Name:_______________________________________
Full name

Test Two

Open Book, Open Notes

You may use the internet, including http://integral-table.com, if necessary

1. Consider a renewal process where the distribution function for the inter-renewal time is given by

$$F(t) = 1 - \exp(-3t) - 3t\exp(-3t) \qquad \text{for } t\geq 0.$$

Using the fact that the Laplace-Stieltjes transform of the distribution function, F, is given by $(3/(3+s))^2$, it is possible to show that the renewal function is given by

$$m(t) = 1.5t - 0.25\times(1 - \exp(-6t)) \qquad \text{for } t\geq 0.$$

(I give the Laplace transform in case you are curious as to the type of distribution that F is. From the Laplace transform you should see that F is an Erlang type-2 distribution. This information and knowledge of the Laplace-Stieltjes transform are not needed to work the problem.)

Assume that the time units for t is in terms of *days*. In addition, assume each day has three 8-hour shifts with a shift starting at midnight. Call the shift starting at midnight Shift 1. Answer the following questions with respect to this renewal process.

Using the integral-table.com website, you should have $\int x \exp(ax)\,dx = (x/a - 1/a^2)\exp(ax)$. (I assume you don't need the website for $\int \exp(ax)\,dx = (1/a)\exp(ax)$.)

First step is to get mean using the above integrals: $\mu = \int_0^\infty [\exp(-3t) - 3t\exp(-3t)]\,dt$ = 2/3. (The material about the Erlang distribution was so that you could do a quick check of your work if you were unsure. Since the Erlang type 2 is the sum of two exponentials each with a mean of 1/3, the mean of the inter-renewals must be 2/3. Of course, if you had some confidence in yourself, you might of used that fact directly and skipped the integration.)

a. What is the expected number of renewals during Shift 1 on the initial day?

Must determine m(t) at t=1/3 which yields 0.5 – 0.25 (1 – 0.1353) = 0.2838 or rounded = 0.284

b. What is the long-run expected number of renewals per shift? Hint: consider Blackwell's Thm.

Use Blackwell's thm.: $\lim_{t\to\infty}[m(t+1/3) - m(t)] = (1/3) / (2/3) =$ 0.5

c. Time t=0 refers to midnight. Assume it is now at the beginning of the first shift change, i.e., it is the now 8AM, the beginning of Shift 2.

i. What is the expected time, in hours, until the next renewal?

$E[V(t)]$ for t=1/3 yielding $\mu(m(1/3) + 1) - 1/3 = 0.5225$ days ➔ 12.54 hr

ii. Using a 12-hour clock (i.e., using AM/PM) give the expected time at which the renewal is expected to occur, rounded to the nearest minute.

Adding 12.54 hr to the current time of 8AM yields 8:32PM.

d. What is the long-run probability that there will be at least one renewal during a given shift? Another way to ask the same question is as follows: assume we are at the some point in time in the "far" future, what is the probability that the time until the next renewal is less than or equal to eight hours from now?

First observe that $\lim_{t\to\infty}P\{V(t) > 1/3\} = (1/\mu)\int_{1/3}^\infty [\exp(-3t) - 3t\exp(-3t)]\,dt = 1.5\,e^{-1} = 0.5518$

Therefore, $\lim_{t\to\infty}P\{V(t) \leq 1/3\} = 1 - 0.5518$ which rounds to 0.448

2. Passengers arrive to a train station according to a Poisson process having rate λ. A train (which is large enough to pick up all waiting passengers) arrives at the train station in accordance to a recurrent renewal process with an aperiodic inter-arrival distribution F. At a train arrival time, all passengers waiting will get onto the train. Let X(t) denote the number of passengers waiting at the station at time t. Assume at time 0, a train just left and thus the train station is empty.

   a. What type of process is $\{X(t); t\geq 0\}$? Regenerative process.

   b. Obtain an expression for $P\{X(t) = k\}$ for $t\geq 0$ and $k\geq 0$.

Let {Sn} be the process of truck arrival times.

$$P\{X(t) = k\} = P\{X(t) = k, S1 > t\} + P\{X(t) = k, S1 <= t\}$$

$$= (\exp(-\lambda t)(\lambda t)^k/k!)\,[1 - F(t)] + \int_{[0,t]} F(ds) P\{X(t-s) = k\}$$

$$= (\exp(-\lambda t)(\lambda t)^k/k!)\,[1 - F(t)] + \int_{[0,t]} m(ds) \left(\frac{\exp(-\lambda t)(\lambda(t-s))^k}{k!}\right) [1 - F(t-s)]$$

Where $m = \Sigma F_n$.

   c. Obtain an expression for $\lim_{t\to\infty} P\{X(t) = k\}$ for $k\geq 0$.

$$\lim_{t\to\infty} P\{X(t) = k\} = (1/\mu) \int_0^\infty \left(\frac{\exp(-\lambda t)(\lambda t)^k}{k!}\right) [1 - F(t)]dt$$

where $\mu = \int_0^\infty [1 - F(t)]dt$