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Prob 1 : EX 6.2 in C & B

Let $X_1, \cdots, X_n$ ind with $f_{X_i}(x|\theta) = \begin{cases} e^{i\theta - x} & x \geq i\theta \\ 0 & x < i\theta \end{cases}$

Prove that $T = \min_i (X_i/i)$ is a sufficient statistic for θ.

proof: Suppose $X := (X_1, \cdots, X_n)$, $x := (x_1, \cdots, x_n)$, $x_{(1)} = T(x)$

$$P_\theta(X=x, T(X)=T(x)) = P_\theta(X=x) = f_X(x|\theta) = f_X(x_1, \cdots, x_n|\theta) \overset{X_1,\cdots,X_n}{\underset{\text{independent}}{=}} \prod_{i=1}^n f_{X_i}(x_i|\theta) = \prod_{i=1}^n \mathbb{1}_{\{x_i \geq i\theta\}} \cdot e^{i\theta - x_i}$$

$$= \prod_{i=1}^n \mathbb{1}_{\{x_i/i \geq \theta\}} e^{i\theta - x_i}$$

$$= \mathbb{1}_{[\theta,\infty)}(T(x)) \cdot \prod_{i=1}^n e^{i\theta} \cdot \prod_{i=1}^n e^{-x_i}$$

$$= \mathbb{1}_{[\theta,\infty)}(T(x))\, e^{\frac{(1+n)n}{2}\theta} \cdot e^{-\sum_{i=1}^n x_i}$$

$$P_\theta(T(X)=T(x)) = \sum_{y \in A_{T(x)}} P_\theta(X=y, T(X)=y) = \sum_{y \in A_{T(x)}} f_X(y|\theta) = \sum_{y \in A_{T(x)}} \mathbb{1}_{[\theta,\infty)}(y)\, e^{\frac{(1+n)n\theta}{2}} \cdot e^{-\sum_{i=1}^n y_i}$$

$$= \mathbb{1}_{[\theta,\infty)}(T(x)) \cdot e^{\frac{(1+n)n}{2}\theta} \cdot \sum_{y \in A_{T(x)}} e^{-\sum_{i=1}^n y_i}$$

$$P(X=x|T(X)=T(x)) = \frac{P_\theta(X=x, T(X)=T(x))}{P_\theta(T(X)=T(x))} = \frac{e^{-\sum_{i=1}^n x_i}}{\sum_{y \in A_{T(x)}} e^{-\sum_{i=1}^n y_i}}$$

$\because P(X=x|T(X)=T(x))$ is free of θ

$\therefore T = \min_i (X_i/i)$ is a sufficient statistic for θ

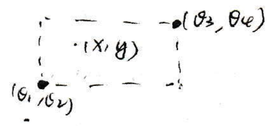
$g(t;\theta) := \mathbb{1}_{[\theta,\infty)}(t) \cdot e^{\frac{n(1+n)}{2}\theta}$, $T(x) := \min_i (x_i/i)$

$h(x) := e^{-\sum_{i=1}^n x_i}$

By Factorization Theorem, T is sufficient statistic for θ

Prob 2 : EX 6.7 in C&B

$f(x, y \mid \theta_1, \theta_2, \theta_3, \theta_4)$ bivariate pdf for the uniform distribution on the rectangle with lower left corner (θ_1, θ_2) & upper right corner (θ_3, θ_4) in $\mathbb{R}^2$.



$\theta_1 < \theta_3$, $\theta_2 < \theta_4$; $(X_1, Y_1) \cdots (X_n, Y_n)$ iid with f distribution

Find a four-dimensional sufficient statistic for $(\theta_1, \theta_2, \theta_3, \theta_4)$

proof: Assume : $T\big((X_1,Y_1),(X_2,Y_2),\cdots,(X_n,Y_n)\big) := \big(\min_i(X_i), \max_i(X_i), \min_i(Y_i), \max_i(Y_i)\big)$

Suppose : $\mathbb{Z} := \big((X_1 Y_1), (X_2 Y_2), \cdots, (X_n Y_n)\big)$; $z := \big((x_1, y_1), \cdots, (x_n, y_n)\big)$

$x_{(1)} = \min_i x_i$, $x_{(n)} = \max_i x_i$, $y_{(1)} = \min_i y_i$, $y_{(n)} = \max_i y_i$

$$P_\theta(\mathbb{Z}=z, T(\mathbb{Z})=T(z)) \overset{iid}{=} \prod_{i=1}^n f(x_i, y_i \mid \theta) = \prod_{i=1}^n \frac{1}{(\theta_3-\theta_1)(\theta_4-\theta_2)} \mathbb{1}_{(\theta_1,\theta_3)}(x_i) \cdot \mathbb{1}_{(\theta_2,\theta_4)}(y_i)$$

$$= \frac{1}{(\theta_3-\theta_1)^n(\theta_4-\theta_2)^n} \mathbb{1}_{(\theta_1,\theta_3)}(x_{(1)}) \mathbb{1}_{(\theta_1,\theta_3)}(x_{(n)}) \mathbb{1}_{(\theta_2,\theta_4)}(y_{(1)}) \mathbb{1}_{(\theta_2,\theta_4)}(y_{(n)})$$

$$P_\theta(T(\mathbb{Z})=T(z)) = \sum_{s \in A_{T(z)}} P_\theta(\mathbb{Z}=s, T(\mathbb{Z})=T(z))$$

$$= \sum_{s \in A_{T(z)}} \frac{1}{(\theta_3-\theta_1)^n(\theta_4-\theta_2)^n} \mathbb{1}_{(\theta_1,\theta_3)}(x_{(1)}) \mathbb{1}_{(\theta_1,\theta_3)}(x_{(n)}) \mathbb{1}_{(\theta_2,\theta_4)}(y_{(1)}) \mathbb{1}_{(\theta_2,\theta_4)}(y_{(n)})$$

$$P(\mathbb{Z}=z \mid T(\mathbb{Z})=T(z)) = \frac{\frac{1}{(\theta_3-\theta_1)^n(\theta_4-\theta_2)^n}}{\sum_{s \in A_{T(z)}} \frac{1}{(\theta_3-\theta_1)^n(\theta_4-\theta_2)^n}} = \frac{1}{\sum_{s \in A_{T(z)}} 1}$$

$\because$ $P(\mathbb{Z}=z \mid T(\mathbb{Z})=T(z))$ is free of $\theta := (\theta_1, \theta_2, \theta_3, \theta_4)$

and $T := (\min_i X_i, \max_i X_i, \min_i Y_i, \max_i Y_i)$ has 4 dimension

$\therefore$ T is a four-dimensional sufficient statistic for $(\theta_1, \theta_2, \theta_3, \theta_4)$

$$g(t, \theta) = \frac{1}{(\theta_3-\theta_1)^n(\theta_4-\theta_2)^n} \mathbb{1}_{(\theta_1,\theta_3)}(t_{(1)}) \mathbb{1}_{(\theta_1,\theta_3)}(t_{(2)}) \mathbb{1}_{(\theta_2,\theta_4)}(t_{(3)}) \mathbb{1}_{(\theta_2,\theta_4)}(t_{(4)})$$

$h(x) = 1$

By Factorization Theorem, T is a four-diemensional sufficient statistic for $(\theta_1, \theta_2, \theta_3, \theta_4)$

prob 3:

proof: (a): if $y < 0$

$f(y;\lambda,c) = f_X(y;\lambda) = 0$

if $0 \le y < c$.

$f(y;\lambda,c) = f_X(y;\lambda) = \lambda e^{-\lambda y}$

if $y = c$

$$f(y;\lambda,c) = P_{\lambda,c}(Y_1 = y) = P_{\lambda,c}(Y_1 = c) = P_{\lambda,c}(\min\{X_1, c\} = c)$$
$$= P_{\lambda,c}(X_1 \ge c) = \int_c^\infty f_X(x;\lambda)\,dx = \int_c^\infty \lambda e^{-\lambda x}\,dx$$
$$= \int_c^\infty d\!-\!e^{-\lambda x} = e^{-\lambda c}$$

if $y > c$.

$f(y;\lambda,c) = 0$

$$\Rightarrow f(y;\lambda,c) = \begin{cases} \lambda e^{-\lambda y} & 0 \le y < c \\ e^{-\lambda c} & y = c \\ 0 & y<0 \text{ or } y > c \end{cases}$$

(b): Suppose $\theta := \lambda$

$$\text{Then } f(y;\lambda,c) = (\lambda e^{-\lambda y})^{1_{\{0\le y<c\}}} (e^{-\lambda c})^{1_{\{y=c\}}} \mathbb{1}_{\{0\le y\le c\}}$$
$$= \mathbb{1}_{\{0\le y\le c\}} \exp\left[1_{\{0\le y<c\}}\cdot(-\lambda y + \log\lambda) + 1_{\{y=c\}}(-\lambda c)\right]$$
$$= \mathbb{1}_{\{0\le y\le c\}} \exp\left[\lambda\cdot(-c\cdot 1_{\{y=c\}} - y\, 1_{\{0\le y<c\}}) + \log\lambda\cdot(1_{\{0\le y<c\}})\right]$$

Then $w_1(\theta) = \lambda$, $t_1(y) = -c\cdot 1_{\{y=c\}} - y\cdot 1_{\{0\le y<c\}}$

$w_2(\theta) = \log\lambda$, $t_2(y) = 1_{\{0\le y<c\}}$

$C(\theta) = 1$, $h(y) = \mathbb{1}_{\{0\le y\le c\}}$

$\therefore f(y;\lambda,c)$ belongs to the exponential family

Its natural parametrization $\eta := (\eta_1, \eta_2)$ where $\eta_1 = \lambda$, $\eta_2 = \log\lambda$.

$C^*(\eta) = 1$

dimension of θ is 1, dimension of w is 2, $\overset{1<2}{\Rightarrow}$ curved

(c) Suppose $Y_{(1)} := \min_i Y_i$, $Y_{(n)} := \max_i Y_i$, $Y_{(c)} := \sum_i \mathbb{1}_{\{Y_i = c\}}$, $Y_+ := \sum_{i=1}^n Y_i$

Then $T(Y_1, \cdots, Y_n) := (Y_{(1)}, Y_{(n)}, Y_{(c)}, Y_+)$

proof: $P_\theta((Y_1 \cdots Y_n) = (y_1, \cdots, y_n), T(Y_1, \cdots, Y_n) = T(y_1, \cdots, y_n))$

$$= P_\theta((Y_1 \cdots Y_n) = y_1 \cdots y_n)$$

$$\overset{iid}{=} \prod_{i=1}^n f_{Y_i}(y_i; \lambda, c)$$

$$= \prod_{i=1}^n (\lambda e^{-\lambda y_i})^{\mathbb{1}_{[0,c)}(y_i)} (e^{-\lambda c})^{\mathbb{1}\{y_i = c\}} \cdot \mathbb{1}_{[0,c]}(y_i)$$

$$= \mathbb{1}_{[0,c]}(y_{(1)}) \cdot \mathbb{1}_{[0,c]}(y_{(n)}) \prod_{i=1}^n \lambda^{1-\mathbb{1}\{y_i = c\}} e^{-\lambda y_i}$$

$$= \mathbb{1}_{[0,c]}(y_{(1)}) \mathbb{1}_{[0,c]}(y_{(n)}) \lambda^{n - y_{(c)}} e^{-\lambda y_+}$$

$$P(Y_1 \cdots Y_n = y_1 \cdots y_n \mid T(Y_1 \cdots Y_n) = T(y_1 \cdots y_n))$$

$$= \frac{P_\theta(Y_1 \cdots Y_n = y_1 \cdots y_n, T(Y_1 \cdots Y_n) = T(y_1 \cdots y_n))}{\sum_{(x_1, \cdots x_n) \in A_{T(y_1, \cdots, y_n)}} P_\theta(Y_1 \cdots Y_n = x_1, \cdots, x_n)}$$

$$= \frac{\mathbb{1}_{[0,c]}(y_{(1)}) \mathbb{1}_{[0,c]}(y_{(n)}) \lambda^{n - y_{(c)}} e^{-\lambda y_+}}{\sum_{(x_1 \cdots x_n) \in A_T(y_1, \cdots y_n)} \mathbb{1}_{[0,c]}(y_{(1)}) \mathbb{1}_{[0,c]}(y_{(n)}) \lambda^{n - y_{(c)}} e^{-\lambda y_+}}$$

$$= \frac{1}{\sum_{(x_1 \cdots x_n) \in A_T(y_1 \cdots y_n)} 1} \quad \text{free of } \theta := (\lambda, c)$$

$\Rightarrow T := (Y_{(1)}, Y_{(n)}, Y_{(c)}, Y_+)$ sufficient statistics of Y

(c) By theorem, $Y_1, \cdots, Y_n$ from exponential family,

Then $T(Y) = (\sum_{i=1}^n t_1(y_i), \sum_{i=1}^n t_2(y_i))$ is sufficient for $\theta = (\lambda)$

Namely $T(Y) = (\sum_{i=1}^n [-c \mathbb{1}_{\{y_i = c\}} - y_i \cdot \mathbb{1}_{\{0 \le y_i < c\}}], \sum_{i=1}^n \mathbb{1}_{\{0 \le y_i < c\}})$