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EX 8.51: H_0 vs H_1, $W(\underline{X})$ test stat. Suppose $\forall\ 0 \le \alpha \le 1$, chose C_α s.t. $\{\underline{x}: W(\underline{x}) \ge C_\alpha\}$ is the rejection region of a size α test of H_0. show that p-value $p(\underline{x})$ in (8.3.9) is the smallest α level at which we could reject H_0, having observed the data $\underline{x}$.

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proof: ① if $\alpha < p(\underline{x})$, $\sup_{\theta \in \Theta_0} P_{\theta_0}(W(X) \ge C_\alpha) = \alpha < p(\underline{x}) = \sup_{\theta \in \Theta_0} P(W(X) \ge W(\underline{x}))$

$\Rightarrow W(\underline{x}) < C_\alpha$, couldn't reject H_0 at level α, with observation $\underline{x}$.

② if $\alpha \ge p(\underline{x})$, $\sup_{\theta \in \Theta_0} P_\theta(W(X) \ge C_\alpha) = \alpha \ge p(\underline{x}) = \sup_{\theta \in \Theta_0} P(W(X) \ge W(\underline{x}))$

1) if $W(\underline{x}) < C_\alpha$, let $C'_\alpha = W(\underline{x})$ then $\alpha = \sup_{\theta \in \Theta_0} P_\theta(W(X) \ge C'_\alpha) = p(\underline{x})$, reject H_0 at size α

2) if $W(\underline{x}) \ge C_\alpha$, then reject H_0 at level α

By 1) 2) $\Rightarrow$ reject H_0 at level α having observed $\underline{x}$

EX 9.13. X a single observation from the beta$(\theta,1)$ pdf.

(a) $Y = -(\log X)^{-1}$. Evaluate the confidence coefficient of the set $[y/2, y]$

(b) Find a pivotal quantity & use it to set up a confidence interval having the same confidence coefficient as the interval in part (a)

(c) Compare the two confidence intervals

Proof. (a) $Y = -(\log X)^{-1}$, $X \sim \text{Beta}(\theta, 1) \Rightarrow f_Y(y) = \frac{\theta}{y^2} e^{-\theta/y}$, $0<y<\infty$

$$P\left(\frac{Y}{2} \le \theta \le Y\right) = P(\theta \le Y \le 2\theta) = \int_{\theta}^{2\theta} \frac{\theta}{y^2} e^{-\theta/y} dy = e^{-\theta/y}\Big|_{\theta}^{2\theta} = e^{-1/2} - e^{-1} = 0.239$$

(b) $f_X(x) = \theta x^{\theta-1}$, $0<x<1$, $Q(t,\theta) := t^{\theta}$, $g(Q(t,\theta)) \equiv 1$

$f(t|\theta) = \theta t^{\theta-1} = 1 \cdot \frac{\partial}{\partial t} Q(t,\theta) = g(Q(t,\theta)) \cdot |\partial/\partial t \, Q(t,\theta)|$

$\Rightarrow Q(X,\theta) := X^{\theta}$ is the Pivotal quantity, with $f_T(t|\theta) \equiv 1$ $\forall\, 0<t<1$

$\Rightarrow$ Suppose CI : $C(X) := \{\theta : a \le Q(X,\theta) \le b\}$

Then $P_\theta(\theta \in C(X)) = P(a \le Q(X,\theta) \le b) \overset{f_T(t|\theta)=1}{=\!=} b-a$

since $Q(t,\theta) = t^{\theta}$ $\Rightarrow$ $C(X) = \{\theta : \frac{\log b}{\log X} \le \theta \le \frac{\log a}{\log X}\}$

$\Rightarrow b-a \overset{(a)}{=} 0.239$, $\Rightarrow$ CI : $\left[\frac{\log(a+0.239)}{\log X}, \log a / \log X\right]$

(c) part (a): $C_1(X) = \{\theta : \frac{Y}{2} \le \theta \le Y\} = \{\theta : -\frac{1}{2\log X} \le \theta \le -\frac{1}{\log X}\}$

By (a) $C_1(X)$ with confidence coefficient 0.239

part (b): a generalization CI of part (a)

$C_2(X) = \{\theta : \frac{\log b}{\log X} \le \theta \le \frac{\log a}{\log X}\}$

Suppose $C_2(X)$ with confidence coefficient $1-\alpha$

then $b-a = 1-\alpha$, $b = 1-\alpha+a$.

give X, $\min_{0\le a\le\alpha} \frac{\log a - \log b}{\log X} \underset{0<X<1}{\Longleftrightarrow} \min \log b - \log a \overset{b=1-\alpha+a}{\Longleftrightarrow} \min \log\left(\frac{1-\alpha}{a}+1\right)$

$\Rightarrow a^* = \alpha$, $b^* = 1$ $\Rightarrow$ best $1-\alpha$ pivotal interval is $\theta \in \left[0, \frac{\log\alpha}{\log X}\right]$

which is better than $C_1(X)$ in part (a), by set $1-\alpha = 0.239$

the $\left[0, \frac{\log(1-0.239)}{\log X}\right]$ is better than $\left[-\frac{1}{2\log X}, -\frac{1}{\log X}\right]$

EX3:

proof: (a) type I error $= P(P_n(X_1,\cdots,X_n) \le \alpha \mid \mu=0)$

$= P(1-\Phi(\sqrt{n}\,\bar{X}) \le \alpha \mid \mu=0)$

$= P(1-\alpha \le \Phi(\sqrt{n}\,\bar{X}) \mid \mu=0)$

$= P(\sqrt{n}\,\bar{X} \ge Z_\alpha \mid \mu=0)$

$= P\left(\frac{\bar{X}-\mu}{\sqrt{n}} \ge \frac{Z_\alpha}{n} - \frac{\mu}{\sqrt{n}} \mid \mu=0\right)$

$= 1-\Phi\left(\frac{Z_\alpha}{n}-\frac{\mu}{\sqrt{n}}\right) \overset{\mu=0}{=} 1-\Phi\left(\frac{Z_\alpha}{n}\right)$

type II error $= P(P_n(X_1,\cdots,X_n) > \alpha \mid \mu=\mu_1)$

$= P(1-\alpha > \Phi(\sqrt{n}\,\bar{X}) \mid \mu=\mu_1)$

$= P\left(\frac{\bar{X}-\mu}{\sqrt{n}} < \frac{Z_\alpha}{n} - \frac{\mu}{\sqrt{n}} \mid \mu=\mu_1\right)$

$= \Phi\left(\frac{Z_\alpha}{n} - \frac{\mu_1}{\sqrt{n}}\right)$

(b) type I error $= 1-P\left(P_{10}(X_{10l+1},\cdots,X_{10l+10}) > \alpha,\ \forall\, l \in \{0,\cdots,L-1\} \mid \mu=0\right)$

$= 1-\prod_{l=0}^{L-1} P(1-\Phi(\sqrt{n}\,\bar{X}_l) > \alpha \mid \mu=0)$

$= 1-\prod_{l=0}^{L-1} \Phi\left(\frac{Z_\alpha}{10}\right)$

$= 1-\left[\Phi\left(\frac{Z_\alpha}{10}\right)\right]^L.$

type II error $= P(P_{10}(X_{10l+1},\cdots,X_{10l+10}) > \alpha \mid \mu=\mu_1)$

$= \prod_{l=0}^{L-1} P(1-\Phi(\bar{X}_l) > \alpha \mid \mu=\mu_1)$

$= \prod_{l=0}^{L-1} P\left(\frac{\bar{X}-\mu}{\sqrt{10}} < \frac{Z_\alpha}{10} - \frac{\mu}{\sqrt{10}} \mid \mu=\mu_1\right)$

$= \left[\Phi\left(\frac{Z_\alpha}{10}-\frac{\mu_1}{\sqrt{10}}\right)\right]^L.$

Suppose $\hat{p} := P(P_{10}(X_1,\cdots,X_{10}) \le \alpha) \overset{X_1,\cdots,X_{10}\text{ ind}}{=\!=} 1-\Phi\left(\frac{Z_\alpha}{10}-\frac{\mu}{\sqrt{10}}\right)$

Then $E[N] = 10\cdot\hat{p} + 2\cdot 10\cdot(1-\hat{p})\hat{p} + 3\cdot 10\cdot(1-\hat{p})^2\hat{p} + \cdots + L\cdot 10\cdot(1-\hat{p})^{L-1}\cdot\hat{p}$
$+ L\cdot 10(1-\hat{p})^L = 10\hat{p}[1+2(1-\hat{p})+3(1-\hat{p})^2+\cdots+L(1-\hat{p})^{L-1}] + L\cdot 10(1-\hat{p})^L$

suppose $S := 1+2(1-\hat{p})+3(1-\hat{p})^2+\cdots+L(1-\hat{p})^{L-1}$ ①

then $(1-\hat{p})S = (1-\hat{p})+2(1-\hat{p})+\cdots+L(1-\hat{p})^L$. ②

① − ② $= \hat{p}S = 1+(1-\hat{p})+\cdots+(1-\hat{p})^{L-1} - L(1-\hat{p})^L = \frac{1-(1-\hat{p})^L}{\hat{p}} - L(1-\hat{p})^L.$

$\Rightarrow E[N] = \frac{10}{\hat{p}}[1-(1-\hat{p})^L] = \frac{10}{1-\Phi\left(\frac{Z_\alpha}{10}-\frac{\mu}{\sqrt{10}}\right)}\cdot\left[1-\left(\Phi\left(\frac{Z_\alpha}{10}-\frac{\mu}{\sqrt{10}}\right)\right)^L\right]$