Name : Lu Sun UIN : 228002579

"An Aggie does not lie, cheat or steal, or tolerate those who do." Lu Sun

Problem 1:

1. Suppose $Y_i := |X_i|$, then $\{Y_i\} \overset{iid}{\sim} U[0,\theta]$ $Y_{(n)} := \max_{1\le i\le n} Y_i = \max_{1\le i\le n} |X_i|$

$Y_{(n)}$ is complete for $\theta$, $P(Y_{(n)} \le x) = \left(\frac{x}{\theta}\right)^n$, $f_{Y_{(n)}}(x|\theta) = \frac{n x^{n-1}}{\theta^n}$

$$E\left[\frac{1}{Y_{(n)}}\right] = \int_0^\theta \frac{1}{x} \cdot \frac{n x^{n-1}}{\theta^n} dx = \frac{n}{\theta^n} \int_0^\theta d\frac{x^{n-1}}{n-1} = \frac{n}{n-1}\frac{1}{\theta}$$

$$E[Y_{(n)}] = \int_0^\theta x \frac{n x^{n-1}}{\theta^n} dx = \frac{n}{\theta^n}\int_0^\theta d\frac{x^{n+1}}{n+1} = \frac{n}{n+1}\theta$$

$$E\left[\frac{n+1}{n} Y_{(n)} + \frac{n-1}{n}\frac{1}{Y_{(n)}}\right] = \theta + \frac{1}{\theta}$$

By the theorem of uniqueness of UMVUE with complete statistics

we have : $\frac{n+1}{n} Y_{(n)} + \frac{n-1}{n}\frac{1}{Y_{(n)}}$ is the UMVUE of $\theta + \frac{1}{\theta}$

2. (a) $$f(X|\eta) = \frac{e^{-\sum_{i=1}^n i\eta} \prod_{i=1}^n (i)^{x_i} \eta^{\sum_{i=1}^n x_i}}{\prod_{i=1}^n (x_i!)} = \frac{e^{-\frac{n(n+1)}{2}\eta} \cdot \prod_{i=1}^n (i)^{x_i} \eta^{\sum_{i=1}^n x_i}}{\prod_{i=1}^n (x_i)!}$$

$$\log L(\eta|X) = -\frac{n(n+1)}{2}\eta + \sum_{i=1}^n x_i \log i + \sum_{i=1}^n x_i \log\eta - \sum_{i=1}^n \log(x_i)!$$

$$\frac{\partial \log L(\eta|X)}{\partial\eta} = -\frac{n(n+1)}{2} + \frac{\sum_{i=1}^n x_i}{\eta}, \quad \frac{\partial^2 \log L}{\partial\eta^2} = -\frac{\sum_{i=1}^n x_i}{\eta^2} \le 0 \Rightarrow \hat\eta_{MLE} = \frac{2\sum_{i=1}^n x_i}{n(n+1)}$$

$$MSE_\eta[\hat\eta_{MLE}] = Var\,\hat\eta_{MLE} \overset{x_i\ iid}{=} \frac{4}{n^2(n+1)^2}\cdot\sum_{i=1}^n Var X_i \overset{Poisson}{=} \frac{4}{n^2(n+1)^2}\cdot\sum_{i=1}^n i\eta = \frac{2}{n(n+1)}\eta$$

(b) $$\frac{\partial}{\partial\eta}\log f(X;\eta) = -\frac{n(n+1)}{2} + \frac{1}{\eta}\sum_{i=1}^n x_i$$

$$I(\eta) = E_\eta\left[\left(-\frac{n(n+1)}{2} + \frac{1}{\eta}\sum_{i=1}^n x_i\right)^2\right] = \frac{n^2(n+1)^2}{4} - \frac{2n(n+1)}{2}\frac{1}{\eta}E\left[\sum_{i=1}^n x_i\right] + \frac{1}{\eta^2}E\left[\left(\sum_{i=1}^n x_i\right)^2\right]$$

$$= \frac{n^2(n+1)^2}{4} - \frac{n(n+1)}{\eta}\sum_{i=1}^n i\eta + \frac{1}{\eta^2}\left[\sum_{i=1}^n E X_i^2 + \sum_{i=1}^n\sum_{\substack{j=1\\j\ne i}}^n E X_i E X_j\right]$$

$$= \frac{n^2(n+1)^2}{4} - \frac{n^2(n+1)^2}{2} + \frac{1}{\eta^2}\left[\sum_{i=1}^n (i\eta + (i\eta)^2) + \sum_{i=1}^n\sum_{\substack{j=1\\j\ne i}}^n i\eta\cdot j\eta\right]$$

$$= -\frac{n^2(n+1)^2}{4} + \frac{1}{\eta^2}\left[\frac{(1+n)n}{2}\eta + \eta^2\sum_{i=1}^n\sum_{j=1}^n ij\right]$$

$$= \frac{n(n+1)}{2\eta}$$

Cramér-Rao Lower Bound : $\frac{|\eta'|^2}{I(\eta)} = \frac{1}{\frac{n(n+1)}{2\eta}} = \frac{2\eta}{n(n+1)}$

(c): $$\frac{\partial}{\partial\eta}\log L(\eta|X) = -\frac{n(n+1)}{2} + \frac{\sum_{i=1}^n x_i}{\eta} = \frac{n(n+1)}{2\eta}\left(\frac{2\sum_{i=1}^n x_i}{n(n+1)} - \eta\right)$$

$a(\eta) := \frac{n(n+1)}{2\eta}$ $W(X) := \frac{2\sum_{i=1}^n x_i}{n(n+1)}$ $\tau(\eta) := \eta$

By the attainment theorem, MLE for $\eta$ attain the CRLB

Problem 2:

1. $f(X;\theta)\cdot f(\theta) \propto \theta^n e^{-\theta \Sigma X_i} \theta^{c-1} e^{-\lambda\theta} \propto \theta^{n+c-1} e^{-\theta(\Sigma X_i+\lambda)}$

$\pi(\theta|X) \propto \theta^{n+c-1} e^{-\theta(\Sigma X_i+\lambda)} \Rightarrow \theta \sim \text{Gamma}(n+c, \frac{1}{\Sigma X_i+\lambda})$

$\min_{\hat\theta} \int_\Theta R(\theta,\hat\theta)\pi(\theta)d\theta \Leftrightarrow \min_{\hat\theta} \int_\Theta L(\theta,\hat\theta)\pi(\theta|X)d\theta$

$\int_0^\infty (\theta^2-2a\theta+a^2)\cdot f(\theta|X)d\theta = E_X(\theta^2)-2aE_X(\theta)+a^2.$

$\hat\theta_{Bayes} = \frac{-(-2E_X(\theta))}{2} = E_X(\theta) \overset{\text{Gamma}}{=} \frac{n+c}{\Sigma X_i+\lambda}.$

2. According to 1.,

$\min_a \int_0^\infty 1_{\{|\theta-a|>\varepsilon\}} f(\theta|X)d\theta = 1-\int_0 1_{\{|\theta-a|\le\varepsilon\}}\cdot f(\theta|X)d\theta$

$= \begin{cases} 1-\int_{-\varepsilon+a}^{\varepsilon+a} f(\theta|X)d\theta & a>\varepsilon \\ 1-\int_0^{\varepsilon+a} f(\theta|X)d\theta & -\varepsilon\le a\le\varepsilon \\ 1 & a<-\varepsilon \end{cases}$

$\Leftrightarrow \max_a \begin{cases} \int_{-\varepsilon+a}^{\varepsilon+a} f(\theta|X)d\theta & a>\varepsilon \\ \int_0^{\varepsilon+a} f(\theta|X)d\theta & -\varepsilon\le a\le\varepsilon \end{cases}$

$\frac{\partial}{\partial a}\int_{-\varepsilon+a}^{\varepsilon+a} f(\theta|X)d\theta = f(\varepsilon+a|X)-f(-\varepsilon+a|X)\le 0$, decrease with a ? $\hat\theta_{MLE}=\varepsilon.$

$\int_0^{\varepsilon+a} f(\theta|X)d\theta = F(\varepsilon+a|X)$ increase with a, $\hat\theta_{MLE}=\varepsilon.$

$\Rightarrow \hat\theta_{Bayes}=\varepsilon.$

Problem 3:

1. $\alpha = P_{\theta=0}(\text{Reject } H_0) = P(X_{(1)} \geq K \text{ or } X_{(n)} \geq 1 \mid \theta = 0)$

$= P(X_{(1)} \geq K \text{ or } X_{(n)} = 1 \mid \theta = 0)$

$P(X_{(1)} \geq K \mid \theta=0) = (1-K)^n$

$\alpha = P_{\theta=0}(\text{Reject } H_0) = (1-K)^n$

$\Rightarrow K = 1 - \frac{\log \alpha}{n}$

2. ① $L(\theta \mid X) = \theta^n e^{-\theta \sum_{i=1}^n X_i}$

$\frac{\partial \log L(\theta \mid X)}{\partial \theta} = \frac{\partial}{\partial \theta}\left(n \log \theta - \theta \sum_{i=1}^n X_i\right) = \frac{n}{\theta} - \sum_{i=1}^n X_i$

$\frac{\partial^2 \log L}{\partial \theta^2} = -\frac{n}{\theta^2} < 0 \Rightarrow \hat{\theta}_{MLE} = \frac{n}{\sum_{i=1}^n X_i}$

if $\frac{n}{\sum X_i} < 1$, $\lambda(X) = 1$ $\quad \hat{\theta}_{RMLE} = 1$

if $\frac{n}{\sum X_i} \geq 1 > \theta$, $\frac{\partial \log L(\theta \mid X)}{\partial \theta} = \frac{n}{\theta} - \sum_{i=1}^n X_i \geq 0$, $L(\theta \mid X)$ increase with $\theta$,

$$\lambda(X) = \frac{e^{-\sum_{i=1}^n X_i}}{\left(\frac{n}{\sum_{i=1}^n X_i}\right)^n e^{-n}} = \left(\frac{\sum_{i=1}^n X_i}{n}\right)^n e^{-(\sum_{i=1}^n X_i + n)}$$

$$\Rightarrow \text{LRT: } \lambda(X) = \begin{cases} 1 & \frac{n}{\sum_{i=1}^n X_i} < 1 \\ n^{-n}\left(\sum_{i=1}^n X_i\right)^n e^{-(\sum_{i=1}^n X_i + n)}, & \frac{n}{\sum_{i=1}^n X_i} \geq 1 \end{cases} \Longleftrightarrow \begin{cases} 1, & \bar{X} > 1 \\ (\bar{X})^n e^{-n\bar{X}-n}, & \bar{X} \leq 1 \end{cases}$$

②

$$\frac{\partial \lambda(X)}{\partial \bar{X}} = n(\bar{X})^{n-1} e^{-n\bar{X}-n} \underbrace{(1-\bar{X})}_{>0} \geq 0$$

$\Rightarrow \lambda(X)$ increase with $\bar{X}$ increase.

Rejection Region $R: \{\lambda(X) \leq C\} \Longleftrightarrow R: \{\bar{X} \leq C^*\}$ for some $C^*$ such that $(C^*)^n e^{-nC^*-n} = C$

power function: $\pi(\theta) = P_\theta(R) = P_\theta(\bar{X} \leq C^*) \overset{\bar{X}=X_1}{=\!=} \int_0^{C^*} \theta e^{-\theta x} dx = 1 - e^{-\theta C^*}$.

$\sup_{\theta \leq 1} \pi(\theta) = 1 - e^{-C^*} = \alpha \Rightarrow C^* = -\lg(1-\alpha) \approx \alpha = 0.05$.

$\bar{X} = X_1 = 0.1 > C^* = 0.05 \Rightarrow$ not to reject $H_0$.

not in Reject Region R