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Robust Approximation to Multiperiod Inventory Management

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We propose a robust optimization approach to address a multiperiod inventory control problem under ambiguous demands, that is, only limited information of the demand distributions such as mean, support, and some measures of deviations. Our framework extends to correlated demands and is developed around a factor-based model, which has the ability to incorporate business factors as well as time-series forecast effects of trend, seasonality, and cyclic variations. We can obtain the parameters of the replenishment policies by solving a tractable deterministic optimization problem in the form of a second-order cone optimization problem (SOCP), with solution time; unlike dynamic programming approaches, it is polynomial and independent on parameters such as replenishment lead time, demand variability, and correlations. The proposed truncated linear replenishment policy (TLRP), which is piecewise linear with respect to demand history, improves upon static and linear policies, and achieves objective values that are reasonably close to optimal.

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1. Introduction

Inventory ties up working capital and incurs holding costs, reducing profit every day that excess stock is held. Good inventory management has hence become crucial to businesses as they seek to continually improve their customer service and profit margins, in the heat of global competition and demand variability. The ability to incorporate more realistic assumptions about product demand into inventory models is one key factor to profitability. Practical models of inventory would need to address the issue of demand forecasting while staying sufficiently immunized against uncertainty and maintaining tractability. In most industrial contexts, demand is uncertain. Many demand histories have factors that behave like random walks that evolve over time with frequent changes in their directions and rates of growth or decline. In practice, for such demand processes, inventory managers often rely on forecasts based on a time series of prior demands, which are often correlated over time. For example, a product demand may depend on factors such as market outlook, oil prices, and so forth, and contains effects of trend, seasonality, cyclic variation, and randomness.

In this paper, we address the problem of optimizing multiperiod inventory using factor-based stochastic demand models, where the coefficients of the random factors can be forecasted statistically, perhaps using historical time-series data. We assume that the demands may be correlated and are ambiguous, that is, limited information of the demand distributions (only the mean, support, and some measures of deviations) are available. Using robust optimization techniques, we develop a tractable methodology that uses past demand history to adaptively control multiperiod inventory. Our model also includes a range of features such as delivery delay and capacity limit on order quantity.

Our work is closely related to the multiperiod inventory control problem, a well-studied problem in operations research. For the single-product inventory control problem with history-independent demands, it is well known that the base-stock policy based on a critical fractile is optimum. See Scarf (1959, 1960), Azoury (1985), Miller (1986), and Zipkin (2000). For correlated demands, Veinott (1965) characterized conditions under which a myopic policy is optimal. Extending the results of Johnson and Thompson (1975) considered an autoregressive, moving-average (ARMA) demand process, zero replenishment lead time and no backlogs, and showed the optimality of a myopic policy when the demand in each period is bounded. Lovejoy (1990) showed that a myopic critical-fractile policy is optimum or near optimum in some inventory models with adaptive demand processes, citing exponential smoothing on the demand process and Bayesian updating on uniformly distributed demand as examples. Song and Zipkin (1993) addressed the case of Poisson demand, where the transition rates between states are governed by a Markov process.

Although optimum policies can be characterized in many interesting variants of inventory control problems, it is not easy to compute them efficiently, that is, in polynomial time with respect to the input size of the problem. In this paper, we use the term *tractable replenishment policy* if the parameters of the policy are polynomial in size and can be obtained in polynomial time. For instance, the celebrated optimum base-stock policy may not necessarily be a tractable one. Sampling-based approximation has been applied to the inventory control problem; see, for instance, Levi et al. (2007a). Using marginal cost accounting and cost-balancing techniques, Levi et al. (2007b) proposed an elegant two-approximation algorithm for the inventory control problem. Other sampling-based approaches include infinitesimal perturbation analysis (see Glasserman and Tayur 1995), which uses stochastic gradient estimation technique, and the concave adaptive value estimation procedure, which successively approximates the objective cost function with a sequence of piecewise-linear functions (see Godfrey and Powell 2001 and Powell et al. 2004). More recently, Iida and Zipkin (2006) and Lu et al. (2006) developed approximate solutions for demand following the martingale model of forecast evolution.

One of the fundamental assumptions of stochastic models, which has recently been challenged, is the availability of probability distributions in characterizing uncertain parameters. Bertsimas and Thiele (2006) illustrated that an optimum inventory control policy that is heavily tuned to a particular demand distribution may perform poorly against another demand distribution bearing the same mean and variance. Assuming a demand distribution tacitly implies that we are able to obtain exact estimates of all the moments, which is practically prohibitive. It is a common practice to estimate the first two moments from data and fit the parameters to an assumed distribution. By doing so, we are artificially extrapolating the rest of the moments using only the information from lower partial moments. Errors in estimating the first two moments will naturally propagate to higher moments. Therefore, it is not surprising that policies that are derived from assuming demand distributions may be less robust. One approach to account for distributional ambiguity is to consider a family of demand distributions, which can be characterized by their descriptive statistics such as partial moments information, support, and so forth. Research on inventory control under ambiguous demand distributions dates back to Scarf (1958), where he considered a newsvendor problem and determined orders that minimize the maximum expected cost over all possible demand distributions with the same first and second moments and with nonnegative support. Various extensions of Scarf's single-period results have been studied by Gallego and Moon (1993). Although the solutions to these single-period models are in the form of second-order cone optimization problems (SOCP), which are polynomial-time solvable, the minimax approach does not scale well computationally with the number of periods. Nevertheless, the optimum policies for multiperiod inventory control problems under various forms of demand ambiguity have been characterized by Kasugai and Kasegai (1960) and Gallego et al. (2001).

In recent years, robust optimization has witnessed an explosive growth and has become a dominant approach to address the optimization problem under uncertainty. Traditionally, the goal of robust optimization is to immunize uncertain mathematical optimization problems against infeasibility while preserving the tractability of the models. See, for instance, Ben-Tal and Nemirovski (1998, 1999, 2000), Bertsimas and Sim (2003, 2004), Bertsimas et al. (2004), El-Ghaoui and Lebret (1997), and El-Ghaoui et al. (1998). Many robust optimization approaches have the following two important characteristics:

(a) The model of data uncertainty in robust optimization permits distributional ambiguity. Data uncertainty can also be completely distributional free and specified by an uncertainty set parameterized by the "Budget of Uncertainty," which controls the size of the uncertainty set. Another model of uncertainty is to consider uncertain parameters whose distributions are unknown but are confined to a family of distributions that would generate the same descriptive statistics on the data, such as known means and variances.

(b) The solution (or approximate solution) to a robust optimization model can be obtained by solving a tractable deterministic mathematical optimization problem such as SOCP, whose associated solvers are commercially available, robust, and efficiently optimized. Robust optimization methodology often decouples model formulation from the optimization engine, which enables the modeler to focus on modeling the actual problem and not to be hindered by algorithm design.

Based on the framework of robust optimization, Bertsimas and Thiele (2006) developed a new approach to address demand ambiguity in a multiperiod inventory control problem, which has the advantage of being computationally tractable. They considered a family of demand distributions similar to Scarf and enforced independence across time periods. Bertsimas and Thiele mapped the demand uncertainty model into a "Budget of Uncertainty" model of Bertsimas and Sim (2004) and proposed an open-loop inventory control approach in which the solutions can be obtained by solving a tractable linear optimization problem. They showed that the optimum solution of their robust model has a base-stock structure and the tractability of the problem readily extends to problems with capacity constraints and over a supply chain network, and their paper characterized the optimum policies for these cases. The analysis of the robust models and computational experiments for independent demands suggests that robust approaches compare well against an optimum model under exact distribution and is yet immunized against distributional ambiguity. Using a similar approach, Adida and Perakis (2006) proposed a deterministic robust optimization formulation for dealing with demand uncertainty in a dynamic pricing and inventory control problem for a make-to-stock manufacturing system. They developed a demand-based fluid model and showed that the robust formulation is not much harder to solve than the nominal problem. Other related work in the robust inventory control literature includes Bienstock and Ozbay (2008), where they proposed a robust model focusing on base-stock policy structure. Song et al. (2007) adopted a data-driven approach to robust inventory management.

To address the inadequacy of open-loop robust optimization models involving multistage decision process, Ben-Tal et al. (2004) introduced the concept of adjustable robust counterpart, which permits decisions to be delayed until the availability of information. Unfortunately, with the additional flexibility in modeling, adjustable robust counterpart models are generally *NP*-hard, and the authors have proposed and advocated the use of linear decision rule as a tractable approximation. Ben-Tal et al. (2005) applied their model to a multiperiod inventory control problem and showed by means of computational studies the advantages of the linear replenishment policy over the open-loop model in which the replenishment policy is static. We emphasize that in contrast to stochastic models, the uncertainty considered in adjustable robust counterpart is completely distribution free, that is, the data uncertainty is characterized only by its support. Due to the different models of uncertainty, it is meaningless to compare adjustable robust counterpart models vis-à-vis stochastic ones.

To bridge the gap between robust optimization and stochastic models, Chen et al. (2007) introduced the notions of directional deviations known as *forward and backward deviations* and proposed computationally tractable robust optimization models for immunizing linear optimization problems against infeasibility, which enhanced the modeling power of robust optimization in the characterization of ambiguous distributions. In a parallel work, Chen et al. (2008) proposed several piecewise-linear decision rules for approximating stochastic linear optimization problems that improve upon linear rules. These approaches have been unified by Chen and Sim (2009), where they proposed a general family of distributions characterized by the mean, covariance, directional deviations and support and showed how it can be extended to approximate the solution for a two-period stochastic model under a satisficing objective.

In this paper, we extend these new ideas of robust optimization to the multiperiod inventory control problem. Instead of the "Budget of Uncertainty" demand model, we focus on uncertain demands being robustly characterized by their descriptive statistics. The former requires specification on the size of the uncertainty set, which, as exemplified in Bertsimas and Sim (2006), can be dependent on the types of stochastic optimization problem we are addressing. We feel that the "Budget of Uncertainty" approach to uncertainty, although it has its strengths, is less appealing when we compare it vis-a-vis with stochastic demand models. The main thrust of this paper is to pursue an approach to optimize multiperiod inventory under ambiguity in demand distributions. Our contributions over the related works of Bertsimas and Thiele (2006) and Ben-Tal et al. (2005) can be summarized as follows:

(a) Our proposed robust optimization approximation is based upon a comprehensive factor-based demand model that can capture correlations such as the autoregressive nature of demand, the effect of external factors, as well as trends and seasonality, among others. In addition, we provide for distributional ambiguity in the underlying factors by considering a family of distributions characterized by the mean, covariance, support, and directional deviations. In contrast, the robust optimization model of Bertsimas and Thiele (2006) is restricted to independent demands with an identical mean and variance, whereas the model of Ben-Tal et al. (2005) is confined to completely distribution-free demand uncertainty.

(b) We propose a new policy called the truncated linear replenishment policy and show that it gives improved approximation to the multiperiod inventory control problem over static and linear decision rules used in the robust optimization proposals of Bertsimas and Thiele (2006) and Ben-Tal et al. (2005), respectively. We also *do not* restrict the policy structure to base stock. We develop a new bound on a nested sum of expected positive values of random variables and show that the parameters of the truncated linear replenishment policy can be obtained by solving a tractable deterministic mathematical optimization problem in the form of SOCP, whose solution time is independent on replenishment lead time, demand variability, and correlations, among others. We study the computational performance of the static, linear, and truncated linear replenishment policies against the optimum policy.

This paper is organized as follows. In §2 we describe a stochastic inventory model. We formulate our robust inventory models in §3 and discuss extensions in §4. We conclude the paper in §5. In addition, the computational studies and proofs can be found in an online companion to this paper. It is available at http://or.journal.informs.org/.

Notations. Throughout this paper, we denote a random variable with the tilde sign such as $\tilde{y}$ and vectors with boldface lower-case letters such as $\mathbf{y}$. We use $\mathbf{y}'$ to denote the transpose of vector $\mathbf{y}$. Also, denote $y^+ = \max(y, 0)$, $y^- = \max(-y, 0)$, and $\|\mathbf{y}\|_2 = \sqrt{\sum y_i^2}$.

2. Stochastic Inventory Model

The stochastic inventory model involves the derivation of replenishment decisions over a discrete planning horizon consisting of a finite number of periods under stochastic demand. The demand for each period is usually a sequence of random variables that are *not* necessarily identically distributed and *not* necessarily independent. We consider an inventory system with T planning horizons from $t = 1$ to $t = T$. External demands arrive at the inventory system, and the system replenishes its inventory from some central warehouse (or supplier) with ample supply. The timeline of events is as follows:

1. At the beginning of the tth time period, before observing the demand, the inventory manager places an order of x_t at unit cost c_t for the product to arrive after a (fixed) order lead time of L periods. Orders placed at the *beginning* of the tth time period will arrive at the *beginning* of $t+L$th period. We assume that replenishment ceases at the end of the planning horizon, so that the last order is placed in period $T-L$. Without loss of generality, we assume that purchase costs for inventory are charged at the time of order. The case where purchase costs are charged at the time of delivery can be represented by a straightforward shift of cost indices.

2. At the beginning of each time period t, the inventory manager faces an initial inventory level y_t and receives an order of x_{t-L}. The demand of inventory for the period is realized at the end of the time period. After receiving a demand of d_t, the inventory level at the end of the period is $y_t + x_{t-L} - d_t$.

3. Excess inventory is carried to the next period, incurring a per-unit overage (holding) cost. On the other hand, each unit of unsatisfied demand is backlogged (carried over) to the next period with a per-unit underage (backlogging) penalty cost. At the last period, $t = T$, the penalty of lost sales can be accounted through the underage cost.

We assume an inventory manager whose objective is to determine the dynamic ordering quantities x_t from period $t = 1$ to period $t = T - L$ so as to minimize the total expected ordering, inventory overage (holding), and inventory underage (backlog) costs in response to the uncertain demands. Observe that for $L \geqslant 1$, the quantities x_{t-L}, $t = 1, \ldots, L$ are known values. They denote orders made before period $t = 1$ and are inventories in the delivery pipeline when the planning horizon starts.

We introduce the following notations:

$\tilde{d}_t$: stochastic exogenous demand at period t;

$\tilde{\mathbf{d}}_t$: a vector of random demands from period 1 to t, that is, $\tilde{\mathbf{d}}_t = (\tilde{d}_1, \ldots, \tilde{d}_t)$;

$x_t(\tilde{\mathbf{d}}_{t-1})$: order placed at the beginning of the tth time period after observing $\tilde{\mathbf{d}}_{t-1}$. The first-period inventory order is denoted by $x_1(\tilde{\mathbf{d}}_0) = x_1^0$;

$y_t(\tilde{\mathbf{d}}_{t-1})$: inventory level at the beginning of the tth time period;

h_t: unit inventory overage (holding) cost charged on excess inventory at the end of the tth time period;

b_t: unit underage (backlog) cost charged on back-logged inventory at the end of the tth time period;

c_t: unit purchase cost of inventory for orders placed at the tth time period; and

S_t: the maximum amount that can be ordered at the tth time period.

We use $x_t(\tilde{\mathbf{d}}_{t-1})$ to represent the nonanticipative replenishment policy at the beginning of period t. That is, the replenishment decision is based solely on the observed information available at the beginning of period t, which is given by the demand vector $\tilde{\mathbf{d}}_{t-1} = (\tilde{d}_1, \ldots, \tilde{d}_{t-1})$. Given the order quantity $x_{t-L}(\tilde{\mathbf{d}}_{t-L-1})$ and stochastic exogenous demand $\tilde{d}_t$, the inventory level at the *end* of the t time period (which is also the inventory level at start of $t+1$ time period) is given by

$$y_{t+1}(\tilde{\mathbf{d}}_t) = y_t(\tilde{\mathbf{d}}_{t-1}) + x_{t-L}(\tilde{\mathbf{d}}_{t-L-1}) - \tilde{d}_t, \quad t = 1, \ldots, T. \tag{1}$$

In resolving the initial boundary conditions, we adopt the following notations:

- The initial inventory level of the system is $y_1(\tilde{\mathbf{d}}_0) = y_1^0$.
- When $L \geqslant 1$, the orders that are placed before the planning horizon starts are denoted by

$$x_t(\tilde{\mathbf{d}}_{t-1}) = x_t^0, \quad t = 1-L, \ldots, 0.$$

Note that Equation (1) can be written using the cumulative demand up to period t and cumulative order received as follows:

$$\begin{aligned} y_{t+1}(\tilde{\mathbf{d}}_t) = {} & \underbrace{y_1^0}_{\text{initial inventory}} + \underbrace{\sum_{\tau=1}^{\min\{L,t\}} x_{\tau-L}^0}_{\text{committed orders}} \\ & + \underbrace{\sum_{\tau=L+1}^{t} x_{\tau-L}(\tilde{\mathbf{d}}_{\tau-L-1})}_{\text{order decisions}} - \underbrace{\sum_{\tau=1}^{t} \tilde{d}_\tau}_{\text{cumulative demands}}. \end{aligned} \tag{2}$$

Observe that positive (respectively, negative) value of $y_{t+1}(\tilde{\mathbf{d}}_t)$ represents the total amount of inventory overage (respectively, underage) at the end of the period t after meeting demand. Thus, the total expected cost, including ordering, inventory overage, and inventory underage charges is equal to

$$\sum_{t=1}^{T} \big(\mathrm{E}(c_t x_t(\tilde{\mathbf{d}}_{t-1})) + \mathrm{E}(h_t (y_{t+1}(\tilde{\mathbf{d}}_t))^+) + \mathrm{E}(b_t (y_{t+1}(\tilde{\mathbf{d}}_t))^-) \big).$$

Therefore, the multiperiod inventory problem can be formulated as a T stage stochastic optimization model as follows:

$$\begin{aligned} Z_{\mathrm{STOC}} = \min \quad & \sum_{t=1}^{T} \big(\mathrm{E}(c_t x_t(\tilde{\mathbf{d}}_{t-1})) + \mathrm{E}(h_t (y_{t+1}(\tilde{\mathbf{d}}_t))^+) \\ & \qquad + \mathrm{E}(b_t (y_{t+1}(\tilde{\mathbf{d}}_t))^-) \big). \\ \text{s.t.} \quad & y_{t+1}(\tilde{\mathbf{d}}_t) = y_t(\tilde{\mathbf{d}}_{t-1}) + x_{t-L}(\tilde{\mathbf{d}}_{t-L-1}) - \tilde{d}_t \\ & \qquad t = 1, \ldots, T \\ & 0 \leqslant x_t(\tilde{\mathbf{d}}_{t-1}) \leqslant S_t \quad t = 1, \ldots, T-L. \end{aligned} \tag{3}$$

The aim of the stochastic optimization model is to derive a feasible replenishment policy that minimizes the expected ordering and inventory costs. That is, we seek a sequence of action rules that advises the inventory manager of the action to take in time t as a function of demand history. Unfortunately, the decision variables in problem (3), $x_t(\tilde{\mathbf{d}}_{t-1})$, $t = 1, \ldots, T-L$ and $y_t(\tilde{\mathbf{d}}_{t-1})$, $t = 2, \ldots, T+1$ are functionals, which means that problem (3) is an optimization problem with an infinite number of variables and constraints, and hence generally intractable.

The stochastic optimization problem (3) can also be formulated as a dynamic programming problem. For simplicity, assuming zero lead time, the dynamic programming requires the following updates on the value function:

$$\begin{aligned} &J_t(y_t, d_1, \ldots, d_{t-1}) \\ &\quad = \min_{x\in[0,S_t]} \mathrm{E}\big(c_t x + G_t(y_t + x - \tilde{d}_t) \\ &\qquad\qquad + J_{t+1}(y_t + x - \tilde{d}_t, d_1, \ldots, d_{t-1}, \tilde{d}_t) \,\big|\, \\ &\qquad\qquad\qquad \tilde{d}_1 = d_1, \ldots, \tilde{d}_{t-1} = d_{t-1}\big), \end{aligned}$$

where $G_t(u) = h_t \max(u, 0) + b_t \max(-u, 0)$. Maintaining the value function $J_t(\cdot)$ is computationally prohibitive, and hence most inventory control literatures identify conditions such that the value functions are not dependent on past demand history, so that the state space is computationally amenable. For instance, it is well known that when the lead time is zero and the demands are independently distributed across time periods, there exists base-stock levels, q_t, such that the following replenishment policy,

$$x_t(\tilde{\mathbf{d}}_{t-1}) = \min\big\{\max\{q_t - y_t(\tilde{\mathbf{d}}_{t-1}), 0\}, S_t\big\} \tag{4}$$

is optimum. Hence, instead of being a function of the entire demand history, the optimum demand policy can be characterized by the inventory level as follows:

$$x_t(y_t) = \min\big\{\max\{q_t - y_t, 0\}, S_t\big\}.$$

2.1. Factor-Based Demand Model

We adopt a factor-based demand model in which the uncertain demand is affinely dependent on zero mean random factors $\tilde{\mathbf{z}} \in \Re^N$ as follows:

$$d_t(\tilde{\mathbf{z}}) \stackrel{\Delta}{=} \tilde{d}_t = d_t^0 + \sum_{k=1}^{N} d_t^k \tilde{z}_k, \quad t = 1, \ldots, T,$$

where

$$d_t^k = 0 \quad \forall\, k \geqslant N_t + 1,$$

and $1 \leqslant N_1 \leqslant N_2 \leqslant \cdots \leqslant N_T = N$. Such an affine factor-based uncertainty model is a common assumption in robust optimization; see, for instance, Ben-Tal and Nemirovski (1998). Under a factor-based demand model, the random factors, $\tilde{z}_k$, $k = 1, \ldots, N$ are realized sequentially. At period t, the factors, $\tilde{z}_k$, $k = 1, \ldots, N_t$ have already been unfolded. In progressing to period $t+1$, the new factors $\tilde{z}_k$, $k = N_t + 1, \ldots, N_{t+1}$ are made available.

Demand that is affected by random noise or shocks can be represented by the factor-based demand model. For independently distributed demand, which is assumed in most inventory models, we have

$$d_t(\tilde{\mathbf{z}}) = d_t^0 + \tilde{z}_t, \quad t = 1, \ldots, T,$$

in which $\tilde{z}_t$ are independently distributed. However, in many industrial contexts, demands across periods may be correlated. In fact, many demand histories behave more like random walks over time, with frequent changes in directions and rate of growth or decline. See Johnson and Thompson (1975) and Graves (1999). In those settings, we may consider standard forecasting techniques such as an ARMA(p, q) demand process (see Box et al. 1994) as follows:

$$d_t(\tilde{\mathbf{z}}) = \begin{cases} d_t^0 & \text{if } t \leqslant 0, \\ \displaystyle\sum_{j=1}^{p} \phi_j d_{t-j}(\tilde{\mathbf{z}}) + \tilde{z}_t + \sum_{j=1}^{\min\{q,\, t-1\}} \theta_j \tilde{z}_{t-j} & \text{otherwise}, \end{cases}$$

where $\phi_1, \ldots, \phi_p, \theta_1, \ldots, \theta_q$ are known constants. Indeed, it is easy to show by induction that $d_t(\tilde{\mathbf{z}})$ can be expressed in the form of a factor-based demand model. Song and Zipkin (1993) presented a world-driven demand model where the demand is Poisson with rate controlled by finite Markov states representing different business environments. However, it may be difficult to determine exhaustively the business states and their state transition probabilities. On the other hand, factor-based models have been used extensively in finance for modeling returns as affine functions of external factors, in which the coefficients of the factors can be determined statistically. In the same way, we can apply the factor-based demand model to characterize the influence of demands with external factors such as market outlook, oil prices, and so forth. Effects of trend, seasonality, cyclic variation, and randomness can also be incorporated.

3. Robust Inventory Model

The stochastic inventory control problem requires full information of the demand distributions, which is practically prohibitive. Furthermore, even if the probability distributions are known, due to computational complexity, we may not be able to obtain the optimum solution. Note that under the factor-based demand model, it is easy to evaluate the demand distribution when the factors are normally distributed factors. However, this is not necessarily the case for other distributions. Nemirovski and Shapiro (2006) noted that evaluating the distribution of a weighted sum of uniformly distributed independent random variables is already *NP*-hard. As such, it would generally be intractable to evaluate the cumulative distributions of

Table 1. Forward and backward deviation of some common probability distributions.

Distribution	σ_f	σ_b
Normal with standard deviation, σ	σ	σ
Uniform with standard deviation, σ	σ	σ
Exponential with standard deviation, σ	∞	σ

the random demand with nonnormally distributed factors. Consequently, it would technically be intractable to compute the myopic critical fractile based on the seemingly benign factor-based demand model. The robust optimization approach we are proposing aims to address these issues collectively.

Instead of assuming full distributions on the factors, which is practically prohibitive, we adopt a modest distributional assumption on the random factors, such as known means, supports, and some aspects of deviations. The factors may be partially characterized using the directional deviations that were recently introduced by Chen et al. (2007).

Definition 1 (Directional deviations). *Given a random variable $\tilde{z}$, the forward deviation is defined as*

$$\sigma_f(\tilde{z}) \stackrel{\Delta}{=} \sup_{\theta>0}\left\{\sqrt{2\ln(\mathrm{E}(\exp(\theta(\tilde{z}-\mathrm{E}(\tilde{z}))))/\theta^2}\right\}, \tag{5}$$

and backward deviation is defined as

$$\sigma_b(\tilde{z}) \stackrel{\Delta}{=} \sup_{\theta>0}\left\{\sqrt{2\ln(\mathrm{E}(\exp(-\theta(\tilde{z}-\mathrm{E}(\tilde{z}))))/\theta^2}\right\}. \tag{6}$$

Table 1 shows the forward and backward deviation of some common probability distributions. We also present, in Table 2, the directional deviations of a truncated exponential random variable, $\tilde{z}$ in $[0, \bar{z}]$ with the following density function:

$$f_{\tilde{z}}(z) = \frac{\exp(-z)}{1-\exp(-\bar{z})}.$$

Although the forward deviation of a pure exponential distributed random variable is infinite, the truncated exponential distribution has a reasonably small forward deviation compared to the support $\bar{z}$. Even when $\bar{z} = 10$, the forward deviation is only slightly more than twice its standard deviation.

Given a sequence of independent samples, we can essentially estimate the magnitude of the directional deviations from (5) and (6). Some of the properties of the directional deviations include:

Proposition 1 (Chen et al. 2007). *Let σ, p, and q be respectively, the standard, forward, and backward deviations of a random variable $\tilde{z}$ with zero mean.*

(a)

$$p \geqslant \sigma \quad q \geqslant \sigma.$$

If $\tilde{z}$ is normally distributed, then $p = q = \sigma$.

(b) *For all $\theta \geqslant 0$,*

$$\mathrm{P}(\tilde{z} \geqslant \theta p) \leqslant \exp(-\theta^2/2);$$

$$\mathrm{P}(\tilde{z} \leqslant -\theta q) \leqslant \exp(-\theta^2/2).$$

Proposition 1(a) shows that the directional deviations are no less than the standard deviation of the underlying distribution, and under the normal distribution, these two values coincide with the standard deviation. As exemplified in Proposition 1(b), the directional deviations provide an easy bound on the distributional tails. The advantage of using the directional deviations is the ability to capture distributional asymmetry and stochastic independence, while keeping the resultant optimization model computationally amicable. We refer the reader to the paper by Natarajan et al. (2008) for the computational experience of using directional derivations derived from real-life data.

In this paper, we adopt the random factor model introduced by Chen and Sim (2009), which encompasses most of the uncertainty models found in the literatures of robust optimization.

Assumption U. We assume that the uncertainties $\{\tilde{z}_j\}_{j=1:N}$ are zero mean random variables, with positive definite covariance matrix, Σ. Let $\mathcal{W}$ be the smallest convex set containing the support of $\tilde{\mathbf{z}}$. We denote a subset, $\mathcal{J} \subseteq \{1, \ldots, N\}$, which can be an empty set, such that $\tilde{z}_j$, $j \in \mathcal{J}$ are stochastically independent. Moreover, the corresponding forward and backward deviations are given by $p_j = \sigma_f(\tilde{z}_j)$ and $q_j = \sigma_b(\tilde{z}_j)$, respectively, for $j \in \mathcal{J}$ and that $p_j = q_j = \infty$ for $j \notin \mathcal{J}$.[1]

The choice of the support set $\mathcal{W}$ can influence the computational tractability of the problem. Henceforth, we assume that the support set is a second-order conic representable set (a.k.a conic quadratic representable set) proposed in Ben-Tal and Nemirovski (1998), which includes polyhedral and ellipsoidal sets. A common support set is the interval set, which is given by $\mathcal{W} = [-\underline{\mathbf{z}}, \bar{\mathbf{z}}]$, in which $\underline{\mathbf{z}}, \bar{\mathbf{z}} > \mathbf{0}$.

Table 2. Directional deviations for truncated exponential variable with support $[0, \bar{z}]$.

$\bar{z}$	4	5	6	7	8	9	10	100
Standard deviation	0.834	0.911	0.954	0.977	0.989	0.995	0.998	1.000
σ_f	1.037	1.239	1.419	1.583	1.733	1.871	2.000	7.000
σ_b	0.834	0.911	0.954	0.977	0.989	0.995	0.998	1.000

For notational convenience, we define the following sets:

$$\mathcal{J}_1 \stackrel{\Delta}{=} \{j\colon p_j < \infty\} \;\; \bar{\mathcal{J}}_1 \stackrel{\Delta}{=} \{j\colon p_j = \infty\}$$
$$\mathcal{J}_2 \stackrel{\Delta}{=} \{j\colon q_j < \infty\} \;\; \bar{\mathcal{J}}_2 \stackrel{\Delta}{=} \{j\colon q_j = \infty\}.$$

Furthermore, if $p_j = \infty$ (respectively, $q_j = \infty$), its product with zero remains zero, that is, $p_j \times 0 = 0$ (respectively, $q_j \times 0 = 0$).

3.1. Bound on $\mathrm{E}((\cdot)^+)$

In the absence of full distributional information, it would be meaningless to evaluate the optimum objective as depicted in problem (3). Instead, we assume that the modeler is averse to distributional ambiguity and aims to minimize a good upper bound on the objective function. Such an approach of soliciting inventory decisions based on partial demand information is not new. In the 1950s, Scarf (1958) considered a min-max newsvendor problem with uncertain demand $\tilde{d}$ given by only its mean and standard deviations. Scarf was able to obtain solutions to the tight upper bound of the newsvendor problem. The central idea in addressing such a problem is to solicit a good upper bound on $\mathrm{E}((\cdot)^+)$, which appears at the objective of the newsvendor problem and also in problem (3). The following result is well known:

Proposition 2 (Scarf 1958). *Let $\tilde{z}$ be a random variable in $[-\mu, \infty)$ with mean μ and standard deviation σ; then, for all $a \geqslant -\mu$,*

$$\mathrm{E}((\tilde{z}-a)^+) \leqslant \begin{cases} \dfrac{1}{2}(-a+\sqrt{\sigma^2+a^2}) & \text{if } a \geqslant \dfrac{\sigma^2-\mu^2}{2\mu} \\[2ex] -a\dfrac{\mu^2}{\mu^2+\sigma^2}+\mu\dfrac{\sigma^2}{\mu^2+\sigma^2} & \text{if } a < \dfrac{\sigma^2-\mu^2}{2\mu} \end{cases}.$$

Moreover, the bound is achievable.

Interestingly, Bertsimas and Thiele (2006) used the bound of Proposition 2 to calibrate the budget of uncertainty parameter in their robust inventory models. Unfortunately, it is generally computationally intractable to evaluate tight probability bounds involving multivariate random variables with known moments and support information (see Bertsimas and Popescu 2002). We adopt the bounds of Chen and Sim (2009) to evaluate the expected positive part of an affine sum of random variables under Assumption U.

Definition 2. *We say a function $f(\mathbf{z})$ is* nonzero crossing *with respect to $\mathbf{z} \in \mathcal{W}$ if at least one of the following conditions holds*:
1. $f(\mathbf{z}) \geqslant 0 \quad \forall\, \mathbf{z} \in \mathcal{W}$,
2. $f(\mathbf{z}) \leqslant 0 \quad \forall\, \mathbf{z} \in \mathcal{W}$.

Theorem 1 (Chen and Sim 2009). *Let $\tilde{\mathbf{z}} \in \Re^N$ be a multivariate random variable under the Assumption U. Then*

$$\mathrm{E}((y_0+\mathbf{y}'\tilde{\mathbf{z}})^+) \leqslant \pi(y_0, \mathbf{y}),$$

where $\pi(y_0, \mathbf{y})$ is given by

$$\begin{aligned} \pi(y_0,\mathbf{y}) = \min\ & r_1+r_2+r_3+r_4+r_5 \\ \text{s.t.}\ & y_{10}+\max_{z\in\mathcal{W}} \mathbf{z}'\mathbf{y}_1 \leqslant r_1 \\ & 0 \leqslant r_1 \\ & \max_{z\in\mathcal{W}} \mathbf{z}'(-\mathbf{y}_2) \leqslant r_2 \\ & y_{20} \leqslant r_2 \\ & \frac{1}{2}y_{30}+\frac{1}{2}\|(y_{30}, \boldsymbol{\Sigma}^{1/2}\mathbf{y}_3)\|_2 \leqslant r_3 \\ & \inf_{\mu>0} \frac{\mu}{e}\exp\left(\frac{y_{40}}{\mu}+\frac{\|\mathbf{u}\|_2^2}{2\mu^2}\right) \leqslant r_4 \\ & u_j \geqslant p_j y_{4j} \quad \forall\, j \in \mathcal{J}_1 \\ & y_{4j} \leqslant 0 \quad \forall\, j \in \bar{\mathcal{J}}_1 \\ & u_j \geqslant -q_j y_{4j} \quad \forall\, j \in \mathcal{J}_2 \\ & y_{4j} \geqslant 0 \quad \forall\, j \in \bar{\mathcal{J}}_2 \\ & y_{50}+\inf_{\mu>0} \frac{\mu}{e}\exp\left(-\frac{y_{50}}{\mu}+\frac{\|\mathbf{v}\|_2^2}{2\mu^2}\right) \leqslant r_5 \\ & v_j \geqslant q_j y_{5j} \quad \forall\, j \in \mathcal{J}_2 \\ & y_{5j} \leqslant 0 \quad \forall\, j \in \bar{\mathcal{J}}_2 \\ & v_j \geqslant -p_j y_{5j} \quad \forall\, j \in \mathcal{J}_1 \\ & y_{5j} \geqslant 0 \quad \forall\, j \in \bar{\mathcal{J}}_1 \\ & y_{10}+y_{20}+y_{30}+y_{40}+y_{50} = y_0 \\ & \mathbf{y}_1+\mathbf{y}_2+\mathbf{y}_3+\mathbf{y}_4+\mathbf{y}_5 = \mathbf{y} \\ & r_i, y_{i0} \in \Re,\ \ \mathbf{y}_i \in \Re^N \\ & \quad i = 1, \ldots, 5,\ \ \mathbf{u}, \mathbf{v} \in \Re^N. \end{aligned} \tag{7}$$

Moreover, the bound is tight if $y_0 + \mathbf{y}'\mathbf{z}$ is a nonzero crossing function with respect to $\mathbf{z} \in \mathcal{W}$. That is, if

$$y_0+\mathbf{y}'\mathbf{z} \geqslant 0 \quad \forall\, \mathbf{z} \in \mathcal{W}$$

we have $\mathrm{E}((y_0+\mathbf{y}'\mathbf{z})^+) = \pi(y_0, \mathbf{y}) = y_0$. Likewise, if

$$y_0+\mathbf{y}'\mathbf{z} \leqslant 0 \quad \forall\, \mathbf{z} \in \mathcal{W},$$

we have $\mathrm{E}((y_0+\mathbf{y}'\mathbf{z})^+) = \pi(y_0, \mathbf{y}) = 0$.

Remark 1. The convexity of $\pi(y_0, \mathbf{y})$ depends on the convexity of the following function

$$f(u_0, \mathbf{u}) = \inf_{\mu>0} \mu \exp\left(\frac{u_0}{\mu}+\frac{\|\mathbf{u}\|_2^2}{\mu^2}\right).$$

It is easy to see that $g(u_0, \mathbf{u}) = \exp(u_0 + \|\mathbf{u}\|_2^2)$ is a convex function, and it is straightforward to check that $h(u_0, \mathbf{u}, \mu) = \mu g(u_0/\mu, \mathbf{u}/\mu)$ is a convex function on domain $\mu > 0$. Hence, $f(u_0, \mathbf{u}) = \inf_{\mu>0} h(u_0, \mathbf{u}, \mu)$ is a convex function. Due to the presence of such a function, the set of constraints in problem (7) is not exactly second-order cone representable (see Ben-Tal and Nemirovski 2001). Fortunately, using a few second-order cones, we can accurately approximate such constraints to a good level of numerical precision. The interested readers can refer to Chen and Sim (2009).

Remark 2. Note that the first and third constraints involving the support set $\mathcal{W}$ take the form of

$$\max_{z\in\mathcal{W}} \mathbf{v}'\mathbf{z} \leqslant r$$

or, equivalently, as

$$\mathbf{v}'\mathbf{z} \leqslant r \quad \forall \mathbf{z} \in \mathcal{W}.$$

Such a constraint is known as the robust counterpart whose explicit formulation under difference choices of tractable support set $\mathcal{W}$ is discussed in Ben-Tal and Nemirovski (1998, 2001). Because $\mathcal{W}$ is a second-order conic representable set, the robust counterpart is also second order cone representable. For instance, if $\mathcal{W} = [-\underline{\mathbf{z}}, \bar{\mathbf{z}}]$, the corresponding robust counterpart is representable by the following linear inequalities:

$$\underline{\mathbf{z}}'\mathbf{t} + \bar{\mathbf{z}}'\mathbf{s} \leqslant r$$

for some $\mathbf{s}, \mathbf{t} \geqslant \mathbf{0}$ satisfying $\mathbf{s} - \mathbf{t} = \mathbf{v}$.

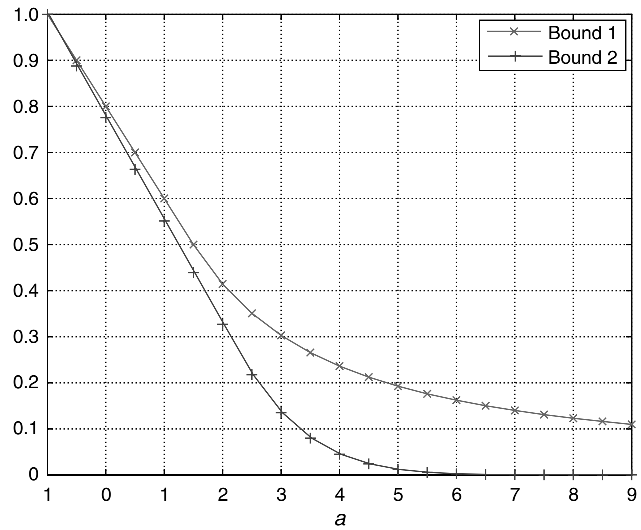
Remark 3. Note that under the Assumption U, it is not necessary to provide all the information, such as the directional deviations. Therefore, whenever such information is unavailable, we can assign an infinite value to the corresponding parameter. For instance, suppose the factor $\tilde{z}_j$ has standard deviation σ and unknown directional deviations; we would set $p_j = q_j = \infty$. When the bounds on p_j and q_j are finite, and not infinity, the results will be better.

Remark 4. In the absence of uncertainty, the nonzero crossing condition ensures that the bound is tight. That is, $y^+ = \mathrm{E}(y^+) = \pi(y, \mathbf{0})$.

Remark 5. The uncertainty model assumes that we have exact estimates of the covariance, means, and deviation measures from data. However, it is possible to consider a model of data uncertainty in which the covariance, means, and deviation measures are uncertain and belong to some uncertainty set. This is could be done by modifying the bound $\pi(y_0, \mathbf{y})$ and applying standard robust optimization techniques such as those of Ben-Tal and Nemirovski (1998) and Bertsimas and Sim (2006).

The robust model of Bertsimas and Thiele (2006) uses Proposition 2. We next show that for a univariate random variable with one-sided support, the bound of Theorem 1 is as tight.

Figure 1. Comparing bounds of $\mathrm{E}((\tilde{z}-a)^+)$.



Proposition 3. *Let $\tilde{z}$ be a random variable in $[-\mu, \infty)$ with mean μ and standard deviation σ, then for all $a \geqslant -\mu$,*

$$\mathrm{E}((\tilde{z}-a)^+) \leqslant \pi(-a, 1)$$
$$= \begin{cases} \dfrac{1}{2}\left(-a + \sqrt{\sigma^2 + a^2}\right) & \text{if } a \geqslant \dfrac{\sigma^2 - \mu^2}{2\mu}, \\ -a\dfrac{\mu^2}{\mu^2+\sigma^2} + \mu\dfrac{\sigma^2}{\mu^2+\sigma^2} & \text{if } a < \dfrac{\sigma^2-\mu^2}{2\mu}. \end{cases}$$

Proof of Proposition 3. See the online companion to this paper.

We can further improve the bound if the distribution of the random variable $\tilde{z}$ is sufficiently light tailed such that the directional deviations are close to its standard deviation, such as those of normal and uniform distributions. Figure 1 compares the bounds of $\mathrm{E}((\tilde{z}-a)^+)$ in which $\mu = 1$ and $\sigma = \sigma_f(\tilde{z}) = \sigma_b(\tilde{z}) = 2$. Bound 1 corresponds to the bound of Proposition 2, whereas Bound 2 corresponds to the bound of Theorem 1. Clearly, despite the lack of tightness results, incorporating the directional deviations can potentially improve the bound on $\mathrm{E}((\tilde{z}-a)^+)$. We will further demonstrate the benefits in our computational experiments.

3.2. Tractable Replenishment Policies

Having introduced the demand uncertainty model, a suitable approximation of the replenishment policy $x_t(\tilde{\mathbf{d}}_{t-1})$ is needed to obtain a tractable formulation. That is, we seek a formulation in which the policy can be obtained by solving an optimization problem that runs in polynomial time and is scalable across time period. We review two tractable replenishment policies, static as well as linear with respect to the random factors of demand, which are decision rules prevalent in the context of robust optimization. We introduce a new replenishment policy known as the truncated linear replenishment policy that improves over these policies.

Static Replenishment Policy. The static replenishment policy, a.k.a the open-loop policy, has order decisions not being influenced by the random factors of demand as follows:

$$x_t(\tilde{\mathbf{d}}_{t-1}) = x_t^0. \tag{8}$$

A tractable model under such a replenishment policy is as follows:

$$\begin{aligned} Z_{\mathrm{SRP}} = \min\ & \sum_{t=1}^{T}(c_t x_t^0 + h_t \pi(y_{t+1}^0, \mathbf{y}_{t+1}) \\ & \quad + b_t \pi(-y_{t+1}^0, -\mathbf{y}_{t+1})) \\ \text{s.t.}\ & y_{t+1}^0 = y_t^0 + x_{t-L}^0 - d_t^0 \quad t=1,\dots,T \\ & y_{t+1}^k = y_t^k - d_t^k \\ & \qquad\qquad k=1,\dots,N,\ t=1,\dots,T \\ & y_{t+1}^k = 0 \quad k \geqslant N_t + 1,\ t=1,\dots,T \\ & 0 \leqslant x_t^0 \leqslant S_t \quad t=1,\dots,T-L, \end{aligned} \tag{9}$$

with y_1^0 being the initial inventory level and $y_1^k = 0$ for all $k = 1, \dots, N$. For $L \geqslant 1$, x_t^0 are the known committed orders made at time periods $t = 1-L, \dots, 0$.

Under Equation (8), it is evident from Equation (1) that the inventory level also takes an affine structure,

$$y_{t+1}(\tilde{\mathbf{d}}_t) = y_{t+1}^0 + \sum_{k=1}^{N} y_{t+1}^k \tilde{z}_k. \tag{10}$$

Using Theorem 1, we can bound the excess inventory level at time period t, that is, $\mathrm{E}((y_{t+1}(\tilde{\mathbf{d}}_t))^+) \leqslant \pi(y_{t+1}^0, \mathbf{y}_{t+1})$. Proceeding similarly for the backlog inventory gives the objective function of problem (9). Equating the coefficients of the constant and $\tilde{z}_k$ term of Equation (1) gives the first two sets of constraints in problem (9), respectively. The last set of constraints enforces the range on order quantity, that is, nonnegativity and upper limit.

THEOREM 2. *The expected cost of the stochastic inventory problem under the static replenishment policy,*

$$x_t^{\mathrm{SRP}}(\tilde{\mathbf{d}}_{t-1}) = x_t^{0*} \quad t = 1, \dots, T-L,$$

in which x_t^{0*}, $t = 1, \dots, T-L$ is the optimum solution of problem (9), *is at most Z_{SRP}.*

PROOF OF THEOREM 2. See the online companion to this paper.

Linear Replenishment Policy. A more refined replenishment policy introduced in Ben-Tal et al. (2005), and Chen et al. (2007) is the linear replenishment policy where the order decisions are affinely dependent on the random factors of demand, that is,

$$x_t^{\mathrm{LRP}}(\tilde{\mathbf{d}}_{t-1}) = x_t^0 + \mathbf{x}_t'\tilde{\mathbf{z}}, \tag{11}$$

in which the vector $\mathbf{x}_t = (x_t^1, \dots, x_t^N)$ satisfies the following nonanticipative constraints:

$$x_t^k = 0 \quad \forall\, k \geqslant N_{t-1} + 1. \tag{12}$$

Because the order decision is made at the beginning of the tth period, the nonanticipative constraints ensure that the linear replenishment policy is not influenced by demand factors that are unavailable up to the beginning of the tth period. The model for the linear replenishment policy is as follows:

$$\begin{aligned} Z_{\mathrm{LRP}} = \min\ & \sum_{t=1}^{T}\big(c_t x_t^0 + h_t \pi(y_{t+1}^0, \mathbf{y}_{t+1}) \\ & \quad + b_t \pi(-y_{t+1}^0, -\mathbf{y}_{t+1})\big) \\ \text{s.t.}\ & y_{t+1}^k = y_t^k + x_{t-L}^k - d_t^k \\ & \qquad\qquad k=0,\dots,N,\ t=1,\dots,T \\ & y_{t+1}^k = 0 \quad k \geqslant N_t + 1,\ t=1,\dots,T \\ & x_t^k = 0 \quad k \geqslant N_{t-1} + 1,\ t=1,\dots,T-L \\ & 0 \leqslant x_t^0 + \mathbf{x}_t'\mathbf{z} \leqslant S_t \\ & \qquad\qquad \forall\, \mathbf{z} \in \mathcal{W}\ t=1,\dots,T-L, \end{aligned} \tag{13}$$

with y_1^0 being the initial inventory level and $y_1^k = 0$ for all $k = 1, \dots, N$. For $L \geqslant 1$, x_t^0 are the known committed orders made at time periods $t = 1-L, \dots, 0$.

Under Equation (11), the inventory level has a structure similar to Equation (10). The objective function and the first set of constraints are hence obtained in a similar manner as problem (9). The last set of constraints ensures that the linear replenishment policy is confined within the ordering capacity for all possible states of random factors. Observe that under the assumption that $\mathcal{W}$ is a tractable conic representable uncertainty set, the robust counterpart

$$0 \leqslant x_t^0 + \mathbf{x}_t'\mathbf{z} \leqslant S_t \quad \forall\, \mathbf{z} \in \mathcal{W}$$

can be represented concisely as tractable conic constraints. Therefore, problem (13) is essentially a tractable conic optimization problem.

THEOREM 3. *The expected cost of the stochastic inventory problem under the linear replenishment policy,*

$$x_t^{\mathrm{LRP}}(\tilde{\mathbf{d}}_{t-1}) = x_t^{0*} + \mathbf{x}_t^{*\prime}\tilde{\mathbf{z}} \quad t = 1, \dots, T-L,$$

in which x_t^{k*}, $k = 0, \dots, N$, $t = 1, \dots, T-L$ is the optimum solution of problem (13), *is at most Z_{LRP}. Moreover, $Z_{\mathrm{LRP}} \leqslant Z_{\mathrm{SRP}}$.*

PROOF OF THEOREM 3. See the online companion to this paper.

Truncated Linear Replenishment Policy. Chen et al. (2008) studied the weakness of linear decision rules (or policy) and showed that carefully chosen piecewise-linear decision rules can strengthen the approximation of stochastic optimization problems. Indeed, a base-stock policy such as Equation (4) can be shown by induction to be piecewise linear with respect to the historical demands. In the same spirit, we introduce a new piecewise-linear replenishment policy that we call the truncated linear replenishment policy. It takes the following form:

$$x_t^{\mathrm{TLRP}}(\tilde{\mathbf{d}}_{t-1}) = \min\{\max\{x_t^0 + \mathbf{x}_t'\tilde{\mathbf{z}}, 0\}, S_t\}, \tag{14}$$

where the vector $\mathbf{x}_t = (x_t^1, \ldots, x_t^N)$ satisfies the following nonanticipative constraints:

$$x_t^k = 0 \quad \forall\, k \geqslant N_{t-1} + 1. \tag{15}$$

Note that the truncated linear replenishment policy is piecewise linear and directly satisfies the ordering range constraint as follows:

$$0 \leqslant x_t^{\mathrm{TLRP}}(\tilde{\mathbf{d}}_{t-1}) \leqslant S_t.$$

Before introducing the model, we present the following bound on a nested sum of expected positive values of random variables:

Theorem 4. *Let $\tilde{\mathbf{z}} \in \Re^N$ be a multivariate random variable under Assumption U. Then*

$$\mathrm{E}\left(\left(y^0 + \mathbf{y}'\tilde{\mathbf{z}} + \sum_{i=1}^{p}\left(x_i^0 + \mathbf{x}_i{}'\tilde{\mathbf{z}}\right)^+\right)^+\right) \leqslant \eta\big((y^0, \mathbf{y}), (x_1^0, \mathbf{x}_1), \ldots, (x_p^0, \mathbf{x}_p)\big) \tag{16}$$

where

$$\eta\big((y^0, \mathbf{y}), (x_1^0, \mathbf{x}_1), \ldots, (x_p^0, \mathbf{x}_p)\big) = \min_{w_i^0, \mathbf{w}_i, i=1,\ldots,p}\left\{\pi\left(y^0 + \sum_{i=1}^{p} w_i^0, \mathbf{y} + \sum_{i=1}^{p}\mathbf{w}_i\right) + \sum_{i=1}^{p}\left(\pi(-w_i^0, -\mathbf{w}_i) + \pi(x_i^0 - w_i^0, \mathbf{x}_i - \mathbf{w}_i)\right)\right\}.$$

Moreover, the bound is tight if $y^0 + \mathbf{y}'\mathbf{z} + \sum_{i=1}^{p}(x_i^0 + \mathbf{x}_i{}'\mathbf{z})^+$ and $x_i^0 + \mathbf{x}_i{}'\mathbf{z}$, $i = 1, \ldots, p$ are nonzero crossing functions with respect to $\mathbf{z} \in \mathscr{W}$.

Proof of Theorem 4. See the online companion to this paper.

Remark 1. It is easy to establish that

$$\begin{aligned}\mathrm{E}\left(\left(y^0 + \mathbf{y}'\tilde{\mathbf{z}} + \sum_{i=1}^{p}(x_i^0 + \mathbf{x}_i{}'\tilde{\mathbf{z}})^+\right)^+\right) &\leqslant \mathrm{E}((y^0 + \mathbf{y}'\tilde{\mathbf{z}})^+) + \sum_{i=1}^{p}\mathrm{E}((x_i^0 + \mathbf{x}_i{}'\tilde{\mathbf{z}})^+) \\ &\leqslant \pi\left(y^0, \mathbf{y}\right) + \sum_{i=1}^{p}\pi(x_i^0, \mathbf{x}_i).\end{aligned}$$

However, this is a weaker bound, considering the fact that

$$\begin{aligned}&\eta\big((y^0, \mathbf{y}), (x_1^0, \mathbf{x}_1), \ldots, (x_p^0, \mathbf{x}_p)\big) \\ &= \min_{w_i^0, \mathbf{w}_i, i=1,\ldots,p}\left\{\pi\left(y^0 + \sum_{i=1}^{p} w_i^0, \mathbf{y} + \sum_{i=1}^{p}\mathbf{w}_i\right) + \sum_{i=1}^{p}\left(\pi(-w_i^0, -\mathbf{w}_i) + \pi(x_i^0 - w_i^0, \mathbf{x}_i - \mathbf{w}_i)\right)\right\} \\ &\leqslant \pi\left(y^0, \mathbf{y}\right) + \sum_{i=1}^{p}\pi(x_i^0, \mathbf{x}_i).\end{aligned}$$

The model for the truncated linear replenishment policy can be formulated as follows:

$$\begin{aligned}Z_{\mathrm{TLRP}} = \min\ & \sum_{t=1}^{T} c_t\pi(x_t^0, \mathbf{x}_t) \\ & + \sum_{t=1}^{L}\big(h_t\pi(y_{t+1}^0, \mathbf{y}_{t+1}) + b_t\pi(-y_{t+1}^0, -\mathbf{y}_{t+1})\big) \\ & + \sum_{t=L+1}^{T}\Big(h_t\eta\big((y_{t+1}^0, \mathbf{y}_{t+1}), (-x_1^0, -\mathbf{x}_1), \ldots, (-x_{t-L}^0, -\mathbf{x}_{t-L})\big) \\ & \qquad + b_t\eta\big((-y_{t+1}^0, -\mathbf{y}_{t+1}), (x_1^0 - S_t, \mathbf{x}_1), \ldots, (x_{t-L}^0 - S_t, \mathbf{x}_{t-L})\big)\Big) \\ \text{s.t. }\ & y_{t+1}^k = y_t^k + x_{t-L}^k - d_t^k \quad k = 0, \ldots, N,\ t = 1, \ldots, T \\ & y_{t+1}^k = 0 \quad k \geqslant N_t + 1,\ t = 1, \ldots, T \\ & x_t^k = 0 \quad k \geqslant N_{t-1} + 1,\ t = 1, \ldots, T - L\end{aligned} \tag{17}$$

with y_1^0 being the initial inventory level and $y_1^k = 0$ for all $k = 1, \ldots, N$. For $L \geqslant 1$, x_t^0 are the known committed orders made at time periods $t = 1 - L, \ldots, 0$.

Under Equation (14), the inventory level $y_{t+1}(\tilde{\mathbf{d}}_t)$ is no longer affinely dependent on $\tilde{\mathbf{z}}$. The terms at the objective function account for the costs associated with excess inventory level and backlog, taking into consideration the piecewise-linear policy. It can be shown that the truncated linear replenishment policy dominates over the linear replenishment policy as follows:

Theorem 5. *The expected cost of the stochastic inventory problem under the truncated linear replenishment policy,*

$$x_t^{\mathrm{TLRP}}(\tilde{\mathbf{d}}_{t-1}) = \min\{\max\{x_t^{0*} + \mathbf{x}_t^{*\prime}\tilde{\mathbf{z}}, 0\}, S_t\} \quad t = 1, \ldots, T - L$$

in which x_t^{k*}, $k = 0, \ldots, N$, $t = 1, \ldots, T - L$ is the optimum solution of problem (17), *is at most Z_{TLRP}. Moreover, $Z_{\mathrm{TLRP}} \leqslant Z_{\mathrm{LRP}}$.*

Proof of Theorem 5. See the online companion to this paper.

We have shown that $Z_{\mathrm{STOC}} \leqslant Z_{\mathrm{TLRP}} \leqslant Z_{\mathrm{LRP}} \leqslant Z_{\mathrm{SRP}}$. The linear replenishment policy improves over the static replenishment policy because it is able to adapt to demand history. Because setting the coefficient of the random factors $\mathbf{x}_t$ to be zero in problem (13) gives problem (9), it is evident from Equation (11) that the linear replenishment policy subsumes the static replenishment policy. Observe that in problem (13), from which the solution of the linear replenishment policy is derived, the set of constraints restricting the ordering quantity

$$0 \leqslant x_t^0 + \mathbf{x}_t'\mathbf{z} \leqslant S_t \quad \forall \mathbf{z} \in \mathscr{W} \;\; t = 1, \ldots, T-L$$

can be overly constraining on the replenishment policy. For the case when the uncertainty set $\mathscr{W}$ is unbounded, such as $\mathscr{W} = \{\mathbf{z}\colon \mathbf{z} \geqslant -\underline{\mathbf{z}}\}$, the decision variables $\mathbf{x}_t$ will be driven to zeros. This means that the ordering decision of Problem (13) degenerates to a static replenishment policy, losing the ability to adapt to the history of random factors. The truncated linear replenishment policy, on the other hand, avoids this issue. Moreover, we also note that in problem (13), information of mean, variance, and directional deviations are not utilized at the set of constraints restricting the ordering quantity. In contrast, the truncation linear replenishment policy is defined to satisfy the ordering constraint. Hence, the robust model of problem (17) does not have the explicit constraints on ordering levels and is able to utilize the additional information via the π and η functions for improving the bound.

It should be noted that establishing the bounds does not necessarily imply the superiority of truncated linear replenishment policy over static and linear ones. Nevertheless, this behavior is observed throughout our computational studies.

4. Other Extensions

In this section, we discuss some extensions to the basic model.

4.1. Fixed Ordering Cost

Unfortunately, with fixed ordering cost the inventory replenishment problem becomes nonconvex and is much harder to address. Using the idea of Bertsimas and Thiele (2006), we can formulate a restricted problem where the time period in which the orders that can be placed is determined at the start of the planning horizon as follows:

$$\begin{aligned} Z_{\mathrm{STOCF}} & \\ = \min \;& \sum_{t=1}^{T} \Big(\mathrm{E}\big(c_t x_t(\tilde{\mathbf{d}}_{t-1}) + K_t \delta_t + h_t(y_{t+1}(\tilde{\mathbf{d}}_t))^+\big) \\ & \qquad\qquad + \mathrm{E}(b_t(y_{t+1}(\tilde{\mathbf{d}}_t))^-)\Big) \\ \text{s.t.} \;& y_{t+1}(\tilde{\mathbf{d}}_t) = y_t(\tilde{\mathbf{d}}_{t-1}) + x_{t-L}(\tilde{\mathbf{d}}_{t-L-1}) - \tilde{d}_t \\ & \qquad\qquad t = 1, \ldots, T \\ & 0 \leqslant x_t(\tilde{\mathbf{d}}_{t-1}) \leqslant S_t \delta_t \quad t = 1, \ldots, T-L \\ & \delta_t \in \{0,1\} \quad t = 1, \ldots, T-L. \end{aligned} \tag{18}$$

In problem (18), inventory can only be replenished at a period where the corresponding binary variable δ_t takes the value of one. We can then incorporate the tractable replenishment policies developed in the previous section. The resulting optimization model is a conic integer program, which is already addressed in commercial solvers such as CPLEX 11.2. Admittedly, algorithms for solving conic integer programs are still in their infancy. On the theoretical front, Atamtürk and Narayanan (2010) recently developed general-purpose conic mixed-integer rounding cuts based on polyhedral conic substructures of second-order conic sets, which can be readily incorporated in branch-and-bound algorithms that solve continuous conic optimization problems at the nodes of the search tree. Their preliminary computational experiments suggest that the new cuts are quite effective in reducing the integrality gap of continuous relaxations of conic mixed-integer programs.

4.2. Supply Chain Networks

The models we have presented in the preceding section can also be extended to more complex supply chain networks such as the series system or, more generally, the tree network. These are multistage systems where goods transit from one stage to the next stage, each time moving closer to their final destination. In many supply chains, the main storage hubs, or the sources of the network, receive their supplies from outside manufacturing plants in a tree-like hierarchical structure and send items throughout the network until they finally reach the stores, or the sinks of the network. The extension to tree structure uses the concept of echelon inventory and closely follows Bertsimas and Thiele (2006). We refer interested readers to their paper.

5. Conclusions

In this paper, we propose a robust optimization approach to address a multiperiod inventory control problem under ambiguity in demand distributions. Our proposed approach has the advantage of being able to obtain the replenishment policy by solving a tractable polynomial-time solvable SOCP of modest size. The preliminary computational studies suggest that the truncated linear replenishment policy performs better than linear and static ones. Moreover, the robustness of the truncated linear replenishment policy is exemplified by performing reasonably well against optimal policies despite using significantly less information. Although it is possible to use our proposed approach for solving hard inventory problems with known demand distribution, it is premature to comment on the performance. In particular, we need to compare against the heuristics such as those of Levi et al. (2007a), Iida and Zipkin (2006), and Lu et al. (2006) among others in order to understand the relative strengths and weaknesses.

6. Electronic Companion

An electronic companion to this paper is available as part of the online version that can be found at http://or.journal.informs.org/.

Endnote

1. It will be shown subsequently that the bound on expectation using directional deviations is valid only when the factors are stochastically independent. For dependent uncertainties, we set $p_j = q_j = \infty$ for $j \notin \mathcal{I}$.

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References

Adida, E., G. Perakis. 2006. A robust optimization approach to dynamic pricing and inventory control with no backorders. *Math. Programming* **107**(1–2) 97–129.

Atamtürk, A., V. Narayanan. 2010. Conic mixed-integer rounding cuts. *Math. Programming Ser. A* **122**(1) 1–20.

Azoury, K. S. 1985. Bayes solution to dynamic inventory models under unknown demand distribution. *Management Sci.* **31**(9) 1150–1160.

Ben-Tal, A., A. Nemirovski. 1998. Robust convex optimization. *Math. Oper. Res.* **23** 769–805.

Ben-Tal, A., A. Nemirovski. 1999. Robust solutions to uncertain programs. *Oper. Res. Lett.* **25** 1–13.

Ben-Tal, A., A. Nemirovski. 2000. Robust solutions of linear programming problems contaminated with uncertain data. *Math. Programming* **88** 411–424.

Ben-Tal, A., A. Nemirovski. 2001. Lectures on modern convex optimization: Analysis, algorithms, and engineering applications. *MPR-SIAM Series on Optimization*. SIAM, Philadelphia.

Ben-Tal, A., B. Golany, A. Nemirovski, J. Vial. 2005. Supplier-retailer flexible commitments contracts: A robust optimization approach. *Manufacturing Service Oper. Management* **7**(3) 248–273.

Ben-Tal, A., A. Goryashko, E. Guslitzer, A. Nemirovski. 2004. Adjusting robust solutions of uncertain linear programs. *Math. Programming* **99** 351–376.

Bertsimas, D., I. Popescu. 2002. On the relation between option and stock prices: A convex optimization approach. *Oper. Res.* **50** 358–374.

Bertsimas, D., M. Sim. 2003. Robust discrete optimization and network flows. *Math. Programming* **98** 49–71.

Bertsimas, D., M. Sim. 2004. Price of robustness. *Oper. Res.* **52**(1) 35–53.

Bertsimas, D., M. Sim. 2006. Tractable approximations to robust conic optimization problems. *Math. Programming* **107**(1) 5–36.

Bertsimas, D., A. Thiele. 2006. A robust optimization approach to inventory theory. *Oper. Res.* **54**(1) 150–168.

Bertsimas, D., D. Pachamanova, M. Sim. 2004. Robust linear optimization under general norms. *Oper. Res. Lett.* **32** 510–516.

Bienstock, D., N. Ozbay. 2008. Computing robust base-stock levels. *Discrete Optim.* **5**(2) 389–414.

Box, G. E. P., G. M. Jenkins, G. C. Reinsel. 1994. *Time Series Analysis Forecasting and Control*, 3rd ed. Holden-Day, San Francisco, 110–114.

Chen, W., M. Sim. 2009. Goal-driven optimization. *Oper. Res.* **57**(2) 342–357.

Chen, X., M. Sim, P. Sun. 2007. A robust optimization perspective on stochastic programming. *Oper. Res.* **55**(6) 1058–1071.

Chen, X., M. Sim, P. Sun, J. Zhang. 2008. A linear decision-based approximation approach to stochastic programming. *Oper. Res.* **56**(2) 344–357.

El-Ghaoui, L., H. Lebret. 1997. Robust solutions to least-square problems to uncertain data matrices. *SIAM J. Matrix Anal. Appl.* **18** 1035–1064.

El-Ghaoui, L., F. Oustry, H. Lebret. 1998. Robust solutions to uncertain semidefinite programs. *SIAM J. Optim.* **9** 33–52.

Gallego, G., I. Moon. 1993. The distribution free newsboy problem: Review and extensions. *J. Oper. Res. Soc.* **44** 825–834.

Gallego, G., J. Ryan, D. Simchi-Levi. 2001. Minimax analysis for finite horizon inventory models. *IIE Trans.* **33** 861–874.

Glasserman, P., S. Tayur. 1995. Sensitivity analysis for base-stock levels in multiechelon production-inventory systems. *Management Sci.* **41** 263–282.

Godfrey, G. A., W. B. Powell. 2001. An adaptive, distribution-free algorithm for the newsvendor problem with censored demands, with applications to inventory and distribution. *Management Sci*. **47** 1101–1112.

Graves, S. 1999. A single-item inventory model for a nonstationary demand process. *Manufacturing Service Oper. Management* **1**(1) 50–61.

Iida, T., P. Zipkin. 2006. Approximate solutions of a dynamic forecasting inventory model. *Manufacturing Service Oper. Management* **8**(4) 401–425.

Johnson, G. D., H. E. Thompson. 1975. Optimality of myopic inventory policies for certain dependent demand processes. *Management Sci.* **21**(11) 1303–1307.

Kasugai, H., T. Kasegai. 1960. Characteristics of dynamic maximin ordering policy. *J. Oper. Res. Soc. Japan* **3** 11–26.

Levi, R., R. O. Roundy, D. B. Shmoys. 2007a. Provably near-optimal sampling-based policies for stochastic inventory control models. *Math. Oper. Res.* **32**(4) 821–839.

Levi, R., M. P al, R. O. Roundy, D. B. Shmoys. 2007b. Approximation algorithms for stochastic inventory control models. *Math. Oper. Res.* **32**(2) 284–302.

Lovejoy, W. S. 1990. Myopic policies for some inventory models with uncertain demand distributions. *Management Sci.* **36**(6) 724–738.

Lu, X. W., J. S. Song, A. Regan. 2006. Inventory planning with forecast updates: Approximate solutions and cost error bounds. *Oper. Res.* **54**(6) 1079–1097.

Miller, B. 1986. Scarf's state reduction method, flexibility and a dependent demand inventory model. *Oper. Res.* **34**(1) 83–90.

Natarajan, K., D. Pachamanova, M. Sim. 2008. Incorporating asymmetric distributional information in robust value-at-risk optimization. *Management Sci.* **54**(3) 573–585.

Nemirovski, A., A. Shapiro. 2006. Convex approximation of chance constrained programs. *SIAM J. Optim.* **17**(4) 969–996.

Powell, W., A. Ruszczynski, H. Topaloglu. 2004. Learning algorithms for separable approximations of discrete stochastic optimization problems. *Math. Oper. Res.* **29**(4) 814–836.

Scarf, H. 1958. A min-max solution of an inventory problem. *Studies in the Mathematical Theory of Inventory and Production*. Stanford University Press, Stanford, CA.

Scarf, H. 1959. Bayes solutions of the statistical inventory problem. *Ann. Math. Statist.* **30** 490–508.

Scarf, H. 1960. Some remarks on Bayes solution to the inventory problem. *Naval Res. Logist.* **7** 591–596.

Song, J., P. Zipkin. 1993. Inventory control in a fluctuating demand environment. *Oper. Res.* **41** 351–370.

Song, K., D. Klabjan, D. Simchi-Levi. 2007. Robust stochastic lot-sizing by means of histograms. Submitted manuscript.

Veinott, A. F., Jr. 1965. Optimal policy for a multi-product, dynamic, nonstationary inventory problem. *Management Sci.* **12** 206–222.

Zipkin, P. 2000. *Foundations of Inventory Management*. McGraw-Hill Higher Education, New York.