Example: Second derivative test for convexity CSCE 633

Let $\mathbf{x} = [x_1, x_2] \in \mathbb{R}^2$ be a 2-dimensional vector and $f(x) = (x_1 - 3x_2)^2$.

(1) Compute the first-order derivative and Hessian of the function of $f(\mathbf{x}) = (x_1 - 3x_2)^2$.

$$\frac{\theta f(\mathbf{x})}{\theta \mathbf{x}} = \left[\frac{\theta f(\mathbf{x})}{\theta x_1} \; , \; \frac{\theta f(\mathbf{x})}{\theta x_2}\right]^T = [2(x_1 - 3x_2) \; , \; -6(x_1 - 3x_2)]^T \in \mathbb{R}^2$$

$$\mathbf{H}_{f(\mathbf{x})} = \begin{bmatrix} \frac{\theta^2 f(\mathbf{x})}{\theta x_1^2} & \frac{\theta^2 f(\mathbf{x})}{\theta x_1 \theta x_2} \\ \frac{\theta^2 f(\mathbf{x})}{\theta x_2 \theta x_1} & \frac{\theta^2 f(\mathbf{x})}{\theta x_2^2} \end{bmatrix} = \begin{bmatrix} \frac{\theta}{\theta x_1}\left(2(x_1 - 3x_2)\right) & \frac{\theta}{\theta x_1}\left(-6(x_1 - 3x_2)\right) \\ \frac{\theta}{\theta x_2}\left(2(x_1 - 3x_2)\right) & \frac{\theta}{\theta x_2}\left(-6(x_1 - 3x_2)\right) \end{bmatrix} = \begin{bmatrix} 2 & -6 \\ -6 & 18 \end{bmatrix} \in \mathbb{R}^{2\times 2}$$

(2) Show that the function $f(\mathbf{x}) = (x_1 - 3x_2)^2$ is convex by showing that its Hessian is positive semi-definite.

We will perform the second derivative test.

For any vector $\mathbf{u} = [u_1 \; , \; u_2]^T \in \mathbb{R}^2$, we have:

$$\begin{aligned} \mathbf{u}^T \mathbf{H}_{f(\mathbf{x})} \mathbf{u} &= [u_1 \; , \; u_2] \begin{bmatrix} 2 & -6 \\ -6 & 18 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \\ &= [2u_1 - 6u_2 \; , \; -6u_1 + 18u_2] \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \\ &= 2(u_1^2 - 6u_1 u_2 + 9u_2^2) \\ &= 2(u_1 - 3u_2)^2 \geq 0 \end{aligned}$$

Therefore $\mathbf{H}_{f(\mathbf{x})}$ is positive semi-definite, so $f(\mathbf{x})$ is convex.

(3) An alternative way to show that the function $f(\mathbf{x}) = (x_1 - 3x_2)^2$ is convex is by showing that the eigenvalues of its Hessian are non-negative.

We find the eigenvalues of $\mathbf{H}_{f(\mathbf{x})}$ by setting its determinant to 0.

$$\begin{aligned} &Det\left(\mathbf{H}_{f(\mathbf{x})}\right) = 0 \Rightarrow \\ &\begin{vmatrix} \lambda - 2 & -6 \\ -6 & \lambda - 18 \end{vmatrix} = 0 \Rightarrow \\ &\lambda^2 - 20\lambda + 36 - 36 = 0 \Rightarrow \\ &\lambda(\lambda - 20) = 0 \Rightarrow \\ &\lambda = 0 \;\text{ or }\; \lambda = 20 \end{aligned}$$

Both eigenvalues are non-negative, therefore $\mathbf{H}_{f(\mathbf{x})}$ is positive semi-definite, thus f is convex.