Markov Processes on Finite State Space

Only consider stationary processes with no instantaneous states

Two properties of a Markov process

1. Sojourn times are exponential with mean time depending on current state only:

   $P\{T_{n+1} - T_n > t \mid X_0, X_1, \ldots, X_n, X_{n+1}, \ldots\} = P\{T_{n+1} - T_n > t \mid X_n\}$

   $P\{T_{n+1} - T_n > t \mid X_n=i\} = e^{-\lambda(i)t}$ for $t \geq 0$.

2. There is an imbedded Markov chain $\{Xn\}$ with $Xn = Y(Tn)$ and

   $P\{X_{n+1} = j \mid X_n=i\} = Q(i,j)$

   for Markov matrix Q and with the diagonal elements zero.

A *generator* matrix, G, is formed by $G(i,j) = \lambda(i)\, Q(i,j)$ for $i \neq j$ and $G(i,i) = -\lambda(i)$ for the diagonal elements. Note that some authors call this the *rate* matrix.

Two properties of the generator matrix.

1. $G(i,j) \geq 0$ for $i \neq j$. (That is, off-diagonal elements are nonnegative.)
2. $\sum_j G(i,j) = 0$ for each fixed i. (That is diagonal elements are nonpositive.)

Intuitively, $G(i,j)$ for $i \neq j$ is the rate of going from i to j in one transition, $G(i,i)$ is the rate of leaving state i. If the diagonal element of G is zero, then the state associated with that row is absorbing.

The Markov process is called irreducible, recurrent if the imbedded Markov chain is irreducible, recurrent. The probabilities associated with the Markov process are given by

1. $P_t(i,j) = P\{Y(t) = j \mid Y(0) = i\} = e^{Gt}(i,j)$ for $t \geq 0$.
2. For an irreducible, recurrent process, $\lim_{t \to \infty} P_t(i,j) = p(i)$, where $pG=\mathbf{0}$ and $p\mathbf{1}=1$.