Prelim 2:

$\boxed{Q1.}$ (1). Complete statistics: $\max_{1\le i\le n} |X_i|$.

Let $Y_i := |X_i|$. then $Y_{(n)}$ is complete for θ.

$$\mathbb{E}_\theta(Y_{(n)}) = \int_0^\theta \left(1-\left(\frac{y}{\theta}\right)^n\right) dy = \frac{n\theta}{n+1}.$$

(Or $\mathbb{P}_\theta(Y_{(n)} \le y) = (y/\theta)^n$. $\Rightarrow$ $\mathbb{E}_\theta(Y_{(n)}) = \int_0^\theta \theta^{-n} \cdot n y^{n-1} \cdot y \, dy$).

$\Rightarrow$ $\frac{n+1}{n} Y_{(n)}$ is unbiased for θ.

Also: $$\mathbb{E}_\theta(Y_{(n)}^{-1}) = \int_0^\theta \theta^{-n} \cdot n y^{n-1} y^{-1} dy$$
$$= \frac{n}{n-1}\theta^{-1}.$$ $\Rightarrow$ $\frac{n-1}{n} Y_{(n)}^{-1}$ is unbiased for θ^{-1}.

$\Rightarrow$ $\frac{1}{n}(n+1) Y_{(n)} + \frac{1}{n}(n-1) Y_{(n)}^{-1}$ is the UMVUE.

(2). (a) Note that the MLE for η is:
$$\hat{\eta} = \frac{2\sum_{i=1}^n X_i}{n(n+1)}$$, unbiased for η.

$$\Rightarrow \mathrm{MSE}_\eta(\hat\eta) = \mathrm{Var}_\eta(\hat\eta) = \mathrm{Var}_{\lambda^2}\left(\frac{2\sum_{i=1}^n X_i}{n(n+1)}\right).$$
$$= \left(\frac{2}{n(n+1)}\right)^2 \sum_{i=1}^n \mathrm{Var}_{\lambda^2}(X_i).$$
$$= \left(\frac{2}{n(n+1)}\right)^2 \sum_{i=1}^n i\lambda^2 = \frac{2\lambda^2}{n(n+1)}.$$

(b). CRLB: $$\frac{\partial^2}{\partial \eta^2} \log L(\eta; \underset{\sim}{x}) = -\frac{1}{\eta^2}\sum_{i=1}^n X_i.$$
$$\Rightarrow \mathbb{E}_\eta\left(-\frac{\partial^2}{\partial\eta^2}\log L(\eta; \underset{\sim}{x})\right) = \frac{1}{\eta^2}\sum_{i=1}^n \mathbb{E}(X_i) = \frac{n(n+1)}{2\eta}.$$
$$\Rightarrow \mathrm{CRLB} = \frac{2\eta}{n(n+1)}.$$ ✓ attained.

(c). Attainment thm.

$$\frac{\partial}{\partial \eta} \log L(\eta;\underline{x}) = \frac{1}{\eta}\sum_{i=1}^{n} X_i - \frac{n(n+1)}{2}$$

$$= \frac{1}{\eta}\left(\sum_{i=1}^{n} X_i - \frac{n(n+1)}{2}\eta\right)$$

$$= \frac{n(n+1)}{2\eta}(\hat{\eta}-\eta)$$

Set $a(\eta) = \frac{n(n+1)}{2\eta}$. $\Rightarrow$ CRLB is attainable.

$\boxed{Q2}$: $f(\theta|\underline{X}) \propto \pi(\theta)\prod_{i=1}^{n} f(x_i|\theta)$.

$$= \theta^{c-1}e^{-\lambda\theta}\cdot\theta^{n}e^{-\theta\sum_{i=1}^{n}X_i}$$

$$= \theta^{c+n-1}e^{-\theta(\lambda+\sum_{i=1}^{n}X_i)}$$

$\Rightarrow$ Posterior dist$^{\underline{n}}$: $Gam(c+n, \sum_{i=1}^{n}X_i+\lambda)$

① When $L(\theta,a) = (\theta-a)^2$. the Bayes risk is minimized at the posterior mean:

$$E(\theta|\underline{X}) = \frac{c+n}{\lambda+\sum_{i=1}^{n}X_i}$$

② When $L(\theta,a) = 1_{\{|\theta-a|>\varepsilon\}}$. the Bayes risk is minimized at the posterior mode:

$$\frac{\partial}{\partial\theta}\log f(\theta|\underline{x}) = \cancel{\text{Constant}+}\ \frac{\partial}{\partial\theta}\left((c+n-1)\log\theta - \theta(\lambda+\sum_{i=1}^{n}X_i)\right)$$

$$= \frac{c+n-1}{\theta} - \left(\lambda+\sum_{i=1}^{n}X_i\right) \overset{set}{=} 0.$$

$$\Rightarrow \hat{\theta} = \frac{c+n-1}{\lambda+\sum_{i=1}^{n}X_i}$$

$$\frac{\partial^2}{\partial\theta^2}\log f(\theta|\underline{x}) = -\frac{c+n-1}{\theta^2} < 0.$$ $\Rightarrow$ the posterior mode is $\frac{c+n-1}{\lambda+\sum_{i=1}^{n}X_i}$.

$\boxed{Q3}$. (1). $\Theta_0 = \{0\}$. Set $\theta_0 = 0$.

$\therefore$ Under H_0, $\mathbb{P}_{\theta_0}(X_{(n)} \geq 1) = 0$.

$$\Rightarrow \sup_{\theta \in \Theta_0} \mathbb{P}_\theta (X_{(1)} \geq k \text{ or } X_{(n)} \geq 1)$$

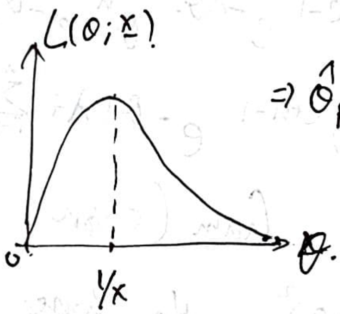
$$= \mathbb{P}_{\theta_0}(X_{(1)} \geq k) = (1-k)^n, \quad 0 \leq k \leq 1.$$

Set $(1-k)^n = \alpha$. $\Rightarrow$ $k = 1 - \alpha^{1/n}$. to have a size α test.

(2). $L(\theta; \underline{x}) = \theta e^{-\theta x}$.

Note that

$$\frac{\partial L(\theta; x)}{\partial \theta} = (1-\theta x) e^{-\theta x} = \begin{cases} \geq 0 & \text{if } \theta \leq 1/x \\ < 0 & \text{if } \theta > 1/x \end{cases}$$



$$\Rightarrow \hat\theta_{RMLE} = \frac{1}{X} 1_{\{1/X \leq 1\}} + 1 \cdot 1_{\{1/X > 1\}}.$$

$$\hat\theta_{MLE} = \frac{1}{X}.$$

$$\lambda(\underline{x}) = \frac{\sup_{0 \leq \theta \leq 1} \theta e^{-\theta x}}{\sup_{\theta \geq 0} \theta e^{-\theta x}}$$

$$= \begin{cases} 1 & \text{if } \frac{1}{x} \leq 1 \\ x e^{-(x-1)} & \text{if } \frac{1}{x} > 1 \end{cases}$$

$$= \begin{cases} 1 & \text{if } x \geq 1 \\ x e^{-(x-1)} & \text{if } x < 1. \end{cases}$$

$\Rightarrow$ By LRT: Reject if $\lambda(\underline{x}) \leq c$.

$$\frac{d}{dx}(x e^{-(x-1)}) = (1-x) e^{-(x-1)} \geq 0 \text{ for } \forall x \leq 1.$$

$\Rightarrow \lambda(\underline{x})$ is $\uparrow$ in x.

$\therefore \{\underline{x}: \lambda(\underline{x}) \leq c\} = \{\underline{x}: x \leq c^*\}$.

Set $\alpha = 0.05 = \sup_{\theta \leq 1} \mathbb{P}_\theta (X \leq c^*) = \sup_{\theta \leq 1} (1 - e^{-\theta \cdot c^*})$

$$= 1 - e^{-c^*}.$$

$$\Rightarrow c^* = -\log 0.95.$$

i.e. Reject if $x \leq -\log 0.95$.

Power fc^n: $\pi(\theta) = \mathbb{P}_\theta (X \leq -\log 0.95)$
$= 1 - (0.95)^\theta$.

Since $-\log 0.95 \approx 0.05$.

then with $X_1 = 0.1$, $n=1$, we will <u>NOT</u> reject H_0 at $\alpha = 0.05$.