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EX1: $f(x;\theta) := \theta^{-1} e^{-(x-\theta)/\theta} 1_{\{x \geq \theta\}}$

(a) a $\boxed{\text{stat}}$ $\boxed{\text{min suff}}$ for θ

(b) Is the min suff stat in (a) complete?

(a): Suppose $T(\mathbf{X}) := (X_+, X_{(1)})$, here $X_{(1)} := \min_{i=1}^{n} X_i$, $X_+ := \sum_{i=1}^{n} X_i$

Want to show $T(X)$ is minimal sufficient statistic for θ

$$f(\mathbf{x};\theta) \overset{iid}{=} \prod_{i=1}^{n} f(X_i;\theta) = \prod_{i=1}^{n} \theta^{-1} e^{-(X_i-\theta)/\theta} 1_{\{X_i \geq \theta\}}$$

$$= \theta^{-n} e^{-(\sum_{i=1}^{n} X_i)/\theta + n} \cdot \prod_{i=1}^{n} 1_{\{X_i \geq \theta\}}$$

$$= \theta^{-n} e^{-(\sum_{i=1}^{n} X_i)/\theta + n} 1_{\{\min_{i=1}^{n} X_i \geq \theta\}}$$

Suppose $\mathbf{x} := (x_1, \cdots, x_n)$ $\mathbf{y} := (y_1, \cdots y_n)$ are any two samples

$$A_{\mathbf{x}} := \{x : T(x) = T(\mathbf{x})\}, \quad A_{\mathbf{y}} := \{x : T(x) = T(\mathbf{y})\}$$

Then $f(\mathbf{x};\theta)/f(\mathbf{y};\theta) = \dfrac{e^{-\frac{1}{\theta}(\sum_{i=1}^{n} x_i - \sum_{i=1}^{n} y_i)} 1_{\{\min_{i=1}^{n} x_i \geq \theta\}}}{1_{\{\min_{i=1}^{n} y_i \geq \theta\}}}$

If $f(\mathbf{x};\theta)/f(\mathbf{y};\theta)$ doesn't depend on θ,

then $\begin{cases} -\frac{1}{\theta}(\sum_{i=1}^{n} x_i - \sum_{i=1}^{n} y_i) = 0 \\ 1_{\{\min_{i=1}^{n} x_i \geq \theta\}} / 1_{\{\min_{i=1}^{n} y_i \geq \theta\}} = \text{constant} \end{cases} \Rightarrow \begin{cases} \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i \\ \min_{i=1}^{n} x_i = \min_{i=1}^{n} y_i \end{cases} \Rightarrow T(\mathbf{x}) = T(\mathbf{y})$

If $T(\mathbf{x}) = T(\mathbf{y})$, $\Rightarrow x_+ = y_+$, $x_{(1)} = y_{(1)}$ $\Rightarrow f(\mathbf{x};\theta)/f(\mathbf{y};\theta) = e^0 \cdot 1 = 1$

independ on θ

$\Rightarrow$ By theorem 6.2.13 in C & B, $T(x) = (\sum_{i=1}^{n} X_i, \min_{i=1}^{n} X_i)$ is minimal sufficient statistic for θ

(b) $T(\underline{X})$ is not complete

proof: suppose $g(t_1, t_2) := t_1/n - t_2$

The $g(T(X)) := X_+/n - X_{(1)}$

$$g(T(X_1+b, X_2+b, \cdots, X_n+b)) = \sum_{i=1}^{n}(X_i+b)/n - \min_{i=1}^{n}(X_i+b)$$

$$= X_+/n + nb/n - X_{(1)} - b$$

$$= X_+/n - X_{(1)}$$

$$= g(T(X_1, \cdots, X_n))$$

$\Rightarrow g \circ T$ is a location invariant statistic.

$\Rightarrow g \circ T$ is ancillary

$\Rightarrow g \circ T$ distribution doesn't depend on θ

$\Rightarrow E_\theta[g \circ T(x)] = \text{constant}$

Without loss of generation, assume $E_\theta[g \circ T(X)] = c$

Asumme $h(T(x)) := g(T(x)) - c$

The $E_\theta[h(T(x))] := E_\theta[g(T(x))] - c = c - c = 0$

However, $h(T(x)) = 0 \Rightarrow g(T(x)) = c \Rightarrow X_+/n - X_{(1)} = c$.

doesn't have probability 1, namely

$P(X_+/n - X_{(1)} = c) \neq 1$, since $X_+/n - X_{(1)}$ is not a constant

is a random variable

$\Rightarrow \exists\, h$, s.t. $E_\theta[h(T(x))] = 0, \forall \theta$,

while, $P_\theta[h(T(x)) = 0] \neq 1, \forall \theta$

which is controdict to the definition of completeness

$\Rightarrow T(\underline{X})$ is not complete

EX2: $f(x;P) := P^x (1-P)^{1-x}$, $x \in \{0,1\}$, $P \in (0,1)$

$X_{ij} \sim$ Bernoulli (P_{ij}), $X_{ij}=1$, : link exists, $X := \begin{pmatrix} 0 & X_{12} & X_{13} & \cdots & X_{1n} \\ X_{12} & \ddots & & & \vdots \\ \vdots & & \ddots & & X_{n-1,n} \\ X_{1n} & \cdots & & X_{n-1,n} & 0 \end{pmatrix}$ sys, 0-diag

(a) a min suffi stat. for $\theta := (P_{ij})_{i<j}$. proof suff & min. (expo family, natural)

(b) β_i: popular of i, $P_{ij} := \dfrac{\exp(\beta_i+\beta_j)}{1+\exp(\beta_i+\beta_j)}$ a min suff. stat for $\theta := (\beta_1, \cdots, \beta_n)$

proof suff. & min, (i) $\dfrac{P_{ij}}{1-P_{ij}} = \exp(\beta_i+\beta_j)$ (ii) simple stat $X_{ij} = X_{ji}$)

(c) Is (b) complete ?

(a) $X_{ij} \sim$ Bernoulli (P_{ij}). $f(x;P) = P^x(1-P)^{1-x} = e^{\ln(P^x(1-P)^{1-x})} = (1-P) \cdot e^{x(\ln P - \ln(1-P))}$

$$f(\underline{X};\theta) \overset{iid}{=} \prod_{i=1}^{n-1} \prod_{j=i+1}^{n} f(X_{ij};P_{ij}) = \prod_{i=1}^{n-1} \prod_{j=i+1}^{n} (1-P_{ij}) \cdot e^{X_{ij}(\ln \frac{P_{ij}}{1-P_{ij}})}$$

$$= \left(\prod_{i=1}^{n-1} \prod_{j=i+1}^{n} (1-P_{ij}) \right) \cdot e^{\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} X_{ij} \ln \frac{P_{ij}}{1-P_{ij}}}$$

$\theta := (P_{ij})_{i<j}$ $\dim(\theta) = 1+2+\cdots+n-1 = \frac{n(n-1)}{2}$

$C(\theta) := \prod_{i=1}^{n-1} \prod_{j=i+1}^{n} (1-P_{ij})$ $h(\underline{X}) = 1$, $W_{ij}(\theta) := \ln \frac{P_{ij}}{1-P_{ij}}$, $t_{ij}(\underline{X}) := X_{ij}$

$\Rightarrow$ $\underline{X}$ is in exponential family with pdf $f(\underline{X};\theta)$

natural parametrization $\eta := (\ln \frac{P_{ij}}{1-P_{ij}})_{i<j}$, $C^*(\eta) = \prod_{i=1}^{n-1} \prod_{j=i+1}^{n} \frac{1}{e^{\eta_{ij}}+1}$

$\dim(\eta) = 1+2+\cdots+n-1 = \frac{n(n-1)}{2}$

(since P_{ij} linear independent, η_{ij} only generate by P_{ij}

$W_{ij}: \theta_{ij} \to \eta_{ij}$ one to one $\Rightarrow \eta_{ij}$ also linear independent)

$\Rightarrow \dim(\eta) = \dim(W) = \dim(\theta)$ $\Rightarrow$ full rank.

Then, by theorem: $T(\underline{X}) := (t_{ij}(\underline{X}))_{i<j} = (X_{ij})_{i<j}$

is sufficient for θ

$\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \alpha_{ij} \ln \frac{P_{ij}}{1-P_{ij}} = 0 \Rightarrow \alpha_{ij} = 0 \quad \forall i,j$

$\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \alpha_{ij} X_{ij} = 0 \Rightarrow \alpha_{ij} = 0 \quad \forall i,j$

$\gg$ $(W_{ij})_{i<j}$ linear independent, $(X_{ij})_{i<j}$ linear independent

$\Rightarrow$ (otherwise, if $\exists\, \alpha_{ij} \neq 0$, then $\ln \frac{P_{ij}}{1-P_{ij}} = 0$, $X_{ij} = 0 \Rightarrow P_{ij} = \frac{1}{2}$, $X_{ij} = 0$ contradict to P_{ij}, X_{ij} are parameter & random variable)

$\Rightarrow$ $T(\underline{X})$ in exponential family is minimal sufficient statastic for $\theta := (P_{ij})_{i<j}$

$\overset{\shortparallel}{(X_{ij})_{i<j}}$

(b) $\because P_{ij} = \dfrac{\exp(\beta_i+\beta_j)}{1+\exp(\beta_i+\beta_j)}$ $\therefore \dfrac{P_{ij}}{1-P_{ij}} = \exp(\beta_i+\beta_j)$, $1-P_{ij} = \dfrac{1}{\exp(\beta_i+\beta_j)+1}$

Suppose $X := (X_{ij})$ a sysmetric $n\times n$, with zero-diagonal, $\theta := (\beta_1, \cdots, \beta_n)$

Then $f(\underline{X};\theta) \overset{X_{ij}\text{ iid}}{\underset{X_{ij}=X_{ji}}{=}} \left(\prod_{i=1}^{n-1}\prod_{j=i+1}^{n} \dfrac{1}{\exp(\beta_i+\beta_j)+1}\right)\cdot e^{\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} X_{ij}(\beta_i+\beta_j)}$

$$\overset{X_{ij}=X_{ji}}{=} \left(\prod_{i=1}^{n-1}\prod_{j=i+1}^{n} \frac{1}{\exp(\beta_i+\beta_j)+1}\right)\cdot e^{\frac{1}{2}\left(\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} X_{ij}(\beta_i+\beta_j) + \sum_{j=1}^{n-1}\sum_{i=j+1}^{n} X_{ij}(\beta_i+\beta_j)\right)}$$

$$\overset{X_{ii}\equiv 0}{=} \left(\prod_{i=1}^{n-1}\prod_{j=i+1}^{n} \frac{1}{\exp(\beta_i+\beta_j)+1}\right) e^{\frac{1}{2}\left(\sum_{i\neq j} X_{ij}(\beta_i+\beta_j) + \sum_{i=j} X_{ij}(\beta_i+\beta_j)\right)}$$

$$= \left(\prod_{i=1}^{n-1}\prod_{j=i+1}^{n} \frac{1}{\exp(\beta_i+\beta_j)+1}\right) e^{\frac{1}{2}\left(\theta X \mathbb{1}_n + \theta\cdot X^T \mathbb{1}_n\right)}$$

$$\overset{X\text{ sysmetric}}{=} \left(\prod_{i=1}^{n-1}\prod_{j=i+1}^{n} \frac{1}{\exp(\beta_i+\beta_j)+1}\right) e^{\theta\cdot X\cdot \mathbb{1}_n}$$

Exponentral family, $C(\theta) := \prod_{i=1}^{n-1}\prod_{j=i+1}^{n} \dfrac{1}{\exp(\beta_i+\beta_j)+1}$ $\quad h(X)=1$

$W_i(\theta) = \beta_i \quad 1\le i\le n$, $\quad t_i(X) = \sum_{j=1}^{n} X_{ij} \overset{X_{ii}=0}{\underset{X_{ij}=X_{ji}}{=}} \sum_{j=i+1}^{n} X_{ij} + \sum_{j=1}^{i-1} X_{ji}$

By theorem: $T(\underline{X}) := \left(\sum_{j=i+1}^{n} X_{ij} + \sum_{j=1}^{i-1} X_{ji}\right)_{i=1}^{n}$ is sufficient for θ

$\dim(\theta) = n = \dim(W)$ $\quad$ ($\beta_1 \cdots \beta_n$ linear independent,

$W_i(\theta) = \beta_i$ linear independent)

$\sum_i \alpha_i W_i(\theta) = 0 \Rightarrow \sum_i \alpha_i \beta_i = 0 \overset{\beta_i\text{ ind}}{\Longrightarrow} \alpha_i = 0$

$\sum_i \alpha_i t_i(X) = 0 \Rightarrow \sum_i \alpha_i \left(\sum_{j=i+1}^{n} X_{ij} + \sum_{j=1}^{i-1} X_{ji}\right)_{i=1}^{n} = 0 \Rightarrow \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} X_{ij}(\alpha_i+\alpha_j) = 0$

$\overset{X_{ij}\text{ ind}}{\Longrightarrow} (\alpha_i+\alpha_j) = 0 \quad \forall\, i=1,\cdots,n-1,\ j=i+1,\cdots,n \quad \Rightarrow \alpha_1 = -\alpha_2 = -\alpha_3 = \cdots = -\alpha_n$

$\alpha_2 = -\alpha_3 = -\alpha_4 = \cdots = -\alpha_n$

$\Rightarrow \alpha_1 = \cdots = \alpha_n = 0 \Rightarrow t_i(X)$ linearly independent

$\Rightarrow T(\underline{X}) := \left(\sum_{j=i+1}^{n} X_{ij} + \sum_{j=1}^{i-1} X_{ji}\right)_{i=1}^{n}$ is minimal sufficient for θ.

(c) In (b), we have known $f(\underline{X};\theta)$ exponenial family of full rank.

By theorem $\Rightarrow T(X)$ is (b) is complete and sufficient for θ.