STAT 611 600: Theory of Inference **Spring 2021**

Homework 10: due Thursday, April 29, 2021, 11:59 pm CDT

Professor: Tiandong Wang *Name:* *UIN:*

Instructions:

- Whether you write out the solution by hand or in a text document, be sure that they are neat, legible and in order (even if you choose to solve them in different order). We highly recommend that you write your solutions in **LaTeX** and print them to a **PDF file**.
- Write/Type your name, UIN at the top of the first page. Otherwise, your submission will not be graded.
- Either scan or print your solutions to a **PDF file** under 15MB in size. It must be in a single file, not separate files for separate pages. Do not take a photo of each page and then paste them into a document - this will make your file too big and the results will generally not be very readable anyway.
- All students should login to their eCampus account to upload your file. You must do this by **11:59 pm U.S. Central time**, on the due date. You can make multiple submissions, but only the last submission will be graded.
- Write down all of your problem-solving process and cite any resources you have used in addition to lecture notes and the textbook.
- It is prohibited to share or distribute the content in this document.

1. 10.9 in C&B.

2. The Hardy-Weinberg law in genetics says that the proportions of genotypes AA, Aa, and aa are θ^2, $2\theta(1-\theta)$, and $(1-\theta)^2$, respectively, where $\theta \in [0,1]$. Suppose that in a sample of n from the population (small relative to the size of the population). We observe X_1 individuals of type AA, X_2 individuals of type Aa, and X_3 individuals of type aa.

   (a) What is the MLE of θ.

   (b) Suppose that $X_1 = 10$, $X_2 = 68$, and $X_3 = 112$ individuals of the three genotypes are observed. Find a LRT with asymptotic size α for testing

   $$H_0 : \theta = 1/2, \qquad \text{v.s.} \qquad H_1 : \theta \neq 1/2.$$

3. **(Variance stabilizing functions)** If $\mathbb{E}(X_i) = \mu$ is the parameter of interest, the central limit theorem gives

   $$\sqrt{n}(\bar{X}_n - \mu) \xrightarrow{d} N\big(0, \sigma^2(\mu)\big),$$

   i.e. the variance of the limiting distribution is a function of μ. This is a problem if we wish to do inference for μ, because ideally the limiting distribution should not depend on the unknown μ. The delta method gives a possible solution: Since

   $$\sqrt{n}\big(g(\bar{X}_n) - g(\mu)\big) \xrightarrow{d} N\big(0, (g'(\mu))^2\sigma^2(\mu)\big),$$

   we may search for a transformation $g(x)$ such that $(g'(\mu))^2\sigma^2(\mu)$ is a constant. Such a transformation is called a variance stabilizing transformation.

   (a) Suppose that $X_1, \dots, X_n$ are iid normal random variables with mean 0 and variance σ^2. Prove the asymptotic normality of $\frac{1}{n}\sum_{i=1}^n X_i^2$ and find a variance-stabilizing transformation for $\frac{1}{n}\sum_{i=1}^n X_i^2$.

   (b) Suppose $X_1, \dots, X_n$ are iid Bernoulli(p), where $0 < p < 1$. Show that $h(x) = \arcsin(\sqrt{x})$ is a variance stabilizing transformation for $\bar{X}_n$.

Optional exercises from C&B: 10.5 and 10.23.