Practice Final, STAT 611

Name:

1. Let $X_1, \cdots, X_n$ be independent identically distributed random variables with pdf

$$f(x) = \frac{1}{\lambda} \exp\left[-\left(1 + \frac{1}{\lambda}\right) \log(x)\right]$$

where $\lambda > 0$ and $x \geq 1$

   (a) Find the maximum likelihood estimator of λ. Log-likelihood is

$$l(\lambda; \boldsymbol{x}) = -n \log \lambda - \left(1 + \frac{1}{\lambda}\right) \sum_{i=1}^{n} \log x_i.$$

   After checking both first and second order conditions (details omitted here!!!), we have

$$\widehat{\lambda} = \frac{1}{n} \sum_{i=1}^{n} \log x_i$$

   is the MLE.

   (b) What is the maximum likelihood estimator of λ^8? Explain your answer. By the invariance property of MLE, the MLE of λ^8 is

$$\widehat{\lambda}^8 = \left(\frac{1}{n} \sum_{i=1}^{n} \log x_i\right)^8.$$

2. Let $X_1, \ldots, X_n$ be independent identically distributed from a $\mathrm{N}\left(\mu, \sigma^2\right)$ population, where σ^2 is known. Let $\overline{X}$ be the sample mean.

   (a) Calculate the MSE of the maximum likelihood estimator of μ. Does this estimator attain the Cramer-Rao lower bound?

   (b) Find the UMVUE of μ^2.

   Class example; check lecture notes.

3. Let $X_1, \ldots, X_n$ be iid random variables from the Pareto(a, b) distribution. The PDF is $f_X(x) = ba^b x^{-b-1}$ for $x > a$ and $f_X(x) = 0$ for $x \leq a$. The parameters a, b satisfy $a > 0, b > 2$.

(a) Find the moment estimator for a, b. Provide brief derivation. [5]

$$E(X) = \frac{ab}{b-1}, \qquad E(X^2) = \frac{ba^2}{(b-1)^2(b-2)} + \frac{a^2b^2}{(b-1)^2} = \frac{ba^2}{b-2}$$

Thus, solve

$$\widehat{m}_1 = \frac{ab}{b-1}, \qquad \widehat{m}_2 = \frac{ba^2}{b-2}.$$

We obtain

$$\widehat{a} = \frac{\widehat{m}_1\sqrt{\widehat{m}_2}}{\sqrt{\widehat{m}_2 - \widehat{m}_1^2} + \sqrt{\widehat{m}_2}}, \qquad \widehat{b} = 1 + \sqrt{\frac{\widehat{m}_2}{\widehat{m}_2 - \widehat{m}_1^2}}.$$

(b) Is your estimator strongly consistent? Briefly justify your answers.[5]

Strongly consistent. Theorem on moment estimator and the continuous function of it.

(c) Find the MLE for a, b. Provide brief derivation. [5]

$$l = n\log(b) + nb\log(a) - (b+1)\sum_{i=1}^{n}\log(x_i)$$

Hence, $\widehat{a} = x_{(1)}$. From

$$l'_b = n/b + n\log(a) - \sum_{i=1}^{n}\log(x_i) = 0,$$

we obtain $\widehat{b} = n/\{\sum_{i=1}^{n}\log(x_i) - n\log(x_{(1)})\}$. Obviously $l''_b < 0$. So this is MLE.

(d) Is your estimator strongly consistent? Briefly justify your answer.[5]

Strongly consistent due to the theorem about the asymptotic properties of MLE

4. Let $X_1, \cdots, X_n$ be independent identically distributed random variables from a $N(\mu, \sigma^2)$ distribution where the variance σ^2 is known. We want to test $H_0 : \mu = \mu_0$ against $H_1 : \mu \neq \mu_0$

(a) Derive the likelihood ratio test

(b) Let λ be the likelihood ratio. Show that $-2\log\lambda$ is a function of $(\bar{X} - \mu_0)$

(c) Derive the distribution for $-2\log\lambda$.

Solutions:

(a) The likelihood function is

$$L(\mu)=\left(2\pi\sigma^2\right)^{-n/2}\exp\left[\frac{-1}{2\sigma^2}\sum\left(x_i-\mu\right)^2\right]$$

and the MLE for μ is $\hat{\mu}=\overline{x}$. Thus the numerator of the likelihood ratio test statistic is $L\left(\mu_0\right)$ and the denominator is $L(\overline{x})$. So the test is reject H_0 if $\lambda(\boldsymbol{x})=L\left(\mu_0\right)/L(\overline{x})\leq c$ where $\alpha=P_{\mu_0}(\lambda(\boldsymbol{X})\leq c)$

(b) $\log\lambda=\log L\left(\mu_0\right)-\log L(\overline{X})=-\frac{1}{2\sigma^2}\left[\sum\left(X_i-\mu_0\right)^2-\sum\left(X_i-\overline{X}\right)^2\right]=\frac{-n}{2\sigma^2}\left[\overline{X}-\mu_0\right]^2$ since $\sum\left(X_i-\mu_0\right)^2=\sum\left(X_i-\overline{X}+\overline{X}-\mu_0\right)^2=\sum\left(X_i-\overline{X}\right)^2+n\left(\overline{X}-\mu_0\right)^2$.
So $-2\log\lambda=\frac{n}{\sigma^2}\left[X-\mu_0\right]^2$

(c) $-2\log\lambda\sim\chi_1^2$

5. Let $X\sim$ Binomial(n,p), where the positive integer n is large and $0<p<1$.

(a) Find the asymptotic distribution of X/n.
Note that $X\stackrel{d}{=}\sum_{i=1}^n Y_i$, where $Y_i, i\geq 1$ are iid Bernoulli r.v. with parameter p. Then by CLT,

$$\sqrt{n}\left(\frac{X}{n}-p\right)\xrightarrow{d}N(0,p(1-p)).$$

(b) Find the asymptotic distribution of $(X/n)^2$.
Using the delta method, we have

$$\sqrt{n}\left(\left(\frac{X}{n}\right)^2-p^2\right)\xrightarrow{d}N(0,4p^3(1-p)).$$

6. Let $X_1,\ldots,X_n$ be a random sample from a uniform $(0,\theta)$ distribution. Let $Y=\max\left(X_1,X_2,\ldots,X_n\right)$.

(a) Find the pdf of Y/θ .

(b) Find a pivotal quantity and use it to construct a $(1-\alpha)\%$ confidence interval for θ.

Solution:

(a) (a) Let $W_i\sim U(0,1)$ for $i=1,\ldots,n$ and let $T_n=Y/\theta$. Then $P\left(\frac{Y}{\theta}\leq t\right)=P\left(\max\left(W_1,\ldots,W_n\right)\leq t\right)=P\left(\text{all } W_i\leq t\right)=\left[F_{W_i}(t)\right]^n=t^n$ for $0<t<1$. It follows that the pdf of T_n is $f_{T_n}(t)=\frac{d}{dt}t^n=nt^{n-1}$ for $0<t<1$

(b) (b) The distribution of $T_n=Y/\theta$ does not depend on θ. Let $W_i=X_i/\theta\sim U(0,1)$ which has cdf $F_Z(t)=t$ for $0<t<1$. Let $W_{(n)}=X_{(n)}/\theta=\max\left(W_1,\ldots,W_n\right)$. Then

$$F_{W_{(n)}}(t)=P\left(\frac{X_{(n)}}{\theta}\leq t\right)=t^n$$

for $0 < t < 1$. Find c_n such that $P\left(c_n \leq \frac{X_{(n)}}{\theta} \leq 1\right) = 1 - \alpha$ for $0 < \alpha < 1$. So $1 - F_{W_{(n)}}(c_n) = 1 - c_n^n = 1 - \alpha$. So $c_n = \alpha^{1/n}$ Therefore $\left(X_{(n)}, \frac{X_{(n)}}{\alpha^{1/n}}\right)$ is an exact $100(1-\alpha)\%$ CI for θ

7. Let $X_1, \ldots, X_n$ be a random sample from a location-exponential family with density

$$f(x; \theta) = \exp^{-(x-\theta)}, \quad \text{if}, \quad x \geq \theta, -\infty < \theta < \infty$$

and CDF,

$$F(x; \theta) = 1 - \exp^{-(x-\theta)}, \quad \text{if}, \quad x \geq \theta, -\infty < \theta < \infty$$

(a) Write down the likelihood function of θ. (Pay attention to the range/support of θ)
The likelihood function is

$$L(\theta; \boldsymbol{x}) = \exp\left\{-\sum_{i=1}^{n}(x_i - \theta)\right\} \mathbf{1}_{\{x_{(1)} \geq \theta\}},$$

where $x_{(1)}$ is the minimum of x_i's.

(b) Derive the likelihood ratio test and calculate the power function for the test

$$H_0 : \theta \leq \theta_0 \quad \text{versus} \quad H_1 : \theta > \theta_0$$

The likelihood ratio is

$$\lambda(\boldsymbol{X}) = \exp\left\{-n(X_{(1)} - \theta_0)\right\}.$$

Hence, the rejection region is:

$$\{\boldsymbol{x} : \lambda(\boldsymbol{x}) \leq C\} \Leftrightarrow \{\boldsymbol{x} : x_{(1)} \geq \theta_0 + C'\}.$$

Since

$$\mathbb{P}_{\theta_0}(X_{(1)} \geq \theta_0 + C') = e^{-nC'} \overset{\text{set}}{=} \alpha,$$

then $C' = -\frac{1}{n}\log\alpha$ and the LRT is to reject H_0 if $X_{(1)} \geq \theta_0 - \frac{1}{n}\log\alpha$.
Power function:

$$\mathbb{P}_{\theta}(X_{(1)} \geq \theta_0 - \frac{1}{n}\log\alpha) = \alpha e^{-n(\theta_0 - \theta)}.$$

(c) Construct a $100(1-\alpha)\%$ confidence interval of θ.
Inverting the LRT in the previous part gives the $100(1-\alpha)\%$ for θ:

$$\left[X_{(1)} + \frac{1}{n}\log\alpha, X_{(1)}\right].$$

8. Review exercises on other topics which are not covered in the previous 7 questions.