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Dynamic Traffic Assignment under Uncertainty: A Distributional Robust Chance-Constrained Approach

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Abstract This paper provides a chance-constrained programming approach for transportation planning and operations under uncertainty. The major contribution of this paper is to approximate a joint chance-constrained Cell Transmission Model based System Optimum Dynamic Traffic Assignment with only partial distributional information about uncertainty as a linear program which is computationally efficient. Numerical experiments have been conducted to show the performance of the proposed approach compared with other two workable approaches based on a cumulative distribution function and a sampling method. This new approach can be used as a pragmatic tool for system optimum dynamic traffic control and management.

Keywords Dynamic traffic assignment · Transportation planning · Chance-constrained programming · Joint chance constraint · Data uncertainty

1 Introduction

Traditional dynamic traffic assignment (DTA) models require deterministic input parameters. However, realistically, the time-dependent traffic demand is usually uncertain. Thus, it is important to incorporate uncertainty into DTA from a pragmatic perspective. Recently, chance-constrained programming has been applied as one approach to formulate and analyze a system optimum DTA (SO DTA) model with

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uncertainty (Waller and Ziliaskopoulos (2006), Yazici and Ozbay (2010)). In these works, the exact probability distribution of uncertainty is assumed to be known in prior. Unfortunately, in practice, only the partial distributional information of uncertainty, such as support, moments, etc., may be available. Therefore, there is a need to develop a new approach for the chance-constrained SO DTA model under uncertainty when only partial distributional information is available.

This paper presents a safe and computationally tractable approximation method of the chance-constrained SO DTA model based on known moments of the uncertain time-dependent traffic demand, aiming to provide a robust and tractable framework for dynamic traffic planning and control. Numerical experiments are presented to demonstrate the value of our safe approximation scheme in the context of stochastic dynamic traffic assignment. Moreover, computational viability is illustrated for the proposed framework. This paper adds to the body of knowledge in stochastic DTA by effectively considering issues related to robustness and partial distributional information of uncertainty.

1.1 Literature review

DTA has been recognized as a key element to analyze network design and transportation planning problems. DTA has evolved substantially and appealed to a large group of researchers who are interested in modeling and analyzing transportation operations and engineering issues (see Peeta and Ziliaskopoulos (2001) for a review). Among the mathematical programming approaches, Cell Transmission Model (CTM) proposed by Daganzo (1994, 1995) is a traffic flow propagation model which simulates traffic flows by decomposing a transportation network into cells. Based on the CTM, a linear programming (LP) formulation is proposed by Ziliaskopoulos (2000) for a single destination SO DTA problem. The LP based CTM model can be interpreted as an optimistic approximation since it allows vehicle holding and can provide an improved system optimal solution compared to the original CTM. Nonetheless, LP based CTM formulation is attractive since it is computationally tractable. Recently, Nie (2011) addressed the vehicle holding issue by developing an equivalent Merchant-Nemhauser model for the SO DTA and adopting a successive readjustment scheme.

Although most research in DTA assume deterministic input parameters, increasing efforts are focusing on incorporating uncertainty in the CTM based SO DTA model. Specifically, stochastic programming and robust optimization techniques have been applied to cope with uncertainty. Karoonsoontawong and Waller (2006) applied Monte Carlo bounding techniques to solve the stochastic SO DTA and the user-optimal (UO) DTA. They showed that it is more valuable to solve a stochastic model than a deterministic model. A scenario based robust optimization (RO), originally proposed by Mulvey et al. (1995), was applied by Karoonsoontawong and Waller (2007) to investigate the trade-off between optimality robustness and model robustness under uncertainty. Karoonsoontawong and Waller (2008) developed a scenario based RO formulation of an integrated model which simultaneously solves continuous network capacity expansion, traffic signal optimization and dynamic traffic assignment. Yao et al. (2009) developed a robust linear programming model based on CTM formulation to investigate evacuation transportation planning under uncertainty. Yazici and Ozbay (2010) formulated a CTM based SO DTA model with probabilistic road capacity constraints and a p-level efficient points method is applied. Assuming that the distribution of the uncertain demands is known and independent, Waller and Ziliaskopoulos (2006) applied individual chance-constrained programming (ICCP) to reformulate the CTM based SO DTA and derived deterministic equivalents of the chance constraints by using cumulative distribution function (CDF) of the uncertainty. The solutions to these models have been shown to be computationally tractable. Our paper differs from the aforementioned literature in two aspects. First, we apply joint chance-constrained programming (JCCP) to formulate the CTM based SO DTA and use a decomposition method to approximate the joint chance-constraint as a system of individual chance constraints. Second, we provide an approximation method to solve the individual chance-constrained program based on only partial distributional information.

The constraint with a restricted violation risk is known as a chance constraint or a probability constraint, which was introduced by Charnes et al. (1958). Since an original chance constraint is computationally intractable, a natural course of action is to formulate and solve its deterministic equivalent or tractable approximation. In general, due to potential non-convexity of the constraint and difficulty in calculating the violation probability of a constraint, it would be difficult to formulate the deterministic equivalent of a chance constraint. Therefore, most efforts have been focusing on deriving a computationally tractable approximation whose feasible solutions remain feasible to the original chance constraint. Such an approximation is viewed as "safe" and tractable. Calafiore and El Ghaoui (2006) showed that a chance constraint is second-order cone representable based on moment, support or symmetric information of the uncertainty. Nemirovski and Shapiro (2006) incorporated moment generating functions to provide less conservative approximation of a chance constraint. Chen and Sim (2009) developed a bounding scheme for a conditional-value-at-risk (CVaR) measure so that the CVaR measure can be employed to approximate chance constraint. For research on approximation using sampling techniques, see Calafiore and Campi (2005, 2006), Ergodan and Iyengar (2006) and Luedtke and Ahmed (2008).

In this paper, we focus on a stochastic CTM based SO DTA with partial distributional information (mean and variance) about the uncertain demand. Incorporating uncertainty makes the original model computationally intractable. We develop a safe and tractable convex approximation of the chance-constrained problem and reformulate the model as a deterministic linear program.

The paper is organized as follows. In Section 2, a deterministic SO DTA model is presented and reformulated as a joint chance-constrained program after incorporating uncertainty. In Section 3, Boole's inequality is applied to decompose the joint chance constraint into individual chance constraints and a safe and tractable approximation is derived to reformulate the chance-constrained SO DTA model as a deterministic linear program. In Section 4, numerical experiments are presented and the results are analyzed. Finally, Section 5 concludes the paper.

Notations Random variables or vectors are represented with the tilde sign, such as $\widetilde{d}$ or $\widetilde{g}$. In addition, we use $E(\cdot)$ and var$(\cdot)$ to represent the expectation and variance respectively.

2 Deterministic model and chance-constrained formulation

In this section, we first present a generalized form of CTM based SO DTA. Then, we introduce uncertain demand to this deterministic linear programming (DLP) model and reformulate it as a chance-constrained program.

With defining $A = [a_{ij}]$ as the adjacency matrix where

$$a_{ij} = \begin{cases} 1 & cell\ i\ is\ connected\ to\ cell\ j \\ 0 & else \end{cases}$$

and the notations in Table 1, the CTM based SO DTA model can be presented as follows:

$$\min_{x,y} \sum_{t\in\Im} \sum_{i\in\{C/C_s\}} c_i^t x_i^t (DLP)$$

subject to

$$x_i^t - x_i^{t-1} - \sum_{k\in C} a_{ki} y_{ki}^{t-1} + \sum_{j\in C} a_{ij} y_{ij}^{t-1} = d_i^{t-1} \quad \forall i \in C_R,\ t \in \Im \tag{1}$$

$$x_i^t - x_i^{t-1} - \sum_{k\in C} a_{ki} y_{ki}^{t-1} + \sum_{j\in C} a_{ij} y_{ij}^{t-1} = 0 \quad \forall i \in \{C\backslash C_R\},\ t \in \Im \tag{2}$$

$$\sum_{k\in C} a_{ki} y_{ki}^t \le Q_i^t \quad \forall i \in C, t \in \Im \tag{3}$$

Table 1 Notations

Notation	Meaning
$\Im$	set of time intervals, {1, …, T}
C	set of cells, {1, …, I}
C_s	set of sink cells
C_R	set of source cells
A	adjacency matrix representing transportation network connectivity
c_i^t	travel cost in cell i at time t
x_i^t	number of vehicles contained in cell i at time t
y_{ij}^t	number of vehicles flowing from cell i to cell j at time t
d_i^t	demand generated in cell i at time t
N_i^t	capacity of cell i at time t
Q_i^t	inflow/outflow capacity of cell i at time t
δ_i^t	ratio of the free-flow speed over the backward propagation speed of cell i at time t
$\hat{x}_i$	initial number of vehicles in cell i

$$\sum_{k\in C} a_{ki}y_{ki}^t + \delta_i^t x_i^t \le \delta_i^t N_i^t \quad \forall i \in C, t \in \Im \tag{4}$$

$$\sum_{j\in C} a_{ij}y_{ij}^t \le Q_i^t \quad \forall i \in C, t \in \Im \tag{5}$$

$$\sum_{j\in C} a_{ij}y_{ij}^t - x_i^t \le 0 \quad \forall i \in C,\ t \in \Im \tag{6}$$

$$x_i^0 = \hat{x}_i \quad \forall i \in C \tag{7}$$

$$y_{ij}^0 = 0 \quad \forall (i,\ j) \in \{C \times C\} \tag{8}$$

$$x_i^t \ge 0 \quad \forall i \in C,\ t \in \Im \tag{9}$$

$$y_{ij}^t \ge 0 \quad \forall (i,\ j) \in \{C \times C\},\ t \in \Im \tag{10}$$

The objective function of this model is to minimize the total travel cost. Both constraint (1) and (2) refer to the flow conservation constraints in cell i at time t. Since only the source cells generate demand, we set the right-hand-side of (1) equal to d_i^{t-1} and the right-hand-side of (2) equal to 0. Constraint (3) bounds the total inflow into a cell by an inflow capacity. Constraint (4) ensures that the total inflow into a cell is bounded by the remaining capacity of the cell. Similarly, total outflow from a cell is bounded by outflow capacity and the current occupancy of that cell, which are represented by constraint (5) and (6) respectively. The remaining constraints, from (7) to (10), reflect the initial conditions and non-negativity conditions.

Since the objective function is a minimization problem, we can rewrite constraint (1) by changing the equality into an inequality as follows (Waller and Ziliaskopoulos 2006):

$$x_i^t - x_i^{t-1} - \sum_{k\in C} a_{ki}y_{ki}^{t-1} + \sum_{j\in C} a_{ij}y_{ij}^{t-1} \ge d_i^{t-1} \quad \forall i \in C_R,\ t \in \Im \tag{11}$$

It is easy to identify that the optimal solution to the problem without constraint (11) is zero for all x and y. The introduction of (11) makes it no longer the optimal solution to the problem. Thus, constraint (11) is always binding at the optimal solution and it can replace constraint (1). Note that vehicle holding phenomenon can also happen in this model since constraint (11) is always binding and two equations are equivalent in this model.

When we substitute d_i^{t-1} in constraint (11) by random variable $\widetilde{d}_i^{t-1}$, it is natural to reformulate (11) as a joint chance constraint (12) with a violation risk or a confidence parameter $\varepsilon \in (0,1)$:

$$\Pr\left(x_i^t - x_i^{t-1} - \sum_{k \in C} a_{ki} y_{ki}^{t-1} + \sum_{j \in C} a_{ij} y_{ij}^{t-1} - \tilde{d}_i^{t-1} \geq 0 \quad \forall i \in C_R,\ t \in \Im \right) \geq 1 - \varepsilon \tag{12}$$

Note that the violation of Eq. (12) means that that more demand is realized than that is used for prediction.

Then, the whole model can be reformulated as a JCCP as follows:

$$\min_{x,\,y} \sum_{t \in \Im} \sum_{i \in \{C \backslash C_s\}} c_i^t x_i^t \quad (JCCP)$$

subject to

$$\begin{aligned}
&\Pr\left(x_i^t - x_i^{t-1} - \sum_{k \in C} a_{ki} y_{ki}^{t-1} + \sum_{j \in C} a_{ij} y_{ij}^{t-1} - \tilde{d}_i^{t-1} \geq 0 \quad \forall i \in C_R,\ t \in \Im \right) \geq 1 - \varepsilon \\
&x_i^t - x_i^{t-1} - \sum_{k \in C} a_{ki} y_{ki}^{t-1} + \sum_{j \in C} a_{ij} y_{ij}^{t-1} = 0 \quad \forall i \in \{C \backslash C_R\},\ t \in \Im \\
&\sum_{k \in C} a_{ki} y_{ki}^t \leq Q_i^t \quad \forall i \in C,\ t \in \Im \\
&\sum_{k \in C} a_{ki} y_{ki}^t + \delta_i^t x_i^t \leq \delta_i^t N_i^t \quad \forall i \in C,\ t \in \Im \\
&\sum_{j \in C} a_{ij} y_{ij}^t \leq Q_i^t \quad \forall i \in C,\ t \in \Im \\
&\sum_{j \in C} a_{ij} y_{ij}^t - x_i^t \leq 0 \quad \forall i \in C,\ t \in \Im \\
&x_i^0 = \hat{x}_i \quad \forall i \in C \\
&y_{ij}^0 = 0 \quad \forall (i,j) \in \{C \times C\} \\
&x_i^t \geq 0 \quad \forall i \in C,\ t \in \Im \\
&y_{ij}^t \geq 0 \quad \forall (i,j) \in \{C \times C\},\ t \in \Im
\end{aligned}$$

3 Safe and tractable approximation of the chance-constrained so DTA

A standard method to approximate a joint chance constraint is to decompose it into a certain number of individual chance constraints using Boole's inequality $\left(\Pr\left(\bigcup_i A_i\right) \leq \sum_i \Pr(A_i)\right)$ and then formulate a deterministic equivalent or a safe approximation of each individual chance constraint. Assuming that the exact probability distribution of $\tilde{d}_i^{t-1}$ is known, Waller and Ziliaskopoulos (2006) proposed to use the cumulative distribution function (CDF) of $\tilde{d}_i^{t-1}$ to reformulate each individual constraint in a SO DTA model as an equivalent deterministic linear constraint. However, in practice, it is more likely for the decision maker to know only partial distributional information about the uncertain variable, such as the supports, mean and variance, which can be computed from historical data. Thus, in this section, we provide a theorem to approximate each individual chance constraint based on known mean and variance of $\tilde{d}_i^{t-1}$.

First, we decompose the JCCP by Boole's inequality. The Eq. (12) is equivalent to (13) since there are in total $H \times I_R$ constraints in the joint chance constraint (12), where H is time intervals and I_R is the number of source cells contained in set C_R.

$$\Pr\left(\bigcup_{i\in C_R,\ t\in\Im}\left\{x_i^t - x_i^{t-1} - \sum_{k\in C} a_{ki}y_{ki}^{t-1} + \sum_{j\in C} a_{ij}y_{ij}^{t-1} - \widetilde{d}_i^{t-1} < 0\right\}\right) \leq \varepsilon. \tag{13}$$

By Boole's inequality, we have

$$\begin{aligned}&\Pr\left(\bigcup_{i\in C_R,\ t\in\Im}\left\{x_i^t - x_i^{t-1} - \sum_{k\in C} a_{ki}y_{ki}^{t-1} + \sum_{j\in C} a_{ij}y_{ij}^{t-1} - \widetilde{d}_i^{t-1} < 0\right\}\right)\\ &\leq \sum_{i\in C_R,\ t\in\Im}\Pr\left\{x_i^t - x_i^{t-1} - \sum_{k\in C} a_{ki}y_{ki}^{t-1} + \sum_{j\in C} a_{ij}y_{ij}^{t-1} - \widetilde{d}_i^{t-1} < 0\right\}.\end{aligned}$$

So given $\sum_{i\in C_R,\ t\in\Im}\varepsilon_i^t \leq \varepsilon$, the system of constraints (14) becomes a conservative approximation of (12).

$$\Pr\left\{x_i^t - x_i^{t-1} - \sum_{k\in C} a_{ki}y_{ki}^{t-1} + \sum_{j\in C} a_{ij}y_{ij}^{t-1} - \widetilde{d}_i^{t-1} < 0\right\} \leq \varepsilon_i^t \quad \forall i\in C_R,\ t\in\Im \tag{14}$$

By multiplying negative one on both sides of the inequality inside the probability function, (14) can be rewritten as

$$\Pr\left\{x_i^{t-1} - x_i^t + \sum_{k\in C} a_{ki}y_{ki}^{t-1} - \sum_{j\in C} a_{ij}y_{ij}^{t-1} + \widetilde{d}_i^{t-1} > 0\right\} \leq \varepsilon_i^t \quad \forall i\in C_R,\ t\in\Im$$

which is also equivalent to (15)

$$\Pr\left\{x_i^{t-1} - x_i^t + \sum_{k\in C} a_{ki}y_{ki}^{t-1} - \sum_{j\in C} a_{ij}y_{ij}^{t-1} + \widetilde{d}_i^{t-1} \leq 0\right\} \geq 1-\varepsilon_i^t \quad \forall i\in C_R,\ t\in\Im \tag{15}$$

If we keep ε_i^t as a design variable satisfying $\sum_{i\in C_R,\ t\in\Im}\varepsilon_i^t \leq \varepsilon$ and $\varepsilon_i^t \geq 0 \quad \forall i \in C_R,\ t\in\Im$, the problem after approximation is still non-convex. To resolve this issue, the simplest way is to arbitrarily set $\varepsilon_i^t = \frac{\varepsilon}{H\times I_R}$ (see Nemirovski and Shapiro (2006) for details). Whence, the joint chance constraint (12) can be decomposed into (16) with $H \times I_R$ individual chance constraints.

$$\Pr\left\{x_i^{t-1} - x_i^t + \sum_{k\in C} a_{ki}y_{ki}^{t-1} - \sum_{j\in C} a_{ij}y_{ij}^{t-1} + \widetilde{d}_i^{t-1} \leq 0\right\} \geq 1-\frac{\varepsilon}{H\times I_R} \quad \forall i\in C_R,\ t\in\Im \tag{16}$$

After the decomposition of the JCCP, we present the following theorem to derive the deterministic approximation of the individual chance constraint based on known moments (mean and variance).

Theorem 1 *Assuming that $\widetilde{d}_i^{t-1}$ follows an unknown probability distribution with mean μ_i^t and variance $\left(\sigma_i^t\right)^2$, the set of individual chance constraints (16) in a stochastic DTA model can be approximated by the following set of deterministic* *linear constraints*

$$x_i^{t-1} - x_i^t + \sum_{k \in C} a_{ki} y_{ki}^{t-1} - \sum_{j \in C} a_{ij} y_{ij}^{t-1} + \mu_i^{t-1} + \sigma_i^{t-1} \sqrt{\frac{H \times I_R}{\varepsilon} - 1} \le 0 \quad \forall i \in C_R,\ t \in \Im \tag{17}$$

in which each constraint is safe and computationally tractable.

Proof: Let x be a $(H \times I_R + H \times I_R + 1) \times 1$ column vector:

$$\left(x_1^0 \cdots x_1^H \cdots\cdots x_{I_R}^0 \cdots x_{I_R}^H\ y_{11}^0 \cdots y_{11}^H \cdots\cdots y_{I_R I}^0 \cdots y_{I_R I}^H\ 1 \right)^T$$

By defining the random vector

$$\widetilde{g}_i^t \equiv \left[\left(b_i^t\right)^T, \widetilde{d}_i^{t-1} \right]^T \in R^{H \times I_R + H \times I_R + 1}$$

where $b_i^t \in R^{H \times I_R + H \times I_R}$ is the coefficient vector and $\widetilde{d}_i^{t-1}$ is the last element of the vector $\widetilde{g}_i^t$ such that

$$\left(\widetilde{g}_i^t\right)^T x \equiv \left[\left(b_i^t\right)^T, \widetilde{d}_i^{t-1} \right] x = x_i^{t-1} - x_i^t + \sum_{k \in C} a_{ki} y_{ki}^{t-1} - \sum_{j \in C} a_{ij} y_{ij}^{t-1} + \widetilde{d}_i^{t-1} \quad \forall i \in C_R,\ t \in \Im.$$

Thus, (16) can be represented by (18):

$$\Pr\left\{ \left(\widetilde{g}_i^t\right)^T x \le 0 \right\} \ge 1 - \frac{\varepsilon}{H \times I_R} \quad \forall i \in C_R,\ t \in \Im. \tag{18}$$

With known mean and variance for $\widetilde{d}_i^{t-1}$, we can calculate the mean and variance for $\widetilde{g}_i^t$ as follows:

$$E\left(\widetilde{g}_i^t\right) = \left[\left(b_i^t\right)^T, E\left(\widetilde{d}_i^{t-1}\right) \right]^T = \left[\left(b_i^t\right)^T, \mu_i^{t-1} \right]^T \quad \forall i \in C_R,\ t \in \Im$$

$$\operatorname{var}\left(\widetilde{g}_i^t\right) = \operatorname{var}\left(\left[\left(b_i^t\right)^T, d_i^{t-1} \right]^T \right) = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \left(\sigma_i^{t-1}\right)^2 \end{pmatrix} \ge 0 \quad \forall i \in C_R,\ t \in \Im$$

Note that (18) can be enforced robustly by

$$\inf_{\widetilde{g}_i^t} \Pr\left\{ \left(\widetilde{g}_i^t\right)^T x \le 0 \right\} \le 1 - \frac{\varepsilon}{H \times I_R} \quad \forall i \in C_R,\ t \in \Im \tag{19}$$

Following Calafiore and El Ghaoui (2006), inequality (19) can be reformulated as an equivalent inequality (20)

$$\kappa_\varepsilon \sigma_i^t(x) + E\left[\left(\widetilde{g}_i^t\right)^T x \right] \le 0, \quad \kappa_\varepsilon = \sqrt{\left(1 - \frac{\varepsilon}{H \times I_R}\right) \Big/ \frac{\varepsilon}{H \times I_R}}, \quad \forall i \in C_R,\ t \in \Im \tag{20}$$

where
$$\begin{cases} E\left[\left(\widetilde{g}_i^t\right)^T x \right] = E\left[\left(\widetilde{g}_i^t\right)^T \right] x = \left[\left(b_i^t\right)^T\ \mu_i^{t-1} \right] x \\ \sigma_i^t(x) = \sqrt{\operatorname{var}\left[\left(\widetilde{g}_i^t\right)^T x \right]} = \sqrt{x^T \operatorname{var}\left(\widetilde{g}_i^t\right) x} \end{cases} \quad \forall i \in C_R,\ t \in \Im.$$

After simplification, (20) is equivalent to (17). Since each constraint in (17) is linear and more conservative than its corresponding original individual chance constraint in (16), (17) is a safe and computationally tractable approximation. ■

Finally, the joint chance constrained model can be approximated by the following deterministic linear program, A-JCCP:

$$\min_{x,y} \sum_{t\in\Im} \sum_{i\in\{C\backslash C_s\}} c_i^t x_i^t \quad (A-JCCP)$$

subject to

$$\begin{aligned}
& x_i^{t-1} - x_i^t + \sum_{k\in C} a_{ki} y_{ki}^{t-1} - \sum_{j\in C} a_{ij} y_{ij}^{t-1} + \mu_i^{t-1} + \sigma_i^{t-1}\sqrt{\tfrac{H\times I_R}{\varepsilon} - 1} \le 0 \quad \forall i \in C_R,\ t\in\Im \\
& x_i^t - x_i^{t-1} - \sum_{k\in C} a_{ki} y_{ki}^{t-1} + \sum_{j\in C} a_{ij} y_{ij}^{t-1} = 0 \quad \forall i\in\{C\backslash C_R\},\ t\in\Im \\
& \sum_{k\in C} a_{ki} y_{ki}^t \le Q_i^t \quad \forall i\in C,\ t\in\Im \\
& \sum_{k\in C} a_{ki} y_{ki}^t + \delta_i^t x_i^t \le \delta_i^t N_i^t \quad \forall i\in C,\ t\in\Im \\
& \sum_{j\in C} a_{ij} y_{ij}^t \le Q_i^t \quad \forall i\in C,\ t\in\Im \\
& \sum_{j\in C} a_{ij} y_{ij}^t - x_i^t \le 0 \quad \forall i\in C,\ t\in\Im \\
& x_i^0 = \hat{x}_i \quad \forall i\in C \\
& y_{ij}^0 = 0 \quad \forall (i,j)\in\{C\times C\} \\
& x_i^t \ge 0 \quad \forall i\in C,\ t\in\Im \\
& y_{ij}^t \ge 0 \quad \forall (i,j)\in\{C\times C\},\ t\in\Im
\end{aligned}$$

4 Numerical examples

In order to employ chance-constrained programming, distributional information is required. If only partial information of the uncertainty is available, the real distribution can be any distribution with the given mean and variance. In this case, one safe approach is to first solve the chance-constrained program with uncertain data following all possible distributions. Then, a robust solution can be chosen in order to guarantee the feasibility of the solution. But obviously, this is impractical due to its complexity. A simpler way is to just arbitrarily assume one certain distribution with the given mean and variance for the uncertainty, and then either use cumulative distribution function (CDF) based approach or scenario based approach to solve the chance-constrained program. The purpose of the numerical experiments is to show how the performance can be negatively impacted by such an assumption and the advantage of the safe approximation proposed. The value of our approach stands out when comparing it with the CDF based approach and the sampling based approach by two numerical tests.

4.1 Test network

Aforementioned approaches are tested on an example network shown in Fig. 1. This network consists of 20 nodes and 25 links. The links can be classified into two groups: freeway links (links in the center of the network), and arterial links (the outer and crosslink). The characteristics of the network include length, the number of lanes, speed limit and capacity limit of the roads. By using this network information and setting 6 s as the length of a time interval, we can further illustrate this network as an equivalent cell transmission model consisting of 62 cells. Among them, there

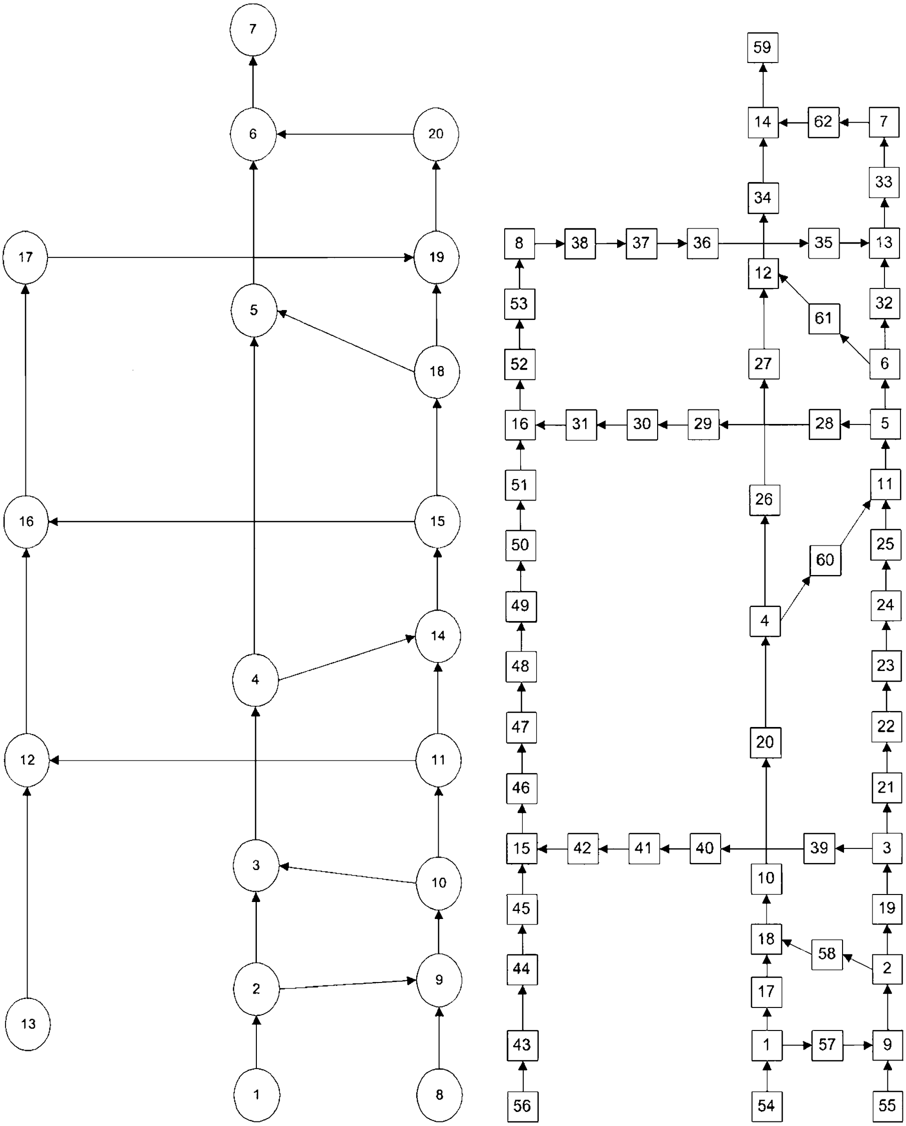

Fig. 1 Test Network (Li et al. 2003)

are three source cells and one sink cell. The others are either arterial cells or freeway cells. The characteristics of the cells in the test network are outlined in Table 2.

We assume that the planning horizon (T) is 100 and cost coefficient c_i^t is 1 except the one at the end of the planning horizon (t=100). c_i^T is set to 10 to penalize the unmet demand. Travel demand is only generated at the three source cells (cell 54, 55 and 56) at time 0. The mean and variance of the uncertain demands are set to 63.6 and 3.84 respectively.

4.2 Numerical test 1: Comparison with CDF based approach

For the CDF based approach, a specific probability distribution is required to reformulate the problem. In this example, we assume that a transportation planner selects beta distribution for the primitive uncertainty $\widetilde{z}$. Using beta(4,1) for $\widetilde{z}$, we can obtain the same mean (63.6) and variance (3.84) using the following equation: $d = 12^*\widetilde{z} + 54$. When the CDF based approach is applied, the chance-constrained programming (CCP) problem can be reformulated as an equivalent deterministic problem (See, e.g., Waller and Ziliaskopoulos (2006) for details). After getting the solutions to A-JCCP and the CDF based deterministic problem, we calculate the total cost consisting of travel cost and penalty cost with 1000 random demand scenarios generated from normal distribution. Since we treat decision variables as deterministic, the following simple heuristic rules are applied for the numerical experiments to make the solution physically meaningful. 1) If there is less demand than expected, then demand is allocated to each path proportionately. 2) If the realized demand is more than expected, then the excess demand remains at the source cell to penalize the infeasibility. The mean, maximum and standard deviation of total travel costs are summarized in Table 3. Also, feasible probability is calculated by counting the cases when the travel demand cannot be satisfied with the solution from each approach.

In general, A-JCCP provides better mean than the CDF based CCP approach. As we can see from the maximum and standard deviation, the proposed method generates more stable and robust solution and works well in worst case scenario, especially with larger violation risk. Most of all, it is found that the safe approximation guarantees the feasible probability within the given confidence level. However, if the probability distribution used for CDF based CCP approach is different from the underlying true distribution, the feasible probability can be greatly deviated from the expected value, 1–ε. For example, when we set the violation risk as 1%, we expect the flow conservation constraint to be feasible with 99% probability. The simulation results show that the constraint holds only with 71.5% probability.

Table 2 Characteristics of the cells

Cell	Source cells	Sink cell	Freeway cells	Arterial cells
N_i	inf	Inf	20	10
Q_i	inf	Inf	12	3
δ_i^t	1	1	1	1

Table 3 Simulation results of numerical test 1

	1-ε	99%	95%	90%	80%	70%	60%
Mean	A-JCCP	4,666.7	4,638.0	4,638.5	4,636.5	4,633.5	4,633.8
	CDF_CCP	4,658.9	4,658.5	4,659.8	4,662.4	4,665.8	4,669.4
Maximum	A-JCCP	5,201.7	5,172.3	5,172.5	5,170.3	5,297.9	5,417.0
	CDF_CCP	5,775.4	5,779.1	5,786.2	5,800.6	5,815.4	5,830.7
Standard Deviation	A-JCCP	117.3	117.7	117.6	117.7	119.4	121.7
	CDF_CCP	157.0	158.5	160.0	163.2	166.4	170.0
Feasible Probability	A-JCCP	100.0%	100.0%	100.0%	100.0%	99.5%	98.3%
	CDF_CCP	71.5%	70.5%	69.0%	65.4%	63.7%	60.3%

We extend our simulation studies to check the feasible probability of the proposed approximation by considering more scenarios about the underlying uncertainty. First, the mean and variance of normal distribution are increased to change demand level (or congestion level). Table 4 shows that the change of mean or variance does not affect the feasible probability as long as the same underlying distribution is assumed. Next, in Table 5, we take into account various underlying distributions without changing mean and variance. We assumed that demand uncertainty ($\tilde{d}$) can be characterized by mean (63.6) and variance (3.48), and the primitive uncertainty ($\tilde{z}$) follows normal distribution, uniform distribution and beta distribution respectively. It is observed that the feasible probability depends on the probability distribution of the primitive uncertainty. Also, in all cases, our approximation approach guarantees the satisfaction of the violation probability of the chance constraints.

4.3 Numerical test 2: Comparison with scenario based approach

In this experiment, we compare our method with a scenario based approach. Scenario approach based on Monte Carlo simulation is one approach to approximate chance constraints. In particular, scenarios of random data vector are generated from a trial distribution to approximate an underlying probability distribution. Then, the system of equations with the generated scenario becomes constraints of the problem. An important theoretical question for the approach is how to determine the sample

Table 4 Feasible probability with normal distribution

Mean	Variance	1–ε					
		99%	95%	90%	80%	70%	60%
63.6	3.84	100.0%	100.0%	100.0%	100.0%	99.5%	98.3%
83.6	3.84	100.0%	100.0%	100.0%	100.0%	99.9%	98.9%
63.6	6.00	100.0%	100.0%	100.0%	100.0%	99.6%	99.1%
124	24.00	100.0%	100.0%	100.0%	100.0%	99.3%	98.4%

Table 5 Feasible probability with various distribution (mean=63.6, variance=3.84)

Primitive Uncertainty (z)	Demand Uncertainty (d)	$1-\varepsilon$					
		99%	95%	90%	80%	70%	60%
Normal Dist. N(63.6,3.84)	$\tilde{d} = \tilde{z}$	100.0%	100.0%	100.0%	100.0%	99.5%	98.3%
Uniform Dist. U (0,1)	$\tilde{d} = 6.7882^*\tilde{z}+60.2059$	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%
Beta Dist. Beta(1,4)	$d = 12^*\tilde{z} + 61.2$	100.0%	100.0%	100.0%	99.9%	97.6%	94.6%
Beta Dist. Beta(1,9)	$\tilde{d} = 21.6441^*\tilde{z}+61.4336$	100.0%	100.0%	99.9%	98.3%	95.4%	93.0%

size N in order to ensure that the optimal solution is feasible for joint chance constraints with reliability $1-\delta$ According to Calafiore and Campi (2006), if constraints are componentwise convex in decision variable, sample size N should be greater than N*

$$N \geq N^* \equiv ceil\left[2n\varepsilon^{-1}\log\left(\frac{12}{\varepsilon}\right) + 2\varepsilon^{-1}\log\left(\frac{2}{\delta}\right) + 2n\right], \tag{21}$$

where n is the dimension of decision variable. When constraints are affine in uncertain data following nice distributions such as uniform or normal distribution, the sample size can be calculated using Eq. (22) (Nemirovski and Shapiro 2006):

$$N = O(1)\left[\log\left(\frac{1}{\delta}\right) + dm^2\log\left(d\left(\log\left(\frac{1}{\varepsilon}\right)\right)\right)\right], \tag{22}$$

where m is the number of chance constraints. However, an intrinsic drawback of the scenario based approach is that, for practical problems, the required sample size is too large, especially with low violation risk. For example, 9,515,162 and 6,735,200 samples are required according to Eq. (21) and (22) respectively in order to solve this example with ε=0.01 and δ=0.01s.

Instead of calculating the sample size using (21) and (22), we simply set the number of samples to 50 and run the test. Similar to the previous section, beta distribution (beta(4,1)) is assumed for scenario based approach and normal distribution is used for the simulation. Table 6 shows the simulation results for the scenario based approach and A-JCCP. Since A-JCCP is an approximation, it can provide a conservative solution when ε is small. In this example, except the mean

Table 6 Simulation results of numerical test 2

	Scenario	A-JCCP					
ε=0.01		99%	95%	90%	80%	70%	60%
Mean	4,660.9	4,666.7	4,638.0	4,638.5	4,636.5	4,633.5	4633.8
Maximum	5,779.5	5,201.7	5,172.3	5,172.5	5,170.3	5,297.9	5,417.0
Standard Deviation	160.5	117.3	117.7	117.6	117.7	119.4	121.7
Feasible Probability	71.0%	100.0%	100.0%	100.0%	100.0%	99.5%	98.3%

value when $\varepsilon = 0.01$, A-JCCP outperforms scenario based approach by providing better objective value, especially in worst case, and less standard deviation. Also, the satisfaction of the probability constraints is guaranteed.

5 Conclusion

A chance-constrained approach has been developed for a CTM based SO DTA problem, in which the flow conservation constraint can be violated within a given confidence level due to uncertain demand. Under the assumption that only partial distributional information such as mean and variance of uncertain data is given, the joint chance-constrained problem was approximated and reformulated as a linear programming problem. Numerical experiments showed that the proposed approach outperforms the other two workable approaches (CDF based approach and sampling based approach), even though our approach is conservative when the violation risk is low. Moreover, it was shown that the confidence level cannot be guaranteed when the assumed distribution for stochastic programming approach is different from the underlying true but unknown distribution. However, we do not argue that the proposed approach always outperforms CDF based approach. When a decision maker has full distributional information on uncertain parameter, CDF based approach should be a better choice. The proposed approximation method is favorable only when partial information is available. Also, the proposed model, A-JCCP, is computationally tractable and can be solved efficiently using commercial solvers while scenario based approximation may not be practically applicable due to the large sample size required.

Future research is needed to develop less conservative approximation of a JCCP. One potential solution is to address dynamic control by adopting a tractable linear decision rule to describe the relationship between the time dependent control variable and the uncertainty which enables an approximation of the impact of demand correlations on SO assignment. Moreover, in order to improve the reality of the model which further improves the utility, uncertain road capacity uncertainty can be considered together with the demand uncertainty in the future research. In addition, it would be very interesting to develop less conservative and computationally efficient approximations of a JCCP under various types of partial distributional information (e.g. range, symmetry, etc.) of the uncertainty.

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