Lu Sun 228002579

8.30: $f(x|\theta) = \frac{\theta}{\pi} \frac{1}{\theta^2+x^2}$, $-\infty < x < \infty$, $\theta > 0$. Cauchy scale pdf

(a) show that MLR doesn't exist

(b) $X \sim f(x|\theta)$. show $|X|$ is suff for θ and dist of $|X|$ does have MLR.

---

proof: (a) $\frac{f(x|\theta_2)}{f(x|\theta_1)} = \frac{\theta_2(\theta_1^2+x^2)}{\theta_1(\theta_2^2+x^2)}$, $\theta_2 > \theta_1$

$\frac{d}{dx} \frac{f(x|\theta_2)}{f(x|\theta_1)} = \frac{\theta_2}{\theta_1} \frac{\theta_2^2-\theta_1^2}{(\theta_2^2+x^2)^2} x$, when $x \in (-\infty, 0)$ $\frac{f(x|\theta_2)}{f(x|\theta_1)}$ monotone decreases

$x \in (0, \infty)$ $\frac{f(x|\theta_2)}{f(x|\theta_1)}$ monotone increases

$\Rightarrow$ MLR doesn't exist

(b) $f(x|\theta) = g(|x| | \theta) = \frac{\theta}{\pi} \frac{1}{\theta^2+|x|^2}$ By factorization theorem, $|X|$ is sufficient for θ

$f_{|X|}(y|\theta) = \frac{2\theta}{\pi} \frac{1}{\theta^2+y^2}$ $y \in [0, +\infty)$, $\theta > 0$

$\frac{d}{dy} \frac{f(y|\theta_2)}{f(y|\theta_1)} = \frac{\theta_2}{\theta_1} \frac{\theta_2^2-\theta_1^2}{(\theta_2^2+y^2)^2} y$, $\theta_2 > \theta_1$, $\theta_2^2-\theta_1^2 \geq 0$, $y \geq 0 \Rightarrow \frac{d}{dy} \frac{f(y|\theta_2)}{f(y|\theta_1)} \geq 0$

$\Rightarrow \frac{f(y|\theta_2)}{f(y|\theta_1)}$ monotone increases in y $\forall$ $\theta_2 > \theta_1$

$\Rightarrow$ $|X|$ does have MLR.

8.33 $\{X_i\} \sim U(\theta, \theta+1)$. $H_0: \theta=0$, $H_1: \theta>0$, reject H_0 if $Y_n \ge 1$ or $Y_1 \ge k$, $Y_1 := X_{(1)}$, $Y_n := X_{(n)}$

(a) Determine k s.t. test has size α

(b) power function in (a)

(c) Prove that the test is UMP size α

(d) Find n, k, s.t. UMP .10 level test has power at least .8 if $\theta > 1$

(a) under H_0, $P(Y_n \ge 1) = 0$, $\Rightarrow \alpha = P(Y_1 \ge k \mid \theta=0) = (1-k)^n$

$\Rightarrow k = 1 - \alpha^{1/n}$

(b) $\Rightarrow f_{Y_1, Y_n}(y, x \mid \theta) = n(n-1)(x-y)^{n-2}$, $\theta < y < x < \theta+1$, $f_{Y_1}(y \mid \theta) = n(1-(y-\theta))^{n-1}$, $0 < y < \theta+1$

when $\theta \le k-1$, $\beta(\theta) = 0$,

when $k-1 < \theta \le 0$, $\beta(\theta) = \int_k^{\theta+1} n(1-(y-\theta))^{n-1}\,dy = (1-k+\theta)^n$

when $0 < \theta \le k$, $\beta(\theta) = \int_k^{\theta+1} n(1-(y-\theta))^{n-1}\,dy + \int_\theta^k \int_1^{\theta+1} n(n-1)(x-y)^{n-2}\,dx\,dy = \alpha + 1 - (1-\theta)^n$

when $\theta > k$, $\beta(\theta) = 1$

(c)
$$\frac{f(y,x \mid \theta)}{f(y,x \mid 0)} = \begin{cases} \begin{cases} 0, & 0 < y_1 \le \theta,\ y_1 < y_n < 1 \\ 1, & 0 < y_1 < y_n < 1 \\ \infty, & 1 \le y_n < \theta+1,\ \theta < y_1 < y_n \end{cases} & 0 < \theta < 1 \\ \begin{cases} 0, & \text{if } y_1 < y_n < 1 \\ \infty, & \text{if } \theta < y_1 < y_n < \theta+1 \end{cases} & \theta \ge 1 \end{cases}$$

① when $0 < \theta < k$, $R = \{Y_n \ge 1, Y_1 \ge k\} \subset \{Y_n \ge 1, Y_1 > \theta\}$

$$\frac{f(y,x \mid \theta)}{f(y,x \mid 0)} = \infty \quad \forall \theta \in (0,\infty),\ \forall \{y,x\} \in R$$

Set reject if $\frac{f(y,x \mid \theta)}{f(y,x \mid 0)} > 1$ and accepts if $\frac{f(y,x \mid \theta)}{f(y,x \mid 0)} < 1$

By Neyman-Pearson lemma, the test is UMP size α.

② when $\theta \ge k$, $R = \{Y_n \ge 1, Y_1 \ge k\} \supset \{Y_n \ge 1, Y_1 > \theta\}$

$$\frac{f(y,x \mid \theta)}{f(y,x \mid 0)} \ge 0, \quad \forall \theta \in (0,\infty),\ \forall \{y,x\} \in R$$

Set reject if $\frac{f(y,x \mid \theta)}{f(y,x \mid 0)} > 0$ and accepts if $\frac{f(y,x \mid \theta)}{f(y,x \mid 0)} < 0$

By Neyman-Pearson Lemma, the test is UMP size α

(d) $k = 1 - \alpha^{1/n} = 1 - (0.1)^{1/n} \le 1$

in (b), $\theta > 1 \ge k$. $\Rightarrow \beta(\theta) = 1 > 0.8$

$\Rightarrow$ for any n, k, UMP 0.1 level test has power at least 0.8 if $\theta > 1$

3. ① $\Lambda_K \le \gamma_0$ Accept H_0 $\Rightarrow$ $f(X_1, \cdots, X_K; P_0) > 1-\alpha$.

$f(X_1, \cdots, X_K; P_1) \le 1-\beta$

$\Lambda_K < \frac{1-\beta}{1-\alpha}$. $\sup \Lambda_K = \frac{1-\beta}{1-\alpha} \le \gamma_0$

② $\Lambda_K \ge \gamma_1$ reject H_0 $\Rightarrow$ $f(X_1, \cdots, X_K; P_0) \le \alpha$

$f(X_1, \cdots, X_K; P_1) > \beta$

$\Lambda_K > \frac{\beta}{\alpha}$, $\inf \Lambda_K = \frac{\beta}{\alpha} \ge \gamma_1$

$\Rightarrow \gamma_1 \le \frac{\beta}{\alpha}$, $\gamma_0 \ge \frac{1-\beta}{1-\alpha}$