STAT 611 Homework 6 Solutions

1. (a) The log-likelihood function is

$$l(\theta, \nu) = n \log \theta_n \theta \log \nu - (\theta + 1) \log \left(\prod_{i=1}^{n} x_i \right)$$

for $\nu \leq x_{(1)}$. For a given fixed value of θ, $l(\theta, \nu)$ is increasing in ν for $\nu \leq x_{(1)}$ so that the MLE of ν is $\hat{\nu} = x_{(1)}$. For the MLE of θ, start with

$$\frac{\partial}{\partial \theta} l(\theta, \nu) = \frac{n}{\theta} + n \log x_{(1)} - \log \left(\prod_{i=1}^{n} x_i \right)$$

Setting this to zero and noting that $n \log x_{(1)} = \log x_{(1)}^n$, we have that the MLE of θ is

$$\hat{\theta} = \frac{n}{\log \left(\frac{\prod_{i=1}^{n} x_i}{x_{(1)}^n} \right)} = \frac{n}{T}$$

where we've assigned $T = \log \left(\frac{\prod_{i=1}^{n} x_i}{x_{(1)}^n} \right)$. Also,

$$\frac{\partial^2}{\partial^2 \theta} l(\theta, \nu) = -\frac{n}{\theta^2} < 0$$

for all θ so $\hat{\theta}$ is the MLE.

(b) Under H_0, the MLE of θ is $\hat{\theta}_0 = 1$. The MLE of ν is still $\hat{\nu} = x_{(1)}$. The likelihood ratio statistic is

$$\begin{aligned}
\lambda(x) = \frac{L(\hat{\theta}_0, \hat{\nu} | \mathbf{x}}{L(\hat{\theta}, \hat{\nu} | \mathbf{x})} &= \frac{\frac{x_{(1)}^2}{\left(\prod_{i=1}^{n} x_i\right)^2}}{\left(\frac{n}{T}\right)^2 \frac{x_{(1)}^{n^2/T}}{\left(\prod_{i=1}^{n} x_i\right)^{n/T+1}}} \\
&= \left(\frac{T}{n}\right)^n \frac{e^{-T}}{\left(e^{-T}\right)^{n/T}} \\
&= \left(\frac{T}{n}\right)^n e^{-T+n}
\end{aligned}$$

Now,

$$\frac{\partial}{\partial T} \log \lambda(\mathbf{x}) = \frac{n}{T} - 1$$

so that $\lambda(\mathbf{x})$ is increasing if $T \leq n$ and decreasing if $T \geq n$. Thus, $T \leq c$ is the same as $T \leq c_1$ and $T \geq c_2$ for some appropriately chosen constants c_1 and c_2.

2. (a) We have that

$$\begin{aligned}
\lambda(\mathbf{x}, \mathbf{y}) &= \frac{\sup_{\Theta_0} L(\theta | \mathbf{x}, \mathbf{y})}{\sup_{\Theta} L(\theta | \mathbf{x}, \mathbf{y})} \\
&= \frac{\sup_{\theta} \prod_{i=1}^{n} \frac{1}{\theta} e^{-x_i/\theta} \prod_{i=1}^{m} \frac{1}{\theta} e^{-y_i/\theta}}{\sup_{\theta, \mu} \prod_{i=1}^{n} \frac{1}{\theta} e^{-x_i/\theta} \prod_{i=1}^{m} \frac{1}{\mu} e^{-y_i/\mu}} \\
&= \frac{\sup_{\theta} \frac{1}{\theta^{m+n}} \exp\left(-\frac{1}{\theta}\left[\sum_{i=1}^{n} x_i + \sum_{i=1}^{m} y_i\right]\right)}{\sup_{\theta, \mu} \frac{1}{\theta^n} \exp\left(-\frac{1}{\theta} \sum_{i=1}^{n} x_i\right) \exp\left(-\frac{1}{\mu} \sum_{i=1}^{m} y_i\right)}
\end{aligned}$$

Differentiation of the numerator tells us that

$$\hat{\theta}_0 = \frac{\sum_{i=1}^n x_i + \sum_{i=1}^m y_i}{n+m}$$

while that for the denomiator tells us that $\hat{\theta} = \bar{x}$ and $\hat{\mu} = \bar{y}$. Hence,

$$\begin{aligned}
\lambda(\mathbf{x}, \mathbf{y}) &= \frac{\left(\frac{n+m}{\sum_{i=1}^n x_i + \sum_{i=1}^m y_i}\right)^{n+m} \exp\left(-\left(\frac{n+m}{\sum_{i=1}^n x_i + \sum_{i=1}^m y_i}\right)\left(\sum_{i=1}^n x_i + \sum_{i=1}^m y_i\right)\right)}{\left(\frac{n}{\sum_{i=1}^n x_i}\right)^n \exp\left(-\left(\frac{n}{\sum_{i=1}^n x_i}\right)\sum_{i=1}^n x_i\right)\left(\frac{m}{\sum_{i=1}^m x_i}\right)^n \exp\left(-\left(\frac{m}{\sum_{i=1}^m y_i}\right)\sum_{i=1}^m y_i\right)} \\
&= \frac{(n+m)^{n+m}}{n^n m^m} \frac{\left(\sum_{i=1}^n x_i\right)^n \left(\sum_{i=1}^m y_i\right)^m}{\left(\sum_{i=1}^n x_i + \sum_{i=1}^n y_i\right)^{n+m}}
\end{aligned}$$

The LRT rejects H_0 if $\lambda(\mathbf{x}, \mathbf{y}) \le c$.

(b) We have

$$\lambda = \frac{(n+m)^{n+m}}{n^n m^m} \left(\frac{\sum_{i=1}^n x_i}{\sum_{i=1}^n x_i + \sum_{i=1}^n y_i}\right)^n \left(\frac{\sum_{i=1}^m y_i}{\sum_{i=1}^n x_i + \sum_{i=1}^n y_i}\right)^m = \frac{(n+m)^{n+m}}{n^n m^m} T^n (1-T)^m$$

and so λ is a function of T. λ is a unimodal function in T that is maximized at $T = n/(n+m)$. Rejection for $\lambda \le c$ is equivalent to rejection when $T \le a$ or $T \ge b$ where a and b are constants that satisfy $a^n(1-a)^m = b^n(1-b)^m$.

(c) When H_0 is true, $\sum_{i=1}^n X_i \sim \text{Gamma}(n, \theta)$ and $\sum_{i=1}^m Y_i \sim \text{Gamma}(m, \theta)$. Since they are independent, $T \sim \text{Beta}(n, m)$.