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Robust empirical optimization is almost the same as mean–variance optimization

Jun-ya Gotoh [a], Michael Jong Kim [b,*], Andrew E.B. Lim [c]

[a] *Department of Industrial and Systems Engineering, Chuo University, Japan*
[b] *Sauder School of Business, University of British Columbia, Canada*
[c] *Department of Analytics and Operations & Department of Finance, NUS Business School, National University of Singapore, Singapore*

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ABSTRACT

We formulate a distributionally robust optimization problem where the deviation of the alternative distribution is controlled by a ϕ-divergence penalty in the objective, and show that a large class of these problems are essentially equivalent to a mean–variance problem. We also show that while a "small amount of robustness" always reduces the in-sample expected reward, the reduction in the variance, which is a measure of sensitivity to model misspecification, is an order of magnitude larger.

1. Introduction

Empirical optimization (or sample average approximation) is a data-driven approach to decision making where decisions are chosen by optimizing the (in-sample) expected reward under the empirical distribution. While it has some desirable asymptotic properties, out-of-sample performance can be poor when the sample size is not sufficiently large or the statistical assumptions that form the basis of empirical optimization are not valid, and robust (worst-case) empirical optimization has emerged as one approach to addressing these limitations [2,5,6,12,15,21]. We add to this literature by (i) providing insight into *how* worst-case empirical optimization comes up with decisions with lower sensitivity to model misspecification for a broad class of models, (ii) characterizing the impact of robustness on the solution of the robust problem, and (iii) quantifying the impact of "robustifying" an empirical optimization problem on the mean and variance of the reward.

Specifically, we show that a large class of robust empirical optimization problems are (almost) equivalent to an empirical mean–variance problem with the equivalence becoming exact in the regime of vanishing model uncertainty. Intuitively, a decision is susceptible to model misspecification if a small mismatch between the in- and out-of-sample reward distributions results in a large difference between the respective expected rewards, which can occur if the in-sample distribution has a large spread and there is a mismatch between the tails of the two distributions. The mean–variance relationship shows that robust optimization reduces sensitivity to misspecification by favoring decisions that reduce the spread of the reward distribution. Expanding on this insight, we show that a "small amount of robustness" adds a bias to the solution of the (risk-neutral) empirical problem, which we characterize, and show that the corresponding reduction in the variance is an order of magnitude larger than the loss in expected reward. This also explains why substantial variance reduction has been observed in a number of applications in the literature [3,5,7,11]. More generally, while robust optimization has been criticized by some for producing excessively conservative solutions, our results show that this is not the case when the robustness parameter is properly chosen.

The results in this paper are related to [10,19] where certain robust portfolio choice problems are shown to be equivalent to regularized empirical risk minimization. Unlike the typical regularizer in (e.g.) LASSO or Ridge regression which focuses on properties of the solution (e.g. sparsity), the variance penalizes the spread of the *reward distribution*. Indeed, it is easy to construct examples where regularizing an arbitrary (concave) empirical maximization problem using a LASSO or Ridge penalty can result in solutions that have poor out-of-sample robustness properties. Our paper also shows that the variance regularization approach in [1] is essentially equivalent to robust mean-CVaR (Conditional Value-at-Risk) optimization.

* Corresponding author.
E-mail addresses: jgoto@indsys.chuo-u.ac.jp (J.-y. Gotoh), mike.kim@sauder.ubc.ca (M.J. Kim), andrewlim@nus.edu.sg (A.E.B. Lim).

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While expansions of worst-case objectives are also developed in the recent papers of Duchi et al. [8] and Lam [13], there are several important differences. Owen's empirical likelihood theory is generalized in [8] to develop confidence intervals for the optimal objective value and consistency of the solutions, while [13] studies the sensitivity of the objective function under worst-case perturbations in the context of simulation. In contrast, we study the impact of robustness on the solution and the reward. We also adopt the penalty formulation of the robust optimization problem while [8,13] adopt a constrained model. These are related though the penalty framework facilitates our analysis of the structure of the robust solution and other extensions such as the derivation of higher order terms in the expansion of the robust objective and similar expansions for robust regular risk measures [18] (which includes CVaR as a special case). The expansion in [8,13] differs from ours because the robust models are different—theirs involves the standard deviation whereas ours involves the variance.

The results in this paper have implications for how the robustness parameter (or the confidence level of an uncertainty set) for a robust model should be chosen. This is explored in the context of an out-of-sample analysis in [9].

2. Empirical optimization

We would like to maximize the expected reward

$$\psi(x) := \mathbb{E}_{\mathbb{P}}\left[f(x, Y)\right], \tag{1}$$

over x. We do not know the distribution $\mathbb{P}$ of Y but have n historical data points $Y_1, \ldots, Y_n$. Since $\mathbb{P}$ and $\psi(x)$ are not known, it is common to optimize the empirical estimate

$$\hat{\psi}_n(x) := \frac{1}{n}\sum_{i=1}^{n} f(x, Y_i) = \mathbb{E}_{\hat{\mathbb{P}}_n}[f(x, Y)],$$

in place of $\psi(x)$, where $\hat{\mathbb{P}}_n$ is the empirical distribution of Y. If the distribution $\mathbb{P}$ of Y does not depend on the decision x and $Y_1, \ldots, Y_n$ are drawn i.i.d. from $\mathbb{P}$, then $\hat{\psi}_n(x)$ is an unbiased estimate of $\psi(x)$, namely, $\psi(x) = \mathbb{E}_{\mathbb{P}}[f(x, Y)] = \mathbb{E}_{\mathbb{P}}[\hat{\psi}_n(x)]$. Indeed, by the strong law of large numbers $\hat{\psi}_n(x)$ will converge almost surely to $\psi(x)$ for any decision x and the variance of $\hat{\psi}_n(x)$, by the Central Limit Theorem, decreases like $O(1/n)$. If any of these assumptions are violated, or the number of data points is small, then $\hat{\psi}_n(x)$ may not be a good estimate of $\psi(x)$. More generally, the true distribution for Y may differ substantially from the empirical which diminishes our confidence in the out-of-sample performance of the in-sample optimal decision.

3. Robust optimization ≈ mean-variance optimization

To account for uncertainty about the true distribution $\mathbb{P}$, we consider a worst-case empirical optimization problem. We begin by defining ϕ-divergence. Let ϕ be a closed convex function such that $\phi : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ and $\phi(z) \geq \phi(1) = 0$ for all z. We assume throughout that $\phi(z) = +\infty$ when $z < 0$. Let $\hat{\mathbb{P}}_n \equiv [\hat{p}_i]_i$ be the empirical distribution and $\mathbb{Q} \equiv [q_i]_i$ be an alternative probability distribution. Here $\hat{p}_i = \hat{\mathbb{P}}_n\{Y = y_i\}$ and $q_i = \mathbb{Q}\{Y = y_i\}$, where we assume throughout that the probability distribution $\mathbb{Q}$ absolutely continuous with respect to $\hat{\mathbb{P}}_n$; that is, $q_i = 0$ if $\hat{p}_i = 0$. (Equivalently, the support of $\mathbb{Q}$ is a subset of the support of $\hat{\mathbb{P}}_n$). The ϕ-divergence of $\mathbb{Q}$ relative to $\hat{\mathbb{P}}_n$ is defined by

$$\mathcal{H}_\phi(\mathbb{Q} \mid \hat{\mathbb{P}}_n) := \begin{cases} \displaystyle\sum_{i:\hat{p}_i>0} \hat{p}_i \phi\left(\frac{q_i}{\hat{p}_i}\right), & \displaystyle\sum_{i:\hat{p}_i>0} q_i = 1, q_i \geq 0, \\ +\infty, & \text{otherwise.} \end{cases} \tag{2}$$

Observe that $\mathcal{H}_\phi(\mathbb{Q} \mid \hat{\mathbb{P}}_n)$ is non-negative and convex in $\mathbb{Q}$ and is equal to zero if $\mathbb{Q}$ equals $\hat{\mathbb{P}}_n$. It plays the role of a distance measure between $\mathbb{Q}$ and $\hat{\mathbb{P}}_n$.

When $\phi(z) = z\ln z - z + 1$, $\mathcal{H}_\phi(\mathbb{Q} \mid \hat{\mathbb{P}}_n)$ corresponds to relative entropy and satisfies the conditions of our main equivalence result Theorem 3.2. Other examples include Berg entropy ($\phi(z) = -\ln z + z - 1$), J-divergence ($\phi(z) = (z-1)\ln z$), χ^2-divergence ($\phi(z) = \frac{1}{z}(z-1)^2$), modified χ^2-divergence ($\phi(z) = (z-1)^2$), and Hellinger distance ($\phi(z) = (\sqrt{z}-1)^2$), where in all these examples we assume, as in [2], that $\phi(z) = +\infty$ when $z < 0$.

We now consider the following robust empirical optimization problem

$$\max_x \min_{\mathbb{Q}} \left\{\mathbb{E}_{\mathbb{Q}}\left[f(x, Y)\right] + \frac{1}{\delta}\mathcal{H}_\phi(\mathbb{Q} \mid \hat{\mathbb{P}}_n)\right\}, \tag{3}$$

where constant $\delta > 0$ is the ambiguity parameter. In the language of robust optimization, $\hat{\mathbb{P}}_n$ is the nominal and the decision maker is optimizing against a potential "worst-case" alternative $\mathbb{Q}$, where the penalized divergence term controls the deviation of the alternative from the nominal. More generally, robust optimization accounts for model misspecification by optimizing against worst-case deviations from the nominal. We take the perspective that robust optimization is just a way of parameterizing a family of "increasingly robust" solutions with $\delta = 0$ corresponding to empirical optimization and the amount of robustness increasing in δ. We study this family of solutions and the associated reward distribution when δ is small.

We define the robust objective function associated with the ϕ-divergence penalty as

$$g_{\delta,\phi}(x) := \min_{\mathbb{Q}} \left\{\mathbb{E}_{\mathbb{Q}}\left[f(x, Y)\right] + \frac{1}{\delta}\mathcal{H}_\phi(\mathbb{Q} \mid \hat{\mathbb{P}}_n)\right\}. \tag{4}$$

Definition 1 (*Regular Measure of Deviation [18]*). Given any probability space $(\Omega, \mathcal{F}, \mathbb{P})$, let $\mathcal{L}^2(\Omega)$ denote the space of square-integrable random variables, i.e., $\mathbb{E}_{\mathbb{P}}\left[X^2\right] < \infty$. A functional $\mathcal{D} : \mathcal{L}^2(\Omega) \to [0, \infty]$ is said to be *a regular measure of deviation* if it is closed convex and satisfies

1. $\mathcal{D}(c) = 0$ for any constant $c \in \mathbb{R}$.
2. $\mathcal{D}(Z) > 0$ for any (non-constant) random variable $Z \in \mathcal{L}^2(\Omega)$.

A deviation measure is a measure of non-constancy of a random variable, generalizing the notion of standard deviation or variance. We have the following connection between the robust objective and a mean-deviation objective.

Proposition 3.1. *Let $\phi : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ be a closed convex function such that $\phi(z) \geq \phi(1) = 0$ for all z and is differentiable around $z = 1$ with $\phi(z) = +\infty$ when $z < 0$. Let ϕ^* denote the conjugate of ϕ, i.e., $\phi^*(\zeta) := \sup_z\{z\zeta - \phi(z)\}$. Then, for any random variable $Z \in \mathcal{L}^2(\Omega)$,*

$$\begin{aligned}\mathcal{D}(Z) &= \mathcal{D}_{\delta,\phi,\mathbb{P}}(Z \mid \mathbb{E}_{\mathbb{P}}[Z]) \\ &:= \min_c \left\{c + \frac{1}{\delta}\mathbb{E}_{\mathbb{P}}\left[\phi^*\left(\delta(\mathbb{E}_{\mathbb{P}}[Z] - Z - c)\right)\right]\right\}\end{aligned} \tag{5}$$

is a regular measure of deviation. Furthermore, the following two objective functions are equal:

1. $g_{\delta,\phi}(x) = \min_{\mathbb{Q}}\left\{\mathbb{E}_{\mathbb{Q}}[f(x, Y)] + \frac{1}{\delta}\mathcal{H}_\phi(\mathbb{Q} \mid \hat{\mathbb{P}}_n)\right\}$. *(robust empirical objective function)*
2. $\hat{\psi}_n(x) - \mathcal{D}_{\delta,\phi,\hat{\mathbb{P}}_n}\left(f(x, Y) \,\middle|\, \hat{\psi}_n(x)\right)$. *(mean-deviation objective function)*

While this follows from general results in [18], we provide a simple and direct proof in Appendix.

We now show that deviation measure (5) is approximately equal to the sample variance of the reward.

Theorem 3.2. *Suppose that $\phi : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ is a closed, convex function such that $\phi(z) \geq \phi(1) = 0$ for all z, is twice continuously differentiable around $z = 1$ with $\phi''(1) > 0$, and $\phi(z) = +\infty$ when $z < 0$. Then*

$$\mathcal{D}_{\delta,\phi,\hat{\mathbb{P}}_n}(f(x,Y) \mid \hat{\psi}_n(x)) = \frac{\delta}{2\phi''(1)} \mathbb{V}_{\hat{\mathbb{P}}_n}[f(x,Y)] + o(\delta). \tag{6}$$

where

$$\mathbb{V}_{\hat{\mathbb{P}}_n}[f(x,Y)] := \mathbb{E}_{\hat{\mathbb{P}}_n}\big[f(x,Y) - \mathbb{E}_{\hat{\mathbb{P}}_n} f(x,Y)\big]^2$$

is the sample variance of the reward $f(x, Y)$.

Note that all the ϕ-divergences mentioned above (Kullback–Leibler, Burg entropy, J-divergence, χ^2-distance, modified χ^2-distance, and Hellinger distance) satisfy the smoothness conditions of Theorem 3.2 [2].

Proof. We begin with some preliminary results. Let $\phi^*(\zeta) = \sup_z \{z\zeta - \phi(z)\}$ be the convex conjugate associated with the function $\phi(z)$. The first order conditions associated with the optimization problem in this definition are $\zeta - \phi'(z) = 0$. Let $z(\zeta)$ denote the solution of these equations when it exists. Clearly, $z(0) = 1$ is the unique solution of this equation when $\zeta = 0$ where uniqueness follows from the assumption that $\phi''(1) > 0$. Since $\phi'(z)$ is continuously differentiable in the neighborhood of $z = 1$ (which follows from the twice continuous differentiability of $\phi(z)$ in the neighborhood of $z = 1$), it follows from the Implicit Function Theorem that $z(\zeta)$ is continuously differentiable in a neighborhood of $\zeta = 0$ and we can write $\phi^*(\zeta) = z(\zeta)\zeta - \phi(z(\zeta))$ in this neighborhood of $\zeta = 0$.

Observing that $\frac{d\phi^*}{d\zeta}(\zeta) = z(\zeta)$ and $\frac{d^2\phi^*}{d\zeta^2}(\zeta) = z'(\zeta)$, it follows that $\phi^*(\zeta)$ is twice continuously differentiable in a neighborhood of $\zeta = 0$ and $\phi^*(0) = 0$, $\frac{d\phi^*}{d\zeta}(0) = z(0) = 1$, $\frac{d^2\phi^*}{d\zeta^2}(0) = \frac{1}{\phi''(z(0))}$. Twice continuous differentiability in a neighborhood of $\zeta = 0$ allows us to write

$$\phi^*(\zeta) = \zeta + \frac{1}{2\phi''(1)}\zeta^2 + o(\zeta^2)$$

and the Taylor series expansion of $\phi^*(\zeta)$ around $\zeta = 0$ is given by

$$\begin{aligned}
&\mathcal{D}_{\delta,\phi,\hat{\mathbb{P}}_n}(f(x,Y) \mid \hat{\psi}_n(x)) \\
&= \min_c \left\{ c + \frac{1}{\delta}\mathbb{E}_{\hat{\mathbb{P}}_n}\left[\delta\left(\hat{\psi}_n(x) - f(x,Y) - c\right)\right.\right. \\
&\quad \left.\left. + \frac{\delta^2}{2\phi''(1)}\left(\hat{\psi}_n(x) - f(x,Y) - c\right)^2 + o(\delta^2)\right]\right\} \\
&= \min_c \frac{\delta}{2\phi''(1)}\mathbb{E}_{\hat{\mathbb{P}}_n}\left[\left(\hat{\psi}_n(x) - f(x,Y) - c\right)^2\right] + o(\delta).
\end{aligned}$$

Clearly, $c = 0$ is optimal and (6) follows. ■

In summary, for fixed $\delta > 0$, the robust optimization problem (3) is essentially equivalent to a mean–variance problem:

$$\begin{aligned}
&\max_x \min_{\mathbb{Q}} \left\{\mathbb{E}_{\mathbb{Q}}[f(x,Y)] + \frac{1}{\delta}\mathcal{H}_\phi(\mathbb{Q} \mid \hat{\mathbb{P}}_n)\right\} \\
&= \max_x \left\{\hat{\psi}_n(x) - \frac{\delta}{2\phi''(1)}\mathbb{V}_{\hat{\mathbb{P}}_n}[f(x,Y)] + o(\delta)\right\}.
\end{aligned} \tag{7}$$

Generalizations

We provide three generalizations of our main results. Proposition 3.3 provides a higher order expansion of the deviation measure in terms of the variance, skewness and a generalization of the notion of kurtosis, of the reward $f(x, Y)$, while Propositions 3.4 and 3.5 provide expansions for worst-case Conditional Value-at-Risk and worst-case regular measures of risk. Proofs are in the Appendix.

Higher order expansions

Proposition 3.3. *Suppose that $\phi : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ is a closed, convex function such that $\phi(z) \geq \phi(1) = 0$ for all z, is four times continuously differentiable around $z = 1$ with $\phi''(1) > 0$, and $\phi(z) = +\infty$ when $z < 0$. Then*

$$\begin{aligned}
&\mathcal{D}_{\delta,\phi,\hat{\mathbb{P}}_n}(f(x,Y) \mid \hat{\psi}_n(x)) \\
&= \frac{\delta}{2}\left[\frac{1}{\phi''(1)}\right]\mathbb{V}_{\hat{\mathbb{P}}_n}[f(x,Y)] - \frac{\delta^2}{3!}\left[\frac{\phi^{(3)}(1)}{[\phi''(1)]^3}\right]\hat{s}(x) \\
&\quad + \frac{\delta^3}{4!}\left[\frac{1}{[\phi''(1)]^3}\left(3\frac{[\phi^{(3)}(1)]^2}{[\phi''(1)]^2} - \frac{\phi^{(4)}(1)}{\phi''(1)}\right)\right]\hat{\kappa}(x) + o(\delta^3),
\end{aligned} \tag{8}$$

where

$$\begin{aligned}
\hat{s}(x) &= \mathbb{E}_{\hat{\mathbb{P}}_n}\left[\left(f(x,Y) - \hat{\psi}(x)\right)^3\right], \\
\hat{\kappa}(x) &= \mathbb{E}_{\hat{\mathbb{P}}_n}\left[\left(f(x,Y) - \hat{\psi}(x)\right)^4\right] \\
&\quad - 3\left(\frac{[\phi^{(3)}(1)]^2}{3[\phi^{(3)}(1)]^2 - \phi^{(4)}(1)\phi''(1)}\right)\left[\mathbb{V}_{\hat{\mathbb{P}}_n}[f(x,Y)]\right]^2.
\end{aligned}$$

Proposition 3.3 tells us that when δ is small, robust optimization makes tradeoffs between the mean, variance, skewness and kurtosis of the reward.

Robust CVaR and robust risk measures

Let L be a loss with empirical distribution $\hat{\mathbb{P}}_n$. The Conditional Value-at-Risk (CVaR) of a risk L at risk level $\beta \in (0, 1)$ is given by

$$\begin{aligned}
\mathrm{CVaR}_{\beta,\hat{\mathbb{P}}_n}[L] &:= \min_\alpha \left\{\alpha + \frac{1}{1-\beta}\mathbb{E}_{\hat{\mathbb{P}}_n}\left[[L-\alpha]^+\right]\right\} \\
&= \alpha_0 + \frac{1}{1-\beta}\mathbb{E}_{\hat{\mathbb{P}}_n}\left[[L-\alpha_0]^+\right],
\end{aligned} \tag{9}$$

where $[x]^+ = \max\{x, 0\}$ and α_0 is a minimizer, which satisfies $\alpha_0 \in \left[\mathrm{VaR}_{\beta,\hat{\mathbb{P}}_n}[L], \mathrm{VaR}^+_{\beta,\hat{\mathbb{P}}_n}[L]\right]$, where $\mathrm{VaR}_{\beta,\hat{\mathbb{P}}_n}[L] = \min\big\{x : \hat{\mathbb{P}}_n\{L > x\} \leq 1-\beta\big\}$ and $\mathrm{VaR}^+_{\beta,\hat{\mathbb{P}}_n}[L] = \inf\big\{x : \hat{\mathbb{P}}_n\{L \geq x\} \leq 1-\beta\big\}$. The robust CVaR at risk level β with respect to the empirical distribution $\hat{\mathbb{P}}_n$ is defined as

$$\mathrm{RCVaR}^{\delta,\phi}_{\beta,\hat{\mathbb{P}}_n}[L] := \max_{\mathbb{Q}}\left\{\mathrm{CVaR}_{\beta,\mathbb{Q}}[L] - \frac{1}{\delta}\mathcal{H}_\phi(\mathbb{Q} \mid \hat{\mathbb{P}}_n)\right\}, \tag{10}$$

where $\mathcal{H}_\phi(\mathbb{Q} \mid \hat{\mathbb{P}}_n)$ is ϕ-divergence (2). The following result generalizes Theorem 3.2 to the case of robust CVaR.

Proposition 3.4. *Suppose that $\phi : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ is a closed, convex function such that $\phi(z) \geq \phi(1) = 0$ for all z, is twice continuously differentiable around $z = 1$ with $\phi''(1) > 0$, and $\phi(z) = +\infty$ when $z < 0$. Then*

$$\begin{aligned}
&\mathrm{RCVaR}^{\delta,\phi}_{\beta,\hat{\mathbb{P}}_n}[L] \\
&= \mathrm{CVaR}_{\beta,\hat{\mathbb{P}}_n}[L] + \frac{\delta}{2\,\phi''(1)}\mathbb{V}_{\hat{\mathbb{P}}_n}\left[\frac{[L-\alpha_0]^+}{1-\beta}\right] + o(\delta).
\end{aligned} \tag{11}$$

Proposition 3.4 shows that to order δ, robust CVaR is equal to (classical) CVaR plus a penalty on the variance of the loss, given that it exceeds the Value-at-Risk. Observe that the regularization of CVaR on the right-hand side of (11) was introduced in [1] to address the poor out-of-sample performance of mean-CVaR problems documented in [16]. Proposition 3.4 shows that this variance-regularized approach is essentially robust optimization.

The above perspective can be extended to a wide class of risk measures. Let

$$\rho_{v,\hat{\mathbb{P}}_n}[L] := \min_\alpha\{\alpha + \mathbb{E}_{\hat{\mathbb{P}}_n}[v(L-\alpha)]\}, \tag{12}$$

where v is a convex function on $\mathbb{R}$ such that $v(0) = 0$ and $v(x) > x$ when $x \neq 0$. This is a regular measure of risk [18] (or optimized certainty equivalent [4]) which is a generalization of CVaR. Note that (12) includes the mean–variance $\rho_{v,\hat{\mathbb{P}}_n}[L] = \mathbb{E}_{\hat{\mathbb{P}}_n}[L] + \lambda \mathbb{V}_{\hat{\mathbb{P}}_n}[L]$ (with $v(z) = z + \lambda z^2$ and $\lambda > 0$), the log-sum-exp $\rho_{v,\hat{\mathbb{P}}_n}[L] = \ln \mathbb{E}_{\hat{\mathbb{P}}_n}[\exp(L)]$ (with $v(z) = \exp(z) - 1$), as well as CVaR (with $v(z) = \max\{0, z\}/(1-\beta)$) (see [18] for other examples).

Define a robust risk measure as

$$\mathrm{Rob}\rho^{\delta,\phi}_{v,\hat{\mathbb{P}}_n}[L] := \max_{\mathbb{Q}} \left\{ \rho_{v,\mathbb{Q}}[L] - \frac{1}{\delta}\mathcal{H}_\phi(\mathbb{Q} \mid \hat{\mathbb{P}}_n) \right\}. \tag{13}$$

Proposition 3.5. *Suppose that $\phi : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ is a closed, convex function such that $\phi(z) \geq \phi(1) = 0$ for all z, is twice continuously differentiable around $z = 1$ with $\phi''(1) > 0$, and $\phi(z) = +\infty$ when $z < 0$. Suppose that v is a convex function on $\mathbb{R}$ such that $v(0) = 0$ and $v(x) > x$ when $x \neq 0$ and is continuously differentiable around $x = 0$. Then*

$$\mathrm{Rob}\rho^{\delta,\phi}_{v,\hat{\mathbb{P}}_n}[L] = \rho_{v,\hat{\mathbb{P}}_n}[L] + \frac{\delta}{2\phi''(1)} \mathbb{V}_{\hat{\mathbb{P}}_n}[v(L - \alpha_0)] + o(\delta),$$

where α_0 is a minimizer of (12).

Proposition 3.5 shows that to order δ, robust risk measure is equal to the empirical risk plus a penalty on the variance of $v(L-\alpha_0)$. For example, if we robustify the mean–variance objective via (13), it implies the addition of the variance term of the form $\frac{\lambda\delta}{2\phi''(1)} \mathbb{V}_{\hat{\mathbb{P}}_n}[(L - \bar{L} + \frac{1}{2\lambda})^2]$, since $\alpha_0 = \bar{L} := \mathbb{E}_{\hat{\mathbb{P}}_n}[L]$.

4. Robust optimal solution

We have shown that "robustifying" an empirical optimization problem is almost the same as regularizing the objective using the variance of the reward and now study the impact of worst-case optimization on the solution of the robust problem. The proof is in the Appendix.

Theorem 4.1. *Suppose that $\phi : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ is a closed, convex function such that $\phi(z) \geq \phi(1) = 0$ for all z, is twice continuously differentiable around $z = 1$ with $\phi''(1) > 0$, and $\phi(z) = +\infty$ when $z < 0$. Suppose that $f(x, Y)$ is twice continuously differentiable and strictly concave in x. For any $\delta > 0$, let*

$$x^*(\delta) = \operatorname*{argmax}_x \min_{\mathbb{Q}} \left\{ \mathbb{E}_{\mathbb{Q}}[f(x, Y)] + \frac{1}{\delta}\mathcal{H}_\phi(\mathbb{Q} \mid \hat{\mathbb{P}}_n) \right\} \tag{14}$$

be the solution of the robust optimization problem and

$$x^*(0) = \arg\max_x \mathbb{E}_{\hat{\mathbb{P}}_n}[f(x, Y)]$$

solve the empirical optimization problem. Then $x^*(\delta)$ is continuously differentiable and

$$x^*(\delta) = x^*(0) + \delta\pi + o(\delta)$$

where

$$\pi = \arg\max_x \left\{ x'\Sigma_f x - \frac{2}{\phi''(1)}\mu_f' x \right\} = \frac{1}{\phi''(1)}\Sigma_f^{-1}\mu_f \tag{15}$$

with

$$\begin{aligned} \mu_f &= \mathbb{E}_{\hat{\mathbb{P}}_n}\Big[f(x^*(0), Y)\,\nabla_x f(x^*(0), Y)\Big] \\ &= \frac{1}{2}\nabla_x \mathbb{V}_{\hat{\mathbb{P}}_n}\big[f(x^*(0), Y)\big], \\ \Sigma_f &= \mathbb{E}_{\hat{\mathbb{P}}_n}\Big[\nabla_x^2 f(x^*(0), Y)\Big]. \end{aligned} \tag{16}$$

Theorem 3.2 shows that to a first order, robust optimization is an exercise in controlling the variability of the reward distribution. This makes intuitive sense: the expected reward is sensitive to errors in the tail of the reward distribution if it has a large spread, and robust optimization reduces sensitivity by controlling this spread (when δ is small). Theorem 4.1 elaborates on this insight by showing how robustness affects solutions. As δ increases from zero, the robust solution $x^*(\delta)$ moves from the empirical optimal $x^*(0)$ in the direction π that solves the optimization problem (15). To interpret this equation, note that the impact of any perturbation of $x^*(0)$ on the mean and variance of the reward is

$$\mathbb{E}_{\hat{\mathbb{P}}_n}[f(x^*(0) + \delta\pi, Y)] - \mathbb{E}_{\hat{\mathbb{P}}_n}[f(x^*(0), Y)] = \frac{\delta^2}{2}\pi'\Sigma_f\pi + o(\delta^2)$$

and

$$\mathbb{V}_{\hat{\mathbb{P}}_n}\Big[f(x^*(0) + \delta\pi, Y)\Big] - \mathbb{V}_{\hat{\mathbb{P}}_n}\Big[f(x^*(0), Y)\Big] = 2\delta\pi'\mu_f + o(\delta),$$

so the bias π in (15) optimizes the trade-off between variability reduction $-\pi'\mu_f$ and the impact of bias on the expected reward $\pi'\Sigma_f\pi$.

5. Mean and variance of the reward

Theorems 3.2 and 4.1 characterize the impact of a small amount of robustness on the empirical optimization problem and the optimal solution. We now close-the-loop by studying the impact of robustness on the reward.

Theorem 5.1. *Suppose that $\phi : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ is a closed, convex function such that $\phi(z) \geq \phi(1) = 0$ for all z, is twice continuously differentiable around $z = 1$ with $\phi''(1) > 0$, and $\phi(z) + \infty$ when $z < 0$. Suppose that $f(x, Y)$ is twice continuously differentiable and strictly concave in x. Then*

$$\begin{aligned} \mathbb{E}_{\hat{\mathbb{P}}_n}[f(x^*(\delta), Y)] &= \mathbb{E}_{\hat{\mathbb{P}}_n}[f(x^*(0), Y)] \\ &\quad + \frac{\delta^2}{2[\phi''(1)]^2}\mu_f'\Sigma_f^{-1}\mu_f + o(\delta^2), \end{aligned} \tag{17}$$

$$\begin{aligned} \mathbb{V}_{\hat{\mathbb{P}}_n}\Big[f(x^*(\delta), Y)\Big] &= \mathbb{V}_{\hat{\mathbb{P}}_n}\Big[f(x^*(0), Y)\Big] \\ &\quad + \frac{2\delta}{\phi''(1)}\mu_f'\Sigma_f^{-1}\mu_f + o(\delta), \end{aligned} \tag{18}$$

where μ_f and Σ_f are defined by (16).

Since Σ_f in (16) is negative definite when $f(x, Y)$ is strictly concave in x, Theorem 5.1 shows that the "cost of robustness", as measured by the reduction in the expected reward (17), is an order of magnitude smaller than the reduction in variance when δ is small (i.e. order $O(\delta)$ v $O(\delta^2)$). Alternatively, while robust optimization has been criticized for producing excessively conservative solutions, this result shows that this is not the case when the robustness parameter is small, where the benefit of robustness (variance/sensitivity reduction) is an order of magnitude larger than the cost (reduction in expected reward).

6. Example

Consider the problem of robust logistic regression where

$$f((x, x_0), (Y, Z)) = \ln(1 + \exp(-Y(x^\top Z + x_0))). \tag{19}$$

Here (x_0, x) are the decision variables corresponding to the intercept and coefficients of the classification model, and (Y, Z) is the data where $Y \in \{-1, 1\}$ is the label and Z is the vector of covariates. The Pima Indians Diabetes data set obtained from [14] is used, which consists of $n = 768$ samples with 8 covariates.

Fig. 1 plots the in-sample mean and variance of the log-likelihood corresponding to solutions of the robust optimization problem and the mean–variance problem for different values of δ, where the two types of optimization problems are solved with

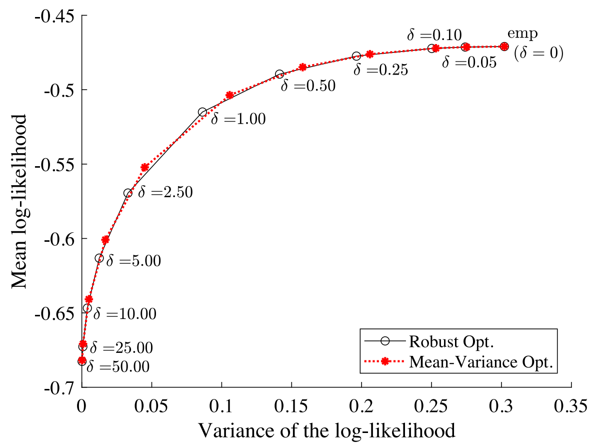

Fig. 1. Robust mean–variance frontier.

RNUOPT (NTT DATA Mathematical Systems Inc.), a nonlinear optimization solver package, and the optimal solution to the robust optimization is used as the initial solution to the mean–variance optimization with the same δ. Consistent with Theorem 5.1, the frontier associated with robust solutions shows significant reduction in the (in-sample) variance of the reward when δ is small with minimal impact on the mean. For example, the variance of the in-sample distribution of the reward under the robust solution corresponding to $\delta = 0.5$ is about half of that of empirical optimization ($\delta = 0$) with relatively little loss in the log-likelihood. More generally, Theorem 5.1 shows that the frontier plots corresponding to solutions of all robust optimization problems have this property. Observe too that when δ is small (e.g. less than $\delta = 0.25$) points on the frontier corresponding to solutions of the mean–variance problem are close to those associated with the worst-case problem, which is consistent with Theorem 3.2. Interestingly, though these points begin to diverge as δ increases, which reflects the impact of the higher-order terms in the expansion of the robust objective as derived in Theorem 4.1, they stay relatively close to each other and eventually converge (e.g. for values of δ exceeding 10). It is worth noting that both frontiers remain close throughout.

The robust mean–variance frontier in Fig. 1 can be used to select δ. For example, one can set a threshold for the expected in-sample reward and choose the value of δ that minimizes the sensitivity to model uncertainty (i.e. the variance) while not violating this constraint.

A more complete picture of the problem of calibrating δ, which accounts for the effects of data variability, is obtained by studying the out-of-sample properties of the robust solution. Interested readers should refer to [9].

7. Conclusions

We show that a large class of robust empirical optimization problems are essentially equivalent to an in-sample mean–variance problem. This makes intuitive sense because reducing variability of the reward also controls the sensitivity of its expected value to misspecification that may affect its tail. We show that a "small amount of robustness" adds a bias to the empirical solution, and while this reduces the in-sample expected reward, the reduction in the in-sample variance is an order of magnitude larger. Our analysis is in the regime where the ambiguity parameter δ is small and provides insight into the value of adding a "little bit of robustness" to the empirical model.

The results in this paper have implications for how the ambiguity parameter δ (or equivalently, the confidence level of an uncertainty set) should be chosen to optimize the out-of-sample performance. This issue is explored in the context of an out-of-sample analysis in [9].

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Appendix A. Supplementary data

Supplementary material related to this article can be found online at https://doi.org/10.1016/j.orl.2018.05.005.

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