Lu Sun 228002579

EX 8.13 $X_1, X_2 \overset{iid}{\sim} U(\theta, \theta+1)$. $H_0: \theta = 0$, vs $H_1: \theta > 0$ $\Rightarrow$ $\phi_1(X_1)$: Reject H_0 if $X_1 > 0.9$
$\phi_2(X_1, X_2)$: Reject H_0 if $X_1 + X_2 > C$

(a) Find C. s.t. $\phi_2 \overset{size}{=} \phi_1$

(b) Calculate the power function of each test. Draw a well-labled graph of each power function

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(a): ① if $\theta \le -0.05$, $P_\theta(X_1 > 0.95) = 0$

if $\theta > 0.95$, $P_\theta(X_1 > 0.95) = 1$

if $-0.05 < \theta \le 0.95$, $P_\theta(X_1 > 0.95) = \theta + 1 - 0.95 = \theta + 0.05$

$\Rightarrow \alpha_1 = P(X_1 > 0.95 \mid \theta = 0) = 0 + 0.05 = 0.05$

② if $c \ge 2\theta + 2$, $P_\theta(X_1 + X_2 > c) = 0$

if $c < 2\theta$. $P_\theta(X_1 + X_2 > c) = 1$

if $2\theta \le c < 2\theta + 1$, $P_\theta(X_1 + X_2 > c) = 1 - P_\theta(X_1 + X_2 \le c)$

$$= 1 - \int_{2\theta}^{c} f_{X_1+X_2}(y)\, dy$$

X_1, X_2 iid

$$= 1 - \int_{2\theta}^{c} \int_{\theta}^{\theta+1} f_{X_1}(t) \cdot f_{X_2}(y-t)\, dt\, dy$$

$$= 1 - \int_{2\theta}^{c} \int_{\theta}^{y-\theta} 1 \cdot dt\, dy$$

$$= -\frac{c^2}{2} + 2\theta c - 2\theta^2 + 1$$

if $2\theta + 1 \le c < 2\theta + 2$. $P_\theta(X_1 + X_2 > c) = 1 - \int_{2\theta}^{c} f_{X_1+X_2}(y)\, dy$

$$= 1 - \int_{2\theta}^{c} \int_{\theta}^{\theta+1} f_{X_1}(t) f_{X_2}(y-t)\, dt\, dy$$

$$= 1 - \int_{2\theta}^{2\theta+1} \int_{\theta}^{y-\theta} 1\, dt\, dy - \int_{2\theta+1}^{c} \int_{y-\theta-1}^{\theta+1} 1\, dt\, dy$$

$$= 1 - \frac{1}{2} - \frac{(c - 2\theta - 1)(2\theta + 3 - c)}{2}$$

$$= \frac{1 - [1 + (c - 2\theta - 2)][1 - (c - 2\theta - 2)]}{2} = \frac{1 - 1 + (c - 2\theta - 2)^2}{2} = \frac{(c - 2\theta - 2)^2}{2}$$

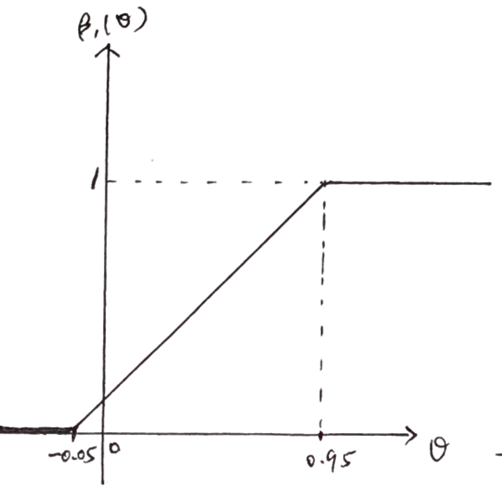
$$\Rightarrow \alpha_2 = P(X_1 + X_2 > c \mid \theta = 0) = \begin{cases} 0 & c \ge 2 \\ 1 & c < 0 \\ -\frac{c^2}{2} + 1 & 0 \le c < 1 \\ \frac{(c-2)^2}{2} & 1 \le c < 2. \end{cases}$$

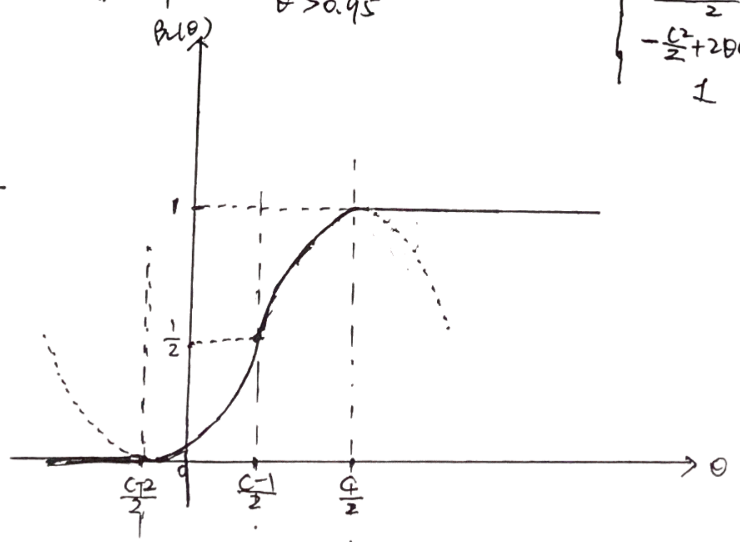
$$\alpha_2 = \alpha_1 \Rightarrow \begin{cases} -\frac{c^2}{2} + 1 = 0.05, & 0 \le c < 1 \\ \frac{(c-2)^2}{2} = 0.05, & 1 \le c < 2 \end{cases} \Rightarrow \begin{cases} c = \sqrt{1.9} > 1 & 0 \le c < 1 \\ c = 2 \pm \sqrt{0.1} & 1 \le c < 2 \end{cases} \Rightarrow c = 2 - \sqrt{0.1} \approx 1.68$$

(b): by (a), we have:

$$\beta_1(\theta) := \begin{cases} 0 & \theta \le -0.05 \\ \theta + 0.05 & -0.05 < \theta \le 0.95 \\ 1 & \theta > 0.95 \end{cases}$$

$$\beta_2(\theta) := \begin{cases} 0, & \theta \le \frac{c-2}{2} \\ \frac{(c - 2\theta - 2)^2}{2}, & \frac{c-2}{2} < \theta \le \frac{c-1}{2} \\ -\frac{c^2}{2} + 2\theta c - 2\theta^2 + 1, & \frac{c-1}{2} < \theta \le \frac{c}{2} \\ 1, & \frac{c}{2} < \theta \end{cases}$$





EX 8.14: $\{X_i\}$ iid Bernoulli(p), $H_0: p=0.49$ vs $H_1: p=0.51$
Determine sample size s.t. type I / II error are both about 0.01
Use a test function that rejects H_0 if $\sum_{i=1}^{n} X_i$ is large.

By CLT, $Y := (\sum_i X_i - np)/\sqrt{np(1-p)} \sim N(0,1)$

then ① $P(\text{type I error}) = P(\text{reject } H_0 \mid p=0.49) = P(Y > y \mid p=0.49)$

$$= P\left(Y > \frac{c-n\,0.49}{\sqrt{n\cdot 0.49\cdot 0.51}}\right) = 0.01$$

$$\Rightarrow P\left(Y \le \frac{c-n\,0.49}{\sqrt{n\cdot 0.49\cdot 0.51}}\right) = \Phi\left(\frac{c-n\,0.49}{\sqrt{n\cdot 0.49\, 0.51}}\right) = 1-0.01 = 0.99$$

$$\Rightarrow \frac{c-n\cdot 0.49}{\sqrt{n\cdot 0.49\cdot 0.51}} = \Phi^{-1}(0.99) = 2.33 \quad ①$$

② $P(\text{type II error}) = P(\text{accept } H_0 \mid p=0.51) = P(Y \le y \mid p=0.51)$

$$= P\left(Y \le \frac{c-n\cdot 0.51}{\sqrt{n\cdot 0.51\cdot 0.49}}\right) = 0.01$$

$$\Rightarrow \frac{c-n\cdot 0.51}{\sqrt{n\cdot 0.49\cdot 0.51}} = \Phi^{-1}(0.01) = -2.33 \quad ②$$

By ①② $\Rightarrow \begin{cases} 2c = n \\ \dfrac{c-2c\cdot 0.49}{\sqrt{2c\cdot 0.49\cdot 0.51}} = 2.33 \end{cases} \Rightarrow \begin{cases} c \approx 6783.4105 \approx 6783 \\ n \approx 13566.8211 \approx 13567 \end{cases}$

EX 8.18 $\{X_i\}$ iid $n(\theta, \sigma^2)$, σ^2 known. $H_0: \theta = \theta_0$, vs $H_1: \theta \neq \theta_0$, reject H_0 if $|\bar{X} - \theta_0|/(\sigma/\sqrt{n}) > C$.

(a) Find an expression for power function

(b) Type I ~~Error~~ $= 0.05$, max Type II Error $= 0.25$. at $\theta = \theta_0 + \sigma$. Find n & C.

(a):
$$\beta(\theta) = P_\theta(\text{reject } H_0) = P_\theta\left(|\bar{X} - \theta_0|/(\sigma/\sqrt{n}) > c\right) = 1 - P_\theta\left(|\bar{X} - \theta_0|/(\sigma/\sqrt{n}) \le c\right)$$

$$= 1 - P_\theta\left(-\frac{c\sigma}{\sqrt{n}} + \theta_0 \le \bar{X} \le \frac{c\sigma}{\sqrt{n}} + \theta_0\right) = 1 - P_\theta\left(-c + \frac{\theta_0 - \theta}{\sigma/\sqrt{n}} \le \frac{\bar{X} - \theta}{\sigma/\sqrt{n}} \le c + \frac{\theta_0 - \theta}{\sigma/\sqrt{n}}\right)$$

$$\overset{Y := \frac{\bar{X}-\theta}{\sigma/\sqrt{n}}}{=} 1 - P\left(-c + \frac{\theta_0 - \theta}{\sigma/\sqrt{n}} \le Y \le c + \frac{\theta_0 - \theta}{\sigma/\sqrt{n}}\right) \underset{Y \sim N(0,1)}{\overset{\text{By CLT}}{=}} 1 - \left[\Phi^{-1}\left(c + \frac{\theta_0 - \theta}{\sigma/\sqrt{n}}\right) - \Phi^{-1}\left(-c + \frac{\theta_0 - \theta}{\sigma/\sqrt{n}}\right)\right]$$

$$= 1 - \Phi^{-1}\left(c + \frac{\theta_0 - \theta}{\sigma/\sqrt{n}}\right) + \Phi^{-1}\left(-c + \frac{\theta_0 - \theta}{\sigma/\sqrt{n}}\right)$$

$$\Rightarrow \beta(\theta) := 1 - \Phi^{-1}\left(c + \frac{\theta_0 - \theta}{\sigma/\sqrt{n}}\right) + \Phi^{-1}\left(-c + \frac{\theta_0 - \theta}{\sigma/\sqrt{n}}\right)$$

(b)
$$P(\text{Type I Error}) = \beta(\theta_0) = 1 - \Phi^{-1}(c) + \Phi^{-1}(-c) = 0.05$$

$$\Rightarrow 1 - \Phi^{-1}(c) + (1 - \Phi^{-1}(c)) = 0.05 \quad \Rightarrow \Phi^{-1}(c) = 1 - \frac{0.05}{2} = 0.975$$

$$\Rightarrow C = 1.96$$

$$\max_{\theta \neq \theta_0} P(\text{Type II Error}) \overset{\text{Accept } H_0}{=} \max_{\theta \neq \theta_0} 1 - \beta(\theta) \le 0.25 \quad \text{and} \quad 1 - \beta(\theta_0 + \sigma) = 0.25$$

$$\Rightarrow 0.75 = \beta(\theta_0 + \sigma) = 1 - \Phi^{-1}(c - \sqrt{n}) + \Phi^{-1}(-c - \sqrt{n})$$

$$= 1 - \Phi^{-1}(1.96 - \sqrt{n}) + \Phi^{-1}(-1.96 - \sqrt{n})$$

$$\Phi^{-1}(-1.96 - \sqrt{n}) < \Phi^{-1}(-1.96) = 0.05 \Rightarrow \Phi^{-1}(-1.96 - \sqrt{n}) \approx 0$$

$$\Rightarrow \Phi^{-1}(1.96 - \sqrt{n}) \approx 0.25$$

$$\Rightarrow 1.96 - \sqrt{n} \approx -0.675$$

$$\Rightarrow n \approx (2.635)^2 = 6.943225 \approx 7$$

$$\Rightarrow c = 1.96,\ n = 7$$