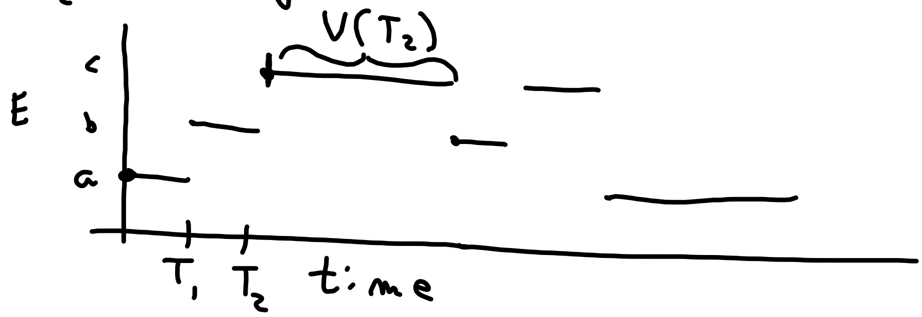
Markov Processes $\{Y(t)\}$

$$P\{Y(t+s)=j \mid Y(u);\ u\le t\} = P\{Y(t+s)=j \mid Y(t)\}$$



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$$P_t(i,j) = P\{Y(t)=j \mid Y(0)=i\}$$

$$P_s(i,j) = P\{Y(t+s)=j \mid Y(t)=i\}$$ ← does not depend on t ⇒ stationary ⇒ time homogenoues

$t \to P_t(i,j)$ transtion functons

For each t and s

$$P_t(i,j) \ge 0$$

$$\sum_k P_t(i,k) = 1$$

$$\sum_k P_t(i,k)\, P_s(k,j) = P_{t+s}(i,j)$$

Chapman-Kolmogorov equation

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Let $V(t) = \inf\{s > 0 : Y(t+s) \neq Y(t)\}$

time to next jump

$P\{V(t) > u \mid Y(t) = i\} = e^{-\lambda(i)u}$ for $u \geq 0$

Def. A state i is called absorbing if $\lambda(i) = 0$.
State i is stable if $0 < \lambda(i) < \infty$.
State i is instantaneous if $\lambda(i) = \infty$.

$T_0 = 0$

$T_{n+1} = T_n + V(T_n)$

$X_n = Y(T_n)$

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$P\{X_{n+1} = j, T_{n+1} - T_n > u \mid X_0, \dots, X_n; T_0, \dots, T_n\} = \underline{Q(i,j)}e^{-\lambda(i)u}$

$P\{T_{n+1} - T_n > u \mid X_n = i, X_{n+1} = j\} = P\{T_{n+1} - T_n > u \mid X_n = i\} = e^{-\lambda(i)u}$

$P\{X_{n+1} = j \mid X_n = i\} = Q(i,j)$

For small t, $P_t(i,i) \approx e^{-\lambda(i)t}$

$$\lim_{t \to 0} \frac{1 - P_t(i,i)}{t} = \lim_{t \to 0} \frac{1 - e^{-\lambda(i)t}}{t} = \lambda(i)$$

Cor. $i \neq j$ $\displaystyle\lim_{t \to 0} \frac{P_t(i,j)}{t} = \lambda(i)Q(i,j)$

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Kolmogorov backward eq.

move to LHS

$$P_{t+s}(i,j) = \sum_k P_s(i,k)\,P_t(k,j)$$

$$= \sum_{k\neq i} P_s(i,k)\,P_t(k,j) + P_s(i,i)\,P_t(i,j) + P_t(i,j) - P_t(i,j)$$

$$\frac{P_{t+s}(i,j) - P_t(i,j)}{s} = \sum_{k\neq i} \frac{P_s(i,k)}{s} P_t(k,j) - \frac{\left[1 - P_s(i,i)\right]}{s} P_t(i,j)$$

Form new matrix

$$G(i,j) = \begin{cases} \lambda(i)\,Q(i,j) & \text{if } i \neq j \\ -\lambda(i) & \text{if } i = j \end{cases}$$

take limit as $s \to 0$

$$\frac{d}{dt} P_t = G P_t \quad \text{also} \quad \frac{d}{dt} P_t = P_t \cdot G$$

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three cities a, b, c

when in a, flips coin $H \to b$, $T \to c$

when not in a, flips two coins
$HH \to$ other town
otherwise $\to a$

Avg. time in a is 2 weeks
b is 1 week
c is $1\frac{1}{2}$ weeks

Q, given $\lambda(a), \lambda(b), \lambda(c)$

$G \leftarrow$ generator matrix

$$G = \begin{bmatrix} -\frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{3}{4} & -1 & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{6} & -\frac{2}{3} \end{bmatrix}$$

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$$f'(t) = a f(t) \Rightarrow f(t) = e^{at}$$

$$P'(t) = G P(t) \Rightarrow P(t) = e^{Gt}$$

$$\boxed{P_t(i,j) = e^{Gt}(i,j)}$$

$$e^{a} = \sum_{n=0}^{\infty} \frac{a^n}{n!} \qquad e^{A} = \sum_{n=0}^{\infty} \frac{A^n}{n!}$$

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If the imbedded Markov chain is recurrent, then the Markov process is called recurrent.

For a recurrent Markov process

$$\lim_{t \to \infty} P\{Y(t) = j \mid Y(0) = i\} = p(j) \text{ where}$$

$$pG = 0 \text{ and } \sum p(j) = 1$$

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t f(Y(s))\,ds = p \cdot f$$

total reward in $[0,t]$

where $f(i)$ is rate of reward while is state i

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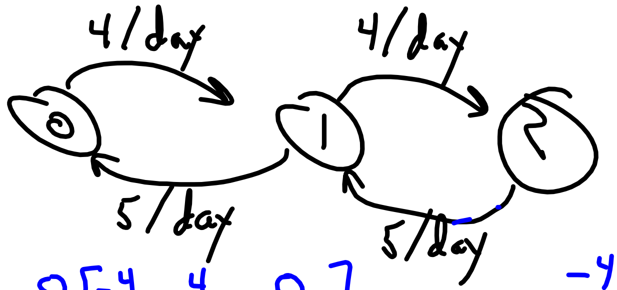
Repair shop, only two machines can
be in shop at one time.
Arrival according to Poisson process of 4/day
Repair can be 5/day if always busy
state space is $\{0, 1, 2\}$



birth-death

$$G = \begin{array}{c} 0 \\ 1 \\ 2 \end{array} \begin{bmatrix} -4 & 4 & 0 \\ 5 & -9 & 4 \\ 0 & 5 & -5 \end{bmatrix} \longrightarrow$$

$$-4p_0 + 5p_1 = 0$$
$$4p_0 - 9p_1 + 5p_2 = 0$$
$$\cancel{4p_1 - 5p_2 = 0}$$
$$p_0 + p_1 + p_2 = 1$$

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