Homework #4

1. Joe and Pete each have two cents in their pockets. They have decided to match pennies; that is, they will each take one of their own pennies and flip them. If the pennies match (two heads or two tails), Joe gets Pete's penny; if the pennies do not match, Pete gets Joe's penny. They will keep repeating the game until one of them has four cents, and the other one is broke. Although they do not realize it, all four pennies are biased. The probability of tossing a head is 0.6, and the probability of a tail is 0.4. Let X be a Markov chain where X_n denotes the amount that Joe has after the n^{th} play of the game. Define the Markov matrix for X and answer the following:

First note that P(0,0) = P(4,4) = 1, P(1,0) = 0.48, and P(1,2) = 0.52.

(a) What is the probability that Joe will have four pennies after the second toss? $P^2(4,2)$ = 0.2704

(b) What is the probability that Pete will be broke after three or less tosses? $P^3(4,2)$ = 0.2704

(c) What is the probability that the game will be over before the third toss? P{X2=0 or 4} = 0.2304+2704 = 0.5008

2. At the start of each week, the condition of a machine is determined by measuring the amount of electrical current it uses. According to its amperage reading, the machine is categorized as being in one of the following four states: low, medium, high, failed. A machine in the low state has a probability of 0.05, 0.03, and 0.02 of being in the medium, high, or failed state, respectively, at the start of the next week. A machine in the medium state has a probability of 0.09 and 0.06 of being in the high or failed state, respectively, at the start of the next week (it cannot, by itself, go to the low state). And, a machine in the high state has a probability of 0.1 of being in the failed state at the start of the next week (it cannot, by itself, go to the low or medium state). If a machine is in the failed state at the start of a week, repair is immediately begun on the machine so that it will (with probability 1) be in the low state at the start of the following week. Define the Markov matrix for X, where X_n is the state of the machine at the start of week n, and answer the following:

The first row of the Markov matrix is (0.9, 0.05, 0.03, 0.02) and the last row is (1, 0, 0, 0).

(a) A new machine always starts in the low state. What is the probability that the machine is in the failed state three weeks after it is new? P^3(low, failed) = 0.0277

(b) What is the probability that a machine has at least one failure three weeks after it is new?

You must replace the fourth row with an absorbing state so that the chain does not return to the low state. Using the revised Markov matrix and cubing it, we get 0.0713.

(c) Each week that the machine is in the low state, a profit of \$1,000 is realized; each week that the machine is in the medium state, a profit of \$500 is realized; each week that the machine is in the high state, a profit of \$400 is realized; and the week in which a failure is fixed, a cost of \$700 is incurred. What is the total profit from the first three weeks of operation. (Remember, for the first week, the machine is in the low state so it yields a profit of \$100 at week 0. Also, profit realized at the end of the week is based on the state at the beginning of the week.)

First note that $f = (1000, 500, 400, -700)^T$ ➔ $P^0f(\text{low}) + P^1f(\text{low}) + P^2f(\text{low})$ = 2803.35.