STAT 611 Homework 9 Solutions

1. We show that for any interval $[a, b]$ and $\varepsilon > 0$, the probability content of $[a-\varepsilon, b-\varepsilon]$ is greater (as long as $b - \varepsilon > a$). Write

$$\begin{aligned}\int_b^a f(x)dx - \int_{a-\varepsilon}^{b-\varepsilon} f(x)dx &= \int_{b-\varepsilon}^{b} f(x)dx - \int_{a-\varepsilon}^{a} f(x)dx \\ &\le f(b-\varepsilon)[b-(b-\varepsilon)] - f(a)[a-(a-\varepsilon)] \\ &\le \varepsilon[f(b-\varepsilon) - f(a)] \le 0\end{aligned}$$

All the inequalities follow because $f(x)$ is decreasing. So moving the interval toward zero increases the probability, and it is therefore maximized by moving a all the way to zero.

2. The sample pdf is

$$f(x_1,\ldots,x_n|\theta) = \prod_{i=1}^n e^{i\theta - x_i}\mathbf{1}_{(i\theta,\infty)}(x_i) = \exp\left(\sum_i (i\theta - x_i)\right)\mathbf{1}_{(\theta,\infty)}(\min x_i/i)$$

Thus $T = \min(X_i/i)$ is sufficient by the Factorization Theorem, and

$$P(T > t) = \prod_{i=1}^n P(X_i > it) = \prod_{i=1}^n \int_{it}^{\infty} e^{i\theta - x}dx = \prod_{i=1}^n e^{i(\theta = t)} = \exp\left(-\frac{n(n+1)}{2}(t-\theta)\right)$$

and

$$f_T(t) = \frac{n(n+1)}{2}\exp\left(-\frac{n(n+1)}{2}(t-\theta)\right)$$

for $t \ge \theta$. Clearly, θ is a location parameter and $Y = T - \theta$ is a pivot. To find the shortest confidence interval of the form $[T + a, T + b]$, we must minimize $b - a$ subject to the constraint $P(-b \le Y \le -a) = 1 - \alpha$. Now the pdf of Y is strictly decreasing, so the interval length is shortest if $-b = 0$ and a satisfies

$$P(0 \le Y \le -a) = \exp\left(-\frac{n(n+1)}{2}a\right) = 1 - \alpha$$

So $a = 2\log(1-\alpha)/(n(n+1))$.

3. (a) We have that

$$P_\theta(|X_{(n)} - \theta| > \varepsilon) = P_\theta(X_{(n)} < \theta - \varepsilon) = (1 - \varepsilon/\theta)^n \to 0 \text{ as } n \to \infty$$

So $X_{(n)} \to \theta$ in probability.

(b) Since

$$P_\theta(-n(X_{(n)} - \theta) \le x) = P(X_{(n)} \ge \theta - x/n) = 1 - (1 - x/n\theta)^n \to 1 - e^{-x/\theta}$$

then $-n(X_{(n)} - \theta) \xrightarrow{d} \text{Exp}(\theta^{-1})$ which is not a normal distribution.