X, Y joint pdf $\rightarrow f(\cdot, \cdot)$

$$f_X(x) = \int_{y=-\infty}^{\infty} f(x, y)\, dy$$

$$f_Y(y) = \int_{x=-\infty}^{\infty} f(x, y)\, dx$$

$f(x \mid y) \leftarrow$ conditional of x given $Y = y$

$$f(x \mid y) = \frac{f(x, y)}{f_Y(y)}$$

$$E[X \mid Y=y] = \int_x x\, f(x \mid y)\, dx$$

Notice $\rightarrow$ $E[X \mid Y]$ is a random variable

$$E[\, E[X \mid Y]\,] = E[X]$$

$$E[X] = E[\,E[X \mid Y]\,] = \int_y E[X \mid Y=y]\, f_Y(y)\, dy$$

$$\int_y \left[\int_x x \frac{f(x,y)}{\cancel{f_Y(y)}}\, dx \right] \cancel{f_Y(y)}\, dy$$

$$\int_x x \left[\int_y f(x,y)\, dy \right] dx = \int_x x\, f_X(x)\, dx = E[X]$$

Let A be subset of Reals and let
Denote the indicator function of A
by $\mathbb{1}_A(t) = \begin{cases} 1 & \text{if } t \in A \\ 0 & \text{if } t \notin A \end{cases}$

Let X be R.V., the $\mathbb{1}_A(X)$ is a R.V.

$$E[\mathbb{1}_A(X)] = P\{X \in A\}$$

$$E[P\{X \leq t \mid Y\}] = P\{X \leq t\}$$

Let $P\{X \le t\} = F(t)$, $P\{Y \le t\} = G(t)$

Let X & Y be independent, nonnegative

$$P\{X+Y \le t\} = E[P\{X+Y \le t \mid Y\}]$$

$$P\{X+Y \le t \mid Y=y\} = P\{X \le t-y\}$$

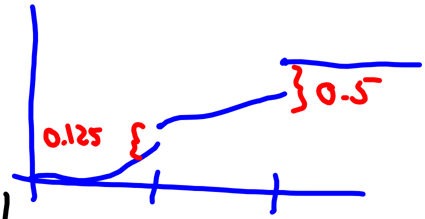
$$E[P\{X \le t-Y\}] = \int_{y=[0,\infty)} P\{X \le t-y\}\, dG(y)$$

$$= \int_{y\in[0,t]} F(t-y)\, dG(y)$$

but $F(t-y)=0$ if $t-y<0 \Rightarrow t<y$

Need $t-y \geq 0 \Rightarrow y \le t$

Find $E[X] = \int x \, dF(x)$



$$F(x) = \begin{cases} 0 & \text{for } x < 0 \\ \frac{1}{2}x^2 & \text{for } 0 \le x < \frac{1}{2} \\ \frac{1}{2}x & \text{for } \frac{1}{2} \le x < 1 \\ 1 & \text{for } x \ge 1 \end{cases}$$

$$E[X] = \frac{1}{2} \times 0.125 + 1 \times 0.5 + \int_0^{1/2} x \cdot x \, dx + \int_{\frac{1}{2}}^{1} x \cdot \frac{1}{2} \, dx$$

Let $X_1, X_2, \cdots$ be i.i.d. and let N be a nonnegative discrete R.V. independent of $\{X_n ; n = 1, 2, \cdots\}$

$$S = X_1 + \cdots + X_N \quad \text{where if } N = 0,\ S = 0$$

$$E[S] = E[E[S \mid N]]$$

$$E[S \mid N = n] = E[X_1 + \cdots + X_n] = nE[X_1]$$

$$E[E[S \mid N]] = E[NE[X_1]] = E[X_1]E[N]$$

$$Var(S) = E[S^2] - E[S]^2$$

$$E[S^2] = E[E[S^2 | N]]$$

$$E[S^2 | N=n] = E[(X_1 + \cdots + X_n)^2]$$

$$\sum_{i=1}^{n} E[X_i^2] + \sum_{i=1}^{n} \sum_{j \neq i} X_i X_j$$

$$= n E[X_1^2] + n(n-1) E[X_1]^2$$

$$Var(S) = E[N] Var(X_1) + Var(N) E[X_1]^2$$

$$P\{X+Y\le t\} = \int_{y\in[0,t]} F(t-y)\,dG(y) = F*G(t)$$

$$= \int_{y\in[0,t)} G(t-y)\,dF(y) = G*F(t)$$

Convolution

$$P\{X-Y\le t\}=E\left[P\{X-Y\le t \mid Y\}\right]$$