1. Sojourn times are exponential with mean time depending on current state only:

   $P\{T_{n+1} - T_n > t \mid X_0, X_1, \ldots, X_n, X_{n+1}, \ldots\} = P\{T_{n+1} - T_n > t \mid X_n\}$

   $P\{T_{n+1} - T_n > t \mid X_n = i\} = e^{-\lambda(i)t}$ for $t \geq 0$.

2. There is an imbedded Markov chain $\{X_n\}$ with $X_n = Y(T_n)$ and

   $P\{X_{n+1} = j \mid X_n = i\} = Q(i,j)$

   for Markov matrix Q and with the diagonal elements zero.

A *generator* matrix, G, is formed by $G(i,j) = \lambda(i)\, Q(i,j)$ for $i \neq j$ and $G(i,i) = -\lambda(i)$ for the diagonal elements. Note that some authors call this the *rate* matrix.

Two properties of the generator matrix.

1. $G(i,j) \geq 0$ for $i \neq j$. (That is, off-diagonal elements are nonnegative.)
2. $\sum_j G(i,j) = 0$ for each fixed i. (That is diagonal elements are nonpositive.)

Intuitively, $G(i,j)$ for $i \neq j$ is the rate of going from i to j in one transition, $G(i,i)$ is the rate of leaving state i. If the diagonal element of G is zero, then the state associated with that row is absorbing.

The Markov process is called irreducible, recurrent if the imbedded Markov chain is irreducible, recurrent. The probabilities associated with the Markov process are given by

1. $P_t(i,j) = P\{Y(t) = j \mid Y(0) = i\} = e^{Gt}(i,j)$ for $t \geq 0$.
2. For an irreducible, recurrent process, $\lim_{t\to\infty} P_t(i,j) = p(i)$, where $pG=\mathbf{0}$ and $p\mathbf{1}=1$.

For a fixed scalar x, the characteristic polynomial of A is defined by

$$\varphi(x) = \det(xI - A).$$

The polynomial φ is of degree n; therefore, there are n roots to the equation $\varphi(x) = 0$. Call these (possibly complex and not necessarily unique) roots, λ_1, …, λ_n eigenvalues. If an eigenvalue is unique among the list of n eigenvalues, it is called a simple eigenvalue.

A vector, v, such that $Av = \lambda v$ is called an eigenvector and v is unique up to a multiplicative constant for each eigenvalue. Sometimes this eigenvector is called a right eigenvector.

A row vector, π, such that $\pi A = \lambda\pi$ is called a left eigenvector. The following information is relevant for Markov chains and/or Markov processes.

1. If each row sums to the same value, call the sum s, then s is an eigenvalue.
2. If each column sums to the same value, call the sum s, then s is an eigenvalue
3. The trace of A, denoted tr(A), is the sum of its diagonal elements and

   a. $tr(A) = \lambda_1 + \ldots + \lambda_n$
   b. $tr(A^k) = \lambda_1^k + \ldots + \lambda_n^k$ for each $k = 1, 2, \ldots$

4. For an irreducible, aperiodic Markov matrix, P, the value 1 is a simple eigenvalue of P. If λ is an eigenvalue of P with $\lambda \neq 1$, then $|\lambda| < 1$.
5. Let P be an irreducible, aperiodic Markov matrix and let π be the left eigenvector associated with the eigenvalue of 1 with $\pi\mathbf{1} = 1$. Let $\beta = \max \{ |\lambda_j| : \lambda_j \neq 1 \text{ and } \lambda_j \text{ is an eigenvalue of P} \}$, then there exists a constant α such that

   $$| P^k(i,j) - \pi(j) | = \alpha\, \beta^k \quad \text{for } k = 1, 2, \ldots$$

6. If all the eigenvalues of A are unique, then A is diagonalizable. Let $v_1, \ldots, v_n$ be the eigenvectors associated with $\lambda_1, \ldots, \lambda_n$. Form the matrix N by letting its k^{th} column be v_k. Let the matrix D be a matrix such that $D(i,j)=0$ if $i \neq j$ and $D(i,i) = \lambda_i$ for i=1, ..., n. Then

   $A = N D N^{-1}$.

   It might also be noted that the rows of N^{-1} are left eigenvectors of A.

7. If A is diagonalizable, then

   $e^A = N e^D N^{-1}$ where

   $e^D(i,j) = 0$ if $i \neq j$ and $e^D(i,i) = e^{\lambda i}$ for i = 1, …, n. Note that $e^A = \sum_{k=0}^{\infty} A^k/k!$

Consider a machine repair problem. The life time of an electronic component in a machine is according to an exponential random variable with mean one week. The component is difficult to replace and it takes, on the average, one day to replace the component although the time to replace it is actually according to an exponential distribution. (Assume the company that uses the machine is open 24/7.) Let Z(t) be a process with state space {w, r} so that Z(t) = w if the component is working at time t and Z(t) = r if the component is being replaced at time t. Let weeks be the time units.

Note: the process {Z(t)} is also a regenerative process so there is a choice in which method to use in analyzing this machine-replacement process. Today, we practice our knowledge of Markov processes.

1. Notice that $\{Z(t)\}$ is a Markov process. Form the generator matrix for this process.
2. What is $\lim_{t\to\infty} P\{Z(t) = w \mid Z(0) = w\}$?
3. Given that the component is working at the beginning of today, what is the probability that it will be working at the beginning of tomorrow?
4. Given that the component is working at the beginning of today, what is the probability that it will be working at the beginning of next week?