2 z_{im}{}^2 + \sum_{k \in K_i} \hat{b}_{ik}{}^2 z_{ik}{}^2 + \hat{p}_i{}^2 z_{i0}{}^2} \leq 0$$

Notice that $x_0 =- 1$, so the final robust counterpart formulation 6.10 is obtained:

$$\sum_m a_{im} x_m + \sum_k b_{ik} y_k + \sum_{m \in M_i} \hat{a}_{im} |x_m - z_{im}| + \sum_{m \in K_i} \hat{b}_{ik} |y_k - z_{ik}| + \hat{p}_i |1 + z_{i0}| + \Omega \sqrt{\sum_{m \in M_i} \hat{a}_{im}{}^2 z_{im}{}^2 + \sum_{k \in K_i} \hat{b}_{ik}{}^2 z_{ik}{}^2 + \hat{p}_i{}^2 z_{i0}{}^2} \leq p_i$$

B.5. Proof of Property 6.5. Applying Property 4.5 on 6.3, we obtain the following equivalent problem

$$\begin{cases} \sum_j A_{ij} X_j + \sum_{j \in J_i} w_{ij} + \Gamma z_i \leq 0 \\ z_i + w_{ij} \geq \hat{A}_{ij} |X_j| \quad \forall\ j \in J_i \\ z_i \geq 0, w_{ij} \geq 0 \end{cases}$$

Expand the above constraints using the definition in equations 6.2, and the resulting robust counterpart formulation is

$$\begin{cases} p_i x_0 + \sum_m a_{im} x_m + \sum_k b_{ik} y_k + [\sum_{m \in M_i} w_{im} + \sum_{k \in K_i} w_{ik} + w_{i0} + z_i \Gamma] \leq 0 \\ z_i + w_{im} \geq \hat{a}_{im} |x_m| \quad \forall\ m \in M_i \\ z_i + w_{ik} \geq \hat{b}_{ik} |y_k| \quad \forall\ k \in K_i \\ z_i + w_{i0} \geq \hat{p}_i |x_0| \end{cases}$$

Notice that $x_0 = -1$, so the final robust counterpart formulation 6.12 is obtained:

$$\begin{cases} \sum_m a_{im} x_m + \sum_k b_{ik} y_k + [z_i \Gamma_i + \sum_{m \in M_i} w_{im} + \sum_{k \in K_i} w_{ik} + w_{i0}] \leq p_i \\ z_i + w_{im} \geq \hat{a}_{im} |x_m| \quad \forall\ m \in M_i \\ z_i + w_{ik} \geq \hat{b}_{ik} |y_k| \quad \forall\ k \in K_i \\ z_i + w_{i0} \geq \hat{p}_i \end{cases}$$

B.6. Proof of Property 6.6. Applying Property 4.6 on 6.3, we obtain the following equivalent problem

$$\begin{cases} \sum_j A_{ij} X_j + \left[\sum_{j \in J_i} |q_{ij}| + \Omega \sqrt{\sum_{j \in J_i} w_{ij}{}^2} + \Gamma z_i \right] \leq 0 \\ z_i \geq |\hat{A}_{ij} X_j - q_{ij} - w_{ij}| \quad \forall\ j \in J_i \end{cases}$$

Expand the above constraints using the definition in equations 6.2, and the resulting robust counterpart formulation is

$$\begin{cases} p_i x_0 + \sum_m a_{im} x_m + \sum_k b_{ik} y_k + \sum_{m \in M_i} |q_{im}| + \sum_{k \in K_i} |q_{ik}| + |q_{i0}| \\ \quad + \Omega \sqrt{\sum_{m \in M_i} w_{im}^2 + \sum_{k \in K_i} w_{ik}^2 + w_{i0}^2} + z_i \Gamma \leq 0 \\ z_i \geq |\hat{a}_{im} x_m - q_{im} - w_{im}| \quad \forall\ m \in M_i \\ z_i \geq |\hat{b}_{ik} y_k - q_{ik} - v_{ik}| \quad \forall\ k \in K_i \\ z_i \geq |\hat{p}_i x_0 - q_{i0} - w_{i0}| \end{cases}$$

Notice that $x_0 = -1$, so the final robust counterpart formulation 6.14 is obtained:

$$\begin{cases} \sum_m a_{im} x_m + \sum_k b_{ik} y_k + z_i \Gamma + \sum_{m \in M_i} |q_{im}| + \sum_{k \in K_i} |q_{ik}| + |q_{i0}| \\ \quad + \Omega \sqrt{\sum_{m \in M_i} w_{im}{}^2 + \sum_{k \in K_i} w_{ik}{}^2 + w_{i0}{}^2} \leq p_i \\ z_i \geq |\hat{a}_{im} x_m - q_{im} - w_{im}| \quad \forall\ m \in M_i \\ z_i \geq |\hat{b}_{ik} y_k - q_{ik} - w_{ik}| \quad \forall\ k \in K_i \\ z_i \geq |\hat{p}_i + q_{i0} + w_{i0}| \end{cases}$$

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■ ACKNOWLEDGMENT

The authors gratefully acknowledge financial support from the National Science Foundation (CMMI-0856021) and the National Institute of Health (5R01LM009338).

■ NOMENCLATURE FOR THE PROCESS SCHEDULING MODEL (7.1)

$i \in I$ = tasks
I_s = tasks which produce or consume state (s)
I_j = tasks which can be performed in unit (j)
$j \in J$ = units
J_i = units which are suitable for performing task (i)
$n \in N$ = event points representing the beginning of a task
$s \in S$ = states
S_p = states belong to products
S_r = states belong to raw materials
price_s = price of state (s)
STI_s = initial amount of state (s)
STF_s = final amount of state (s)
$d_{s,n}$ = amount of state (s) delivered to the market at event point (n)
$wv_{i,j,n}$ = binary, whether or not task (i) in unit (j) start at event point (n)
$st_{s,n}$ = continuous, amount of state (s) at event point (n)
$\rho^P_{s,i}, \rho^C_{s,i}$ = proportion of state (s) produced, consumed by task(i), respectively
$b_{i,j,n}$ = amount of material undertaking task (i) in unit (j) at event point (n)
$st^{\max}{}_s$ = available maximum storage capacity for state (s)
$V^{\min}_{i,j}, V^{\max}_{i,j}$ = minimum amount, maximum capacity of unit (j) when processing task (i)
r_s = market demand for state (s) at the end of the time horizon

$Tf_{i,j,n}$ = time at which task (i) finishes in unit (j) while it starts at event point (n)
$Ts_{i,j,n}$ = time at which task (i) starts in unit (j) at event point (n)
$\alpha_{i,j}$, $\beta_{i,j}$ = variable term of processing time of task (i) in unit (j)
H = time horizon

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