Prelim 2 Practice Solution.

Q1:

$$\log L(\theta;x) = x-\theta-2\log(1+e^{x-\theta})$$

$$\Rightarrow \frac{\partial}{\partial\theta}\log L(\theta;x) = -1+\frac{2e^{x-\theta}}{1+e^{x-\theta}} \overset{\text{set}}{=} 0.$$

$$\Rightarrow \hat{\theta} = X. \quad (\text{Check: } \hat{\theta}=X \text{ is the MLE}).$$

$$\Rightarrow \lambda(X) = \frac{L(0;x)}{L(\hat{\theta};x)} = \frac{4e^{x}}{(1+e^{x})^2}:$$

<u>Check</u>: $\lambda(x)$ is $\downarrow$ in x if $x>0$.

$\therefore$ Rejection region: $\{x: \lambda(x)\le c\} = \{x: X > c^*\}$.

$$\Rightarrow \alpha = \mathbb{P}_{\theta_0}(X>c^*) = \int_{c^*}^{\infty}\frac{e^x}{(1+e^x)^2}dx = -(1+e^x)^{-1}\Big|_{c^*}^{\infty}$$

$$= (1+e^{c^*})^{-1} \quad \Rightarrow \quad c^* = \log\left(\frac{1}{\alpha}-1\right).$$

i.e. Reject H_0 if $X > \log\left(\frac{1}{\alpha}-1\right)$.

Q2. (a). When $\bar{X}\le 2$. $\lambda(\underline{x})=1$.

When $\bar{X}>2$. $\lambda(\underline{x}) = \dfrac{\exp\{-\frac{1}{2}\sum(x_i-2)^2\}}{\exp\{-\frac{1}{2}\sum(x_i-\bar{x})^2\}} = \exp\{-\frac{n}{2}(\bar{x}-2)^2\}$

then $\lambda(\underline{x})\le c \Leftrightarrow \dfrac{\bar{X}-2}{1/\sqrt{n}} \ge c^*$.

If $\alpha=0.05$. $\Rightarrow 0.05 = \mathbb{P}_{\mu=2}(\sqrt{n}(\bar{X}-2)\ge c^*)$ $\Rightarrow c^* = z_{0.05} = 1.645$.

i.e. $R=\{\underline{x}: \frac{\bar{x}-2}{1/\sqrt{n}} \ge 1.645\}$.

(b) Power fc^{n}: $\pi(\mu) = \mathbb{P}_\mu(\sqrt{n}(\bar{X}-2)\ge c^*) = \mathbb{P}_\mu(\sqrt{n}(\bar{X}-\mu)\ge c^*+\sqrt{n}(2-\mu))$

$= 1-\Phi(c^*+\sqrt{n}(2-\mu))$ which is $\uparrow$ in μ.

(c) $\pi(2.5) = 1-\Phi(1.645+\sqrt{n}(2-2.5)) = \Phi(\frac{1}{2}\sqrt{n}-1.645) = 0.9$. by the symmetry of normal pdf.

$\Rightarrow \frac{\sqrt{n}}{2}-1.645 = z_{0.1} = 1.28$. $\Rightarrow n = (2(z_{0.05}+z_{0.1}))^2 = 35$.

(d) <u>p-value</u>: $\mathbb{P}_{\mu=2}\left(\frac{\bar{X}-2}{1/\sqrt{n}} \ge \frac{2.5-2}{1/\sqrt{n}}\right) = 1-\Phi(1.5) > 1-\Phi(z_{0.05}) = 0.05$.

$\Rightarrow$ Not reject H_0 at $\alpha=0.05$.

$\boxed{Q3}$: Posterior:

$$p(\lambda|\underline{x}) \propto p(\lambda)\, p(\underline{x}|\lambda) \propto \lambda^{\alpha-1+\sum X_i} e^{-\lambda(n+\beta)}$$

$\hookrightarrow \text{Gam}(\alpha+\sum X_i,\ n+\beta)$

$$\Rightarrow \mathbb{E}(\lambda|\underline{x}) = \frac{\alpha+\sum X_i}{n+\beta}$$

$$\mathbb{E}(\lambda^2|\underline{x}) = (\alpha+\sum X_i)(1+\alpha+\sum X_i)\frac{1}{(n+\beta)^2}$$

Expected loss:

$$\mathbb{E}(L(\lambda,\lambda')|\underline{x}) = \mathbb{E}(\lambda^3|\underline{x}) - 2\,\mathbb{E}(\lambda^2|\underline{x})\cdot\lambda' + \mathbb{E}(\lambda|\underline{x})(\lambda')^2$$

$$\Rightarrow \text{minimized at } \hat{\lambda}_B(\underline{x}) = \frac{\mathbb{E}(\lambda^2|\underline{x})}{\mathbb{E}(\lambda|\underline{x})} = \frac{1+\alpha+\sum X_i}{n+\beta}$$

$\boxed{Q4}$: (a). $\bar{X}$ is the complete stat. and $\mathbb{E}_\lambda(\bar{X}) = \lambda$

$\therefore \bar{X}$ is the UMVUE.

(b). $MSE_\lambda(\bar{X}) = Var_\lambda(\bar{X}) = \frac{\lambda^2}{n}$

(c). $I(\lambda) = -\mathbb{E}_\lambda\left(\frac{\partial^2}{\partial\lambda^2}\log L(\lambda;\underline{x})\right)$

$$= -\frac{n}{\lambda^2} + \frac{2}{\lambda^3}\mathbb{E}_\lambda\left(\sum_{i=1}^n X_i\right) = \frac{n}{\lambda^2}$$

$$\frac{\partial}{\partial\lambda}\log L(\lambda;\underline{x}) = -\frac{n}{\lambda} + \frac{1}{\lambda^2}\sum_{i=1}^n X_i = -\frac{n}{\lambda^2}(\bar{x}-\lambda)$$

Let $a(\lambda) = -\frac{n}{\lambda^2}$.

$\Rightarrow$ Attainment thm applies.