

Advanced Microeconomic Theory¹

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¹This lecture notes drawn from some chapters of my book draft is for my teaching of ECON 630 (Microeconomic Theory II) and convenience of my students in class. Please not distribute it.

Contents

Preface

I Preliminary Knowledge and Methods	3
1 Nature of Modern Economics	7
1.1 Economics and Modern Economics	7
1.1.1 What is economics about?	7
1.1.2 Four Basic Questions in Economics	8
1.1.3 What is Modern Economics?	9
1.1.4 Economics vs. Natural Science	11
1.2 Two Categories of Economic Theory	12
1.2.1 Benchmark Theory and Relatively Realistic Theory .	13
1.2.2 The Domain and Scientific Rigor of Economics	18
1.2.3 The Roles of Economic Theory	21
1.2.4 Microeconomic Theory	22
1.3 Economics and Market System	23
1.3.1 Market and Market Mechanism	23
1.3.2 Three Functions of Price	26
1.3.3 The Superiority of the Market System	29
1.4 Government, Market, and Society	35
1.4.1 Three Elements of State Governance and Development	35
1.4.2 Good Market Economies vs. Bad Market Economies	37
1.4.3 The Boundaries of Government-Market-Society . . .	40
1.5 Comprehensive Governance by the Three Arrangements . .	42
1.5.1 Governance on Rules	43

1.5.2	Incentive Mechanism	44
1.5.3	Social Norms	45
1.5.4	The Hierarchical Structure of the Three Arrangements	47
1.6	Ancient Chinese Thoughts on the Market	49
1.7	A Cornerstone Assumption in Economics	55
1.7.1	Self-love, Selfishness, and Self-interest	55
1.7.2	Practical Rationality of Self-interested Behavior . . .	56
1.7.3	Boundaries of Self-interest and Altruism	57
1.8	Key Points in Economics	59
1.8.1	Scarcity of Resources	60
1.8.2	Information Asymmetries and Decentralization . . .	60
1.8.3	Economic Freedom and Voluntary Exchange	62
1.8.4	Acting under Constraints	63
1.8.5	Incentives and Incentive Compatibility	65
1.8.6	Property Rights as an Incentive Scheme	66
1.8.7	Equality of Opportunity and Equity in Outcome . .	67
1.8.8	Efficient Allocation of Resources	68
1.9	Understanding Economics Properly	69
1.9.1	On the Scientification of Economics	70
1.9.2	On the Mathematical Feature of Economics	72
1.9.3	Misunderstandings on Economic Theory	72
1.9.4	On Experiments in Economics	75
1.10	Basic Analytical Framework of Modern Economics	80
1.10.1	Specifying Economic Environment	82
1.10.2	Making Behavioral Assumptions	85
1.10.3	Setting-up Institutional Arrangements	87
1.10.4	Determining Equilibrium	89
1.10.5	Making Evaluations	90
1.11	Basic Research Methodologies in Economics	92
1.11.1	Setting up a Benchmark	93
1.11.2	Establishing a Reference System	93
1.11.3	Building Studying Platforms	95
1.11.4	Developing Analytical Tools	97
1.11.5	Constructing Rigorous Analytical Models	97

1.11.6	Making Positive and Normative Analysis	98
1.12	Practical Role of the Analytical Framework and Methodologies	99
1.13	Requirements for Learning Economics	102
1.14	Distinguishing Sufficient and Necessary Conditions	103
1.15	The Role of Mathematics and Statistics in Economics	104
1.16	Conversion between Economic and Mathematical Languages	108
1.17	Biographies	109
1.17.1	Adam Smith	109
1.17.2	David Ricardo	111
1.18	Exercises	112
1.19	References	118
2	Preliminary Knowledge and Methods of Mathematics	125
2.1	Basic Set Theory	125
2.1.1	Set	125
2.1.2	Mapping	126
2.2	Basic Linear Algebra	128
2.2.1	Matrix and Vector	128
2.2.2	Matrix Operations	129
2.2.3	Linear Dependence of Vectors	131
2.2.4	Transpose and Inverse of Matrix	132
2.2.5	Solving a Linear System	133
2.2.6	Quadratic Form and Matrix	137
2.2.7	Eigenvalues, Eigenvectors, and Traces	139
2.3	Basic Topology	141
2.3.1	Topological Space	141
2.3.2	Metric Space	142
2.3.3	Open Sets, Closed Sets, and Compact Sets	143
2.3.4	Connectedness of Sets	147
2.3.5	Sequences and Convergence	147
2.3.6	Convex Set and Convexity	148
2.4	Single-Valued Function	149
2.4.1	Continuity of functions	149
2.4.2	Upper Semi-continuity and Lower Semi-continuity	151

2.4.3	Transfer Upper and Lower Continuity	152
2.4.4	Differentiation and Partial Differentiation of Functions	152
2.4.5	Mean Value Theorem and Taylor Expansion	154
2.4.6	Homogeneous Functions and Euler's Theorem	156
2.4.7	Implicit Function Theorem	156
2.4.8	Concave and Convex Function	158
2.4.9	Quasi-concave and Quasi-convex Function	159
2.4.10	Separating and Supporting Hyperplane Theorems . .	163
2.5	Multi-Valued Function	164
2.5.1	Point-to-Set Mappings	164
2.5.2	Upper Hemi-continuous and Lower Hemi-continuous Correspondence	166
2.5.3	Open and Closed Graphs of Correspondence	169
2.5.4	Transfer Closed-valued Correspondence	170
2.6	Static Optimization	173
2.6.1	Unconstrained Optimization	173
2.6.2	Optimization with Equality Constraints	180
2.6.3	Optimization with Inequality Constraints	184
2.6.4	The Envelope Theorem	186
2.6.5	Maximum Theorems	188
2.6.6	Continuous Selection Theorems	190
2.6.7	Fixed Point Theorems	191
2.6.8	Variation Inequality	194
2.6.9	FKKM Theorems	195
2.7	Dynamic Optimization	197
2.7.1	Calculus of Variation	197
2.7.2	Optimal Control	201
2.7.3	Dynamic Programming	204
2.8	Differential Equations	208
2.8.1	Existence and Uniqueness Theorem of Solutions for Ordinary Differential Equations	210
2.8.2	Some Common Ordinary Differential Equations with Explicit Solutions	211

2.8.3	Higher Order Linear Equations with Constant Coefficients	214
2.8.4	System of Ordinary Differential Equations	218
2.8.5	Stability of Simultaneous Differential Equations	222
2.8.6	The Global Stability of Dynamical System	224
2.9	Difference Equations	225
2.9.1	First-order Difference Equations	227
2.9.2	Second-order Difference Equation	230
2.9.3	Difference Equations of Order n	231
2.9.4	Stability of n th-Order Difference Equations	232
2.9.5	Difference Equations with Constant Coefficients	233
2.10	Basic Probability	234
2.10.1	Probability and Conditional Probability	234
2.10.2	Mathematical Expectation and Variance	235
2.10.3	Continuous Distributions	236
2.10.4	Common Probability Distributions	237
2.11	Stochastic Dominance and Affiliation	239
2.11.1	Order Stochastic Dominance	239
2.11.2	Hazard Rate Dominance	243
2.11.3	Reverse Hazard Rate Dominance	244
2.11.4	Likelihood Ratio Dominance	245
2.11.5	Order Statistics	246
2.11.6	Affiliation	247
2.12	Biographies	250
2.12.1	Friedrich August Hayek	250
2.12.2	Joseph Alois Schumpeter	252
2.13	Exercises	254
2.14	References	267

III Game Theory and Market Theory 271

6	Non-Cooperative Game Theory	277
6.1	Introduction	277
6.2	Basic Concepts	277

6.2.1	Strategic Form Representation of Games	278
6.2.2	Extensive Form Representation of Games	281
6.2.3	Mixed Strategies and Behavior Strategies	286
6.3	Static Games with Complete Information	288
6.3.1	Dominant and Dominated Strategies	289
6.3.2	Rationalization of Strategies	297
6.3.3	Nash Equilibrium	299
6.3.4	Refinements of Nash Equilibrium	305
6.4	Dynamic Games with Complete Information	308
6.4.1	Subgame Perfect Nash Equilibrium	310
6.4.2	Backward Induction	311
6.5	Static Games with Incomplete Information	319
6.5.1	Bayesian Game	320
6.5.2	Bayesian-Nash Equilibrium	323
6.6	Dynamic Games with Incomplete Information	326
6.6.1	Weak Perfect Bayesian Equilibrium	329
6.6.2	Sequential Equilibrium	333
6.6.3	Forward Induction	338
6.6.4	Signaling Game	341
6.6.5	Dominated Strategy in Equilibrium	346
6.6.6	Intuitive Criterion	347
6.7	Existence of Nash Equilibrium	350
6.7.1	Existence of Nash Equilibrium in Continuous Games	350
6.7.2	Existence of Nash Equilibrium in Discontinuous Games	351
6.8	Biographies	357
6.8.1	John Forbes Nash Jr.	357
6.8.2	John C. Harsanyi	358
6.9	Exercises	361
6.10	References	377
7	Repeated Games and Reputation Mechanism	381
7.1	Introduction	381
7.2	Examples of Repeated Games	384
7.3	Repeated Games with Perfect Monitoring	390

7.3.1	Feasible and Individually Rational Payoffs	391
7.3.2	One-Shot Deviation Principle	396
7.3.3	Automaton Representation of Strategy Profiles	399
7.3.4	Credible Continuation Promises	401
7.3.5	Self-Generation of Repeated Games	405
7.4	Folk Theorems with Perfect Monitoring	410
7.5	Some Variations of Repeated Games	418
7.5.1	Long-Run Players and Short-Run Players	418
7.5.2	Overlapping Generations Games	420
7.5.3	Community Constraints and Social Norms	422
7.6	Repeated Games with Imperfect Public Monitoring	426
7.6.1	Basics of Repeated Games with Imperfect Public Monitoring	426
7.6.2	The Recursive Method of Repeated Games with Imperfect Monitoring	430
7.6.3	Potential Efficiency Loss in Repeated Games with Imperfect Monitoring	436
7.6.4	Private Strategy	438
7.7	Reputation Mechanism	443
7.8	Biographies	447
7.8.1	John Richard Hicks	447
7.8.2	Thomas Schelling	449
7.9	Exercises	451
7.10	References	466
8	Cooperative Game Theory	471
8.1	Introduction	471
8.2	The Core	472
8.2.1	Coalitional Game with Transferable Payoff	472
8.2.2	The Existence Theorem on Nonempty Cores	478
8.2.3	Coalitional Game without Transferable Payoff	481
8.3	Application of the Core: Market Design	481
8.3.1	Transaction of Homogeneous Goods	482
8.3.2	Matching of Heterogeneous Goods	485

8.3.3	Two-sided Matching: Marriage Market	489
8.4	Stable Set, Bargaining Set, and Shapley Value	493
8.4.1	Stable Set	494
8.4.2	Bargaining Set, Kernel, and Nucleolus	496
8.4.3	Shapley Value	502
8.5	Biographies	505
8.5.1	Robert J. Aumann	505
8.5.2	Reinhard Selten	507
8.6	Exercises	509
8.7	References	516
9	Market Theory	521
9.1	Introduction	521
9.2	Perfect Competition	523
9.2.1	The Competitive Firm	523
9.2.2	The Competitive Firm's Short-Run Supply Function	524
9.2.3	Single-Commodity Market Equilibrium	526
9.2.4	Competitive Market and Returns to Scale of Production Technology	527
9.2.5	Long-Run Equilibrium	529
9.2.6	Social Welfare under Perfect Competition	530
9.3	Pure Monopoly	532
9.3.1	Monopoly in the Product Market	532
9.3.2	Monopoly in the Long Run	535
9.3.3	Disadvantage of Monopoly: Social Welfare Losses	536
9.3.4	Advantage of Monopoly: Corporate Innovation	537
9.3.5	Price Discrimination of Monopolies	540
9.3.6	First-Degree (Perfect) Price Discrimination	545
9.3.7	Second-Degree (Self-Selection) Price Discrimination	547
9.3.8	Third-Degree (Multi-Market) Price Discrimination	553
9.3.9	Monopolist of Durable Goods	556
9.3.10	Monopoly in an Input Market	560
9.4	Monopolistic Competition	561

9.4.1	Long-run Equilibrium under Monopolistic Competition	562
9.4.2	Social Welfare in Monopolistic Competition	563
9.4.3	Dixit-Stiglitz Model of Monopolistic Competition	563
9.5	Oligopoly	566
9.5.1	Price Competition: Bertrand Model	567
9.5.2	Price Competition with Production Capacity Constraints	570
9.5.3	Quantity Competition: Cournot Model	572
9.5.4	Sequential Quantity Competition: Stackelberg Model	576
9.5.5	Dynamic Price Competition and Firm Collusion	577
9.5.6	Price Competition under Horizontal Product Differentiation: Hotelling Model	579
9.5.7	Vertical Product Differentiation Model	581
9.5.8	Market Entry Deterrence	583
9.5.9	Price Competition with Asymmetric Information	586
9.5.10	Limit Pricing with Asymmetric Information: Dynamic Market Structure	589
9.5.11	Concluding Remarks on the Oligopolistic Market	595
9.6	Biographies	596
9.6.1	Harold Hotelling	596
9.6.2	George J. Stigler	597
9.7	Exercises	600
9.8	References	610

V Externalities and Public Goods 613

14 Externalities 619

14.1	Introduction	619
14.1.1	Consumption Externality	619
14.1.2	Production Externality	621
14.2	Consumption Externality	623
14.3	Production Externality	632
14.4	Solutions to Externalities	634
14.4.1	Pigovian Tax	635

14.4.2	Coase's Approach	636
14.4.3	Missing Market	649
14.4.4	The Compensation Mechanism	650
14.5	Emissions Trading and Efficient Allocation of Pollution Rights	654
14.5.1	Emission Reduction without Trading Market	655
14.5.2	Emissions Trading	655
14.6	Biographies	659
14.6.1	Arthur Pigou	659
14.6.2	Ronald Coase	660
14.7	Exercises	662
14.8	References	670
15	Public Goods	675
15.1	Introduction	675
15.2	Notations and Basic Settings	676
15.3	Discrete Public Goods	679
15.3.1	Efficient Provision of Public Goods	679
15.3.2	Free-Rider Problem	681
15.3.3	Voting for a Discrete Public Good	682
15.4	Continuous Public Goods	684
15.4.1	Efficient Provision of Public Goods	684
15.4.2	Lindahl Mechanism and Equilibrium	687
15.5	Welfare Properties of Lindahl Equilibrium	692
15.5.1	The First Fundamental Theorem of Welfare Economics	692
15.5.2	Economic Core in the Presence of Public Goods	694
15.5.3	The Second Fundamental Theorem of Welfare Eco- nomics	695
15.6	Free-Rider Problem	698
15.7	Biographies	701
15.7.1	Ludwig Mises	701
15.7.2	Douglass North	703
15.8	Exercises	705
15.9	References	715

Preface

This book is based on my lecture notes, which I used to teach courses at many universities, including Texas A&M University, Shanghai University of Finance and Economics, Hong Kong University of Science and Technology, Tsinghua University, and Renmin University of China. A Chinese version was published in 2016. This book is rich in content, and comprises a full range of topics, including the most typical themes in modern microeconomic theory up to those at the frontier, such as dynamic mechanism design, auction theory and matching theory that were mainly developed in the last three decades, but not covered in standard microeconomics textbooks. It also constitutes an integration of my study, research, and teaching of microeconomic theory for more than 30 years. The content in this book is suitable for multiple purposes. By selecting different chapters and/or sections, it can serve as the text for a graduate or an advanced undergraduate course in microeconomics, a sequential course of advanced microeconomics for doctoral students, a course in advanced topics in microeconomic theory, or even a course in mathematical economics. It can also be used as an important reference book for researchers.

Economics is a discipline that is seemingly simple, but hard to learn and master well. The main reason why economic issues are challenging to understand and solve is that, besides the common behavior pattern that economic agents at any level (as nation, firm, household, or person) pursues their own interests under normal circumstances, there is information asymmetry in the vast majority of cases given that economic agents are usually privately informed about relevant economic characteristics. For example, a person may say something, but whether or not he or she is telling the

truth is unclear; and even if someone is staring directly at you, seeming to be listening attentively, one cannot be certain if he or she is really listening. Indeed, how best to deal with these two most basic objective realities, and what kinds of economic systems, institutions, incentive mechanisms, or policies should be adopted have become core issues and topics in all branches of economics.

At the same time, economics frequently involves normative statements, i.e., subjective value judgments. Different individuals have dissimilar values and varied opinions. For instance, some people emphasize the efficiency of resource allocations, while others highlight the equality of resource allocations. Individuals also frequently hold divergent views on economic reforms and policies. Consequently, it is easy to cause major controversy, making it challenging to understand and master economics.

In addition, the use of economics is likely to generate strong externality, either positive or negative. Unlike in the case of medical treatment, those who suffer, or even die, from poor medical skills are only a few individuals, while poor economic application may influence all aspects of the economy. As a consequence, a rigorous and systematic training of economics, and especially the main content of microeconomics discussed in this textbook, is not only important for theoretical innovations, but also critical for practical applications. Once faults are made and inappropriate economic policies or rules are implemented, not only will individuals be harmed, but also economic development at the national level. As such, in addition to learning economics well, when making policy suggestions, we should not make decisions with too much concern about your personal interests. Indeed, we should not only be courageous in taking responsibility for being thorough, but also possess a strong sense of social conscience and responsibility.

Modern economics constitutes a dynamically developing and markedly inclusive open discipline. Indeed, it has far surpassed neo-classical economics, and economic practice can provide rich realities for the innovative development of economic theory. From the author's perspective, as long as rigorous inherent logical analysis (not necessarily a mathematical model) is used and rational assumptions (including bounded rationality assumptions) are adopted, such research fits in the category of economics. Under the

framework of modern economics, microeconomics primarily concerns the theory of how individuals make decisions, thereby laying the microfoundation of macroeconomics, as well as almost all other branches of economics. It is a theory about how markets operate, as well as how specific market mechanisms should be modified in the case of market failure. It also focuses on the study of how limited resources are allocated among different uses to better meet the various needs of humanity. Therefore, it can assist one to understand the decisive role of market in resource allocation, and the critical role of government in making the market orderly and efficient, providing public goods and services, and guaranteeing social fairness.

Knowing what an economic theory is, it is also necessary to be cognizant of the scope of its applications. If this is not the case, once it is broadly used to guide the formulation of economic policies, it will produce highly negative social impacts. With limited training of theoretical logic and empirical quantifications, and the lack of understanding in the premise of theory, blind applications will indeed lead to negation of the role of economics and the assertion that the basic assumptions of mainstream economics are too strong, and too much focus is given to mathematics and rigor. Consequently, a false conclusion will be reached that economic theories cannot explain and solve practical problems. In extreme cases, one will disavow the basic role of economics in economic development and market-oriented reforms.

In fact, in most cases, people who hold this view do not know the preconditions of a theory, and thus they are unaware of the scope of application of the theory. Once proven to be incorrect, they blame the theory for being fallacious and of no value. In fact, just like the theories of any discipline, any rigorous economic theory provides preconditions, i.e., it is not valid in all situations. Therefore, unless a theory itself possesses one or more logical inconsistencies, there is nothing right or wrong about the theory, but only whether or not the theory is suitable for a particular economic environment. If there is no rigorism, how can one always reach the result of inherent logic? This is as pointed out by Dani Rodrik ². These accusation-

²DANI RODRIK. "Economists vs. Economics" , <http://www.project-syndicate.org/commentary/economists-versus-economics-by-dani-rodrik-2015-09>.

s usually come from laymen or certain unorthodox marginalists. Indeed, individuals who hold such opinions are often those who have a limited understanding of economic theory and its methodology. Overall, whether the goal is to conduct original research or applied research, it is necessary to learn economics well, thoroughly understand its analytical framework and methods, and adapt the rigor of logical analysis.

As a high-level course in microeconomics, the purpose of advanced microeconomics teaching is to elucidate the inherent logic existing behind common principles and concepts, and to cultivate students' ability and way of thinking to analyze theoretical issues in a rigorous manner. It also aims to teach students how to characterize the nature of complex economic behaviors and economic phenomena (i.e., through modeling) in order to carry out rigorous economic analyses. This book systematically expounds on the content of modern microeconomics from basic theory, benchmark theory, analytical framework, research methods, to the latest frontier topics. Therefore, all or part of the chapters and sections can be selected for use by doctoral, postgraduate, and senior undergraduate students majoring in economics, finance, statistics, management, applied mathematics, and related disciplines according to the course requirements. This book can also serve as a useful reference book for teachers and scholars engaged in economics teaching and research.

The characteristics of this book

While the mainstream of the economics profession prior to World War II focused mainly on economic thoughts and lacked scientific rigor, the most attention is currently paid to techniques and rigor, and the profound economic thoughts that exist behind economics are largely neglected. It is the case that many people become lost in mathematical models, and do not know the underlying assumptions and profound insights of these economic theories. Why can we not achieve the dialectical unity of rigorous academics "with deep thoughts and for deep thoughts"? In fact, rigorous and original research can achieve this unity, numerous highly technical theories and methods also contain profound economic thoughts, and mod-

els can reflect them (e.g., general equilibrium theory, mechanism design theory). As such, when studying economics, one should know well not only the academic contents of economics, but also its systems, in order to master the underlying profound thoughts and wisdom. Although it may not be crucial for it to become a norm for the public to understand these thoughts in a mature modern market system, it remains extremely important in institutional transitional countries where the direction of economic and social transformation is not clear. Combining both, this textbook attempts to advocate pursuing academics both “with deep thoughts and for deep thoughts” .

I have comprehensively compiled the biographies of 44 economists who made pioneering contributions to the development of modern microeconomics, in order to enhance understanding of the origin, development, and inheritance of the various economic theories (including the economic theory expressed in advanced mathematics) discussed in this textbook. This was also done to increase readers’ interest in learning economic theory, to expand the comprehensiveness of knowledge, and to balance scientific rigor and profound thoughts well.

This textbook also connects with the profound philosophy and wisdom of ancient China, in an effort to achieve more comprehensive ideological and academic thinking, and realize ideological and organic academic integration. To learn economics, it is not sufficient to just be proficient at economics itself, but also to grasp its inherent logic, master its profound ideas, and consequently attain pragmatic wisdom.

Structure of this textbook

Microeconomics focuses on the study of economic phenomena and economic issues from the analysis of individual economic behavior, and then develops various theories and results under given or designed institutional arrangements, especially under the market system. This textbook arranges thematic chapters according to this logic. Prior to this, however, the textbook presents an introduction to the requisite knowledge and methods to study advanced microeconomic theories.

The textbook is divided into seven parts. Part I comprises the general introduction and preparatory knowledge of the textbook. Parts II-IV mainly introduce the benchmark models and theories in ideal economic environments in which a spontaneous market works well, as well as analytical frameworks, methods, and tools. Parts V-VII address market failure in terms of efficiency. They primarily examine how to modify the market in the presence of economic externalities, public goods, and especially private information, in order to achieve efficient allocation of resources. The contents of the textbook are as follows:

There are two chapters in Part I. Chapter 1 primarily introduces the nature and methods of economics, the scope of the textbook, as well as the preparatory knowledge and methods of mathematics. It commences with an overview of the nature, scope, thoughts, analytical framework, and research methods of economics, especially microeconomic theory, as well as the similarities between economic thoughts and Chinese wisdom. These contents aim to increase readers' inclusiveness to all subdisciplines of economics, which is not only essential for the development of various branches of economics, but also within disciplines, such as benchmark theories and relatively realistic theories.

Chapter 2 provides almost all of the commonly used mathematical analytical tools and methods in economics, and particularly in this textbook. It can be used as the basic text or primary reference textbook for courses of mathematical economics for studying advanced macro/micro economics. It can also serve as a manual reference for basic mathematic tools that are needed to study economics.

Part II comprises three chapters, which largely discuss individual decision-making, including consumer theory, producer theory, and individual choice under uncertainty. The individual decision-making theory constitutes the micro foundation for the establishment of numerous theoretical models in economics, and it occupies a central position in the way that economists think about problems. Many choices are made in risk and uncertain situations. Since people frequently need to avoid some uncertainties, for example, by purchasing insurance, choice under risk and uncertainty is a critical aspect of economics.

Part III consists of four chapters. It mainly discusses game theory and market theory, including basic game theory, repeated game and reputation mechanism, cooperative game, and market theory of various market structures. Game theory has become an extremely important subdiscipline in mainstream economics, a core field in microeconomics, and the most foundational analytical tool for analyzing various interactive decision problems in economics. For instance, monopolistic competition, and especially the discussion of oligopolistic markets, necessitates the use of a great amount of knowledge and results of game theory, and thus it is discussed together as applications.

Part IV comprises five chapters. It mainly discusses the benchmark market theory in the frictionless situation of perfect competition: general equilibrium theory and social welfare, including the positive theory of competitive equilibrium, the normative theory of competitive equilibrium, economic core, fair allocation, social choice theory, and general equilibrium theory under certainty. General equilibrium theory is one of the most important theories in the history of economic theory development in the past 100 years. It provides a core reference and benchmark for better studying and solving practical problems. This part describes the nature of competitive equilibrium, and explores how to achieve the fair allocation of resources, in order to further demonstrate the universality, optimality, and rationality of the market economy system.

The above four parts mainly describe benchmark models and theories under ideal economic environments in which the market results in efficient allocations, as well as the analytical framework, methods, and tools. However, in many cases, the market is not omnipotent and frequently fails in term of efficiency, especially from the micro- and information-perspective. Indeed, it can face numerous problems, leading to “market failure”. Therefore, it is very important to elucidate where the market fails and precisely what corrective actions should be taken by governments, regulators, and/or designers. As a consequence, the next three parts of this textbook focus on how to deal with problems caused by market failures, which is closer to reality.

Part V consists of two chapters. It discusses theories of externalities

and public goods, including the typical market failures of externalities and public goods provision. In this part, it will be shown that, in general, these “non-market goods” or “harmful goods” will result in Pareto inefficient allocations, and produce market failure. Due to externalities and the provision of public goods, a primary market is generally not an ideal mechanism for allocating resources.

Part VI has five chapters. The most important three keywords in economics are information, incentive, and efficiency. Therefore, this part is devoted to discussing incentive, information and economic mechanism design, including principal-agent theory under hidden information and moral hazard, mechanism design under complete information and incomplete information, and dynamic mechanism design. Mechanism design does not attempt to change human nature, which is essentially immutable. As such, mechanism design theory investigates whether and how to design a set of mechanisms (i.e., rules of games or systems) to achieve desired goals under the conditions of individuals pursuing their self-interest, free choice, voluntary exchange, incomplete information and decentralized decision-making, and to compare and assess the advantages and disadvantages of a particular mechanism. Whether or not information is symmetric and incentive is compatible constitute the root causes of different performances of alternative competing mechanisms.

Part VII comprises two chapters, which discusses market design of auction theory and matching theory that are the two frontier subfields of microeconomic theory. Market design, as a relatively new field, can be regarded as the specific expansion and extension of the general mechanism design theory in Part VI, and has a wide range of applications in the real world.

Teaching tips

As mentioned above, the content of this textbook may be taught at many different levels, and is intended for courses of microeconomics for graduate and advanced undergraduate students, topics of advanced microeconomic theory, or courses in mathematical economics. The rich and fully

expounded-upon content in this textbook also provides a broad range of choices and space for the teaching of microeconomics at different levels, and for the aspects that teachers consider to be the most important. Moreover, teachers can flexibly select relevant chapters and sections to teach according to their teaching needs. The second chapter, about which concerns the knowledge of mathematics, can be also be used as a teaching material or an important reference for mathematical economics.

The following are some suggestions for instructors who choose to use this textbook. (1) For microeconomics sequence I for graduate students and senior undergraduates, consider the following chapters for the teaching of one semester: Chapters 3-5 on individual decision-making and Chapters 6-8 on game theory. (2) For microeconomics II for graduate students, consider the following chapters for the teaching of one semester: Chapter 9 on market theory, Chapters 10-13 on general equilibrium theory, Chapters 14-15 on externalities and public goods, and Chapters 16-17 on principal-agent theory. (3) For advanced topics in microeconomic theory for graduate students, depending on different foci, the instructor can choose Chapters 18-20 on mechanism design theory or Chapters 21-22 on market design. Of course, the selection of chapters needs to be based on the preference of instructors, research interests, and constraints on course time. In addition, in teaching advanced microeconomics I, II, or advanced topics in microeconomic theory, it is crucial for students to know the context of Chapter 1 on the nature, category, analytical framework, thoughts and methodologies of economics, and thus students should first either self-study Chapter 1 or be taught it by instructors. In addition, for undergraduates to obtain a general understanding of the scope, ideas, analytical framework, and research methods of economics, they are strongly recommended to read Chapter 1 and the introduction sections of the other chapters.

Doing exercises is the most reliable and effective way to master the content of teaching materials. The exercises at the end of each chapter were written by myself, adapted from classical textbooks and the Doctoral Qualification Examination Question Bank of the Economics Department of world-class universities, or are examples or basic conclusions of original academic papers, indicating their sources whenever possible. I would al-

so like to express my sincere gratitude to the anonymous authors of many exercises in this textbook.

Research tips

This textbook can also be used as a reference for research. Whether it is to carry out original theoretical, applied or policy-oriented research, it is necessary to learn economics well, master its basic analytical framework and research methods, and establish a solid theoretical and methodological foundation. From the content of textbook, it is possible to learn the most advanced microeconomic theory and engage in relevant research. In general, the study of economics can be broadly divided into two categories. The first category is the study of basic, original, and common theories and tools, most of which have no borders and can generally be applied to any region. The various theories introduced in this textbook basically fall into this category. The second is the attempt to solve pragmatic or realistic issues, i.e., applying the basic principles, analytical frameworks, research methods, and analytical tools of economics to study and solve real-world problems of a nation or region. Indeed, these two are dialectically unified, and it is crucial not to negate the latter by the former, or vice versa. This is the same situation as in basic research in the natural sciences and technological innovation in industry: they are complementary, equally important, and indispensable.

In addition, it should be pointed out that economics, and especially microeconomic theory, mainly provides two categories of theories, both of which possess stringent prerequisites. The first category is to provide various benchmark theories. Most of the theories introduced in Parts II-IV of this textbook belong in this category. The second category is to deliver various relatively more realistic theories. Parts V-VII are more closely associated with reality, and are primarily theories that are proposed to solve practical problems. The first category of theories is primarily based on the economic environments of mature market economies or nations, and provides basic theory in ideal situations. Although it serves an important role in guiding improvement or reform orientation, it deviates from reality, e-

specially concerning economic environments that are still in the process of transition. As a consequence, it is necessary to revise the benchmark theory and consider situations that have frictions and are closer to reality, in order to develop a relatively more realistic economic theory that solves specific practical problems, draws accurate conclusions, and makes predictions with intrinsic logic. For these reasons, these two categories of theories exist in a progressive relationship in the development of disciplines. The basic analytical framework, thoughts, and research methods of economics described in this textbook are not only common to the study of these two types of theories, but can also be employed to better apply economic theory to investigate real-world economic problems. They can even establish a solid theoretical and methodological foundation for the development of economic theories that are suitable for studying transitional economies. This requires the special attention of researchers when using this textbook.

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ticipated in the practice of the reform and development of China's economic education, including the introduction of a large-scale, organized-system of overseas high-level talents, and the reform of the curriculum system with advanced international standards. By now, the School of Economics at SUFE is internationally well-known. According to the Worldwide Economics Departments Research Ranking by Tilburg University, SUFE ranks constantly the first among all the universities in mainland China, top three in Asia, and 50s worldwide in last eight years. During that period, almost every year, I have given doctoral students at SUFE course, entitled Advanced Microeconomics II. Without this opportunity, this textbook would not have been possible.

In short, this textbook is a crystallization of many people's wisdom in a sense. Of course, I am fully responsible for the deficiencies and errors of this textbook.

Guoqiang Tian

In my Xingkong Study in Texas

December, 2020

Part I

Preliminary Knowledge and Methods

In order for readers to grasp the content in this textbook more effectively, learn economics, and rigorously understand its profound economic thoughts, theoretical models and proofs, this part introduces the preliminary knowledge and methods of economics and mathematics.

Chapter 1 briefly discusses the nature, essence, category, thoughts, and methods of the discipline of economics. We will fully discuss the ideas and methods of economics, especially those that are involved in this textbook. While the mainstream of the economics profession prior to World War II focused largely on qualitative analysis and economic concepts, to a great extent lacking scientific rigor and quantitative analysis, it currently seems that primary attention is given to techniques and rigor, and the profound economic thoughts behind economics are largely neglected. When studying economics, one should know not only the academic contexts of economics well, but also its systems, in order to master its deep thoughts and wisdom. Combining both, this textbook attempts to advocate pursuing academics “with deep thoughts and for deep thoughts” .

Chapter 2 introduces the basic knowledge and results of mathematics that are indispensable for studying economics, in general, and advanced microeconomic theory, in particular. They are used for rigorous analysis of economic problems and for derivation and proofs of theoretical results. In other words, they provide requisite mathematical knowledge and tools for formalization, axiomatization, and scientification of microeconomic theory.

Chapter 1

Nature of Modern Economics

In this chapter, we discuss the nature, role, and methodology of economics, categories of theories, and the scope, preliminary knowledge, concepts, and methods involved in the textbook. We will present the basic terminologies, core assumptions, standard analytical frameworks, methodologies and techniques used in economics, and discuss its research object of the market system, its connection with ancient Chinese economic thought, as well as some key points.

Methodologies and techniques for studying economics include the following: providing benchmarks, establishing reference systems, setting up studying platforms, developing analytical tools, conducting positive and normative analyses, mastering the basic requirements of learning economic theory, understanding the role of economic theory, clarifying necessary and sufficient conditions for a statement, elucidating the role of mathematics and statistics in economics, and becoming familiar with the conversion between economic and mathematical languages.

1.1 Economics and Modern Economics

1.1.1 What is economics about?

To learn economics well, one must first know its definition and understand its nature, connotation, scope, and concerns.

Economics is a social science that studies how people interact with value in the face of resource scarcity and/or information asymmetries. Specifically, it studies economic behavior and phenomena, and how rational individuals (agents, households, firms, nations, organizations, and government agencies) make trade-off choices with limited resources.

In fact, economics could only come into being due to the fundamental inconsistency and conflict that exists between resource scarcity and individuals' unlimited desires (or wants). The core idea is that individuals, who are under the basic constraint of limited resources (i.e., limited information, capital, time, capacity, freedom etc.) and driven by unlimited desires, must make trade-off choices in resource allocation to make optimal utilization of limited resources to maximize satisfaction of their needs.

As a discipline of social science, economics investigates the problem of choices based on logical analysis and scientific tools, and establishes itself via systematic exploration of the specific topic of choice. Such exploration not only involves the building of theory, but also provides analytical tools for the testing of economic data.

1.1.2 Four Basic Questions in Economics

For any economic system, regardless of whether it is a planned economy, wherein the government plays a decisive role, a free economy, wherein the market plays a decisive role, or a semi-market and semi-planned mixed economy, wherein the state-owned economy plays a leading role, all face the following four basic questions regarding the allocation of resources:

- (1) What should be produced, and in what quantity?
- (2) How should the product be produced?
- (3) For whom should it be produced, and how should it be distributed?
- (4) Who makes the decisions?

Although these questions must be answered in all economic systems, different economic institutional arrangements provide varied answers. Whether

an institutional arrangement can effectively resolve these problems depends on whether it can effectively deal with the issues induced by information and incentives, and result in efficient or equitable allocations of resources.

Two basic economic institutional arrangements have been used in the real world:

- (1) The institutional arrangement of planned economies: All of the four questions are answered by the government, who determines most economic activities and monopolizes decision-making processes and all sectors. The government makes all decisions on market access, product catalogue, infrastructure investment allocation, individual job assignment, product pricing and wages, and it bears all of the risk.
- (2) The institutional arrangement of market economies: Most economic activities are organized through the free market. The decisions about what to produce, how to produce and for whom to produce are mainly made by decentralized firms and consumers, and the risk is borne by individuals.

While almost every real-world economic system exists somewhere in between these two extremes, the key factor is which extreme is in the dominant position. Due to the pursuit of personal interests and the presence of private or incomplete information, the fundamental flaw of the planned economic system is that it cannot effectively resolve problems induced by information and incentives, which in turn results in inefficient allocation of resources. On the other hand, the free-market economic system provides a viable solution in these respects in most situations. This is the fundamental reason why countries that once adopted a planned economic system inevitably failed, and why China carried out market-oriented reforms and strove to have the market play the decisive role in resource allocation.

1.1.3 What is Modern Economics?

Modern economics, which has developed rapidly since the 1940s and was constructed on the basic recognition of individuals pursuing their self-interest,

systematically studies individuals' economic behavior and economic phenomena by intensively using mathematical tools and adopting scientific methods for rigorous thinking. Specifically, it makes historical and empirical observations of the real world, utilizes the observations towards the formation of theory through rigorous logical analysis, and then again tests the theory in the continuing real world. As a consequence, it is a branch of science equipped with a scientific analytical framework and research methodology. This systematic inquiry not only involves the form of theory, but also provides analytical tools for testing economic data. For brevity, modern economics is simply termed as economics. In the following we will use the both terms interchangeably.

Scientific economic analysis, especially aimed at studying and solving major practical problems affecting the overall situation, is inseparable from **“three dimensions and six natures”**, among which the **“three dimensions”** are **“theoretical logic, practical knowledge, and historical perspective”** and the **“six natures”** are **scientific, rigorous, realistic, pertinent, forward-looking, and thought-provoking**. Since social economic issues generally cannot be studied by only using real society and performing experiments on it, we need not only theoretical analysis with inherent logical inferences, but also empirical quantitative analysis or tests with appropriate tools, such as statistics and econometrics. However, only using theory and practice is insufficient, and may cause shortsightedness, because the short-term optimum does not necessary equate to the long-term optimum. As a consequence, historical comparisons from a broad perspective are also requisite for gaining experience and drawing lessons. Indeed, only through the three dimensions of **“theoretical logic, practical knowledge, and historical perspective”** can we guarantee that its conclusions or reform measures satisfy the **“six natures”**. Therefore, the **“three dimensions and six natures”** are indispensable.

Indeed, all knowledge is presented as history, all science is exhibited as logics, and all judgment is understood in the sense of statistics. This is why Joseph Schumpeter (see Section 2.12.2 for his biography) asserted that the difference between an economic scientist and a general economist lies in whether he or she adopts the following three elements when conducting

economic analysis: the first element is theory for logical analysis; the second is history for historical analysis; and the third is statistics for empirical analysis with data.¹

For theoretical innovations and practical applications, it is of critical importance to correctly understand and master general knowledge of economics and the content of this textbook. It is useful for studying and analyzing economic problems, interpreting economic phenomena and individuals' economic behavior, setting up goals, and identifying the direction of improvements. More importantly, with the support of comparative analysis from the historical perspective and quantitative analysis based on data, we can draw reliable conclusions of inherent logic and make relatively accurate predictions through rigorous inference and analysis.

Economics — a key part of social sciences, is referred to as the “crown” of social sciences due to its extremely general analytical framework, research methods, and analytical tools. Its basic ideas, analytical framework, and research methodologies are potent for studying economic problems and phenomena that occur in different countries, regions, customs and cultures, and can be applied to almost all social sciences. It can even be beneficial if one strives to be a good leader with strong leadership ability, management and work ethic. Indeed, it is lightheartedly referred to as “economic imperialism” or an “omnipotent discipline” due to a bunch of influential works by Gary S. Becker (1930-2014, see Section 13.7.2 for his biography), who applied economic analysis to the entire spectrum of human behavior, including areas previously considered more or less the exclusive domain of sociology, psychology, criminology, demography and education.

1.1.4 Economics vs. Natural Science

There are three major differences between economics and natural science:

(1) Economics studies human behavior and needs to impose certain be-

¹In his inaugural speech titled “Science and Ideology” when assuming the position of President of the American Economic Association in 1949, Schumpeter pointed out that “Science is knowledge processed by special skills. Economic analysis, i.e., scientific economics, involves skills of history, statistics, and economic theory.” See Schumpeter, Joseph A. (1984). “Science and Ideology” in Daniel M. Hausman, eds., *The Philosophy of Economics*, Cambridge: Cambridge University Press, 260-275.

havioral assumptions; whereas, natural science, in general, does not involve the behavior of human beings (of course, such distinction is not absolute; for example, biology and medicine sometimes involve human behavior. However, these involvements are not from the perspective of rationality, while economics considers human behavior primarily from the perspective of utilitarianism). Once individuals are involved, information is highly incomplete and private (asymmetric) and easy to obscure because their behavior is unpredictable or they reveal information strategically, making it highly challenging to deal with.

(2) In the discussion and study of economic problems, positive analysis of description and normative analysis of value judgment are both needed. As people possess dissimilar values and self-interests, controversies frequently emerge, while natural science generally makes descriptive positive analysis only and the conclusions can be verified through experiments or practice.

(3) Society cannot be simply experimented upon or subjected to tests to form conclusions in economics because policies have broad impacts and large externalities. However, this does not constitute a problem for almost all branches of natural science.

These three differences may make the study of economics more complex and harder. In order to study and solve practical economic problems, one must start from the reality, combine theory with practice, establish the overall and systematic thinking, and adhere to the comprehensive governance concept with general equilibrium analysis as the core, rather than simple controlled experiment (although it is the first step of scientific research). It is essential to adopt the research methodology of “three dimensions and six natures” mentioned above.

1.2 Two Categories of Economic Theory

Modern economic theory is an axiomatic way to study economic issues. Similar to mathematics, it relies on logic deductions from presupposed assumptions. It further consists of assumptions/conditions, analytical frameworks and models, and conclusions (interpretations and/or predictions).

Since these conclusions are strictly derived from the assumptions and analytical frameworks and models used, it constitutes an analytical method with inherent logic. This analysis method is highly advantageous for clearly elucidating the problem and can avoid unnecessary complexities and disputes. Economics aims to explain and evaluate observed economic phenomena and make predictions based on economic theory.

1.2.1 Benchmark Theory and Relatively Realistic Theory

Economic theory can be divided into two categories by function. One is benchmark economic theory, which provides various benchmarks or reference systems², which is relatively remote from reality and deals with ideal situations. Parts II-IV of the textbook provide such benchmark theories. The second category is a relatively realistic theory that aims to solve practical issues, so that assumptions are closer to reality, which are usually modifications to the benchmark theory. Parts V-VII provide such relatively applied theories. As such, both types of theories are essential, and can be used to draw logical conclusions and make predictions. In addition, a progressive and complementary relationship of development and extension exists between these two categories. The second category of realistic theories is developed by revising the first category of benchmark theories as the reference frame, thus making the theoretical system of economics complete and proximal to the real world.

The benchmark theories are largely built on the economic environment of mature market economies and ideal situations. Their great significance should not be underestimated, misunderstood, or denied. They have demonstrated their critical importance in at least two aspects.

Firstly, although theoretical results of this category do not exist and cannot be realized in practice, they do play a critical role in providing guidance, orientation, and benchmarks. When we tackle a problem, it is necessary to first determine what to do and whether it should be done, and then proceed to the question of how to do it. Benchmark theories answer

²We will come back to discuss the role of the benchmark or reference system in more detail.

the question of what to do, or provide the direction and goals of improvement towards the ideal situation. Although it is the case that sometimes only a relatively better result can be achieved, the optimal outcome can be approached through the process of continually comparing the results with the benchmark or reference system. This is why it is correctly claimed that it is only through comparing our results with the best and learning from the best, can we perpetually improve. Therefore, the benchmark theory provides necessary standards for judging what is better and whether it constitutes the right direction, without which what we are doing may not be moving us towards our goals at all. .

Secondly, it establishes the necessary foundations for developing the other category of realistic theories. Any theory, conclusion, or statement can only be considered relatively; otherwise, there will be no basis for analysis or evaluation. It is for this reason that benchmark theories are required. This is true for both physics, which is a natural science, and economics, which is a social science. For instance, a world with friction is relative to a world with no friction, information asymmetry is relative to information symmetry, monopoly is relative to competition, technological progress and institutional changes are relative to technological and institutional lock-in, etc. Consequently, we must first develop the benchmark theory under rather ideal situations. This is analogous to basic laws and principles in physics, which only hold under an ideal situation without friction, and do not exist in reality, but nevertheless remain fundamentally important because they provide requisite benchmarks for solving physics problems in reality. Similarly, to study real economic behaviors and phenomena, which include “friction” , it is necessary to first be clear about the ideal situation without “friction” , and then use it as a benchmark and reference system. Indeed, the rapid development of economics would be impossible without benchmark economic theories.

As an important part of economics, neoclassical economics assumes the regularity conditions of complete information, zero transaction costs, and convexities of consumer preference and production sets, and thus falls into the first category of benchmark economic theory. Neoclassical economics considers ideal situations; although there is no artificially designed so-

cial goal, it contends that, as long as individuals are self-interested, the market of free competition will naturally lead to the efficient allocation of resources. This is regarded as a rigorous statement of the “invisible hand” proposed by Adam Smith (1723-1790, see 1.17.1 for his biography). It is thus set as the reference system for us to determine the direction and goals for reforms in order to improve the economic, political, and social environment, establish the competitive market system, and let the market play a decisive role in the allocation of resources. The boundary conditions for the market to work well also inform us about when the market will fail, at which time the government will need to step in and play a guiding role.

One may assert that an ideal reference system is far removed from the real economy, and thus deny the role of neoclassical economics and reject the instructive role of economics in economic reform. This, however, constitutes a serious misunderstanding, as it fails to acknowledge that the great gap between reality and the benchmark/reference system only shows the necessity for a nation, such as China, to implement market-oriented reforms and to continuously improve the efficiency of resource allocation. This kind of opinion, which refutes the role of benchmark theory, is similar to the denial of physics by a junior high school student who has just learned several formulas of Newton’s three laws and criticizes them for postulating conditions that are totally dissimilar from those in the real world. In these cases, however, the role of benchmark theory is not understood correctly. Indeed, without the benchmark theory in physics concerning free fall and uniform motion, how could we know the magnitude of frictional force so that we could construct a house that is stable? Furthermore, how could we determine how much frictional force should be overcome to solve problems regarding the taking-off and landing of airplanes or the launching of satellites? It is clear that, without benchmark theory, applied physics cannot be developed. The study of economics follows the same logic, and thus we need directions, structures, goals, and certain fundamentals, which is especially the case for implementing reforms in transitional economies.

To facilitate reforms for transitional economies, such as China, goals must be established, and thus benchmarks and reference systems are required for the reform to orientate itself. For the social economic develop-

ment of a country, it is necessary to transcend rational thinking, theoretical discussion and theoretical innovation, and determine the direction and goals of the reform in the first place. Moreover, it must be acknowledged that the fundamental institutions that determine the rule of collective decision-making, legal systems, strategies and policies play decisive roles in this process. If basic institutions of the rule of law, legal systems, politics, economy, society, and culture that concern a nation's path of development and long-term stability are not determined, economic theories in the present state-of-the-art may accomplish nothing, and may even have deleterious effects. In the discipline of economics, there is not a universal economic theory that is always applicable for all development stages, but rather there is an optimal one that is best suited for certain development stage under certain institutional environments.

For market-oriented reforms, it is natural and necessary to set neoclassical economic theory - especially economic theory of the first category, such as general equilibrium theory, that demonstrates the market as the optimal economic system - as a benchmark and the competitive market as a reference system for the orientation of these reforms. In this way, the results of the reforms will be continuously improved towards reaching the best possible outcome.

According to the economic environment defined by these benchmarks, reforms of deregulation and delegation for competition neutrality must be carried out, including ownership neutrality, liberalization, privatization, and marketization or reforms against government monopoly of resources and control of market access. In particular, the general equilibrium theory defines the applicable range of market mechanisms and identifies the environments under which the market may fail. With such knowledge, it is possible to identify the areas in which governments could establish rules and institutions to correct market failures.

Therefore, the study of economic problems and reforms, and especially determination of the direction of reforms, must commence from the benchmark of economics. Reforms that run counter to common sense in economics will end up in failure. The benchmark and reference system present the premises on which the market will lead to a more efficient allocation

and result in a more prosperous market economy, thereby revealing the direction of the reforms. The fourth part of this textbook stresses the Arrow-Debreu general equilibrium theory (Kenneth Joseph Arrow, 1921-2017, see Section 10.8.2 for his biography; Gerald Debreu, 1921-2004, see Section 11.9.2 for his biography) and the rational expectations theory of macroeconomics (referred to as neoclassical macroeconomics), both of which are standard theories of neoclassical economics and rigorously demonstrate that markets of free competition lead to efficient allocation of resources.

Alternative benchmarks and reference systems under different value judgements and goals could lead to markedly divergent outcomes. For example, when students regard a “pass” grade as their benchmark, the result is frequently a failure because a test comprised of questions is a random variable to the students. Just as Confucius and Sun Zi asserted, those who aim at the superior get the medium, those who aim at the medium get the inferior, and those who aim at the inferior lose entirely, which illustrates the critical importance of the choice of benchmark. Meanwhile, given that many benchmark theories are established under ideal conditions, they cannot be simply adopted to solve real problems in practice. In other words, a well-trained economist will not mechanically apply economic theories in the first category. However, concerning the economic reforms implemented in some transitional economies, we may come across some economists who do not analyze the dynamics in the transition, consider only developed countries but not developing countries, and neglect the objective law and constraints in special development stages. As such, both the target and the path selection and schedule arrangement for accomplishing the target are of crucial importance in practice. In addition, although some short-term economic and social problems might be resolved through promoting growth and development per se, market-oriented reforms should not be delayed; otherwise, an emerging economy is likely to be trapped in the transitional status.

It should be pointed out that a benchmark economic theory is established under an exogenously given economic environment that constitutes a relatively ideal situation. In so doing, we are able to address the key issues and draw some benchmark conclusions; otherwise, no question can be

scientifically discussed if we do not control for many factors. Therefore, in neoclassical theories and numerous other economic theories, an economic environment that comprises basic institutions, individual preferences and production technologies must be exogenously-given. Elaborating further, if achieving the market system under the ideal situation is our goal, it is logical to set it as a given institutional arrangement in order to thoroughly elucidate its desirable properties. However, this does not mean that economics only investigates situations in which the institution is given. In fact, many theories in economics are specialized in addressing how the economic environment changes, such as the study of institutional evolution, economic transition, endogenous preferences and technological progress. As such, one should not misinterpret economics as a sort of discipline that provides only benchmark economic theories established under ideal situations.

Theories in the second category constitute relatively more realistic economic theories that aim how to solve practical problems, and are constructed on presupposed assumptions that more closely approach reality and are modifications of the benchmark theories. According to their functions, they can be further divided into two types: the first kind provides an analytical framework, method, or tools for solving practical problems, such as game theory, mechanism design theory, principal-agent theory, auction theory, matching theory; and the second kind offers specific policy suggestions, such as the Keynesian theory, rational expectations theory, and growth theory in macroeconomics.

1.2.2 The Domain and Scientific Rigor of Economics

It can be seen from the definition of the above two categories of economic theories, modern economics is a highly inclusive and open discipline in dynamic development, which far exceeds the scope of neoclassical economics. Through weakening the assumptions of benchmark theories and standardized axiomatic formulation of descriptive theories, economics continuously develops the second category of economic theories, which grants itself great insight, explanatory power, and predictability. In the author's

opinion, as long as a study involves rigorous logical analysis (not necessarily using mathematical models) and adopts rationality assumption (bounded rationality assumption included), it falls into the category of (modern) economics.

Modern economics originated from classical economics, which was developed based on integration of Adam Smith's work by Thomas Robert Malthus (1766-1834, see Section 4.6.1 for his biography) and David Ricardo (1772-1823, see Section 1.17.2 for his biography), including not only benchmark theories, such as neoclassical marginal analysis economics established by Alfred Marshall (1842-1924, see Section 3.11.1 for his biography) and Arrow-Debreu's general equilibrium theory, but also many more realistic economic theories. For instance, the new institutional economics by Douglass C. North (1920-2015, see Section 5.5.1 for his biography) and mechanism design theory by Leonid Hurwicz (1917-2008, see Section 16.10.2 for his biography) have both developed neoclassical theory in a revolutionary manner. Specifically, while neoclassical theory takes institutions as given, North and Hurwicz endogenized institutions, viewing them as changeable, shapeable and designable, and thus formulated various institutional arrangements by complying with human nature for different environments. Indeed, they have both become crucial components of economics. Furthermore, the development of new political economics has, to a large extent, borrowed the analytical methods and tools of the second category of economic theory.

It is important to note that, because theories of the second category strive to develop analytical frameworks, methods, and tools for solving practical problems and offer specific policy recommendations largely based on a mature modern market system, their application must be approached cautiously. In fact, every rigorous economic theory in modern economics possesses a self-consistent logic and consequently must have boundaries and scopes within which they are applicable. This is true for not only original theory but also analytical tools, and for not only the first category of theory that offers benchmarks or reference systems but also the second category of theory that aims to solve practical economic problems. As a consequence, much mathematics is frequently needed, which incurs a common criticism of economics' overemphasis on model details and increas-

ingly heavy involvement of mathematics and statistics, making economic questions and conclusions per se even more opaque and challenging to understand.

The major reason that modern economics uses a substantial amount of mathematics and statistics is that economists must scientifically (both qualitatively and quantitatively) identify the applicable scope of an economic theory, which is especially the case when proposing economic policies based on the economic theory. Once a theory is adopted for making policy, great negative externalities may come into being if the boundary conditions of the theory are not known. In particular, it is the economists who make policy proposals, rather than the policy-makers and the public, that must have a good knowledge of the details or premises of a rigorous theoretical analysis. To this end, mathematics is needed to thoroughly identify the boundary conditions and applicable scopes of economic theories; simultaneously, economists equipped with the knowledge of mathematics can lay a strong foundation for continuous theoretical innovations. Moreover, the application of a theory or the formulation of a policy will usually require the use of statistics and econometrics for quantitative analyses or empirical tests.

In most cases, as real society cannot be simply used for experiments, a larger historical perspective is useful for viable vertical and horizontal comparisons. In addition, many superfluous disputes can be avoided in the exploration and discussion of certain questions. Hurwicz, for example, believed that the biggest shortcoming of traditional economic theories is the imprecise explanation of concepts, while the greatest significance of the axiomatic method lies in its formalization of the theory, providing a commensurable research paradigm and analytical framework for both discussion and criticism.

As a result, as the basic theoretical foundation for market economies, economics relies heavily on the introduction of research methodology and analytical framework of natural sciences to study social economic behaviors and phenomena. Indeed, using mathematical models as basic analytical tools, it stresses the inherent logic from assumptions and derivations to conclusions, along with the use of statistics, econometrics and comput-

er simulations for data-driven empirical research, laboratory experimental research, as well as field research. As such, comparing to other humanities and social sciences, modern economics exhibits strong characteristics of positivism and pragmatism, and is more likely to have the flavor of natural sciences.

1.2.3 The Roles of Economic Theory

Economic theory has at least three roles.

The first role is to provide various benchmarks and reference systems to establish desired goals in order to create directions to be pursued for improvement. Through reforms, transitions and innovations guided by theory, the economy in the real world is driven increasingly proximal to the ideal state.

The second role is to be used to learn and understand the real economic world, and to explain economic phenomena and economic behaviors in order to solve real problems. Indeed, this is the major content of economics.

The third role is to be used to make logically inherent inferences and predictions. Practice is the sole criterion for testing truth, but not the sole criterion for predicting it. In many cases, problems may still arise if only historical examination and existing data are employed for economic prediction, and thus theoretical analysis with inherent logic is imperative. Through logical analysis of economic theory, it is possible to make logically inherent inferences and predictions on possible outcomes under given economic environments. In so doing, we can solve real economic problems in a better way. As long as the pre-assumptions in a theoretical model are roughly met, scientific conclusions can be obtained and essentially correct predictions and inferences can be made accordingly, so that we may know the outcomes. For instance, the theoretical inference that a planned economy is unfeasible proposed in 1920s by Friedrich Hayek (1899-1992, see Section 2.12.1 for his biography) possesses this kind of insight. A good theory can deduce logically inherent results without requiring social experiments, which can somehow overcome the shortcoming that economics cannot carry out experiments on real society to a great extent. What is necessary,

however, is to check whether the assumptions made on economic environments are reasonable (experimental economics, which has become popular in recent years, is mainly engaged in fundamental theoretical research, such as testing individual behavioral assumptions). For example, we are not allowed to issue currency just for examining the relationship between inflation and unemployment. Similar to the case of astronomers and biologists, most of the time economists can only rely on existing data and phenomena to test and revise theories.

Of course, as indicated above, it would not be helpful to exaggerate the role of economic theory, and expect it to solve key and fundamental problems. What is fundamental, key, and decisive is the basic constitution and institution that determine a nation's fundamental path of development. If the underlying system governing the direction of the country and long-term prosperity in politics, economy, society, and culture is not yet built, the application of economic theory may even lead to deleterious results.

1.2.4 Microeconomic Theory

A notable feature of microeconomic theory is that it sets up theoretical models for economic activities of self-interested individuals, especially in market economies, conducts rigorous analysis and examines how the market works on such a basis.

Microeconomics deals with the core issue of pricing. It focuses on such questions as: which factors affect pricing? Do enterprises have pricing power? How can an enterprise get the power of pricing? How can an enterprise set the optimal price? To elucidate the answers to such a large issue, it is necessary to study the demand, supply, characteristics and functions of the market, and pricing in all kinds of markets and economic environments. As a result, microeconomics is also called price theory.

Microeconomics constitutes the core of economics and the theoretical foundation of all branches of economics. It enables us to employ simplified assumptions for in-depth analyses of various aspects of the complex world in order to get some useful insights. It also assists us to extract the most useful information from unrelated entities and consider various issues us-

ing the method of economics to develop explanations and predictions that conform to reality. It is in this sense that all other branches of economics, such as macroeconomics, finance, applied economics, etc., call for support from microeconomic theory.

1.3 Economics and Market System

A main purpose of economics is to investigate the objective laws of market and individuals' (e.g., consumers and firms) behavior in the market. Specifically, it examines how self-interested individuals coordinate their economic activities and make optimal choices in the market, how the market allocates social resources, how economic stability and sustainable growth are achieved, etc. Therefore, for the purpose of studying economics comprehensively, one should possess a general understanding of the functions and advantages of modern market mechanisms.

1.3.1 Market and Market Mechanism

Here, we briefly introduce the operation and basic functions of the market and how the market coordinates individuals' economic activities without requiring excessive participation or intervention of the government.

Market: The market constitutes a modality of trade in which buyers and sellers conduct voluntary exchanges. It refers not only to the location where buyers and sellers conduct exchanges, but also to all forms of trading activity, such as auction and bargaining mechanisms.

When studying microeconomics, it is crucial to keep in mind that any transaction in the market has both buyers and sellers. In other words, for a buyer of any good, there is a corresponding seller. The final outcome of the market process is determined by the rivalry of relative forces of sellers and buyers in the market. Three forms of competition exist in such a rivalry: consumer-producer competition, consumer-consumer competition, and producer-producer competition. Throughout this textbook, readers will find that the bargaining position of consumers and producers in the market is circumscribed by these three sources of competition in economic

transactions. Competition in any form is like a disciplinary mechanism that guides the market process and has varied impacts on different markets.

Market mechanism: The market mechanism or price mechanism is an economic institution in which individuals make decentralized decisions guided by price. It is worth noting, however, that this is usually a narrow definition of market mechanism. In fact, a mature market mechanism or market system constitutes the set of all systems and mechanisms closely related with the market (including the system of market laws and regulations). As a form of economic organization featuring decentralized decision-making, as well as voluntary cooperation and voluntary exchange of products and services, it is one of the greatest inventions in human history and by far the most successful means for human beings to solve their economic problems. Indeed, the establishment of the market mechanism is not a conscious, purposeful human design, but rather a product of the natural process of evolution. In the opinion of Hayek, market order is a spontaneous order of the economy which has evolved through long-term choices and processes of trial and error. The emergence, development, and further extension of economics are mainly based on the study of the market system. In aggregate, the operation of the market appears wonderful and beyond comprehension. It is genuinely awe-inspiring that, in the market system, decisions on resource allocation are independently made by producers and consumers who pursue their own interests under the guidance of market price without the imposition of any command or order. The market system unknowingly solves the previously mentioned four basic questions which must be faced by all economic systems: what to produce, how to produce, for whom to produce, and who makes the decisions.

Under the market system, firms and individuals make the decisions on voluntary exchange and cooperation. Consumers seek maximal satisfaction of their demands, while firms pursue profits. In order to maximize profits, firms must have meticulous plans for the most efficient utilization of resources. In other words, for resources with similar usage or quality, firms will choose the ones with the lowest possible cost. Although “making the best use of everything” may differ in meaning from the point of view of firms and that of the economy, price makes them related which,

as a result, harmonizes the interests of firms and those of the entire society, and leads to the efficient allocation of resources. *The price level reflects the supply and demand of resources in the economy and the degree of scarcity of resources.* For example, in the case of an inadequate timber supply and ample steel supply in the economy, timber will be expensive while steel will be inexpensive; consequently, to reduce expenses and make more profits, firms will strive to use more steel and less timber. In doing so, firms do not take the interests of society into consideration, but the outcome is precisely in accordance with social interests, and it is the role of resource price that achieves this. Resource price coordinates the interests of firms and those of the overall society, and solves the problem of how to produce. The price system also guides firms to make production decisions in the interest of society. Indeed, it is the consumer who has the final say about what to produce. Firms only need consider how to produce products that have a higher price. Yet, in the market system, the price level exactly reflects social needs. For instance, poor harvests and the corresponding rising grain price will encourage farmers to produce more grain. As such, profit-pursuing producers “come to the rescue” under this guiding force, and the problem of what to produce is solved. Moreover, the market system also addresses the problem of how to distribute products among consumers. If a consumer really needs a shirt, he or she will offer a higher price for it than will others. Profit-pursuing producers will certainly aim to sell the shirt to the consumer who offers the highest price. In this way, the problem of for whom to produce is solved. Furthermore, all of these decisions are made by producers and consumers in a decentralized manner, and thus the problem of who makes the decision is also resolved.

As such, the market mechanism easily coordinates seemingly incompatible individual interests and public interests. As early as over 200 years ago, Adam Smith, the Father of modern Economics, identified the harmony and wonder of the market mechanism in his masterpiece, *The Wealth of Nations* (Adam Smith, 1776). He regarded the competitive market mechanism as an “invisible hand”. Under the guidance of this invisible hand, individuals solely pursuing their own interests move towards a common goal, and thus achieve maximization of social welfare:

“... every individual necessarily labours to render the annual revenue of the society as great as he can. He generally, indeed, neither intends to promote the public interest, nor knows how much he is promoting it ... he intends only his own gain, and he is in this, as in many other cases, led by an invisible hand to promote an end which was no part of his intention. Nor is it always the worse for the society that it was no part of it. By pursuing his own interest, he frequently promotes that of the society more effectually than when he really intends to promote it.”

Smith meticulously examined how the market system combines the self-interestedness of individuals with social interests and the division and cooperation of labor. The core of Smith's paradigm is that, if the division of labor and exchange of goods are totally voluntary, then exchange will only occur when we realize that the result of the exchange is mutually beneficial to both parties of exchange. Indeed, as long as there are benefits, individuals driven by self-interest will cooperate voluntarily. In other words, external pressure is not a requisite condition for cooperation. Even if language barriers exist, as long as mutual benefits can be obtained, exchange can take place. In most cases, the market mechanism works so harmoniously that individuals are not even cognizant of its existence. With the metaphor of “the invisible hand”, Smith highlighted the importance of voluntary cooperation and voluntary exchange in economic activities. It is worth noting, however, that the claim that the welfare of society can be achieved by the market system is not yet recognized universally, nor was it during Smith's lifetime. The Arrow-Debreu general equilibrium theory, which will be discussed in this textbook, contains a formal statement of Smith's “invisible hand”, and rigorously demonstrates how the market of free competition can lead to the maximization of social welfare, and proves the optimality of the market in allocating resources.

1.3.2 Three Functions of Price

As discussed above, the normal operation of the market system is realized via the price mechanism. As analyzed by Milton Friedman (1912-2006, see Section 4.6.2 for his biography), the Nobel Laureate in Economics, price performs three functions in organizing rapidly changing economic activi-

ties involving hundreds of millions of individuals:

- (1) Transmitting information: price transmits production and consumption information in the most efficient manner;
- (2) Providing incentive: price provides incentives for individuals to carry out consumption and production in an optimal way;
- (3) Determining income distribution: endowment of resources, price, and the efficiency of economic activities determine the income distribution.

In fact, as early as the Han dynasty of China, Sima Qian (a Chinese historian of the Han dynasty who is considered the father of Chinese historiography) observed and summarized the law of commodity price fluctuation, stating that, for all commodities, “when an article has become extremely expensive, it will surely fall in price, and when it has become extremely cheap, then the price will begin to rise” . Therefore, in the effort to become rich, individuals will make good use of this law to “look for a profitable time to sell” .

Function 1 of Price: Transmitting Information

Price guides the decision-making of participants and transmits information about changes in supply and demand. When the demand for a certain commodity increases, sellers will notice the increase of sales and thus place more orders with wholesalers. These wholesalers will then place more orders with manufacturers, causing the price to rise, and the manufacturers will then invest more factors of production to produce this commodity. In this way, the message of increasing demand for this commodity is received by all related parties.

The price system also transmits information in a highly efficient manner, and it only transmits information to those who need it. Moreover, the price system not only transmits information, but also produces a certain incentive to ensure the smooth transmission of information so that information will not remain in the hands of those who do not need it. Those

who transmit information are internally motivated to look for those who are in need of that information, while those who need information are internally driven to acquire information. For example, ready-to-wear apparel manufacturers are continually striving to obtain the best kind of cloth and looking for new suppliers. Meanwhile, cotton cloth manufacturers are also always reaching out to clients to attract them with the high quality and inexpensive price of their products by various means of marketing and publicity. Those who are not involved in such activities will surely be indifferent to the prices and supply and demand of cotton cloth. The mechanism design theory, as discussed in Chapter 18 of this textbook, will demonstrate that the competitive market mechanism is the most efficient mechanism in the utilization of information because it requires the least amount of information and thus the lowest transaction cost. In the 1970s, Hurwicz and his collaborators had already proved that, for the neoclassical economic environment of pure exchange, no other economic mechanism can achieve as efficient resource allocation using less information than does the competitive market mechanism.

Function 2 of Price: Providing Incentive

Price can also provide incentives, so that individuals will react to changes of supply and demand. When the demand of a commodity decreases, an economic society should provide certain incentives so that manufacturers of the commodity will increase production. One of the advantages of the market price system is that prices not only transmit information, but also provide incentives for individuals to respond to the information voluntarily out of self-interest, so that consumers are driven to consume in an optimal way while producers are driven to conduct production in the most efficient manner. The incentive function of price is closely related to the third function of price: determining income distribution. As long as the increased gain brought by increased production (i.e., marginal revenue) exceeds the increased cost (i.e., marginal cost), producers will continue to increase production until the two are equal, and thus maximum profits are realized.

Function 3 of Price: Determining Income Distribution

In a market economy, an individual's income depends on the resource endowment that he or she owns (e.g., assets, labor) and the outcomes of economic activities in which he or she is engaged. Concerning income distribution, it is always desirable to separate price's function of income distribution from its other functions, in the aim of attaining a more equal income distribution without affecting the other two functions of transmitting information and providing incentive. The three functions, however, are closely related and complementary. Indeed, once price no longer influences income, its functions of transmitting information and providing incentive will disappear. If one's income does not depend on the price of labor or commodities that he or she offers to others, why would he or she bother to acquire the information of price and market demand and supply and respond to such information? If one receives the same income irrespective of how much work he or she performs, then who would strive to do an excellent job? If no benefits are given for innovation and inventions, who would be willing to invest effort in this endeavor? If price has no impact on income distribution, it will also lose its other two critical functions.

1.3.3 The Superiority of the Market System

The modern market system is a sophisticated and delicate economic institution that has emerged, gradually taken shape, and been constantly improved in the long-term evolution of human society. The fundamental and decisive role of the market mechanism in resource allocation is the key to the market economy's capacity for optimal resource allocation. Optimality here has the same meaning as the Pareto optimality (efficiency) proposed by Vilfredo Pareto (1848–1923, see his biography in Section 11.9.1) which will be discussed in Chapter 11 in more detail. It means that, under given resource constraints, no other feasible allocation of resources exists that can make some participants better off without harming the welfare of others. Even though Pareto optimality fails to consider the issue of social fairness and justice in terms of equality, it provides a basic criterion of whether or not a resource is wasted concerning social benefit for an economic system,

and evaluates social economic effects regarding feasibility. According to this criterion, if an allocation is not efficient, space exists for such allocation to be improved.

Two fundamental theorems of welfare economics, which we will discuss in Part IV on general equilibrium theory, provide a rigorous formal expression of Adam Smith's assessment. As the formal statement of his 'invisible hand', the theorems prove that the free competitive market can maximize social welfare and achieve market optimality in terms of resource allocation. The First Fundamental Theorem of Welfare Economics demonstrates that when individuals pursue their own self-interest, and if economic agents have unlimited or locally non-satiable desire, the competitive market system can achieve Pareto-efficient allocation for economic environments with private divisible goods, complete information, and no externality. The Second Fundamental Theorem of Welfare Economics, on the other hand, proves that for neo-classical economic environments, any Pareto-efficient allocation can be achieved by reallocating initial endowments and competitive market equilibrium without the need to introduce other economic systems to replace the market mechanism. Precise statements and rigorous proofs of these theorems will be given in Chapter 11.

The economic core equivalence theorem in Chapter 12, from another perspective, proves that the market system can benefit social stability, and is optimal and unique in terms of resource allocation, being the result of natural selection with objective inherent logic regarding economic activities. The competitive market mechanism can not only lead to efficient allocation of resources, but also solve the problem of social stability and orderliness well. Moreover, it is the product of free and full competition. The basic connotation of the economic core is that, when the allocation of social resources possesses the core property, there will not be any coalition (i.e., a group of agents) that is dissatisfied with the allocation, and wants to improve their welfare by controlling and utilizing their own resources. In this sense, no powers or groups exist that pose a threat to society, and thus society will be relatively stable.

Under the fundamental fact of individuals' pursuit of self-interest, the economic core equivalence theorem reveals that: the equilibrium allocation

by competitive market mechanism has the core property; on the contrary, under some regularity conditions, such as monotonicity, continuity, and convexity (diminishing marginal rate of substitution) of preference, as long as individuals are given enough economic freedom (i.e., the freedom to cooperate and exchange voluntarily) and perfect competition, the outcome will be identical to that of competitive market equilibrium without establishing any institutional arrangements in advance, so they are equivalent. Therefore, the market system is not an invention, but rather an inherent economic rule and a spontaneous order, which is as objective and reliable as any law of nature. The policy implication of this conclusion is that the market should be allowed to fully play its role when the competitive market mechanism is able to attain optimal allocation. Indeed, it is only under the circumstances in which the competitive market is incapable will other mechanisms be designed to compensate for market failures.

Furthermore, the competitive market mechanism is not only optimal and unique concerning social stability maintenance and efficient resource allocation, but also effective in the transmission of information. In the 1970s, Hurwicz and his collaborators proved that, for neoclassical pure exchange economies, there is no other economic mechanism which can lead to such efficient allocation of resources using less information than the competitive market mechanism. In 1982, Jordan further proved that, in pure exchange economies, the market mechanism is the only mechanism that achieves efficient allocation using the least amount of information. Tian (2006) also demonstrated that this conclusion is true not only in pure exchange economies, but also for economies with production, and that the market mechanism is unique. Consequently, an important inference follows: irrespective of whether in a command planned economy, state-owned economy or mixed economy, the amount of information that is needed to realize the efficient allocation of resources is more than that in a competitive market mechanism, and thus those economic systems are not informationally efficient. This conclusion provides a key theoretical explanation for why China needs market-oriented economic reform and privatization of its state-owned economy. The uniqueness result of information efficiency will be explored further in Chapter 18.

Even though the market mechanism cannot perfectly solve the problem of social fairness manifested in the large income and wealth gap between the rich and the poor, as long as the government strives to provide a level playing field with equality of opportunity and equal value of resources for all individuals and allows the market to play its role instead of controlling it, equity and efficient resource allocation can be achieved by the market, as the fairness theorem in Chapter 12 indicates. The above statements and proofs about the optimality, uniqueness, and fairness of the modern competitive free market system in resource allocation and its contribution to social stability are all core aspects of the general equilibrium theory.

Joseph Schumpeter also discussed the optimality of the market mechanism from the perspective of how interactions (dynamic game) between competition and monopoly lead to innovation-driven growth. His innovation theory informed us that valuable competition is not merely price competition, but more importantly, competition in new commodities, new technologies, new markets, new supply sources, and new combinations of ideas, knowledge, and resources. As a consequence, the root of the long-term vitality of the market economy is innovation and creativity, which stems from entrepreneurship and entrepreneurs' constant, creative destabilization of the market equilibrium, which he refers to as 'creative destruction'. Profit-pursuing entrepreneurs and private economies are necessary to cultivate the soil for innovation, and to encourage and protect innovation.

In fact, competition and monopoly, like supply and demand, can form an astonishing unity of opposites through the power of the market, thus revealing the true beauty and power of the market system. Indeed, market competition and enterprise innovation are inseparable. If there is no competition, there will also be no motivation for innovation; this is what occurs in state-owned enterprises in state monopolies. It is the case, of course, that competition results in profit decline. Overall, the fiercer the competition becomes, the more rapidly corporate profits decrease. This provides enterprises with strong incentives to innovate in order to survive. Innovating enterprises may gain a monopoly position, which implies monopoly profits, which will attract more enterprises to participate in the competi-

tion. As a consequence, a repeated cycle of “competition → innovation → monopoly profit → competition” is established, in which market competition tends to achieve an equilibrium, but innovation disrupts it. The market continually goes through such cycles to inspire enterprises to pursue innovation. Through this dynamic process, the market maintains its vitality, and greater economic development and social welfare are obtained. Therefore, in order to encourage innovation, the government should strictly enforce laws regarding intellectual property rights protection.

While people may realize the importance of entrepreneurs and entrepreneurship, not all fully recognize the fundamental importance of the institutional basis for the emergence of entrepreneurs and entrepreneurship, and that innovation and development need institutional support. This is because entrepreneurs and entrepreneurship do not appear randomly, but rather entrepreneurship is derivative and superficial. Therefore, it must be built on a basic meta-institution, which necessitates a conducive institutional environment as a prerequisite. As such, Baumol (1990) extended Schumpeter's innovation theory, and argued that *innovation and entrepreneurship depend on institutional choice, and are therefore endogenous variables*. If the rules of the game that affect the choice of entrepreneurial behaviors are abnormal, innovation and entrepreneurship cannot achieve their full potential. Indeed, so far, three industrial revolutions have occurred in the world: the industrial revolution in Britain; the second industrial revolution led by the United States and Germany; and the third industrial revolution, which takes artificial intelligence manufacturing as the core. The occurrence and development of each industrial revolution are closely related to institutional innovation, which provides critical support for the smooth development of industrial revolutions.

For instance, in order to encourage competition and form positive externalities of technological innovation, anti-monopoly legislation should be enacted and enforced. In addition, protection of intellectual property rights should not last forever, but rather should be limited to a certain number of years so that they will not establish a perpetual oligopoly or monopoly. Therefore, technological innovations operate on the basis of institutional innovations. These two kinds of innovations behave like an action-reaction

pair. Specifically, a good institution can reduce the transaction costs of innovation, create conditions for cooperation, provide incentives for innovation, and facilitate internalization of the benefits of innovation. One goal of constructing a technological innovation system is to promote effective interaction and cooperation among innovative elements.

Innovation comprises transcending rules and regulations, which inevitably poses high risks. High-tech innovation, in particular, means high risks and an extremely low possibility of success for venture capital investment; when it succeeds, however, it brings considerable, and possibly parabolic, returns, which then attracts more investment. Nevertheless, it is impossible for state-owned enterprises to take such high risks due to the fact that they inherently lack the incentive mechanisms that would enable the assumption of such risks. In contrast, it is private enterprises that typically dare to take the most risk out of a strong incentive to pursue their self-interest, and consequently they are the most creative and innovative entities. Therefore, entrepreneurial innovation (not fundamental scientific research) largely takes place within the context of private business. In fact, Sima Qian, a major Chinese philosopher, also affirmed that competition and survival of the fittest constituted fully natural tendencies. He believed that it was not certain trades that were more likely to produce wealth, and that wealth was not exclusively attained by specific people. He asserted that a capable person would accumulate wealth; whereas, incapable people would forfeit it.

What should be noted is, with the emergence of innovation in financial technology and big data method, the deviation between the real economic situation and the ideal state will be decreased. Innovation will push the real market economy toward the ideal state of market economy described by Adam Smith, Friedrich Hayek, Kenneth Joseph Arrow, Gerard Debrue, and Ronald H. Coase (1910-2013, see Section 14.6.2 for his biography). Indeed, the market can be proven to be optimal, irrespective of whether it defines the role of the competitive market as the “invisible hand” à la Adam Smith, general equilibrium in the perfectly competitive market by Arrow and Debrue, the theory with the zero transaction cost in the perfectly competitive market by Coase, or the assertion that competition benefits inno-

vation from Schumpeter. The basic conclusion of these theories is that the perfectly competitive market leads to efficient allocation of resources and social welfare maximization. Of course, the perfectly competitive market merely provides a reference system or an ultimate goal, which means that the more competitive the market is, the better it is; and the more symmetric the information is, the better it is. Such a perfectly competitive market system, however, does not exist in reality because communication costs, transaction costs, and financing costs cannot be zero (although they may approach it).

With the Internet as a medium for finance and big data method, transaction costs are becoming increasingly diminutive. Due to the disruptive innovation and development of financial technology and big data method, to some extent, the perfectly competitive market, as considered by the first category of economic theory, is not just an ideal state but also tends to increasingly approach reality. In particular, financial technology and big data method will greatly reduce the cost of information communication in reality in order to make market economic activities closer to the ideal state of perfect competition and thus more efficient.

1.4 Government, Market, and Society

The theoretical conclusion concerning market optimality relies on an implicit assumption about the critical importance of fundamental institutions. In other words, there should be a mature governance structure to regulate the government, the market, and the society.

1.4.1 Three Elements of State Governance and Development

State governance involves three dimensions: **government, market, and society**. The market mechanism may give an incorrect impression that, in a market economy, one can do whatever one desires in the pursuit of self-interest; this, however, is not the case. In the world, there is no completely laissez-faire market economy that is totally independent of the government. A well-functioning market requires appropriate and effective integration

of government, market and society, which constitutes a three-dimensional structure of state governance. A completely laissez-faire market without governance and regulation is also not omnipotent. As we shall discuss in Parts V-VII of this textbook, the market frequently fails in certain circumstances, such as monopoly, unfair income distribution, polarization of rich and poor, externality, unemployment, inadequate supply of public goods and information asymmetry, thus resulting in inefficient allocation of resources and various social problems.

The three basic institutional arrangements of government, market and society are the *three elements of state governance and benign development*: 1) inclusive economic institution; 2) state capacity to plan and implement policies and laws; and 3) an inclusive and transparent civil society with democracy, the rule of law, fairness and justice. Either in short-term response or in long-term governance, the three are all indispensable and essentially necessary and sufficient conditions for a nation's sustainable and benign economic development, social harmony and stability, and long-term peace and stability. Indeed, the practice at all times and all over the world has repeatedly shown that all economic and social achievements or progress were due to the improvement of some aspects of these three elements, and the problems were inevitably caused by the lack of some of them. As such, these three elements are the most fundamental comprehensive governance elements to identify whether a state's governance system and capacity are good or not and whether it can cope with crisis in the short term and maintain stability in the long term. A modern state governance system must be a system that properly deals with the relationship between government and market and between government and society, so that each functions in its respective position and interacts effectively with one another.

Benign development and governance exhibit an inherent dialectical relationship, and must be accurately understood. Economic development primarily focuses on the improvement of a nation's hard power; whereas, governance stresses the construction of soft power. Of course, governance should be all-dimensional from different aspects, including governance systems of the government and the market, social equity and justice, culture, values, etc. How the relationship between the government and the

market and between the government and the society is handled frequently determines the effect of state governance and development. If they cannot be well balanced, a series of major problems and crises may occur, including poor development, an excessive gap between the rich and the poor, unequal opportunities, etc., preventing an inclusive market economy and a tolerant and harmonious society from coming into existence. In this way, in the logic of governance, there is a “good” kind and a “bad” kind of governance that will lead to a good or bad market economy and good or bad social norms, respectively. Therefore, governance should not be taken as being equivalent to rules, controls or regulations, or regarded as the opposition of development, making it difficult to attend to both governance and development simultaneously. To achieve and maintain an efficient market and harmonious society, a limited government should be built that is capable, accountable, effective and caring, leading towards a desired governance that features the principle of the rule of law.

1.4.2 Good Market Economies vs. Bad Market Economies

A market economy can be classified into “**good market economy**” and “**bad market economy**”. Whether it is good or bad depends on the system of state governance and whether the governance boundaries among the government, the market, and the society are clearly and appropriately delineated. In a good market system, the government enables the market to fully play its role, and in case of market failures, the government can take certain remediative actions. This does not mean that the government should directly intervene in economic activities, but rather the government is expected to enact appropriate the design of rules or institutions to correct market failures in order to achieve incentive-compatible outcomes so that individual interests and social interests are consistent. One of the most successful examples of institutional design is the enactment of the basic constitution of the U.S. at its founding, which made the U.S. become the most powerful nation in the world within approximately 100 years.

A good, inclusive, and efficient modern market economy should protect the private interests of individuals to the greatest extent through insti-

tutions or laws, and simultaneously limit and counterbalance the government and its public powers as much as possible. In this sense, it is a contractual and rule-of-law economy constrained by an agreement regarding commodity exchange, rule of market operation, and reputation. Under the constraints of individuals' pursuit of self-interests, of resources and of information asymmetries in an economic society, to build a strong state with prosperous people, individuals should first be endowed with private rights, the core of which includes the basic rights to survival, freedom of choice for one to pursue happiness, and private property rights. Through their participation in full competition, voluntary cooperation, and voluntary exchange of the market mechanism and their pursuit of self-interest, efficient allocation of resources and maximization of social welfare can be attained. Therefore, the modern market economy is established upon the basis of the rule of law, which works in two ways: first, it is of fundamental importance to restrict arbitrary government intervention in market economic activities; and second, it further supports and promotes the market in certain ways, including the definition and protection of property rights, enforcement of contracts and laws, maintenance of fair market competition, etc., in order to allow the market to play the decisive role in resource allocation and give full range to the three basic functions of price, i.e., transmitting information, providing incentives, and determining income distribution. In addition, a good market calls for good social norms, for one's pursuit of self-interest should exist on the premise of respect for others' pursuit of self-interest, and self-interest and fair competition function in parallel to one another with no conflict. The spirit of compromise and respect for others' standards of values constitute the foundation for the normal process of exchange.

On the other hand, in a bad market economy, in which there is a lack of adequate ruling and governance capacity in economic and social transitions, and the government is unable to provide sufficient public goods and services to compensate for market failures. The government's excessive economic activities result in public powers that are not effectively counterbalanced, property rights of state-owned enterprises that are not clearly defined, and the government that is involved in numerous rent-seeking and corrupt behaviors so that equity and justice are greatly diminished. This

breeds the so-called “**State Capture**” , which refers to the phenomenon that, by providing personal interests for government officials, economic agents interfere in decisions on laws, rules and regulations, and thus, without going through fair competition, they convert their personal interests to the basis of rules of the game of the whole market economy. This then leads to policies that produce high monopoly profits for specific individuals at the expense of enormous social costs and the decrease of government credibility. As a result, an inefficient balance in public choice continues over a long period of time. The behavior of striving for social and governmental resources by means of unfair rent-seeking instead of fair competition will not only produce market failures, but more importantly gradually result in bad social norms in the long term. These poor social norms produce a distortion of resource allocation and social values, the decline in moral values, an absence of good faith, the popularity of “false, big, and empty” words and deeds, the frivolity of society and increased factors of instability, which finally result in enormous explicit and implicit transaction costs. Some sociologists refer to such social state as “**social corruption**” , meaning that the social cells of the social organism are dead and experiencing functional failures.

Therefore, in the three-dimensional framework of government, market and society, the government, as an institutional arrangement with great positive and negative externalities, plays a vital role. Indeed, it can make the market efficient, become the impetus for economic development, assist to construct a harmonious society, and realize sustainable development. On the other hand, it may also make the market inefficient, lead to various social conflicts, offer tremendous resistance against the benign development of the society and economy, and exert deleterious social impacts. Although almost all countries in the world have adopted a market economy, a majority have not achieved sound and rapid development. Among numerous reasons for this, the most fundamental one is the lack of reasonable and clearly delineated governance boundaries between the government, the market and the society, so that there is over-playing, under-playing, and mis-playing of the government role. Only when the government appropriately tightens its omnipresent “visible hand” , and the functions

and governance scope of the government are appropriately and reasonably defined, can it be expected to define a proper and scientific governance boundary between the government, the market, and the society.

1.4.3 The Boundaries of Government-Market-Society

How can governance boundaries be best defined among the government, the market, and the society? The answer is to allow the market to do whatever it can do well, while the government does not participate in economic activities directly (however, it is necessary for the government to maintain market order and guarantee strict implementation of contracts and rules). Regarding what the market cannot achieve, or in circumstances in which it is not appropriate for the market to be involved, such as in cases affecting national security, the government can directly participate in economic activities. In other words, when considering the construction of an inclusive and transparent civil society and benign development of an economy, or when transforming government functions and innovating modes of management, the boundaries of the government, the market, and the society should be carefully considered. For instance, the government should exit from competitive sectors. Only in the case of market failure should the government solve problems by itself or be in cooperation with the market. However, the basic dictum is that the government should not directly intervene in economic activities, but instead enact conducive rules and institutions to correct market failures. Because individuals' pursuit of self-interest and private information are at the core of many economic activities, direct intervention in economic activities (e.g., large numbers of state-owned enterprises, and arbitrary restriction of market access and interference with commodity prices) would not frequently generate desired outcomes. In this respect, mechanism design theory can play a key role in making the market more efficient and solving the problem of market failures. In Hurwicz's opinion, "law-making by the U.S. Congress or other legislative bodies equals to designing new mechanisms" .

Under a modern market economy, the basic and sole functions of government can be distilled as "maintenance" and "service" , i.e., making

fundamental rules to ensure national security, social stability and economic order, as well as providing public goods and services. Just as Hayek asserted, the government has two basic functions: firstly, the government must be responsible for law enforcement and defense against its enemies; and secondly, the government must provide services that the market is unable to provide or unable to fully provide. He also stated that “it is indeed most important that we keep clearly apart these altogether different tasks of government and do not confer upon it in its service functions the authority which we concede to it in the enforcement of the law and defence against enemies.”³ This requires that, in addition to undertaking necessary functions, the government should separate its powers regarding market and society. Abraham Lincoln provided the following clear and incisively-defined functions of the government:

*“The legitimate object of government, is to do for a community of people, whatever they need to have done, but can not do, at all, or can not, so well do, for themselves—in their separate, and individual capacities. In all that the people can individually do as well for themselves, government ought not to interfere.”*⁴

Meanwhile, a good-inclusive-efficient modern market economy and state governance mode require an independent autonomous civil society with a strong ability to coordinate interests as an auxiliary informal institutional arrangement. Otherwise, the explicit and implicit transaction costs of economic activities would be prohibitive, and it would be very challenging to establish the most basic norm of social trust.

In summary, a reasonable and clearly defined governance boundary among the government, the market, and the society is a prerequisite for establishing a good-inclusive-efficient market economic system and achieving benign development that is characterized by efficiency, equity, and harmony. Of course, the transition to an effective modern market system often constitutes a long and arduous process. Due to various constraints, governance boundaries of the government, the market, and the society cannot be clearly defined in a single attempt, but rather a series of transitional institu-

³See Von Hayek F. *Law, Legislation, and Liberty* (Vol. 3). Chicago, IL: The University of Chicago Press, 1979, pp42.

⁴Abraham Lincoln's Quotes on Government (1854).

tional arrangements are frequently requisite. However, with the deepening of transitions, some transitional institutional arrangements may decline in efficiency, and may even degenerate into invalid institutional arrangements or negative ones. If governance boundaries of the government, the market, and the society cannot be timely and appropriately clarified while some temporary, transitional institutional arrangements (e.g., government-led economic development) function as permanent and ultimate institutional arrangements, it is then impossible to achieve an efficient market and an inclusive, transparent society. With the development of modern economics, its analytical framework and research methods play an essential role in the study of how to reasonably and clearly define governance boundaries among the government, the market and the society, and how to carry out comprehensive governance.

1.5 Comprehensive Governance by the Three Arrangements

As discussed above, in order to establish a well-functioning and efficient modern market system, it is necessary to coordinate and integrate the relationship of the three basic institutional arrangements, i.e., the government, the market and the society, in order to regulate and guide individuals' economic behavior and conduct comprehensive governance. The government, the market, and the society correspond precisely to the three basic elements of governance, incentive, and social norms in an economy, respectively. Mandatory public governance and formal institutional arrangements, such as market mechanism as an incentive scheme, as two elements of comprehensive governance which overlap, through long-term interactions, may assist to guide and form normative informal institutional arrangements, enhance the predictability and certainty of social and economic activities, and markedly decrease transaction costs. The informal institutional arrangement mentioned here is social norms and culture. The same as with enterprises, the first-class enterprise performs branding, in which its corporate culture plays the key role, the second-class enterprise devel-

1.5. COMPREHENSIVE GOVERNANCE BY THE THREE ARRANGEMENTS⁴³

ops technologies, and the third-class enterprise produces. Similarly, it is the position of the government that plays the crucial role in the establishment of good or bad social norms and market economy.

Expressed in another way, the three basic institutional arrangements for comprehensive governance are employed to “enlighten with reason, guide with benefits, and persuade with emotions”, all of which are realized and implemented mainly by the government, the market, and the society, respectively, on the level of state governance. To “enlighten with reason” means to work through the stimulus of legal principles and reason; to “guide with benefits” means to link up economic activities with revenue through the stimulus of rewards and penalties, thus becoming an incentive mechanism; and to “persuade with emotions” means to work through the stimulus of emotions and shared beliefs, as sometimes relationships, friendships and emotions, and especially shared beliefs and concepts, can assist to solve large problems, which as a kind of social culture will greatly decrease transaction costs.

1.5.1 Governance on Rules

Governance on rules, as the basic institutional arrangement and management rule, is about enforcement. The basic criterion for whether such rules and regulations shall be formulated is whether or not they facilitate a clear definition (according to the possibility of information transparency and symmetry), and whether the costs of information acquisition, supervision, and enforcement are high. If a regulation is too costly concerning supervision, it will not be feasible for enforcement. Protection of property rights, contract implementation, and appropriate supervision all call for relevant rules, which thus require a third party to oversee enforcement of those rules. The third party is then a government agency. In order to maintain market order, the role of the government is inevitable. As the government is also an economic agent, it functions as both a judge and a player, and thus it exerts an enormous influence. This requires procedures and rules to constrain the behavior of the government. Regulations on other economic agents and the market should be the opposite of this. Due to information

asymmetry, regulations in this regard should not be too detailed, so as not to interfere too much with the freedom of choice of economic agents.

Institutional arrangements that “enlighten with reason” and deter people with the threat of sanctions are similar to the thoughts of Legalism in ancient China. However, a major problem of Legalism was that it only aimed to govern economic agents, but applied no restrictions on the government. In this way, it is rule by law, but not rule of law. Moreover, this kind of institutional arrangement only considered cruelty in individuals’ struggle for power and profit, but neglected the influence of individuals’ consciences and emotions upon behavior. The employment of the heavy hand of Legalism without considering other institutional arrangements will frequently lead to coercive powers, and hinders the formation of a modern market economy.

1.5.2 Incentive Mechanism

Incentive mechanisms, such as market mechanisms, are inducing by their nature, have the widest applications, and are a main concern of this textbook. Due to information asymmetry and the high cost of information acquisition, as well as the fact that human nature, especially the nature of self-interest, can hardly be altered, specific operation rules need to mobilize individuals’ enthusiasm through incentive mechanisms, such as market mechanisms, to realize incentive compatibility in order to make individuals, who pursue their own interests, strive to achieve the aims of the mechanism designer, as well. As shown in Chapter 7 on repeated game theory, reputation and integrity, we will show that reputation and integrity under the market incentive mechanism are one kind of punishment incentive mechanism. Integrity is essential in engaging in business activities; this does not, however, mean that business owners behave with integrity out of their own will, but rather that they have no choice because, otherwise, they will be forced out of the market. It is also the case that integrity can save economic costs and lower transaction costs.

Institutional arrangements that “guide with benefits” are similar to the precepts of Taoism in ancient China. It asserts that individuals are all

1.5. COMPREHENSIVE GOVERNANCE BY THE THREE ARRANGEMENTS⁴⁵

self-interested and have a limited ideological realm, and these two assertions should be universally recognized. Although it is not difficult to stop “talking” about interests, in reality, it is challenging not to “value” interests. Taoism contends that “the law of nature is beneficial, but not harmful” , and emphasizes compliance with natural tendency and governance by non-interference. However, it neglects two necessary conditions for governing by non-interference, i.e., the establishment of basic institutions and the proper role of government.

1.5.3 Social Norms

A key factor in understanding institutions and other long-term relationships is the role of shared expectations of behavioral norms and cultural beliefs, as well as the role of sanctions in ensuring compliance with the “rules” . Behavioral norms and cultural beliefs mentioned here are the so-called social norms. They are informal institutional arrangements that are implemented by neither forces nor incentives. Solving problems with mandatory laws and inducing incentives in the long term will become a kind of social norms, values, beliefs, and culture that requires neither enforcement nor incentive, such as corporate culture, folk customs, religious faith, ideology, sentiments and concepts. This is an efficient way to minimize transition costs. Especially when the force of collective consciousness is well developed, problems will be much easier to resolve, and working efficiency will be greatly augmented. Otherwise, even if one problem is solved through mandatory commands, inducing incentive mechanisms or personal relationships, new problems will continue to arise in the same way, which will produce large implementation costs. Chapter 7 will discuss the important role of social norms in stimulating voluntary cooperation among individuals.

Even so, social norms that espouse morality and “persuade with emotions” remain largely constrained by the reality of individuals’ ideological limitations. Relying on improvement of humanity, social norms lack the power of constraint and possess a limited scope of governance. This kind of institutional arrangement is similar to the philosophy of Confucianism in

ancient China. The concept of the “rule of virtue” in Confucianism over-emphasizes ethical relations among people, and deliberately overlooks economic relations. Overall, it is successful in managing families or small communities, but biased in the governance of a nation. Benevolence and morality can be dominant, or at least considerably important, in a family or a small group, but may not be very effective in the lives of general people. Indeed, benevolence is highly personal, and its effect diminishes with the enlargement of the realm. Those who rely on others’ benevolence to acquire the necessities of life cannot be satisfied, in most cases. Although all people may need assistance from others, they cannot depend on benevolence alone to meet their needs. Consequently, the experience of managing a family cannot be simply extended to all economic and social activities; otherwise, manifold problems and even disasters may ensue. Especially under the environment of a modern market economy and people’s limited ideological development, reliance solely on internal ethical norms, and an absence of external laws and incentive mechanisms, will cause the market economy to move towards a perilous state.

Therefore, all three of these mentioned institutional arrangements have two sides: positive and negative. They also possess varied functions, ranges of application, and limitations. Moreover, if emphasis is placed too heavily on only one of them, serious negative consequences will occur; it is necessary for the three of them to play their own specific roles and complement each other. This condition can be elucidated with the example of friendship. If friends are made solely on the basis of interests, the friendship will disintegrate when the interests are exhausted; if friends are made based on influence, the friendships will end when the influence fails; if friends are established on the basis of power, the friendships will fail when the power dissipates. Of course, only when one makes friends with a sincere heart can the friendship last forever. Making friends with a sincere heart is the best way, but it can be very difficult, and even many couples may not be able to sustain this.

1.5.4 The Hierarchical Structure of the Three Arrangements

Even though this textbook primarily discusses the problem of economic incentives, governance on rules or institution remains the most essential and fundamental among the three kinds of arrangements since it establishes the most basic institutional environment, has strong positive and negative externalities, and determines whether or not the role of government is appropriate, thereby determining the effect of incentive mechanism design and the formation of good or bad social norms. In addition, for the formulation of institutional arrangements of both regulatory governance and incentive mechanisms, the principle should not, and essentially cannot, change the self-interested nature of human beings. Instead, it should make use of individuals' immutable self-interestedness to guide them to perform actions that are beneficial to society. In other words, the design of an institution should conform to the self-interested nature of individuals, rather than attempting to alter it. Moreover, individuals' self-interestedness cannot be simply deemed to be either good or evil, but instead what kinds of institutions that are employed and towards what directions they are guided must be taken carefully into account. Different institutional arrangements will result in individuals' different responses to incentives and various trade-off choices, thereby leading to markedly dissimilar consequences. As Deng Xiaoping contended, "Good institutions can make bad people unable to run amok arbitrarily, and bad institutions can make good people unable to do good enough, or even go to the opposite side."⁵ Expressed in a colloquial way, bad institutions can turn good people into bad people, but good institutions can turn even bad people to do good things.

Therefore, the utilization of "reason, interest, and emotion," should be synthesized, and vary with individuals, matter, place, and time to analyze and solve problems case by case. The criterion for deciding which aspect to use is determined by the importance of regulations, the degree of information symmetry, and the cost of supervision and law enforcement. All three institutional arrangements possess their own boundary conditions.

⁵*Selected Works of Deng Xiaoping (Second Version)*. Beijing: People's Publishing House. Volume 2, 333.

To “enlighten with reason” should depend on the availability of information symmetry and difficulty of legal supervision. Indeed, the law will be meaningless if its cost of supervision and execution is too high.

To sum up this and the previous sections, a well-functioning market needs the government, the market, and the society to be optimally situated so that the three-dimensional structure of state governance can be effectively interconnected and integrated. Defining the boundaries between the government and the market and between the government and the society involves two levels. The first level is defining boundaries. It is first necessary to know the appropriated boundaries among them. The prerequisite for an efficient market and a normal society is to build a limited government that is capable, accountable, effective and caring, and thus the reasonable position of government is vital. The principle here is that the market should be allowed to do whatever it can do, while the government should do what the market cannot do or cannot do well. Consequently, the function of government can be generalized as maintenance and service.

The second level is the identification of priorities. What is the top priority? The answer is fundamental institutions. After knowing the boundaries of the three, it is then necessary to sort them out. Who can best sort them out? The answer is the government. Then, what is the best entity to regulate the position of the government? It must be the *rule of law that mainly restricts the government, but not rule by law that mainly restricts the people*. Regulatory governance (or institutions) is the most important and foundational arrangement, which establishes the most basic institutional environment, possesses great positive or negative externalities, assesses whether the position of the government is appropriate, and thus determines the effect of the incentive mechanism design and the formation of good or bad social norms. However, is the government willing to limit its power? In general, absolutely not. Therefore, power requires further partitioning, and separation of the responsibilities and power of administrative departments, law-making departments, and judicial departments.

For all of these reasons, the underlying governance system is of determinant importance. Only when the governance boundaries between the government and the market and between the government and the society

are reasonably defined through comprehensive governance by the three dimensions of institution, the rule of law and civil society, can government power be regulated, restricted, and supervised, problems of economic efficiency, social equity and justice be resolved, the phenomena of corruption and bribery be eradicated, and healthy relations among the government, the market, the society, enterprises, and individuals be established. Indeed, in this way, relations of benign interactions are established between all of them. When benign interactions are realized, the government can then augment the efficiency of the market through continuous enactment and enforcement of laws, in order to truly promote the long-term peace and stability of a nation.

1.6 Ancient Chinese Thoughts on the Market

Many of the basic concepts of the market economy and conclusions of economics, including the idea that commodity prices are determined by the market and the “invisible hand” of Adam Smith, were stated in a profound way thousands of years ago. Indeed, numerous fundamental ideas, core assumptions and basic conclusions of economics, such as the behavioral assumption of pursuing self-interest, economic freedom, governance by the invisible hand, the social division of labor, the intrinsic relationship between national prosperity and individual wealth and between development and stability, and the relationship between the government and the market have all been discussed by ancient Chinese philosophers. Some examples of this are given as follows:

As early as over 3,000 years ago, Jiang Shang (also known as Jiang Ziya and Jiang Taigong, an ancient Chinese strategist and adviser) believed that “averting risks and pursuing interests” is the innate nature of human beings, i.e., *“In general, people hate death and take pleasure in life. They love virtue and incline to profit.”* He proposed the people-centered concept of dialectical unity between the wealth of the people and the stability and strength of the state, and the fundamental law of state governance, by stating that *“the state is not the property of one man but of all people. The man who shares interests with all men will win the state”*, and provided the fundamental s-

strategy of state governance, i.e., the government should take the common interests, risks, welfare and livelihood of the populace as its own, in order to achieve the incentive-compatible outcome in which the populace shares the same interests and risks as the government. Jiang Shang also provided an incisive answer to the relationship and priority of the wealth of the state and the wealth of the people: *“The true king enriches the people. The hegemon enriches the gentry. The state which barely survives enriches its grand ministers. The state which perishes enriches its coffers and fills up its treasuries.”* His advice was followed by King Wen of the Zhou dynasty, who ordered the granary to be opened to assist the poor and reduce taxes to enrich the people. As a consequence, the Western Zhou became a growing power.

Over 2,600 years ago, Guan Zhong (a Legalist chancellor and reformer of the State of Qi in ancient China) had deep insights on numerous economic issues. The core of his economic thought was *“the theory of self-interest”*. In *Guan Zi: Jinzang* (On Maintaining Restraint), he vividly explained social economic activities with individuals pursuing their interests: *“All men pursue interests and avert harm. When doing business, merchants hasten on the way day and night and make light of traveling from afar because, for them, interests are on their way. When fishermen go fishing in the sea, though the sea is hundreds of meters deep, they sail against the current for hundreds of miles day and night because, for them, interests are in water. So as long as there is interest, people would climb the mountain regardless of its height and go to sea regardless of its depth. Therefore, if those who know well how to govern the origin of interest, people will naturally admire the state and settle. The governor does not need to push them to go or lead them to come. Without being bothered or disturbed, people will get rich in a natural course. It is like a bird incubating eggs, the process of which is invisible and silent, but the result is noticeable when it is done.”* Essentially, this constitutes a clear demonstration of Adam Smith’s “invisible hand” more than 2,000 years earlier. In his book *Guan Zi*, Guan Zhong presented the law of demand by stating that *“The devaluation comes from the excess, while the value from the scarcity”*, and also drew the basic conclusion that individuals’ wealth leads to national stability, security, prosperity, and power by saying that *“Only at times of plenty will people observe the etiquette. Only when they are well-clad and fed will they have a sense of honor and shame.”* He further

pointed out that “*State governance must start with enriching the people. When the people become well-off, the state will be easy to govern. If they are in poverty, the state will be hard to govern.*” . . . “*Usually, an orderly state is abundant in prosperity, while a disorderly one is deep in poverty. So, a king versed in ruling a state must give priority to making people wealthy over governance itself.*” . Furthermore, comprehensive governance is another essential point in Guan Zhong’s thought of state management. For example, with respect to vassal kings, Guan Zhong suggested “*restraining them with interest, associating with them with trust, admonishing them with military power*” so that vassal kings “*won’t dare to defy the king and will accept his interest, trust his benevolence, and fear his military force.*” Indeed, it does not require much effort to discern that certain corresponding relations exist between “*restraining them with interest, associating with them with trust, admonishing them with military power*” mentioned here and the three institutional arrangements previously discussed.

More than 2,500 years ago, Sun Tzu (a military general, strategist, and philosopher in ancient China)’s book entitled *The Art of War* focused on military strategies and tactics, but its principles and ideas are highly similar to the market behavior and decision-making of firms. The first chapter “**Detail Assessment and Planning**” of the book coincides, to a large extent, with the basic analytical framework of economics and can be fully adopted in the context of accomplishing an endeavor. It serves as essential guidance for accomplishing big goals, making optimal decisions, and winning competitions in governing a state and managing an enterprise or organization. He also provided the basic conclusion of information economics: it is possible to achieve the optimal outcome (“the best is first best”) only under complete information; under information asymmetry, we can, at most, obtain a suboptimal outcome (“the best is second best”): “*if you know your enemies and know yourself, you will not be imperiled in a hundred battles; if you do not know your enemies but do know yourself, you will win one and lose one; if you do not know your enemies nor yourself, you will be imperiled in every single battle.*”

In the same period, a more remarkable fact was that Lao Tzu (a famous philosopher of ancient China, the founder of Taoism) presented the

supreme law of comprehensive governance: “Rule a kingdom by the Normal. Fight a battle by (abnormal) tactics of surprise. Win the world by doing nothing.” (Chapter 57, *Tao Te Ching*) This is the essential way of governing a state or administering an organization, which can be abstracted in common parlance as being righteous in deed, flexible in practice and minimal in intervention, and thus the government realizes governing by non-intervention. Lao Tzu considered “Tao” to be the invisible inner law of nature (that is, the fundamental rules and laws of doing things), while “Te” (meaning inherent character, integrity, virtue) to be the concrete embodiment of Tao. He further deemed that the governance of the state and the people should follow the Way of Heaven (referring to the objective laws governing nature or the manifestations of heavenly will), the Virtue of Earth and the Principle of Non-intervention, by stating that “*Man models himself after the Earth; The Earth models itself after Heaven; The Heaven models itself after Tao; Tao models itself after nature.*” (Chapter 25, *Tao Te Ching*) In addition, he also pointed out that “*The difficult (problems) of the world must be dealt with while they are yet easy; the great (problems) of the world must be dealt with while they are yet small.*” (Chapter 63, *Tao Te Ching*) In other words, in whatever we do, success lies in the details. All of the above-mentioned statements demonstrate that Lao Tzu’s thought of non-intervention does not mean “doing nothing” as is commonly believed. In fact, the non-intervention discussed by Lao Tzu is a relative concept, which requires non-intervention in major aspects, but action and care in specific aspects. Expressed in another way, we should never lose sight of the general goal, and begin by taking action to solve small, immediate practical problems.

Over 2,300 years ago, Shang Yang (an important Chinese statesman) of the State of Qin used the example of the hare to expound on the utmost importance of establishing private property rights and how clearly well-defined property rights can “determine ownership and settle disputes” and assist to establish market order. He came to this conclusion 2,300 years earlier than Coase. In *Shang Jun Shu (The Book of Lord Shang)*, Shang Yang wrote that “*when a hare is running, a hundred men chase after it; this is not because the hare can be divided into one hundred shares, but its owner-*

ship is not yet determined. On the other hand, hares are sold on the market, but even a thief dare not take one because ownership has been determined. Thus, if ownership is not definite, sages like Yao, Shun, Yu, and Tang would also chase after it; when ownership is definite, even a greedy thief will not dare to take it.” The hare is chased because people are driven to strive for its ownership, and even sages would do the same. In contrast, ownership of a captured hare in the market is determined, and thus others cannot simply take it.

Approximately 2,100 years ago, Sima Qian (a Chinese historian of the Han dynasty who is considered the father of Chinese historiography) made a remarkable statement in his work *Records of the Grand Historian: Biographies of the Money-makers*, “Jostling and joyous, the whole world comes after profit; racing and rioting, after profit the whole world goes,” which succeeded Guan Zhong’s concept of self-interest. He also demonstrates the economic paradigm of achieving social welfare through the social division of labor based on self-interest, which is similar to that of Adam Smith. Sima Qian investigated the development of social and economic life, and realized the importance of the social division of labor. He wrote that “All of them are commodities coveted by the people, who according to their various customs use them for their bedding, clothing, food, and drink, fashioning from them the goods needed to supply the living and bury the dead.” Therefore, “Society obviously must have farmers before it can eat; foresters, fishermen, miners, etc., before it can make use of natural resources; craftsmen before it can have manufactured goods; and merchants before they can be distributed.” Moreover, he believed that the entire social economy, composed of agriculture, forestry, industry and commerce, should develop in a natural manner without the constraint of administrative orders.

Also in the *Biographies of the Money-makers*, Sima Qian continued to write that “What need is there for government directives, mobilizations of labour, or periodic assemblies? Each man has only to be left to utilize his own abilities and exert his strength to obtain what he wishes. Thus, when a commodity is very cheap, it invites a rise in price; when it is very expensive, it invites a reduction. When each person works away at his own occupation and delights in his own business then, like water flowing downward, goods will naturally flow forth ceaselessly day and night without having been summoned, and the people will produce commodities

without having been asked. Does this not tally with reason? Is it not a natural result?"

In addition, Sima Qian's thinking contains great wisdom regarding the philosophy of state governance, the importance of economic freedom, and the priority of several basic institutional arrangements. Sima Qian provided an insightful conclusion in the *Biographies of the Money-makers*, "*The highest type of ruler accepts the nature of the people, the next best leads the people to what is beneficial, the next gives them moral instruction, the next forces them to be orderly, and the very worst kind enters into competition with them.*"

Confucius affirmed that the pursuit of personal material interests on the premise of social ethics is justifiable. He stated, "*When a country is well governed, poverty and a mean condition are things to be ashamed of. When a country is ill governed, riches and honor are things to be ashamed of.*" (*The Analects: Tai Bo*) By saying so, Confucius encouraged people to pursue a sufficient amount of material wealth. *The Analects* also recorded Confucius's compliments on his disciple Zigong (Duanmu Ci), who was a merchant. In *The Analects: Xian Jin*, it states that "*The Master said, 'There is Hui! He has nearly attained to perfect virtue. He is often in want. Ci does not acquiesce in the appointments of Heaven, and his goods are increased by him. Yet his judgments are often correct.'*" Here, Confucius compared Yan Hui, his favorite disciple, with Zigong. The former was almost perfect in morality, but frequently lived in poverty, which did not seem to be the right way of living; whereas, the latter, who did not follow the arrangement of destiny and went into business, turned out to be excellent in predicting the market.

These ancient Chinese economic thoughts are profound and historically important. Indeed, what Adam Smith discussed had already been addressed by ancient Chinese philosophers much earlier. Yet, as those ancient Chinese statements just constituted summaries of experience, they did not form rigorous scientific subjects or disciplines, provide boundary conditions and scopes for conclusions, or make logically inherent analyses. As a result, little is currently known about them in the outside world.

In the remaining parts of this chapter, we will present a rough discussion on core assumptions, key points, analytical frameworks, and research methodology of economics to assist you to understand the rigorous analy-

sis of the content covered by this textbook.

1.7 A Cornerstone Assumption in Economics

Every social science discipline imposes assumptions on individual behavior as the logical starting point of its theoretical system. As discussed above, the essential distinction of social science and natural science is that the former studies individuals' behavior and needs to make certain assumptions about individuals' behavior, while the latter investigates natural environments and objects. Economics is an important social science discipline since it not only studies and elucidates economic phenomena and enables positive analyses, but also closely examines individual behavior in order to make accurate predictions and value judgments.

1.7.1 Self-love, Selfishness, and Self-interest

When discussing individuals' behavior, three terms are often mentioned: self-love, selfishness, and self-interest, which are related to, and also distinct from, each other. Self-love means one's esteem and affection for oneself, which can be positive, as it encourages individuals to live a righteous and productive life, and can also be negative, since it may lead to a tendency towards narcissism, and even self-harm or self-deceit. Self-love can also generate self-interestedness and selfishness.

Selfishness refers to caring only for one's own welfare or advantage at the expense of or in disregard of others. It can lead to manifold harms. For example, being selfish makes one greedy; being greedy makes one overly ambitious; being overly ambitious makes one vain and arrogant; and being vain makes individuals lose themselves, while being arrogant may make individuals ruthless and offensive.

Self-interest means benefiting oneself without at the expense of others while it may or may not benefit others. Therefore, self-interest makes one judicious and rational, while selfishness produces greed. In other words, it may be altruism for the sake of self-interest. To pursue and obtain one's interest, individuals have to choose altruistic behavior. This constitutes the

self-interest assumption adopted in the study of economics. When self-love and self-interest complement one another, individuals will possess clear self-knowledge; whereas, when self-love is related with selfishness, individuals will experience moral decline. Consequently, when individuals are dominated by self-love, it does not necessarily mean that they will disregard others because self-love may also be combined with self-interest.

1.7.2 Practical Rationality of Self-interested Behavior

In this textbook, especially in the proof that competitive market mechanisms lead to optimal resource allocation, a key assumption is that individual behavior is driven by self-interest in normal situations. In fact, this is the most basic assumption in economics and forms its cornerstone. Indeed, economics claims that this is also the most accurate description of reality.

This assumption also applies to the handling of relations among nations, groups, households and individuals, being an objective reality or constraint that must be taken into account when studying and solving political, social, and economic problems. For example, when dealing with the relation between two nations, as a citizen, one must protect the interests of his or her own nation and speak and act from the standpoint of that nation, and may be subject to penalties if he or she divulges state secrets. When dealing with the relation between enterprises, as an employee, one must protect the interests of his or her organization, and if he or she leaks firm secrets to competitors, he or she may face negative consequences. The self-interest assumption is frequently questioned with the following: why are there families if individuals are rationally self-interested and pursue personal interests? In fact, concerning the family, individuals act in the interest of their own families. In other words, under normal circumstances, individuals care about their own families more than they do about the families of others. The discussion on the relationships between individuals follows the same reasoning. In practice, misunderstandings of this assumption are common, one of which is that this applies to only individuals in every case.

It is necessary to assume that individuals are self-interested because this conforms to the basic reality, and more importantly, the risk of doing so is

minimal. Specifically, even if the self-interest assumption is incorrect, serious consequences will not ensue. In contrast, if one adopts the altruistic behavior assumption, once it is proven to be incorrect, the consequences may be quite dire. In fact, the rules of the game adopted under the self-interest assumption also apply to altruistic individuals in most cases, and institutional arrangements or game rules and individuals' trade-offs under the altruistic behavior assumption are much simpler. Furthermore, the choice of behavior assumption is also critical for making optimal judgments of individual behavior in daily life. For example, the costs incurred could be enormous if a selfish, cunning person is misperceived as a simple, selfless and honest person, and trusted with important responsibilities or information.

Acknowledging that individuals are self-interested shows a realistic and responsible attitude towards solving social and economic problems. This is why we need certain government legislation to prevent opportunists from taking advantage of loopholes in institutions under the altruism assumption. As already discussed, if altruism is used as the premise to solve social and economic problems, the consequences may be disastrous. As an additional example, in the organization of production, if we deny the crucial importance of individuals' self-interest and only motivate individuals by emphasizing their contribution to the nation and the group, the result would be that everyone would aim to take advantage of the institutions to benefit from others' contributions, thereby leading to the free-rider problem. In this case, how could any nation become prosperous?

1.7.3 Boundaries of Self-interest and Altruism

It should be specially noted that, although the self-interest assumption holds true in most cases, it exhibits certain boundaries in application. For instance, under abnormal circumstances, such as natural or man-made disasters, wars, earthquakes and other major crises, individuals will often demonstrate their altruistic and selfless character, make sacrifices to fight for the nation, and go to great lengths to assist others. This constitutes another form of rationality, i.e., altruism, with which individuals are willing

to sacrifice their lives (even certain animals possess this instinct). For example, when a nation is invaded by another nation, many citizens are willing to sacrifice their lives to protect their country. Under a normal and peaceful environment, however, when engaged in economic activities, individuals normally pursue their own interests. These illustrations demonstrate that self-interest and altruism are not oppositional to each other, but rather constitute natural responses to different situations and environments.

Therefore, we can view self-interest and altruism as relative terms. In fact, such duality can also be witnessed in animals. For example, when wild goats are chased to the edge of a cliff, the old goats sacrifice themselves by making the first jump, so that the goatlings could jump on them and have a greater chance of survival. Adam Smith not only wrote the foundational work of *The Wealth of Nations*, but also wrote *The Theory of Moral Sentiments*, which contends that individuals should have sympathy and a sense of justice. These two works complement each other in the overall philosophy of Adam Smith. It is the case that, under the reality of an individual's self-love and self-interest, morality should be a kind of balance, an equilibrium outcome, and a convention realized through the social division of labor and cooperation. Under the guidance of appropriate institutions, individuals voluntarily divide the work by choosing varied specializations and cooperate in order to establish a harmonious, civilized, stable, and orderly society. It is against human nature to regard self-interest as immoral and runs counter to reality; it is not selfish. In fact, the organic combination of moral ethics and self-interest can actually promote social civilization and individuals' decency. The biggest advantage of the modern market is its utilization of the power of self-interest to counteract the weakness of benevolence so that hard-workers can be rewarded. As a consequence, we should not neglect the role of benevolence and morals in the formation of the modern market system.

Overall, self-interested individuals can be benevolent, altruistic, and moral. However, "self-interest" should not be "at others' cost". There are limitations and boundaries for self-interest and altruism, while selfishness that benefits oneself at the expense of others is the origin of maleficence and greed. Rationally self-interested behavior will conform to social

norms as a requisite constraint. As a society, we collectively agree to educate individuals to pursue their personal interests without violating public order and to protect the public interest based on individual rationality. However, we also disagree with economic idealism that grounds policies in ignorance of the driving force of personal interest in a wayward attempt to protect the public interest. It is the case that self-interested behavior under appropriate legal constraints must be distinguished from selfishness that violates laws and/or harms the interests of others.

It should also be mentioned that, even in the case of self-interested behavior, there are differences in extent. Under ideal situations, the less self-interested are individuals, the better are the results produced. However, it is also impossible to eliminate self-interest completely. We can prudently state that self-interest is the logical starting point for economics. If all human beings are unselfish and always considerate to others, then economics involving human behavior would be useless, and industrial engineering or input-output analysis may be sufficient. It is from the starting point that self-interest is an objective reality and individuals tend to pursue their own interests concerning economic activities that China has carried out the reform and opening-up, and made the transition from a planned economy to a market economy.

1.8 Key Points in Economics

When investigating economic issues, economists commonly use the following basic assumptions, constraints, axioms, and principles:

- (1) **Scarcity of resources;**
- (2) **Information asymmetry and decentralization:** individuals prefer decentralized decision-making;
- (3) **Economic freedom: voluntary cooperation and voluntary exchange;**
- (4) **Decision-making under constraints;**
- (5) **Incentive compatibility:** economic institutions or mechanisms should be used to solve the problem of conflicting

interests among individuals or economic units, i.e., provide individuals with the incentive to do what the institution or mechanism designer wants them to do;

(6) **Well-defined property rights;**

(7) **Equality of opportunity;**

(8) **Efficient allocation of resources.**

The above assumptions and constraints are crucial because relaxing any one of them may result in different outcomes. The consideration and application of these assumptions, constraints, and principles are also useful for people in their daily lives. Although they may appear to be simple, thoroughly understanding and skillfully utilizing them in reality is not easy. In the following, we briefly discuss these key points, conditions, axioms, and principles, respectively.

1.8.1 Scarcity of Resources

Economics stems from the fact that resources are limited in the world (at least the Earth's mass is finite). As long as an individual is self-interested, and his or her material desire is unlimited (i.e., the more one possesses, the better), it is impossible to realize distribution according to wants. Therefore, the problem of precisely how to employ limited resources to optimally satisfy wants must be addressed, and economics is needed to achieve this.

1.8.2 Information Asymmetries and Decentralization

In addition to the major objective reality of the self-interested nature of individuals, another fundamental objective reality is that, in most cases, information is private or asymmetric among economic agents, so that the effect of institutional arrangements adopted may be inadvertently neutralized. This is a fundamental reason why some core economic problems are difficult to solve. For example, although a person's words may be righteous, it may be challenging to discern if he or she actually means what is said; listeners may appear to concentrate on what you are saying, but you do not know if the message is actually being well-received by them.

The fundamental reason for such phenomena is the presence of private information. Private information, together with the self-interested nature of individuals, frequently leads to conflicts of interest among economic agents. If there is no appropriate governance system to equitably reconcile these differences, dishonesty, cheating, and selfishness in fighting for limited resources may become rampant. This is also the reason why the social sciences, and especially economics, are more complex and challenging to study and master than the natural sciences. Also due to information asymmetry, centralized decision-making is often inefficient; whereas, decentralized decision-making, such as the use of market mechanisms, is required to solve economic problems.

Only when complete information is acquired can the outcome be the first best. However, private information is frequently hard to obtain, and thus an incentive mechanism is needed to obtain truthful information. Since information acquisition also incurs costs, only the second-best outcome can be obtained in most situations. This is a basic conclusion in principal-agent theory, optimal contract theory and optimal mechanism design theory, which will be discussed in Part VI of this textbook. Without appropriate institutional arrangements, incentive distortion will exist, in which inducing information revelation inevitably incurs costs. As a consequence, it is particularly important to achieve information symmetry for the first-best outcome, without which many misunderstandings and inefficiencies may arise. By communicating openly with others, you enable others to understand you (signaling) and also get to know them (screening), which makes information more symmetric, clears up misunderstandings and enables consensus, which is the requisite condition of obtaining desired outcomes.

Excessive intervention of governments in economic activities and the over-playing of governments' role will lead to inefficiency, which is essentially resultant from information asymmetry. Many problems exist regarding governments' information acquisition and discrimination. If decision-makers are able to possess all relevant information, centralized decision-making featuring direct control would not be problematic, i.e., all that would be required is optimal decision-making. However, it is impossible for decision-

makers to have all related information. This is why we prefer decentralized decision-making. This is also the reason why economists call for the design of various incentive mechanisms, and that a decentralized decision-making method featuring indirect control should be used to stimulate individuals to do as decision-makers desire or to achieve the goals that decision-makers want to be attained. We will focus on the issue of information and incentive in Parts VI and VII.

It is worth mentioning that centralized decision-making also offers certain advantages, especially concerning decision-making involving rapid major changes. For instance, centralized decision-making is more efficient when a nation, group, or enterprise is establishing a vision, orientation and/or strategies, or making major decisions. Such major changes, however, might bring about enormous successes or disasters. For example, the decision of adopting the reform and opening-up policy has led to the rapid development of the Chinese economy. In contrast, the decision of the Cultural Revolution almost drove the Chinese economy to collapse. One viable solution to this problem is to take public opinion into deep consideration when making decisions and select great leaders.

1.8.3 Economic Freedom and Voluntary Exchange

Due to economic agents' pursuit of their own interests and information asymmetry, institutional arrangements of the mandatory "stick" style are often not effective. Consequently, it is necessary to provide individuals with more freedom of economic choice, which is the most important right among the three private rights (right to survival, freedom of choice for one to pursue happiness, and private property rights). To achieve this, we should mobilize economic agents with economic freedom based on voluntary cooperation and exchange through inductive incentive mechanisms, such as the market. Therefore, the freedom of economic choice ("deregulation") plays a vital role in market mechanisms, with decentralized decision-making ("decentralization") being a prerequisite for the normal operation of market mechanisms and also a core precondition to ensure efficient allocation of resources under competitive market mecha-

nisms.

In fact, the Economic Core Equivalence Theorem, which will be discussed in Chapter 12, reveals that once full economic freedom is given and free competition, voluntary cooperation and exchange are allowed, even without the establishment of any institutional arrangement in advance, the outcome of resource allocation driven by the self-interested behavior of individuals will be theoretically consistent with equilibrium allocation of a perfectly competitive market. The essence of the Economic Core Equivalence Theorem can be summarized as follows: under the rationality assumption, as long as economic freedom and competition are given, even if institutional arrangements are not considered, the economic core obtained will constitute a competitive market equilibrium.

China's reform and opening-up over the past 40 years have confirmed this assertion in practice. When analyzing the reasons for China's remarkable economic achievements, the critical factor is the provision to individuals of more freedom of economic choice. Indeed, reform practices from rural to urban areas indicate that wherever there are looser policies and a greater degree of economic freedom provided for producers and consumers, higher levels of economic efficiency prevail. China's so-called miraculous economic growth stems from the government's delegation of powers to the market; whereas, its imperfect market today is the result of excessive government intervention and inadequate or inappropriate government regulation and institutional arrangements.

1.8.4 Acting under Constraints

Acting under constraints is one of the most fundamental principles in economics, and is well expressed in the saying that "one must bow under the eaves". In fact, everything has its own objective constraints. Individuals, then, make trade-off choices under these existing constraints. Moreover, individuals' choices are determined by both objective constraints and subjective preferences. Constraints include material constraints, information constraints and incentive constraints, all of which can make it difficult for economic agents to achieve their goals. In economics, one embodiment

t of the basic idea of constraints is the budget set (or opportunity set) of consumer theory, as will be discussed in Chapter 3, which states that an individual's budget is constrained by the prices of the commodities and his or her income. For an enterprise, constraints include available technologies and prices of inputs, under which the goal of maximum profit requires firms to determine the quantity of production, technologies to be adopted, the quantity of each input, pricing for products, responses to competitors' decisions, etc. The development of a person, or even a nation, must contend with various constraints, including political, social, cultural, environmental, and resource constraints. Furthermore, if constraint conditions are not clearly identified and understood, it is difficult to perform tasks and achieve goals.

When introducing a reform measure or an institutional arrangement, it is essential to consider feasibility and meet the objective constraints. In addition, the implementation risk is expected to be reduced to a minimum so that social, political, and economic turmoil will not result. Therefore, feasibility is a requisite condition to judge whether a reform measure or institutional arrangement is conducive to economic development and the smooth transformation of economic systems. In a nation's economic transition, to make a feasible institutional arrangement, it must conform to the institutional environment of the specific stage of the country's development.

Participation constraint is critical when considering optimal contract design (cf. Chapters 16 and 17), which means that an economic agent can benefit, or at least will not experience harm, from economic activities; otherwise, he or she will not participate in, or may even oppose, the rules or policies to be implemented. Individuals who pursue the maximization of self-interest will not automatically accept an institutional arrangement, but instead will make a choice between acceptance and refusal. Only under an institutional arrangement in which the individual's benefit is not less than his or her reserve level (or he or she does not accept the arrangement) will the individual be willing to work, produce, trade, distribute, and consume. Moreover, if a reform measure or an institutional arrangement does not meet the participation constraint, individuals may give up. Of course, if everyone is reluctant to accept the reform measure or institutional arrange-

ment, it cannot be successfully implemented. Mandatory reform may also arouse opposition and cause social instability, so that development will not be possible. Consequently, participation constraint is closely related to social stability, and is an essential factor of social stability in development.

1.8.5 Incentives and Incentive Compatibility

The incentive is one of the core concepts in economics. Each individual has his or her own self-interest; to obtain interests from some activity, one must also pay the corresponding cost. Through a comparison between benefits and costs, individuals may be willing (have an incentive) to get something done or do it well, or be reluctant or unwilling to get it done or do it well, and thus will have a rational incentive response to the rules of the game. This, however, frequently leads to incentive-incompatible conflicts of interest among individuals or between individuals and society, and produces chaos. The concept of incentive compatibility has been discussed by Adam Smith in *The Theory of Moral Sentiments*, in which he stated that “...in the great chess-board of human society, every single piece has a principle of motion of its own, altogether different from that which the legislature might choose to impress upon it. If those two principles coincide and act in the same direction, the game of human society will go on easily and harmoniously, and is very likely to be happy and successful. If they are opposite or different, the game will go on miserably, and the society must be at all times in the highest degree of disorder.”⁶ The reason for this is that, under given institutional arrangements or rules of the game, individuals will make optimal choices according to their own interests, but such choices will not automatically satisfy the interests or goals of others and society. In addition, information asymmetry makes it challenging to implement social optimum by command. A good institutional arrangement or rule is able to guide self-interested individuals to act subjectively for themselves, but objectively for others, making individuals’ social and economic behavior beneficial to the nation and the individuals, as well as to themselves. This is core content of economics.

Everything that an individual does involves interests and costs (i.e.,

⁶Adam Smith: *The Theory of Moral Sentiments*.

benefits and costs), making incentive a ubiquitous issue that must be dealt with in daily work and life. As long as the benefits and costs are not equal, there will be different incentive reactions. To maximize profits, an enterprise has the incentive to use resources in the most efficient way and provide incentives to guide employees to expend the greatest effort. Outside of the enterprise, changes of profits provide an incentive for resource holders to modify their ways of using the resource; whereas, inside of the enterprise, incentive influences the way that the resource is used and the effort that employees invest in the work. To make management efficient, the role of incentive in organizations must be clear, as well as how to construct incentives to guide subordinates to exert the greatest effort in the work.

Since the interests of individuals, society and economic organizations cannot be identical, how can self-interest, mutual benefits, and social interests be organically combined? This requires incentive compatibility, with which the reform measures and institutional arrangements adopted can drive individuals' incentive for production and work. As a consequence, to implement a goal of one's own or that of society, appropriate rules of the game must be defined, under which when individuals pursue their self-interest, the goal can also be achieved. In other words, it unifies the self-interest of individuals and mutual benefits among individuals so that when pursuing one's self-interest, each individual can assist in attaining the goal intended by society or other individuals. We will focus on the issue of how to achieve incentive compatibility in Part V of the textbook.

1.8.6 Property Rights as an Incentive Scheme

Property rights are an important component of a market economy. In the previous section about ancient Chinese thoughts on the market, we mentioned that over 2,300 years ago, Shang Yang of the State of Qin utilized the example of the hare to expound on the crucial importance of establishing private property rights, and how a clear definition of property rights can “determine ownership and settle disputes” , and assist to establish the market order.

Property rights determine how a resource or economic good is owned

and used. A clear definition of property rights will enable a clear definition of the attribution of profits, thus providing incentives for property owners to consume and produce in the most efficient way, to provide quality products and good services, to build reputation and credibility, and to maintain their own commodities, housing, and equipment. If property rights are not unequivocally defined, however, the enterprises' incentive will be diminished, giving rise to incentive distortion and moral risk. For example, unclear property rights in state-owned enterprises will lead to inefficiency, induce wide-spread corruption, such as rent-seeking and interest transfer, squeeze the private economy, impede innovation, and lead to unfair competition. In the market mechanism, incentives are given to individuals mainly in forms of property possession and profit acquisition. The Coase Theorem, which will be discussed in Chapter 14, is a benchmark theorem in property rights theory. It claims that when there is neither transaction cost nor income effect, as long as property rights are clearly defined, an efficient allocation of resources can be achieved through voluntary coordination and cooperation.

1.8.7 Equality of Opportunity and Equity in Outcome

“**Equality in outcome**” is a goal that an ideal society aims to achieve. However, for human society with self-interested behavior, this “outcome equality” often produces low efficiency. Then, in what sense can equality or equity be consistent with economic efficiency? The answer is that if one uses “**equality of opportunity**” as the value judgment standard, equity and efficiency can be consistent. “Equality of opportunity” means that no barrier should exist to hinder individuals from pursuing their goals, and there should be a level playing field for every individual. The Fairness Theorem, to be discussed in Chapter 12, informs us that as long as individuals' initial endowments are of equal value, through the operation of a competitive market, allocation of resources that is not only efficient, but also equitable, can be attained even if individuals pursue self-interest. A concept similar to “equality of opportunity” is “individual equality” (also known as “all men are created equal”), which means that, al-

though individuals are born with different values, genders, physical conditions, cultural backgrounds, capacities, ways of life, etc., “individual equality” requires deep respect for such individual differences.

As individuals hold heterogeneous preferences, the seemingly equal distribution of, for example, milk and bread, may not satisfy everyone, and then the difference between equality and equity must be emphasized. Although both promote fairness, equality achieves this through treating everyone the same irrespective of need, while equity achieves this through treating people differently depending on need. Therefore, beside equal allocation that defines equality as absolute equalitarianism, the concept of **equity** in other senses should also be used in the discussion of economic issues. For instance, **equitable allocation**, which will be discussed in Chapter 12, considers both subjective needs and objective factors, such as income, which means that everyone is satisfied with their own allotment.

1.8.8 Efficient Allocation of Resources

Whether resources are efficiently allocated is a basic criterion to evaluate the effectiveness of an economic system. In economics, efficient allocation of resources usually refers to Pareto efficiency/optimality, which means that no other feasible allocation exists, such that at least one individual is better off without making any one worse off. As such, it requires not only efficient consumption and production, but also production of products that can best meet the needs of consumers.

It is worth mentioning that, concerning economic efficiency, we should distinguish between three types of efficiency: a firm’s production efficiency; industrial production efficiency; and allocation efficiency of an economy. By stating that a firm’s production is efficient, we mean that, with a given input, the output is a maximum, or with a given output, its inputs are a minimum. Industry is the aggregation of all firms’ production for a particular commodity, the efficiency of which can be similarly defined. However, the efficiency of a firm does not imply the efficiency of an industry. The reason for this is that if the production materials of firms with outdated technologies are given to those firms with advanced technologies,

there will be more outputs for the whole industry. At the same time, even if production of the entire industry is efficient, the allocation of resources may not be (Pareto) efficient.

The concept of Pareto efficiency in allocating resources is applicable to any economic institution. Indeed, it provides a basic criterion of value judgment for an economic institution from the perspective of social welfare, and assesses the economic performance from the perspective of feasibility. It can also be applied to a planned economy, a market economy, or a mixed economy. The First Fundamental Theorem of Welfare Economics, which will be introduced in Chapter 11, proves that when individuals pursue their own self-interest, a perfectly competitive market will lead to the efficient allocation of resources.

1.9 Understanding Economics Properly

An accurate and thorough understanding of economics can assist individuals to correctly use basic principles and analytical methods of economics to study various economic issues under different economic environments, behavioral assumptions, and institutional arrangements. The different schools and theories of economics per se show inclusiveness, specificity, universality, and generality of the analytical framework and methodologies of economics. Under different economic environments, various assumptions and specific models are required. Only in this way will the developed theory be able to explain different economic phenomena and individuals' behavior, and more importantly, make logically inherent analyses, draw conclusions of inherent logic, and make scientific predictions and reasoning under various economic environments that are close to the theoretical assumptions. However, due to the complexity of economic environments, for the sake of semantic and logic clarity, economics employs various rigorous mathematical tools to construct economic models in order to develop various economic theories. Rigorous mathematical tools are often difficult to master, which results in frequent misunderstandings and criticisms of economics. In addition to the common misunderstandings of benchmark theories discussed previously, misunderstandings exist about

economics concerning the following several aspects.

1.9.1 On the Scientification of Economics

One of the primary misunderstandings is that economics is not a science and includes numerous conflicting theories. A common criticism is that too many different economic theories exist that come from different economic schools in economics, making it difficult to discern which is correct and which is not. Those who hold such an opinion, however, fail to fully comprehend that all economic theories do not diverge from the two basic categories of theory discussed previously. Indeed, it is precisely due to the complexity of objective reality, and dissimilar economic, political, social and cultural environments in different countries and regions, different thoughts and preferences of people, and various economic goals that people pursue, that different theoretical economic models and economic institutional arrangements need to be developed.

It may be easy to understand that different economic theories or models should be developed for different economic, social, and political environments. However, it is challenging for many to comprehend why different economic theories are developed under the same economic environment. Consequently, some individuals, while disavowing economics and its scientific characteristics, sharply criticize economists by stating that “100 economists will have 101 opinions”. Actually, it is they who do not realize that it is analogous to the fact that we need different maps for different purposes, such as traffic, travel, military, etc., even though there is only one Earth. It is the case that, in the same given economic environment, we may need to develop distinct economic theories and different economic institutional arrangements for various goals.

The fact that different opinions of economists will arise for the same problem merely demonstrates the precision and thoroughness of economics because, when the premises and the environment change, the conclusions should vary accordingly; this is especially true in the case of the second category of economic theory that aims to solve practical problems. Furthermore, as different individuals will also have dissimilar subjective judge-

ments of values, there are few universally correct conclusions that satisfy everyone and apply in all situations. Just as medicines will vary according to the disease, when considering economic problems, case-by-case analyses must be conducted according to the specific time, place, people, and occasion involved. The major difference in the analogy to medicine is that, while an incorrect prescription for a condition may result in serious health problems or the death of an individual, an incorrect choice of economic policy, with large externalities, will influence an entire group of individuals or even a nation.

Although different economic theories and models exist, both the benchmark economic theory, which provides a benchmark or reference system, and the second category of economic theory, that aims to solve practical problems, they definitely do not constitute different “economics” .

The basic analytical framework and methodologies of (modern) economics, just like those of mathematics, physics, chemistry, engineering, etc., are not bounded by regions or nations. The foundational principles, methodologies, and analytical framework can be used to investigate a variety of economic issues under all economic environments and institutions, and to study economic behavior and phenomena in specific areas and time periods. The analytical framework and methodologies that will be introduced later can be used to conduct comparative analyses on almost every economic phenomenon and issue. In fact, this is precisely where the power and wonder of the analytical framework of economics lies: its essence and core require that the economic, political, and social environment conditions at a specific time and place must be considered and clearly defined when carrying out research. Economics can be used to investigate economic issues and phenomena under human behavior as manifested in different nations, regions, and cultures. Its basic analytical framework and methodologies can also be applied to elucidate other social phenomena and human decision-making processes. Indeed, it has been proven that, due to the universality and generality of the analytical framework and methodologies of economics, in the past few decades, various analytical methods and theories have been successfully extended to other disciplines, including political science, sociology, and the humanities.

1.9.2 On the Mathematical Feature of Economics

In addition to the criticisms that the assumptions of economic theory do not conform with reality and economics is not a science, another common criticism is that economics pays too much attention to details and involves an increasing amount of mathematics, statistics and models, and this makes questions more incomprehensible. However, the reason that economics uses so much mathematics and statistics is to achieve rigor and quantifiability of empirical studies. Although decision-makers in government and the general public do not need to understand details or premises of rigorous theoretical analysis, economists who make policy suggestions must thoroughly understand these. As economic theory will generate great externalities once adopted, blind application without considering premises may produce serious problems and perhaps catastrophic consequences. This is why mathematics should be employed to rigorously define the boundary conditions of a theory. In addition, the application of a theory or enactment of a policy often requires the tools of statistics and econometrics to carry out quantitative analyses or empirical tests. Moreover, because the real economic society is so complicated, the use of mathematical models in economic theories can assist to depict the real economic world for people to understand problems to be solved in reality. We will discuss the role of mathematics in economics later in this chapter.

1.9.3 Misunderstandings on Economic Theory

Each theory or model in economics that aims to solve practical problems adopts axiomatic analysis, which comprises a set of presupposed assumptions on economic environments, behavior patterns and institutions, and conclusions based on these assumptions. Considering the complexity of economic environments and the diversity of individual preferences in the real world, the more general are the presupposed assumptions of a theory, the more powerful is the theory. If the presupposed assumptions of a theory are too restrictive, the theory will lose generality, and thus offer less utility in reality. As economics is employed to serve the society and the government, the theory must be of some breadth and depth. Consequent-

ly, a necessary requirement for a good theory is its generality, in which the more general it is, the larger explanatory power it will have and the more useful it will be. The general equilibrium theory, which studies competitive markets, constitutes such a theory. It proves that the existence of competitive equilibrium leads to efficient resource allocation under the condition of very general individual preferences and production technologies.

Even so, theories of social sciences, and especially those of economics, like all theorems in mathematics, are subject to boundary conditions. As discussed previously, because of the great externalities of economic theories, when discussing or applying an economic theory, it is necessary to focus on its presupposed assumptions and application scope. This is because the conclusions of any economic theory are not absolute, but rather only hold when the assumptions are satisfied. Whether or not this point is recognized when discussing economic issues is a basic criterion of whether an economist is well-trained. As economic issues are closely correlated with daily life, even ordinary people can give their opinions about economic problems, such as inflation, business climate, balance or imbalance of supply and demand, unemployment, the stock market, and the housing market. For this reason, many people do not regard economics as a science. Indeed, economics would not be a science if it did not take into account constraints or base itself on accurate data and rigorous theoretical logical analysis. A well-trained economist always discusses issues based on economic theories and is fully cognizant of the boundary conditions of the relationship among economic variables and the inherent logic of the corresponding conclusions. It is crucial to fully understand the boundary conditions of economic theories; otherwise, one will not be able to distinguish between the theory and the reality, but instead tend towards one of two extremes: either simply applying the theory to reality, irrespective of constraints in reality; or completely denying the value of economic theory.

The first extreme viewpoint overestimates the role of theory and misuses it. For example, some people disregard the realistic objective constraints that face a nation. They blindly or mechanically apply the two categories of economic theories to solve the nation's problem, and indiscriminately copy models to study the problem, assuming that the inclusion of mathematical

models will produce valid theories. Conclusions and suggestions obtained by copying economic theories or models, whether they be benchmark economic theory, or the second category of economic theory that closely reflects reality, if blindly applied without taking into consideration various constraints and boundary conditions created by the nation's actual situation and economic institutional environment will frequently lead to serious problems. In fact, irrespective of how general the theory and behavioral assumption are, they always have application boundaries and limitations, and thus must not be used indiscriminately. Indeed, especially for those theories developed based on an ideal state that exists far from reality and primarily for the purposes of establishing a reference system, benchmark and goal, they should not be applied directly or incorrect conclusions will result. In addition, without a sense of social responsibility or strong training in economics, a person may overestimate the role of theory and blindly apply economic theory to the real economy without taking the premises into account, which could bring severe consequences, negatively influence social and economic development, and lead to great negative externalities. For instance, the conclusion of the First Fundamental Theorem of Welfare Economics that a competitive market leads to the efficient allocation of resources is subject to a series of preconditions, and its misapplication will produce severe policy errors and damage to the real economy.

The other extreme viewpoint completely denies the role of theory. People holding such a point of view underestimate or deny the pragmatic utility of economics, including its behavioral assumptions, analytical framework, basic principles, and research methodology. In fact, just as in the case of the benchmark theory of economics, no discipline in the world exists whose assumptions and principles coincide with reality perfectly (like the concepts of free fall without air resistance and fluid motion without friction in physics). However, this is not a reason to deny the scientific features and usefulness of a discipline, including economics. We learn economics to acquire not only its basic principles and utility, but also its approaches towards thinking, asking, and solving questions. As discussed previously, the value of benchmark theories lies not in the direct explanation of reality, but in the provision of a study platform and reference system for develop-

ing new theories to explain the real world. With these methods, economists can become enlightened on how to solve economic problems in reality. Furthermore, as discussed in the previous section, a theory applicable to one nation or region may not be applicable to another due to different environments. Instead of applying the theory mechanically and indiscriminately, it is necessary to modify the original theory to develop new theories according to given economic environments and individual behavior patterns.

There is also another extreme viewpoint, in which certain people contend that market failures may occur under any circumstance, and that the market has externalities. In claiming this, they deny the practical significance of economic theory, believing that economics is highly hypothetical, and that these assumptions are superfluous because the market does not have boundaries.

It is also sometimes stated that a certain theory or conclusion has been overturned. As not all of the conditions of a certain theory are in accordance with reality, the theory is deemed to be incorrect and thus supplanted. In general, this statement is not correct. Assumptions, even those in the second category of theories that aim to solve practical problems, cannot fully coincide with reality or cover every possible case. Indeed, a theory may be applicable to the economic environment of one location but inapplicable to that of another. As long as there is no inherent logic error present, however, it cannot be concluded that the theory is fallacious and needs to be abandoned. It may only be stated that it is not applicable to a certain place or a particular time.

Another common mistake is attempting to draw a general theoretical conclusion based solely on certain specific examples. This constitutes a methodological error. Of course, we do not deny the unique role of history, culture, and paradigms of each country in the establishment of its own discourse of economics.

1.9.4 On Experiments in Economics

Another criticism about economics is that it is not an experimental science at all, and thus the scientific feature of economics is negated. Such view-

point constitutes a misunderstanding. First of all, as with the rapid development of experimental economics in recent years, economics is using increasing numbers of laboratory experiments, field experiments, and computer simulation experiments to investigate various economic problems. Experimental economics also tests individuals' behavior and rationality of behavioral assumptions through these experimental methods, and consequently constitutes an important tool to determine whether or not an economic theory fits objective reality. Theorists have also obtained crucial information from experiments in order to promote the advancement of theories (important discussions of numerous economists on how to understand experiments in economics can be found on the website of Al Roth). Furthermore, experiments in economics are already transitioning more from the laboratory toward field experiments (see relevant discussion by John List).

Indeed, from an empirical perspective, in real economic activities, experiments in economics provide an indispensable advantage to verify policies and systems, especially concerning the need for institutional transitions. After continual exploration by early scholars, and the systematic synthesis of methodologies and tools of experiments in economics by Vernon Smith, winner of the 2002 Nobel Memorial Prize in Economic Sciences, experimental economics as an important empirical tool has received increasing attention in market mechanism design. When external environments change rapidly and new technologies abound, reform becomes inevitable; however, the strategic risks and social costs of various policy suggestions must be carefully considered. Therefore, it is a difficult and key objective in institutional reform to identify a way to comprehensively examine problems in advance that might arise out of new proposals regarding institutions. For instance, in the earlier days of the reform and opening-up, China took various measures, including the "special economic zone policy", "pioneering pilot scheme", "typical example as the lead", etc. Although economic experiments are consistent with such measures in the guiding aim of lowering the risk and cost of the reform, major differences in methodology remain between economic experiments and pilot experiments for the accumulation of experience.

In comparison with the pilot method, firstly, the economic experiment is liable to focus on a single research question, so that each experiment examines the effects of only one policy and the characteristics of only one mechanism. Secondly, the economic experiment employs relatively normal techniques and tools. Indeed, there are multiple influential factors in real economic activities, while the methodology of economic experiments is able to control factors that are irrelevant to the research question in order to focus more closely on the effect of one factor on certain economic phenomena. In comparison with the pilot method, it is also less costly to carry out economic experiments. It is also worth noting that the relation between genetics and economic behavior is meaningful and promising. Once the relation between the two is accurately determined, the foundation will be established for economics to become a discipline of science, just as a natural science.

It must also be admitted that some economic theories, such as general equilibrium theory that can be used for comprehensive analysis cannot easily be tested by social experiments since policy mistakes may result in enormous risks to economy and society. This is the greatest difference from natural sciences. As in natural sciences, natural phenomena and objects can be studied through experiments, and theories can be tested and further developed in the laboratory. Astronomy might be the only exception to this, but it involves no individual behavior. In all cases, once individual behavior is involved, the situation will at least become relatively more complicated. Moreover, extreme precision can be attained in the application of theories in natural sciences. For instance, in the construction of buildings or bridges, and the manufacturing of missiles or nuclear weapons, accuracy of any degree is attainable since all of the parameters are controllable, and the interrelationship between variables is amenable to experimentation. In economics, however, numerous factors affecting economic phenomena are uncontrollable.

Economists are also frequently criticized for inaccurate economic forecasts. This can be explained from two perspectives. From the subjective perspective, it is due to whether or not specific economists have had systematic and rigorous training in economics. If not, they may be unable

to discern the main causes of problems or make correct logical analyses and inferences when they discuss and attempt to solve economic problems, and thus will prescribe solutions to economic problems that will be unsuccessful. From the objective perspective, some economic factors that influence economic results may suddenly and uncontrollably change, thus causing predictions to be inaccurate, even though they may be made by well-trained economists with excellent economic insight. Moreover, an economic issue involves not only human behavior, which increases complexity, but numerous other uncontrollable factors, as well. Although an economist might be knowledgeable and talented, his or her predictions may still deviate because certain uncontrollable factors that influence economic results may change. For instance, even for a good economist with sound judgment, his or her economic forecasts might become inaccurate once sudden changes occur in the economic, political, or social environment. It is also the case that some people assert that no matter how economics develops and what the reason may be, inaccurate prediction is the norm, and accurate prediction is attributable to chance. In principle, this is true since economic fluctuations are random variables. However, since the probability for an event to occur and the capability of economists vary, a well-trained economist can better judge the probability of an event and be more likely to be accurate in his or her prediction. This constitutes the subjective factor of inaccurate prediction as mentioned above.

How can the problem that economic theory cannot experiment on society under many circumstances be overcome? The answer is logically inherent analysis, based on which inherent conclusions and inferences can be drawn, and comparisons and empirical data tests can be carried out through both horizontal and vertical perspectives of history. In this way, when conducting economic analysis or providing policy suggestions, according to the three dimensions of economic analysis discussed previously, there should first be theoretical analysis of inherent logic to define the applicable boundary conditions and scope. Meanwhile, tools of statistics, econometrics and/or experimental economics should be used to perform empirical quantitative analyses or tests, and the invaluable perspective of history should be taken into account to conduct vertical and horizon-

tal comparative analysis. Therefore, when carrying out economic analysis or identifying policy suggestions, there must be theoretical analysis of inherent logic, historical comparative analysis, and empirical quantitative analysis based on data and statistics; and all three are indispensable, which makes the analysis have six natures: scientific, rigorous, realistic, pertinent, forward-looking, and thought-provoking. This three-in-one study methodology, to a large extent, overcomes the challenge that many economic theories cannot or cannot easily experiment on society.

The method of logically inherent analysis of economics aims to fully understand and characterize relevant circumstances (economic environment, situation, and status quo) for a problem to be solved to ascertain what the problem is and its causes, apply appropriate economic theories accordingly, draw conclusions, and make accurate forecasts and inferences or give the direction of reforms. As long as the status quo accords with the causes (economic environments and behavioral assumptions) presupposed in the economic model, logically inherent conclusions can be reached according to economic theories, and thus solutions can be obtained or certain institutional arrangements can be suggested for different circumstances (varying with time, place, individual, and case). The method of logically inherent analysis can assist to make academic inferences on possible results under circumstances in which real economic and social environments, behavior patterns of economic agents, and economic institutional arrangements are given, and thus provide useful guidance for solving real economic problems. In other words, once the problem and its causes are clearly identified, and the appropriate economic theory is applied, if such a theory exists, then the optimal solution can be implemented, and logically inherent conclusions can be drawn in order to make accurate predictions and inferences or provide appropriate reform measures. Otherwise, severely harmful consequences might arise.

It is true that the result of an economic theory may not be tested by experiments on society under many circumstances, and data do not provide the sole basis for analysis. Although practice is the sole criterion for testing truth, but not for predicting it. It is logically inherent analysis and historic comparison that should be relied upon, and thus theory becomes

indispensable. Like a doctor prescribing medications for a patient or a mechanic repairing an automobile, the most challenging and crucial task is the accurate diagnosis of the disease or the cause of the failure. The criterion for a good physician lies in whether he or she can accurately find the true cause of the problem. Once the cause is identified and a remedy identified, it is relatively easy to administer appropriate treatment. For economic problems, however, the prescription is economic theories. Once we truly understand the characteristics of an economic environment, thoroughly investigate the situation, and reasonably characterize individuals' behavior, the likelihood of success should rise dramatically.

1.10 Basic Analytical Framework of Modern Economics

For performing most tasks in life, basic laws exist. The way that economics studies and solves problems is similar to how people deal with personal, household, economic, political, and social affairs. In order to do something well and establish and maintain good relationships with others, the first task is to understand national conditions and customs, i.e., to know the real environment, behavior, and personality of the persons with whom you interact. On such a basis, one determines the optimal way of dealing with them and performing tasks, and make an incentive response after weighing the advantages and disadvantages to obtain the best outcome. Finally, one must make a value judgement on the choice, and evaluate the rules of the particular game being played. The basic analytical framework and research methods of economics follow this mode precisely to study economic phenomena, human behavior, and how people assess trade-offs and make decisions. Of course, a major difference between these two is the rigorous reasoning of economics, which uses formal models to identify the logical relationship between presupposed assumptions and conclusions. Such analytical framework offers great generality and consistency.

A standard academic work needs first to delineate all of the problems to be studied and/or resolved, or the economic phenomena to be elucidat-

1.10. BASIC ANALYTICAL FRAMEWORK OF MODERN ECONOMICS 81

ed. In other words, economists should first identify research objectives and their significance, provide readers with information regarding an overview and progress of the issues under investigation through a literature review, and illustrate the work's innovation concerning technical analyses and/or theoretical conclusions. Subsequent to this, they should discuss how to address the issues raised and draw conclusions.

Although economic issues under study may be quite dissimilar, the basic analytical framework used is essentially identical, which is an axiomatic approach to investigate economic problems. The analytical framework for a standard economic theory in economics consists of the following five steps: (1) specifying economic environments; (2) making behavioral assumptions; (3) establishing institutional arrangements; (4) determining equilibrium outcomes; and (5) making evaluations. All economics papers written with clarity and logical consistency comprise these five parts, especially the first four parts, irrespective of the conclusion and whether or not the author realized it. In this way, writing an economics paper constitutes innovative writing with a logical structure and analysis in such steps. Once these components are understood, the basic writing pattern of academic economics papers is known, and it will be substantially easier to learn economics. These five steps are also quite beneficial for understanding economic theory and its proofs, and finding research topics and conducting research.

Prior to discussing the five components in detail, it is first necessary to define the term "institution". An institution is usually defined as a set of rules related to social, political, and economic activities that dominate and restrict the behavior of various social classes (Schultz, 1968; Ruttan, 1980; North, 1990). When people consider an issue, it is normal to consider certain factors as exogenously-given variables or parameters, and others as endogenous or dependent variables. These endogenous variables depend on the exogenous variables, and thus are functions of those exogenous variables. In line with the classification method of Davis-North (1971, pp 6-7) and the issue to be studied, any institution can be divided into two categories: institutional environment and institutional arrangement. An institutional environment is the set of a series of basic economic, political, social,

and legal rules that form the basis for establishing production, exchange, and distribution rules. Among these rules, the basic rules and policies that govern economic activities, and property and contract rights, constitute the economic institutional environment. An institutional arrangement is the set of rules that dominate potential cooperation and competition existing among economic participants. It can be interpreted as the generally known rules of the game, with different rules of the game leading to dissimilar incentive reactions of individuals. In the long term, institutional environment and institutional arrangement will affect each other and evolve. Yet, in most cases, as Davis-North points out, people usually regard economic institutional environments as an exogenously-given variable, and consider economic institutional arrangements (e.g., the market system) as exogenous or endogenous, depending on the issue under study or discussion.

1.10.1 Specifying Economic Environment

The primary component of the analytical framework of economics aims to specify the economic environment where the issue or object to be studied lies. An economic environment is usually composed of economic agents, their characteristics, the institutional environment of the economic society, the information structure, etc., which are treated as exogenous variables and parameters. They are the embodiment of the basic idea of constraints.

How can we best specify the economic environment? It can be divided into the following two levels: (1) objective and realistic description of economic environment; and (2) concise and acute characterization of the essential features. The former constitutes science and the latter is art, and the two must be combined and balanced. Overall, the more clear and accurate is the description of an economic environment, the greater is the likelihood of reaching correct theoretical conclusions. In addition, the more refined and acute is the characterization of an economic environment, the easier it is to reach and fully comprehend the theoretical conclusions. Only by combining these two levels together can the essence of issues under study be elucidated, as discussed below:

Description of economic environment: The first step in every econom-

ic theory of economics is to give an approximately objective description of the economic environment where the issue or object to be studied is situated. A reasonable, useful economic theory should accurately describe the specific economic environment. Although different nations and areas have distinct economic environments, which usually lead to dissimilar conclusions, the basic analytical framework and methodologies used are identical. As stated previously, a common initial task of studying economic issues is to describe the economic environment under investigation. Overall, the more clear and accurate is the description of the economic environment, the higher is the probability of obtaining correct theoretical conclusions.

Characterization of economic environment: When describing the economic environment, a question that is equal in importance to a clear and accurate description of the economic environment is how to concisely characterize the economic environment in order to capture the essence of a problem. Because most facts and phenomena in reality may be largely irrelevant to the economic issue to be analyzed and solved, a completely objective description of the economic environment may not only be unhelpful, but may also be confusing and overwhelming. Of course, if we precisely depict all aspects, a highly accurate and truthful description of the economic environment is achieved, but this result presents innumerable irrelevant facts without identifying the key points of the problem to be investigated. In order to avoid trivial aspects and focus on the most critical and central issues, it is necessary to characterize the economic environment specifically according to the demands of the issue to be studied. For example, when discussing consumer behavior in Chapter 3, we simply describe consumers as a composition of preference relation, consumption space, and income (or initial endowment) regardless of gender, age, race, or beauty. When discussing producer theory in Chapter 4, a producer is characterized as the production possibilities set. When studying transitional economies, we cannot simply mimic and apply conclusions derived under a mature market economic environment, but must determine the basic features of transitional economies, even though the basic analytical framework and methodologies of economics can still be applied.

A common criticism of economics is that it is useless because it relies

on a few simple assumptions to summarize complex situations. In fact, this is also the basic research methodology of physics. In the study of the relationship between two physical variables, both theoretical research and experimental operations fix other variables that will influence the object of study. In many cases, clarifying every aspect (especially unrelated aspects) is superfluous and may even lead to a loss of focus. This is similar to the previously described analogy of different kinds of maps. A tourist map may be needed for travelling, a traffic map may be useful for driving, and a military map may be appropriate for military exercises. Each of these maps describes only certain characteristics of a region, and does not present an aggregate picture of the real world. Indeed, people need a tourist map, traffic map, and military map according to different purposes. If we were to depict the entire real world into one map, although it completely describes the objective reality, how can it be a useful map for any one purpose?

Therefore, economics is not only a science of objective description, but also the art of abstracting and depicting real economic environments. Economics utilizes concise and profound characterizations of economic environments to describe the causes of problems and conduct inherently logical analyses, and thus obtain logical conclusions and inferences. A good economist should be able to accurately grasp the most essential characteristics of the current economic situation in his or her study. It is only when we truly clarify the causes and current situation that we can address specific problems with appropriate solutions (the economic theory adopted). Of course, to achieve this, one needs to have basic training in economics.

Determining how to describe the economic environment frequently leads to divergent theoretical results, and can even produce new schools of thought. Because economic environments are markedly complex, in many cases, economics can not only carry out descriptive analysis, like in natural science, but must also refine and characterize the behavior of the economic environment in an abstract manner to identify the most important characteristics. This, however, often makes economists possess a degree of subjective judgment. Different subjective judgments will lead to dissimilar specifications of economic environments, which will in turn lead to various economic theories, economic schools, and/or theoretical results. For

*1.10. BASIC ANALYTICAL FRAMEWORK OF MODERN ECONOMICS*85

example, there are many schools in macroeconomics: the Keynesian school, the post-Keynesian school, the rational expectations school (or neoclassical macroeconomics), the monetarism school, the supply school, and the new institutional school. In fact, the antagonism between these schools is not as great as commonly believed by non-economists and the media. Indeed, the schools have numerous factors in common: the same basic analytical framework, the same research method (using the economic model and market equilibrium to analyze the market), and the same object (studying the interaction and change law of macroeconomic variables under the arrangement of the market system). It is also the case that they all believe in the market system. Moreover, it is agreed upon that, in terms of the long-term or general trend of economic operation, there will be an optimal market equilibrium. The differences among these theories are primarily resultant from the differences in describing the economic environment, especially whether the impact on or interference to the economic system comes from the demand side or the supply side, whether information about economic fluctuation is sufficient, and whether the differences are caused by different assumptions on the time effect of the interference, such as lag or instant.

1.10.2 Making Behavioral Assumptions

The second basic component of the analytical framework of economics is to make assumptions regarding the behavior mode of economic agents. This is the key difference between economics and natural science. Indeed, the assumption is of critical importance and constitutes the foundation of economics. Whether an economic theory is convincing and has practical value and whether an institutional arrangement or economic policy is conducive to sustainable and rapid economic development primarily depends on whether the individual behavior assumed truly reflects the behavior of most individuals. It also depends on whether the institutional arrangement and individuals' behavior are incentive-compatible, i.e., whether individuals' reaction to the incentive is also beneficial to others or society.

In general, under a given environment and rules of the game, individuals will make trade-off choices according to their behavioral disposition.

Therefore, when deciding the rules of the game, policies, regulations or institutional arrangements, it is necessary to take into account the behavioral pattern of participants and make correct judgments. Just as when dealing with different individuals in our daily lives, it is necessary to know whether or not they are selfless and honest. Different rules of the game should be instituted when faced with different people. When interacting with an honest person who tends to tell the truth, the best way to deal with him or her, or the rules of the game imposed on him or her, may be comparatively simple. When facing totally selfless individuals, the rules to deal with them can be even simpler, since it is not necessary to take precautions or invest much energy (in designing the rules of game) to interact with them, and the rules may seem not as important. In contrast, when encountering a cunning and dishonest person, the best way to deal with him or her will be quite different and requires substantial energy, and thus the rules will be much more complicated. As such, making accurate judgments about individuals' behavior is a crucial step for the study of how individuals react to incentives and make trade-off choices. When investigating economic problems, such as economic choice, and interactions between economic variables and how they change, it is also important to determine the behavioral pattern of economic agents.

As mentioned above, under normal circumstances, a logical and realistic assumption about individuals' behavior used by economists is the self-interest assumption, or the stronger **rationality** assumption, i.e., economic agents pursue the maximization of benefits. Bounded rationality means to make the best choice according to the knowledge and information possessed by an agent, which belongs to the category of rationality assumption. In consumer theory, which will be discussed later, we assume that consumers pursue utility/satisfaction maximization; in producer theory, we assume that producers pursue profit maximization; and in game theory, various equilibrium solution concepts have been introduced to describe the behavior of economic agents, which are given based on different behavioral assumptions. Overall, any economic agent, in his or her contact with others, implicitly assumes others' behavior.

The assumption of (bounded) rationality is largely reasonable. From

*1.10. BASIC ANALYTICAL FRAMEWORK OF MODERN ECONOMICS*87

a practical point of view, as mentioned previously, there are three basic kinds of institutional arrangements: mandatory regulation (for situations with small operation costs and relatively easy information symmetry); incentive mechanism (for cases of information asymmetry); and social norms (composed of ideology, beliefs, morals, customs, etc., that are norms of self-discipline). If individuals are all selfless and highly developed ideologically, there will be no need for rigid “stick-style” regulations that “enlighten with reason” or the flexible market system that “guides with interest” .

1.10.3 Setting-up Institutional Arrangements

The third basic component of the analytical framework of economics is to set up economic institutional arrangements, which are usually referred to as institutions or rules of the game. Different institutions or rules of the game should be taken for different situations, environments, and individuals with varied goals and behavioral patterns. When the situation or environment changes, the rules of the game will also change accordingly. When an economic environment is given, agents need to decide on the economic rules of the game, which is termed the economic institutional arrangement. Determination of institutional arrangement is important for accomplishing anything in this context. Economics studies and provides various economic institutional arrangements or economic mechanisms according to different economic environments and behavioral assumptions. Depending on the issue under discussion, an economic institutional arrangement could be exogenously-given (in which case it will degenerate into the institutional environment, such as a perfectly competitive market under which we study consumer or producer problems) or endogenously-determined.

As discussed previously, there are three basic kinds of institutional arrangements that guide individuals' behavior: mandatory regulatory governance or government intervention; institutional norm of incentive mechanism; and didactic social norm. The three means play different roles and possess respective applicable ranges and limitations. Didactic social norm relies on the improvement of humanity and lacks a constraining force;

mandatory regulatory governance or government intervention incurs high information costs, and over-intervention will harm individual freedom; compared with the other two means, the incentive mechanism is the most effective. This is why economists focus closely on institutions.

Therefore, for the enactment of an institutional arrangement, irrespective of whether regulatory governance or incentive mechanism is present, the purpose is not to change the self-interested nature of individuals, but rather to make use of such immutable self-interestedness in order to guide individuals to engage in action that will objectively benefit society. In other words, mechanism design should follow the nature of individuals, but not attempt to change such nature. There is no good or evil in the discussion of individuals' self-interestedness, and the key lies in how to guide it with institutions. Different institutional arrangements will induce dissimilar incentive reactions and trade-off choices, and thus could produce markedly divergent results.

Any theory of economics involves economic institutional arrangements. Standard economics mainly focuses on the market system and studies how individuals make trade-off decisions in a market system (e.g., consumer theory, producer theory, and general equilibrium theory) and under what economic environments market equilibrium exists. It also makes value judgments on the results of resource allocation under different market structures (the criterion is based on whether resource allocation is efficient and equitable). In these investigations, the market system is normally assumed to be exogenously-given. In this way, it is possible to simplify the problem in order to focus on the study of individuals' economic behavior and how individuals make trade-off choices.

Of course, as the exogeneity assumption of institutional arrangements is not entirely reasonable in many cases, different economic institutional arrangements should be established, depending on particular economic environments and individuals' behavioral patterns. As will be discussed in Parts V-VII of the textbook, there will be market failures (i.e., inefficient allocation of resources and non-existence of market equilibrium) in many situations, and thus we will need to find an alternative or better economic mechanism. In that case, it is necessary to treat institutional arrangements

as endogenously-determined by the economic environment and individual behavior. As a consequence, economists should provide a range of alternative economic institutional arrangements for various purposes.

When studying the economic behavior and choice issues of a specific economic organization, economic institutional arrangements should especially be endogenously-determined. New institutional economics, transition economics, theory of the firm, and in particular the economic mechanism design theory, information economics, optimal contract theory, auction theory, and matching theory that have developed in the last few decades, provide various economic institutional arrangements for a broad range, from the state to individuals, according to different economic environments and behavioral assumptions. Part VI presents detailed discussions on the issue of incentive design in economic institutional arrangements.

1.10.4 Determining Equilibrium

The fourth basic component of the analytical framework of economics is to make trade-off choices and determine the “optimal” outcome. Given the economic environment, institutional arrangement (rules of the game) and other constraints that must be adhered to, individuals will respond to incentives based on their own behavior, and assess and choose an outcome from available and feasible outcomes. Such an outcome is termed equilibrium. Equilibrium means that, among various feasible and available choices, the one that is ultimately chosen is called the equilibrium. Those who are self-interested will choose the optimal one for themselves; whereas, those who are altruistic may choose an outcome that is favorable to others. For this reason, equilibrium, which refers to a state without deviation incentives for all economic agents, is a static concept.

The equilibrium defined above may be the most general definition in economics. It embraces equilibria in textbooks that are reached by independent decisions under the drive of self-interested motivation and manifold technology or budget constraints. For instance, under the market system, for the producer, a profit-maximizing production plan under the constraint of production technology is called an equilibrium production plan; for

the consumer, a utility-maximizing consumption bundle established under the budget constraint is called an equilibrium consumption bundle. When producers, consumers, and their interactions reach a state at which there is no incentive for deviations, a competitive market equilibrium for each commodity is obtained.

It should be noted that equilibrium is also a relative concept. The equilibrium outcome depends on economic environments, participants' behavior patterns (whether regarding rationality assumption, bounded rationality assumption, or other behavioral assumptions), and the rules of the game by which individuals react to incentives. Indeed, it constitutes the "optimal" choice relative to these factors. Moreover, due to bounded rationality, it may not be the optimal choice in objective reality, but is nevertheless the "optimal" one chosen by individuals according to their preferences and the information that they possess at the time.

1.10.5 Making Evaluations

The fifth basic component of the analytical framework of economics is to make evaluations and value judgments on equilibrium outcomes and institutional arrangements. After making their choices, individuals usually hope to evaluate the equilibrium that then arises and compare it with the ideal outcome (e.g., efficient allocation, equitable allocation, incentive compatibility, informational efficiency, etc.), in order to make further assessments and value judgments on economic institutional arrangements. They also strive to determine whether or not the adopted economic institutional arrangement has led to a certain "optimal" outcome, and test whether the theoretical result is consistent with the empirical evidence, whether it can provide accurate predictions, and whether it is of pragmatic significance. Finally, they assess the economic institution and rules adopted to discern if there is room for improvement. Overall, in order to achieve better results, after completing the task, we should evaluate the effects, whether it is worth continuing, and whether there is a possibility for improvement. Therefore, it is necessary to make evaluations and value judgments on equilibrium outcomes under institutional arrangements and trade-off choices in

order to identify the institutions that are best suited to the development of a nation.

When making evaluations on economic mechanisms or institutional arrangements, one of the most important criterion adopted in economics is whether the institutional arrangement is in accordance with the principle of efficiency. Certainly, as economic environments and individuals' behavioral patterns, as well as science and technology, continue to change, the precise Pareto optimality may never be fully realized. Just like Newton's three laws of motion, free fall, and fluid flow without friction in physics, Pareto optimality is an ideal state and provides the direction for improvement regarding economic efficiency. As long as the improvement of economic efficiency is desired, individuals will continually strive to approach this goal as closely as possible. With the ideal standard of Pareto optimality, we have a benchmark against which to compare, measure, and evaluate various economic institutional arrangements in the real world. Furthermore, it enables us to determine how far they are from this ideal goal for improvement of economic efficiency in order to continue to approach Pareto optimality.

Nonetheless, Pareto optimality is not the sole criterion for social value, equality or equity is also used. The market system achieves efficient allocation of resources, but it also faces numerous problems, such as social injustice resultant from an enormous wealth gap. There are a variety of definitions of equality, equity, and fairness. Equitable allocation, which will be introduced in Chapter 12, takes both objective equality and subjective factors into consideration, and more importantly, it can achieve equitable and efficient outcomes simultaneously. This is the basic conclusion of the Fairness Theorem, which will be discussed in Chapter 12. Another important criterion for evaluating an economic institutional arrangement is incentive compatibility.

In summary, the five components discussed above constitute the analytical framework underlying almost all standard economic theories and models, regardless of how much mathematics is used, or whether the insti-

tutional arrangement is exogenously-given or endogenously-determined. In the study of economic issues, one should first define the economic environment, and then examine how the self-interested behavior of individuals affects each other under exogenously-given or endogenously-determined mechanisms. Economists usually take “equilibrium”, “efficiency”, “information”, “incentive compatibility” and “equity” as key aspects to observe the effects of different mechanisms on individual behavior and economic organizations, explain how individual behavior achieves equilibrium, and evaluate and compare the equilibrium. Using such a basic analytical framework in the analysis of economic issues is not only compatible in methodology, but may also lead to surprising (but logically consistent) conclusions.

1.11 Basic Research Methodologies in Economics

We have discussed the five components of the basic analytical framework of economics: (1) specifying economic environment; (2) making behavioral assumptions; (3) establishing institutional arrangements; (4) determining equilibrium; and (5) making evaluations. In general, any economic theory consists of these five aspects. Discussion of the five components naturally leads to the question of how to combine them appropriately, gradually deepen the study of various economic phenomena, and develop new economic theories. This is what we will discuss in this section: the basic research methodology and key points, which include setting up benchmarks, establishing reference systems, building studying platforms, developing analytical tools, constructing rigorous models, and conducting positive and normative analyses.

The research methodology of economics aims to firstly provide basic studying platforms for all levels and aspects, and then establish benchmarks and reference systems in order to present the criteria to evaluate equilibrium outcomes and institutional arrangements. Building a studying platform, setting benchmarks, and establishing reference systems are of great importance to the construction and development of any discipline, and economics is no exception to this.

1.11.1 Setting up a Benchmark

Evaluations or judgements can only be relative, and not absolute, and thus there should be a benchmark, which also applies to the discussion of economic issues. In economics, *benchmark refers to a relatively ideal state or simple economic environment*. As discussed in the first section of this chapter, to investigate a realistic economic issue and develop a new theory, we usually need to first consider it under a relatively ideal economic environment to develop a simple result or theory. Subsequently, we discuss the result in a non-ideal economic environment, which is closer to reality, develop a more general theory, and compare it with that developed under the benchmark situation.

In this sense, benchmarks are relative to non-ideal economic environments and new theories to be developed that are closer to reality. For instance, a complete information environment is the benchmark for the study of incomplete information. When investigating economic issues under private information, the situation of complete information must first be understood (even though it is highly unrealistic). Only when we are clear about the situation of complete information can we adeptly study economic issues taking place under the circumstance of private information. This is the case with theoretical research in economics. We start from the ideal state or simple scenarios prior to considering more realistic or general scenarios. In addition, we learn from others' research results before innovating the existing theories. In fact, new theories are always developed on the basis of prior research findings and results. An example of this is that Newton's mechanics makes Einstein's theory of relativity possible, while the theory of relativity makes it possible for Chen-Ning Yang and Tsung-Dao Lee to put forward the non-conservation of parity.

1.11.2 Establishing a Reference System

A reference system refers to economic theories or systems generated in an ideal situation, such as the general equilibrium theory, i.e., a perfectly competitive market will lead to efficient resource allocation. Establishing a reference system is of key importance to the construction and development of a new

discipline, including economics. Although economic theories set as the reference system may include many assumptions that do not accord with reality, at least they can assist in the following: (1) simplifying the issue and capturing its characteristics directly; (2) establishing a comparative measurement criterion that is conducive to the assessment of the gap between the existing system and the ideal system, understanding the reality, and establishing the direction of improvement; and (3) studying how to reform or carry out further theoretical innovation, which can be used as a reference system for additional analysis.

Although an economic theory as a reference system might possess numerous unrealistic assumptions, they remain highly useful and can serve as a reference for further analysis. The importance of the reference system does not lie in whether or not it accurately describes the real world, but rather in establishing the measurement for a better understanding of reality. This is similar to our practice of setting role models in life. In addition, like a mirror, it assists to reveal the gap between theoretical models or realistic economic institutions and the ideal state. It is critically important in the sense that it identifies the direction of efforts and adjustments, and the extent of those adjustments. If a person has no goal and is unaware of the gap and approximate direction that should be followed, how can he or she make improvements or reach any goal?

General equilibrium theory is such a reference system. As we know, a perfectly competitive market will lead to efficient allocations of resources. Even if there is no such market in reality, if we make efforts in that direction, efficiency will be enhanced. This is why we have institutional arrangements, such as anti-trust laws, to protect market competition. By virtue of the reference system with perfect competition and complete information as the benchmark, we can study what outcomes can be obtained from economic institutional arrangements that are closer to reality (e.g., some monopolistic or transitional economic institutional arrangements) where assumptions in the general equilibrium theory are not valid (incomplete information, imperfect competition, externalities), and compare them with the results obtained from general equilibrium theory in the ideal state. In this way, we will know whether an economic institutional arrangement (be

it theoretical or realistic) is efficient in allocating resources and using information, and how far away the economic institutional arrangement adopted in reality is from the ideal situation. Based on this, we can make appropriate policy suggestions or reform measures. In this sense, general equilibrium theory also serves as a reference theory that provides a criterion for evaluating institutional arrangements and the corresponding economic policies or pointing out the direction of reforms in practice.

Therefore, we should hold lofty ideals, which let us know our goals and the best direction to go. Even if we cannot get there eventually, by only learning from the best and comparing our performance with the best, we can do better and better, and continue to at least approach the ideal.

1.11.3 Building Studying Platforms

A studying platform in economics consists of certain basic economic theories, models or methods, which provide the basis for deeper analysis. The methodology of economics is very similar to that of physics, i.e., simplifying the issue first to capture the core essence of the issue. In cases in which many factors produce an economic phenomenon, the impact of every factor must be elucidated. This can be achieved by investigating the effect of one factor at a time, while holding all other factors constant. The theoretical foundation of economics is microeconomics, and the most fundamental theory in microeconomics is individual choice theory; individual choice theory further comprises consumer theory and producer theory, which are the basic studying platform and cornerstone of economics. This is why almost all economics textbooks start from the discussion of consumer theory and firm theory. They provide the fundamental theories that explain how individuals make choices as consumers and firms, and establish the studying platform for further study of individual choice.

In general, the equilibrium choice of an individual depends not only on one's own choice, but also on others' choices. In order to investigate individual choice, it is necessary to determine what are the most important factors in an individual's decision making in the absence of influence by other agents. Consumer theory and producer theory are developed through this

approach. It is assumed that economic agents are in the institutional arrangement of a perfectly competitive market. Therefore, every agent will take price as given, and individual choice will not be influenced by others' choices. The optimal choice is determined by subjective factors (e.g., the pursuit of utility or profit maximization) and objective factors (e.g., budget line or production constraints).

This methodology that involves simplification and idealization of the question establishes a basic platform for deepening research. It is like the approach in physics: in order to study a problem, we examine the essence first, start with the simplest situation that excludes frictions from consideration, and then gradually deepen the research and consider more general and complicated cases. In microeconomics, theories of market structures, such as monopolies, oligopolies and monopolistic competition, are generated from more general cases, in which producers can influence each other. To study the choice issue under the more general situation in which economic agents can influence each other's decision-making, economists have developed a very powerful analytical tool: game theory.

General equilibrium theory is a more sophisticated studying platform based on consumer theory and producer theory. Consumer theory and producer theory provide a fundamental platform for investigating individual choice problems; whereas, general equilibrium theory provides a core platform for analyzing how to reach market equilibrium through interactions of individuals and all markets. Mechanism design theory provides an even higher-level platform for studying, designing, and comparing various institutional arrangements and economic mechanisms. It can be used to study and prove the optimality and uniqueness of perfectly competitive market mechanisms in resource allocation and information requirement, and more importantly, in the case of market failure, it offers methods of how to design alternative mechanisms. Under some regularity conditions, the institutional arrangement of a perfectly competitive market will not only lead to efficient allocation of resources, but also prove the most efficient concerning information requirements (mechanism operation cost, transaction cost), as it requires the least amount of information. Under other circumstances of market failure, it is necessary to design a variety of alterna-

tive mechanisms for different economic environments. Furthermore, the studying platform also creates conditions for providing reference systems for evaluating various kinds of economic institutional arrangements. In other words, it provides a criterion for measuring the gap between reality and the ideal state.

1.11.4 Developing Analytical Tools

For the research of economic phenomena and economic behavior, we also need various analytical tools besides the analytical framework, benchmark, reference system, and studying platform. Economics necessitates not only qualitative analysis, but also quantitative analysis, to find the boundary conditions for each theory to be true, so that the theory will not be misapplied. Consequently, a series of powerful analytical tools should be provided, which are usually given as mathematical models or diagrams. The power of these tools lies in their ability to support us to deeply analyze intricate economic phenomena and economic behavior through a simple and clear diagram or mathematical structure. Examples of this include the demand-supply curve, game theory, principal-agent theory for investigating information asymmetry, Paul A. Samuelson's (1915-2009, see Section 3.11.2 for his biography) overlapping generations model, dynamic optimality theory, etc. Of course, exceptions exist which are not expressed with analytical tools. For instance, Coase's theory is established and demonstrated through words and basic logical deduction only.

1.11.5 Constructing Rigorous Analytical Models

Logical and rigorous theoretical analysis is needed when we explain economic phenomena or economic behavior, and make conclusions or economic inferences. As mentioned above, each theory holds true under certain conditions. We need to establish rigorous analytical models that clearly identify the conditions under which a theory holds true. A lack of related mathematical knowledge will make it difficult to achieve an accurate understanding of the connotations of a definition, or to have discussions on related issues, as well as defining boundary conditions or constraints.

Therefore, it is not surprising that mathematics and mathematical statistics are used as basic analytical tools; indeed, they are also among the most important research methods in economics.

1.11.6 Making Positive and Normative Analysis

According to the research method, economic analysis can be divided into two categories: positive or descriptive analysis; and normative or value judgement analysis. Another major difference between economics and natural science is that the latter essentially only performs positive analysis, while the former involves both positive and normative analysis.

Positive analysis explains how an economy operates. It only gives facts and provides explanations (thus verifiable), but does not make value judgments about economic phenomena or offer means of revision. For example, an important task of economics is to describe, compare, and analyze such phenomena as production, consumption, unemployment and price, and to predict possible outcomes of different policies. Consumer theory, producer theory, and game theory are typical examples of the positive approach.

Normative analysis, on the other hand, makes value judgments on economic phenomena. It not only explains how an economy operates, but also attempts to identify the means of revision. As a consequence, it always involves the value judgments and opinions of the economists, and is thus not verifiable through facts. For instance, certain economists may place more emphasis on economic efficiency, while others may focus on equality or social justice. If we are cognizant of the differences between the two methods, many disputes can be avoided when discussing economic issues. Economic mechanism design theory is a typical example of the normative approach.

Positive analysis is the foundation for normative analysis, while normative analysis is the extension of positive analysis. In this sense, the foremost task of economics is to make positive analysis, and then normative analysis follows. General equilibrium theory includes both the former (e.g., the existence, stability, and uniqueness of competitive equilibrium) and the latter (the First and Second Fundamental Theorems of Welfare Economics).

1.12 Practical Role of the Analytical Framework and Methodologies

The most basic analytical framework and research methods of economics that have been discussed are of practical utility. Even though these analytical frameworks and methodologies may appear simple, it is not easy to comprehend and use them in our lives, studies, and research. However, once the idea is mastered, lifelong benefits will ensue. For example, one will be able to think scientifically. It will also be possible to investigate esoteric pure economic theories, as well as come up with viable solutions to solve practical issues in one's life and work.

First of all, from the perspective of studying economics, if the analytical framework and methodologies are mastered, abstract models and intricate mathematics will not be confusing. This is because, irrespective of the profundity of the mathematics used, how many formulae are employed, and how complicated are the economic models that are used in an economic theory, it still uses the above analytical framework and methodologies. If the basic analytical framework and methodologies are grasped and employed as the main paradigm, you will be able to maintain your focus and understand its general idea. Therefore, incomprehensible technical details can be temporarily put aside, and the framework and conclusion understood first; only then can we fully comprehend the details. In other words, it is necessary to first grasp the primary point and general idea, know its objectives, line of thinking as well as conclusions, and then consider the details. Moreover, once the basic analytical framework and methodologies are mastered, one possesses the correct idea of economics and cannot be misled. This, of course, can have a markedly positive effect on your study of economics. The reason why some people criticize economics and its methodologies is that their judgements are mostly not based on methods of scientific analysis, and may even rely only on subjective assumptions. If the basic analytical framework and methodologies are not understood, it is possible that their judgments will make you confused, lose direction when studying economics, and/or resist or even disregard the study of economics.

Second, regarding economic research, once the basic analytical framework and methodologies are understood, it will be easier to carry out research. It is the case that many people who strive to do economic research, even though they understand economics quite well and have read a large number of related papers, they still find it difficult to conduct research. Either they do not know how to do research, or their research findings are not significant or widely recognized. In fact, research would become relatively easier, as long as the basic analytical framework and methodologies are understood, and one possesses basic mathematic knowledge and the ability to carry out logical analysis. Conducting research is, to a certain extent, step-by-step writing with inherent logic according to the five components of the basic analytical framework. The basic analytical framework and methodologies can greatly assist you to improve your research and innovative abilities.

For example, if you are interested in a specific economic issue or phenomenon, or you want to put forward a new theory with stronger explanatory power to guide the resolution of a pragmatic economic issue, it is necessary to reasonably and precisely describe the economic environment and economic agent's behavior, employ existing analytical tools or develop new ones to build as simple a model as possible, and then perform the deduction and proof. On the other hand, if you only intend to extend and improve the original theory, it is necessary to determine whether the original assumptions about the economic environment, behavior, and models fit the reality and whether or not the assumptions can be relaxed to derive novel or more general conclusions or even pathbreaking theories. In this case, it might be easier to carry out the work of extensions and improvements. Of course, it is also possible to modify the specification of economic environments or other components in order to reach a different, or perhaps more important, conclusion. This is how the school of macroeconomics and numerous theories under information asymmetry were developed. If an economic theory is to be criticized, one should criticize which parts of its analytical framework are unreasonable, illogical, or unrealistic, and in what way, rather than censuring economics and its methodologies in aggregate.

Finally, understanding economics, its methodologies, and analytical frame-

work may assist us to have a greatly improved thinking process, and deal with everyday concerns and other individuals in a much better manner. Indeed, it can make you more thoughtful, insightful, and capable in your work. It is frequently stated that economics is esoteric and metaphysical since it involves so much difficult mathematics that seems remote from practice. Some people wonder “What will it be used for?” In fact, the basic approach of dealing with others and events in our daily lives is similar to the basic framework in economic analysis. For example, when one is in a new place preparing to perform a task or cooperate with others, the first thing one needs to do is to become familiar with the local environment and situation, which is similar to “specifying the economic environment” in the framework. It is then necessary to know the local culture and customs, the behavioral patterns and personalities of your counterparts, etc., which is similar to “making behavioral assumptions”. Subsequently, by taking all of the information together, one can decide on one’s rules for dealing with people, which is similar to “establishing economic institutional arrangements”. The next step is to choose the optimal scheme by making trading-offs among feasible options, which is similar to the “determining equilibrium”. The final step is to summarize and reflect on your decisions, actions, and your ways of dealing with people, circumstances, and events to assess whether they constitute the most effective approaches, whether they achieve the best outcomes, whether they are fair and reasonable, whether individuals’ enthusiasm is engendered, whether people have reactions to the incentive, and whether the intended goal of incentive compatibility is achieved, which is similar to “making evaluations”. In addition, when the environment and conditions are changed, or the subject with which you are working is altered, the rules should be changed accordingly. If you can act in accordance with the five aspects, and adjust the rules with changes in conditions, better results will certainly be produced. Not only may this be one of the best ways to deal with daily life and work, certain conclusions of economic theories can also assist you to better think about and solve problems.

1.13 Requirements for Learning Economics

There are three basic requirements for learning and mastering economic theory:

1. The first requirement is to master the basic concepts and definitions, which is a reflection of logical thinking and a clear mind. This is the prerequisite not only for the discussion and analysis of questions and logically inherent analysis, but also for a strong command of economics. Otherwise, different definitions of terms may produce substantial confusion and lead to unnecessary controversies.

2. One should be able to clearly state all theorems or propositions, and also be definitive about the basic conclusions and their conditions. Otherwise, even a small misunderstanding about an economic theory in application to the analysis of issues may lead to serious problems. Since any theory or institution has its proper scope of application, if we go beyond that, problems are liable to arise, bringing about tremendous negative externalities. In many cases, social or economic problems occur solely because economists misapply some theories without a solid understanding of their applicable conditions. In this sense, a good economist is similar to a qualified physician who needs to master the properties and efficacies of different kinds of medicines when prescribing for his or her patients.

3. One must also grasp how the basic theorems or propositions are proven (ideas and processes). A good economist, like a good physician, should know what the problem is and why it is present, understand its pathology, as well as be able to determine appropriate medicines. Then, he or she can gain a deeper understanding and a better command of the theories that he or she has learned.

If these requirements are met, it would be easy to refresh your memory, even if the proofs of some conclusions are forgotten. Economics relies primarily upon the analysis of inherent logic, which also constitutes the power of economic theory. This is why it is so crucial to accurately grasp the theories and their scopes of application.

1.14 Distinguishing Sufficient and Necessary Conditions

When discussing economic issues, it is also important to distinguish between sufficient conditions and necessary conditions, which can assist people to think clearly and avoid superfluous debates. A necessary condition is a condition that is indispensable in order for an assessment to be true. A sufficient condition, on the other hand, is a condition that guarantees that the assessment is true. For instance, some people often negate the market economy based on examples of severely underdeveloped countries, which adopt a market economy but remain poor. From this, they contend that a nation should not embark on the path of a market economy. These people, however, do not realize the difference between sufficient and necessary conditions: the adoption of a market economy is a necessary condition, rather than a sufficient condition, for a nation to become developed and prosperous. In other words, if a country aims to be prosperous, it must adopt a market economy. Indeed, it is the case that, without exception, all wealthy nations in the world are market economies. One must also admit, however, that the market mechanism does not necessarily lead to national prosperity. To be sufficient, other supplementary systems are needed, such as the rule of law, strong state capacity, a suitable political system, etc.

Indeed, as discussed previously, a distinction exists between a good market economy and a bad market economy. The reason for this is that, although (based on observations of reality) the market mechanism is critical for national prosperity, many other factors also influence the prosperity of a nation, such as the degree of government intervention, the political system, the legal framework, religions, cultures and social structures, all of which contribute to the labelling of the market mechanism as either good or bad.

1.15 The Role of Mathematics and Statistics in Economics

Mathematics and statistics are of extreme significance for people to possess a thorough knowledge of nature and to successfully manage their daily affairs. As the well-known statistician, C. Radhakrishna Rao, pointed out, mathematics is a type of logic employed to deduce results on a given premise; whereas, statistics is a rational method acquired through experience and a kind of logic used to verify the premise with a given result. Rao believes that “*All knowledge is, in final analysis, history. All sciences are, in the abstract sense, mathematics. All judgements are, in their rationale, statistics.*”⁷ This assertion profoundly and comprehensively depicts the significance of mathematics and statistics, and their respective connotations.

Mathematics and statistics are also essential in economics. Almost every branch in economics uses mathematics, statistics, and econometrics to a greater or lesser extent. The reasons for this include the following: economics is increasingly becoming a science; mathematical analytical tools are being increasingly used; and social systems are becoming increasingly complex and influential. Therefore, when investigating economic problems, it is necessary to have a rigorous theoretical model for inherently logical analysis, and determine the range for a theoretical conclusion to be true; then, empirical tests for the given results must be performed by means of econometric analysis. As a consequence, it is not surprising that mathematics and mathematical statistics are used as basic analytical tools, and have become the most important analytical tools in economics. Those who study and conduct research on economics must possess a thorough knowledge of both mathematics and mathematical statistics.

Modern economics mainly adopts the mathematical language to make assumptions about economic environments and individual behavior patterns, uses mathematical expressions to illustrate logical relations between economic variables and economic rules, constructs mathematical models to study economic issues, and finally follows the logic of mathematical lan-

⁷C. R. Rao, *Statistics and Truth: Putting Chance to Work*, World Scientific, Singapore, 2nd edition, 1997.

guage to deduce conclusions. Without related mathematical knowledge, it is difficult to grasp the essence of concepts and discuss related issues, and certainly conduct research and determine necessary boundaries or constraints when reaching conclusions. Consequently, it is necessary to master sufficient mathematical knowledge if one intends to learn economics well, engage in research on economics, and become a good economist.

People with little knowledge of mathematics are unlikely to be able to master the basic theories and analytical tools of economics or understand advanced economic textbooks or papers. In fact, they may use certain excuses, such as it is more important to produce economic thoughts, or mathematics is not generally needed to approach practical economic issues. Of course, no one could refute the importance of economic thoughts since they constitute the output of research. Without the tools of mathematics, however, how could the boundary conditions and the applicable scope of economic thoughts or conclusions be identified? Without knowledge of conditions and scope, how could we protect against the misapplication of economic thoughts or conclusions? Moreover, what are the odds for us to develop such profound economic thoughts without using mathematical models, as Adam Smith and Ronald H. Coase did? Even so, economists have never ceased investigating what conditions are required for their conclusions to hold. Without strict arguments, the thoughts or conclusions could not be widely acknowledged. As mentioned above, the economic thoughts presented by philosophers in ancient China, such as Jiang Shang, Lao Tzu, Sun Tzu, Guan Zhong and Sima Qian, are extremely profound, and some of the ideas of Adam Smith had already been put forward by these philosophers much earlier. However, their thoughts have never been known to the outside world because they were just observations or conclusions of experience that did not form a scientific system or involve logically inherent analyses using scientific methods.

There is another misunderstanding that research of economic issues with mathematics is remote from reality. In fact, however, most mathematical knowledge is developed on the basis of practical demands. People who possess basic knowledge of physics, physical science history, or history of mathematical thought will know that both primary and advanced

mathematics originated from the demands of scientific development and reality. As such, why cannot mathematics be used to investigate practical economic issues? The foundation of mathematics and economics is absolutely essential for one to be a good economist. If one understands mathematics well and masters fundamental analytical frameworks and research methodologies of economics, economics can be learned more easily and study efficiency will be markedly improved.

The primary functions of mathematics in the theoretical analysis of economics are as follows: (1) it makes the language more precise and the statement of assumptions clearer, which can reduce superfluous disputes resulting from ambiguous definitions. (2) It makes the analytical logic more rigorous and makes it possible to definitively state the boundary, application scope and conditions for a conclusion to hold true, and accurately identify the direction of theoretical innovation beyond the limitations of the original theory. Otherwise, the abuse of a theory may occur. For example, when discussing the issue of property rights, some people may quote the Coase theorem, assuming that as long as the transaction cost is zero, there will be efficient allocation of resources. Surprisingly, many people still exist (including Coase himself when providing his assessments) who do not know that this conclusion is normally false if the utility (payoff) function is not quasi-linear. (3) Mathematics can assist to obtain results that cannot be easily attained through intuition. For example, from intuition, according to the law of supply and demand, competitive markets will achieve market equilibrium through the adjustment of market prices according to the “invisible hand” as long as the supply and the demand are not equal. Yet, this conclusion does not always hold. For example, Scarf (1960) gave counterexamples of market instability. (4) It helps to improve and extend existing economic theory. Examples of this are manifold in the study of economic theory. For instance, economic mechanism design theory constitutes an improvement and extension of general equilibrium theory.

Qualitative theoretical analysis and quantitative empirical analysis are both requisite for studying economic problems. Statistics and econometrics play a key role in these analyses. Statistics focuses more on data collection, description, sorting and providing statistical methods; econometrics

identifies economic structures through economic theory, and focuses more on testing economic theory, evaluating economic policy, making economic forecasts, and identifying causal relationships between economic variables; big data analysis can process data faster, better and more efficiently, and use these data to discover new opportunities, provide new products and services, and has great cost advantages. In order to better estimate economic models and reach more accurate predictions, theoretical econometricians and statisticians have continually developed increasingly powerful econometric tools and big data analytic tools.

It is, however, worth noting that economics is not mathematics. Mathematics in economics is used as a tool to elucidate economic problems. Economists employ mathematics to express their opinions and theories more rigorously, and to assess interdependent relationships among economic variables. With the metrication of economics and the precision of various assumptions, economics has become a veritable social science with a rigorous system.

However, a person with a strong knowledge of mathematics does not necessarily become a good economist. This also requires a deep understanding of the analytical frameworks and research methodologies of economics, and acute intuitions of actual economic environments and economic issues. The study of economics not only calls for the understanding of terms, concepts and results from the perspective of mathematics (including geometry), but more importantly, one must strive vigorously to grasp economic meanings and underlying economic thoughts. This is, of course, also the case when those are given by mathematical language and/or geometric figures. As a consequence, confusion must be avoided regarding mathematical formulas or symbols in the study of economics. For all of these reasons, we often state that, in order to become a good economist, one needs to pursue academics with a good mastery of both scientific rigor and underlying profound thoughts.

1.16 Conversion between Economic and Mathematical Languages

The product of economic research is economic inferences and conclusions. A standard economics paper usually consists of three parts: (1) it raises questions, states the significance, and identifies the research objective; (2) it establishes economic models and rigorously expresses and proves the inferences; and (3) it uses non-technical language to explain the conclusions and provides policy suggestions. In other words, an economic conclusion is usually obtained through the following three stages: non-mathematical language stage \rightarrow mathematical language stage \rightarrow non-mathematical language stage. The first stage proposes economic ideas, concepts or conjectures, which may stem from economic intuition or historical and foreign experience. As they have not yet been proven by theories, they can be regarded as primary products of general production. The first stage is critical because it is the origin of theoretical research and creation.

The second stage verifies whether or not the proposed economic ideas or conjectures hold true. The verification requires economists to give formal and rigorous proofs through axiomatic analysis by introducing economic models and analytical tools, and if possible, to test them with empirical data. The conclusions and inferences obtained are usually expressed in mathematical language or technical terms, which may not be understandable to non-experts. Therefore, they may not be adopted by the public, government administrations, or policy-makers. For this reason, these conclusions expressed by technical language can be regarded as intermediate products of general production.

Economic studies should serve the real economic world. Therefore, the third stage aims to express the conclusions and inferences in non-expert language rather than expert language, making them more decipherable to the general public. Policy implications and profound meanings of conclusions and insightful inferences conveyed through non-expert language constitute the final products of economics. It is notable that, in both the first and the third stages, economic ideas and conclusions are presented in common, non-technical and non-mathematical language; whereas, the

third phase is a kind of enhancement of the first phase. In fact, the three-stage form of common language - technical language - common language is a normal research method that is widely adopted by numerous disciplines.

1.17 Biographies

1.17.1 Adam Smith

Adam Smith (1723-1790) is well-known as the father of economics. Adam Smith finished his study of Latin, Greek, mathematics, and ethics at the University of Glasgow in Britain. Subsequently, he worked at the University of Glasgow as a Professor in Logics and Moral Philosophy, and held the honorary position of Lord Rector. *The Wealth of Nations*, published in 1776, is Smith's most influential work and also a great contribution to the establishment of economics as an independent discipline. This book is widely considered to be the most influential work among all publications in the field of economics. His main academic thinking was strongly influenced by Bernard de Mandeville (1670-1731), Francis Hutcheson (1694-1764), David Hume (1711-1776), J. Vanderlint (year of birth unknown, died in 1740), George Berkeley (1685-1753), and others.

Smith suggested that the economic development of human society was the outcome of the spontaneous actions of tens of millions of individuals whose behavior followed the power of instinct and was driven by their self-interested nature. Smith regarded this power as the “invisible hand”, which is his idea of allowing the rule of the market to play the decisive role in the organization of economic society. *The Wealth of Nations* denied the attention to land in physiocracy, but valued labor as the most important aspect and believed that the division of labor could increase the efficiency of production.

Thomas Robert Malthus and David Ricardo focused on summarizing Smith's theories into a theory known as classical economics in the twentieth century (where economics thus originated). Malthus further extended Smith's theories to the problem of surplus of population. Ricardo, on the other hand, put forward the iron law of wages, suggesting that the surplus

of population may lead to the consequence that workers' livelihood cannot be guaranteed. Smith assumed that the increase in wages would accompany the increase in production, which seems more correct from today's perspective. Theories involved in his book not only set up the division of labor theory, but also pioneered in certain areas, including monetary theory, theory of value, theory of distribution, capital accumulation theory, theory of taxation, etc. In addition, Marx's labor theory of value, which was built upon the basis of Ricardo's political economy, also received indirect influence from Smith's theory.

Before the foundational work, *The Wealth of Nations*, Smith wrote *The Theory of Moral Sentiments* (first published in 1759). In this book, he mainly argued that people should have sympathy and a sense of justice, which may occur particularly under unusual circumstances (for example, when people are suffering or when a nation is invaded). He strove diligently to prove how self-interested individuals controlled their emotions and behavior, and especially their selfish sentiment and behavior, so that incentive compatibility between social interest and self-interest could be achieved. Indeed, the system of economic theory built by Smith in *The Wealth of Nations* is based on his arguments in *The Theory of Moral Sentiments*.

Smith worked on the two works simultaneously, and revised them repeatedly until his death. They became two organic and complementary components in his academic thinking system. *The Theory of Moral Sentiments* states the problem of morality, while *The Wealth of Nations* states the problem of economic development. Smith regarded *The Wealth of Nations* as a continuation of his thinking in *The Theory of Moral Sentiments*. The two books, however, differ in mood, scope of discussion, structural arrangement, and emphasis. For example, the control for self-interested behavior in *The Theory of Moral Sentiments* relied on sympathy and a sense of justice; whereas, in *The Wealth of Nations*, it relied on the competitive mechanism. Nonetheless, the discussions about the motivation of self-interested behavior were essentially identical. It can be suggested that individuals' sympathy, sense of justice, and pursuit of self-interest just constitute different reactions to different circumstances (unusual or usual). In *The Theory of Moral Sentiments*, Smith regarded "sympathy" as the core of judgemen-

t, but when it is viewed as the motivation of an individual's behavior, a completely different outcome will result.

1.17.2 David Ricardo

David Ricardo (1772-1823), a representative of Classical Political Economy, together with Thomas Robert Malthus (1766-1834), integrated Smith's theory into classical economics. Ricardo was born to a Jewish family, and his father was a stock broker. He attended a business college in the Netherlands at 12-years-old and worked on a stock exchange with his father at 14. He was engaged in stock exchange work independently in 1793, and by the time he was 25, he had amassed a wealth of two million pounds. After that, he began to study mathematics and physics. When he first read Adam Smith's *The Wealth of Nations* in 1799, he began to study economic issues. At the age of 37, he published his first paper in economics and did very well in this field. During the 14 years of his short academic life, he left numerous works, papers, notes, letters, and speeches. Among those, the most famous one is the *Principles of Political Economy and Taxation* published in 1817. It has been claimed that Ricardo was somewhat conceited. He stated that his point of view was different from Smith and Malthus, who enjoyed great prestige then and, in Britain, there would be less than 25 people who could fully comprehend his book. In 1819, Ricardo was elected as a member of parliament.

Starting with Bentham's version of utilitarianism, Ricardo established a theoretical system based on the labor theory of value and centered on the theory of distribution. He insisted in the principle which stated that the value of a commodity was determined by the labor cost contained in it. He also criticized Smith's theory of value by contending that labor that could decide value constituted socially necessary labor. In addition, labor that could decide the value of a commodity was not only direct living labor, but also labor in factors of production. Ricardo suggested that all value was produced by labor and was distributed among three classes: wages that were decided by the value of the essential means of subsistence of workers; profit was the surplus when wages were deducted; and land rent was the

surplus when wages and profit were deducted.

Based on the labor theory of value, Ricardo established the theory of comparative advantage. In *On the Principles of Political Economy and Taxation*, he clearly stated that “The value of a commodity, or the quantity of any other commodity for which it will exchange, depends on the relative quantity of labour which is necessary for its production” . He further asserted that “the exchangeable value of these commodities, or the rule which determines how much of one shall be given in exchange for another, depends almost exclusively on the comparative quantity of labour expended on each” . The profit of each party in international trade is also fully correlated with the exchangeable value of all commodities in the international market, i.e., the relative price level.

Ricardo regarded the free flow of factors of production, such as capital and labor, among regions and industries within a country as the fundamental reason of equalized rate of profit. The flow of factors between nations, however, would inevitably be interrupted by force or even totally stopped due to various reasons. Ricardo concluded that it was the immobility of factors between countries that decided “the same rule which regulates the relative value of commodities in one country, does not regulate the relative value of the commodities exchanged between two or more countries” . Since there are numerous reasons for different relative values of one commodity in different countries, there is room for profit for all of the participating countries in international trade. Its main premise, however, is that each country is cognizant of its advantages compared with others, i.e., they are certain about their own comparative advantages.

1.18 Exercises

Exercise 1.1 (Economics and the three dimensions of scientific economic analysis)

Answer the following questions:

1. What is the definition of economics?
2. What are the two great objective realities facing the study of economic problems?

3. What is modern economics?
4. What are the “three dimensions and six natures” ?
5. Why does scientific economic analysis need the “three dimensions and six natures” ?

Exercise 1.2 (Differences between economics and natural science) Answer the following questions:

1. What are the main differences between economics and natural science?
2. Why do these differences make economic research more complex and difficult?

Exercise 1.3 (Two basic categories of economic theory) Answer the following questions:

1. Which two categories of economic theories can be classified according to their functions?
2. Describe the connotation and function of each category, as well as the relationship between these two categories.
3. How should we correctly regard and deal with the interaction between these two kinds of economic theory?

Exercise 1.4 (The fundamental functions of economic theory) Answer the following questions:

1. What are the three main functions of economic theory?
2. Why is there not a best kind of economic theory that is always right and fits every development stage, but instead a kind that fits certain institutional environments the best?
3. What are two common misunderstandings of economic theories?

Exercise 1.5 (Market and market mechanism) Answer the following questions:

1. From the perspectives of information and incentive, what are the advantages of the market economic system compared with the planned economic system?
2. Under the condition of the market economy, what are the three basic functions of price?
3. What is the superiority of the market system?
4. What are the three development stages that an economy will experience? How can efficiency-driven and further innovation-driven development be realized? What is the basic economic institution behind this?

Exercise 1.6 (The dialectical relationship between competition and monopoly)

Answer the following questions:

1. Why do people want to introduce the competitive mechanism in the view of social resource allocation, while enterprises desire monopolies? Please also state the dialectical relationship between competition and monopoly.
2. What does the *Innovation Theory* of Schumpeter tell us? Please state the importance of innovation-driven development.
3. Why do innovation and entrepreneurship depend on institutional choice, and therefore constitute endogenous variables?

Exercise 1.7 (The boundaries among the government, the market, and the society)

Answer the following questions:

1. Why is it necessary to reasonably define the boundaries between the government and the market and between the government and the society?
2. How should the boundaries between the government and the market and between the government and the society be generally defined?

3. Why does a well-governed nation need to reasonably define not only the boundaries between the government and the market, but also those between the government and the society?

Exercise 1.8 (The three basic institutional arrangements for state governance and benign development)

Answer the following questions:

1. What are the three elements of state governance and benign development?
2. What are the three basic institutional arrangements for state governance?
3. State the range of application and limitation of each of the three basic institutional arrangements. Which one is the most basic and important?

Exercise 1.9 (The logic of development and governance) Answer the following questions:

1. How shall we correctly understand the logic of development and governance and the dialectical relationship between the two?
2. Explain the achievements and limitations of economic reform in China according to this framework.

Exercise 1.10 (Ancient Chinese Economic Thought) Answer the following questions:

1. Provide five examples that indicate the thought of the market economy in ancient China.
2. Why could not those numerous deep economic thoughts in ancient China form a scientific economic theory?

Exercise 1.11 (The cornerstone assumptions of economics) Answer the following questions:

1. State the relationship and distinction between self-love, selfishness, and self-interest.

2. Why does economics use the self-interest assumption as the most basic, important, and central assumption?
3. How shall we regard self-interestedness and altruism?

Exercise 1.12 (Key points in economics) Answer the following questions:

1. What are the key points of economics? Please state each of them generally.
2. State the meanings, advantages, and limitations of centralized and decentralized decision-making.
3. Why are economic freedom and competition crucial to economic development?
4. Why is acting under constraints one of the most fundamental principles in economics?
5. What is the relationship between incentive and information?
6. Why are clearly defined property rights crucial to the efficient allocation of resources?
7. Discuss the differences between equity in outcome and equality of opportunity. Which one does not conflict with efficiency?

Exercise 1.13 (Proper understanding of economics) Answer the following questions:

1. How shall we regard the scientific nature of economics?
2. How shall we regard the mathematical nature of economics?
3. How shall we regard economic theory correctly?
4. How shall we regard the critics that contend that economics cannot be tested through experimentation?

Exercise 1.14 (Basic analytical framework of (modern) economics) Answer the following questions:

1. What are the components that constitute the basic analytical framework of a standard modern economic theory?
2. Why do different economic environments need different economic theories?
3. Why are different economic theories needed even for the same economic reality or environment under many circumstances?
4. Why should evaluation be included in the analytical framework?
5. Taking Coase theorem as an example, expound on its economic analytical framework.
6. What are practical usages of the basic analytical framework and research methodologies of economics?

Exercise 1.15 (Benchmark and reference system) Answer the following questions:

1. What are the definitions of benchmark and reference system?
2. Why are setting-up benchmarks and establishing reference systems the premise of discussing economic problems?
3. What are typical examples of reference systems?

Exercise 1.16 (Methodologies) Answer the following questions:

1. When considering the problem of economic reform, why is it important to distinguish necessary conditions from sufficient conditions?
2. Why are both positive and normative analysis needed when discussing economic problems?
3. How shall we regard the role of mathematics and statistics in economics?
4. How shall we complete the conversion between economic and mathematical languages?

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Chapter 2

Preliminary Knowledge and Methods of Mathematics

This chapter provides self-contained mathematical knowledge and results that constitute indispensable tools of modern economics, in general, and advanced microeconomic theory, in particular. It includes basic knowledge and results of topology, linear algebra, mathematical analysis, fixed point theory, static optimization, dynamic optimization, differential equations, difference equations, probability theory, and stochastic dominance and affiliation. Readers with more mathematics training will also find this chapter useful as an important reference course for most of the principal mathematical theorems that arise in modern economics.

We begin with the basic mathematical knowledge that you will frequently encounter.

2.1 Basic Set Theory

This section introduces some basic concepts and results of set theory.

2.1.1 Set

A **set** S is a collection of elements. According to the number of elements, a set can be a **finite set**, for example, $S = \{1, 3, 5, 7, 9\}$; an **infinite countable set**, for example, $S = \mathcal{N}$, where \mathcal{N} is the set of all natural numbers; or an

infinite uncountable set, for example, $S = \mathcal{R}$, where \mathcal{R} is the set of all real numbers. A countable set can be either finite or infinite. A set can also be described with some properties, for example, $S = \{1, 3, 5, 7, 9\} = \{x : x < 10, x \in \mathcal{N}, \frac{x}{2} \notin \mathcal{N}\}$. The **empty set** \emptyset is a set consisting of no elements.

A **subset** T of a set S is also a set, and any element in T belongs to S , denoted by $T \subseteq S$. If T is a subset of S and S has at least one element that does not belong to T , then T is a **proper subset** of S . If T and S are subsets of each other, then these two sets are equal, i.e., $T = S$.

The **union** of two sets T and S is denoted by $T \cup S = \{x : x \in T \text{ or } x \in S\}$; the **intersection** of two sets T and S is denoted by $T \cap S = \{x : x \in T \text{ and } x \in S\}$.

The **complement set** of S in the universal set U is denoted by $S^c = \{x : x \in U, x \notin S\}$. The complement set of the universal set U is the empty set, and the complement of the empty set is the universal set. The **difference** between sets S and T is denoted by $S \setminus T$ or $S - T$, which is defined as $S \setminus T = S - T = \{x : x \in S, x \notin T\}$.

The complement of the union or the intersection of any number of sets satisfies De Morgan's law:

$$\left(\bigcup_{i \in I} A_i \right)^c = \bigcap_{i \in I} A_i^c;$$

$$\left(\bigcap_{i \in I} A_i \right)^c = \bigcup_{i \in I} A_i^c.$$

The **product** of sets T and S is denoted by $S \times T = \{(s, t) | s \in S, t \in T\}$.

The n -dimensional real space is defined as $\mathcal{R}^n = \underbrace{\mathcal{R} \times \mathcal{R} \times \mathcal{R} \times \cdots \times \mathcal{R}}_{\text{product of } n \text{ spaces}}$.

2.1.2 Mapping

Definition 2.1.1 (Mapping) Given sets A and B , if for each element x in A , there always exists an element y in B related to it, then this relation is called *mapping* or *function*, denoted by $f : A \rightarrow B$. The set A is called *domain* of f , and the set B is called *range*.

The **image** of f is the set of points in the range into which some points

in the domain are mapped, i.e., $I = \{y : y = f(x), \text{ for some } x \in A\}$.

The following definition gives the types of mapping.

Definition 2.1.2 Given sets A and B , and a mapping $f : A \rightarrow B$, we call f a *surjection* if $\{y \in B : f(x) = y, x \in A\} \equiv f(A) = B$; an *injection* if $f(x) \neq f(x')$ holds for all $x \neq x'$; a *bijection* if f is both a surjection and an injection.

If there is a bijection f between sets A and B , then A and B are **equivalent**, denoted by $A \sim B$. Next, we will discuss the number of elements of a set.

Definition 2.1.3 Let $J_n = \{1, 2, \dots, n\}$ be the set of the first n positive integers, and J the set of all positive integers.

- (1) A set A is a *finite set*, if there exists a certain n such that $A \sim J_n$;
- (2) A set A is a *countably infinite set*, if $A \sim J$;
- (3) A set A is *countable*, if it is either a finite set or an countably infinite set;
- (4) A set A is *uncountable*, if it is not countable.

Both sets of natural and rational numbers are countable sets, but the real number set is not. The following conclusion shows that the set of all binary numbers is not countable.

Theorem 2.1.1 Suppose that A is a set consisting of all sequences made up of 0 and 1. Then, it is uncountable.

Refer to the proof of Theorem 2.14 in Rudin (1976)'s Principles of Mathematical Analysis.

Since real numbers can be represented in binary, the set of real numbers is equivalent to the set A above, which is uncountable. Any interval (a, b) is equivalent to the real space since $f = \frac{y}{1 - |y|}$ with

$$y = \frac{x - \frac{(a+b)}{2}}{\frac{(b-a)}{2}}$$

is a bijection of them, and thus any real interval is uncountable.

2.2 Basic Linear Algebra

2.2.1 Matrix and Vector

We use \mathcal{R}^n to represent a set of all n -tuple real numbers. Its elements are called **points** or **vectors**. $\mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ represents a column vector, and x_i

is the i th component of the vector \mathbf{x} . $\mathbf{x}' = (x_1, \dots, x_n)$, a row vector, is defined as the **transpose** of \mathbf{x} . If not expressly specified, vectors refer to column vectors, in general.

The inequality signs \geq , \geq and $>$ about vectors are defined as follows. Let $\mathbf{a}, \mathbf{b} \in \mathcal{R}^n$, then $\mathbf{a} \geq \mathbf{b}$ represents that $a_s \geq b_s$ for all $s = 1, \dots, n$; $\mathbf{a} \geq \mathbf{b}$ represents that $\mathbf{a} \geq \mathbf{b}$, but $\mathbf{a} \neq \mathbf{b}$; $\mathbf{a} > \mathbf{b}$ represents that for all $s = 1, \dots, n$, $a_s > b_s$.

In economics, it is usually required to solve the system of linear equations, which can now be easily expressed and solved by linear algebra.

We consider a system of m linear equations with n variables (x_1, x_2, \dots, x_n) :

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= d_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= d_2 \\ \dots & \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= d_m \end{aligned}$$

where the letter with double subscript, a_{ij} , denotes the coefficient appearing in i th equation and attached to the j th variable x_j , and d_j denotes the constant term on the right side of the j th equation.

This can be expressed more succinctly in the following matrix form:

$$A\mathbf{x} = \mathbf{d},$$

where $A, \mathbf{x}, \mathbf{d}$ are, respectively:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{pmatrix}$$

$$\mathbf{d} = \begin{pmatrix} d_1 \\ d_2 \\ \cdots \\ d_m \end{pmatrix}$$

A is called the **coefficient matrix** of a $m \times n$ system of equations, which consists of m rows and n columns; \mathbf{x} is called a **variable vector**, and \mathbf{d} is a **constant vector**. An n -dimensional vector can be viewed as a special $n \times 1$ matrix.

As a shorthand device, the array in matrix A can be written more simply as

$$A = [a_{ij}]_{m \times n} \quad (i = 1, 2, \dots, m; j = 1, 2, \dots, n).$$

2.2.2 Matrix Operations

Here, we provide a brief introduction to some common matrix operations.

Equality of Two Matrices: $A = B$ if and only if $a_{ij} = b_{ij}$ holds for all $i = 1, 2, \dots, m, j = 1, 2, \dots, n$.

Addition and Subtraction of Matrices: $A \pm B = [a_{ij}] \pm [b_{ij}] = [a_{ij} \pm b_{ij}]$.

Note that addition and subtraction make sense only if the dimensions of matrices are identical.

Scalar Multiplication of Matrices: $\lambda A = \lambda[a_{ij}] = [\lambda a_{ij}]$.

Matrix Multiplication: Given two matrices $A_{m \times n}$ and $B_{p \times q}$, matrix multiplication requires a **compatibility condition**: *the number of columns*

in matrix A is the same as the number of rows in matrix B , i.e., $n = p$. If the compatibility condition is satisfied, the dimension of the product of AB is $m \times q$. AB is defined as:

$$AB = C$$

with $c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj} = \sum_{l=1}^n a_{il}b_{lj}$. Obviously, the matrix product of AB is not necessarily equal to BA .

Identity Matrix

A square matrix is a matrix with the same number of rows and columns, which is assumed to be n .

An **identity matrix** of order n , denoted by I_n , is a square matrix with ones in its principal diagonal and zeros everywhere else.

It possesses the following properties:

Property 1:

$$I_m A_{m \times n} = A_{m \times n} I_n = A_{m \times n}$$

Property 2:

$$A_{m \times n} I_n B_{n \times p} = (A_{m \times n} I_n) B_{n \times p} = A_{m \times n} B_{n \times p}$$

Property 3:

$$(I_n)^k = I_n$$

This matrix is analogous to 1 in real space.

Idempotent Matrices: A matrix A is said to be **idempotent** if $AA = A$.

Null Matrix

A null or zero matrix (not necessarily a square matrix) –denoted by 0 , plays the role of the number 0.

A $m \times n$ **null matrix** is simply a matrix whose elements are all zero.

Null matrices obey the following rules of operation:

$$A_{m \times n} + 0_{m \times n} = A_{m \times n};$$

$$A_{m \times n} 0_{n \times p} = 0_{m \times p};$$

$$0_{q \times m} A_{m \times n} = 0_{q \times n}.$$

Remark 2.2.1 (1) $CD = CE$, $C \neq 0$ does not imply that $D = E$,
e.g.,

$$C = \begin{bmatrix} 2 & 3 \\ 6 & 9 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \quad E = \begin{bmatrix} -2 & 1 \\ 3 & 2 \end{bmatrix}.$$

(2) Even if A and $B \neq 0$, we still have $AB = 0$, e.g.,

$$A = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}, B = \begin{bmatrix} -2 & 4 \\ 1 & -2 \end{bmatrix}.$$

2.2.3 Linear Dependence of Vectors

One of the most important properties among vectors is linear dependence.

Definition 2.2.1 (Linear Dependence) A set of vectors v^1, \dots, v^n is *linearly dependent*, if and only if there exists a vector v^i , which is a linear combination of the others, i.e., $v^i = \sum_{j \neq i} \alpha_j v^j$.

Example 2.2.1 The following three vectors

$$v^1 = \begin{bmatrix} 2 \\ 7 \end{bmatrix}, v^2 = \begin{bmatrix} 1 \\ 8 \end{bmatrix}, v^3 = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

are linearly dependent, since

$$3v^1 - 2v^2 = \begin{bmatrix} 6 \\ 21 \end{bmatrix} - \begin{bmatrix} 2 \\ 16 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix} = v^3$$

or

$$3v^1 - 2v^2 - v^3 = \mathbf{0}$$

where $\mathbf{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is a zero vector.

2.2.4 Transpose and Inverse of Matrix

The transpose of a $m \times n$ matrix A is a matrix which is obtained by interchanging the rows and columns of the matrix A . Formally, we have

Definition 2.2.2 (Transpose of Matrix) $B = [b_{ij}]_{n \times m}$ is said to be the *transpose* of the matrix $A = [a_{ij}]_{m \times n}$, denoted by A' or A^T , if $a_{ji} = b_{ij}$ for all $i = 1, \dots, n$ and $j = 1, \dots, m$.

Definition 2.2.3 We have the following types of matrices regarding the transpose of a matrix:

The matrix A is said to be *symmetric* if $A' = A$.

The matrix A is said to be *antisymmetric* if $A' = -A$.

The matrix A is said to be *orthogonal* if $A'A = I$.

Properties of Transpose:

- a) $(A')' = A$;
- b) $(A + B)' = A' + B'$;
- c) $(\alpha A)' = \alpha A'$, where α is a real number;
- d) $(AB)' = B'A'$.

We now discuss the **inverse** of a square matrix. The inverse of matrix A , denoted by A^{-1} , should satisfy

$$AA^{-1} = A^{-1}A = I.$$

While the transpose of matrix always exists, the inverse does not necessarily exist.

Remark 2.2.2 The following statements are true:

- 1) Not every square matrix has an inverse, i.e., squareness is a necessary, but not sufficient, condition for the existence of an inverse. If a square matrix A has an inverse, A is said to be

nonsingular. If A has no inverse, it is said to be a **singular** matrix.

- 2) If A is nonsingular, then A and A^{-1} are inverse of each other, i.e., $(A^{-1})^{-1} = A$.
- 3) If A is $n \times n$, then A^{-1} is also $n \times n$.
- 4) The inverse of A is unique.
- 5) $AA^{-1} = I$ implies that $A^{-1}A = I$.
- 6) Suppose that A and B are nonsingular matrices with dimension $n \times n$.

$$(a) (AB)^{-1} = B^{-1}A^{-1}$$

$$(b) (A')^{-1} = (A^{-1})'$$

2.2.5 Solving a Linear System

Consider a system of n equations with n unknowns:

$$Ax = d.$$

If A is nonsingular, then multiplying both sides by A^{-1} gives:

$$A^{-1}Ax = A^{-1}d.$$

Therefore, $x = A^{-1}d$ is the unique solution of the linear system $Ax = d$, where A^{-1} is unique.

Prior to applying the method of inverse matrix to solve linear systems, it is necessary to first determine whether or not the matrix is nonsingular. Secondly, we shall solve it by the Cramer's rule.

There are two ways to test the nonsingularity of a square matrix A . The first is to determine whether or not the row or column vectors of a matrix are linearly independent; the second is to discern whether the determinant of a square matrix is equal to zero.

An $n \times n$ square matrix A could be written as a set of vectors in terms

of rows.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} = \begin{bmatrix} \mathbf{v}^{1'} \\ \mathbf{v}^{2'} \\ \cdots \\ \mathbf{v}^{n'} \end{bmatrix}$$

where $\mathbf{v}^{i'} = [a_{i1}, a_{i2}, \dots, a_{in}]$, $i = 1, 2, \dots, n$. Whether a square matrix A is nonsingular is determined by whether the vectors $\mathbf{v}^{i'}$, $i = 1, 2, \dots, n$ are linearly independent.

Determinant of a Matrix

The determinant of an n th order square matrix $A = (a_{ij})$, denoted by $|A|$ or $\det(A)$, is a uniquely defined scalar associated with that matrix. Determinants are defined only for square matrices. Prior to giving the definition of an n th order square matrix A , we give the definition of A with 2th and 3th orders, respectively.

For a 2×2 matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix},$$

its determinant is defined as follows:

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

In view of the dimension of matrix A , $|A|$ as defined in the above is called a **second-order determinant**.

For a 3×3 matrix A , its **third-order determinant** is defined as

$$\begin{aligned} |A| &= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\ &= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}. \end{aligned}$$

From the definition of the determinant for matrix A with 2th or 3th order, it is defined as the sum of all possible products, in which each product

consists of elements in different rows and columns. This is also true for a general n th order square matrix A . Then, for a $n \times n$ matrix A , its determinants is defined as:

$$|A| = \sum_{(\alpha_1, \dots, \alpha_n)} (-1)^{I(\alpha_1, \dots, \alpha_n)} a_{1\alpha_1} a_{2\alpha_2} \cdots a_{n\alpha_n},$$

where $(\alpha_1, \dots, \alpha_n)$ is the permutation of $(1, \dots, n)$, and $I(\alpha_1, \dots, \alpha_n)$ is the number of inversion times when reordering $(1, \dots, n)$.

For example, $(2, 1, 3)$ is reordered once by $(1, 2, 3)$, and $(2, 3, 1)$ is reordered twice by $(1, 2, 3)$.

There is a simple method to calculate the determinant of a matrix, i.e., the **Laplace expansion**:

$$|A| = \sum_{k=1}^n (-1)^{l+k} a_{lk} \times \det(M_{lk}), \text{ for any } l \in \{1, \dots, n\},$$

where M_{lk} is the $n - 1$ th order square matrix that results from A by deleting the l -th row and the k -th column, called the **minor of** a_{lk} .

The Laplace expansion of an n th-order determinant will reduce the problem to one of evaluating n minors, each of which is of the $(n - 1)$ th order, and the repeated application of the process will methodically lead to increasingly lower orders of determinants, eventually culminating in the basic second-order determinants. Then, the value of the original determinant is calculated.

Even though one can expand $|A|$ by any row or any column, concerning the numerical calculation, a row or column with largest number of 0's or 1's is always preferable for this purpose, because 0 times its cofactor is simply 0.

Basic Properties of Determinants

1. The determinant of a matrix A has the same value as that of its transpose A' , i.e., $|A| = |A'|$. Therefore, row independence is equivalent to column independence.
2. The multiplication of any one row (or column) by a scalar k

will change the value of the determinant k -fold.

3. The interchange of any two rows (columns) will alter the sign, but not the numerical value, of the determinant.
4. A scale of any row (column) added to any other row (column) does not change the value or the sign of the determinant.
5. If two rows (or columns) are proportional, i.e., they are linearly dependent, then the determinant will vanish.
6. $\det(AB) = \det(A)\det(B)$.
7. $\det(A^{-1}) = \frac{1}{\det(A)}$. As a consequence, a necessary condition for the existence of A^{-1} is that $\det(A) \neq 0$.

Using these properties, we can simplify the matrix (e.g., by applying Property 4, we can make a row or column have as many zero elements as possible), and the Laplace expansion of a determinant will become a much more manageable task.

Next, we provide a formula for solving the inverse of a nonsingular square matrix. Let $A^{-1} = (d_{ij})$, then

$$d_{ij} = \frac{1}{\det(A)} (-1)^{i+j} \det(M_{ij}).$$

The **Cramer's Rule** given below summarizes how to solve a linear system. For a system of linear equations:

$$A\mathbf{x} = \mathbf{d},$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix},$$

$$\mathbf{d}' = (d_1, \cdots, d_n),$$

$$\mathbf{x}' = (x_1, \cdots, x_n).$$

The solution is:

$$x_j = \frac{\det(A_j)}{\det(A)},$$

where:

$$A_j = \begin{bmatrix} a_{11} & \cdots & a_{1j-1} & b_1 & a_{1j+1} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2j-1} & b_2 & a_{2j+1} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & \cdots & a_{nj-1} & b_n & a_{nj+1} & \cdots & a_{nn} \end{bmatrix}$$

which is obtained by replacing the j th column of $|A|$ with constant terms d_1, \dots, d_n . This result is the statement of Cramer's rule.

2.2.6 Quadratic Form and Matrix

A function q with n variables is called a **quadratic form** if it has the following expression:

$$\begin{aligned} q(x_1, x_2, \dots, x_n) = & a_{11}x_1^2 + 2a_{12}x_1x_2 + \cdots & + 2a_{1n}x_1x_n \\ & + a_{22}x_2^2 + \cdots & + 2a_{2n}x_2x_n \\ & \cdots & \\ & & + a_{nn}x_n^2. \end{aligned}$$

Let $a_{ji} = a_{ij}$, $i < j$, and then $q(x_1, x_2, \dots, x_n)$ can be written as

$$\begin{aligned} q(x_1, x_2, \dots, x_n) = & a_{11}x_1^2 + a_{12}x_1x_2 + \cdots + a_{1n}x_1x_n \\ & + a_{12}x_2x_1 + a_{22}x_2^2 + \cdots + a_{2n}x_2x_n \\ & \cdots \\ & + a_{n1}x_nx_1 + a_{n2}x_nx_2 + \cdots + a_{nn}x_n^2 \\ = & \sum_{i=1}^n \sum_{j=1}^n a_{ij}x_i x_j \\ = & \mathbf{x}' A \mathbf{x}, \end{aligned}$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

is called a **matrix of quadratic form**. Since $a_{ij} = a_{ji}$, A is an n th-order symmetric square matrix.

Definition 2.2.4 For a matrix of quadratic form A , the quadratic form $q(u_1, u_2, \dots, u_n) = u' Au$ is said to be

- (a) *positive definite (PD)* if $q(u) > 0$ for all $u \neq 0$;
- (b) *positive semidefinite (PSD)* if $q(u) \geq 0$ for all $u \neq 0$;
- (c) *negative definite (ND)* if $q(u) < 0$ for all $u \neq 0$;
- (d) *negative semidefinite (NSD)* if $q(u) \leq 0$ for all $u \neq 0$.

Otherwise, q is called *indefinite (ID)*.

Sometimes, we say that a matrix D is, for instance, positive definite if the corresponding quadratic form $q(u) = u' Du$ is positive definite.

A necessary and sufficient condition for a matrix of quadratic form A to be positive definite is that all of its minors are positive, i.e.,

$$\begin{aligned} |A_1| &= A_{11} > 0; \\ |A_2| &= \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0; \\ &\dots \\ |A_n| &= \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} > 0. \end{aligned}$$

A necessary and sufficient condition for a quadratic form A to be negative definite is that its minors alternate in sign starting from negative, namely,

$$|A_1| < 0, |A_2| > 0, |A_3| < 0, \dots, (-1)^n |A_n| > 0.$$

2.2.7 Eigenvalues, Eigenvectors, and Traces

If a square matrix A and a real number λ satisfy the equation $Ax = \lambda x$, then λ is called the **eigenvalue** of A , and the vector x is called the **eigenvector** of A belonging to the eigenvalue λ .

Eigenvalues and some properties of matrix, such as positive or negative definiteness, have close connections. The following theorem characterizes the relation between the eigenvalues and positive (or negative) definiteness.

Theorem 2.2.1 *A Matrix of quadratic form A is*

positive definite, if and only if eigenvalues $\lambda_i > 0$ for all $i = 1, 2, \dots, n$;

negative definite, if and only if eigenvalues $\lambda_i < 0$ for all $i = 1, 2, \dots, n$;

positive semi-definite, if and only if eigenvalues $\lambda_i \geq 0$ for all $i = 1, 2, \dots, n$;

negative semi-definite, if and only if eigenvalues $\lambda_i \leq 0$ for all $i = 1, 2, \dots, n$;

indefinite, if at least one eigenvalue is positive and at least one eigenvalue is negative.

For a symmetric matrix A , there is a convenient decomposition method. Matrix A is **diagonalizable** if there exist a non-singular matrix P and a diagonal matrix D , such that

$$P^{-1}AP = D.$$

Matrix U is **orthogonal matrix** if $U' = U^{-1}$.

Theorem 2.2.2 (The Spectral Theorem for Symmetric Matrices) *Suppose that A is a symmetric matrix of order n and $\lambda_1, \dots, \lambda_n$ are its eigenvalues. Then, there exists an orthogonal matrix U , such that*

$$U^{-1}AU = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

or

$$A = U \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} U'.$$

Usually, U is the orthogonal matrix formed by eigenvectors. It has the property $U'U = I$. “Orthogonal” means that for any column u of the matrix U , $u'u = 1$.

The power operation of symmetric matrix has a convenient form:

$$A^k = U \begin{bmatrix} \lambda_1^k & & 0 \\ & \ddots & \\ 0 & & \lambda_n^k \end{bmatrix} U'.$$

If the eigenvalues of A are nonzero real numbers, then the inverse of A can be reformulated as follows:

$$A^{-1} = U \begin{bmatrix} \lambda_1^{-1} & & 0 \\ & \ddots & \\ 0 & & \lambda_n^{-1} \end{bmatrix} U'.$$

Another common concept about square matrix is the **trace**. The trace of an n th-order A is $tr(A) = \sum_{i=1}^n a_{ii}$. It also has following properties:

- (1) $tr(A) = \lambda_1 + \cdots + \lambda_n$;
- (2) If A and B have the same dimension, then $tr(A + B) = tr(A) + tr(B)$;
- (3) If a is a real number, $tr(aA) = a \cdot tr(A)$;
- (4) $tr(AB) = tr(BA)$, if AB is a square matrix;
- (5) $tr(A') = tr(A)$;
- (6) $tr(A'A) = \sum_{i=1}^n \sum_{j=1}^n a_{ij}^2$.

2.3 Basic Topology

Topology is a branch of mathematics that studies the basic properties of topological spaces and various kinds of mathematical structures defined on them. This branch is originated from the study of the set of points on real axis, manifolds, metric spaces, and functional analysis.

There are two branches of topology. The one that focuses on using the analysis method is called general topology, point-set topology, or analytic topology. If further subdivided, point-set topology also has a branch: differential topology. The other branch emphasizes the use of algebraic method, called algebraic topology. However, these branches tend to be unified. Topology is broadly applied in functional analysis, Lie Group, differential geometry, differential equations, and many other branches of mathematics.

Here, we provide a very brief introduction to basic knowledge of point-set topology, and apply it to establish some important conclusions about the properties of sets and continuous mappings between sets.

2.3.1 Topological Space

Definition 2.3.1 Suppose that X is a nonempty set. We call a family of subsets of X , denoted by τ , a *topology* of X if

- (1) both X and the empty set belong to τ ;
- (2) the union of any number of members in τ is in τ ;
- (3) the intersection of a finite number of members in τ is in τ .

The set X together with its topology τ is called a **topological space**, denoted by (X, τ) ; members in τ are called the **open sets** of this topological space.

Example: Examples of topological spaces:

- (1) **(Discrete Topology)** Suppose that X is a nonempty set. $\tau = 2^X$ gives a discrete topology.
- (2) **(Trivial Topology)** Suppose that X is a nonempty set. $\tau = \{X, \emptyset\}$ gives a trivial topology.

- (3) **(Euclidean Topology)** Suppose that R is the set of all real numbers. τ , defined as a collection of open sets, gives the Euclidean topology (see the following definition).
- (4) **(Quotient Topology)** Suppose that X is a nonempty set. By a given equivalence relation R , we partition X into disjoint subsets, all of which make up a new collection, denoted by X/R . We specify the subset U of X/R as an open set, and then X/R gives a quotient topology if and only if the union of any elements of U is an open set belonging to X .

Although the study object of topology can be an arbitrary type of sets, for convenience of understanding and application, in the following, we primarily introduce some commonly used topological spaces, especially metric spaces in a finite dimensional real space.

2.3.2 Metric Space

We first illustrate the definitions of metric and metric space. **Metric** is a measure of distance. A **metric space** (X, d) is composed of a set X and the metric d defined on the elements of X . The metric space may be of finite or infinite dimensions, depending on the topology structure defined on X .

The metric should satisfy three basic assumptions. For any $p, q, r \in X$, we have

- (1) $d(p, q) > 0$ if and only if $p \neq q$;
- (2) $d(p, q) = d(q, p)$;
- (3) $d(p, q) \leq d(p, r) + d(r, q)$.

Remark 2.3.1 If the metrics on the same set are different, then the corresponding metric spaces are different. For example,

- (1) Metric Space 1: $(X = \mathcal{R}^n, d_1)$, $\forall x^1, x^2 \in X, d_1(x^1, x^2) = \sqrt{\sum_i (x_i^1 - x_i^2)^2}$, and it is called the **n -dimensional Euclidean space**.
- (2) Metric Space 2: $(X = \mathcal{R}^n, d_2)$, $\forall x^1, x^2 \in X, d_2(x^1, x^2) = \sum_i |x_i^1 - x_i^2|$.

- (3) Metric Space 3: $(X = \mathcal{R}^n, d_3), \forall \mathbf{x}^1, \mathbf{x}^2 \in X, d_3(\mathbf{x}^1, \mathbf{x}^2) = \max\{|x_1^1 - x_1^2|, \dots, |x_n^1 - x_n^2|\}$.

Although the remaining discussions in this section are also true for general metric spaces, we mainly focus on Euclidean spaces for convenience of statement.

2.3.3 Open Sets, Closed Sets, and Compact Sets

With the concept of metric, we can clearly define proximity between points. In an n -dimensional Euclidean space, given $\mathbf{x}^0 \in \mathcal{R}^n$, the set of all points of distances less than ϵ from \mathbf{x}^0 is called an **open ball** with radius ϵ and center \mathbf{x}^0 , denoted by $B_\epsilon(\mathbf{x}^0)$. A related concept is **closed ball**, which is given by the set of all points of distances less than or equal to ϵ , denoted by $B_\epsilon^*(\mathbf{x}^0)$.

Next, we give the definition of closed sets and compact sets.

Definition 2.3.2 The set $S \subseteq \mathcal{R}^n$ is an *open set* if for any $\mathbf{x} \in S$, there always exists an $\epsilon > 0$, such that $B_\epsilon(\mathbf{x}) \subseteq S$.

Based on the definition of open sets, the following theorem gives some basic properties of open sets:

Theorem 2.3.1 (Open Sets in \mathcal{R}^n) *In terms of open sets, the following conclusions are true.*

- (1) *The empty set \emptyset is an open set.*
- (2) *The universal space \mathcal{R}^n is an open set.*
- (3) *The union of open sets is an open set.*
- (4) *The intersection of a finite number of open sets is an open set.*

PROOF. (1) Since \emptyset has no elements, the proposition “for each point in \emptyset , there is an ϵ, \dots ”, satisfies the definition of an empty set.

(2) For any point \mathbf{x} in \mathcal{R}^n and any $\epsilon > 0$, according to the definition of an open ball, the set $B_\epsilon(\mathbf{x})$ is a subset in \mathcal{R}^n . Therefore, $B_\epsilon(\mathbf{x}) \subseteq \mathcal{R}^n$, and then \mathcal{R}^n is open.

(3) For all $i \in I$, let S_i be an open set. We need to show that $\cup_{i \in I} S_i$ is an open set. Suppose that $\mathbf{x} \in \cup_{i \in I} S_i$. Then, for some $i' \in I$, we have $\mathbf{x} \in S_{i'}$.

Since $S_{i'}$ is open, we have $B_\epsilon(x) \subseteq S_{i'}$ for an $\epsilon > 0$. It then follows that $B_\epsilon(x) \subseteq \cup_{i \in I} S_i$, and thus $\cup_{i \in I} S_i$ is open.

(4) Suppose that $B = \bigcap_{k=1}^n B_k$. If $B = \emptyset$, it is clear that B is an open set. If $B \neq \emptyset$, for any $x \in B$, obviously, we have: for any $k \in \{1, \dots, n\}$, $x \in B_k$. Since B_k is an open set, there must exist an $\epsilon_k > 0$, such that $B_{\epsilon_k}(x) \subseteq B_k$. Let $\epsilon = \min\{\epsilon_1, \dots, \epsilon_n\}$. Then, for any $k \in \{1, \dots, n\}$, $B_\epsilon(x) \subseteq B_k$, and thus $B_\epsilon(x) \subseteq B$. Therefore, B is an open set. \square

The following theorem shows the relationship between open sets and open balls.

Theorem 2.3.2 (Each open set is a union of open balls) Suppose that $S \subseteq \mathcal{R}^n$ is an open set. Then, for each $x \in S$, there exists an $\epsilon_x > 0$, such that $B_{\epsilon_x}(x) \subseteq S$, and also

$$S = \bigcup_{x \in S} B_{\epsilon_x}(x).$$

PROOF. Suppose that $S \subseteq \mathcal{R}^n$ is an open set. Then, it follows from the definition of open sets that for any $x \in S$, there exists an $\epsilon_x > 0$, such that $B_{\epsilon_x}(x) \subseteq S$. We now need to show that $x' \in S$ implies that $x' \in \cup_{x \in S} B_{\epsilon_x}(x)$, and $x' \in \cup_{x \in S} B_{\epsilon_x}(x)$ implies $x' \in S$.

If $x' \in S$, then it follows from the definition of open balls with centre x' that $x' \in B_{\epsilon_{x'}}(x')$. But x' belongs to any union containing this open ball. Therefore, we have $x' \in \cup_{x \in S} B_{\epsilon_x}(x)$.

If $x' \in \cup_{x \in S} B_{\epsilon_x}(x)$, then $x' \in B_{\epsilon_{x'}}(x')$. Since $B_{\epsilon_{x'}}(x') \subseteq S$, it follows that $x' \in S$. \square

We now discuss closed sets, and first provide the definition of closed sets based on the definition of open sets.

Definition 2.3.3 (Closed Sets in \mathcal{R}^n) If the complement of S , i.e., S^c , is an open set, then S is a *closed set*.

We also have some conclusions about the basic properties of closed sets.

Theorem 2.3.3 (Closed Sets in \mathcal{R}^n) In terms of closed sets, the following conclusions are true.

- (1) The empty set \emptyset is a closed set.

- (2) *The universal space \mathcal{R}^n is a closed set.*
- (3) *The intersection of any collection of closed sets is a closed set.*
- (4) *The union of a finite number of closed sets is a closed set.*

PROOF. (1) Since $\emptyset = \{\mathcal{R}^n\}^c$, and \mathcal{R}^n is an open set, it follows from the definition of closed sets that \emptyset is a closed set.

(2) Since $\{\mathcal{R}^n\}^c = \emptyset$, and \emptyset is an open set, it follows from the definition of closed sets that \mathcal{R}^n is a closed set.

(3) Suppose that for all $i \in I$, S_i is a closed set in \mathcal{R}^n . Then, it is necessary to show that $\cap_{i \in I} S_i$ is closed. Since S_i is closed, its complement S_i^c is an open set. The union $\cup_{i \in I} S_i^c$ is also open. It follows from the De Morgan's laws that $i \in I$, $(\cup_{i \in I} S_i^c)^c = \cap_{i \in I} S_i$ holds. Since $\cup_{i \in I} S_i^c$ is open, then its complement $\cap_{i \in I} S_i$ is closed.

(4) Let C_1 and C_2 be closed sets, and denote $C = C_1 \cup C_2$. Since C_1 and C_2 are closed, $C_k^c = B_k$, $k = 1, 2$, are open. It follows from the properties of open sets above that $B_1 \cap B_2$ is an open set, and thus $C = (B_1 \cap B_2)^c$ is a closed set. \square

Next, we discuss the concept of point sets related to open and closed sets.

Definition 2.3.4 For set S in \mathcal{R}^n , a point $x \in \mathcal{R}^n$ is called *limit point* (or *cluster point* or **accumulation point**) of S if for any $\epsilon > 0$, $B_\epsilon(x) \cap S \neq \emptyset$, i.e., every neighbourhood of x contains at least one point of S different from x itself. The collection of all limit points of set S is denoted by ∂S . For set S , a point $x \in S$ is called an **interior point** of S , if there is an $\epsilon > 0$, such that $B_\epsilon^*(x) \subseteq S$.

Now, we can redefine the open set as follows: a set is open if every element in the set is an interior point. Similarly, closed sets can also be defined as follows: a set is called a closed set if all limit points of the set belong to itself. In addition, for any set S in a metric space, the smallest closed set containing S is called the **closure** of the set, denoted by $\bar{S} = S \cup \partial S$ or $\text{cl}S$. Obviously, if S is closed, $\bar{S} = S$.

Definition 2.3.5 (Bounded Set) If a set S in \mathcal{R}^n is contained in a ball (open or closed ball) with radius ϵ , then S is called *bounded*. In other words, if for some $x \in \mathcal{R}^n$, there is an $\epsilon > 0$, such that $S \subseteq B_\epsilon(x)$, and then S is *bounded*.

Definition 2.3.6 (Compact Set) If a set $S \subseteq \mathcal{R}^n$ is closed and bounded, then it is *compact*.

The compact set is a crucial concept in mathematical analysis, but the definition of compact sets given above only applies to finite dimensional spaces. The definition of compact sets for infinite dimensional spaces is based on the concept of open cover. Whether a set is in finite or infinite dimension, another way to define compact sets exists, i.e., a set is compact if each open cover of a set has a finite subcover.

We first introduce the concept of open covering.

Definition 2.3.7 (Open Cover) For a set S and a collection of open sets $\{G_\alpha\}$ in metric space X , if $S \subseteq \bigcup_\alpha G_\alpha$, then $\{G_\alpha\}$ is called an **open cover** of S ; if the index set $\{\alpha\}$ is finite, it is called a finite open cover.

Next, we discuss an important feature of the compact set. The following Heine-Borel theorem, also known as the finite covering theorem, proves that the above two ways of definition are consistent for compact sets in finite dimensional spaces.

Theorem 2.3.4 (Heine-Borel Theorem or Finite Covering Theorem) For a set $S \subseteq \mathcal{R}^n$, the following two arguments are consistent:

- (1) S is a bounded closed set;
- (2) Any open cover of S has a finite subcover $\{G_\alpha\}$. In other words, for $\{G_\alpha\}$, there is a finite set $\{1, \dots, n\} \subseteq \{\alpha\}$, such that $S \subseteq \bigcup_{i=1}^n G_i$.

Refer to the proof of Theorem 2.41 in Rudin's *Principles of Mathematical Analysis*.

2.3.4 Connectedness of Sets

Definition 2.3.8 (Connected Set) For a set S in metric spaces, if there do not exist two sets A and B , such that $A \cap \bar{B} = B \cap \bar{A} = \emptyset$, and $S \subseteq A \cup B$, then S is called a *connected set*.

The following theorem illustrates the characteristic of connected sets.

Theorem 2.3.5 *The set $S \subseteq \mathcal{R}^1$ is connected if and only if it satisfies the following property: for any $x, y \in S$, if $x < z < y$, then $z \in S$.*

Refer to the proof of the Theorem 2.47 in Rudin's *Principles of Mathematical Analysis*. Obviously, the whole real space is connected, and intervals in real space, such as (a, b) and $[a, b]$, are all connected sets.

2.3.5 Sequences and Convergence

Definition 2.3.9 (Sequence in \mathcal{R}^n) Let Z be the set of positive integers. A *sequence* in \mathcal{R}^n is a function which maps Z into \mathcal{R}^n , represented by $\{x^k\}_{k \in Z}$, and for each $k \in Z$, $x^k \in \mathcal{R}^n$.

For all sufficiently large k , if each element of sequence $\{x^k\}$ can arbitrarily approach a point in \mathcal{R}^n , then we conclude that the sequence converges to this point. Formally, we have the following definition:

Definition 2.3.10 (Convergent Sequence) If for each $\epsilon > 0$, there is a \bar{k} , such that for all $k \in Z$ larger than \bar{k} , $x^k \in B_\epsilon(x)$, then we call that the sequence $\{x^k\}_{k \in Z}$ *converges to* $x \in \mathcal{R}^n$.

Like subsets of a set, we have the concept of subsequences of a sequence.

Definition 2.3.11 (Subsequence) If J is an infinite subset of Z , then $\{x^k\}_{k \in J}$ is called a *subsequence* of $\{x^k\}_{k \in Z}$ in \mathcal{R}^n .

Definition 2.3.12 (Bounded Sequences) If for $M \in \mathcal{R}$ and any $k \in Z$, $\|x^k\| \leq M$, then the sequence $\{x^k\}_{k \in Z}$ in \mathcal{R}^n is *bounded*.

The following is a property of the subsequence of a bounded sequence.

Theorem 2.3.6 (Bounded Sequences) *Each bounded sequence in \mathcal{R}^n has a convergent subsequence.*

2.3.6 Convex Set and Convexity

The convex set is an important type of set, which is widely used in economics. For example, sets of budget constraints of indivisible goods are generally convex sets and possess a strong economic meaning.

We first define convex sets.

Definition 2.3.13 If for any two elements $\mathbf{x}^1, \mathbf{x}^2 \in S$ and any $t \in [0, 1]$, we have $t\mathbf{x}^1 + (1 - t)\mathbf{x}^2 \in S$, then the set $S \subseteq \mathcal{R}^n$ is a *convex set*.

If $\mathbf{z} = t\mathbf{x}^1 + (1 - t)\mathbf{x}^2, t \in (0, 1)$, then point \mathbf{z} is called the weighted average or convex combination of \mathbf{x}^1 and \mathbf{x}^2 . If $\mathbf{z} = \sum_{l=1}^k \alpha^l \mathbf{x}^l, \mathbf{x}^l \in S, \alpha^l \in [0, 1], l \in \{1, \dots, k\}, \sum_l \alpha^l = 1$, then \mathbf{z} is also a convex combination of $\{\mathbf{x}^l\}$.

We have the following theorems about convex sets.

Theorem 2.3.7 If both sets S and T are convex, then their intersection $T \cap S$ is also convex.

Any set can be convexified, i.e., it has a convex hull, denoted by $co S$.

Definition 2.3.14 The *convex hull* of a set $S \subseteq \mathcal{R}^n$ is the smallest convex set containing S , denoted by $co S$.

The following theorem illustrates how to convexify a set.

Theorem 2.3.8 For a set $S \subseteq \mathcal{R}^n$, its convex hull is

$$co S = \left\{ \mathbf{y} \in \mathcal{R}^n : \mathbf{y} = \sum_{l=1}^k \alpha^l \mathbf{x}^l, \mathbf{x}^l \in S, \forall \alpha^l \in [0, 1], l \in \{1, \dots, k\}, \sum_l \alpha^l = 1 \right\},$$

i.e., the convex hull of S is formed by the set of all convex combinations of finite points in S .

The points of the convex hull are made up of convex combinations of finite points. The following Caratheodory theorem simplifies the way of convexification in finite dimensional real space.

Theorem 2.3.9 (Caratheodory Theorem) If the set is in a finite dimensional real space, i.e., $S \subseteq \mathcal{R}^n$, the points of its convex hull $co S$ can be written as the convex combination of at most $n + 1$ points in S .

The following theorems show that the convex hull of a compact set is a compact set.

Theorem 2.3.10 *If $S \subseteq \mathcal{R}^n$ is a compact set, then its convex hull $\text{co } S$ is also a compact set.*

See A3.1 of Kreps (2013) for the proof of the above three theorems.

Every point in a convex hull is a convex combination of finite points in a set, but this does not mean that it must be a convex combination formed by other points. If a point is not a convex combination formed by other points, we define such a point as the extreme point. For compact sets, the structure of the convex hull will be more simplified. The following Krein-Milman theorem characterizes the convex hull of compact sets.

Theorem 2.3.11 (Krein-Milman Theorem) *If a set S is a compact set of a finite dimensional real space, and $EX(S)$ is the set of the extreme points of set S , then $\text{co } S = \text{co } EX(S)$, which means that the convex hull of a compact set is composed of finite convex combinations of all of the extreme points.*

2.4 Single-Valued Function

2.4.1 Continuity of functions

Continuity of functions can be defined in any topological space X .

Definition 2.4.1 (Continuity) A function $f : X \rightarrow \mathcal{R}$ is *continuous* at $x_0 \in X$ if

$$\lim_{x \rightarrow x_0} f(x) = f(x_0),$$

or equivalently, **the upper contour set** of f at x_0

$$U(x_0) \equiv \{x' \in X : f(x') \geq f(x_0)\}$$

and its **lower contour set**

$$L(x_0) \equiv \{x' \in X : f(x') \leq f(x_0)\}$$

are both the closed subsets of X .

When we suppose $X \subseteq \mathcal{R}^n$, it can be equivalently defined: for any $\epsilon > 0$, there is $\delta > 0$, such that for any $\mathbf{x} \in X$ with $|\mathbf{x} - \mathbf{x}_0| < \delta$, we have

$$|f(\mathbf{x}) - f(\mathbf{x}_0)| < \epsilon.$$

In economics, it is usually assumed that $X \subseteq \mathcal{R}^n$. If f is continuous at any $\mathbf{x} \in X$, we say the function $f : X \rightarrow \mathcal{R}$ is continuous on X .

Although the three definitions of continuity are all equivalent, the second definition is easier to verify. The idea of continuity is quite intuitive. If we draw the function, the curve has no disconnected point.

Since the function is continuous, then the change of $f(x)$ is small when x changes slightly.

The following theorem illustrates the relationship between the continuity of functions and the open sets.

Theorem 2.4.1 (Continuity and Inverse Image) *Let D be a subset of \mathcal{R}^m . Then, the following conditions are equivalent.*

- (1) $f : D \rightarrow \mathcal{R}^n$ is continuous.
- (2) For each open ball B in \mathcal{R}^n , $f^{-1}(B)$ is also open in D .
- (3) For each open set S in \mathcal{R}^n , $f^{-1}(S)$ is also open in D .

PROOF. We will show that (1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (1).

(1) \Rightarrow (2). Suppose that (1) holds, and B is an open ball in \mathcal{R}^n . Picking any $\mathbf{x} \in f^{-1}(B)$, we have $f(\mathbf{x}) \in B$. Since B is open in \mathcal{R}^n , then there is an $\epsilon > 0$, such that $B_\epsilon(f(\mathbf{x})) \subseteq B$, and it follows from the continuity of f that there is a δ , such that $f(B_\delta(\mathbf{x}) \cap D) \subseteq B_\epsilon(f(\mathbf{x})) \subseteq B$. Therefore, $B_\delta(\mathbf{x}) \cap D \subseteq f^{-1}(B)$. Since $\mathbf{x} \in f^{-1}(B)$ is arbitrary, it can be seen that $f^{-1}(B)$ is open in D , and thus (2) is established.

(2) \Rightarrow (3). Suppose that (2) holds, and S is open in \mathcal{R}^n . Then, S can be written as a union of open balls $B_i (i \in I)$, such that $S = \cup_{i \in I} B_i$. Therefore, $f^{-1}(S) = f^{-1}(\cup_{i \in I} B_i) = \cup_{i \in I} f^{-1}(B_i)$. It follows from (2) that each set $f^{-1}(B_i)$ is open in D , and then $f^{-1}(S)$ is the union of open sets in D . Therefore, $f^{-1}(S)$ is also open in D . Since S is an arbitrary open set in \mathcal{R}^n , (3) is established.

(3) \Rightarrow (1). Suppose that (3) holds. Take $\mathbf{x} \in D$ and $\varepsilon > 0$. Then, since $B_\varepsilon(f(\mathbf{x}))$ is open in \mathcal{R}^n , it follows from (3) that $f^{-1}(B_\varepsilon(f(\mathbf{x})))$ is open in D . Since $\mathbf{x} \in f^{-1}(B_\varepsilon(f(\mathbf{x})))$, there is a $\delta > 0$, such that $B_\delta(\mathbf{x}) \cap D \subseteq f^{-1}(B_\varepsilon(f(\mathbf{x})))$, which means that $f(B_\delta(\mathbf{x}) \cap D) \subseteq B_\varepsilon(f(\mathbf{x}))$. Therefore, f is continuous at \mathbf{x} . Since \mathbf{x} is arbitrary, (1) is established. \square

We have the following conclusion for a continuous function whose domain is a compact set.

Theorem 2.4.2 (The continuous image of a compact set is a compact set) Suppose that $f : D \subseteq \mathcal{R}^m \rightarrow \mathcal{R}^n$ is a continuous function. If $S \subseteq D$ is a compact set in D , then its image $f(S) \subseteq \mathcal{R}^n$ is compact in \mathcal{R}^n .

2.4.2 Upper Semi-continuity and Lower Semi-continuity

The **upper semi-continuity** and **lower semi-continuity** of functions are weaker than continuity. Suppose that X is an arbitrary topological space.

Definition 2.4.2 A function $f : X \rightarrow \mathcal{R}$ is said to be *upper semi-continuous* at point $\mathbf{x}_0 \in X$ if we have

$$\limsup_{\mathbf{x} \rightarrow \mathbf{x}_0} f(\mathbf{x}) \leq f(\mathbf{x}_0);$$

or equivalently, the upper contour set $U(\mathbf{x}_0)$ of f is a closed set of X .

When we suppose $X \subseteq \mathcal{R}^n$, it can be equivalently defined: for any $\epsilon > 0$, there is a $\delta > 0$, such that for any $\mathbf{x} \in X$ with $|\mathbf{x} - \mathbf{x}_0| < \delta$, we have

$$f(\mathbf{x}) < f(\mathbf{x}_0) + \epsilon.$$

A function $f : X \rightarrow \mathcal{R}$ is said to be *upper semi-continuous* on X if f is upper semi-continuous at every point $\mathbf{x} \in X$.

Definition 2.4.3 A function $f : X \rightarrow \mathcal{R}$ is said to be *lower semi-continuous* on X if $-f$ is upper semi-continuous.

It is clear that a function $f : X \rightarrow \mathcal{R}$ is continuous on X if and only if it is both upper and lower semi-continuous.

2.4.3 Transfer Upper and Lower Continuity

A weaker concept of continuity is **transfer continuity**. It is used to completely characterize the extreme values of functions or preferences (see a series of papers by Tian (1992, 1993, 1994), Tian & Zhou (1995), and Zhou & Tian (1992)). Suppose that X is an arbitrary topological space.

Definition 2.4.4 A function $f : X \rightarrow \mathcal{R}$ is said to be *transfer (weakly) upper continuous* on X , if for any points $x, y \in X$, $f(y) < f(x)$ means that there exists a point $x' \in X$ and a neighbourhood $\mathcal{N}(y)$ of y , such that $f(z) < f(x')$ ($f(z) \leq f(x')$) for all $z \in \mathcal{N}(y)$.

Definition 2.4.5 A function $f : X \rightarrow \mathcal{R}$ is said to be *transfer (weakly) lower continuous* on X , if $-f$ is transfer (weakly) upper continuous on X .

Remark 2.4.1 It is clear that the upper (lower) semi-continuity of a function implies transfer upper (lower) continuity (let $x' = x$); while transfer upper (lower) continuity implies the transfer weakly upper (lower) continuity of the function, and the converse may not be true. We will then prove that a function f has the maximal (minimal) value on the compact set X if and only if f is transfer weakly upper continuous on X , and the set of maximal (minimal) points of f is compact if and only if f is transfer upper (lower) continuous on X .

2.4.4 Differentiation and Partial Differentiation of Functions

The differentiability of a function in one-dimensional real space measures sensitivity to the change of a function's value with respect to a change in an independent variable. Let X be a subset of \mathcal{R} .

Definition 2.4.6 (Derivative) The *derivative* of $f : X \rightarrow \mathcal{R}$ at point $x_0 \in X$ is defined as

$$f'(x_0) = \lim_{\Delta x \rightarrow 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x},$$

where $\Delta x = x - x_0$.

Obviously, if a function has a derivative at a point, then it must be continuous; however, this may not be true for the converse.

We may use the derivatives to find the limit of a continuous function of which the numerator and denominator approach to zero (or infinity), i.e., we have the following L'Hopital rule:

Theorem 2.4.3 (L'Hopital Rule) Suppose that $f(x)$ and $g(x)$ are differentiable on an open interval I , except possibly at c . If $\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} g(x) = 0$ or $\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} g(x) = \pm\infty$, $g'(x) \neq 0$ for all x in I with $x \neq c$, and $\lim_{x \rightarrow c} \frac{f'(x)}{g'(x)}$ exists. Then,

$$\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \lim_{x \rightarrow c} \frac{f'(x)}{g'(x)}.$$

Higher order derivatives and partial derivatives are widely used in economics.

Definition 2.4.7 (Higher Order Derivative) The n th order derivative of $f : X \rightarrow \mathcal{R}$ at $x_0 \in X$ is defined as

$$f^{[n]}(x_0) = \lim_{\Delta x \rightarrow 0} \frac{f^{[n-1]}(x_0 + \Delta x) - f^{[n-1]}(x_0)}{\Delta x}.$$

In a multidimensional real space $X \subseteq \mathcal{R}^n$, we can define the concept of partial differentiation of a function $f : X \rightarrow \mathcal{R}$, $f(x_1, \dots, x_n)$, to measure the degree of change of the function value with respect to a change of one of n independent variables, with the others being held constant.

Definition 2.4.8 (Partial Derivative) The partial derivative of $f : X \rightarrow \mathcal{R}$, $X \subseteq \mathcal{R}^n$ with respect to x_i at $\mathbf{x}^0 = (x_1^0, \dots, x_n^0) \in X$ is defined as

$$\frac{\partial f(\mathbf{x}^0)}{\partial x_i} = \lim_{\Delta x_i \rightarrow 0} \frac{f(x_1^0, \dots, x_i^0 + \Delta x_i, \dots, x_n^0) - f(\mathbf{x}^0)}{\Delta x_i}.$$

We characterize the degree of change of a multidimensional function in different directions in the way of the matrix, which is called the gradient vector.

Definition 2.4.9 (Gradient Vector) Let f be a function defined on \mathcal{R}^n that has partial derivatives. Define the gradient of f as a vector

$$Df(\mathbf{x}) = \left[\frac{\partial f(\mathbf{x})}{\partial x_1}, \frac{\partial f(\mathbf{x})}{\partial x_2}, \dots, \frac{\partial f(\mathbf{x})}{\partial x_n} \right].$$

Suppose that f has second-order partial derivative. We define the Hessian matrix of f at \mathbf{x} as an $n \times n$ matrix $D^2 f(\mathbf{x})$:

$$D^2 f(\mathbf{x}) = \left[\frac{\partial^2 f(\mathbf{x})}{\partial x_i \partial x_j} \right].$$

If all of the second-order partial derivatives are continuous, then

$$\frac{\partial^2 f(\mathbf{x})}{\partial x_i \partial x_j} = \frac{\partial^2 f(\mathbf{x})}{\partial x_j \partial x_i},$$

and thus the above matrix is a symmetric matrix.

2.4.5 Mean Value Theorem and Taylor Expansion

Theorem 2.4.4 (Férmat lemma) *Let X be a subset of \mathcal{R} . Suppose that*

- (i) $f : X \rightarrow \mathcal{R}$ is well-defined in a neighborhood $N(x_0)$ of x_0 , and $f(x) \leq f(x_0)$ or $f(x) \geq f(x_0)$ in this neighborhood;
- (ii) $f(x)$ is derivable at point x_0 .

Then, we have

$$f'(x_0) = 0.$$

Theorem 2.4.5 (Rolle Theorem) *Suppose that f is continuous on $[a, b]$, differentiable on (a, b) , and $f(a) = f(b)$. Then, there exists at least one point $c \in (a, b)$, such that $f'(c) = 0$.*

From the Roll Theorem, we can have the well-known and useful Lagrange's Theorem or the Mean-Value Theorem.

Theorem 2.4.6 (The Mean-Value Theorem or the Lagrange Formula) *Suppose that $f : [a, b] \rightarrow \mathcal{R}$ is continuous on $[a, b]$ and differentiable on (a, b) . Then, there exists $c \in (a, b)$, such that*

$$f'(c) = \frac{f(b) - f(a)}{(b - a)}.$$

The above mean value theorem is also true for multivariate \mathbf{x} . If function $f : \mathcal{R}^n \rightarrow \mathcal{R}$ is differentiable, then there is $\mathbf{z} = t\mathbf{x} + (1 - t)\mathbf{y}$ with $0 \leq t \leq 1$, such that

$$f(\mathbf{y}) = f(\mathbf{x}) + Df(\mathbf{z})(\mathbf{y} - \mathbf{x}).$$

Proof. Let $g(x) = f(x) - \frac{f(b)-f(a)}{b-a}x$. Then, g is continuous on $[a, b]$, differentiable on (a, b) , and $g(a) = g(b)$. Therefore, by Rolle or Férmat's Theorem, there exists a point $c \in (a, b)$, such that $g'(c) = 0$, and therefore $f'(c) = \frac{f(b)-f(a)}{b-a}$.

A variation of the above mean-value theorem is in the form of integral calculus:

Theorem 2.4.7 (Mean-Value Theorem of Integral Calculus) *If $f : [a, b] \rightarrow \mathcal{R}$ is continuous on $[a, b]$, then there exists a number $c \in (a, b)$, such that*

$$\int_a^b f(x)dx = f'(c)(b - a).$$

The second variation of the mean-value theorem is the generalized mean-value theorem:

Theorem 2.4.8 (Cauchy's Theorem or the Generalized Mean-Value Theorem)

If f and g are continuous on $[a, b]$ and differentiable on (a, b) , then there exists a point $c \in (a, b)$, such that $(f(b) - f(a))g'(c) = (g(b) - g(a))f'(c)$.

Taylor's expansion is a useful method for solving approximation.

Consider a continuously differentiable function $f : \mathcal{R}^n \rightarrow \mathcal{R}$, $\mathbf{x}, \mathbf{y} \in \mathcal{R}^n$. By the mean-value theorem, we know that there exist $\mathbf{z}, \mathbf{w} \in co(\mathbf{x}, \mathbf{y})$, such that the following two equations hold:

$$f(\mathbf{y}) = f(\mathbf{x}) + Df(\mathbf{z})(\mathbf{y} - \mathbf{x}),$$

$$f(\mathbf{y}) = f(\mathbf{x}) + Df(\mathbf{x})(\mathbf{y} - \mathbf{x}) + \frac{1}{2}(\mathbf{y} - \mathbf{x})' D^2 f(\mathbf{w})(\mathbf{y} - \mathbf{x}),$$

where $(\mathbf{y} - \mathbf{x})'$ is the transpose of the vector $(\mathbf{y} - \mathbf{x})$.

Generally, we have the following theorem:

Theorem 2.4.9 (Taylor's Theorem) *Given any function $f(x) : \mathcal{R} \rightarrow \mathcal{R}$, if there exists $(n + 1)$ th order derivative at x_0 , then the function can be expanded*

at x_0 :

$$\begin{aligned} f(x) &= f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2!}f''(x_0)(x - x_0)^2 + \cdots + \frac{1}{n!}f^{(n)}(x_0)(x - x_0)^n + R_n \\ &\equiv P_n + R_n, \end{aligned}$$

where P_n represents the n -th order polynomial, and R_n is the Lagrange's remainder:

$$R_n = \frac{f^{(n+1)}(P)}{(n+1)!}(x - x_0)^{n+1},$$

where P is a point between x and x_0 , and $n!$ is the factorial of n :

$$n! \equiv n(n-1)(n-2) \cdots (3)(2)(1).$$

We have the following approximation of function by Taylor's expansion. If \mathbf{y} approximates \mathbf{x} , then

$$\begin{aligned} f(\mathbf{y}) &\approx f(\mathbf{x}) + Df(\mathbf{x})(\mathbf{y} - \mathbf{x}), \\ f(\mathbf{y}) &\approx f(\mathbf{x}) + Df(\mathbf{x})(\mathbf{y} - \mathbf{x}) + \frac{1}{2}(\mathbf{y} - \mathbf{x})' D^2 f(\mathbf{x})(\mathbf{y} - \mathbf{x}). \end{aligned}$$

2.4.6 Homogeneous Functions and Euler's Theorem

Definition 2.4.10 Let $X = \mathcal{R}^n$. A function $f : X \rightarrow \mathcal{R}$ is said to be *homogeneous of degree k* if for any t , $f(t\mathbf{x}) = t^k f(\mathbf{x})$.

An important result concerning homogeneous function is Euler's theorem.

Theorem 2.4.10 (Euler's Theorem) A function $f : \mathcal{R}^n \rightarrow \mathcal{R}$ is homogeneous of degree k if and only if

$$kf(\mathbf{x}) = \sum_{i=1}^n \frac{\partial f(\mathbf{x})}{\partial x_i} x_i.$$

2.4.7 Implicit Function Theorem

If a variable y is clearly expressed as a function of \mathbf{x} , we call $y = f(x_1, x_2, \dots, x_n)$ an **explicit function**. In many cases, y is not an explicit function, and the

relationship between y and x_1, \dots, x_n is expressed by an equation:

$$F(y, x_1, x_2, \dots, x_n) = 0.$$

For a domain D , if for each vector $\mathbf{x} \in D$, there is a unique determined value y satisfying the above equation, then y is an **implicit function** of \mathbf{x} , denoted by $y = f(x_1, x_2, \dots, x_n)$. Then, the pertinent question is how to determine whether there is a unique value y satisfying this equation for every \mathbf{x} in a certain domain. The following implicit function theorem indicates that, under certain conditions, the implicit function $y = f(x_1, x_2, \dots, x_n)$ determined by $F(y, x_1, x_2, \dots, x_n) = 0$ not only exists, but is also differentiable.

Theorem 2.4.11 (Implicit Function Theorem) *Let $X = \mathcal{R}^n$. Suppose that a function $F(y, x_1, x_2, \dots, x_n) = 0$ satisfies the following four conditions:*

- (a) $F_y, F_{x_1}, F_{x_2}, \dots, F_{x_n}$ are continuous in the domain X containing $(y^0, x_1^0, x_2^0, \dots, x_n^0)$;
- (b) $F(y, x_1, x_2, \dots, x_n)$ has continuous partial derivatives with respect to \mathbf{x} and y in the domain X ;
- (c) $F(y^0, x_1^0, x_2^0, \dots, x_n^0) = 0$;
- (d) The partial derivative F_y of $F(y, x_1, x_2, \dots, x_n)$ with respect to y at $(y^0, x_1^0, x_2^0, \dots, x_n^0)$ is not equal to zero.

Then:

- (1) In a neighbourhood $N(\mathbf{x}^0)$ of $(x_1^0, x_2^0, \dots, x_n^0)$, the function $y = f(x_1, x_2, \dots, x_n)$ of (x_1, x_2, \dots, x_n) can be defined implicitly, which satisfies $F(y(x_1, \dots, x_n), x_1, x_2, \dots, x_n) = 0$ and $y^0 = f(x_1^0, x_2^0, \dots, x_n^0)$.
- (2) $y = f(x_1, x_2, \dots, x_n)$ is continuous in $N(\mathbf{x}^0)$.
- (3) $y = f(x_1, x_2, \dots, x_n)$ has continuous partial derivatives in $N(\mathbf{x}^0)$, which is given by:

$$\frac{\partial y}{\partial x_i} = -\frac{F_i}{F_y}, \quad i = 1, \dots, n.$$

2.4.8 Concave and Convex Function

Concave functions, convex functions, and quasi-concave functions are common functions in economics and possess strong economic significance. They also hold a special position in optimization problems.

Let $X \subseteq \mathcal{R}^n$ be a convex set.

Definition 2.4.11 For a function $f : X \rightarrow \mathcal{R}$, if for any $\mathbf{x}, \mathbf{x}' \in X$ and any $t \in [0, 1]$, we have

$$f(t\mathbf{x} + (1-t)\mathbf{x}') \geq tf(\mathbf{x}) + (1-t)f(\mathbf{x}'),$$

then, f is said to be *concave* on X .

If for all $\mathbf{x} \neq \mathbf{x}' \in X$ and $0 < t < 1$, we have

$$f(t\mathbf{x} + (1-t)\mathbf{x}') > tf(\mathbf{x}) + (1-t)f(\mathbf{x}'),$$

then f is said to be *strictly concave* on X .

Definition 2.4.12 If $-f$ is (strictly) concave on X , then $f : X \rightarrow \mathcal{R}$ is called a (strictly) *convex* function on X .

Remark 2.4.2 We have the following results:

- (1) A linear function is both convex and concave.
- (2) The sum of two concave (convex) functions is still concave (convex).
- (3) The sum of a concave (convex) function and a strictly concave (convex) function is strictly concave (convex).

The statement that the function $f : X \rightarrow \mathcal{R}$ is concave on X is equivalent to the statement that for any $\mathbf{x}_1, \dots, \mathbf{x}_m \in X$ and any $t_i \in [0, 1]$, we have

$$f(t_1\mathbf{x}_1 + t_2\mathbf{x}_2 + \dots + t_m\mathbf{x}_m) \geq t_1f(\mathbf{x}_1) + \dots + t_mf(\mathbf{x}_m).$$

This formula is also called **Jensen's inequality**. If t_i is regarded as the probability of \mathbf{x}_i , when $f : X \rightarrow \mathcal{R}$ is concave on X , Jensen's inequality

ty implies that the expectation of function value with respect to a random variable is not greater than the function value with respect to the expectation of the random variable, i.e.,

$$f(E(X)) \geq E(f(X)).$$

When a function is twice differentiable, its convexity or concavity can be determined by whether the second-order partial derivative matrix is positive (negative) definite.

Remark 2.4.3 A function f defined on X has a continuous second-order partial derivative. Then, it is a concave (convex) function if and only if its **Hessian matrix** $D^2f(x)$ is negative (positive) semi-definite on X . It is strictly concave (convex) if and only if its Hessian matrix $D^2f(x)$ is negative (positive) definite on X .

Remark 2.4.4 The strict concavity of the function $f(x)$ can be determined by testing whether the principal minors of the Hessian matrix change signs alternately, i.e.,

$$\begin{vmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{vmatrix} > 0, \\ \begin{vmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{vmatrix} < 0,$$

and so on, where $f_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}$. This algebraic condition is very useful for testing second-order conditions of optimality.

2.4.9 Quasi-concave and Quasi-convex Function

Quasi-concave functions are frequently used in economics, which are weaker than concavity.

Let $X \subseteq \mathcal{R}^n$ be a convex set.

Definition 2.4.13 For a function $f : X \rightarrow \mathcal{R}$, if for any $\mathbf{x}, \mathbf{x}' \in X$ and any $t \in [0, 1]$, we have

$$f(t\mathbf{x} + (1 - t)\mathbf{x}') \geq \min\{f(\mathbf{x}), f(\mathbf{x}')\},$$

then f is called *quasi-concave* on X .

If for any $\mathbf{x}, \mathbf{x}' \in X$ and any $t \in [0, 1]$, we have

$$f(t\mathbf{x} + (1 - t)\mathbf{x}') > \min\{f(\mathbf{x}), f(\mathbf{x}')\},$$

and then f is *strictly quasi-concave* on X .

If $-f$ is (strictly) quasi-concave on X , then the function $f : X \rightarrow \mathcal{R}$ is (strictly) **quasi-convex** on X .

Remark 2.4.5 The following facts are clear:

- (1) If a function f is (strictly) concave (convex), then it is (strictly) quasi-concave (convex);
- (2) The function f is (strictly) quasi-concave if and only if $-f$ is (strictly) quasi-convex;
- (3) An arbitrary (strictly) monotone function defined on an interval of one-dimensional real number space is both (strictly) quasi-concave and (strictly) quasi-convex;
- (4) The sum of two quasi-concave (convex) functions is generally not a quasi-concave (convex) function.

The following theorem correlates the quasi-concavity of a function to the convexity of upper contour set.

Theorem 2.4.12 (Quasi-concavity and Upper Contour Sets) $f : X \rightarrow \mathcal{R}$ is a quasi-concave function if and only if for any $y \in \mathcal{R}$, the upper contour set $S(y) \equiv \{\mathbf{x} \in X : f(\mathbf{x}) \geq y\}$ is a convex set.

PROOF. Necessity: Let \mathbf{x}^1 and \mathbf{x}^2 be two points of $S(y)$ (if $S(y)$ is the empty set, then it is clearly convex). We need to show: all convex combinations $\mathbf{x}^t \equiv t\mathbf{x}^1 + (1 - t)\mathbf{x}^2$, $t \in [0, 1]$ are in $S(y)$.

Since $x^1 \in S(y)$ and $x^2 \in S(y)$, by the definition of upper contour set, we have $f(x^1) \geq y$ and $f(x^2) \geq y$.

Now, for any x^t , since f is quasi-concave, then:

$$f(x^t) \geq \min[f(x^1), f(x^2)] \geq y.$$

Therefore, $f(x^t) \geq y$, and then $x^t \in S(y)$. Consequently, $S(y)$ must be a convex set.

Sufficiency: we need to show: if for all $y \in \mathcal{R}$, $S(y)$ is a convex set, then $f(x)$ is a quasi-concave function. Let x^1 and x^2 be two arbitrary points in X . Without loss of generality, suppose $f(x^1) \geq f(x^2)$. Since for all $y \in \mathcal{R}$, $S(y)$ is a convex set, then $S(f(x^2))$ must be convex. It is also clear that $x^2 \in S(f(x^2))$, and since $f(x^1) \geq f(x^2)$, we have $x^1 \in S(f(x^2))$. As such, for any convex combination of x^1 and x^2 , we must have $x^t \in S(f(x^2))$. It follows from the definition of $S(f(x^2))$ that $f(x^t) \geq f(x^2)$. As a consequence, we must have

$$f(x^t) \geq \min[f(x^1), f(x^2)].$$

Therefore, $f(x)$ is quasi-concave. □

The following theorem characterizes the properties of quasi-concave functions, i.e., quasi-concavity is robust to monotonic transformations.

Theorem 2.4.13 *Suppose that the function $f : X \rightarrow \mathcal{R}$ is quasi-concave on X , and $h : \mathcal{R} \rightarrow \mathcal{R}$ is a monotonically non-decreasing function. Then, the composite function $h(f(x))$ is also quasi-concave. If f is strictly quasi-concave and h is strictly increasing, then the composite function is strictly quasi-concave.*

Similar to concavity, when a function is differentiable, we have the following result.

Proposition 2.4.1 *Suppose that $f : \mathcal{R} \rightarrow \mathcal{R}$ is differentiable. Then, f is quasi-concave if and only if for any $x, y \in \mathcal{R}$, we have*

$$f(y) \geq f(x) \Rightarrow f'(x)(y - x) \geq 0. \quad (2.4.1)$$

When there are two or more variables, the above proposition becomes:

Proposition 2.4.2 *Suppose that $f : \mathcal{R}^n \rightarrow \mathcal{R}$ is differentiable. Then, f is quasi-concave if and only if for any $\mathbf{x}, \mathbf{y} \in \mathcal{R}$, we have*

$$f(\mathbf{y}) \geq f(\mathbf{x}) \Rightarrow \sum_{j=1}^n \frac{\partial f(\mathbf{x})}{\partial x_j} (y_j - x_j) \geq 0. \quad (2.4.2)$$

When a function f defined on a convex set X has continuous second-order partial derivatives, the **bordered Hessian determinant** is defined as follows:

$$|B| = \begin{vmatrix} 0 & f_1 & f_2 & \cdots & f_n \\ f_1 & f_{11} & f_{12} & \cdots & f_{1n} \\ f_2 & f_{21} & f_{22} & \cdots & f_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ f_n & f_{n1} & f_{n2} & \cdots & f_{nn} \end{vmatrix}$$

The principal minors of the bordered Hessian determinant B are as follows:

$$|B_1| = \begin{vmatrix} 0 & f_1 \\ f_1 & f_{11} \end{vmatrix}, \quad |B_2| = \begin{vmatrix} 0 & f_1 & f_2 \\ f_1 & f_{11} & f_{12} \\ f_2 & f_{21} & f_{22} \end{vmatrix}, \quad \cdots, \quad |B_n| = |B|.$$

Then, the necessary condition for $f : X \rightarrow \mathcal{R}$ to be a quasi-concave function is

$$|B_1| \leq 0, \quad |B_2| \geq 0, \quad |B_3| \leq 0, \quad \cdots, \quad (-1)^n |B_n| \geq 0.$$

The sufficient condition for $f : X \rightarrow \mathcal{R}$ to be a strictly quasi-concave function is:

$$|B_1| < 0, \quad |B_2| > 0, \quad |B_3| < 0, \quad \cdots, \quad (-1)^n |B_n| > 0.$$

The necessary condition for $f : X \rightarrow \mathcal{R}$ to be a quasi-convex function is:

$$|B_1| \leq 0, \quad |B_2| \leq 0, \quad \cdots, \quad |B_n| \leq 0.$$

The sufficient condition for $f : X \rightarrow \mathcal{R}$ to be a strictly quasi-concave function is:

$$|B_1| < 0, |B_2| < 0, \dots, |B_n| < 0.$$

2.4.10 Separating and Supporting Hyperplane Theorems

The separating hyperplane theorem also has crucial applications in economics. First, recall that if $X \subseteq \mathcal{R}^n$ is a **compact set**, then it is bounded and closed. X is **convex set**, if for any $x, x' \in X$ and any $0 \leq t \leq 1$, $tx + (1-t)x' \in X$. The convex set implies that the connections between any two points in the set belong to this set.

Theorem 2.4.14 (Separating Hyperplane Theorem) Suppose that $A, B \subseteq \mathcal{R}^m$ are convex, and $A \cap B = \emptyset$. Then, there is a vector $p \in \mathcal{R}^m, p \neq 0$ and $c \in \mathcal{R}$, such that

$$px \leq c \leq py \quad \forall x \in A, \forall y \in B.$$

Moreover, suppose that $B \subseteq \mathcal{R}^m$ is convex and closed, $A \subseteq \mathcal{R}^m$ is convex and compact, and $A \cap B = \emptyset$. Then, there is a vector $p \in \mathcal{R}^m, p \neq 0$ and $c \in \mathcal{R}$, such that A, B are strictly separated, i.e.,

$$px < c < py \quad \forall x \in A, \forall y \in B.$$

Theorem 2.4.15 (Supporting Hyperplane Theorem) Suppose that $A \subseteq \mathcal{R}^m$ are convex, and $y \in \mathcal{R}^m$ is not an interior of A (i.e., $y \notin \text{int}A$). Then, there is a vector $p \in \mathcal{R}^m$ with $p \neq 0$, such that

$$px \leq py \quad \forall x \in A.$$

Unlike the Separating Hyperplane Theorem, the Supporting Hyperplane Theorem above does not need to assume that the intersection of two sets A and $\{y\}$ is an empty set.

Definition 2.4.14 Let $C \subseteq \mathcal{R}^m$. C is called a *cone* if for any $x \in C$ and $\lambda \in \mathcal{R}$, we have $\lambda x \in C$.

Proposition 2.4.3 A cone C is convex if and only if $x, y \in C$ implies that $x + y \in C$.

Proposition 2.4.4 Let $C \subseteq \mathcal{R}^m$ be a closed and convex cone, and $K \subseteq \mathcal{R}^m$ be a compact and convex cone. Then, $C \cap K \neq \emptyset$ if and only if for any $\mathbf{p} \in C$, there is $\mathbf{z} \in K$, such that

$$\mathbf{p} \cdot \mathbf{z} \leq 0.$$

2.5 Multi-Valued Function

Set-valued mapping refers to the situation in which the image of a mapping may not be a single point, but rather a set.

2.5.1 Point-to-Set Mappings

Suppose that X and Y are two subsets of a topological vector space (e.g., the Euclidean space).

Point-to-set mapping is also called **correspondence** or **multi-valued function**. A correspondence F maps points \mathbf{x} in a domain X into sets in Y (e.g., maps point \mathbf{x} in $X \subseteq \mathcal{R}^n$ into the range $Y \subseteq \mathcal{R}^m$), denoted by $F : X \rightarrow 2^Y$. One also uses $F : X \rightrightarrows Y$ or $F : X \rightarrow\rightarrow Y$ to denote the multi-valued mapping $F : X \rightarrow 2^Y$.

Definition 2.5.1 Let $F : X \rightarrow 2^Y$ be a correspondence.

- (1) If $F(\mathbf{x})$ is non-empty for every $\mathbf{x} \in X$, then the correspondence F is said to be *non-empty valued*;
- (2) If $F(\mathbf{x})$ is a convex set for every $\mathbf{x} \in X$, then the correspondence F is said to be *convex valued*;
- (3) If $F(\mathbf{x})$ is a closed set for every $\mathbf{x} \in X$, then the correspondence F is said to be *closed valued*;
- (4) If $F(\mathbf{x})$ is compact for every $\mathbf{x} \in X$, then the correspondence F is said to be *compact valued*;
- (5) If $F(\mathbf{x})$ is open for every $\mathbf{x} \in X$, then the correspondence F is said to have *open upper sections*;
- (6) If the preimage $F^{-1}(\mathbf{y}) = \{\mathbf{x} \in X : \mathbf{y} \in F(\mathbf{x})\}$ is open for every $\mathbf{y} \in Y$, then the correspondence F is said to have *open lower sections*.

Definition 2.5.2 Let $F : X \rightarrow 2^X$ be a correspondence from X to X itself.

- (1) F is said to be *FS-convex* if for any $x_1, \dots, x_m \in X$ and its convex combination $x_\lambda = \sum_{i=1}^m \lambda_i x_i$,¹ we have

$$x_\lambda \in \bigcup_{i=1}^m F(x_i).$$

- (2) F is said to be *SS-convex* if for any $x \in X$, $x \notin co F(x)$.²

Remark 2.5.1 It is easy to verify that correspondence $P : X \rightarrow 2^X$ is SS-convex if and only if correspondence $G : X \rightarrow 2^X$ defined by $G(x) = X \setminus P(x)$ is FS-convex.

Specially, for function $f : X \rightarrow \mathcal{R}$, define upper contour set

$$U_w(x) = \{y \in X : f(y) \geq f(x)\}, \forall x \in X,$$

strict upper contour set

$$U_s(x) = \{y \in X : f(y) > f(x)\}, \forall x \in X,$$

lower contour set

$$L_w(x) = \{y \in X : f(y) \leq f(x)\}, \forall x \in X,$$

and strict lower contour set

$$L_s(x) = \{y \in X : f(y) < f(x)\}, \forall x \in X.$$

The following equivalence results are used later.

Proposition 2.5.1 *The following arguments are equivalent:*

- (1) $f : X \rightarrow \mathcal{R}$ is quasi-concave;
- (2) $U_w : X \rightarrow 2^X$ is a convex-valued correspondence;

¹The concept of FS-convex is introduced by Fan (1984) & Sonnenschein (1971), and thus it is called FS-convex.

²The concept of SS-convex is introduced by Shafer & Sonnenschein (1975), and thus it is called SS-convex.

(3) $U_s : X \rightarrow 2^X$ is a convex-valued correspondence;

(4) $U_s : X \rightarrow 2^X$ is SS-convex;

(5) $U_w : X \rightarrow 2^X$ is FS-convex.

PROOF. It is clear that: (1) implies (2); (2) implies (3); (3) implies (4); and (5) implies (1). We just need to show that (4) implies (5). Suppose that this is not the case, and there is a finite set $\{x_1, x_2, \dots, x_m\} \subset X$ and certain convex combination, $x_\lambda = \sum_{j=1}^m \lambda_j x_j$, such that $x_\lambda \notin \cup_{j=1}^m U(x_j)$. Therefore, for all j , we have $x_\lambda \in L_s(x_j)$, i.e., $x_j \in U_s(x_\lambda)$, and thus $x_\lambda \in co U_s(x_\lambda)$, which is a contradiction. \square

2.5.2 Upper Hemi-continuous and Lower Hemi-continuous Correspondence

Intuitively, a correspondence is continuous if a small change in x only leads to a small change in the set $F(x)$. Unfortunately, giving a formal definition of continuity for correspondences is not so simple. Figure 2.1 shows a continuous correspondence.

The notions of **hemi-continuity** are usually defined in terms of sequences (see Debreu (1959) and Mask-Collell et al. (1995)). Although they are relatively easy to verify, they depend on the assumption that a correspondence is compact-valued. The following definitions are more formal (see Border, 1985).

Definition 2.5.3 For a correspondence $F : X \rightarrow 2^Y$ and a point x , F is said to be *upper hemi-continuous* at x if for each open set U containing $F(x)$, there is an open set $N(x)$ containing x , such that $F(x') \subseteq U$ for all $x' \in N(x)$.

F is said to be *upper hemi-continuous* on X if the correspondence F is *upper hemi-continuous* at every $x \in X$, or equivalently, for every open subset V of Y , $\{x \in X : F(x) \subset V\}$ is always an open subset of X .

Remark 2.5.2 Upper hemi-continuity captures the idea that $F(x)$ should not “suddenly contain new points” when passing through a point x , i.e., $F(x)$ does not jump if x changes slightly. In other words, if one starts at a point x and moves slightly to x' , upper hemi-continuity at x implies that there is no point in $F(x')$ that is not close to some points in $F(x)$.

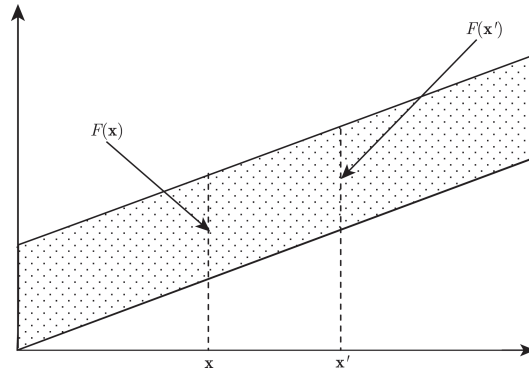


Figure 2.1: Continuous correspondence

Definition 2.5.4 For a correspondence $F : X \rightarrow 2^Y$ and a point x , correspondence F is said to be *lower hemi-continuous* at x if for every open set V , $F(x) \cap V \neq \emptyset$, there exists a neighborhood $N(x)$ of x , such that $F(x') \cap V \neq \emptyset$ for all $x' \in N(x)$.

If F is lower hemi-continuous at every x , or equivalently, the set $\{x \in X : F(x) \cap V \neq \emptyset\}$ is open in X for every open set V of Y , then F is said to be *lower hemi-continuous* on X .

Remark 2.5.3 Lower hemi-continuity captures the idea that any element in $F(x)$ can be “approached” from all directions, i.e., $F(x)$ does not suddenly become much smaller if one changes x slightly. In other words, if one starts at x and $y \in F(x)$, lower hemi-continuity at x implies that if one moves slightly from x to x' , there is some $y' \in F(x')$ that is close to y .

Combining the concepts of upper and lower hemi-continuity, we can define the continuity of a correspondence.

Definition 2.5.5 A correspondence $F : X \rightarrow 2^Y$ is said to be *continuous at* $x \in X$ if it is both upper hemi-continuous and lower hemi-continuous at $x \in X$. The correspondence $F : X \rightarrow 2^Y$ is said to be *continuous on* X if it is both upper hemi-continuous and lower hemi-continuous on X .

Figure 2.2 shows a correspondence that is upper hemi-continuous, but not lower hemi-continuous. To see why it is upper hemi-continuous, imagine an open interval U that encompasses $F(x)$. Now, consider moving s-

lightly to the left of x to a point x' . Clearly, $F(x') = \{\hat{y}\}$ is in the interval. Similarly, if we move to a point x' slightly to the right of x , then $F(x)$ will be in the interval so long as x' is sufficiently close to x . Therefore, it is upper hemi-continuous. On the other hand, it is not lower hemi-continuous. To see this, consider the point $y \in F(x)$, and let U be a very small interval around y that does not include \hat{y} . If we take any open set $N(x)$ containing x , then it will contain some point x' to the left of x . However, then $F(x') = \{\hat{y}\}$ will contain no points near y , i.e., it will not intersect U . Therefore, the correspondence is not lower hemi-continuous.

Figure 2.3 shows a correspondence that is lower hemi-continuous, but not upper hemi-continuous. To see why it is lower hemi-continuous: For any $0 < x' < x$, note that $F(x') = \{\hat{y}\}$. Let $x_n = x' - 1/n$, $y_n = \hat{y}$. Then, $x_n > 0$ for sufficiently large n , $x_n \rightarrow x'$, $y_n \rightarrow \hat{y}$, and $y_n \in F(x_n) = \{\hat{y}\}$. It is thus lower hemi-continuous. It is also clearly lower hemi-continuous for $x_i > x$. Consequently, it is lower hemi-continuous on X . On the other hand, it is not upper hemi-continuous. If we start at x by noting that $F(x) = \{\hat{y}\}$, and make a small move to the right to a point x' , then $F(x')$ suddenly contains many points that are not close to \hat{y} . Therefore, this correspondence fails to be upper hemi-continuous.

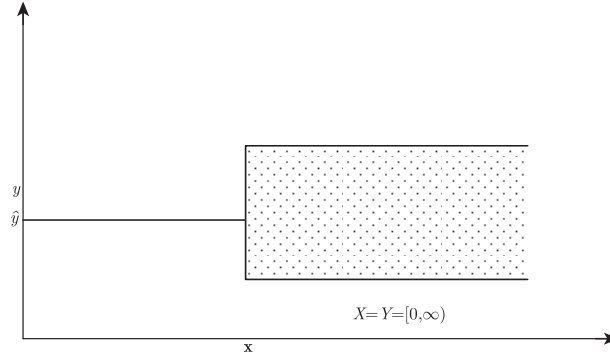


Figure 2.2: The correspondence is upper hemi-continuous, but not lower hemi-continuous

Remark 2.5.4 In fact, the notions of upper and lower hemi-continuous correspondence both reduce to the standard notion of continuity for a function if $F(\cdot)$ is a single-valued correspondence, i.e., a function. In other words,

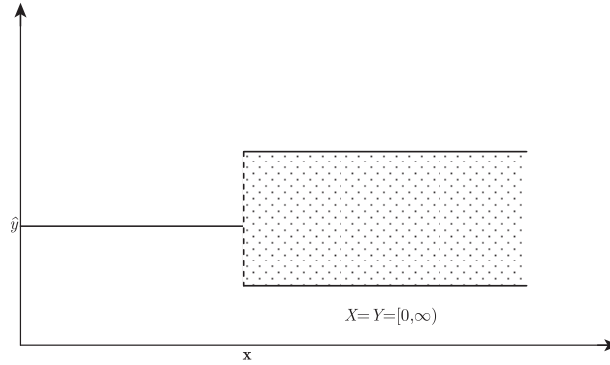


Figure 2.3: The correspondence is lower hemi-continuous, but not upper hemi-continuous

$F(\cdot)$ is a single-valued upper (or lower) hemi-continuous correspondence if and only if it is a continuous function.

Remark 2.5.5 Based on the following two facts, both notions of hemi-continuity can be characterized by sequences.

- (a) If a correspondence $F : X \rightarrow 2^Y$ is compact-valued, then it is upper hemi-continuous if and only if for any $\{x_k\}$ and $\{y_k\}$, where $x_k \rightarrow x$, $y_k \in F(x_k)$, there exists a converging subsequence $\{y_{k_m}\}$, such that $y_{k_m} \rightarrow y$ and $y \in F(x)$.
- (b) A correspondence $F : X \rightarrow 2^Y$ is lower hemi-continuous at x if and only if for any $\{x_k\}$ and $y \in F(x)$, where $x_k \rightarrow x$, there is a sequence $\{y_k\}$, such that $y_k \rightarrow y$ and $y_k \in F(x_k)$.

2.5.3 Open and Closed Graphs of Correspondence

Definition 2.5.6 A correspondence $F : X \rightarrow 2^Y$ is said to be *sequentially closed* at x if for any $\{x_k\}$ and $\{y_k\}$, where $x_k \rightarrow x$ and $y_k \rightarrow y$, $y_k \in F(x_k)$, we have $y \in F(x)$. F is said to be *sequentially closed* or has a closed graph if F is sequentially closed for all $x \in X$, or equivalently graph

$$Gr(F) = \{(x, y) \in X \times Y : y \in F(x)\} \text{ is closed.}$$

Regarding the relationship between upper hemi-continuity and closed graph, we have the following results.

Proposition 2.5.2 *Let $F : X \rightarrow 2^Y$ be a correspondence.*

- (i) *Suppose that Y is compact, and $F : X \rightarrow 2^Y$ is closed-valued. If F has a closed graph, it is upper hemi-continuous.*
- (ii) *Suppose that X and Y are closed, and $F : X \rightarrow 2^Y$ is closed-valued. If F is upper hemi-continuous, then it has a closed graph.*

Because of fact (i), a correspondence with a closed graph is sometimes used to define a hemi-continuous correspondence in the literature. However, one should keep in mind that they are not the same, in general. For example, let $F : \mathcal{R}_+ \rightarrow 2^{\mathcal{R}}$ be defined by

$$F(x) = \begin{cases} \left\{ \frac{1}{x} \right\}, & \text{if } x > 0, \\ \{0\}, & \text{if } x = 0. \end{cases}$$

The correspondence is sequentially closed, but not upper hemi-continuous. Moreover, define $F : \mathcal{R}_+ \rightarrow 2^{\mathcal{R}}$ by $F(x) = (0, 1)$. Then, F is upper hemi-continuous, but not sequentially closed.

Definition 2.5.7 A correspondence $F : X \rightarrow 2^Y$ is said to be *open* if its graph

$$Gr(F) = \{(x, y) \in X \times Y : y \in F(x)\} \text{ is open.}$$

Proposition 2.5.3 *Let $F : X \rightarrow 2^Y$ be a correspondence. Then,*

- (1) *if a correspondence $F : X \rightarrow 2^Y$ has an open graph, then it has open upper and lower sections.*
- (2) *If a correspondence $F : X \rightarrow 2^Y$ has open lower sections, then it must be lower hemi-continuous.*

2.5.4 Transfer Closed-valued Correspondence

The concepts of transfer closedness, transfer openness, transfer convexity, and others for multivalued mapping (correspondence) introduced in Tian

(1992, 1993) and Zhou and Tian (1992) weaken the conditions for establishing some basic mathematical theorems in nonlinear analysis and the existence of equilibrium solutions of optimization problems. They can be employed to obtain many characterization results, such as necessary and sufficient conditions for the existence of the maximal element of preference relations and the existence of Nash equilibrium. These conclusions are provided in the corresponding chapters of this textbook.

Denote by $\text{int } D$ and $\text{cl } D$ the set of interior points, and the closure of set D , respectively.

Definition 2.5.8 A correspondence $G : X \rightarrow 2^Y$ is said to be *transfer closed-valued* on X if for any $x \in X$, $y \notin G(x)$ implies that there is an $x' \in X$, such that $y \notin \text{cl } G(x')$.

Definition 2.5.9 A correspondence $P : X \rightarrow 2^Y$ is said to have *transfer open upper sections* on X if for any $x \in X$ and $y \in Y$, $y \in P(x)$ implies that there is a point $x' \in X$, such that $y \in \text{int } P(x')$.

Remark 2.5.6 If a correspondence is closed-valued, then it is a transfer closed-valued (it is obtained by letting $x' = x$); if a correspondence has open upper sections, then it has the transfer open upper sections (let $x' = x$). Furthermore, the correspondence $P : X \rightarrow 2^Y$ has transfer open upper sections in X if and only if $G : X \rightarrow 2^Y$ defined by $G(x) = Y \setminus P(x)$ is transfer closed-valued in X .

Remark 2.5.7 For any function $f : X \rightarrow \mathcal{R}$, the correspondence $G : X \rightarrow 2^Y$ defined by

$$G(x) = \{y \in X : f(y) \geq f(x)\}, \forall x \in X$$

is transfer closed-valued if and only if f is transfer upper continuous on X .

The following proposition significantly weakens the various continuity conditions involved when proving many optimization problems.

Proposition 2.5.4 (Tian (1992)) Let X and Y be two topological spaces, $G : X \rightarrow 2^Y$ be a correspondence from point to set. Then,

$$\bigcap_{x \in X} cl G(x) = \bigcap_{x \in X} G(x)$$

if and only if G is transfer closed-valued on X .

PROOF. Sufficiency: We need to show

$$\bigcap_{x \in X} cl G(x) = \bigcap_{x \in X} G(x).$$

It is clear that

$$\bigcap_{x \in X} G(x) \subseteq \bigcap_{x \in X} cl G(x),$$

and thus we just need to show that

$$\bigcap_{x \in X} cl G(x) \subseteq \bigcap_{x \in X} G(x).$$

Suppose that this is not the case. Then, there is a y , such that $y \in \bigcap_{x \in X} cl G(x)$, but $y \notin \bigcap_{x \in X} G(x)$. Therefore, there is a $z \in X$, such that $y \notin G(z)$. Note that G is transfer closed-valued on X , and then there exists a $z' \in X$, such that $y \notin cl G(z')$, and thus $y \notin \bigcap_{x \in X} cl G(x)$, which is a contradiction.

Necessity: Suppose that

$$\bigcap_{x \in X} cl G(x) = \bigcap_{x \in X} G(x).$$

If $y \notin G(x)$, then

$$y \notin \bigcap_{x \in X} cl G(x) = \bigcap_{x \in X} G(x),$$

and thus $y \notin cl G(x')$ for some $x' \in X$. Consequently, G is a transfer closed-valued correspondence on X . \square

Similarly, we can define transfer convexity.

Definition 2.5.10 (Transfer FS-convex) Let X be a topological space, and Z be a convex subset of X . A correspondence $G : X \rightarrow 2^Z$ is said to be *transfer FS-convex* on X if for any finite set $\{x_1, x_2, \dots, x_n\} \subseteq X$, there is

a corresponding finite set $\{y_1, y_2, \dots, y_n\} \subseteq Z$, such that for any subset $\{y_{i1}, y_{i2}, \dots, y_{is}\} (1 \leq s \leq n)$, we have

$$co \{y_{i1}, y_{i2}, \dots, y_{is}\} \subseteq \bigcup_{r=1}^s G(y_{ir}).$$

Definition 2.5.11 (Transfer SS-convex) Let X be a topological space, and Z be a convex subset of the topological space. A correspondence $P : Z \rightarrow 2^X$ is said to be *transfer SS-convex* on X if for any finite set $\{y_1, y_2, \dots, y_n\} \subseteq X$, there exists a finite set $\{y_1, y_2, \dots, y_n\} \subseteq Z$, such that for any subset $\{y_{i1}, y_{i2}, \dots, y_{is}\} (1 \leq s \leq n)$ and any $y_{i0} \in co \{y_{i1}, y_{i2}, \dots, y_{is}\}$, and we have $x_{ir} \notin P(y_{i0})$.

Remark 2.5.8 Unlike FS-convex and SS-convex, when defining the transfer FS-convex and transfer SS-convex, we do not assume that correspondences are mapping from itself to itself. It is clear that when $X = Z$ and picking $y_i = x_i$, FS-convex implies transfer FS-convex and SS-convex implies transfer SS-convex. Similarly, it is not difficult to verify that the correspondence $P : X \rightarrow 2^Z$ is transfer SS-convex if and only if $G : X \rightarrow 2^X$ defined by $G(x) = Z \setminus P(x)$ is transfer FS-convex.

2.6 Static Optimization

The optimization problem constitutes a core issue in economics. Rationality is a key assumption about individual decision-makers in economics. Individuals pursue maximizing their personal interests, and the basic analysis is to solve optimization problems. This section introduces various methods for solving static optimization problems.

2.6.1 Unconstrained Optimization

The optimization problem discusses whether an objective function can reach the maximum or minimum on a given set. Let X be an arbitrary topological space. First, we give the following concepts:

Definition 2.6.1 (Local Optimum) If $f(x^*) \geq f(x)$ ($f(x^*) > f(x)$) for all x in some neighbourhood of x^* , then the function is said to have *local maximum (unique local maximum)* at point x^* .

If $f(\tilde{x}) \leq f(x)$ ($f(\tilde{x}) < f(x)$) for all $x \neq \tilde{x}$ in some neighbourhoods of \tilde{x} , then the function is said to have *local minimum (unique local minimum)* at \tilde{x} .

Definition 2.6.2 (Global Optimum) If $f(x^*) \geq f(x)$ ($f(x^*) > f(x)$) for all x in the domain of the function, then the function is said to have *global (unique) maximum* at x^* ; if $f(x^*) \leq f(x)$ ($f(x^*) < f(x)$) for all x in the domain of the function, then the function is said to have *global (unique) minimum* at x^* .

A classical conclusion about global optimization is the so-called **Weierstrass theorem**.

Theorem 2.6.1 (Weierstrass Theorem) Any upper (lower) semi-continuous function must reach its maximum (minimum) on a compact set, and the set of maximal points is compact.

Transfer continuity can be used to generalize the Weierstrass Theorem by providing sufficient and necessary conditions for a function f to reach global maximum (minimum) on a compact set X , sufficient and necessary conditions for the set of global maximal (minimal) points to be compact, and characterising a function that has a global maximum (minimum) value on arbitrary sets in Tian (1992, 1993, 1994), Tian & Zhou (1995), and Zhou & Tian (1992).

Theorem 2.6.2 (Tian-Zhou Theorem I) Suppose that X is a compact set in an arbitrary topological space. The function $f : X \rightarrow \mathcal{R}$ has a maximum (minimum) on X if and only if f is transfer weakly upper (lower) continuous on X .

PROOF. Since f is transfer weakly upper continuous on X if and only if $-f$ is transfer weakly lower continuous, we just need to show the case in which the function has a maximal point.

Sufficiency: We prove it by contradiction. Suppose that f does not have a maximum on X . Then, for each $y \in X$, there is $x \in X$, such that $f(x) >$

$f(\mathbf{y})$. It follows from the transfer weak upper continuity of f that there is a $\mathbf{x}' \in X$ and a neighbourhood $\mathcal{N}(\mathbf{y})$ of \mathbf{y} , such that $f(\mathbf{x}') \geq f(\mathbf{y}')$ for all $\mathbf{y}' \in \mathcal{N}(\mathbf{y})$. Therefore, we have $X = \cup_{\mathbf{y} \in X} \mathcal{N}(\mathbf{y})$. Since X is compact, there is a finite number of points $\{\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_n\}$, such that $X = \cup_{i=1}^n \mathcal{N}(\mathbf{y}_i)$. Let \mathbf{x}'_i be the corresponding points, such that $f(\mathbf{x}'_i) \geq f(\mathbf{y}')$ for all $\mathbf{y}' \in \mathcal{N}(\mathbf{y}_i)$. f must have the maximum in the finite subset $\{\mathbf{x}'_1, \mathbf{x}'_2, \dots, \mathbf{x}'_n\}$. Without loss of generality, suppose \mathbf{x}'_1 satisfying $f(\mathbf{x}'_1) \geq f(\mathbf{x}'_i)$ for $\forall i = 1, 2, \dots, n$. It follows from the previous assumption that f has no maximum on X , i.e., \mathbf{x}'_1 is not the maximal point of f on X . Therefore, there exists $\mathbf{x} \in X$, such that $f(\mathbf{x}) > f(\mathbf{x}'_1)$. However, since $X = \cup_{i=1}^n \mathcal{N}(\mathbf{y}_i)$, there is j , such that $\mathbf{x} \in \mathcal{N}(\mathbf{y}_j)$, and then $f(\mathbf{x}'_j) \geq f(\mathbf{x})$. Therefore, $f(\mathbf{x}) > f(\mathbf{x}'_1) \geq f(\mathbf{x}'_j) \geq f(\mathbf{x})$, which is a contradiction. As a consequence, f has a maximum on X .

Necessity: We prove this in a straightforward manner. Let \mathbf{x}' be a maximal point of f . Then, $f(\mathbf{x}') \geq f(\mathbf{y}')$ holds for all $\mathbf{y}' \in X$.

□

In many cases, when proving the existence of competitive equilibrium and the existence of equilibrium in a game, we not only need to prove the existence of optimal points, but also prove that the set of the optimal points is compact.

Theorem 2.6.3 (Tian-Zhou Theorem II) *Suppose that X is a compact set in an arbitrary topological space, and $f : X \rightarrow \mathcal{R}$ is a function. The set of maximal (minimal) points of f on X is nonempty and compact if and only if f is transfer upper (lower) continuous on X .*

PROOF. We only need to prove the case with a set of maximal points.

Necessity: Suppose that the set of maximal points of f on X is nonempty and compact. If $f(\mathbf{y}) < f(\mathbf{x})$ for any $\mathbf{x}, \mathbf{y} \in X$, then \mathbf{y} cannot be a maximal point of f on X . It follows from the compactness of the set of maximal points that there is a neighbourhood $\mathcal{N}(\mathbf{y})$ of \mathbf{y} that does not contain any maximal points of f on X . Let \mathbf{x}' be a maximal point of f on X , and then $f(\mathbf{z}) < f(\mathbf{x}')$ for all $\mathbf{z} \in \mathcal{N}(\mathbf{y})$. Therefore, f is transfer upper continuous on X .

Sufficiency: First, note that $G : X \rightarrow 2^Y$ defined by

$$G(x) = \{y \in X : f(y) \geq f(x)\}, \forall x \in X$$

is a transfer closed-valued correspondence if and only if f is transfer upper continuous on X . Since f is transfer upper continuous on X , according to Proposition 2.5.4, we have $\bigcap_{x \in X} cl G(x) = \bigcap_{x \in X} G(x)$, and thus the set of maximal points is closed.

Since f has a maximal point on any finite subset $\{x_1, x_2, \dots, x_m\} \subseteq X$, let $f(x_1) \geq f(x_i)$ hold for all $\forall i = 1, \dots, m$. Then, we have $x_1 \in G(x_i)$ for all $i = 1, \dots, m$, and thus

$$\emptyset \neq \bigcap_{i=1}^m G(x_i) \subseteq \bigcap_{i=1}^m cl G(x_i),$$

i.e., the class of sets $\{cl G(x) : x \in X\}$ has the property of finite intersection on X . Since $\{cl G(x) : x \in X\}$ is a collection of closed sets in compact set X , $\emptyset \neq \bigcap_{x \in X} cl G(x) = \bigcap_{x \in X} G(x)$. This implies that there is $x^* \in X$, such that $f(x^*) \geq f(x)$ for all $x \in X$. Since the set of maximal points $\bigcap_{x \in X} cl G(x)$ is a closed subset of the compact set X , it is also compact. \square

In order to easily determine whether a function has an extreme point, the following gives the method of finding extreme values by the differential method. We first provide the necessary conditions for interior extreme points without constraints, and then give the sufficient conditions.

Necessary Conditions for Optimization

Generally, there are two necessary conditions for the interior extreme point, i.e., first- and second-order necessary conditions.

Theorem 2.6.4 (The first-order necessary condition for interior extreme points)

Suppose that $X \subseteq \mathcal{R}^n$. If a differentiable function $f(x)$ reaches a local maximum or minimum at an interior point $x^* \in X$, then x^* is the solution to the following system of simultaneous equations:

$$\frac{\partial f(x^*)}{\partial x_1} = 0,$$

$$\begin{aligned}\frac{\partial f(\mathbf{x}^*)}{\partial x_2} &= 0, \\ &\vdots \\ \frac{\partial f(\mathbf{x}^*)}{\partial x_n} &= 0.\end{aligned}$$

PROOF. Suppose that $f(\mathbf{x})$ reaches the local extreme value at an interior point \mathbf{x}^* , then we need to prove that $Df(\mathbf{x}^*) = 0$. Although this proof is not the simplest one, it will be very useful when considering the second-order condition.

Choose any vector $\mathbf{z} \in \mathcal{R}^n$, and then construct a familiar univariate function of any scalar t :

$$g(t) = f(\mathbf{x}^* + t\mathbf{z})$$

First, for $t \neq 0$, $\mathbf{x}^* + t\mathbf{z}$ gives a vector that is different from \mathbf{x}^* . For $t = 0$, $\mathbf{x}^* + t\mathbf{z}$ is equal to \mathbf{x}^* , and thus $g(0)$ is exactly the value of f at \mathbf{x}^* . According to the assumption that f attains an extremum at \mathbf{x}^* , $g(t)$ must reach a local extreme at $t = 0$. It follows from the Fermat Theorem given by Proposition 2.4.4 that $g'(0) = 0$. Taking the derivative of $g(t)$ by the Chain Rule gives:

$$g'(t) = \sum_{i=1}^n \frac{\partial f(\mathbf{x}^* + t\mathbf{z})}{\partial x_i} z_i.$$

When $t = 0$ and using $g'(0) = 0$, we have

$$g'(0) = \sum_{i=1}^n \frac{\partial f(\mathbf{x}^*)}{\partial x_i} z_i = Df(\mathbf{x}^*)\mathbf{z} = 0.$$

Since the above equation holds for any vector \mathbf{z} , including the unit vector, this means that each partial derivative of f must equal to zero, i.e.,

$$Df(\mathbf{x}^*) = 0.$$

□

Theorem 2.6.5 (The second-order necessary conditions for interior extreme points)

Suppose that $f(\mathbf{x})$ is twice continuously differentiable on $X \subseteq \mathcal{R}^n$.

(1) If $f(\mathbf{x})$ reaches a local maximum at the interior point \mathbf{x}^* , then the Hessian

matrix $\mathbf{H}(\mathbf{x}^*)$ is negative semi-definite.

(2) If $f(\mathbf{x})$ reaches a local minimum at the interior point $\tilde{\mathbf{x}}$, then $\mathbf{H}(\tilde{\mathbf{x}})$ is positive semi-definite.

PROOF. Let $g(t) = f(\mathbf{x} + t\mathbf{z})$, $\mathbf{z} \in \mathcal{R}^n$ and \mathbf{x} be a stationary point of f . If f attains a stationary point at \mathbf{x} , then g gets a stationary point at $t = 0$. Moreover, for any t , we have

$$g'(t) = \sum_{i=1}^n \frac{\partial f(\mathbf{x} + t\mathbf{z})}{\partial x_i} z_i.$$

We have the second-order derivatives:

$$g''(t) = \sum_{i=1}^n \sum_{j=1}^n \frac{\partial^2 f(\mathbf{x} + t\mathbf{z})}{\partial x_i \partial x_j} z_i z_j.$$

Now, suppose that f reaches maximum at $\mathbf{x} = \mathbf{x}^*$. Since $g''(0) \leq 0$, then the value of $g''(t)$ at \mathbf{x}^* and $t = 0$ is

$$g''(0) = \sum_{i=1}^n \sum_{j=1}^n \frac{\partial^2 f(\mathbf{x}^*)}{\partial x_i \partial x_j} z_i z_j \leq 0,$$

or $\mathbf{z}^T \mathbf{H}(\mathbf{x}) \mathbf{z} \leq 0$. Since \mathbf{z} is arbitrary, this implies that $\mathbf{H}(\mathbf{x}^*)$ is negative semi-definite. Similarly, if f is minimized at $\mathbf{x} = \tilde{\mathbf{x}}$, then $g''(0) \geq 0$ and $\mathbf{H}(\tilde{\mathbf{x}})$ is positive semi-definite. \square

Sufficient Conditions for Optimization

Theorem 2.6.6 (The First-Order Sufficient Conditions for Maximization)

Suppose that $f(\mathbf{x})$ is differentiable on $X \subseteq \mathcal{R}$. Then, we have:

- (1) If $f_i(\mathbf{x}^*) = 0$, and if $f'(x)$ changes its sign from positive to negative from the immediate left of the point x_0 to its immediate right, then $f(\mathbf{x})$ has a local maximum at \mathbf{x}^* .
- (2) If $f_i(\tilde{\mathbf{x}}) = 0$, and $f'(x)$ changes its sign from negative to positive from the immediate left of the point x_0 to its immediate right, then $f(\mathbf{x})$ has a local minimum at $\tilde{\mathbf{x}}$.

- (3) *There is no extreme point if $f'(x)$ has the same sign on some neighborhood.*

Theorem 2.6.7 (The Second-Order Sufficient Conditions for Maximization)

Suppose that $f(x)$ is twice continuously differentiable on $X \subseteq \mathcal{R}^n$. Then, we have:

- (1) *If $f_i(x^*) = 0$, and $(-1)^i D_i(x^*) > 0, i = 1, \dots, n$, then $f(x)$ has a local maximum at x^* .*
- (2) *If $f_i(\tilde{x}) = 0$, and $D_i(\tilde{x}) > 0, i = 1, \dots, n$, then $f(x)$ has a local minimum at \tilde{x} .*

Global Optimization

The local optimum is, in general, not the same as the global optimum. However, under certain conditions, these two are consistent with each other.

Theorem 2.6.8 (Local and Global Optimum) *Suppose that f is a concave and twice continuously differentiable function on $X \subseteq \mathcal{R}^n$, and x^* is an interior point of X . Then, the following three statements are equivalent:*

- (1) $Df(x^*) = 0$.
- (2) f has a local maximum at x^* .
- (3) f has a global maximum at x^* .

PROOF. It is clear that (3) \Rightarrow (2), and it follows from the previous theorem that (2) \Rightarrow (1). We just need to prove that (1) \Rightarrow (3).

Suppose that $Df(x^*) = 0$. Then, that f is concave implies that for all x in the domain, we have:

$$f(x) \leq f(x^*) + Df(x^*)(x - x^*).$$

These two formulas mean that for all x , we must have

$$f(x) \leq f(x^*).$$

Therefore, f reaches a global maximum at x^* . □

Theorem 2.6.9 (Strict Concavity/Convexity and Uniqueness of Global Optimum)

Let $X \subseteq \mathcal{R}^n$.

- (1) If a strictly concave function f defined on X reaches a local maximum value at \mathbf{x}^* , then \mathbf{x}^* is the unique global maximum point.
- (2) If a strictly convex function f reaches a local minimum value at $\tilde{\mathbf{x}}$, then $\tilde{\mathbf{x}}$ is the unique global minimum point.

PROOF. Proof by contradiction. If \mathbf{x}^* is a global maximum point of function f , but not unique, then there is a point $\mathbf{x}' \neq \mathbf{x}^*$, such that $f(\mathbf{x}') = f(\mathbf{x}^*)$. Suppose that $\mathbf{x}^t = t\mathbf{x}' + (1 - t)\mathbf{x}^*$. Then, strict concavity requires that for all $t \in (0, 1)$,

$$f(\mathbf{x}^t) > tf(\mathbf{x}') + (1 - t)f(\mathbf{x}^*).$$

Since $f(\mathbf{x}') = f(\mathbf{x}^*)$,

$$f(\mathbf{x}^t) > tf(\mathbf{x}') + (1 - t)f(\mathbf{x}') = f(\mathbf{x}').$$

This contradicts the assumption that \mathbf{x}' is a global maximum point of f . Consequently, the global maximum point of a strictly concave function is unique. The proof of part (2) is similar, and thus omitted. \square

Theorem 2.6.10 (The sufficient condition for the uniqueness of global optimum)

Suppose that $f(\mathbf{x})$ is twice continuously differentiable on $X \subseteq \mathcal{R}^n$. We have:

- (1) If $f(\mathbf{x})$ is strictly concave and $f_i(\mathbf{x}^*) = 0, i = 1, \dots, n$, then \mathbf{x}^* is a unique global maximum point of $f(\mathbf{x})$.
- (2) If $f(\mathbf{x})$ is strictly convex and $f_i(\tilde{\mathbf{x}}) = 0, i = 1, \dots, n$, then $\tilde{\mathbf{x}}$ is a unique global minimum point of $f(\mathbf{x})$.

2.6.2 Optimization with Equality Constraints**Equality-Constrained Optimization**

An optimization problem with equality-constraints has the following form: Suppose that a function of n variables defined on $X \subseteq \mathcal{R}^n$ with m

constraints, where $m < n$. The optimization problem is:

$$\begin{aligned} & \max_{x_1, \dots, x_n} f(x_1, \dots, x_n) \\ & \text{s.t. } g^1(x_1, \dots, x_n) = 0, \\ & \quad g^2(x_1, \dots, x_n) = 0, \\ & \quad \vdots \\ & \quad g^m(x_1, \dots, x_n) = 0. \end{aligned}$$

The most important conclusion of the equality-constrained optimization problem is the **Lagrange theorem**, which gives a necessary condition for a point to be the solution of the optimization problem.

The Lagrange function of the above equality-constrained problem is defined as:

$$\mathcal{L}(\mathbf{x}, \lambda) = f(\mathbf{x}) + \sum_{j=1}^m \lambda_j g^j(\mathbf{x}), \quad (2.6.3)$$

where $\lambda_1, \dots, \lambda_m$ are called **Lagrange multipliers**.

The following Lagrange theorem presents how to solve optimization problems under equality constraints.

Theorem 2.6.11 (First-Order Necessary Condition for Interior Extremum)

Suppose that $f(\mathbf{x})$ and $g^j(\mathbf{x})$, $j = 1, \dots, m$, are continuously differentiable functions defined on $X \subseteq \mathcal{R}^n$, \mathbf{x}^ is an interior point of X and an extreme point (maximal or minimal point) of f —here f is subject to the constraint of $g^j(\mathbf{x}^*) = 0$, where $j = 1, \dots, m$. If the gradient $Dg^j(\mathbf{x}^*) = 0$, $j = 1, \dots, m$, are linearly independent, then there is a unique λ_j^* , $j = 1, \dots, m$, such that:*

$$\frac{\partial \mathcal{L}(\mathbf{x}^*, \lambda^*)}{\partial x_i} = \frac{\partial f(\mathbf{x}^*)}{\partial x_i} + \sum_{j=1}^m \lambda_j^* \frac{\partial g^j(\mathbf{x}^*)}{\partial x_i} = 0, \quad i = 1, \dots, n.$$

The following proposition gives the sufficient conditions for interior extreme values with equality constraints.

Proposition 2.6.1 (Second-Order Necessary Condition for Interior Extremum)

Suppose that f and g^1, \dots, g^m are twice continuously differentiable functions, and \mathbf{x}^* satisfies the necessary conditions of Theorem 2.6.10. Let the bordered Hessian determinant

$$|\bar{H}_r| = \det \begin{pmatrix} 0 & \cdots & 0 & \frac{\partial g^1}{\partial x_1} & \cdots & \frac{\partial g^1}{\partial x_r} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \frac{\partial g^m}{\partial x_1} & \cdots & \frac{\partial g^m}{\partial x_r} \\ \frac{\partial g^1}{\partial x_1} & \cdots & \frac{\partial g^m}{\partial x_1} & \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_1} & \cdots & \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_r} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial g^1}{\partial x_r} & \cdots & \frac{\partial g^m}{\partial x_r} & \frac{\partial^2 \mathcal{L}}{\partial x_r \partial x_1} & \cdots & \frac{\partial^2 \mathcal{L}}{\partial x_r \partial x_r} \end{pmatrix}, r = m+1, 2, \dots, n$$

take value at \mathbf{x}^* . Thus

- (1) If $(-1)^{r-m+1} |\bar{H}_r(\mathbf{x}^*)| > 0, r = m+1, \dots, n$, then \mathbf{x}^* is the local maximum of the optimization problem.
- (2) If $(-1)^m |\bar{H}_r(\mathbf{x}^*)| < 0, r = m+1, \dots, n$, then \mathbf{x}^* is the local minimum of the optimization problem.

In particular, when there is only one equality constraint, i.e., $m = 1$, the bordered Hessian determinant $|\bar{H}|$ becomes:

$$|\bar{H}| = \begin{vmatrix} 0 & g_1 & g_2 & \cdots & g_n \\ g_1 & \mathcal{L}_{11} & \mathcal{L}_{12} & \cdots & \mathcal{L}_{1n} \\ g_2 & \mathcal{L}_{21} & \mathcal{L}_{22} & \cdots & \mathcal{L}_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ g_n & \mathcal{L}_{n1} & \mathcal{L}_{n2} & \cdots & \mathcal{L}_{nn} \end{vmatrix}.$$

where $\mathcal{L}_{ij} = f_{ij} - \lambda g_{ij}$. The first-order condition is

$$\lambda = \frac{f_1}{g_1} = \frac{f_2}{g_2} = \cdots = \frac{f_n}{g_n}.$$

The principal minors of the bordered Hessian are

$$|\bar{H}_2| = \begin{vmatrix} 0 & g_1 & g_2 \\ g_1 & \mathcal{L}_{11} & \mathcal{L}_{12} \\ g_2 & \mathcal{L}_{21} & \mathcal{L}_{22} \end{vmatrix}, \quad |\bar{H}_3| = \begin{vmatrix} 0 & g_1 & g_2 & g_3 \\ g_1 & \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} \\ g_2 & \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} \\ g_3 & \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} \end{vmatrix}, \quad \dots$$

This leads to the following two conclusions.

Conditions for Maximum with Equality Constraints

- (1) $\mathcal{L}_\lambda = \mathcal{L}_1 = \mathcal{L}_2 = \dots = \mathcal{L}_n = 0$ [First-order Necessary Condition];
- (2) $|\bar{H}_2| < 0, |\bar{H}_3| < 0, |\bar{H}_4| < 0, \dots, |\bar{H}_n| < 0$.

Conditions for Minimum with Equality Constraints

- (1) $\mathcal{L}_\lambda = \mathcal{L}_1 = \mathcal{L}_2 = \dots = \mathcal{L}_n = 0$ [First-order Necessary Condition];
- (2) $|\bar{H}_2| > 0, |\bar{H}_3| < 0, |\bar{H}_4| > 0, \dots, (-1)^n |\bar{H}_n| > 0$.

Note that when the constraint function g is linear, $g(\mathbf{x}) = a_1x_1 + \dots + a_nx_n = c$, all of the twice partial derivatives of g are equal to zero, and thus the bordered determinant $|B|$ and the bordered Hessian determinant have the following relations:

$$|B| = \lambda^2 |\bar{H}|$$

Therefore, the sequential principal minors of the bordered determinant have the same signs. As such, *as long as the objective function is strictly quasi-concave, the first-order necessary condition is also a sufficient condition to have the maximum value.*

2.6.3 Optimization with Inequality Constraints

Consider an optimization problem with inequality constraints:

$$\begin{aligned} & \max f(\mathbf{x}) \\ \text{s.t.} \quad & g_i(\mathbf{x}) \leq d_i, \quad i = 1, 2, \dots, k. \end{aligned}$$

If for a point \mathbf{x} that makes all constraints held with equality, $Dg_1(\mathbf{x})$, $Dg_2(\mathbf{x})$, \dots , $Dg_k(\mathbf{x})$ are linearly independent, then \mathbf{x} is said to satisfy the strong version of **constrained qualification**. Here, the symbol D represents the partial differential operator.

Theorem 2.6.12 (Kuhn-Tucker Theorem) *Suppose that \mathbf{x} solves the inequality-constrained maximization problem and satisfies the constrained qualification condition. Then, there is a set of Kuhn-Tucker multipliers $(\lambda_i \geq 0, i = 1, \dots, k)$, such that*

$$Df(\mathbf{x}) = \sum_{i=1}^k \lambda_i Dg_i(\mathbf{x}).$$

Moreover, we have the complementary slackness conditions:

$$\begin{aligned} \lambda_i &\geq 0, \quad \text{for all } i = 1, 2, \dots, k. \\ \lambda_i &= 0, \quad \text{if } g_i(\mathbf{x}) < d_i. \end{aligned}$$

Comparing the Kuhn-Tucker theorem with Lagrange multipliers in the equality-constrained optimization problem, we see that the major difference is that the signs of the Kuhn-Tucker multipliers are nonnegative, while the signs of the Lagrange multipliers can be positive or negative. This additional information can be useful in various occasions.

The Kuhn-Tucker theorem only provides a necessary condition for a maximum. The following theorem states conditions that guarantee that the above first-order conditions are sufficient.

Theorem 2.6.13 (Kuhn-Tucker Sufficiency) *Suppose that f is concave, and $g_i, i = 1, \dots, k$, are convex. If \mathbf{x} satisfies the Kuhn-Tucker first-order conditions, then \mathbf{x} is a global solution to the constrained maximization problem.*

We can weaken the conditions in the above theorem when there is only

one constraint. Let $C = \{\mathbf{x} \in \mathcal{R}^n : g(\mathbf{x}) \leq d\}$. We have the following propositions.

Proposition 2.6.2 *Suppose that f is quasi-concave, and the set C is convex (this is true if g is quasi-convex). If \mathbf{x} satisfies the Kuhn-Tucker first-order conditions, then \mathbf{x} is a global solution to the constrained maximization problem.*

Sometimes, we require \mathbf{x} to be nonnegative. Suppose that we have the following optimization problem:

$$\begin{aligned} \max \quad & f(\mathbf{x}) \\ \text{s.t.} \quad & g_i(\mathbf{x}) \leq d_i, \quad i = 1, 2, \dots, k, \\ & \mathbf{x} \geq 0. \end{aligned}$$

Then, the Lagrange function in this case is given by

$$L(\mathbf{x}, \lambda) = f(\mathbf{x}) + \sum_{l=1}^k \lambda_l [d_l - g_l(\mathbf{x})] + \sum_{j=1}^n \mu_j x_j,$$

where μ_1, \dots, μ_n are the multipliers associated with constraints $x_j \geq 0$. The first-order conditions are

$$\begin{aligned} \frac{L(\mathbf{x}, \lambda)}{\partial x_i} &= \frac{\partial f(\mathbf{x})}{\partial x_i} - \sum_{l=1}^k \lambda_l \frac{\partial g_l(\mathbf{x})}{\partial x_i} + \mu_i = 0, \quad i = 1, 2, \dots, n. \\ \lambda_l &\geq 0, \quad l = 1, 2, \dots, k. \\ \lambda_l &= 0, \quad \text{if } g_l(\mathbf{x}) < d_l. \\ \mu_i &\geq 0, \quad i = 1, 2, \dots, n. \\ \mu_i &= 0, \quad \text{if } x_i > 0. \end{aligned}$$

Eliminating μ_i , we can equivalently write the above first-order conditions with nonnegative choice variables as

$$\frac{L(\mathbf{x}, \lambda)}{\partial x_i} = \frac{\partial f(\mathbf{x})}{\partial x_i} - \sum_{l=1}^k \lambda_l \frac{\partial g_l(\mathbf{x})}{\partial x_i} \leq 0, \text{ with equality if } x_i > 0, \quad i = 1, 2, \dots, n,$$

or in matrix notation,

$$Df - \lambda Dg \leq 0,$$

$$\mathbf{x}[Df - \lambda Dg] = 0,$$

where we have written the product of two vectors \mathbf{x} and \mathbf{y} as the inner product, i.e., $\mathbf{x}\mathbf{y} = \sum_{i=1}^n x_i y_i$. Therefore, if we are at an interior optimum, we have

$$Df(\mathbf{x}) = \lambda Dg.$$

2.6.4 The Envelope Theorem

The Envelope Theorem without Constraints

Consider the following maximization problem:

$$M(\mathbf{a}) = \max_{\mathbf{x}} f(\mathbf{x}, \mathbf{a}).$$

The function $M(\mathbf{a})$ gives the maximum of the objective function as a function of parameter \mathbf{a} .

Let $\mathbf{x}(\mathbf{a})$ be the value of \mathbf{x} that solves the maximization problem. Then, we can also write $M(\mathbf{a}) = f(\mathbf{x}(\mathbf{a}), \mathbf{a})$. It is often of interest to know how $M(\mathbf{a})$ changes as \mathbf{a} changes. The Envelope Theorem gives us the answer:

$$\frac{dM(\mathbf{a})}{d\mathbf{a}} = \frac{\partial f(\mathbf{x}, \mathbf{a})}{\partial \mathbf{a}} \bigg|_{\mathbf{x}=\mathbf{x}(\mathbf{a})}$$

The conclusion is particularly useful. This expression informs us that the derivative of M with respect to \mathbf{a} is given by the partial derivative of f with respect to \mathbf{a} , holding \mathbf{x} fixed at the optimal choice. This is the meaning of the vertical bar to the right of the derivative. The proof of the envelope theorem is a relatively straightforward calculation.

The Envelope Theorem with Constraints

Now, consider a more general parameterized constrained maximization problem of the form:

$$\begin{aligned} M(\mathbf{a}) &= \max_{x_1, x_2} g(x_1, x_2, \mathbf{a}) \\ \text{s.t.} \quad & h(x_1, x_2, \mathbf{a}) = 0. \end{aligned}$$

The Lagrangian for this problem is

$$\mathcal{L} = g(x_1, x_2, \mathbf{a}) - \lambda h(x_1, x_2, \mathbf{a}),$$

and the first-order conditions for interior points are

$$\begin{aligned} \frac{\partial g}{\partial x_1} - \lambda \frac{\partial h}{\partial x_1} &= 0, \\ \frac{\partial g}{\partial x_2} - \lambda \frac{\partial h}{\partial x_2} &= 0, \\ h(x_1, x_2, \mathbf{a}) &= 0. \end{aligned} \tag{2.6.4}$$

These conditions determine the optimal choice functions $(x_1(\mathbf{a}), x_2(\mathbf{a}))$, which, in turn, determine the maximum value function

$$M(\mathbf{a}) \equiv g(x_1(\mathbf{a}), x_2(\mathbf{a}), \mathbf{a}). \tag{2.6.5}$$

The Envelope Theorem gives us a formula for the derivative of the value function with respect to a parameter in the maximization problem. Specifically, the formula is

$$\begin{aligned} \frac{dM(\mathbf{a})}{d\mathbf{a}} &= \left. \frac{\partial \mathcal{L}(\mathbf{x}, \mathbf{a})}{\partial \mathbf{a}} \right|_{\mathbf{x}=\mathbf{x}(\mathbf{a})} \\ &= \left. \frac{\partial g(x_1, x_2, \mathbf{a})}{\partial \mathbf{a}} \right|_{x_i=x_i(\mathbf{a})} - \lambda \left. \frac{\partial h(x_1, x_2, \mathbf{a})}{\partial \mathbf{a}} \right|_{x_i=x_i(\mathbf{a})} \end{aligned}$$

As previously, special focus should be given to the interpretation of these partial derivatives: they are the derivatives of g and h with respect to \mathbf{a} , holding x_1 and x_2 fixed at their optimal values.

2.6.5 Maximum Theorems

In optimization problems, we usually need to check if an optimal solution is continuous in parameters, e.g., to check the continuity of the demand function. We can apply the so-called **maximum theorem** to these problems.

Berge's Maximum Theorem

Theorem 2.6.14 (Berge's Maximum Theorem) *Let A and X be two topological spaces. Suppose that $f : A \times X \rightarrow \mathcal{R}$ is a continuous function, and the constraint set $F : A \rightarrow 2^X$ is a continuous correspondence with non-empty compact values. Then, the maximum value function (also called the marginal function)*

$$M(\mathbf{a}) = \max_{\mathbf{x} \in F(\mathbf{a})} f(\mathbf{x}, \mathbf{a})$$

is a continuous function on A , and the maximum correspondence

$$\mu(\mathbf{a}) = \arg \max_{\mathbf{x} \in F(\mathbf{a})} f(\mathbf{x}, \mathbf{a})$$

is upper hemi-continuous.

Walker's Maximum Theorem

In many cases of optimization problems, the preference of an economic agent may not be represented by a utility function. Walker (1979) generalized Berge's maximum theorem to the case of maximal element under the open preference relation. Walker's maximum theorem allows the preference relations and constraint sets to vary with parameters.

Theorem 2.6.15 (Walker's Maximum Theorem) *Let A and Y be two topological spaces. Suppose that $U : Y \times A \rightarrow 2^Y$ is a correspondence with an open graph. The constraint set $F : A \rightarrow 2^Y$ is a continuous and non-empty compact-valued correspondence. Define the maximum correspondence $\mu : A \rightarrow 2^Y$ as*

$$\mu(\mathbf{a}) := \{\mathbf{y} \in F(\mathbf{a}) : U(\mathbf{y}, \mathbf{a}) \cap F(\mathbf{a}) = \emptyset\},$$

μ is a compact-valued upper semi-continuous correspondence.

Tian-Zhou Maximum Theorem

Both Berge's and Walker's maximum theorems depend on the continuity (or open graph) of the constraint correspondence and the objective function (preference correspondence).

Tian and Zhou (1995) relaxed these assumptions, and generalized and characterized Berge's and Walker's maximum theorems. We first give the following definition of transfer continuity.

Definition 2.6.3 Let A and Y be two topological spaces, and $F : A \rightarrow 2^Y$ be a correspondence. A function $u : A \times Y \rightarrow \mathcal{R} \cup \{\infty\}$ is said to be *quasi-transfer upper continuous in (a, y) with respect to F* if, for every $(a, y) \in A \times Y$ with $y \in F(a)$, $u(a, z) > u(a, y)$ for some $z \in F(a)$ implies that there is a neighbourhood $\mathcal{N}(a, y)$ of (a, y) , such that for any $(a', y') \in \mathcal{N}(a, y)$ with $y' \in F(a')$, there is a $z' \in F(a')$, such that

$$u(a', z') > u(a', y').$$

The following definition is a natural generalization of transfer upper continuity.

Definition 2.6.4 Let A and Y be two topological spaces, and $F : A \rightarrow 2^Y$ be a correspondence. A function $u : A \times Y \rightarrow \mathcal{R} \cup \{\infty\}$ is said to be *transfer upper continuous on F* if, for every $(a, y) \in A \times Y$ with $y \in F(a)$, $u(a, z) > u(a, y)$ for some $z \in F(a)$ implies that there is a point $z' \in Y$ and a neighbourhood $\mathcal{N}(y)$ of y , such that for any $y' \in \mathcal{N}(y)$ with $y' \in F(a)$, we have $u(a, z') > u(a, y')$ and $z' \in F(a)$.

Theorem 2.6.16 (Tian-Zhou Maximum Theorem) Let A and Y be two topological spaces, and $u : A \times Y \rightarrow \mathcal{R} \cup \{\infty\}$ be a function. Suppose that $F : A \rightarrow 2^Y$ is a compact and closed valued correspondence. Then, the maximum correspondence $\mu : A \rightarrow 2^Y$ is a nonempty, compact-valued and closed correspondence if and only if u is transfer upper continuous in y on F , and quasi-transfer upper continuous in (a, y) with respect to F . Moreover, if F is upper hemi-continuous, then the correspondence of extreme value μ is also upper hemi-continuous.

This theorem relaxes the upper semi-continuity of the objective function and the constraint correspondence in Berge's maximum theorem.

2.6.6 Continuous Selection Theorems

The continuous selection theorem is a powerful tool to prove the existence of equilibrium, and it is closely related to the fixed point theorem, which will be introduced below. The basic conclusion of the continuous selection theorem is that if a correspondence is lower hemi-continuous with non-empty convex values, there is a continuous function so that for all points in the domain, the function value is a subset of the correspondence.

Definition 2.6.5 Let $X \subseteq \mathcal{R}^n$, $Y \subseteq \mathcal{R}^m$ and $F : X \rightarrow 2^Y$ be a correspondence. If for any $x \in X$, we have $f(x) \in F(x)$, then the single valued function $f : X \rightarrow Y$ is said to be a *selection* corresponding to F .

Theorem 2.6.17 (Michael(1956)) Let $X \subseteq \mathcal{R}^n$ be compact. Suppose that $F : X \rightarrow 2^{\mathcal{R}^m}$ is a lower hemi-continuous correspondence with closed and convex values. Then, F has a continuous selection, i.e., there exists a single-valued continuous function $f : X \rightarrow \mathcal{R}^m$, such that $f(x) \in F(x)$ for all $x \in X$.

For the infinite dimension space, we have the following Browder Theorem.

Theorem 2.6.18 (Browder, (1968)) Let X be a Hausdorff compact space, and Y be a locally convex topological vector space. Suppose that $F : X \rightarrow 2^Y$ is a correspondence with open lower sections and convex values. Then, F has a continuous selection, i.e., there is a single-valued continuous function $f : X \rightarrow Y$, such that $f(x) \in F(x)$ for all $x \in X$.

Since the open lower section of a correspondence implies the lower hemi-continuity of the correspondence (see Proposition 2.5.3), we then have the following result.

Corollary 2.6.1 (Yannelis-Prabhakar (1983)) Let $X \subseteq \mathcal{R}^n$. Suppose that $F : X \rightarrow 2^{\mathcal{R}^m}$ is a correspondence with open lower sections and convex values. Then, F has a continuous selection, i.e., there is a single-valued continuous function $f : X \rightarrow Y$, such that $f(x) \in F(x)$ for all $x \in X$.

2.6.7 Fixed Point Theorems

The fixed point theorem plays a crucial role in proving the existence of equilibrium. It is the most commonly used method for determining whether there is a solution of equilibrium equations. John von Neumann (1903-1957, see Section 5.8.1 for his biography) was the first to propose results that are essentially the fixed point theorem in two papers published in 1928 and 1937, respectively.

Definition 2.6.6 Let X be a topological space, and $f : X \rightarrow X$ be a single-valued function from X to itself. If there is a point $x^* \in X$, such that $f(x^*) = x^*$, then x^* is called a *fixed point* of function f .

Definition 2.6.7 Let X be a topological space, and $F : X \rightarrow 2^X$ is a correspondence from X to itself. If there is a point $x^* \in X$, such that $x^* \in F(x^*)$, then x^* is called a *fixed point* of correspondence f .

There are some important fixed point theorems which are widely used in economics.

Brouwer's Fixed Theorem

Brouwer's fixed point theorem is one of the most fundamental and important fixed point theorems.

Theorem 2.6.19 (Brouwer's Fixed Theorem) Let X be a non-empty, compact, and convex subset of \mathcal{R}^m . If a function $f : X \rightarrow X$ is continuous on X , then f has a fixed point, i.e., there is a point $x^* \in X$, such that $f(x^*) = x^*$ (See Figure 2.4).

Example 2.6.1 If $f : [0, 1] \rightarrow [0, 1]$ is continuous, then f has a fixed point x . To see this, let $g(x) = f(x) - x$. Then, we have

$$g(0) = f(0) \geq 0$$

$$g(1) = f(1) - 1 \leq 0.$$

From the mean-value theorem, there is a point $x^* \in [0, 1]$, such that $g(x^*) = f(x^*) - x^* = 0$.

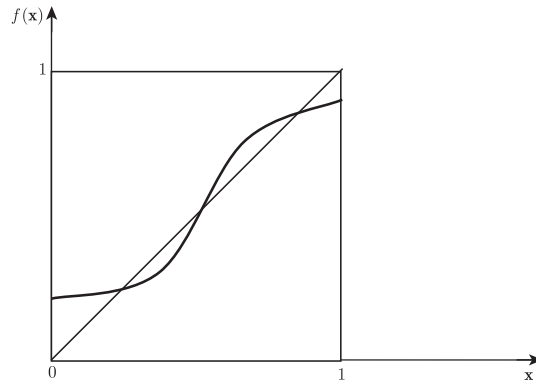


Figure 2.4: The intersection point of 45° line and the curve of a function is a fixed point. There are three fixed points in this example

Kakutani's Fixed Point Theorem

In applications, mapping is often a correspondence, and thus Brouwer's fixed point theorem cannot be used directly, and Kakutani's fixed point theorem is commonly used instead.

Theorem 2.6.20 (Kakutani's Fixed Point Theorem (1941)) *Let $X \subseteq \mathcal{R}^m$ be a non-empty, compact, and convex subset. If a correspondence $F : X \rightarrow 2^X$ is an upper hemi-continuous correspondence with non-empty compact and convex values on X , then F has a fixed point, i.e., there is a point $x^* \in X$, such that $x^* \in F(x^*)$.*

Browder's Fixed Point Theorem

It follows from Theorem 2.6.16 that we have the following Browder's Fixed Point Theorem.

Theorem 2.6.21 (Browder (1968)) *Let $X \subseteq \mathcal{R}^n$ be a compact and convex subset. Suppose that a correspondence $F : X \rightarrow 2^{\mathcal{R}^m}$ is convex-valued with open lower sections. Then, F has a fixed point, i.e., there is a point $x^* \in X$, such that $x^* \in F(x^*)$.*

Michael's Fixed Point Theorem

It follows from Theorem 2.6.17 that Michael's Fixed Point Theorem is given as follows.

Theorem 2.6.22 (Michael (1956)) *Let $X \subseteq \mathcal{R}^n$ be a compact and convex subset. Suppose that $F : X \rightarrow 2^{\mathcal{R}^m}$ is a lower hemi-continuous correspondence with closed and convex values. Then, F has a fixed point, i.e., there is a point $\mathbf{x}^* \in X$, such that $\mathbf{x}^* \in F(\mathbf{x}^*)$.*

Tarsky's Fixed Point Theorem

Tarsky's fixed point theorem is a very different type of fixed point theorem. It does not require the function to have any kind of continuity, but only requires that the function be monotonic and non-decreasing, and be defined on the domain composed of intervals. It is becoming increasingly important in applications of economics, especially in games with a monotonic payoff function.

Theorem 2.6.23 (Tarsky's Fixed Point Theorem (1955)) *Let $[0, 1]^n$ be the n times product of interval $[0, 1]$. If $f : [0, 1]^n \rightarrow [0, 1]^n$ is a non-decreasing function, then f has a fixed point, i.e., there is a point $\mathbf{x}^* \in X$, such that $f(\mathbf{x}^*) = \mathbf{x}^*$.*

Contraction Mapping Theorem

In numerous dynamic economic models, we not only need to prove the existence of equilibrium, but also prove the uniqueness of equilibrium. The contraction mapping principle is an important tool to solve this problem. It is also the most basic and simple theorem of existence in functional analysis. Indeed, many of the existence theorems in mathematical analysis are its special cases. Its basic conclusion is that a contraction mapping from a complete metric space to itself has a unique fixed point.

Definition 2.6.8 Let (X, d) be a complete metric space, and $f : X \rightarrow X$ be a single-valued function from X to itself. If for any point $\mathbf{x}, \mathbf{x}' \in X$, there is $\alpha \in (0, 1)$, such that $d(f(\mathbf{x}), f(\mathbf{x}')) < \alpha d(\mathbf{x}, \mathbf{x}')$, then f is a *contraction mapping*.

Theorem 2.6.24 (Banach Contraction Mapping Theorem) *Suppose that $f : X \rightarrow X$ is a contraction mapping from a complete metric space X to itself. Then, f has a unique fixed point on X .*

Characterization of the Existence of Fixed Points

All of the above fixed-point theorems are only sufficient conditions for the existence of fixed points. Tian (2017) introduced a series of concepts of recursive transfer continuity, and provided a sufficient and necessary condition for the existence of fixed points.

We first introduce the concept of diagonal transfer continuity introduced by Baye, Tian and Zhou (1993).

Definition 2.6.9 A function $\varphi : X \times X \rightarrow R \cup \{\pm\infty\}$ is said to be *diagonally transfer continuous in y* if, whenever $\varphi(x, y) > \varphi(y, y)$ for $x, y \in X$, there exists a point $z \in X$ and a neighborhood $\mathcal{V}_y \subset X$ of y , such that $\varphi(z, y') > \varphi(y', y')$ for all $y' \in \mathcal{V}_y$.

We now define the concept of recursive diagonal transfer continuity.

Definition 2.6.10 (Recursive Diagonal Transfer Continuity) A function $\varphi : X \times X \rightarrow R \cup \{\pm\infty\}$ is said to be *recursively diagonally transfer continuous in y* if, whenever $\varphi(x, y) > \varphi(y, y)$ for $x, y \in X$, there exists a point $z^0 \in X$ (possibly $z^0 = y$) and a neighborhood \mathcal{V}_y of y , such that $\varphi(z, y') > \varphi(y', y')$ for all $y' \in \mathcal{V}_y$ and for any finite subset $\{z^1, \dots, z^m\} \subseteq X$ with $z^m = z$ and $\varphi(z, z^{m-1}) > \varphi(z^{m-1}, z^{m-1})$, $\varphi(z^{m-1}, z^{m-2}) > \varphi(z^{m-2}, z^{m-2})$, \dots , $\varphi(z^1, z^0) > \varphi(z^0, z^0)$ for $m \geq 1$.

Theorem 2.6.25 (Tian's Fixed Point Theorem (2017)) *Let X be a nonempty and compact subset of a metric space (E, d) , and $f : X \rightarrow X$ be a function. Then, f has a fixed point if and only if the function $\varphi : X \times X \rightarrow R \cup \{\pm\infty\}$, defined by $\varphi(x, y) = -d(x, f(y))$, is recursively diagonally transfer continuous in y .*

2.6.8 Variation Inequality

Ky-Fan minimax inequality is one of the most prominent results in nonlinear analysis. It is equivalent to many important mathematical theorems in a

certain sense, such as KKM lemma, Sperner lemma, Brouwer's fixed point theorem, and Kakutani's fixed point theorem (which can be derived from each other). In numerous disciplines, such as variation inequalities, mathematical programming, partial differential equations and economic models, it can be used to prove the existence of equilibrium solutions.

Theorem 2.6.26 (Ky-Fan minimax inequality) *Let $X \subseteq \mathcal{R}^m$ be a nonempty, convex and compact set, and let $\phi: X \times X \rightarrow \mathcal{R}$ be a function that satisfies the following conditions:*

- (1) *for all $x \in X$, $\phi(x, x) \leq 0$;*
- (2) *ϕ is lower semi-continuous in y ;*
- (3) *ϕ is quasi-concave in x .*

Then, there exists a point $y^ \in X$, such that $\phi(x, y^*) \leq 0$ holds for all $x \in X$.*

Ky-Fan inequality has been generalized in various forms in mathematical literature. Tian (2017) fully characterized the existence of solutions to Ky-Fan inequalities, and provided the sufficient and necessary conditions for the existence of Ky-Fan inequalities.

Definition 2.6.11 Let X be a topological space. A function $\phi: X \times X \rightarrow \mathcal{R} \cup \{\pm\infty\}$ is said to be γ -recursively transfer lower semicontinuous in y if, whenever $\phi(x, y) > \gamma$ for $x, y \in X$, there exists a point $z^0 \in X$ (possibly $z^0 = y$) and a neighborhood \mathcal{V}_y of y , such that $\phi(z, \mathcal{V}_y) > \gamma$ for any sequence of points $\{z^1, \dots, z^{m-1}, z\}$ with $\phi(z, z^{m-1}) > \gamma$, $\phi(z^{m-1}, z^{m-2}) > \gamma$, ..., $\phi(z^1, z^0) > \gamma$, $m = 1, 2, \dots$. Here, $\phi(z, \mathcal{V}_y) > \gamma$ means that $\phi(z, y') > \gamma$ for all $y' \in \mathcal{V}_y$.

Theorem 2.6.27 (Tian, 2017) *Let X be a compact subset in a topological space, $\gamma \in \mathcal{R}$, and $\phi: X \times X \rightarrow \mathcal{R} \cup \{\pm\infty\}$ be a function satisfying $\phi(x, x) \leq \gamma, \forall x \in X$. Then, there is a point $y^* \in X$, such that $\phi(x, y^*) \leq \gamma$ for all $x \in X$ if and only if ϕ is γ -recursively diagonally transfer lower hemi-continuous in y .*

2.6.9 FKKM Theorems

The Knaster-Kuratowski-Mazurkiewicz (KKM) lemma is quite basic and is, in certain ways, more useful than Brouwer's fixed point theorem.

Theorem 2.6.28 (KKM Theorem) Let $X \subseteq \mathcal{R}^m$ be a convex set. Suppose that $F : X \rightarrow 2^X$ is a correspondence, such that

- (1) $F(\mathbf{x})$ is closed for all $\mathbf{x} \in X$;
- (2) F is FS-convex, i.e., for any $\mathbf{x}_1, \dots, \mathbf{x}_m \in X$ and its convex combination $\mathbf{x}_\lambda = \sum_{i=1}^m \lambda_i \mathbf{x}_i$, we have

$$\mathbf{x}_\lambda \in \bigcup_{i=1}^m F(\mathbf{x}_i),$$

then

$$\bigcap_{\mathbf{x} \in X} F(\mathbf{x}) \neq \emptyset.$$

The following is a generalized version of the KKM lemma by Ky Fan (1984).

Theorem 2.6.29 (FKKM Theorem) Suppose that $X \subseteq \mathcal{R}^m$ is a convex set, $\emptyset \neq X \subseteq Y$, and $F : X \rightarrow 2^Y$ is a correspondence, such that

- (1) $F(\mathbf{x})$ is closed for all $\mathbf{x} \in X$;
- (2) $F(\mathbf{x}_0)$ is compact for some $\mathbf{x}_0 \in X$;
- (3) F is FS-convex, i.e., for any $\mathbf{x}_1, \dots, \mathbf{x}_m \in X$ and its convex combination $\mathbf{x}_\lambda = \sum_{i=1}^m \lambda_i \mathbf{x}_i$, we have

$$\mathbf{x}_\lambda \in \bigcup_{i=1}^m F(\mathbf{x}_i).$$

Then,

$$\bigcap_{\mathbf{x} \in X} F(\mathbf{x}) \neq \emptyset.$$

This theorem has numerous generalizations. Tian (2017) also provided the sufficient and necessary conditions for establishing the FKKM theorem:

Theorem 2.6.30 (Tian, 2017) Let X be a nonempty compact set in a topological space T , and $F : X \rightarrow 2^X$ be a correspondence satisfying $\mathbf{x} \in F(\mathbf{x})$ for all $\mathbf{x} \in X$.

Then, $\bigcap_{x \in X} F(x) \neq \emptyset$ if and only if the correspondence $\phi: X \times X \rightarrow \mathcal{R} \cup \{\pm\infty\}$ defined by

$$\phi(x, y) = \begin{cases} \gamma, & \text{if } (x, y) \in G, \\ +\infty, & \text{otherwise} \end{cases}$$

is γ -recursively transfer semi-continuous with respect to y , where $\gamma \in \mathcal{R}$ and $G = \{(x, y) \in X \times Y : y \in F(x)\}$.

2.7 Dynamic Optimization

We generally encounter various constraints when making optimal decisions, and the constrained optimization problems in the last section are all among different variables in the same period. However, individuals usually need to make decisions in a dynamic environment, and early decisions will affect decisions in later periods. Dynamic optimization, dynamic programming, and optimal control provide analytical frameworks and tools for solving optimization problems in dynamic environments. In this section, we discuss the calculus of variation, optimal control, and the basic results of dynamic programming. We focus mainly on continuous cases of dynamic optimization problems defined on $X \subseteq \mathcal{R}$.

2.7.1 Calculus of Variation

A general dynamic optimization problem has the following form:

$$\max \int_{t_0}^{t_1} F[t, x(t), x'(t)] dt \quad (2.7.6)$$

$$\text{s.t. } x(t_0) = x_0, x(t_1) = x_1. \quad (2.7.7)$$

The above optimization problem is to choose a function $x(t)$ subject to the constraints in (2.7.7) to maximize the objective function (2.7.6). Calculus of variation is a common method to solve such problems. Let $x^*(t)$ be the solution to the above optimization problem, and the necessary condition is that the solution must satisfy the Euler equation:

$$F_x[t, \mathbf{x}^*(t), \mathbf{x}'^*(t)] = \frac{dF_{x'}[t, \mathbf{x}^*(t), \mathbf{x}'^*(t)]}{dt}, \quad t \in [t_0, t_1]. \quad (2.7.8)$$

Next, we will derive the Euler equation of dynamic optimization.

We say the function satisfying the constraint (2.7.7) is **admissible**. Let $\mathbf{x}(t)$ be admissible, and let $\mathbf{h}(t) = \mathbf{x}(t) - \mathbf{x}^*(t)$ be the difference between $\mathbf{x}(t)$ and the optimal selection. We then have $\mathbf{h}(t_0) = \mathbf{h}(t_1) = 0$.

For any constant a , $\mathbf{y}(t) = \mathbf{x}^*(t) + a\mathbf{h}(t)$ is also admissible. In this way, the dynamic optimization problem can be transformed into solving under what conditions $a = 0$ is the optimal choice under dynamic optimization.

$$\begin{aligned} g(a) &= \int_{t_0}^{t_1} F[t, \mathbf{y}(t), \mathbf{y}'(t)] dt \\ &= \int_{t_0}^{t_1} F[t, \mathbf{x}^*(t) + a\mathbf{h}(t), \mathbf{x}'^*(t) + a\mathbf{h}'(t)] dt. \end{aligned} \quad (2.7.9)$$

The first-order condition of optimization is obtained by differentiating (2.7.9) with respect to a and then is set to 0:

$$\begin{aligned} g'(0) &= \int_{t_0}^{t_1} F_x[t, \mathbf{x}^*, \mathbf{x}'^*(t)] \mathbf{h}(t) + F_{x'}[t, \mathbf{x}^*, \mathbf{x}'^*(t)] \mathbf{h}'(t) dt \\ &= 0. \end{aligned} \quad (2.7.10)$$

Using integration by parts on the second part of the right side of the equation (2.7.10) yields:

$$\int_{t_0}^{t_1} \left\{ F_x[t, \mathbf{x}^*, \mathbf{x}'^*(t)] - \frac{dF_{x'}[t, \mathbf{x}^*(t), \mathbf{x}'^*(t)]}{dt} \right\} \mathbf{h}(t) dt = 0. \quad (2.7.11)$$

If equation (2.7.11) holds for any continuous function $\mathbf{h}(t)$ that satisfies the constraint $\mathbf{h}(t_0) = \mathbf{h}(t_1) = 0$, the Euler equation (2.7.8) also holds (see Kamien & Schwartz (1991)).

Example 2.7.1 (Kamien & Schwartz (1991)) Suppose that an enterprise receives an order, requiring B units of products delivered at time T . Assume that the production capacity of the enterprise is limited, and the unit cost of production is proportional to the output. In addition, completed products

need to be stocked, and the inventory cost per unit is a constant. Business managers need to consider production problems from now (time 0) to delivery date (time T). Suppose that at time $t \in [0, T]$, the inventory of the enterprise is $x(t)$, and the change of inventory depends on the production of the enterprise, i.e., $\dot{x}(t) \equiv x'(t) = y(t)$, where $y(t)$ is the productivity at time t . At t , the cost of the enterprise is $c_1 x'(t)x'(t) + c_2 x(t)$ or $c_1 u(t)u(t) + c_2 x(t)$, where $c_1 u(t)$ is the unit cost of production when the yield is $u(t)$, and c_2 is the unit cost of inventory. The goal of the enterprise is to minimize costs (including both production costs and inventory costs), and therefore the dynamic optimization problem is

$$\begin{aligned} & \min \int_0^T [c_1 x'^2(t) + c_2 x(t)] dt \\ \text{s.t. } & x(0) = 0, x(T) = B, x'(t) \geq 0. \end{aligned} \quad (2.7.12)$$

In expression (2.7.12), $u(t)$ is called a **control variable**, and $x(t)$ is called a **state variable**. Using the calculus of variation to solve the optimization problem, we have

$$F[t, x(t), x'(t)] = c_1 x'^2(t) + c_2 x(t).$$

The Euler equation is:

$$c_2 = 2c_1 x''(t).$$

With the constraint conditions: $x^*(0) = 0, x^*(T) = B$, we solve the above Euler equation and obtain:

$$x^*(t) = \frac{c_2}{4c_1} t(t - T) + Bt/T, \quad t \in [0, T].$$

Integrating the Euler equation (2.7.8) yields:

$$F_x = F_{x't} + F_{x'x}x' + F_{x'x'}x''. \quad (2.7.13)$$

Now, we introduce the Hamilton equations to avoid taking second-order derivatives. Let $p(t) = F_{x'}[t, x(t), x'(t)]$, and the Hamilton equation

is:

$$H(t, \mathbf{x}, \mathbf{p}) = -F(t, \mathbf{x}, \mathbf{x}') + \mathbf{p}\mathbf{x}'. \quad (2.7.14)$$

In equation (2.7.14), $\mathbf{p}(t)$ can be regarded as the shadow price. The total differential of equation (2.7.14) is:

$$dH = -F_t dt - F_x d\mathbf{x} - F_{x'} d\mathbf{x}' + \mathbf{p} d\mathbf{x}' + \mathbf{x}' d\mathbf{p} = -F_t dt - F_x d\mathbf{x} + \mathbf{x}' d\mathbf{p}.$$

The partial derivatives of equation (2.7.14) with respect to \mathbf{x} and \mathbf{p} , respectively, are:

$$\begin{aligned} \partial H / \partial \mathbf{x} &= -F_x; \\ \partial H / \partial \mathbf{p} &= \mathbf{x}'. \end{aligned}$$

Since $-F_x = -(dF_{x'}/dt) = -\mathbf{p}'$, we have two Euler equations under first-order conditions:

$$\begin{aligned} \partial H / \partial \mathbf{x} &= -\mathbf{p}'; \\ \partial H / \partial \mathbf{p} &= \mathbf{x}'. \end{aligned}$$

The Euler equations are only the necessary conditions for solving dynamic optimization, and the sufficient conditions involve the second-order conditions. After deriving the first-order conditions by the calculus of variation, it is clear that the second-order condition is:

$$g''(0) = \int_{t_0}^{t_1} [F_{xx} h^2 + 2F_{xx'} \mathbf{h} \mathbf{h}' + F_{x'x'} (\mathbf{h}')^2] dt \leq 0.$$

It is easy to verify that if the objective function F is concave in \mathbf{x} and \mathbf{x}' , then the second-order condition is satisfied.

Denote $F = F(t, \mathbf{x}, \mathbf{x}')$, $F^* = F(t, \mathbf{x}^*, \mathbf{x}'^*)$, and let $\mathbf{h}(t) = \mathbf{x}(t) - \mathbf{x}^*(t)$.

Then, we have $\mathbf{h}'(t) = \mathbf{x}'(t) - \mathbf{x}'^*(t)$, and thus

$$\begin{aligned} \int_{t_0}^{t_1} (F - F^*) dt &\leq \int_{t_0}^{t_1} [(\mathbf{x} - \mathbf{x}^*) F_{\mathbf{x}}^* + (\mathbf{x}' - \mathbf{x}'^*) F_{\mathbf{x}'}^*] dt \\ &= \int_{t_0}^{t_1} (\mathbf{h} F_{\mathbf{x}}^* + \mathbf{h}' F_{\mathbf{x}'}^*) dt \\ &= \int_{t_0}^{t_1} \mathbf{h} (F_{\mathbf{x}}^* - dF_{\mathbf{x}'}^*/dt) dt = 0. \end{aligned}$$

It can be proven (see Kamien & Schwartz (1991, p.43)) that the first-order condition, i.e., the Euler equation, will be met as long as $F_{\mathbf{x}'\mathbf{x}'} \leq 0$, and thus the dynamic maximization problem is solved. Regarding dynamic minimization, the first-order condition is also a sufficient condition if the second-order condition satisfies $F_{\mathbf{x}'\mathbf{x}'} \geq 0$.

2.7.2 Optimal Control

We have two types of variables in the previous example: state variable and control variable. We can also discuss the dynamic optimization problem using the analytical framework of optimal control.

The optimal control problem can be generally expressed as follows:

$$\max \int_{t_0}^{t_1} f[t, \mathbf{x}(t), \mathbf{u}(t)] dt \quad (2.7.15)$$

$$\text{s.t.} \quad \mathbf{x}'(t) = g(t, \mathbf{x}(t), \mathbf{u}(t)), \quad (2.7.16)$$

$$\mathbf{x}(t_0) = \mathbf{x}_0. \quad (2.7.17)$$

In the above statement, $\mathbf{x}(t)$ is a state variable, $\mathbf{u}(t)$ is a control variable that affects the change of the state variable, and the objective (2.7.15) is a function of the state variable and the control variable.

The necessary and sufficient conditions for optimal control are given below. Analogously to the optimization problem under static constraints, the dynamic Lagrange equation is established as:

$$L = \int_{t_0}^{t_1} \{f[t, \mathbf{x}(t), \mathbf{u}(t)] + \lambda_t [g(t, \mathbf{x}(t), \mathbf{u}(t)) - \mathbf{x}'(t)]\} dt, \quad (2.7.18)$$

where λ_t is the multiplier of the constraint on the change in state at time t ,

commonly known as **costate variable**. Integrating by parts gives:

$$L = \int_{t_0}^{t_1} \{f[t, \mathbf{x}(t), \mathbf{u}(t)] + \lambda_t g(t, \mathbf{x}(t), \mathbf{u}(t)) + \mathbf{x}(t) \lambda_t'\} dt - \lambda_{t_1} \mathbf{x}(t_1) + \lambda_{t_0} \mathbf{x}(t_0).$$

The necessary conditions for the optimal control problem can be derived by using a similar process of deducing the calculus of variation. Assuming that $\mathbf{u}^*(t)$ is the optimal control function, we introduce a new control function $\mathbf{u}^*(t) + a\mathbf{h}(t)$, which reduces to the original optimal control function when $a = 0$. The optimal state function $\mathbf{x}^*(t)$ can be determined by giving the optimal control function $\mathbf{u}^*(t)$ and the initial state $\mathbf{x}(t_0) = \mathbf{x}_0$. Denote the state variable generated by control function $\mathbf{u}^*(t) + a\mathbf{h}(t)$ and initial state \mathbf{x}_0 as $\mathbf{y}(t, a)$, which satisfy: $\mathbf{y}(t, a) = \mathbf{x}^*(t)$, $\mathbf{y}(t, 0) = \mathbf{x}_0$, and $d\mathbf{y}(t, a)/dt = g(t, \mathbf{y}(t, a), \mathbf{u}^*(t) + a\mathbf{h}(t))$. Set the function:

$$\begin{aligned} J(a) &= \int_{t_0}^{t_1} f[t, \mathbf{y}(t, a), \mathbf{u}^*(t) + a\mathbf{h}(t)] dt \\ &= \int_{t_0}^{t_1} \{f[t, \mathbf{y}(t, a), \mathbf{u}^*(t) + a\mathbf{h}(t)] \\ &\quad + \lambda_t [g(t, \mathbf{y}(t, a), \mathbf{u}^*(t) + a\mathbf{h}(t)) + \mathbf{y}'(t, a) \lambda_t']\} dt \\ &\quad - \lambda_{t_1} \mathbf{y}(t_1, a) + \lambda_{t_0} \mathbf{y}(t_0, a). \end{aligned} \quad (2.7.19)$$

The derivative of the function (2.7.19) at $a = 0$ is:

$$J'(a) = \int_{t_0}^{t_1} [(f_x + \lambda g_x + \lambda') \mathbf{y}_a + (f_u + \lambda g_u) \mathbf{h}] dt - \lambda_{t_1} \mathbf{y}'(t_1, 0).$$

Here, $\lambda(t)$ is required to be differentiable, and the optimization needs to satisfy the following three conditions:

The first one is the first-order condition with respect to the control variable:

$$f_u[t, \mathbf{x}(t), \mathbf{u}(t)] + \lambda g_u(t, \mathbf{x}(t), \mathbf{u}(t)) = 0. \quad (2.7.20)$$

The second one is the first-order condition with respect to the costate variable:

$$\lambda'(t) = -f_x[t, \mathbf{x}(t), \mathbf{u}(t)] - \lambda(t) g_x[t, \mathbf{x}(t), \mathbf{u}(t)], \quad \lambda(t_1) = 0. \quad (2.7.21)$$

The third one is the state function:

$$\mathbf{x}'(t) = g(t, \mathbf{x}(t), \mathbf{u}(t)), \quad \mathbf{x}(t_0) = \mathbf{x}_0. \quad (2.7.22)$$

The Hamilton equation for optimal control, which is similar to the Lagrange equation for constrained optimizations, is defined as:

$$H(t, \mathbf{x}(t), \mathbf{u}(t)) \equiv f(t, \mathbf{x}(t), \mathbf{u}(t)) + \lambda(t)g(t, \mathbf{x}(t), \mathbf{u}(t)). \quad (2.7.23)$$

We then have: 1) the optimal control condition,

$$\partial H / \partial \mathbf{u} = 0 : \quad \partial H / \partial \mathbf{u} = f_{\mathbf{u}} + \lambda g_{\mathbf{u}} = 0, \quad (2.7.24)$$

i.e., the equation (2.7.20);

2) the costate (multiplier) equation,

$$-\partial H / \partial \mathbf{x} = \lambda' : \quad \lambda'(t) = -\partial H / \partial \mathbf{x} = -(f_{\mathbf{x}} + \lambda g_{\mathbf{x}}), \quad (2.7.25)$$

i.e., the equation (2.7.21);

3) the state equation,

$$\partial H / \partial \lambda = \mathbf{x}' : \quad \mathbf{x}'(t) = \partial H / \partial \lambda = g, \quad (2.7.26)$$

i.e., the equation (2.7.20).

Example 2.7.2 Consider Example 2.7.1 again. The problem is

$$\begin{aligned} & \min \int_0^T [c_1 u^2(t) + c_2 x(t)] dt \\ \text{s.t.} \quad & x'(t) = u(t), x(0) = 0, x(T) = B, x'(t) \geq 0. \end{aligned}$$

It follows from the above three conditions of optimization that:

$$2c_1 u(t) = -\lambda(t); \lambda'(t) = -c_2; x'(t) = u(t), x(0) = 0, x(T) = B.$$

and thus we have:

$$x^{*''}(t) = \frac{c_2}{2c_1}, \quad t \in [0, T],$$

$$x^*(t) = \frac{c_2}{4c_1}t(t-T) + Bt/T, \quad t \in [0, T],$$

$$u^*(t) = \frac{c_2}{2c_1}t + k, t \in [0, T]; \quad k = \frac{-c_2}{4c_1}T + B/T.$$

The second-order conditions of optimal control can be similarly derived. If the objective function and the state function f and g are concave with respect to x and u , then the first-order necessary conditions are also the sufficient conditions, and one can refer to Kamien & Schwartz (1991) for the proof.

2.7.3 Dynamic Programming

The third method of dealing with dynamic optimization is dynamic programming proposed by Richard Bellman, and its basic logic can be summarized as Bellman's principle of optimality. An optimal path satisfies the property that whatever the states and the control variables are prior to a certain time, the selection of decision function must constitute an optimal policy from now to the end with regard to the current state.

The general form of dynamic programming problems is:

$$\max \int_0^T f(t, \mathbf{x}(t), \mathbf{u}(t))dt + \phi(\mathbf{x}(T), T) \quad (2.7.27)$$

$$\text{s.t.} \quad \mathbf{x}'(t) = g(t, \mathbf{x}(t), \mathbf{u}(t)), \mathbf{x}(0) = a, \quad t \in [0, T]. \quad (2.7.28)$$

Define the value function $J(t_0, \mathbf{x}_0)$ as the maximal value starting at time t_0 in state \mathbf{x}_0 :

$$J(t_0, \mathbf{x}_0) = \max_{\mathbf{u}} \int_{t_0}^T f(t, \mathbf{x}(t), \mathbf{u}(t))dt + \phi(\mathbf{x}(T), T) \quad (2.7.29)$$

$$\text{s.t.} \quad \mathbf{x}'(t) = g(t, \mathbf{x}(t), \mathbf{u}(t)), \mathbf{x}(t_0) = \mathbf{x}_0, t \in [t_0, T]. \quad \forall t_0 \in [0, T].$$

When $t_0 = T$, the value function is $J(T, \mathbf{x}(T)) = \phi(\mathbf{x}(T), T)$.

We can break up the equation (2.7.29) and obtain:

$$J(t_0, \mathbf{x}_0) = \max_{\mathbf{u}} \left\{ \int_{t_0}^{t_0+\Delta t} f dt + \int_{t_0+\Delta t}^T f dt + \phi(\mathbf{x}(T), T) \right\}. \quad (2.7.30)$$

At time $t_0 + \Delta t$, the state changes to $\mathbf{x}_0 + \Delta \mathbf{x}$, and it follows from Bellman's

principle of optimality that the equation (2.7.30) is equivalent to:

$$\begin{aligned} J(t_0, \mathbf{x}_0) &= \max_{\mathbf{u}} \int_{t_0}^{t_0+\Delta t} f dt + \max_{\mathbf{u}} \left(\int_{t_0+\Delta t}^T f dt + \phi(\mathbf{x}(T), T) \right) \\ &= \max_{\mathbf{u}} \int_{t_0}^{t_0+\Delta t} f dt + J(t_0 + \Delta t, \mathbf{x}_0 + \Delta \mathbf{x}), \\ \mathbf{x}' &= g, \mathbf{x}(t_0 + \Delta t) = \mathbf{x}_0 + \Delta \mathbf{x}. \end{aligned} \quad (2.7.31)$$

The equation (2.7.31) depicts Bellman's principle of optimality. Expanding the right side of (2.7.31) by Taylor's theorem yields:

$$\begin{aligned} J(t_0, \mathbf{x}_0) &= \max_{\mathbf{u}} [f(t_0, \mathbf{x}_0, \mathbf{u})\Delta t + J(t_0, \mathbf{x}_0) + J_t(t_0, \mathbf{x}_0)\Delta t \\ &\quad + J_x(t_0, \mathbf{x}_0)\Delta \mathbf{x} + h.o.t]. \end{aligned} \quad (2.7.32)$$

Let $\Delta t \rightarrow 0$, equation (2.7.32) becomes:

$$0 = \max_{\mathbf{u}} [f(t, \mathbf{x}, \mathbf{u}) + J_t(t, \mathbf{x}) + J_x(t, \mathbf{x})\mathbf{x}'],$$

and then we have

$$-J_t(t, \mathbf{x}) = \max_{\mathbf{u}} [f(t, \mathbf{x}, \mathbf{u}) + J_x(t, \mathbf{x})g(t, \mathbf{x}, \mathbf{u})]. \quad (2.7.33)$$

Compared to the method of optimal control, $J_x(t, \mathbf{x})$ on the right side of (2.7.33) plays the role of the costate variable λ . We just define $\lambda(t) = J_x(t, \mathbf{x})$, and thus the economic meaning behind the costate variables is the marginal contribution of states to the value function.

The derivative of (2.7.33) with respect to \mathbf{x} gives:

$$-J_{tx}(t, \mathbf{x}^*) = f_x(t, \mathbf{x}^*, \mathbf{u}^*) + J_x(t, \mathbf{x}^*)g_x. \quad (2.7.34)$$

Since

$$\lambda'(t) = \frac{dJ_x(t, \mathbf{x})}{dt} = J_{tx} + J_{xx}g,$$

together with (2.7.34), we obtain:

$$-\lambda'(t) = f_x + \lambda g_x. \quad (2.7.35)$$

The equation (2.7.35) is just the first-order condition for optimal control with respect to the state variable:

$$-\partial H / \partial \mathbf{x} = \lambda'.$$

The derivative of the right side of (2.7.33) with respect to \mathbf{u} gives:

$$f_{\mathbf{u}} + J_{\mathbf{x}} g_{\mathbf{u}} = 0,$$

and this is the first-order condition for optimal control with respect to control variables:

$$\frac{\partial H}{\partial \mathbf{u}} = f_{\mathbf{u}} + \lambda g_{\mathbf{u}} = 0.$$

Then, optimal control and dynamic programming are essentially consistent.

In the discrete case, the method of dynamic programming may be more convenient. The following results are given only for an infinite time horizon.

Suppose that the state set $S \subseteq \mathcal{R}^n$ is a nonempty and compact set, and $U : S \times S \rightarrow \mathcal{R}$ is a bounded continuous function, which generally represents the utility function in a period. Given the initial state $s_0 = \mathbf{z}$, the general dynamic optimization problem is:

$$\max_{\{s_t\}} \sum_{t=0}^{\infty} \delta^t U(s_t, s_{t+1}) \quad (2.7.36)$$

$$\begin{aligned} \text{s.t.} \quad & s_t \in S, \quad \forall t, \\ & s_0 = \mathbf{z}. \end{aligned} \quad (2.7.37)$$

It can be proven by using the contraction mapping theorem that there is a sequence of maximum points in the problem (2.7.36), and thus there exists a maximum value denoted by $V(\mathbf{z})$. Function $V : S \rightarrow \mathcal{R}$ is called the value function of problem (2.7.36). Like function $U(\cdot, \cdot)$, the value function is also continuous. In addition, if S is a convex set and $U(\cdot, \cdot)$ is concave, then $V(\cdot)$ is also concave, and it is equivalent to Bellman's principle of optimality, i.e.,

it is the solution to the following Bellman equation:

$$V(\mathbf{s}) = \max_{\hat{\mathbf{s}} \in S} U(\mathbf{s}, \hat{\mathbf{s}}) + \delta V(\hat{\mathbf{s}}).$$

The equivalence results provide the basis for solving the dynamic optimization problem by the Bellman method. The following theorem reveals that the value function is the only function satisfying the Bellman equation.

Theorem 2.7.1

$$f(\mathbf{s}) = \max_{\hat{\mathbf{s}} \in S} U(\mathbf{s}, \hat{\mathbf{s}}) + \delta V(\hat{\mathbf{s}}) \quad (2.7.38)$$

i.e., $f(\cdot) = V(\cdot)$.

PROOF. Using (2.7.38) repeatedly gives: for each T ,

$$\begin{aligned} f(\mathbf{z}) &= \max_{\{\mathbf{s}_t\}_{t=0}^T} \sum_{t=0}^{T-1} \delta^t U(\mathbf{s}_t, \mathbf{s}_{t+1}) + \delta^T f(\mathbf{x}_T) \\ \text{s.t.} \quad &\mathbf{s}_t \in S, \quad \forall t, \\ &\mathbf{s}_0 = \mathbf{z}. \end{aligned}$$

When $T \rightarrow \infty$, the contribution of $\delta^T f(\mathbf{x}_T)$ to the above summation is increasingly negligible, and thus $f(\cdot) = V(\cdot)$. \square

The above theorem provides a way to calculate the value function. Starting from any continuous function $f_0(\cdot) : S \rightarrow \mathcal{R}$, one can imagine $f_0(\hat{\mathbf{s}})$ as a trial “value” function which gives the estimated value from time 0. Then, let

$$f_1(\mathbf{s}) = \max_{\hat{\mathbf{s}} \in S} U(\mathbf{s}, \hat{\mathbf{s}}) + \delta f_0(\hat{\mathbf{s}})$$

holds for any $\mathbf{s} \in S$, and thus we obtain a new value function $f_1(\hat{\mathbf{s}})$.

Value function $V(\cdot)$ can be also found by the iterative method. If $f_1(t) = f_0(t)$, then $f_0(t)$ satisfies the Bellman equation. It follows from the above theorem that $f_0(t) = V(t)$. If $f_1(t) \neq f_0(t)$, we obtain a new value function from $f_1(t)$, and also obtain the whole sequence of functions $\{f_r(\cdot)\}_{r=0}^{\infty}$. Then, it can be shown that

$$\lim_{r \rightarrow \infty} f_r(\mathbf{s}) = V(\mathbf{s}),$$

i.e., it will converge to the value function as r increases to infinity.

If the function is differentiable, there is a similar first-order condition called the Euler equation of dynamic optimization:

$$0 = \frac{\partial U(s_t^*, s_{t+1}^*)}{\partial s_{t+1}} + \delta \frac{\partial U(s_{t+1}^*, s_{t+2}^*)}{\partial s_{t+1}}, \quad t = 0, 1, 2, \dots \quad (2.7.39)$$

The first-order condition of optimal decision gives:

$$0 = \frac{\partial U[\mathbf{x}, g(\mathbf{x})]}{\partial g} + \delta V'[g(\mathbf{x})], \quad (2.7.40)$$

where $g(\mathbf{x})$ is the state of the next periods determined by \mathbf{x} following Bellman's principle of optimality. It follows from the envelope theorem that

$$V'(\mathbf{x}) = U_{\mathbf{x}}[\mathbf{x}, g(\mathbf{x})]. \quad (2.7.41)$$

The Euler equation is derived from these two equations above.

2.8 Differential Equations

We first provide the general concept of ordinary differential equations defined on Euclidean spaces.

Definition 2.8.1 An equation,

$$F(x, y, y', \dots, y^{(n)}) = 0, \quad (2.8.42)$$

which constitutes independent variable x , unknown function $y = y(x)$ of the independent variable, and its first derivative $y' = y'(x)$ to the n th order derivative $y^{(n)} = y^{(n)}(x)$, is called an *ordinary differential equation*.

If the highest order derivative in the equation is n , the equation is also called n th-order ordinary differential equation.

If for all $x \in I$, the function $y = \psi(x)$ satisfies

$$F(x, \psi(x), \psi'(x), \dots, \psi^{(n)}(x)) = 0,$$

then $y = \psi(x)$ is called a *solution to the ordinary differential equation* (2.8.42).

Sometimes, the solutions of ordinary differential equations are not unique, and even infinite solutions may exist. For example, $y = \frac{C}{x} + \frac{1}{5}x^4$ is the solution of the ordinary differential equation $\frac{dy}{dx} + \frac{y}{x} = x^3$, where C is an arbitrary constant. Next, we introduce the concept of general solutions and particular solutions of ordinary differential equations.

Definition 2.8.2 The solution of the n th-order ordinary differential equation (2.8.42)

$$y = \psi(x, C_1, \dots, C_n), \quad (2.8.43)$$

which contains n independent arbitrary constants, C_1, \dots, C_n , is called the *general solution* to ordinary differential equation (2.8.42). Here, independence means that the Jacobi determinant

$$\frac{D[\psi, \psi^{(1)}, \dots, \psi^{(n-1)}]}{D[C_1, \dots, C_n]} \stackrel{\text{def}}{=} \begin{vmatrix} \frac{\partial \psi}{\partial C_1} & \frac{\partial \psi}{\partial C_2} & \cdots & \frac{\partial \psi}{\partial C_n} \\ \frac{\partial \psi^{(1)}}{\partial C_1} & \frac{\partial \psi^{(1)}}{\partial C_2} & \cdots & \frac{\partial \psi^{(1)}}{\partial C_n} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial \psi^{(n-1)}}{\partial C_1} & \frac{\partial \psi^{(n-1)}}{\partial C_2} & \cdots & \frac{\partial \psi^{(n-1)}}{\partial C_n} \end{vmatrix}$$

is not identically equal to 0.

If a solution of an ordinary differential equation, denoted $y = \psi(x)$, does not contain any constant, it is called a particular solution. Obviously, a general solution becomes a particular solution when the arbitrary constants are determined. In general, the restrictions of some initial conditions determine the value of any constants. For example, for ordinary differential equation (2.8.42), if there are some given initial conditions:

$$y(x_0) = y_0, y^{(1)}(x_0) = y_0^{(1)}, \dots, y^{(n-1)}(x_0) = y_0^{(n-1)}, \quad (2.8.44)$$

then the ordinary differential equation (2.8.42) and the initial value conditions (2.8.44) are said to be the Cauchy problem or initial value problem for n th-order ordinary differential equations. Then, the pertinent question is

what conditions the function F should satisfy so that the above ordinary differential equations are uniquely solvable. This problem is the existence and uniqueness of solutions for ordinary differential equations.

2.8.1 Existence and Uniqueness Theorem of Solutions for Ordinary Differential Equations

We first consider an ordinary differential equation of first-order $y' = f(x, y)$ that satisfies initial condition (x_0, y_0) , i.e., $y(x_0) = y_0$. Let $y(x)$ be a solution to the differential equation.

Definition 2.8.3 Let a function $f(x, y)$ be defined on $D \subseteq \mathcal{R}^2$. We say that f satisfies the *local Lipschitz condition* with respect to y at the point $(x_0, y_0) \in D$, if there exists a neighborhood $U \subseteq D$ of (x_0, y_0) , and a positive number L , such that

$$|f(x, y) - f(x, z)| \leq L|y - z|, \forall (x, y), (x, z) \in U.$$

If there is a positive number L , such that

$$|f(x, y) - f(x, z)| \leq L|y - z|, \forall (x, y), (x, z) \in D,$$

we say that $f(x, y)$ satisfies *global Lipschitz condition* with respect to y in $D \subseteq \mathcal{R}^2$.

The following lemma characterizes the properties of the function satisfying the Lipschitz condition.

Lemma 2.8.1 Suppose that $f(x, y)$ defined on $D \subseteq \mathcal{R}^2$ is continuously differentiable. If there is an $\epsilon > 0$, such that $f_y(x, y)$ is bounded on $U = \{(x, y) : |x - x_0| < \epsilon, |y - y_0| < \epsilon\}$, then $f(x, y)$ satisfies the local Lipschitz condition with respect to y . If $f_y(x, y)$ is bounded on D , then $f(x, y)$ satisfies the global Lipschitz condition with respect to y .

Theorem 2.8.1 If f is continuous on an open set D , then for any $(x_0, y_0) \in D$, there always exists a solution $y(x)$ of the differential equation, and it satisfies $y' = f(x, y)$ and $y(x_0) = y_0$.

The following is the theorem on the uniqueness of the solution for differential equations.

Theorem 2.8.2 *Suppose that f is continuous on an open set D , and satisfies the global Lipschitz condition with respect to y . Then, for any $(x_0, y_0) \in D$, there always exists a unique solution $y(x)$ satisfying $y' = f(x, y)$ and $y(x_0) = y_0$.*

For n th order ordinary differential equations, $y^{(n)} = f(x, y, y', \dots, y^{(n-1)})$, if the Lipschitz condition is changed to for $y, y', \dots, y^{(n-1)}$ instead of for y , we have similar conclusions about the existence and uniqueness of solution. See Ahmad and Ambrosetti (2014) for the specific proof of existence and uniqueness.

2.8.2 Some Common Ordinary Differential Equations with Explicit Solutions

Generally, we aim to obtain the concrete form of solutions, i.e., explicit solutions, for differential equations. However, in many cases, there is no explicit solution. Here, we present some common cases in which differential equations can be solved explicitly.

Case of Separable Equations

Consider a separable differential equation $y' = f(x)g(y)$, and $y(x_0) = y_0$. It can be rewritten as:

$$\frac{dy}{g(y)} = f(x)dx.$$

Integrating both sides, we then obtain the solution to the differential equation.

For example, for $(x^2 + 1)y' + 2xy^2 = 0$, $y(0) = 1$, using the above solving procedure, we obtain the solution as

$$y(x) = \frac{1}{\ln(x^2 + 1) + 1}.$$

In addition, the differential equation with the form $y' = f(y)$ is called an autonomous system, since y' is only determined by y .

Homogeneous Type of Differential Equation

Some differential equations with constant coefficients have explicit solutions.

Definition 2.8.4 We call the function $f(x, y)$ a *homogeneous function of degree n* if for any λ , $f(\lambda x, \lambda y) = \lambda^n f(x, y)$.

Differential equations have the form of homogeneous functions if $M(x, y)dx + N(x, y)dy = 0$, where $M(x, y)$ and $N(x, y)$ are homogeneous functions with the same order.

By variable transformation $z = \frac{y}{x}$, the above differential equations can be transformed into separable form. Suppose that $M(x, y)$ and $N(x, y)$ are homogeneous functions of degree n , and $M(x, y)dx + N(x, y)dy = 0$ is transformed to $z + x \frac{dz}{dx} = -\frac{M(1, z)}{N(1, z)}$, then the final form is $\frac{dz}{dx} = -\frac{z + \frac{M(1, z)}{N(1, z)}}{x}$, where $z + \frac{M(1, z)}{N(1, z)}$ is a function of z .

Exact Differential Equation

Given a simply connected and open subset $D \subseteq \mathcal{R}^2$ and two functions M and N , which are continuous and satisfy $\frac{\partial M(x, y)}{\partial y} \equiv \frac{\partial N(x, y)}{\partial x}$ on D , then the implicit first-order ordinary differential equation of the form

$$M(x, y)dx + N(x, y)dy = 0$$

is called the **exact differential equation** or the **total differential equation**.

The nomenclature of “exact differential equation” refers to the exact derivative of a function. Indeed, when $\frac{\partial M(x, y)}{\partial y} \equiv \frac{\partial N(x, y)}{\partial x}$, the solution is $F(x, y) = C$, where the constant C is determined by the initial value, and $F(x, y)$ satisfies $\frac{\partial F}{\partial x} = M(x, y)$ or $\frac{\partial F}{\partial y} = N(x, y)$.

It is clear that a separable differential equation is a special case of an exact differential equation $y' = f(x)g(y)$ or $\frac{1}{g(y)}dy - f(x)dx = 0$, and then we have $M(x, y) = -f(x)$, $N(x, y) = \frac{1}{g(y)}$, and $\frac{\partial M(x, y)}{\partial y} = \frac{\partial N(x, y)}{\partial x} = 0$.

For example, $2xy^3dx + 3x^2y^2dy = 0$ is an exact differential equation, of which the general solution is $x^2y^3 = C$, and C is a constant.

When solving differential equations with explicit solutions, we usually convert differential equations into the form of exact differential equations.

First-Order Differential Linear Equation

Consider the first-order linear differential equation of the following form:

$$\frac{dy}{dx} + p(x)y = q(x). \quad (2.8.45)$$

When $q(x) = 0$, the above differential equation (2.8.45) is a separable differential equation, and its solution is assumed to be $y = \psi(x)$.

Suppose that $\psi_1(x)$ is a particular solution of the differential equation (2.8.45). Then, $y = \psi(x) + \psi_1(x)$ is clearly also the solution of the equations (2.8.45).

It is easy to show that the solution to $\frac{dy}{dx} + p(x)y = 0$ is $y = Ce^{-\int p(x)dx}$. Next, we find the general solution to the differential equation (2.8.45).

Suppose that

$$y = c(x)e^{-\int p(x)dx},$$

and differentiating this gives

$$y' = c'(x)e^{-\int p(x)dx} + c(x)p(x)e^{-\int p(x)dx},$$

then substituting this back into the original differential equation, we have

$$c'(x)e^{-\int p(x)dx} + c(x)p(x)e^{-\int p(x)dx} = p(x)c(x)e^{-\int p(x)dx} + q(x),$$

and thus

$$c'(x) = q(x)e^{\int p(x)dx}.$$

We have

$$c(x) = \int q(x)e^{\int p(x)dx}dx + C.$$

Therefore, the general solution is

$$y(x) = e^{-\int p(x)dx} \left(\int q(x)e^{\int p(x)dx} dx + C \right).$$

Bernoulli Equation

The following differential equation is called the **Bernoulli equation**:

$$\frac{dy}{dx} + p(x)y = q(x)y^n, \quad (2.8.46)$$

where n (with $n \neq 0, 1$) is a natural number.

Multiplying both sides by $(1 - n)y^{(-n)}$ gives:

$$(1 - n)y^{(-n)} \frac{dy}{dx} + (1 - n)y^{(1-n)} p(x) = (1 - n)q(x).$$

Let $z = y^{(1-n)}$, and get:

$$\frac{dz}{dx} + (1 - n)zp(x) = (1 - n)q(x),$$

which becomes a first-order linear differential equation whose explicit solution can be obtained.

Differential equations with explicit solutions have other forms, such as some special forms of Riccati equations, and equations similar to $M(x, y)dx + N(x, y)dy = 0$, but not satisfying

$$\frac{\partial M(x, y)}{\partial y} \equiv \frac{\partial N(x, y)}{\partial x}.$$

2.8.3 Higher Order Linear Equations with Constant Coefficients

Consider a differential equation of degree n with constant coefficients

$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = f(x). \quad (2.8.47)$$

If $f(x) \equiv 0$, then the differential equation (2.8.47) is called **constant coefficient homogeneous differential equation of degree n** ; otherwise, it is called **constant coefficients nonhomogeneous differential equation**.

There is a method for finding the general solution $y_g(x)$ of a constant coefficient homogeneous differential equation of degree n . The general solution is the sum of n bases of solutions y_1, \dots, y_n , i.e., $y_g(x) = C_1 y_1(x) + \dots + C_n y_n(x)$, where C_1, \dots, C_n are arbitrary constants. These arbitrary constants are uniquely determined by initial-value conditions. Find a function $y(x)$ satisfying

$$y(x) = y_{0_0}, y'(x) = y_{0_1}, \dots, y^{(n-1)}(x) = y_{0_{n-1}}, \text{ when } x = x_0,$$

where $x_0, y_{0_0}, y_{0_1}, \dots, y_{0_{n-1}}$ are given initial values.

The procedures for solving the fundamental solution of homogeneous differential equations are given below:

- (1) Solve the characteristic equation with respect to λ :

$$\lambda^n + a_1 \lambda^{n-1} + \dots + a_{n-1} \lambda + a_n = 0.$$

Suppose that the roots of the characteristic equation are $\lambda_1, \dots, \lambda_n$. Some roots may be complex, and some are multiple.

- (2) If λ_i is the non-multiple real characteristic root, then the fundamental solution corresponding to this root is $y_i(x) = e^{\lambda_i x}$.

- (3) If λ_i is the real characteristic root of multiplicity k , then there are k fundamental solutions:

$$y_{i_1}(x) = e^{\lambda_i x}, y_{i_2}(x) = x e^{\lambda_i x}, \dots, y_{i_k}(x) = x^{k-1} e^{\lambda_i x}.$$

- (4) If λ_j is the non-multiple complex characteristic root, $\lambda_j = \alpha_j + i\beta_j, i = \sqrt{-1}$, its complex conjugate denoted by $\lambda_{j+1} = \alpha_j - i\beta_j$ is also the characteristic root, and thus there are two fundamental solutions generated by these complex conjugate roots λ_j, λ_{j+1} :

$$y_{j_1} = e^{\alpha_j x} \cos \beta_j x, \quad y_{j_2} = e^{\alpha_j x} \sin \beta_j x.$$

- (5) If λ_j is the complex characteristic root of multiplicity l , $\lambda_j = \alpha_j + i\beta_j$, its complex conjugate is also the complex characteristic root of multiplicity

l , and thus these $2l$ complex roots generate $2l$ fundamental solutions:

$$y_{j_1} = e^{\alpha_j x} \cos \beta_j x, y_{j_2} = x e^{\alpha_j x} \cos \beta_j x, \dots, y_{j_l} = x^{l-1} e^{\alpha_j x} \cos \beta_j x;$$

$$y_{j_{l+1}} = e^{\alpha_j x} \sin \beta_j x, y_{j_{l+2}} = x e^{\alpha_j x} \sin \beta_j x, \dots, y_{j_{2l}} = x^{l-1} e^{\alpha_j x} \sin \beta_j x.$$

The following is a general method for solving nonhomogeneous differential equations.

The general form of solution to nonhomogeneous differential equations is $y_{nh}(x) = y_g(x) + y_p(x)$, where $y_g(x)$ is the corresponding general solution of the homogeneous equation, and $y_p(x)$ is the particular solution of the nonhomogeneous equation.

Next, we will provide some procedures for solving for particular solutions of nonhomogeneous equations.

(1) If $f(x) = P_k(x)e^{bx}$, and $P_k(x)$ is the polynomial of degree k , then the form of particular solutions is:

$$y_p(x) = x^s Q_k(x) e^{bx},$$

where $Q_k(x)$ is also a polynomial of degree k . If b is not a characteristic root corresponding to the characteristic equation, then $s = 0$; if b is a characteristic root of multiplicity m , then $s = m$.

(2) If $f(x) = P_k(x)e^{px} \cos qx + Q_k(x)e^{px} \sin qx$, and $P_k(x)$ and $Q_k(x)$ are all polynomials of degree k , then the form of particular solutions is:

$$y_p(x) = x^s R_k(x) e^{px} \cos qx + x^s T_k(x) e^{px} \sin qx,$$

where $R_k(x)$ and $T_k(x)$ are also polynomials of degree k . If $p + iq$ is not a root of the characteristic equation, then $s = 0$; if $p + iq$ is a characteristic root of multiplicity m , then $s = m$.

(3) A general method for solving nonhomogeneous differential equations is called **the variation of parameters** or **the method of undetermined-coefficients**.

Suppose that the general solution of a homogeneous equation is given

as follows:

$$y_g = C_1 y_1(x) + \cdots + C_n y_n(x),$$

where $y_i(x)$ is the fundamental solution. Regard constants C_1, \dots, C_n as the functions with respect to x , such as $u_1(x), \dots, u_n(x)$. Therefore, the form of particular solutions to the nonhomogeneous equation can be expressed as

$$y_p(x) = u_1(x)y_1(x) + \cdots + u_n(x)y_n(x),$$

where $u_1(x), \dots, u_n(x)$ are the solutions of the following equations

$$u_1'(x)y_1(x) + \cdots + u_n'(x)y_n(x) = 0,$$

$$u_1'(x)y_1'(x) + \cdots + u_n'(x)y_n'(x) = 0,$$

$$\vdots$$

$$u_1'(x)y_1^{(n-2)}(x) + \cdots + u_n'(x)y_n^{(n-2)}(x) = 0,$$

$$u_1'(x)y_1^{(n-1)}(x) + \cdots + u_n'(x)y_n^{(n-1)}(x) = f(x).$$

(4) If $f(x) = f_1(x) + f_2(x) + \cdots + f_r(x)$, and $y_{p1}(x), \dots, y_{pr}(x)$ are the particular solutions corresponding to $f_1(x), \dots, f_r(x)$, then

$$y_p(x) = y_{p1}(x) + \cdots + y_{pr}(x).$$

Here, we provide an example to familiarize the application of this method.

Example 2.8.1 Solve $y'' - 5y' + 6y = t^2 + e^t - 5$.

The characteristic roots are $\lambda_1 = 2$ and $\lambda_2 = 3$. The general solution of the homogeneous equation is thus:

$$y(t) = C_1 e^{2t} + C_2 e^{3t}.$$

Next, to find a particular solution of the nonhomogeneous equation, its form is written as:

$$y_p(t) = at^2 + bt + c + de^t.$$

We first substitute this particular solution in the initial equation to determine the coefficients a, b, c, d :

$$2a + de^t - 5(2at + b + de^t) + 6(at^2 + bt + c + de^t) = t^2 - 5 + e^t.$$

The coefficients of both sides should be consistent, and thus we obtain:

$$6a = 1, \quad -5 \times 2a + 6b = 0, \quad 2a - 5b + 6c = -5, \quad d - 5d + 6d = 1,$$

Therefore, $d = 1/2$, $a = 1/6$, $b = 5/18$, and $c = -71/108$.

Finally, the general solution of the nonhomogeneous differential equation is:

$$y(t) = C_1 e^{2t} + C_2 e^{3t} + \frac{t^2}{6} + \frac{5t}{18} - \frac{71}{108} + \frac{e^t}{2}.$$

2.8.4 System of Ordinary Differential Equations

The general form is:

$$\dot{\mathbf{x}}(t) = A(t)\mathbf{x}(t) + \mathbf{b}(t), \quad \mathbf{x}(0) = \mathbf{x}_0,$$

where t (time) is an independent variable, $\mathbf{x}(t) = (x_1(t), \dots, x_n(t))'$ is a vector of dependent variables, $A(t) = (a_{ij}(t))_{[n \times n]}$ is an $n \times n$ matrix of real varying coefficients, and $\mathbf{b}(t) = (b_1(t), \dots, b_n(t))'$ is an n -dimensional varying vector.

Consider the case that A is a constant coefficient matrix and \mathbf{b} is a constant vector, also called the **system of differential equations with constant coefficients**:

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + \mathbf{b}, \quad \mathbf{x}(0) = \mathbf{x}_0, \tag{2.8.48}$$

where A is assumed to be nonsingular.

The system of differential equations (2.8.48) can be solved by the following two steps.

Step 1: consider the system of homogeneous equations (i.e., $\mathbf{b} = 0$):

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t), \quad \mathbf{x}(0) = \mathbf{x}_0. \tag{2.8.49}$$

Its solution is denoted by $\mathbf{x}_c(t)$.

Step 2: find a particular solution \mathbf{x}_p to the nonhomogeneous equation (2.8.48). The constant vector \mathbf{x}_p is a particular solution so that $A\mathbf{x}_p = -\mathbf{b}$, i.e., $\mathbf{x}_p = -A^{-1}\mathbf{b}$.

Given the general solution of the homogeneous equation and the particular solution to the nonhomogeneous equation, the general solution of the system of differential equations (2.8.49) is:

$$\mathbf{x}(t) = \mathbf{x}_c(t) + \mathbf{x}_p.$$

There are two methods for solving the system of homogeneous differential equations (2.8.49).

The first one is that we can eliminate $n - 1$ dependent variables, and thus the system of differential equations becomes the differential equation of order n , such as the following example.

Example 2.8.2 The system of differential equation is:

$$\begin{cases} \dot{x} = 2x + y, \\ \dot{y} = 3x + 4y. \end{cases}$$

We differentiate the first equation to eliminate y and \dot{y} . Since $\dot{y} = 3x + 4y = 3x + 4\dot{x} - 4 \cdot 2x$, we obtain the corresponding quadratic homogeneous differential equation:

$$\ddot{x} - 6\dot{x} + 5x = 0,$$

thus the general solution is $x(t) = C_1e^t + C_2e^{5t}$. Since $y(t) = \dot{x} - 2x$, $y(t) = -C_1e^t + 3C_2e^{5t}$.

The second method is to rewrite the homogeneous differential equation (2.8.49) as:

$$\mathbf{x}(t) = e^{At}\mathbf{x}_0,$$

where

$$e^{At} = I + At + \frac{A^2t^2}{2!} + \cdots.$$

Now, we solve e^{At} in three different cases.

Case 1: A has different real eigenvalues

Matrix A has different real eigenvalues, which means that its eigenvectors are linearly independent. Therefore, A can be diagonalized, namely,

$$A = P\Lambda P^{-1},$$

where $P = [v_1, v_2, \dots, v_n]$ consists of the eigenvectors of A , and Λ is a diagonal matrix whose diagonal elements are the eigenvalues of A , and thus we have

$$e^A = Pe^{\Lambda}P^{-1}.$$

Therefore, the solution to the system of differential equation (2.8.49) is:

$$\begin{aligned} \mathbf{x}(t) &= Pe^{\Lambda t}P^{-1}\mathbf{x}_0 \\ &= Pe^{\Lambda t}\mathbf{c} \\ &= c_1v_1e^{\lambda_1 t} + \dots + c_nv_ne^{\lambda_n t}, \end{aligned}$$

where $\mathbf{c} = (c_1, c_2, \dots, c_n)$ is a vector of arbitrary constants, and it is determined by the initial value, i.e., $\mathbf{c} = P^{-1}\mathbf{x}_0$.

Case 2: A has multiple real eigenvalues, but no complex eigenvalues

First, consider a simple case in which A has only one eigenvalue of multiplicity m . In this case, there are at most m linearly independent eigenvectors, which means that the matrix P cannot be constructed as a matrix consisting of linearly independent eigenvectors, and thus A can not be diagonalized.

Therefore, the solution has the following form:

$$\mathbf{x}(t) = \sum_{i=1}^m c_i h_i(t),$$

where $h_i(t)$, $\forall i$, are quasi-polynomials, and c_i , $\forall i$, are determined by initial

conditions. For example, when $m = 3$, we have:

$$\begin{aligned}h_1(t) &= e^{\lambda t} v_1, \\h_2(t) &= e^{\lambda t} (t v_1 + v_2), \\h_3(t) &= e^{\lambda t} (t^2 v_1 + 2t v_2 + 3v_3),\end{aligned}$$

where v_1, v_2, v_3 are determined by the following conditions:

$$(A - \lambda I)v_i = v_{i-1}, v_0 = 0.$$

If A has more than one multiple real eigenvalues, then the solution of the differential equation (2.8.49) can be obtained by summing up the solutions corresponding to each eigenvalue.

Case 3: A has complex eigenvalues

Since A is a real matrix, complex eigenvalues will be generated in the form of conjugate pairs.

If an eigenvalue of A is $\alpha + \beta i$, then its conjugate complex $\alpha - \beta i$ is also an eigenvalue.

Now, consider a simple case: A has only one pair of complex eigenvalues, $\lambda_1 = \alpha + \beta i$ and $\lambda_2 = \alpha - \beta i$.

Let \mathbf{v}_1 and \mathbf{v}_2 be the eigenvectors corresponding to λ_1 and λ_2 . Then, we have $\mathbf{v}_2 = \bar{\mathbf{v}}_1$, where $\bar{\mathbf{v}}_1$ refers to the conjugation of \mathbf{v}_1 . The solution of the differential equation (2.8.49) can be expressed as:

$$\begin{aligned}\mathbf{x}(t) &= e^{At} \mathbf{x}_0 \\&= P e^{\Lambda t} P^{-1} \mathbf{x}_0 \\&= P e^{\Lambda t} \mathbf{c} \\&= c_1 v_1 e^{(\alpha + \beta i)t} + c_2 v_2 e^{(\alpha - \beta i)t} \\&= c_1 v_1 e^{\alpha t} (\cos \beta t + i \sin \beta t) + c_2 v_2 e^{\alpha t} (\cos \beta t - i \sin \beta t) \\&= (c_1 v_1 + c_2 v_2) e^{\alpha t} \cos \beta t + i(c_1 v_1 - c_2 v_2) e^{\alpha t} \sin \beta t \\&= h_1 e^{\alpha t} \cos \beta t + h_2 e^{\alpha t} \sin \beta t,\end{aligned}$$

where $h_1 = c_1v_1 + c_2v_2$ and $h_2 = i(c_1v_1 - c_2v_2)$.

If A has many pairs of conjugate complex eigenvalues, then the solution of the differential equation (2.8.49) is obtained by summing up the solutions corresponding to all eigenvalues.

2.8.5 Stability of Simultaneous Differential Equations

Consider the following simultaneous differential equations system:

$$\dot{\mathbf{x}} = f(t, \mathbf{x}), \quad (2.8.50)$$

where t (time) is an independent variable, $\mathbf{x} = (x_1, \dots, x_n)$ are dependent variables, and $f(t, \mathbf{x})$ is continuously differentiable with respect to $\mathbf{x} \in \mathcal{R}^n$ and satisfies the initial condition $\mathbf{x}(0) = \mathbf{x}_0$. Such simultaneous differential equations are called **planar dynamic systems**. If $f(t, \mathbf{x}^*) = 0$, the point \mathbf{x}^* is called the **stationary point** of the above dynamical system.

Definition 2.8.5 A simultaneous differential equation system \mathbf{x}^* is *locally stable* if there is $\delta > 0$ and a unique path of $\mathbf{x} = \phi(t, \mathbf{x}_0)$, such that $\lim_{t \rightarrow \infty} \phi(t, \mathbf{x}_0) = \mathbf{x}^*$ whenever $|\mathbf{x}^* - \mathbf{x}_0| < \delta$.

Consider the case of a simultaneous differential equations system with two variables $x = x(t)$ and $y = y(t)$:

$$\begin{cases} \frac{dx}{dt} = f(x, y), \\ \frac{dy}{dt} = g(x, y). \end{cases}$$

Let \mathcal{J} be the Jacobian

$$\mathcal{J} = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}$$

evaluated at (x^*, y^*) , and λ_1 and λ_2 be the eigenvalues of this Jacobian.

Then, the stability of the stationary point is characterized as follows:

(1) It is a (locally) **stable (or unstable) node** if λ_1 and λ_2 are different real numbers and are negative (or positive);

(2) It is a (locally) **saddle point** if eigenvalues are real numbers but with opposite signs, i.e., $\lambda_1 \lambda_2 < 0$;

(3) It is a (locally) **stable (or unstable) focus** if λ_1 and λ_2 are complex numbers, and $\text{Re}(\lambda_1) < 0$ (or $\text{Re}(\lambda_1) > 0$);

(4) It is a **center** if λ_1 and λ_2 are complex, and $\text{Re}(\lambda_1) = 0$;

(5) It is a (locally) **stable (or unstable) improper node** if λ_1 and λ_2 are real, $\lambda_1 = \lambda_2 < 0$ (or $\lambda_1 = \lambda_2 > 0$), and the Jacobian is not a diagonal matrix;

(6) It is a (locally) **stable (or unstable) star node** if λ_1 and λ_2 are real, $\lambda_1 = \lambda_2 < 0$ (or $\lambda_1 = \lambda_2 > 0$), and the Jacobian is a diagonal matrix.

Figure 2.5 below depicts six types of stationary points.

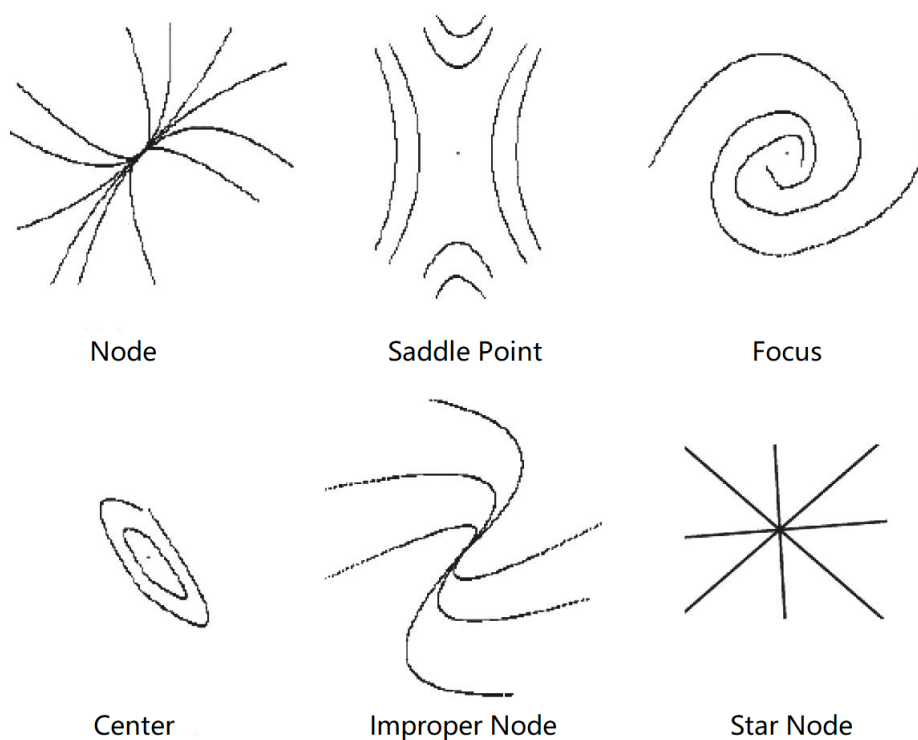


Figure 2.5: Types of stationary points

2.8.6 The Global Stability of Dynamical System

In a dynamic system, the Lyapunov method studies the global stability of stationary points.

Let $\bar{x}(t, x_0)$ be the unique solution of the dynamic system (2.8.50), and $B_r(x) = \{x' \in D : |x' - x| < r\}$ be an open ball of radius r centered at x .

The following is the definition of stability of stationary points.

Definition 2.8.6 *The stationary point x^* of the dynamic system (2.8.50)*

- (1) *is globally stable if for any $r > 0$, there is a neighbourhood U of x^* , such that*

$$\bar{x}(t, x_0) \in B_r(x^*), \forall x_0 \in U.$$

- (2) *is globally asymptotically stable if for any $r > 0$, there is a neighbourhood U' of x^* , such that*

$$\lim_{t \rightarrow \infty} \bar{x}(t, x_0) = x^*, \forall x_0 \in U'.$$

- (3) *is globally unstable if it is neither globally stable nor asymptotically globally stable.*

Definition 2.8.7 *Let x^* be the stationary point of the dynamic system (2.8.50), $Q \subseteq \mathbf{R}^n$ be an open set containing x^* , and $V(x) : Q \rightarrow \mathcal{R}$ be a continuously differentiable function. If it satisfies:*

- (1) $V(x) > V(x^*), \forall x \in Q, x \neq x^*$;
 (2) $\dot{V}(x)$ is defined as:

$$\dot{V}(x) \stackrel{\text{def}}{=} \nabla V(x) f(t, x) \leq 0, \forall x \in Q, \quad (2.8.51)$$

where $\nabla V(x)$ is the gradient of V with respect to x ,

thus it is called a Lyapunov function.

The following is the Lyapunov theorem about the stationary point of dynamic systems.

Theorem 2.8.3 *If there exists a Lyapunov function V for the dynamic system (2.8.50), then the stationary point \mathbf{x}^* is globally stable.*

If the Lyapunov function (2.8.51) of the dynamic system satisfies $\dot{V}(\mathbf{x}) < 0, \forall \mathbf{x} \in Q, \mathbf{x} \neq \mathbf{x}^$, then the stationary point \mathbf{x}^* is asymptotically globally stable.*

2.9 Difference Equations

Difference equations can be regarded as discretized differential equations, and many of their properties are similar to those of differential equations.

Let y be a real-valued function defined on natural numbers. y_t means the value of y at t , where $t = 0, 1, 2, \dots$, which can be regarded as time points.

Definition 2.9.1 The first-order difference of y at t is:

$$\Delta y(t) = y(t+1) - y(t).$$

The second-order difference of y at t is:

$$\Delta^2 y(t) = \Delta(\Delta y(t)) = y(t+2) - 2y(t+1) + y(t).$$

Generally, the n th-order difference of y at t is:

$$\Delta^n y(t) = \Delta(\Delta^{n-1} y(t)), \quad n > 1.$$

Definition 2.9.2 The difference equation is a function of y and its differences $\Delta y, \Delta^2 y, \dots, \Delta^{n-1} y$,

$$F(y, \Delta y, \Delta^2 y, \dots, \Delta^n y, t) = 0, \quad t = 0, 1, 2, \dots \quad (2.9.52)$$

If n is the highest order of nonzero coefficient in the formula (2.9.52), the above equation is called an n th-order difference equation.

If $F(\psi(t), \Delta\psi(t), \Delta^2\psi(t), \dots, \Delta^n\psi(t), t) = 0$ holds for $\forall t$, then we call $y = \psi(k)$ a solution of the difference equation. Similar to differential equations, the solutions of difference equations also have general solutions and

particular solutions. The general solutions usually contain some arbitrary constants that can be determined by initial conditions.

The difference equations can also be expressed in the following form by variable conversion:

$$F(y(t), y(t+1), \dots, y(t+n), t) = 0, t = 0, 1, 2, \dots \quad (2.9.53)$$

The following is mainly about difference equations with constant coefficients. A common expression is written as:

$$f_0 y(t+n) + f_1 y(t+n-1) + \dots + f_{n-1} y(t+1) + f_n y(t) = g(t), t = 0, 1, 2, \dots, \quad (2.9.54)$$

where f_0, f_1, \dots, f_n are real numbers, and $f_0 \neq 0, f_n \neq 0$.

Dividing both sides of the equation by f_0 , and making $a_i = \frac{f_i}{f_0}$ for $i = 0, \dots, n$, $r(t) = \frac{g(t)}{f_0}$, the n th order difference equation can be written in a simpler form:

$$y(t+n) + a_1 y(t+n-1) + \dots + a_{n-1} y(t+1) + a_n y(t) = r(t), t = 0, 1, 2, \dots \quad (2.9.55)$$

Here, we provide three procedures that are usually used to solve n th order linear difference equations:

Step 1: find the general solution of the homogeneous difference equation

$$y(t+n) + a_1 y(t+n-1) + \dots + a_{n-1} y(t+1) + a_n y(t) = 0,$$

and let the general solution be Y .

Step 2: find a particular solution y^* of the difference equation (2.9.53).

Step 3: the solution of the difference equation (2.9.53) is

$$y(t) = Y + y^*.$$

The following are solutions of first-order, second-order, and n th-order difference equations, respectively.

2.9.1 First-order Difference Equations

The first-order difference equation is defined as:

$$y(t+1) + ay(t) = r(t), t = 0, 1, 2, \dots \quad (2.9.56)$$

The corresponding homogeneous difference equation is:

$$y(t+1) + ay(t) = 0,$$

and the general solution is $y(t) = c(-a)^t$, where c is an arbitrary constant.

To obtain a particular solution for a nonhomogeneous difference equation, consider $r(t) = r$, namely, the case that does not change over time.

Obviously, a particular solution is as follows:

$$y^* = \frac{r}{1+a}, a \neq -1,$$

$$y^* = rt, a = -1.$$

Therefore, the solution of the nonhomogeneous difference equation(2.9.56) is:

$$y(t) = \begin{cases} c(-a)^t + \frac{r}{1+a}, & \text{if } a \neq -1, \\ c + rt, & \text{if } a = -1. \end{cases} \quad (2.9.57)$$

If the initial condition $y(0) = y_0$ is known, the solution of the difference equation (2.9.56) is:

$$y(t) = \begin{cases} \left(y_0 - \frac{r}{1+a}\right) \times (-a)^t + \frac{r}{1+a}, & \text{if } a \neq -1, \\ y_0 + rt, & \text{if } a = -1. \end{cases} \quad (2.9.58)$$

If r depends on t , a particular solution is:

$$y^* = \sum_{i=0}^{t-1} (-a)^{t-1-i} r(i),$$

thus the solution of the difference equation (2.9.56) is:

$$y(t) = (-a)^t y_0 + \sum_{i=0}^{t-1} (-a)^{t-1-i} r(i), \quad t = 1, 2, \dots.$$

For a general function $r(t) = f(t)$, the coefficients of A_0, \dots, A_m can be determined by using the method of undetermined-coefficients, namely, considering $y^* = f(A_0, A_1, \dots, A_m; t)$. The following is to solve for a particular solution in a case in which $r(t)$ is a polynomial.

Example 2.9.1 Solve the following difference equation:

$$y(t+1) - 3y(t) = t^2 + t + 2.$$

The homogeneous equation is:

$$y(t+1) - 3y(t) = 0,$$

The general solution is:

$$Y = C3^t.$$

Using the method of undetermined-coefficients to obtain the particular solution of the nonhomogeneous equation, suppose that the particular solution has the following form:

$$y^* = At^2 + Bt + D.$$

Substitute y^* into the nonhomogeneous difference equation, and derive

$$A(t+1)^2 + B(t+1) + D - 3At^2 - 3Bt - 3D = t^2 + t + 2,$$

or

$$-2At^2 + 2(A-B)t + A+B-2D = t^2 + t + 2.$$

Since equality holds for each t , we must have:

$$\begin{cases} -2A = 1 \\ 2(A - B) = 1 \\ A + B - 2D = 2, \end{cases}$$

which gives $A = -\frac{1}{2}$, $B = -1$ and $D = -\frac{3}{4}$, and thus we have a particular solution: $y^* = -\frac{1}{2}t^2 - t - \frac{3}{4}$. As a consequence, a particular solution of the nonhomogeneous equation is $y(t) = Y + y^* = C3^t - \frac{1}{2}t^2 - t - \frac{3}{4}$.

We can also solve the case with an exponential function by using the method of undetermined-coefficients.

Example 2.9.2 Consider the first-order difference equation:

$$y(t+1) - 3y(t) = 4e^t.$$

Suppose that the form of particular solution is $y^* = Ae^t$. Then, substituting it into the nonhomogeneous difference equation gives: $A = \frac{4}{e-3}$. Therefore, the general solution of the first-order difference equation is: $y(t) = Y + y^* = C3^t + \frac{4e^t}{e-3}$.

Here, we provide some of the common ways for finding particular solutions:

- (1) when $r(t) = r$, a usual form of particular solution is: $y^* = A$;
- (2) when $r(t) = r + ct$, a usual form of particular solution is:
 $y^* = A_1t + A_2$;
- (3) when $r(t) = t^n$, a usual form of particular solution is: $y^* = A_0 + A_1t + \cdots + A_nt^n$;
- (4) when $r(t) = c^t$, a usual form of particular solution is: $y^* = Ac^t$;
- (5) when $r(t) = \alpha \sin(ct) + \beta \cos(ct)$, a usual form of particular solution is: $y^* = A_1 \sin(ct) + A_2 \cos(ct)$.

2.9.2 Second-order Difference Equation

The second-order difference equation is defined as:

$$y(t+2) + a_1y(t+1) + a_2y(t) = r(t).$$

The corresponding homogeneous differential equation is:

$$y(t+2) + a_1y(t+1) + a_2y(t) = 0.$$

Then, its general solution depends on the roots of the following linear equation:

$$m^2 + a_1m + a_2 = 0,$$

which is called the **auxiliary equation or characteristic equation** of second-order difference equations. Let m_1 and m_2 be the roots of this equation. Since $a_2 \neq 0$, both m_1 and m_2 are not 0.

Case 1: m_1 and m_2 are different real roots.

The general solution of the homogeneous equation is $Y = C_1m_1^t + C_2m_2^t$, where C_1 and C_2 are arbitrary constants.

Case 2: m_1 and m_2 are the same real roots.

The general solution of the homogeneous equation is $Y = (C_1 + C_2t)m_1^t$.

Case 3: m_1 and m_2 are two complex roots, i.e., $r(\cos \theta \pm i \sin \theta)$ with $r > 0, \theta \in (-\pi, \pi]$. The general solution of the homogeneous equation is $Y = C_1r^t \cos(t\theta + C_2)$.

For a general function $r(t)$, it can be solved by the method of undetermined-coefficients.

2.9.3 Difference Equations of Order n

The general n th-order difference equation is defined as:

$$y(t+n) + a_1y(t+n-1) + \cdots + a_{n-1}y(t+1) + a_ny(t) = r(t), t=0, 1, 2, \dots \quad (2.9.59)$$

The corresponding homogeneous equation is:

$$y(t+n) + a_1y(t+n-1) + \cdots + a_{n-1}y(t+1) + a_ny(t) = 0,$$

and its characteristic equation is:

$$m^n + a_1m^{n-1} + \cdots + a_{n-1}m + a_n = 0.$$

Let its n characteristic roots be m_1, \dots, m_n .

The general solutions of the homogeneous equations are the sum of the bases generated by these eigenvalues, and its concrete forms are as follows:

Case 1: The formula generated by a single real root m is C_1m^k .

Case 2: The formula generated by the real root m of multiplicity p is:

$$(C_1 + C_2t + C_3t^2 + \cdots + C_pt^{p-1})m^t.$$

Case 3: The formula generated by a pair of nonrepeated conjugate complex roots $r(\cos \theta \pm i \sin \theta)$ is:

$$C_1r^t \cos(t\theta + C_2).$$

Case 4: The formula generated by a pair of conjugate complex roots $r(\cos \theta \pm i \sin \theta)$ of multiplicity p is:

$$r^t[C_{1,1} \cos(t\theta + C_{1,2}) + C_{2,1}t \cos(t\theta + C_{2,2}) + \cdots + C_{p,1}t^{p-1} \cos(t\theta + C_{p,2})].$$

The general solution of the homogeneous difference equation is obtained by summing up all formulas generated by eigenvalues.

A particular solution y^* of a nonhomogeneous difference equation can be generated by the method of undetermined-coefficients.

A particular solution is:

$$y^* = \sum_{s=1}^n \theta_s \sum_{i=0}^{\infty} m_s^i r(t-i),$$

where

$$\theta_s = \frac{m_s}{\prod_{j \neq s} (m_s - m_j)}.$$

2.9.4 Stability of n th-Order Difference Equations

Consider an n th-order difference equation

$$y(t+n) + a_1 y(t+n-1) + \cdots + a_{n-1} y(t+1) + a_n y(t) = r(t), \quad t = 0, 1, 2, \dots \quad (2.9.60)$$

The corresponding homogeneous equation is:

$$y(t+n) + a_1 y(t+n-1) + \cdots + a_{n-1} y(t+1) + a_n y(t) = 0, \quad t = 0, 1, 2, \dots \quad (2.9.61)$$

Definition 2.9.3 The difference equation (2.9.55) is *asymptotically stable*, if an arbitrary solution $Y(t)$ of the homogeneous equation (2.9.61) satisfies $Y(t)|_{t \rightarrow \infty} = 0$.

Let m_1, \dots, m_n be the solution of their characteristic equation:

$$m^n + a_1 m^{n-1} + \cdots + a_{n-1} m + a_n = 0. \quad (2.9.62)$$

Theorem 2.9.1 Suppose that the modulus of all eigenvalues of the characteristic equation are less than 1. Then, the difference equation (2.9.60) is asymptotically stable.

When the following inequality conditions are satisfied, the modulus of all eigenvalues of the characteristic equation are less than 1.

$$\begin{vmatrix} 1 & a_n \\ a_n & 1 \end{vmatrix} > 0,$$

$$\begin{vmatrix}
1 & 0 & a_n & a_{n-1} \\
a_1 & 1 & 0 & a_n \\
a_n & 0 & 1 & a_1 \\
a_{n-1} & a_n & 0 & 1
\end{vmatrix} > 0,$$

$$\begin{vmatrix}
1 & 0 & \cdots & 0 & a_n & a_{n-1} & \cdots & a_1 \\
a_1 & 1 & \cdots & 0 & 0 & a_n & a_{n-1} & \cdots & a_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
a_{n-1} & a_{n-2} & \cdots & 1 & 0 & 0 & \cdots & a_n \\
a_n & 0 & \cdots & 0 & 1 & a_1 & \cdots & a_{n-1} \\
a_{n-1} & a_n & \cdots & 0 & 0 & 1 & \cdots & a_{n-2} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
a_1 & a_2 & \cdots & a_n & 0 & 0 & \cdots & 1
\end{vmatrix} > 0.$$

2.9.5 Difference Equations with Constant Coefficients

The difference equation with constant coefficients is defined as:

$$\mathbf{x}(t) = A\mathbf{x}(t-1) + \mathbf{b}, \quad (2.9.63)$$

where $\mathbf{x} = (x_1, \dots, x_n)'$, $\mathbf{b} = (b_1, \dots, b_n)'$. Suppose the matrix A is diagonalizable, the corresponding eigenvalues are $\lambda_1, \dots, \lambda_n$, and the matrix P formed by linearly independent eigenvectors, such that

$$A = P^{-1} \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix} P.$$

Similar to the discussion of obtaining the solutions of ordinary differential equations, we can find the general solutions of the difference equation with constant coefficients. Moreover, a necessary and sufficient condition for the difference equation (2.9.63) to be (asymptotically) stable is that the modulus of all eigenvalues λ_i are less than 1. When the modulus of all eigenvalues λ_i are less than 1, the stationary point $\mathbf{x}^* = \lim_{t \rightarrow \infty} \mathbf{x}(t) =$

$$(I - A)^{-1}b.$$

2.10 Basic Probability

Risk and uncertainty, as well as some of their basic operations, are broadly used in economics. This section briefly introduces knowledge involved in the textbook.

2.10.1 Probability and Conditional Probability

Compared with other fields of mathematics, the development of probability theory occurred relatively late. However, probability theory has developed rapidly and become a very important field in mathematics since its axiomatization.

When dealing with probabilities, we must clearly define the probability space. Classical probability (i.e., explaining probability as the same possibility) is often associated with permutation and combination. Since statistics requires obtaining data from sampling, the randomness in probability theory is revealed.

Let the probability of random variable X_a be X_{as} be π_s , $s \in S$, where S can be either discrete or continuous. When $S = \{1, \dots, n\}$, where n can be finite or infinite, this situation is about a discrete random variable. If S is an interval of the real space, then it is called a continuous random variable.

If there is a correlation between two random variables, then the value of a random variable provides information for the value of the other random variable, which gives the concept of conditional probability.

When the two random variables, X_a and X_b , have the joint probability distribution $\pi_{ss'}$, with the known information that $X_a = X_{as}$, the probability of $X_b = X_{bs'}$ is called a conditional probability:

$$P(X_b = X_{bs'} | X_a = X_{as}) = \frac{\pi_{ss'}}{\sum_{t' \in S} \pi_{st'}}.$$

This formula is also called the Bayes rule.

2.10.2 Mathematical Expectation and Variance

The **(mathematical) expectation** of random variable X_a is the weighted average of all possible values, and is defined and denoted by

$$E(X_a) \equiv \bar{X}_a = \sum_{s \in S} \pi_s X_{as},$$

which in a continuous case is defined by integral instead of summation, and it will be discussed in the next subsection.

The operation rule of expected utility is that if X_a and X_b are two random variables, then we have

$$E(aX_a + bX_b) = a\bar{X}_a + b\bar{X}_b.$$

The **variance** of a random variable X_a measuring the degree of variation of its value is defined as

$$\text{Var}(X_a) \equiv \sigma_{X_a}^2 = \sum_{s \in S} \pi_s (X_{as} - \bar{X}_a)^2.$$

Therefore, the larger is the variance, the greater is the variation degree.

There may be some correlations between the two random variables, X_a and X_b . Suppose that the value space of X_a is $\{X_{as}\}_{s \in S}$, and that of X_b is $\{X_{bs'}\}_{s' \in S'}$. Then, their **covariance** measures the correlations between their values.

Let $\pi_{ss'}$ be the probability of $X_a = X_{as}$ and $X_b = X_{bs'}$. **Covariance**, denoted by $\text{Cov}(X_a, X_b)$, is defined as:

$$\text{Cov}(X_a, X_b) = \sum_{s \in S, s' \in S'} \pi_{ss'} (X_{as} - \bar{X}_a)(X_{bs'} - \bar{X}_b)$$

or

$$\text{Cov}(X_a, X_b) = E(X_a - \bar{X}_a)(X_b - \bar{X}_b) = E(X_a X_b) - E(X_a)E(X_b).$$

If two random variables X_a and X_b are independent, then $\pi_{ss'} = \pi_s \pi_{s'}$, and thus we have $\text{Cov}(X_a, X_b) = 0$.

The following operation is for deriving the variance of linear combinations:

$$\text{Var} \left(\sum_{a \in A} \alpha_a X_a \right) = \sum_{a \in A, b \in A} \alpha_a \alpha_b \text{Cov}(X_a, X_b).$$

2.10.3 Continuous Distributions

When a random variable X takes values over $[a, b]$, its **probability distribution function** F on support $[a, b]$ is defined by

$$F(x) = \text{Prob}[X \leq x],$$

which is the probability that X takes values not exceeding x . By definition, the function F is nondecreasing and satisfies $F(a) = 0$ and $F(b) = 1$. Here, a and b can be any real number, and thus it is possible that $a = -\infty$ and $b = \infty$.

The derivative of F is called the **probability density function** and is denoted by $f \equiv F'$. We assume that f is continuous, and $f(x) > 0$ for all $x \in (a, b)$.

The expectation of X is then defined by

$$E(X) = \int_a^b x f(x) dx.$$

If $u : [a, b] \rightarrow \mathcal{R}$ is an arbitrary function, the expectation of $u(X)$ is defined by

$$E[u(X)] = \int_a^b u(x) f(x) dx,$$

which can also be written as

$$E[u(X)] = \int_a^b u(x) dF(x).$$

The **conditional expectation** of X given that $X < x$ is

$$E[X|X < x] = \frac{1}{F(x)} \int_a^x t f(t) dt,$$

and thus

$$F(x)E[X|X < x] = \int_a^x tf(t)dt = xF(x) - \int_a^x F(t)dt,$$

in which the second equality is obtained by integrating by parts.

2.10.4 Common Probability Distributions

Next, we review some common distributions, as well as their expectations and variances.

Binomial Distribution

Assume that there are many balls in a box with two colors. The proportion of red balls is p , and that of black balls is $1 - p$. The value of the random variable X is 1 if the red ball is drawn; otherwise, it is 0. If it is taken only once, the probability distribution of the random variable is $p(X = 1) = p$ and $p(X = 0) = 1 - p$.

The expectation and variance are:

$$E(X) = p; \quad \text{Var}(X) = p(1 - p).$$

If we draw n times (the ball is put back into the box at each time), the random variable is defined as the number of times that the red ball is drawn.

The probability distribution of random variables is

$$p(X = k) = \frac{n!}{k!(n - k)!} p^k (1 - p)^{n - k}.$$

Its expectation and variance are:

$$E(X) = np; \quad \text{Var}(X) = np(1 - p).$$

Poisson Distribution

If the probability of a random variable X is

$$P(X = k) = e^{-\lambda} \frac{\lambda^k}{k!},$$

then X follows a **Poisson distribution** with parameter λ , and its expectation and variance are:

$$E(X) = \lambda; \quad \text{Var}(X) = \lambda.$$

Uniform Distribution

If the probability density function of a random variable X is

$$f(x) = \frac{1}{b-a}, \quad x \in [a, b],$$

then X follows a **uniform distribution** over $[a, b]$. Its expectation and variance are:

$$E(X) = \frac{b+a}{2}; \quad \text{Var}(X) = \frac{(b-a)^2}{12}.$$

Normal Distribution

If the probability density function of a random variable X is

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad x \in (-\infty, \infty),$$

then X follows a **normal distribution** with parameters (μ, σ^2) .

The expectation and variance are:

$$E(X) = \mu; \quad \text{Var}(X) = \sigma^2.$$

Exponential Distribution

If the probability density function of a random variable X is

$$f(x) = \lambda e^{-\lambda x}, x \in [0, \infty),$$

then X follows an **exponential distribution** with parameter λ , and its expectation and variance are

$$E(X) = \frac{1}{\lambda}; \quad \text{Var}(X) = \frac{1}{\lambda^2}.$$

2.11 Stochastic Dominance and Affiliation

2.11.1 Order Stochastic Dominance

First-Order Stochastic Dominance

Definition 2.11.1 (First-Order Stochastic Dominance) Given two distribution functions F and G with support $[a, b]$, we say that F *first-order stochastically dominates* G if for all $x \in [a, b]$, $F(x) \leq G(x)$.

First-order stochastic dominance means that for any outcome x , the probability of obtaining at least x under $F(\cdot)$ is at least as high as that under $G(\cdot)$. For example, considering two assets, first-order stochastic dominance means that when two assets are greater than a certain constant return, the probability of one asset's return is higher than that of the other. This is analogous to the monotonicity concept under certainty.

There is another test criterion for F to first-order stochastically dominate G . The following theorem shows that these two criteria are equivalent.

Theorem 2.11.1 $F(\cdot)$ *first-order stochastically dominates* $G(\cdot)$ if and only if for any nondecreasing function $u : [a, b] \rightarrow \mathcal{R}$, we have

$$\int_a^b u(z) dF(z) \geq \int_a^b u(z) dG(z).$$

PROOF. Define $H(z) = F(z) - G(z)$. We need to prove that $H(z) \leq 0$ if and only if $\int_a^b u(z) dH(z) \geq 0$ for any increasing and differentiable function $u(\cdot)$.

Sufficiency: We prove this by way of contradiction. Suppose that there is a \hat{z} , such that $H(\hat{z}) > 0$. We choose a weakly increasing and differentiable function $u(z)$ as

$$u(z) = \begin{cases} 0, & z \leq \hat{z}, \\ 1, & z > \hat{z}, \end{cases}$$

then immediately $\int_a^b u(z)dH(z) = -H(\hat{z}) < 0$, which is a contradiction.

Necessity: Since a monotonic function is differentiable almost everywhere, in finding integration, we may just assume that u is differentiable, and we have

$$\int_a^b u(z)dH(z) = [u(z)H(z)]_a^b - \int_a^b u'(z)H(z)dz = 0 - \int_a^b u'(z)H(z)dz \geq 0,$$

in which the first equality is obtained by integration by parts, the second equality is based on

$$F(a) = G(a) = 0, \quad F(b) = G(b) = 1,$$

while the inequality is based on the assumptions that $u(\cdot)$ is weakly increasing ($u'(\cdot) \geq 0$) and $H(z) \geq 0$. \square

For any two probability distributions F and G , as long as an agent's utility is (weakly) increasing in outcomes, he or she prefers the one that first-order stochastically dominates the other one.

Second-Order Stochastic Dominance

Definition 2.11.2 (Second-Order Stochastic Dominance) Given two distribution functions F and G defined on $[a, b]$, which have the same expectation, we say that $F(\cdot)$ *second-order stochastically dominates* $G(\cdot)$ if

$$\int_a^z F(r)dr \leq \int_a^z G(r)dr$$

for all z .

It is clear that first-order stochastic dominance implies second-order stochastic dominance. In addition, second-order stochastic dominance im-

plies not only monotonicity, but also lower risk. To show this, we introduce the notion of “mean-preserving spreads”.

Suppose that X is a random variable with distribution function F . Let Z be a random variable whose distribution conditional on $X = x$, $H(\cdot|X = x)$, is, such that for all x , $E[Z|X = x] = 0$. Suppose that $Y = X + Z$ is the random variable obtained from first drawing X from F and then for each realization $X = x$, drawing a Z from the conditional distribution $H(\cdot|X = x)$ and adding it to X . Let G be the distribution of Y so defined. We will then say that G is a **mean-preserving spread** of F .

While random variables X and Y have the same mean, i.e., $E[X] = E[Y]$, variable Y is “more spread-out” than X since it is obtained by adding a “noise” variable Z to X . Now, suppose that $u : [a, b] \rightarrow \mathcal{R}$ is a concave function. Using Jensen’s inequality, we have

$$\begin{aligned} E_Y[u(Y)] &= E_X[E_Z[u(X + Z)|X = x]] \\ &\leq E_X[u(E_Z[X + Z|X = x])] \\ &= E_X[u(X)]. \end{aligned}$$

As such, similar to Theorem 2.11.1, we have the following conclusion for second-order stochastic dominance.

Theorem 2.11.2 *If distributions $F(\cdot)$ and $G(\cdot)$ defined on $[a, b]$ have the same mean, then the following statements are equivalent.*

- (1) $F(\cdot)$ second-order stochastically dominates $G(\cdot)$;
- (2) for any nondecreasing concave function $u : \mathcal{R} \rightarrow \mathcal{R}$, we have $\int_a^b u(z)dF(z) \geq \int_a^b u(z)dG(z)$;
- (3) $G(\cdot)$ is a mean-preserving spread of $F(\cdot)$.

PROOF. (3) \Rightarrow (2): It is obtained by using

$$\begin{aligned} \int_a^b u(z)dF(z) &= \int_a^b u \left(\int_a^b (x + z)dH_z(x) \right) dF(z) \\ &\geq \int_a^b \left(\int_a^b u(x + z)dH_z(x) \right) dF(z) \\ &= \int_a^b u(z)dG(z), \end{aligned}$$

in which the inequality follows from the concavity of $u(\cdot)$.

(1) \Rightarrow (2): For expositional convenience, we set $b = 1$. We have

$$\begin{aligned} & \int_a^b u(z) dF(z) - \int_a^b u(z) dG(z) \\ &= -u'(1) \int_a^b (F(z) - G(z)) dz + \int_a^b \left(\int_a^z (F(x) - G(x)) dx \right) u''(z) dz \\ &= \int_a^b \left(\int_a^z (F(x) - G(x)) dx \right) u''(z) dz \\ &\geq 0, \end{aligned}$$

in which the inequality follows from the definition of second-order stochastic dominance, i.e.,

$$\int_a^z F(r) dr \leq \int_a^z G(r) dr,$$

and also $u''(\cdot) \leq 0$ for any z . We thus have

$$\int_a^b u(z) dF(z) - \int_a^b u(z) dG(z) \geq 0.$$

(1) \Rightarrow (3): We just show the case with discrete distributions.

Define

$$\begin{aligned} S(z) &= G(z) - F(z), \\ T(x) &= \int_a^x S(z) dz. \end{aligned}$$

By the definition of second-order stochastic dominance, we have $T(x) \geq 0$ and $T(1) \geq 0$, which imply that there exists some \hat{z} , such that $S(z) \geq 0$ for $z \leq \hat{z}$ and $S(z) \leq 0$ for $z \geq \hat{z}$.

Since the random variable follows a discrete distribution, $S(z)$ must be a step function. Let $I_1 = (a_1, a_2)$ be the first interval over which $S(z)$ is positive, and $I_2 = (a_3, a_4)$ be the first interval over which $S(z)$ is negative. If no such $I_1 = (a_1, a_2)$ exists, then $S(z) \equiv 0$, and thus statement (3) is immediate. If $I_1 = (a_1, a_2)$ does exist, then $I_2 = (a_3, a_4)$ must exist, as well.

Therefore, $S(z) \equiv \gamma_1 > 0$ for $z \in I_1$, and $S(z) \equiv -\gamma_2 < 0$ for $z \in I_2$. By $T(x) \geq 0$, we must have $a_2 < a_3$. If $\gamma_1(a_2 - a_1) \geq \gamma_2(a_4 - a_3)$, then there exist $a_1 < \hat{a}_2 \leq a_2$ and $\hat{a}_4 = a_4$, such that $\gamma_1(\hat{a}_2 - a_1) = \gamma_2(\hat{a}_4 - a_3)$.

If $\gamma_1(a_2 - a_1) < \gamma_2(a_4 - a_3)$, then there exists $a_3 < \hat{a}_4 \leq a_4$, such that $\gamma_1(\hat{a}_2 - a_1) = \gamma_2(\hat{a}_4 - a_3)$.

Letting

$$S_1(z) = \begin{cases} \gamma_1, & \text{if } a_1 < z < \hat{a}_2, \\ -\gamma_2, & \text{if } a_3 < z < \hat{a}_4, \\ 0, & \text{otherwise.} \end{cases}$$

If $F_1 = F + S_1$, then F_1 is a mean-preserving spread of F . Letting $S^1 = G - F_1$, we can similarly construct $S_2(z)$ and F_2 . Since $S(z)$ is a step function, then there exists an n , such that $F_0 = F$, $F_n = G$, and F_{i+1} is a mean-preserving spread of F_i . Furthermore, a finite summation of mean-preserving spreads is still a mean-preserving spread. \square

Although a continuous function can be arbitrarily approximated by step functions, the formal proof is complicated. Rothschild and Stiglitz (1971) provided a complete proof for the case with continuous distributions.

2.11.2 Hazard Rate Dominance

Let F be a distribution function with support $[a, b]$. The hazard rate of F is the function $\lambda : [a, b) \rightarrow \mathcal{R}_+$ defined by

$$\lambda(x) \equiv \frac{f(x)}{1 - F(x)}.$$

If we interpret F as the probability that some event will occur prior to time x , then the hazard rate at x represents the instantaneous probability that the event will happen at x , given that it has not occurred until time x . Since the event may be the failure of some component, e.g., a lightbulb, it is sometimes also called the “failure rate”.

Solving for F , we have

$$F(x) = 1 - \exp\left(-\int_a^x \lambda(t)dt\right). \quad (2.11.64)$$

This shows that any arbitrary function $\lambda : [a, b) \rightarrow \mathcal{R}_+$, such that for all $x < b$,

$$\int_a^x \lambda(t)dt < \infty, \quad \lim_{x \rightarrow b} \int_a^x \lambda(t)dt = \infty,$$

is the hazard rate of *some* distribution that is given by (2.11.64).

Definition 2.11.3 (Hazard Rate Dominance) For any two distributions F and G with hazard rates λ_F and λ_G , respectively, we say that F *dominates* G in terms of the hazard rate if $\lambda_F(x) \leq \lambda_G(x)$ for all x . This order is also referred to in shortened form as **hazard rate dominance**.

If F dominates G in terms of the hazard rate, then

$$F(x) = 1 - \exp\left(-\int_a^x \lambda_F(t)dt\right) \leq 1 - \exp\left(-\int_a^x \lambda_G(t)dt\right) = G(x),$$

and thus F first-order stochastically dominates G . Therefore, hazard rate dominance implies first-order stochastic dominance.

2.11.3 Reverse Hazard Rate Dominance

A closely related concept to the hazard rate is the reverse hazard rate $\sigma : (a, b] \rightarrow \mathcal{R}_+$ given by

$$\sigma(x) \equiv \frac{f(x)}{F(x)},$$

and is sometimes referred to as the **inverse of the Mills' ratio**. Similarly, solving for F gives

$$F(x) = \exp\left(-\int_x^b \sigma(t)dt\right). \quad (2.11.65)$$

This shows that any arbitrary function $\sigma : (a, b] \rightarrow \mathcal{R}_+$, such that for all $x > a$,

$$\int_x^b \sigma(t)dt < \infty \text{ and } \lim_{x \rightarrow 0} \int_x^b \sigma(t)dt = \infty.$$

is the “reverse hazard rate” of some distribution that is given by (2.11.65).

Definition 2.11.4 (Reverse Hazard Rate Dominance) For two distributions F and G with reverse hazard rates σ_F and σ_G , we say that F *dominates* G in terms of the reverse hazard rate if $\sigma_F(x) \geq \sigma_G(x)$ for all x . This order is also referred to in shortened form as *reverse hazard rate dominance*.

If F dominates G in terms of the reverse hazard rate, then

$$F(x) = \exp\left(-\int_x^b \sigma_F(t)dt\right) \leq \exp\left(-\int_x^b \sigma_G(t)dt\right) = G(x),$$

and thus, again, F first-order stochastically dominates G . Therefore, reverse hazard rate dominance also implies first-order stochastic dominance.

2.11.4 Likelihood Ratio Dominance

Definition 2.11.5 (Likelihood Ratio Dominance) We say that the distribution function F dominates G in terms of the *likelihood ratio* if for all $x < y$,

$$\frac{f(x)}{g(x)} \leq \frac{f(y)}{g(y)}, \quad (2.11.66)$$

which means that $\frac{f}{g}$ is a nondecreasing function. As such, we refer to this order as *likelihood ratio dominance*.

Rewriting (2.11.66) gives

$$\frac{f(y)}{f(x)} \leq \frac{g(y)}{g(x)},$$

and then for all x , we have

$$\int_x^b \frac{f(y)}{f(x)} dy \leq \int_x^b \frac{g(y)}{g(x)} dy,$$

which, in turn, implies that

$$\frac{1 - F(x)}{f(x)} \leq \frac{1 - G(x)}{g(x)}.$$

Therefore, likelihood ratio dominance implies hazard rate dominance.

Similarly, rewriting (2.11.66) gives

$$\frac{f(x)}{f(y)} \leq \frac{g(x)}{g(y)},$$

and then for all x , we have

$$\int_a^y \frac{f(x)}{f(y)} dx \leq \int_a^y \frac{g(x)}{g(y)} dx,$$

which implies that

$$\frac{F(y)}{f(y)} \leq \frac{G(y)}{g(y)}.$$

Therefore, likelihood ratio dominance implies reverse hazard rate dominance.

Summarizing the above discussions, one can see that likelihood ratio dominance is the strongest, which implies both hazard rate dominance and reverse hazard rate dominance, both of which, in turn, imply first-order stochastic dominance that again implies second-order stochastic dominance.

2.11.5 Order Statistics

Let X_1, X_2, \dots, X_n be n random variables independently and randomly drawn from a distribution F with density f . Let $Y_1^{(n)}, Y_2^{(n)}, \dots, Y_n^{(n)}$ be a rearrangement of these, and thus

$$Y_1^{(n)} \geq Y_2^{(n)} \geq \dots \geq Y_n^{(n)},$$

where $Y_k^{(n)}$, $k = 1, 2, \dots, n$ are referred to as **order statistics**.

Let $F_k^{(n)}$ denote the distribution of $Y_k^{(n)}$, with corresponding probability density function $f_k^{(n)}$. If there is no confusion, we simply denote them as Y_k , F_k and f_k . In auction theory, we will typically be interested in properties of the highest and second highest order statistics, i.e., Y_1 and Y_2 , respectively.

Highest Order Statistic

The distribution of the **highest order statistic** Y_1 can be obtained as follows. The event that $Y_1 \leq y$ is equivalent to the event: $X_k \leq y$ for all k . Since X_k is independently drawn from the same distribution F , we have that

$$F_1(y) = F(y)^n.$$

The density function is

$$f_1(y) = nF(y)^{n-1}f(y).$$

Note that if F stochastically dominates G , and F_1 and G_1 are distributions of the highest order statistics of n draws from F and G , respectively, then F_1 stochastically dominates G_1 .

Second-Highest Order Statistics

The distribution of the **second-highest order statistic** Y_2 can also be easily derived. The event that $Y_2 \leq y$ is the union of the following disjoint events: (1) all X_k 's are less than or equal to y ; and (2) $n - 1$ of the X_k 's are less than or equal to y , and one is greater than y . There are n different ways in which (2) can occur. Therefore, we have

$$\begin{aligned} F_2(y) &= F(y)^n + nF(y)^{n-1}(1 - F(y)) \\ &= nF(y)^{n-1} - (n - 1)F(y)^n. \end{aligned}$$

The probability density function is then

$$f_2(y) = n(n - 1)(1 - F(y))F(y)^{n-2}f(y).$$

Again, one can verify that if F stochastically dominates G and also F_2 and G_2 are distributions of the second-highest order statistics of n draws from F and G , respectively, then F_2 stochastically dominates G_2 .

2.11.6 Affiliation

Affiliation is a basic assumption employed to study auctions with interdependent values in which random variables are non-negatively correlated.

Definition 2.11.6 Suppose that random variables X_1, X_2, \dots, X_n are distributed on some product of intervals $D \subseteq \mathcal{R}^n$ according to the joint density function f . $X = (X_1, X_2, \dots, X_n)$ are said to be *affiliated* if for all $\mathbf{x}', \mathbf{x}'' \in \mathcal{X}$,

$$f(\mathbf{x}' \vee \mathbf{x}'')f(\mathbf{x}' \wedge \mathbf{x}'') \geq f(\mathbf{x}')f(\mathbf{x}''), \quad (2.11.67)$$

in which

$$\mathbf{x}' \vee \mathbf{x} = (\max(x'_1, x_1), \dots, \max(x'_n, x_n))$$

denotes the component-wise **maximum** of \mathbf{x}' and \mathbf{x} , and

$$\mathbf{x}' \wedge \mathbf{x}'' = (\min(x'_1, x''_1), \dots, \min(x'_n, x''_n))$$

denotes the component-wise **minimum** of \mathbf{x}' and \mathbf{x}'' . If (2.11.67) is satisfied, then we also say that f is *affiliated*.

Suppose that the density function $f : D \rightarrow \mathcal{R}_+$ is strictly positive in the interior of D and twice continuously differentiable. One can verify that f is affiliated if and only if, for all $i \neq j$,

$$\frac{\partial^2}{\partial x_i \partial x_j} \ln f \geq 0,$$

which means that the off-diagonal elements of the Hessian of $\ln f$ are non-negative.

Proposition 2.11.1 *Let X_1, X_2, \dots, X_n be random variables, and Y_1, Y_2, \dots, Y_{n-1} be the largest, second largest, ..., smallest order statistics from among X_2, X_3, \dots, X_n . If X_1, X_2, \dots, X_n are symmetrically distributed and affiliated, then we have*

- (1) *variables in any subset of X_1, X_2, \dots, X_n are also affiliated;*
- (2) *$X_1, Y_1, Y_2, \dots, Y_{n-1}$ are affiliated.*

Monotone Likelihood Ratio Property

Suppose that the two random variables X and Y have a joint density $f : [a, b]^2 \rightarrow \mathcal{R}$. If X and Y are affiliated, then for all $x' \geq x$ and $y' \geq y$, we have

$$f(x', y)f(x, y') \leq f(x, y)f(x', y') \Leftrightarrow \frac{f(x, y')}{f(x, y)} \leq \frac{f(x', y')}{f(x', y)} \quad (2.11.68)$$

and

$$\frac{f(y'|x)}{f(y|x)} \leq \frac{f(y'|x')}{f(y|x')},$$

so the **likelihood ratio**

$$\frac{f(\cdot|x')}{f(\cdot|x)}$$

is increasing, and this is referred to as the **monotone likelihood ratio property**.

Using the same arguments as for order stochastic dominance in the previous subsection, it can be deduced that for all $x' \geq x$, $F_Y(\cdot|x')$ dominates

$F_Y(\cdot|x)$ in terms of the likelihood ratio, and the other dominance relationships then follow as usual. We have the following conclusions.

Proposition 2.11.2 *If X and Y are affiliated, the following properties hold:*

- (1) *For all $x' \geq x$, $F(\cdot|x')$ dominates $F(\cdot|x)$ in terms of **hazard rate**, i.e.,*

$$\lambda(y|x') \equiv \frac{f(y|x')}{1 - F(y|x')} \leq \frac{f(y|x)}{1 - F(y|x)} \equiv \lambda(y|x).$$

Or equivalently, for all y , $\lambda(y|\cdot)$ is nonincreasing.

- (2) *For all $x' \geq x$, $F(\cdot|x')$ dominates $F(\cdot|x)$ in terms of the **reverse hazard rate**, i.e.,*

$$\sigma(y|x') \equiv \frac{f(y|x')}{F(y|x')} \leq \frac{f(y|x)}{F(y|x)} \equiv \sigma(y|x),$$

or equivalently, for all y , $\sigma(y|\cdot)$ is nondecreasing.

- (3) *For all $x' \geq x$, $F(\cdot|x')$ **first-order stochastically dominates** $F(\cdot|x)$, i.e.,*

$$F(y|x') \leq F(y|x),$$

or equivalently, for all y , $F(y|\cdot)$ is nonincreasing.

- (4) *For all $x' \geq x$, $F(\cdot|x')$ **second-order stochastically dominates** $F(\cdot|x)$, i.e., for all y*

$$\int_a^y F(r|x')dr \leq \int_a^y F(r|x)dr$$

or equivalently, for all y , $\int_a^y F(y|\cdot)$ is nonincreasing.

All of these results extend in a straightforward manner to the case in which the number of conditional variables is more than one. Suppose that Y, X_1, X_2, \dots, X_n are affiliated and let $F_Y(\cdot|\mathbf{x})$ denote the distribution of Y conditional on $X = \mathbf{x}$. We can then also obtain the above dominance relations.

2.12 Biographies

2.12.1 Friedrich August Hayek

Friedrich August Hayek (1899-1992), one of the greatest economic thinkers of the 20th century and a representative of the Austrian school, won the 1974 Nobel Prize in Economics for his contributions to the theory of money and economic cycles, as well as his penetrating analysis of the interdependence of economic, political, and institutional phenomena. The Nobel Prize Committee believed that Hayek's in-depth analysis of the economic cycle made him one of the very few economists who had warned about a possible great economic depression prior to 1929. In fact, both academically and practically, the 20th century was characterized by competition between the market economic system and the planned economic system, and disputes concerning their respective advantages and disadvantages. Hayek's penetrating analysis of different economic systems led him to point out very early that the planned economy is not feasible from the perspective of information efficiency, incentive compatibility, and resource allocation efficiency. Indeed, pragmatic results proved Hayek's extraordinary judgment and insight. Finally, the planned economic system experienced its demise, which made him one of the most influential economists of the 20th century.

Hayek was born in an intellectual family in Vienna and received a doctorate from the University of Vienna (1921-1923). When Hayek was at the University of Vienna, he attended classes taught by Ludwig von Mises (1881-1973). It was Mises' thorough critique of socialism published in 1922 that eventually pulled Hayek out of the Fabian socialist ideological trend. The best way to understand Hayek's great contribution to economics and classical liberalism is to analyze it from the perspective of Mises' paradigm of social collaboration. Hayek taught at the London School of Economics and Political Science (1931-1950), the University of Chicago (1950-1962), and Freiburg University (1962-1968). At the University of Chicago, Hayek was a professor of social and ethical science in the "Committee on Social Thought" and did not obtain a teaching post in the Department of Economics. Professor Friedman, a friend of his in the economics department,

was also critical of Hayek's books on economics. When he first arrived at the University of Chicago, Hayek conducted political studies and did not engage in economics research, and he held a negative attitude towards some research methods that were being used at the Department of Economics. Even so, Hayek interacted frequently with some of the members of the Chicago School of Economics, and his political views were compatible with many of the Chicago School. Hayek made a remarkable contribution to the University of Chicago. He strongly supported Aaron Director, a Chicago School economist and the founder of law and economics, to carry out the "Law and Society" project at the University of Chicago Law School. Indeed, Professor Director persuaded the University of Chicago Press to publish Hayek's *The Road to Serfdom*, which later became popular globally. Hayek also collaborated with Friedman and others on the establishment of the International Forum of Liberal Economists.

Hayek had two profound debates in his life: one was the "socialist controversy" in the 1920s and 1930s, which was the debate between Mises-Hayek and Lange-Lerner on the theoretical feasibility of efficient allocations under socialism. He criticized the drawbacks of the planned economy from the perspective of information and incentives. He held that the planned economy was theoretically impracticable, and emphasized the importance of a spontaneous social order based on freedom, competition, and rules. This advanced internal logic judgment was verified prior to his death. The second was the theoretical debate with Keynes in the 1930s. He pointedly criticized Keynes's theoretical claims and academic viewpoints put forward in *A Treatise on Money*, and thought that Keynes's economic proposition of achieving full employment by lowering interest rates and increasing the money supply was fundamentally incorrect. In 1947, Hayek advocated for the establishment of the Pilgrimage Mountain Society, an important liberal academic organization. He advocated thorough economic freedom and opposed any form of state intervention, calling for "non-nationalization" of currency issuance.

Hayek's profound philosophy of revealing the importance of institutions will undoubtedly continue to influence and guide the world, especially the next step of reform in China.

2.12.2 Joseph Alois Schumpeter

Joseph Alois Schumpeter (1883-1950), an Austrian American political economist (but not a member of the Austrian School) was profoundly influential, and is hailed as the originator of the Innovation Theory. Known as one of the greatest economists in history, his name is closely associated with most of the concepts and knowledge about market economies and innovation. Indeed, he proposed the four most representative and well-known economic terms, i.e., innovation, entrepreneurship, corporate strategy, and creative destruction. In addition, he believed that “creative destruction” is a double-edged sword which can engender economic growth, but can also impair some values that people have traditionally cherished. He expressed this by stating, “What poverty brings is a tragic life, while it is difficult for prosperity to maintain peace of mind.”

In 1883, Schumpeter was born into the family of a weaving factory owner in Triesch, Habsburg Moravia (now part of Czech, and thus Schumpeter is sometimes considered to be a Czech-American), Austria-Hungary. He enrolled in an elite middle school in Vienna. He studied law and sociology at the University of Vienna from 1901 to 1906, and received his doctoral degree in law in 1906. In 1908, he became an associate professor at the University of Czernowitz through his instructor’s recommendation just at the beginning of his journey as an economist. Czernowitz is a remote city, but a suitable place for learning with its tranquility outside of modern industrial civilization. Here, Schumpeter wrote his first masterpiece, *The Theory of Economic Development* published in 1912, which touches on “innovation” and its role in economic development, and had a great impact in the economics community. According to statistics, the concept of “creative destruction” proposed by Schumpeter was cited frequently, second only to the “invisible hand” of Adam Smith. *The Theory of Economic Development* has become one of the classical economic works of the 20th century. Later, Schumpeter emigrated to the United States, and taught at Harvard University until the end of his life.

In his famous book, *History of Economic Analysis*, Schumpeter argued that the difference between an economic scientist and an average economist

lies in the adoption of the following three elements in the process of economic analysis. The first one is economic theory with an inherent logical analysis. The second is history with analysis from historical perspectives. Finally, the third is statistics with data and empirical analysis. Schumpeter's five innovation concepts are also frequently quoted and mentioned, even to the extent that his name appears in almost every discussion about innovation. Moreover, as the founder of the Innovation Theory and the research of business history, Schumpeter's influence is also currently being "rediscovered" .

Innovation refers to an economic process that recombines and integrates original production factors into new production methods in order to increase efficiency and reduce costs. In Schumpeter's economic model, those who can successfully innovate can survive the dilemmas of diminishing returns; whereas, those who fail to recombine production factors will be the first to be eliminated by the market. The creative destruction of capitalism means that when the economy cycles to the bottom, this is the time when some entrepreneurs have to consider exiting the market and others must innovate to survive. As long as excess competitors are excluded or some successful "innovations" are created, the economy will improve and production efficiency will increase. When an industry becomes profitable again, however, it will attract the investment of new competitors. Then, the process of diminishing returns begins again and returns to the previous state. Therefore, every depression implies the possibility of another technological innovation, or it can alternatively be stated that the result of technological innovation is another expected depression. In Schumpeter's view, the creativity and destructiveness of capitalism are homologous. However, Schumpeter did not believe that the superiority of capitalism is due to its own impetus which can promote its own development continually. Instead, he contended that the capitalist economy will eventually collapse because it cannot withstand the energy of its rapid expansion. The business cycle, also known as the economic cycle, is Schumpeter's most quoted economic term. Schumpeter's concept of creative destruction has had a great influence on the development of modern economics. The combination of the dynamic market mechanism and R&D economics provides economists

with a crucial perspective of endogenous technological change. Schumpeter's technological innovation has become a core element of the theory of endogenous growth in macroeconomics.

In *Capitalism, Socialism and Democracy*, Schumpeter gave the following modern definition of democracy: “the democratic method is that institutional arrangement for arriving at political decisions in which individuals acquire the power to decide by means of a competitive struggle for the people's vote” . He held that democracy constitutes a process in which political elites compete for power, and that the people choose political leaders. The essence of democracy lies in a competitive election process. Political elites occupy political power and implement their rule, but their legitimacy comes from the choice of the people. Schumpeter also took the view that, in the political market with democracy, politicians provide political programs and policies according to the preferences of voters, and compete freely in elections to attract voters. Schumpeter's definition of democracy symbolizes the great transformation of democratic theory from classical democracy directly ruled by the people to the modern election democracy.

The economic development of any country needs to go through three stages: factor-driven, efficiency-driven, and innovation-driven. The ideas and theories of Hayek and Schumpeter play a crucial role in theoretically guiding and clarifying the way that the two transition stages occur.

Hayek's economic thought on the fundamental importance of the market and institutions deeply affected economic development in the 20th century; likewise, Schumpeter's economic thought on the critical importance of innovation will undoubtedly continue to exert remarkable influence, as we have already witnessed during the first 20 years of the 21st century.

2.13 Exercises

Exercise 2.1 Consider an economy with two sectors: the industrial sector and the monetary sector, characterized by the following equations:

$$Y = C + I + G,$$

$$C = a + b(1 - t)Y,$$

$$I = d - ei,$$

$$G = G_0,$$

where Y, C, I and i (i is the interest rate) are endogenous variables, G_0 is an exogenous variable, and a, b, d, e and t are all structure parameters.

In the newly introduced monetary market, we have:

the equilibrium conditions: $M_d = M_s,$

the money demand: $M_d = kY - li,$

and the money supply: $M_s = M_0,$

where M_0 is the exogenous variable of money stock, and k and l are parameters. Given this economy, please solve the following problems: (using Cramer's rule)

1. Equilibrium income Y^* ;
2. Money supply multiplier;
3. Government expenditure multiplier.

Exercise 2.2 Q represents the set of rational numbers, and as a metric space, its distance is defined by $d(p, q) = |p - q|$, where $p \in E = \{p \in Q : 2 < p < 40\} \subseteq Q, p \in E = \{p \in Q : 2 < p < 40\} \subseteq Q$.

1. Prove that E is closed and bounded in Q .
2. Prove that E is not compact.
3. Is E open in Q ? If yes, why?

Exercise 2.3 Given a metric space X , consider a series of open sets $\{E_n\}_{n \in N}$ in X .

1. Prove that $\bigcup_{n \in N} E_n$ is an open set.
2. Prove that it may not be true that the intersection of a series of open sets is open (please provide an example of this).

Exercise 2.4 Prove the following theorems:

1. The difference of an open set and a closed set is also open, while the difference of a closed set and an open set is also closed.
2. Each closed set is the intersection of a countable number of open sets; each open set is the union of a countable number of closed sets.

Exercise 2.5 Let $S \subseteq \mathcal{R}^L$. Prove that the following propositions are equivalent:

1. S is compact.
2. S is bounded and closed.
3. Every sequence in S has a convergent subsequence with the limit point in S .
4. Every infinite subset of S has a cluster point in S .
5. Each closed subset of the set S with finite intersection property (i.e., the intersection over any finite subcollection is nonempty) is nonempty.

Exercise 2.6 Prove the following propositions:

1. Every closed subset of a compact set is compact.
2. If $f : X \rightarrow Y$ is continuous and K is compact in X , then $f(K)$ is compact in Y .
3. S_i is compact, $i \in I$, if and only if $\prod_{i \in I} S_i$ is compact.
4. S_i is compact, $i = 1, 2, \dots, m$, if and only if $\sum_i^m S_i$ is compact.

Exercise 2.7 (Shapley-Folkman Theorem) Prove the theorem: Let S_i , ($i = 1, \dots, n$) be n non-empty subsets of R^m and $S = \sum_{i=1}^n S_i$. Then, each $x \in Co(S)$ has a representation $x = \sum_{i=1}^n x_i$, such that $x_i \in Co(S_i)$ for all i , and $x_i \in S_i$ for at least $(n - m)$ indices i .

Exercise 2.8 Prove the following theorems:

1. If f is a differentiable function defined on \mathcal{R}^1 , then f is concave if and only if the first-order condition $f'(x)$ is non-increasing.
2. If f is a twice differentiable function defined on \mathcal{R}^1 , then f is concave if and only if the second-order condition $f''(x)$ is non-positive.
3. If f is a differentiable function defined on \mathcal{R}^1 , then f is concave if and only if $f(y) \leq f(x) + f'(x)(y - x)$ for any $x, y \in \mathcal{R}^1$.

Exercise 2.9 Suppose that $f(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T A \mathbf{x} + \mathbf{b}^T \mathbf{x} + c$, where $\mathbf{x} \in \mathcal{R}^n$, \mathbf{x}^T is the transpose of vector \mathbf{x} , A is an $n \times n$ symmetric matrix, \mathbf{b} is an n -dimensional vector, and c is a constant.

1. Prove that if A is a positive semi-definite matrix, then $f(\mathbf{x})$ is a convex function.
2. Prove that if A is a positive definite matrix, then $f(\mathbf{x})$ is a strictly convex function.

Exercise 2.10 Determine whether Kuhn-Tucker conditions are applicable for the following optimization problems and solve them.

$$\begin{aligned} & \max x_1 \\ \text{s.t. } & x_1^3 - x_2 \leq 0, \\ & x_2 \leq 0. \end{aligned}$$

Exercise 2.11 Solve the following optimization problems using Kuhn-Tucker conditions.

$$\begin{aligned} & \max xyz \\ \text{s.t. } & x^2 + y^2 + z^2 \leq 6, \\ & x \geq 0, y \geq 0, z \geq 0. \end{aligned}$$

Exercise 2.12 The maximization problem is as follows:

$$\begin{aligned} & \max f(\mathbf{x}) \\ \text{s.t. } & g^1(\mathbf{x}) = 0, \dots, g^m(\mathbf{x}) = 0, \end{aligned}$$

where $f : \mathcal{R}^n \rightarrow \mathcal{R}$ and $g^j : \mathcal{R}^n \rightarrow \mathcal{R}$ are increasing functions with respect to x , and $m < n$. Prove that: If f is quasi-concave and all g^j are quasi-convex functions, then any local optimum is the global optimal solution.

Exercise 2.13 Let $u : \mathcal{R}^n \rightarrow \mathcal{R}$ be a function, $p, x \in \mathcal{R}^n$, and $y \in \mathcal{R}$. Consider the following optimization problem:

$$\begin{aligned} & \max_x u(x) \\ \text{s.t. } & px = y. \end{aligned}$$

Suppose that there is an optimum solution $x^*(p, y) > 0$, such that $v(x, y) = u(x^*(p, y))$.

1. Prove that $v(p, y)$ is homogeneous of degree zero.
2. Prove that $v(p, y)$ is a quasi-concave function.

Exercise 2.14 Suppose that a Cobb-Douglas utility function $u : \mathcal{R}^2 \rightarrow \mathcal{R}$ is defined as:

$$u(x_1, x_2) = x_1^\alpha x_2^\beta, \quad \alpha, \beta > 0.$$

Prove:

1. If $\alpha + \beta \leq 1$, then u is a concave function.
2. If $\alpha + \beta > 1$, then u is a quasi-concave function, but not concave.
3. For any $\alpha > 0$ and $\beta > 0$, $h(x_1, x_2) = \ln(u(x_1, x_2))$ is a concave function.

Exercise 2.15 Suppose that \overline{X} is a nonempty, closed, and convex set in \mathcal{R}^n , $x_0 \notin \overline{X}$. Prove that the following propositions are true.

1. There is a point $a \in \overline{X}$, such that $d(x_0, a) < d(x_0, x)$ for all $x \in \overline{X}$, and $d(x_0, a) > 0$.
2. There is a point $p \in \mathcal{R}^n$, $p \neq 0$, $\|p\| \equiv (\sum_{i=1}^n p_i^2)^{1/2} < \infty$ and $\alpha \in \mathcal{R}$, such that

$$p \cdot x \geq \alpha, \text{ for all } x \in \overline{X} \text{ and } p \cdot x_0 < \alpha.$$

Specifically, \overline{X} and x_0 is separated by a hyperplane $H = \{x : p \cdot x = \alpha, x \in \mathcal{R}^n\}$.

Exercise 2.16 Consider following functions:

(1) $3x^5y + 2x^2y^4 - 3x^3y^3$.

(2) $3x^5y + 2x^2y^4 - 3x^3y^4$.

(3) $x^{3/4}y^{1/4} + 6x + 4$.

(4) $\frac{x^2 - y^2}{x^2 + y^2} + 3$.

(5) $x^{1/2}y^{-1/2} + 3xy^{-1} + 7$.

(6) $x^{3/4}y^{1/4} + 6x$.

1. Find homogeneous functions among them and determine their orders of degree.
2. Test whether the above functions satisfy the Euler theorem.

Exercise 2.17 There is a simple application of upper (lower) hemi-continuity of correspondences. Suppose that $f : X \times Y \longrightarrow \mathcal{R}$,

$$G(x) = \{y \in \Gamma(x) : f(x, y) = \max_{y \in \Gamma(x)} f(x, y)\}.$$

1. Suppose that $X = \mathcal{R}, \Gamma(x) = Y = [-1, 1]$. For all $x \in X, f(x, y) = xy^2$. Draw the graph of $G(x)$ and prove that $G(x)$ is upper hemi-continuous at $x = 0$, but not lower hemi-continuous.
2. Suppose that $x = \mathcal{R}$ and $\Gamma(x) = Y = [0, 4]$ for all $x \in X$. Define that

$$f(x, y) = \max\{2 - (y - 1)^2, x + 1 - (y - 2)^2\}.$$

Draw the graph of $G(x)$ and prove that: $G(x)$ is upper hemi-continuous but not lower hemi-continuous, and specify at which points it is not lower hemi-continuous.

3. Suppose that $X = \mathcal{R}_+, \Gamma(x) = Y = \{y \in \mathcal{R} : -x \leq y \leq x\}$. For all $x \in X$, define that $f(x, y) = \cos(y)$, then draw the graph of

$G(x)$ and prove that: $G(x)$ is upper hemi-continuous but not lower hemi-continuous, and specify at which points it is not lower hemi-continuous.

Exercise 2.18 Let $S = \{x \in \mathcal{R}^2 : \|x\| = 4\}$ be the boundary of a circle with a radius of 2. The mapping $\psi : \mathcal{R}^2 \rightarrow S$ is defined as:

$$\psi(x) = \arg \min_{x' \in S} d(x, x'),$$

specifically, $\psi(x)$ contains the closest point in S to x . Discuss the upper and lower hemi-continuity of $\psi(x)$.

Exercise 2.19 Consider a correspondence $\Gamma : D \subseteq \mathcal{R}^l \rightarrow \mathcal{R}^k$, of which the graph is defined as

$$G(\Gamma) = \{(x, y) \in D \times \mathcal{R}^k : y \in \Gamma(x)\}.$$

If $G(\Gamma)$ is a closed set, then we say Γ has a closed graph; if $G(\Gamma)$ is a bounded and closed set, we call Γ compact-valued. Suppose that Γ is compact-valued. Prove:

1. If Γ is upper hemi-continuous, then it has a closed graph.
2. If Γ is locally bounded and its graph is closed, then Γ is upper hemi-continuous. (Hint: The definition of locally bounded correspondence $\Gamma : G(\Gamma) = \{(x, y) \in D \times \mathcal{R}^k : y \in \Gamma(x)\}$ is **locally bounded**, if for each $x \in D$, there is an $\epsilon > 0$ and a bounded set $Y(x) \subseteq \mathcal{R}^k$, such that for all $x' \in N_\epsilon(x) \cap D$, $\Gamma(x') \subseteq Y(x)$.)

Exercise 2.20 Suppose that $X \subseteq \mathcal{R}_+$ is a nonempty compact set. Prove that:

1. If $f : X \rightarrow X$ is a continuous increasing function, then f has a fixed point.
2. Specially, suppose that $X = [0, 1]$. If $f : X \rightarrow X$ is an increasing function (not necessarily continuous), does f have another fixed point?

Exercise 2.21 Suppose that X is a complete metric space, and T is the mapping from X to X . Denote

$$a_n = \sup_{x \neq x'} \frac{d(T^n x, T^n x')}{d(x, x')}, n = 1, 2, \dots$$

Prove that: If $\sum_{n=1}^{\infty} a_n < \infty$, then the mapping T has a unique fixed point.

Exercise 2.22 Consider $n \in \mathbb{N}$ and an n th order square matrix $A = (a_{ij})_{n \times n}$. For any $x \in \mathbb{R}^n$, we have

$$Ax = \left(\sum_{j=1}^n a_{1j}x_j, \sum_{j=1}^n a_{2j}x_j, \dots, \sum_{j=1}^n a_{nj}x_j \right)^T.$$

Suppose that f is a differentiable mapping from \mathbb{R} to \mathbb{R} , such that

$$s = \sup\{|f'(t)| : t \in \mathbb{R}\} < \infty.$$

Define a mapping F from \mathbb{R}^n to \mathbb{R}^n , namely,

$$F(x) = (f(x_1), \dots, f(x_n))^T.$$

For a given n -dimensional vector w , we can solve the following system of nonlinear equations:

$$z = AF(z) + w. \quad (*)$$

1. Prove that: if $\max\{\sum_{j=1}^n |a_{ij}| : i = 1, \dots, n\} < \frac{1}{s}$, then there is a unique $z \in \mathbb{R}^n$ satisfying the above system of equations $(*)$.
2. Prove that: if $\sum_{i=1}^n \sum_{j=1}^n |a_{ij}| < \frac{1}{s^2}$, then there is a unique $z \in \mathbb{R}^n$ satisfying the above system of equations. $(*)$.

Exercise 2.23 Suppose that h is a mapping from \mathbb{R}_+ to \mathbb{R}_+ , and $H : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$ is a bounded function, such that there is a $K \in (0, 1)$,

$$|H(x, y) - H(x, z)| < K|y - z|, \text{ for any } x \geq 0, y, z \in \mathbb{R}.$$

Prove that there is a unique bounded function, $f : \mathbb{R}_+ \rightarrow \mathbb{R}$, such that

$$f(x) = H(x, f(h(x))), \text{ for any } x \geq 0.$$

Exercise 2.24 Find the extremum curve of the following functional:

1. $V(y) = \int_0^1 (t^2 + y'^2) dt, y(0) = 0, y(1) = 2;$
2. $V(y) = \int_0^1 (y + yy' + y' + 0.5y'^2) dt, y(0) = 2, y(1) = 5;$
3. $V(y) = \int_0^T (1 + y'^2)^{0.5} dt, y(0) = A, y(T) = Z.$

Exercise 2.25 Solve the following optimal control problem:

$$\begin{aligned} \max \int_0^3 (x - 2)^2 (x'(t) - 1)^2 dt \\ \text{s.t. } x(0) = 0, x(3) = 2. \end{aligned}$$

Exercise 2.26 Consider the following optimal control problem, write the Hamilton equation, and solve the optimal function.

$$\begin{aligned} \max \int_0^1 (x + u) dt \\ \text{s.t. } x'(t) = 1 - u^2, x(0) = 1. \end{aligned}$$

Exercise 2.27 Consider the following optimization problem:

$$v(q) = \max_{x \in \mathcal{R}^+} \ln(2x + q) - 6x + 2q,$$

where $q \in (0, 2)$.

1. Solve $v(q)$ and its derivative $v'(q)$.
2. Verify that the Envelope Theorem holds.

Exercise 2.28 Find the general solution to the extremum curve of the following functional:

$$V(y, z) = \int_a^b (y'^2 + z'^2 + 3y'z') dt.$$

Exercise 2.29 In the problem of functional $\int_0^T F(t, y, z, y', z') dt$, suppose that $y(0) = A, z(0) = B, y_T = C, z_T = D, T$ are free, and A, B, C , and D are constants.

1. How many transversal conditions are required for the problem? Why?
2. Write these transversal conditions.

Exercise 2.30 The integrand function of the target functional is $F(t, y, y') = 4y^2 + 4yy' + y'^2$.

1. Write the Euler equation.
2. Is the above Euler equation sufficient for maximization or minimization problems? Why?

Exercise 2.31 Solve the paths of $y(t)$ and $z(t)$ of extremum curves of $V(y, z) = \int_0^T (y'^2 + z'^2)dt$ subject to $y - z' = 0$.

Exercise 2.32 Solve the optimal paths of control variables, state variables, and costate variables as follows:

1. $\max \int_0^T -(t^2 + 2u^2)dt$ subject to $y' = u, y(0) = 2$, and $y(T) = 3$, and T is free.
2. $\max \int_0^T -(u^2 + y^2 + 3uy)dt$ subject to $y' = u, y(0) = y_0$, and $y(t)$ is free.
3. $\max \int_0^4 2ydt$ subject to $y' = y + u, y(0) = 3, y(4) \geq 200$.

Exercise 2.33 Find the optimal consumption path of the following exhaustible resource problem:

$$\begin{aligned} & \max \int_0^T \ln q e^{-\delta t} dt \\ \text{s.t.} \quad & s' = -q, s(0) = s_0, s(t) \geq 0. \end{aligned}$$

Exercise 2.34 Using the revised transversal conditions expressed by the present value Hamilton function, solve the problems

1. with the end curve $y_T = \phi(t)$.
2. with truncated vertical end line.
3. with truncated horizontal end line.

Exercise 2.35 In a maximization problem, there are two known state variables, (y_1, y_2) , two control variables (u_1, u_2) , an inequality constraint, and an inequality integral constraint. The initial state is fixed, but the final state is free at fixed T .

1. State the maximization problem.
2. Define the Hamilton's equation and the Lagrange function.
3. Suppose that there is an interior solution, and then write the conditions of the maximum principle.

Exercise 2.36 Consider the problem of “eating cake” as follows. The agent has $A_0 > 0$ units of the commodity for consumption in period 0 and can save the commodity to the next period without costs, and its utility function is $\sum_{t=0}^{\infty} \beta^t \ln c_t$.

1. Write the Bellman equations of the problem.
2. Define the state variable and control variable.
3. Find the value function.

Exercise 2.37 Consider the following problem of “tree cutting” : the growth of a tree can be represented by the function h , i.e., $k_{t+1} = h(k_t)$, where k_t is the scale of the tree at time t .

There is no cost for cutting trees, and the timber price is $p = 1$. Interest rate r remains unchanged, $\beta = 1/(1 + r)$.

1. Assuming that trees cannot be replanted, we write the maximization problem of present value as $v(k) = \max\{k, \beta v[h(k)]\}$. Under what conditions about h is there a simple rule that can be used to describe when to cut trees?
2. Suppose that another tree can be planted where the original tree is cut down, and the replanting cost $c \geq 0$ remains unchanged for a long period of time. Under what conditions about h and c is there a simple rule that can be used to characterize when to cut trees?

Exercise 2.38 Solve the following dynamic programming problems by three methods: value function iteration, guessing value function, and guessing policy function, respectively:

$$\begin{aligned} & \max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \ln c_t \\ \text{s.t.} \quad & c_t + k_{t+1} = A k_t^{\alpha}, \end{aligned}$$

where k_0 is given.

Exercise 2.39 Solve the following differential equations:

1. $y' = t^2 y$.
2. $y'' - 4y' + 5y = 0$.
3. $y'' - 2y' - 3y = 9t^2$.

Exercise 2.40 Consider the following two dimensional autonomous differential equations:

$$\begin{aligned}\frac{dx}{dt} &= x(4 - x - y), \\ \frac{dy}{dt} &= y(6 - y - 3x).\end{aligned}$$

1. Solve the equilibrium of the power system.
2. Verify the stability of each equilibrium.

Exercise 2.41 Solve the following difference equations:

1. $y(t+1) - 2y(t) = 4^t$.
2. $y(t+2) + 3y(t+1) + 2y(t) = 0$.
3. $y(t+2) - y(t+1) - 6y(t) = t + 2$.

Exercise 2.42 Suppose that X_1, X_2, \dots, X_n are n independent and identically distributed random variables. The distribution function is F , and the probability density function is f . Let $Y_1^{(n)}, Y_2^{(n)}, \dots, Y_n^{(n)}$ be the corresponding order statistics satisfying $Y_1^{(n)} \leq Y_2^{(n)} \leq \dots \leq Y_n^{(n)}$.

1. Find the distribution function and probability density function of $Y_n^{(n)}$.
2. Find $E(Y_n^{(n)})$ and $\text{Var}(Y_n^{(n)})$.
3. Find $\text{Cov}(Y_1^{(n)}, Y_n^{(n)})$.

Exercise 2.43 Let X and Y be two random variables in the range $[a, b]$. Prove:

1. If X first-order stochastically dominates Y , then X necessarily second-order stochastically dominates Y .
2. If X second-order stochastically dominates Y , then $EX \geq EY$.
3. If X second-order stochastically dominates Y and $EX = EY$, then $Eu(X) \geq Eu(Y)$ for all concave and twice differentiable functions (whether increasing or not).
4. If X second-order stochastically dominates Y and $EX = EY$, then $Var(X) \leq Var(Y)$.

Exercise 2.44 Suppose that X is a non-negative random variable, and the distribution function and density function are F and f , respectively. The risk rate of random variable X is defined as

$$\lambda_X : \mathcal{R}_+ \longrightarrow \mathcal{R}_+, \quad \lambda_X(t) = \frac{f(t)}{1 - F(t)}.$$

If $\lambda_X(\cdot) \leq \lambda_Y(\cdot)$, we say that the random variable X stochastically dominates random variable Y in terms of risk rate. Suppose that G and g are, respectively, the distribution function and density function of random variable Y . If $f(\cdot)/g(\cdot)$ is a non-decreasing function, then we say that X stochastically dominates Y in terms of likelihood ratio. Prove the following statements:

1. $\lambda_X(\cdot) \leq \lambda_Y(\cdot)$ if and only if $1 - G(t)/[1 - F(t)]$ is a non-increasing function.
2. If X dominates Y in terms of likelihood ratio, then X must stochastically dominate Y in terms of risk rate.

Exercise 2.45 Prove that: if X_1, X_2, \dots, X_N are correlated, and $\gamma(\cdot)$ is an increasing function, then for $x'_1 > x_1$, we have

$$E[\gamma(Y_1)|X_1 = x'_1] \geq E[\gamma(Y_1)|X_1 = x_1].$$

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Part III

Game Theory and Market Theory

This part discusses game theory and market theory. Game theory has become an extremely important subdiscipline in mainstream economics, a core field in microeconomic theory, and one of the most important analytical tools for investigating various economic issues. Readers will find that the market theory introduced in this part involves abundant knowledge and results in game theory, and thus we will present them as applications of game theory.

Game theory studies the strategic interactions between individuals. In earlier chapters, we studied the optimal choice of an individual, such as a firm or a consumer — the simplest case, and we assumed how individuals make optimal decisions when they are not affected by other individuals' decisions. However, in many situations, this is certainly not realistic. In reality, individuals' decisions are frequently more complicated. One aspect is that their decisions tend to affect each other. The game theory introduced in this part examines how interactions among individuals affect their outcomes or payoffs. There are many aspects from which one can study interactions of decision-makers. For example, one could investigate behaviors from the perspectives of sociology, psychology, biology, etc. Each of these approaches is useful in certain contexts. Game theory emphasizes the study of strictly “rational” decision-making, since this may constitute the most appropriate approach for economic behaviors and activities in which “business is business” .

Game theory can assist us to understand the phenomenon of individuals' interaction and its underlying mechanism. In the previous part, we have discussed the individual's optimal decision, assuming that the decision-maker is rational. In game theory, a stronger assumption is required concerning decision-makers' rationality. It not only requires that a decision-maker be rational, but also requires the decision-maker to assume that other decision-makers are rational and that others also think that he or she is rational. In other words, it is common knowledge that all decision-makers are rational. Game theory studies the strategic interactions among individuals in this context.

Game theory has two branches: non-cooperative game and cooperative game (sometimes called coalition game). These two branches are not divid-

ed literally. The former studies non-cooperative relations of individuals, while the latter investigates cooperation among individuals. In numerous situations, we will find that non-cooperative game frequently studies the mechanism of cooperation of players, and cooperative game often studies non-cooperative behaviors of players. Taking cost-sharing as an example, all players hope that the other party will bear more of the cost. As a consequence, cooperative game and non-cooperative game are not reflected in research objects, but instead are reflected in assumptions.

Cooperative game assumes that individuals communicate prior to making their decisions, and that the decisions are reflected in the choice of contract. Once a contract is chosen, it will be followed by a coalition (i.e., the group of individuals who have signed the contract). In this way, the analysis is based on the collective unit, and thus cooperative games are sometimes called coalition games. Corresponding to this, in non-cooperative games, players' communication, choice of contract, and compliance with contracts are all based on individual rational decision-making. In addition, neither of the theories is superior or inferior to the other. Instead, they analyze different issues on different levels of analysis and in dissimilar contexts. Overall, they are complementary in our understanding of practical problems.

In the development history of game theory, John von Neumann (1903–1957, see his biography in Section 5.8.1) is recognized as the founder of game theory. His work with Morgenstern (1944) marked the occasion when game theory become a subdiscipline. In its early stages, game theory was considered more of a branch of mathematics, and especially a branch of operations research.

Game theory has a broad range of applications. It is an abstract description of how individuals make rational decisions in real life, and thus can be employed to study various aspects of the economy, society, and politics. Any phenomenon that involves strategic interactions, such as competition among firms in the market, voting in the political system, lobbying of interested groups, war and disarmament, and even the evolution of species in ecosystems, can be investigated using game theory. Due to its influence and importance, John Nash, John Harsanyi, and Reinhard Selten in 1994,

Robert J. Aumann and Thomas C. Schelling in 2006, and Lloyd S. Shapley in 2012 were awarded the Nobel Prize in Economics, respectively. Their biographies can be found in Section 6.8.1, Section 6.8.2, Section 8.5.2, Section 8.5.1, Section 20.6.2, and Section 22.5.1.

Game theory, to be discussed in this part, is somewhat technical and abstract. However, various game theoretical models and results are widely used in many fields of economics, including the discussion of market theory that will be also discussed in this part. The core issue of market theory is pricing, i.e., how to determine the market equilibrium price and quantity. We will discuss the basic structure of four major markets: perfect competition, monopoly, monopolistic competition, and oligopoly. Subsequently, we will examine how the actions of consumers and firms affect market efficiency when they interact in the market. Game theory is involved extensively in the study of oligopolistic behaviors.

This part consists of four chapters: Chapter 6 discusses non-cooperative games, Chapter 7 explores repeated games and reputation mechanisms, Chapter 8 examines cooperative games, and Chapter 9 proceeds to market theory.

Chapter 6

Non-Cooperative Game Theory

6.1 Introduction

This chapter is organized as follows: Section 6.2 introduces basic concepts of non-cooperative games, including the components of a game, two form representations of a game, pure strategy, mixed strategies, and behavior strategies. Section 6.3 discusses static games with complete information and their solution concepts, including dominant strategy equilibrium, iterated elimination of strictly dominated strategy equilibrium (IESDSE), Nash equilibrium, and refinements of Nash equilibrium. Section 6.4 discusses dynamic games with complete information and their solution concepts, including subgame perfect equilibrium and backward induction. Section 6.5 explores static games with incomplete information and their solution concepts, including Bayesian game and Bayesian Nash equilibrium. Section 6.6 discusses dynamic games with incomplete information and their solution concepts, including weak perfect Bayesian equilibrium and sequential equilibrium. Section 6.7 explores the existence of Nash equilibrium.

6.2 Basic Concepts

This section presents basic terminologies used in game theory to depict real-world interactions among individuals and discusses the assumptions behind game theory. In game theory, there are usually two ways of de-

scribing the interactions of players, i.e., the **normal form** (also called the strategic form) and the **extensive form**. These two forms possess distinct advantages in expressing different games and are normally interchangeable.

To describe a situation of strategic interaction, we need to know four things:

- (1) **Players:** Who are involved in the game? It is assumed that players are rational, ie., the goal is to maximize their own utilities/payoffs.
- (2) **Rules:** Who moves and when? What information do the players possess when taking actions? What actions can the player choose?
- (3) **Outcomes:** What is the sequence of the game for each possible set of actions by players? A primary purpose of game theory is to determine the outcomes of games according to a solution concept, such as those outcomes of equilibrium strategy profile, or equilibrium action profile.
- (4) **Payoffs:** What are the benefits or utilities over possible outcomes? A payoff profile is the utility levels of all players under a certain outcome.

A central concept of game theory is the notion of player's strategy. A **strategy** is a decision rule of actions or complete contingent plan which depends not only on players' own actions but also on the actions of others. More formally, it is a mapping from the collection of information sets to the set of actions.

It is worth noting that strategy and action are generally two different concepts, because a strategy is the rule of actions but not the actions per se. However, for the normal form game, since each information set is a single element set, a (pure) strategy is identical with an action.

6.2.1 Strategic Form Representation of Games

The strategic/normal form of the game is frequently used to describe players' interactions when making choices. Suppose that every decision-maker

chooses his or her action only once. Then, the strategic form of a game has the following three elements:

- (1) **The set of players** $N = \{1, 2, \dots, n\}$;
- (2) **Strategy space** $S = S_1 \times S_2 \times \dots \times S_n$. Each player $i \in N$ has a set of strategies S_i , and the strategic choices of all players $s = (s_i)_{i \in N}$ constitute a strategy profile, and outcome of the game; and
- (3) **Payoff or utility functions** $u_i : S \rightarrow \mathcal{R}, i \in N$. Payoffs are usually represented by a payoff matrix if strategies are finite and there are two players (more complicated nested payoff matrix forms may be used if there are more than two players).

We describe below the strategic form representation of games through examples.

Example 6.2.1 (Rock-Paper-Scissors Game) Two brothers, A and B , employ the usual Rock-Paper-Scissors game to determine ownership of the 10 dollars given to them by their parents.

The rules of the game: Each one's hand forms one of three shapes (rock, paper, or scissors) simultaneously to determine the winner. Rock beats scissors, scissors beats paper, and paper beats rock. The winner will obtain 10 dollars, and the loser will obtain 0 dollars. If the game is tied, i.e., they choose the same shape, then each A and B will obtain 5 dollars.

		Player B		
		Rock	Paper	Scissors
Player A	Rock	5, 5	0, 10	10, 0
	Paper	10, 0	5, 5	0, 10
	Scissors	0, 10	10, 0	5, 5

Table 6.1: Strategic Form Representation of Rock-Paper-Scissors Game

From this game, we inspect the elements of a game. The set of players is $N = \{A, B\}$. The set of strategies for players A and B are the same: {rock, paper, scissors}, and there are 9 possible strategy profiles in this game (see Table 6.1). The corresponding payoff of each outcome is represented by the

payoff matrix shown in the table. For example, the corresponding payoff of strategy profile (rock, scissors) is $(10, 0)$, where 10 represents the payoff obtained by player A under this outcome, and 0 represents the payoff obtained by player B under this outcome.

Note that here we assume that each player's utility function is $u_i(x(s)) = x$, $i = A, B$, where x is the dollars obtained. If the utility function is $u_i(x(s)) = x^{\frac{1}{2}}$ for $i = A, B$, then under the outcome of strategy profile (scissors, rock), the payoff profile is $(0, 10^{\frac{1}{2}})$.

The game in this example is a **constant-sum game**, i.e., the sum of players' payoffs is a constant; the higher payoff one player obtains means the lower payoff the other player will obtain; these players are in a confrontational relationship in the game. When the constant is zero, the game is called a **zero-sum game**, which is a specific case of a constant-sum game.

In fact, the non-cooperative game can be used to describe many cooperative relationships, such as the following example.

Example 6.2.2 (Meet at the Restaurant) Two individuals, Tom (T) and Schelling (S), decide to meet and have lunch together at noon. They have forgotten the exact place to meet, and only know that there are two possible places, i.e., Restaurant 1 and Restaurant 2. They left home so hurriedly that they forgot to take their mobile phones. They can only choose one place. If they happen to go to the same restaurant, they can eat together, and their utility levels in this situation are both 10; otherwise, they can only eat alone, and their utility levels in this situation are both 0. (see Table 6.2)

The elements of this game are: the set of players is $N = \{T, S\}$; the sets of strategies for Tom (T) and Schelling (S) are both $\{\text{Restaurant 1, Restaurant 2}\}$; there are four possible outcomes. For example, in the outcome (Restaurant 1, Restaurant 1), the payoff of each player is 10.

		Schelling	
		Restaurant 1	Restaurant 2
Tom	Restaurant 1	10, 10	0, 0
	Restaurant 2	0, 0	10, 10

Table 6.2: Strategic Form Representation of Meeting at the Restaurant

In this example, the interaction between these two players is actually a situation of seeking cooperation.

In the above two examples, players' actions happen to be their strategies. However, in many situations, one player may make multiple decisions in one game. A strategy is then a complete contingent action plan for a player in all possible situations. Therefore, if there are multiple decisions, as well as a description of the decisions of different players in different time structures, a more effective representation is the tree-view extensive form.

6.2.2 Extensive Form Representation of Games

The extensive form representation of a game specifies players, decision rules, outcomes, and payoffs: the players of the game, when each player has the move, what each player can do at each of their moves, what each player knows for every move, and the payoff (or utility level) received by each player for every possible outcome.

An extensive form game, denoted as

$$\Gamma_E = (N, \bar{N}, W, X, Z, p, H, \iota(\cdot), (u_i(\cdot))_{i \in N}),$$

like a tree, has the following elements:

- (1) **The set of players.** In addition to the actual participants $N = \{1, 2, \dots, n\}$ involved in the interaction, there may be some external events that are uncertain, and we usually add an additional player “Nature” (\bar{N}) who determines the probability distribution of external events, in addition to the actual players. It can be understood that which event will occur is decided by throwing a dice.
- (2) **Order of moves.** The order of moves is represented by a game tree that consists of a finite set of ordered nodes Γ_E and a precedence relation \prec on the set. The precedence relation \prec describes the order of the nodes, and satisfies asymmetry and transitivity so that it is a partial order. $P(y) = \{y' \in \Gamma_E : y' \prec y\}$ is the set of all nodes preceding $y \in \Gamma_E$, which we call **the set of predecessors** of y ; $S(y) = \{y' \in \Gamma_E :$

$y \vec{\rightarrow} y'$ is the set of all nodes succeeding y , which we call **the set of successors** of y ; $W = \{y \in \Gamma_E : P(y) = \emptyset\}$ (\emptyset represents an empty set) is the **initial node** of the game tree; $Z = \{y \in \Gamma_E : S(y) = \emptyset\}$ is the set of **terminal nodes** of the game tree; $X = \{x \in \Gamma_E : x \notin Z\}$ represents the set of non-terminal nodes, which we call **the set of decision nodes**. Assume that for each $x \in X \setminus W$, there is a *unique* immediate predecessor $p(x) \in P(x)$.

- (3) **The correspondences.** The set of decision nodes to the set of players (including Nature), $\iota : X \rightarrow \{\bar{N}, 1, 2, \dots, n\}$, indicates the player that makes a decision at each decision node.
- (4) **Action sets.** The set of a player's choices at a decision node x is called the **action set** at that node, denoted by $A(x)$, which may be a finite, infinite, or even continuum set.
- (5) **The collection of information sets.** A set of decision nodes among which a player cannot distinguish is called **an information set**, i.e., for any $x \in X$, there is a corresponding non-empty set $h(x)$, such that if $x' \in h(x)$, then $x \in h(x')$. Different information sets contain different nodes. The decision nodes in the same information set are linked with a dashed line, indicating that the decision-maker does not know exactly which decision node to act on. The set of all information sets is denoted as H , which forms a partition of X , i.e., for $h, h' \in H$, either $h = h'$ or $h \cap h' = \emptyset$. If all information sets in a game tree are singletons, then the game is called a **perfect information game**.
- (6) **Outcomes.** The actions chosen by all players in each information set determine the outcome of the game, i.e., a terminal node $z \in Z$. Each player (except Nature) is assigned a payoff at each outcome $u_i(\cdot) : Z \rightarrow \mathcal{R}, \forall i \in N$.
- (7) **External events.** At the initial node W , there is a probability distribution $\rho : W \rightarrow [0, 1]$, which can be interpreted as "Nature" 's choices.

In an extensive form game, a strategy of a player is his or her complete contingent action plan for making decisions on each possible information

set, i.e.,

$$S_i = \prod_{h \in H: \iota(h)=i} A(h),$$

where $A(h)$ is the set of actions at information set h . Then the total number of pure strategies he or she can choose is equal to the multiplication of the numbers of pure strategies of all action sets, i.e.,

$$|S_i| = \prod_{h \in H: \iota(h)=i} |A(h)|.$$

For instance, if player i has two information sets, among which one set has three actions and the other set has two actions to choose, then the pure strategy number of the player's strategy set is 6.

Below, we utilize an example to illustrate that the extensive form representation can describe interaction behaviors in more detail.

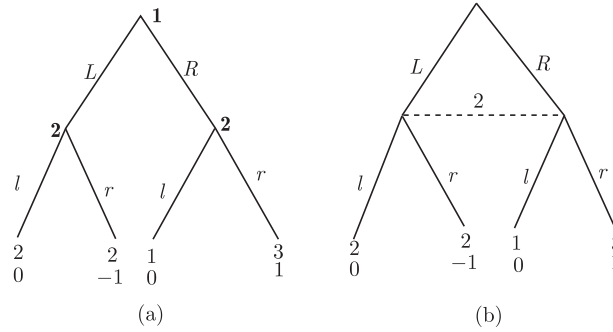


Figure 6.1: (a): Game 1, player 2 knows player 1's choice; (b): Game 2, player 2 does not know player 1's choice.

Example 6.2.3 The following Game 1 and Game 2 describe interactions in two different situations (See Figure 6.1).

Game 1. The set of players contains two elements: player 1 and player 2. The game has two action stages. In the first stage, player 1 makes a decision, and at this stage his action set is L and R . Since player 1 only acts once, his strategy set is exactly the action set. Then, player 2 makes a decision. When making a decision, she can observe player 1's different actions, and thus player 2 has two perfect (single node)

Player2 Player1	$a_2^1 = (l, l)$	$a_2^2 = (r, r)$	$a_2^3 = (l, r)$	$a_2^4 = (r, l)$
$a_1^1 = L$	2,0	2,-1	2,0	2,-1
$a_1^2 = R$	1,0	3,1	3,1	1,0

(a) Game 1

Player2 Player1	$a_2^1 = l$	$a_2^2 = r$
$a_1^1 = L$	2,0	2,-1
$a_1^2 = R$	1,0	3,1

(b) Game 2

Table 6.3: Table (a): the strategic form representation of Game 1 (player 2 knows player 1's choice); Table (b): the strategic form representation of Game 2 (player 2 does not know player 1's choice).

information sets. A strategy of player 2 is constituted by decisions made on each of her information sets. In each information set, player 2's action set is $\{l, r\}$. Therefore, there are four possible outcomes of strategy profiles, i.e., (l, l) , (l, r) , (r, l) , and (r, r) . For example, strategy profile (l, l) indicates that player 2 selects l on both of her information sets. Once players 1 and 2 have chosen their strategies, there will be an payoff profile. For each strategy profile outcome, the corresponding payoffs of players 1 and 2 are assigned. The four terminal nodes in Figure 6.1(a) are the payoff profiles (usually the upper number is player 1's payoff, and the lower number is player 2's payoff).

Game 2. The player set is the same as in Game 1. The game has two action stages. The first stage is the same as in Game 1. However, the second stage differs from that of Game 1. We link the two decision nodes of player 2 together with a dashed line, indicating that player 2 does not know the actual action of player 1 in the first stage when making a decision. Therefore, the information that player 2 has at these

two decision nodes is indistinguishable, i.e., player 2 only has one information set. In other words, when player 2 makes a decision, he or she does not know whether he or she is at the left or the right decision-making node. For this reason, a strategy of player 2 is to choose an action in the unique information set, and thus player 2 only has two possible strategies $\{l, r\}$. Once players 1 and 2 have chosen their respective strategies, an outcome will be reached, along with their corresponding payoffs.

Tables (a) and (b) in Table 6.3 illustrate the strategic forms corresponding to Game 1 and Game 2 in Figure 6.1. Usually, two forms of a game are interchangeable.

From Table 6.3, we know that there are differences between Game 1 and Game 2, where the key difference arises from player 2's strategies, which is the outcome of the information status of player 2 when making her decision.

In an extensive form game, the game tree is common knowledge for each player, i.e., all players know the game tree, all players know that all players know the game tree, etc. In an extensive form game, it is usually necessary to satisfy the requirement of **perfect recall**, i.e., when making the current decisions, players know the information that they had when making the previous decisions, as well as their previous decisions.

Figure 6.2 depicts an **imperfect recall** situation. In Figure 6.2, player 2 cannot distinguish between the two decision-making nodes when making her second decision. This means that player 2 cannot recall whether her first decision was R or L . If player 2 has perfect recall, however, he or she can distinguish them. In addition, in an extensive form game, there may be external uncertainties. Usually, we introduce "Nature" as the decision-maker to select external uncertain events.

Example 6.2.4 (Matching Pennies) There are two players 1 and 2 who play a matching pennies game, i.e., choosing heads or tails. If these two players have the same choice, then player 1 pays one dollar to player 2; otherwise, player 2 pays one dollar to player 1. Suppose that the game is played as

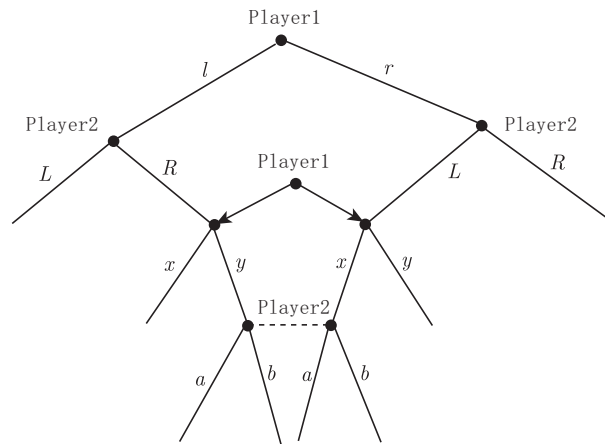


Figure 6.2: Game without Perfect Recall

follows: first, by tossing a coin, if the head side is face-up, player 1 chooses first and then player 2 chooses; if the tail side is face-up, their selection order is reversed. The player selecting later knows the action of the previous player. In this game, there is an external uncertainty, i.e., who chooses first. We introduce a new player “Nature”, and let it act as a decision-maker for external events. Figure 6.3 describes this game.

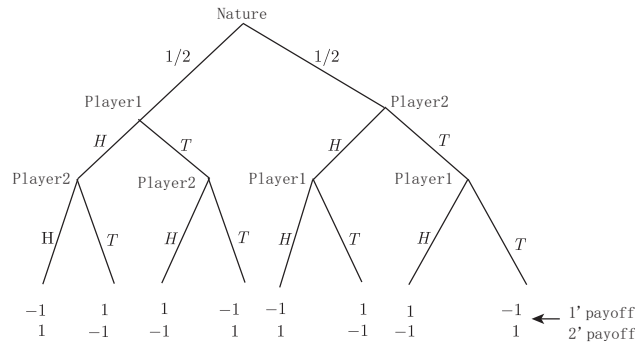


Figure 6.3: “Nature” selects the order of the game.

6.2.3 Mixed Strategies and Behavior Strategies

In the above, a strategy of a player is defined as a complete plan for him in a game. In the Rock-Paper-Scissors game, players A and B both have

3 strategies. Obviously, a player is reluctant to let the other player know his or her choice. In numerous interactive situations, a player introduces random factors to prevent the other players from knowing his or her exact choice. Randomizing pure strategies with these factors generates **mixed strategies**.

Consider a normal form game $\Gamma_N = [N, \{S_i\}, \{u_i(\cdot)\}]$, in which the strategy set of player i is $S_i = \{s_i^1, \dots, s_i^{n_i}\}$, i.e., player i has n_i strategies, and a mixed strategy σ_i of player i is a probability distribution on $S_i = \{s_i^1, \dots, s_i^{n_i}\}$, i.e., $\sigma_i : S_i \rightarrow [0, 1]^{n_i}$, where $\sigma_i(s_i^k) \geq 0$ indicates the probability of player i choosing strategy s_i^k , which satisfies $\sum_{k=1}^{n_i} \sigma_i(s_i^k) = 1$. Consequently, a pure strategy can be seen as a mixed strategy in an extreme situation, i.e., a mixed strategy in which the probability of selecting a certain strategy is 1.

If we use an extensive-form description of a game, there is another way that player i could randomize. Rather than randomizing over the entailed set of pure strategies in S_i , the player could randomize separably over the possible actions at each of his or her information sets H . This way of randomizing actions at each information set is termed a **behavior strategy**. While the concepts of mixed strategy and behavior strategy are very closely related in the context of normal-form games, they have very different implications for extensive-form games. Roughly, a mixed strategy specifies a randomization over complete paths of play through the game tree, while a behavior strategy specifies an independent probability distribution over actions at *each information set* h .

Formally, for each information set $h \in H$, define $\Lambda(A(h))$ as the probability distribution space on action set $A(h)$ on information set h . For each player $i \in N$, the choices in all information sets constitute a behavior strategy, and the behavior strategies of all players constitute a behavior strategy profile $\sigma = (\sigma_h)_{h \in H}$, where σ_h represents the behavior strategy of $i(h)$ on information set h . Starting from a behavior strategy, we can define a mixed strategy for player i , i.e., $\sigma_i = \prod_{h \in H: i(h)=i} \sigma_h$. All action plans over information sets belonging to player i constitute a mixed strategy for player i .

If all players choose mixed strategies, the expected payoff (utility) of

player i is

$$u_i(s) = \sum_{s \in S} [\sigma_1(s_1) \sigma_2(s_2) \dots \sigma_n(s_n)] u_i(s), \quad (6.2.1)$$

i.e., utility from strategy s , times the probability of the occurrence of s , summed over all $s \in S$.

In order to have an intuitive understanding how to get the players' expected payoffs, consider $n = 2$ and the expected payoff of player 1 which is given by

$$\begin{aligned} Eu_1(s) &= \sum_{l=1}^{n_1} \sum_{k=1}^{n_2} \sigma_1(s_1^l) \sigma_2(s_2^k) u_1(s_1^l, s_2^k) \\ &= \sigma_1(s_1)' U_1 \sigma_2(s_2), \end{aligned} \quad (6.2.2)$$

where the payoff matrix of player 1 is

$$U_1 = \begin{bmatrix} u_1(s_1^1, s_2^1) & u_1(s_1^1, s_2^2) & \dots & u_1(s_1^1, s_2^{n_2}) \\ u_1(s_1^2, s_2^1) & u_1(s_1^2, s_2^2) & \dots & u_1(s_1^2, s_2^{n_2}) \\ \dots & \dots & \dots & \dots \\ u_1(s_1^{n_1}, s_2^1) & u_1(s_1^{n_1}, s_2^2) & \dots & u_1(s_1^{n_1}, s_2^{n_2}) \end{bmatrix}.$$

(6.2.2) can be used to easily compute the expected payoffs and solve equilibrium strategies of players.

As different interaction situations have different characteristics, in the introduction of non-cooperative games below, non-cooperative games will be divided into four basic types based on static games and dynamic games of complete or incomplete information. The repeated game is a special type of dynamic game. As this type of game exhibits a special structure, we will discuss it separately in the next chapter.

6.3 Static Games with Complete Information

Static games of complete information are the simplest type of game, in which knowledge about other players is available to all players, (i.e., the payoffs, strategy space and “types” of players are common knowledge), and each player only makes one decision. In this type of game, the action set of a player is precisely his or her strategic set.

6.3.1 Dominant and Dominated Strategies

Individuals are rational in playing a game, which implies that individuals will attempt to avoid unfavorable choices. Therefore, prior to discussing the interactions of players, we first introduce two concepts of strategies—**dominant** and **dominated** strategies, which indicate how a player can circumvent adverse consequences.

Example 6.3.1 (Prisoner’s Dilemma) Two prisoners have been accused of collaborating in a crime. They are in separate jail cells and cannot communicate with each other. Each has been asked to confess, and are subjected to a policy of being “lenient to those who confess their crimes and severe to those who refuse to”. If both prisoners choose to deny, the prosecution’s case will be difficult to make, and then they will be assigned a lesser charge in the insufficiency of evidence. In this case, both will receive 2 years in jail. If only one of them confesses, then the confessor receives lighter sentence and the other one will be punished severely. In this situation, the confessor will receive 1 year in jail, and the other one will go to jail for 10 years. If both confess about the crimes that they committed, and the criminal acts are conclusive, then both will be assigned heavier charges. In this case, both of them will receive 5 years in jail. The participant’s utility levels for t years of imprisonment is $-t$. According to the normal-form representation of games, Table 6.4 describes the interaction between the two suspects.

		Prisoner 2	
		Confess	Deny
Prisoner 1	Confess	-5, -5	-1, -10
	Deny	-10, -1	-2, -2

Table 6.4: The Prisoner’s Dilemma

At first glance, both players should choose to deny, and this is a better choice for both players. However, the rational assumption implies that a players only pursue their own utility maximization. It is not difficult to see that “Confess” is always better than “Deny” because if the other player chooses “Deny”, your choice of “Confess” will result in a payoff of -1 ,

greater than -2 , which is the payoff if you alternatively choose “Deny” ; if the other player chooses “Confess” , your choice of “Confess” will result in a payoff of -5 , greater than -10 , which is the payoff if you alternatively choose “Deny” . In this way, regardless of what the opponent chooses, choosing “Confess” is always in the best interest of a player. This type of strategy is called a **dominant strategy**. For these two prisoners, however, it is better for them to choose “Deny” together. This kind of collective irrationality resulting from individual rationality is known in economics as the Prisoner’s Dilemma, and sometimes is also known as the Prisoner’s Paradox.

Definition 6.3.1 (Strict Dominant Strategy) A strategy $s_i \in S_i$ is a *strictly dominant strategy* for player i in game $\Gamma_N = [N, \{S_i\}, \{u_i(\cdot)\}]$ if for all $s_i' \neq s_i$, we have

$$u_i(s_i, s_{-i}) > u_i(s_i', s_{-i})$$

for all $s_{-i} \in S_{-i} \triangleq S_1 \times \cdots \times S_{i-1} \times S_{i+1} \times \cdots \times S_n$.

Another concept associated with this concept is termed the *dominated strategy*.

Definition 6.3.2 (Strict Dominated Strategy) A strategy $s_i \in S_i$ is a *strictly dominated strategy* for player i in game $\Gamma_N = [N, \{S_i\}, \{u_i(\cdot)\}]$ if there is another strategy $s_i' \neq s_i$, such that for all $s_{-i} \in S_{-i}$,

$$u_i(s_i', s_{-i}) > u_i(s_i, s_{-i}).$$

In this case, we say that strategy s_i' strictly dominates strategy s_i .

With this definition, a strategy $s_i \in S_i$ is a strictly dominant strategy for player i in game $\Gamma_N = [N, \{S_i\}, \{u_i(\cdot)\}]$ if and only if it strictly dominates every other strategy in S_i .

The following is a weak version of a dominant strategy.

Definition 6.3.3 (Weak Dominant Strategy) A strategy $s_i \in S_i$ is a *weakly dominant strategy* in game $\Gamma_N = [N, \{S_i\}, \{u_i(\cdot)\}]$ if it weakly dominates

every other strategy in S_i , i.e., for every $s_i' \neq s_i$,

$$u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i})$$

for all $s_{-i} \in S_{-i}$ with strict inequality for some s_{-i} .

Similarly, we have

Definition 6.3.4 (Weak Dominated Strategy) A strategy $s_i \in S_i$ is *weakly dominated* in game $\Gamma_N = [N, \{S_i\}, \{u_i(\cdot)\}]$ if there is another pure strategy $s_i' \neq s_i$, such that for all $s_{-i} \in S_{-i}$,

$$u_i(s_i', s_{-i}) \geq u_i(s_i, s_{-i})$$

with strict inequality for some s_{-i} . In this case, we say that strategy s_i' *weakly dominates* strategy s_i .

If every player has a strictly dominant strategy, then we call the profile of all players' strictly dominant strategies a **strictly dominant strategy equilibrium**. If every player has a weakly dominant strategy, then we call the profile of all players' weakly dominant strategies a **dominant strategy equilibrium**. Formally, we have

Definition 6.3.5 (Strict Dominant Strategy Equilibrium) A strategy profile (s_1, s_2, \dots, s_n) is a *strictly dominant strategy equilibrium* of $\Gamma_N = [N, \{S_i\}, \{u_i(\cdot)\}]$ if for all i , $s_i \in S_i$ is a strictly dominant strategy.

Definition 6.3.6 (Dominant Strategy Equilibrium) A strategy profile (s_1, s_2, \dots, s_n) is a *dominant strategy equilibrium* of $\Gamma_N = [N, \{S_i\}, \{u_i(\cdot)\}]$ if for all i , $s_i \in S_i$ is a weakly dominant strategy.

Since the players are rational, whenever there is a strictly dominant strategy for player i in a game, player i will choose it. In the prisoner's dilemma example above, since "Confess" is a strictly dominant strategy for both players, these two prisoners will choose the "Confess" strategy in the rational interaction. Therefore, the strategy profile ("Confess" , "Confess") is a strictly dominant strategy equilibrium.

If a player has a strictly dominant strategy in a game, he or she must have only one strictly dominant strategy, and all other strategies are strictly dominated strategies. As long as the player is rational, he or she will not choose strictly dominated strategies. Therefore, when considering players' decision-making behavior, we can narrow their action sets by the **iterated elimination of strictly dominated strategies (IESDS)**, which is exactly what its name suggests: we iteratively eliminate strictly dominated strategies, yielding at each stage a smaller subset of surviving strategies. IESDS is a common technique for solving games that involves iteratively eliminating dominated strategies. If there is only one strategy profile remaining after iterated elimination of strictly dominated strategies, the strategy profile should be the equilibrium outcome of the game and is called the iterated elimination of strictly dominated strategy equilibrium (IESDSE).

Example 6.3.2 Consider the game described at the top of Table 6.5. Player 1 has three strategies $\{T, M, B\}$, and player 2 also has three strategies $\{L, C, R\}$.

In the initial game, since M is a strictly dominated strategy for player 1, it is impossible for player 1 to choose strategy M , and the game after eliminating the strictly dominated strategy is described in Table 6.5(b). In this game, C is a strictly dominated strategy for player 2, and the game after eliminating the strictly dominated strategy is described in Table 6.5(c). In this game, T is a strictly dominated strategy for player 1, and the game after eliminating the strictly dominated strategy is described in Table 6.5(d). In this game, L is a strictly dominated strategy for player 2. In the game after eliminating the strictly dominated strategy, as described in Table 6.5(e), only one strategy profile remains. This strategy profile is the iterated elimination of strictly dominated strategy equilibrium (IESDSE) of the game.

Example 6.3.3 (Father Objecting Daughter's Marriage Game) Imagine a girl in a rich family falling in love with a poor boy. The father does not think it is a good match. He threatens his daughter, and says, "If you marry the poor boy, I will sever family ties with you". A normal-form game can be employed to describe the game between the father and the daughter. Here, the daughter has two strategies: "Give In" and "Don't Give In"; the

		Player 2		
		L	C	R
Player 1	T	1, 1	2, 0	1, 1
	M	0, 0	0, 1	0, 0
	B	2, 1	1, 0	2, 2

		Player 2		
		L	C	R
Player 1	T	1, 1	2, 0	1, 1
	B	2, 1	1, 0	2, 2

		Player 2	
		L	R
Player 1	T	1, 1	1, 1
	B	2, 1	2, 2

		Player 2	
		L	R
Player 1	B	2, 1	2, 2

		Player 2
		R
Player 1	B	2, 2

Table 6.5: Elimination of Strictly Dominated Strategy

father also has two strategies: “Agree” and “Disagree”. If the daughter chooses “Give In” and the father chooses “Agree”, then the father gets the best of both worlds (neither losing his daughter nor accepting the marriage), and the daughter loses her boyfriend. If the daughter chooses “Give In” and the father chooses “Disagree”, then the father loses his daughter and the daughter loses everything. If the daughter chooses “Don’t Give In” and the father chooses “Agree”, then the father has to accept the poor boy, and the daughter gets the best of both worlds (neither losing her boyfriend nor her father). If the daughter chooses “Don’t Give In” and the father chooses “Disagree”, then the father loses his daughter, and the daughter loses her father.

Therefore, the game matrix shown in Table 6.6 can be obtained. What

is the final equilibrium outcome? First, it can be seen that for the daughter, “Give In” is a strictly dominated strategy. Irrespective of what strategy the father chooses, “Don’t Give In” is a strictly dominant strategy for the daughter. If the daughter chooses “Don’t Give In”, then the father’s best response is to choose “Agree”, because accepting the poor boy is always better than losing his daughter!

Father:	Daughter	
	Given In	Don’t Give In
Agree	the best of both, losing boyfriend	enduring poor boy, the best of both
Disagree	losing daughter, losing everything	losing daughter, losing father

Table 6.6: Father Objecting to Daughter’s Marriage Game

This example illustrates why in reality most fathers’ efforts to resist their daughters’ marriage end in failure. The reason for this is that the father’s threat of cutting off family ties is not credible. For the daughter, “Don’t Give In” is a dominant strategy: losing her father is better than losing everything (losing her father and her boyfriend), and the best of both worlds is better than losing her boyfriend. Indeed, a large number of actual phenomena in reality show that once the daughter and the poor boy go home with a grandson, the father often forgives his daughter. This further shows that “Don’t Give In” is the optimal strategy for the daughter whose father opposes her marriage. The ideas shown in this example can also be utilized to study whether the threat of a price war is credible. We will return to the discussion of credibility issues later.

Example 6.3.4 (Boxed Pigs (or Rational Pigs) Game) Imagine that two clever pigs, one Big Pig and one Piglet, live together in a pigpen. There is a lever on one side of the pigpen and a device that can provide food to the pigs on the other side. The device produces food only if the pigs press the lever. Pushing a lever by one pig yields 10 units of food, the other pig will get the chance to run to the food earlier. Pushing “costs” either pig 2 units of food. Because the Big Pig is bigger than the Piglet, it eats faster too. Each pig can choose “Push” or “Not Push”. There are four possible outcomes:

- (1) Both pigs choose to “Not Push” . Then, there is no food, and the payoff is 0 for each pig.
- (2) The Big Pig chooses to “Push” , and the Piglet chooses to “Not Push” . The Big Pig, delayed by the action of pushing, eats quickly and consumes five units of food. The Piglet, not delayed, eats slowly and also consumes five units of food. After deducting the energy cost, the Big Pig gains 3 units of food, and the Piglet who does not pay any physical cost gets 5 units of food as payoff.
- (3) The Piglet chooses to “Push” , and the Big Pig chooses to “Not Push” . When the Piglet arrives at the other end of the pigpen, the Big Pig has already eaten nine units of food. The Piglet can only eat 1 unit of food, but it has to pay 2 units of energy cost. Therefore, the payoff of the Piglet is -1 .
- (4) Both pigs choose to “Push” . In this case, the two pigs come back to eat at the same time. The Big Pig consumes 7 units of food, while the Piglet can only eat 3 units of food. After deducting their costs, the Big Pig and the Piglet gain 5 units and 1 unit of food, respectively.

		Piglet	
		Push	Not Push
Big Pig	Push	5, 1	4, 4
	Not Push	9, -1	0, 0

Table 6.7: Boxed Pig Game

Therefore, we have the payoff matrix shown in Table 6.7 for this Boxed Pig's Game.

In this example, for the Piglet, “Not Push” is a strictly dominant strategy, and “Push” is a strictly dominated strategy. In other words, whether or not the Big Pig pushes the lever, it is better for the Piglet not to push. On the other hand, the Piglet is known not to push the lever, and thus pushing the lever is better for the Big Pig. The basis for the existence of the Boxed

Pig's Game is that both sides cannot escape the coexistence situation, and that there must be a party that has to pay a cost in exchange for the interests of both parties. Once one side's strength is sufficiently strong to disrupt this balance, the coexistence situation no longer exists, the expectation will be reset, and the situation of the Boxed Pig's Game will also be solved.

The Boxed Pig's Game has wide implications. For example, if a new product has just entered the market and its performance and function are not well known, and if there are other firms with more production capacity and stronger marketing capability, then it is not necessary for small firms to invest too much in advertising for product promotion. In this case, small firms need simply to wait for large firms' advertising. As another example, if the internal incentive mechanism of an enterprise is not set properly, a situation will occur in which big pigs do everything while piglets do nothing. Indeed, this kind of situation is ubiquitous in reality: most common tasks or public services are completed by a few people and other people just enjoy the outcomes. There are also many examples of the Boxed Pig's Game in society. For example: the people are big pigs and the government is piglets; private enterprises are big pigs and state-owned enterprises are piglets; reformers are big pigs and status-quo advocates are piglets; and innovators are big pigs and followers are piglets.

The story of the Boxed Pig Game informs the weak participant (Piglet) in a competition that waiting is the Piglet best strategy. However, as far as the society is concerned, since piglets do not participate in competitions and are free riders, the allocation of social resources is not optimal. The above result is, in fact, dependent on the setting of the circumstance and the rules of the game. The problem of the "free rider" may or may not occur, depending on the change in the feeding scheme. We will return to the solution of free rider in Chapter 18 on the theory of mechanism design.

To generally and more rigorously define the strictly dominated strategy, we should also take mixed strategies into account.

Definition 6.3.7 A mixed strategy σ_i is called player i 's *strictly dominated mixed strategy* in game $\Gamma_N = [N, \{\Delta S_i\}, \{u_i(\cdot)\}]$ (where ΔS_i is player i 's mixed strategy space, i.e., all possible probability distributions in pure s-

strategy space S_i), if there exists player i 's another mixed strategy $\sigma_i' \neq \sigma_i$, such that for any $\sigma_{-i} \in \Delta S_{-i} \triangleq \Delta S_1 \times \cdots \times \Delta S_{i-1} \times \Delta S_{i+1} \times \cdots \times \Delta S_n$, we have $u_i(\sigma_i, \sigma_{-i}) < u_i(\sigma_i', \sigma_{-i})$.

		player 2		
		L	R	
player 1	T	2, 0	-1, 0	(a)
	M	0, 0	0, 0	
	B	-1, 0	2, 0	

		player 2		
		L	R	
player 1	T	1, 3	-2, 0	(b)
	M	-2, 0	1, 3	
	B	0, 1	0, 1	

Table 6.8: Mixed Dominated Strategies

Example 6.3.5 Consider the two games described in Table 6.8. In these two games, player 1 has three strategies $\{T, M, B\}$, and player 2 has two strategies $\{L, R\}$. In Table (a), neither player has a strictly dominated pure strategy. However, consider a mixed strategy, such as σ_1 , in which player 1 has the same probability $1/2$ of choosing T and B . Then, M is a strictly dominated mixed strategy for player 1. In Table (b), neither player has a strictly dominated strategy. However, consider a mixed strategy, say σ_1' , in which player 1 has the same probability $1/2$ of choosing T and M . Then, regardless of what player 2 chooses, the utility that σ_1' brings to player 1 is always lower than that brought by pure strategy B . In this way, mixed strategy σ_1' is a strictly dominated (mixed) strategy.

6.3.2 Rationalization of Strategies

Eliminating strictly dominated strategies is actually a rational choice of a player. The rational choice of a player is based on his or her belief in the choice of other players. In a static game, there is a logical consistency

between the rationalization of players' strategic choices and the elimination of strictly dominated strategies.

First, we introduce the concept of best response.

Definition 6.3.8 (Best Response) For game $\Gamma_N = [N, \{\Delta S_i\}, \{u_i(\cdot)\}]$, strategy σ_i is a *best response* of player i to other players' strategy σ_{-i} , if for any $\sigma_i' \in \Delta(S_i)$, we have

$$u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma_i', \sigma_{-i}).$$

If a strategy σ_i is player i 's best response to σ_{-i} , then this strategy is player i 's best choice when he or she assumes that other players will adopt σ_{-i} . If strategy σ_i is never a best response, there is no other players' strategy σ_{-i} , such that σ_i is a best response of player i to σ_{-i} . In other words, σ_i cannot be considered as a rational choice.

Obviously, if a strategy is a strictly dominated strategy, then that strategy can never be a best response. Meanwhile, it can be shown that in a game of finite strategies, if a strategy is not a best response of player i in any situation, then this strategy must also be a strictly dominated strategy (cf. Osborne and Rubinstein (1994)).

Below, we introduce the rationalizability of strategy.

Definition 6.3.9 For game $\Gamma_N = [N, \{\Delta S_i\}, \{u_i(\cdot)\}]$, a (pure) strategy $s_i \in S_i$ is *rationalizable*, if it survives the iterative elimination of those strategies that are never a best response.

Obviously, if player i 's strategy a_i is a strictly dominated strategy, it is not rationalizable. This is because there is no belief $\mu_i(a_i)$ that makes a_i the best response under this belief. Also, despite that strategy a_i is not player i 's best choice, it is a best response for player i under beliefs $\mu_i(a_i)$, and all such supports of these beliefs contain strictly dominated strategies of other players, i.e., for all beliefs $\mu_i(a_i)$ under which a_i is a best response, there is some $j \in N \setminus \{i\}$, such that $\mu_i(a_i)(a_j) > 0$ and a_j is player j 's strictly dominated strategy, then a_i is also not rationalizable. Moreover, if strategy a_i is the best response for player i under beliefs $\mu_i(a_i)$, but all such supports of

these beliefs contain strategies that are not rationalizable for other players, then strategy a_i is also not rationalizable.

Osborne and Rubinstein (1994) proved that for a game with finite strategies, if strategy space $X = \times_{j \in N} X_j$ is the one that is left after repeated elimination of strictly dominated strategies, then for each player $i \in N$, X_i is a set of rationalizable strategies for player i .

Besides rationalizability, one can use best responses to identify the Nash equilibria of a game, as discussed below.

6.3.3 Nash Equilibrium

Rationalization can assist us to constrain individuals' choices in interactions. However, it is a weaker solution concept of equilibrium. In many games, there are too many rationalizable strategies. Then, we make a stronger assumption: The players are rational and their expectations on others are mutually known. In an equilibrium, players will no longer adjust their strategies. Here, we impose an additional constraint on players' beliefs, i.e., the rational expectation constraint.

First, we introduce the concept of Nash equilibrium, which implies that once the Nash equilibrium is reached, no participant has an incentive to deviate from the Nash equilibrium unilaterally.

Definition 6.3.10 (Nash Equilibrium) For game $\Gamma_N = [N, \{\Delta S_i\}, \{u_i(\cdot)\}]$, strategy profile $(\sigma_i^*, \sigma_{-i}^*)_{i \in N}$ is a *Nash equilibrium* if for every $i \in N$, we have

$$u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(\sigma'_i, \sigma_{-i}^*)$$

for all $\sigma'_i \in \Delta S_i$.

If strategic choices are limited to pure strategies, there will be a corresponding definition for pure strategy Nash equilibrium.

Definition 6.3.11 For game $\Gamma_N = [N, \{S_i\}, \{u_i(\cdot)\}]$, strategy profile $(s_i^*, s_{-i}^*)_{i \in N}$ is a *pure strategy Nash equilibrium* if for every $i \in N$, we have

$$u_i(s_i^*, s_{-i}^*) \geq u_i(s'_i, s_{-i}^*)$$

for all $s'_i \in S_i$.

It is clear that, for the game $\Gamma_N = [N, \{\Delta S_i\}, \{u_i(\cdot)\}]$, a strategy profile $(\sigma_i^*, \sigma_{-i}^*)_{i \in N}$ is a Nash equilibrium if and only if for every $i \in N$, σ_i^* is a best response of player i to other players' strategy σ_{-i}^* . The concept of Nash equilibrium means that given opponents' strategic choices, no one will choose to unilaterally deviate from the equilibrium choice. In other words, when a strategy profile is a Nash equilibrium, given the choices of opponents, each player chooses a strategy that is best for himself.

Therefore, the set of Nash equilibria coincides with the set of intersections of all players' best response set, which can be used to find Nash equilibria through finding the intersections. This conclusion provides a simple method to find Nash equilibrium for two-person game; see Example 6.3.6 below.

It is clear that strictly dominant equilibrium, dominant equilibrium, and iterated elimination of strictly dominated strategy equilibrium are all Nash equilibria, and the converse may not be true. Their relationship is in turn extended, i.e., the strictly dominant equilibrium is the strongest, and the Nash equilibrium is the weakest. For example, in the previous Prisoner's Dilemma game, "Confess" is a strictly dominant strategy for each player, and thus both players choosing to "Confess" constitutes a strictly dominant equilibrium and also a Nash equilibrium.

The following Nash equilibrium concept is based on the optimal decision-making of subjective beliefs:

Definition 6.3.12 For game $\Gamma_N = [N, \{\Delta S_i\}, \{u_i(\cdot)\}]$, a Nash equilibrium consists of subjective beliefs μ_i^* (defined on S_j , $j \neq i$) and strategy profile $(\sigma_i^*, \sigma_{-i}^*)_{i \in N}$, such that for any $\sigma_i' \in \Delta(S_i)$, we have

$$\begin{aligned} Eu_i(\sigma_i^* | \mu_i^*) &\geq Eu_i(\sigma_i' | \mu_i^*); \\ \sigma_j^* &= \mu_i^*|_{S_j}, \end{aligned}$$

where $Eu_i(\sigma_i^* | \mu_i^*) = \int_{s_i \in S_i, s_{-i} \in S_{-i}} u(s_i, s_{-i}) d(\sigma_i(s_i)) d(\mu_i(s_{-i}))$ denotes the expected utility of player i choosing σ_i under belief μ_i ; $\mu_i^*|_{S_j}$ represents the marginal distribution of belief μ_i^* on S_j . If every player's strategy is independent, then $\mu_i^* = \times_{j \in N \setminus \{i\}} \mu_i^*|_{S_j}$.

Below, we use examples to find Nash equilibrium.

Example 6.3.6 There are two players 1 and 2, and their game matrix is shown in Table 6.9.

		player 2		
		L	C	R
player 1	T	<u>5</u> , 3	0, 4	3, <u>5</u>
	M	4, 0	<u>5</u> , <u>5</u>	4, 0
	B	3, <u>5</u>	0, 4	<u>5</u> , 3

Table 6.9: Example of Nash Equilibrium

We can find the Nash equilibrium of this game conveniently and quickly by using the conclusion that the set of Nash equilibria is identical with the set of intersections of all players' best response sets. Consider the strategy of player 1, and for each strategy of player 2, find out the best response of player 1. Draw a horizontal line under its corresponding payoff. Similarly, find out the best response of player 2. The strategy profile with both horizontal lines is (M, C) , and such a strategy profile is unique. Therefore, the strategy profile (M, C) is the unique pure strategy Nash equilibrium, and its corresponding Nash equilibrium payoff is $(5, 5)$.

Example 6.3.7 (Chicken Game) Consider the following Chicken Game. There are two equal-strength chickens. Each chicken has two strategies: "Continue to Fight" and "Retreat". If both chickens choose "Continue to Fight", the outcome is a lose-lose, and the payoff of each player is -1 . If both chickens choose "Retreat", however, there is neither victory nor failure, and the payoff of each player is 0 . If one chicken chooses "Continue to Fight" and the other chicken chooses "Retreat", the payoff of the winning chicken is 1 and the payoff of the retreating chicken is 0 . In this way, the payoff matrix is:

It can be seen that both $(A \text{ Retreats}, B \text{ Continues to Fight})$ and $(A \text{ Continues to Fight}, B \text{ Retreats})$ are pure strategy Nash equilibria.

		Chicken B	
		Continue To Fight	Retreat
Chicken A	Continue To Fight	-1, -1	<u>1</u> , <u>0</u>
	Retreat	<u>0</u> , <u>1</u>	0, 0

Table 6.10: Chicken Game

When two chickens are fighting, it is a dilemma to make a choice between advancing and retreating, as the Nash equilibrium has given a best strategy of one winning and the other failing. In many contests, exerting the utmost strength does not ensure success. This is also the logic of the famous guerrilla tactic of Mao Zedong's "The enemy advances, we retreat; the enemy retreats, we pursue" .

This example offers some pertinent implications for two equally powerful firms to get along and compete with each other. Two powerful firms already in the market are likely to consciously follow the Nash equilibrium. When one side attacks, the other side temporarily retreats. Although one side may be temporarily damaged, this is far superior to a lose-lose outcome. However, to maintain this situation, it must be ensured that the next time the earlier damaged party launches an attack, and the other side will also retreat.

For some games, such as the Rock-Paper-Scissors game, there is no pure strategy Nash equilibrium, but there may exist a mixed strategy Nash equilibrium. Then, can we easily solve for the mixed strategy Nash equilibrium? Of course, it can be obtained by the method of maximizing the expected payoff (say, by the first-order condition), but this method is a little bit involved. In fact, there is a more straightforward way.

The following proposition shows that the indifference among strategies played with a positive probability is a general feature of a mixed strategy.

Proposition 6.3.1 *Let $S_i^+ \subseteq S_i$ be the set of pure strategies that player i plays with a positive probability under the mixed strategy profile $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_n)$. Strategy profile $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_n)$ is a mixed strategy Nash equilibrium of game $\Gamma_N = [N, \{\Delta S_i\}, \{u_i(\cdot)\}]$ if and only if, for every $i \in N$, we have:*

$$(1) u_i(s_i, \sigma_{-i}) = u_i(s_i', \sigma_{-i}) \text{ for all } s_i, s_i' \in S_i^+;$$

$$(2) u_i(s_i, \sigma_{-i}) \geq u_i(s_i', \sigma_{-i}) \text{ for all } s_i \in S_i^+, s_i' \notin S_i^+.$$

PROOF. Necessity: If at least one of the conditions (1) and (2) above is not satisfied, then there exists $s_i \in S_i^+, s_i' \in S_i$, such that $u_i(s_i', \sigma_{-i}) > u_i(s_i, \sigma_{-i})$. If player i changes the chosen strategy from s_i to s_i' , the expected payoff of player i can be strictly increased, which means that σ_i is not the best response of σ_{-i} .

Sufficiency: Suppose that both conditions (1) and (2) above are satisfied, but $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_n)$ is not a Nash equilibrium. Then, there exists at least one player i and his or her another strategy σ_i' , such that $u_i(\sigma_i', \sigma_{-i}) > u_i(\sigma_i, \sigma_{-i})$. This means that at σ_i' , there is at least one pure strategy \hat{s}_i chosen by player i with positive probability, and thus $u_i(\hat{s}_i, \sigma_{-i}) > u_i(\sigma_i, \sigma_{-i})$ is established. Since $u_i(\sigma_i, \sigma_{-i}) = u_i(s_i, \sigma_{-i})$, $s_i \in S_i^+$, we have $u_i(\hat{s}_i, \sigma_{-i}) > u_i(s_i, \sigma_{-i})$. However, this contradicts at least one of the conditions (1) and (2). \square

Thus, given the mixed strategy of the opponent, the expected utility of any strategy is the same, and thus players have no incentive to change the probabilities of choosing these strategies, i.e., no players will unilaterally change their mixed strategies. This proposition is very helpful in finding mixed strategy equilibria, and thus it provides a simple method to solve for mixed strategy Nash equilibrium.

		Player B		
		Rock	Paper	Scissors
Player A	Rock	5, 5	0, 10	10, 0
	Paper	10, 0	5, 5	0, 10
	Scissors	0, 10	10, 0	5, 5

Table 6.11: Rock-Paper-Scissors Game

Example 6.3.8 (Rock-Paper-Scissors Game continued) Consider Rock-Paper-Scissors Game once again. Its payoff matrix is given by Table 6.11

Suppose that column's mixed strategy assigns probability weight q_r to Rock, q_p to Paper and $(1 - q_r - q_p)$ to Scissors. Then, using (6.2.2), we have:

row's expected payoff from Rock against $(q_r, q_p, 1 - q_r - q_p)$ is

$$5q_r + 0q_p + 10(1 - q_r - q_p);$$

row's expected payoff from Paper against $(q_r, q_p, 1 - q_r - q_p)$ is

$$10q_r + 5q_p + 0(1 - q_r - q_p);$$

row's expected payoff from Scissors against $(q_r, q_p, 1 - q_r - q_p)$ is

$$0q_r + 10q_p + 5(1 - q_r - q_p).$$

Setting these three expected payoffs equal to one another leads to $q_r = q_p = (1 - q_r - q_p) = 1/3$. By symmetry, this is also the row's optimal mixed strategy. Thus, the mixed strategy Nash equilibrium is ({Rock with probability 1/3, Paper with probability 1/3, Scissors with probability 1/3}) for both players.

In this example, for the mixed strategy Nash equilibrium, $S_i^+ = S_i$ is established for all players.

Some games have both pure strategy Nash equilibrium and mixed strategy Nash equilibrium, as shown in Table 6.12.

Example 6.3.9 (Battle of the Sexes) The battle of the sexes is also a classical example analyzed in game theory. A man and a woman want a date over the weekend, but they cannot agree over what to do. The man prefers to watch a basketball game, whereas the woman wants to watch an opera. The payoff matrix is given by 6.12.

		Male	
		Opera	Basketball
Female	Opera	<u>2</u> , <u>1</u>	0, 0
	Basketball	0, 0	<u>1</u> , <u>2</u>

Table 6.12: Battle of the Sexes

In this game, there are two pure strategy Nash equilibria: (Opera, Opera) and (Basketball, Basketball). Using the above method, we now show

there is also a mixed strategy Nash equilibrium : ({Opera with probability $2/3$, Basketball with probability $1/3$ }, {Opera with probability $1/3$, Basketball with probability $2/3$ }).

Given the man's choice of the mixed strategy $(\sigma_1, 1 - \sigma_1)$, the expected payoff of the woman's choice of opera is $2\sigma_1 + 0(1 - \sigma_1)$ and the expected payoff of choosing basketball is $0 + 1(1 - \sigma_1)$. Equalizing them leads to $\sigma_1 = 1/3$. Similarly, given the woman's choice of the mixed strategy $(\sigma_2, 1 - \sigma_2)$, the expected payoff of the man's choice of opera is $1\sigma_2 + 0(1 - \sigma_2)$ and the expected payoff of choosing basketball is $0 + 2(1 - \sigma_2)$. Equalizing them leads to $\sigma_2 = 2/3$. Thus, the man chooses the mixed strategy {Opera with probability $1/3$, Basketball with probability $2/3$ }, and the woman chooses the mixed strategy {Opera with probability $2/3$, Basketball with probability $1/3$ }.

In the above game and also the Chicken Game, there are two pure strategy Nash equilibria and one mixed Nash equilibrium. In fact, the pattern holds not just for 2×2 games, but for almost all $n \times n$ matrix games. The Oddness Theorem (Wilson 1971) shows that nearly all finite games have an odd number of Nash equilibria. Thus, as long as a game has two pure strategy Nash equilibria, then there must exist a mixed Nash equilibrium.

6.3.4 Refinements of Nash Equilibrium

In many games, there are multiple Nash equilibria. The non-uniqueness of Nash equilibrium makes it impossible to accurately predict the outcome of an interaction or in some sense, some of them are bad Nash equilibria that should be eliminated. As such, numerous approaches to refining Nash equilibria are proposed. Thomas Schelling (1960) put forward the concept of **focal point**, which is a solution that people tend to choose by default in the absence of communication. The context in which the players are located, such as culture, tradition, and practice, will constrain individuals' strategic choice in the interaction process. Indeed, as Schelling point out, "people can often concert their intentions or expectations with others if each knows that the other is trying to do the same" in a cooperative situation, and then their actions will approach a focal point which has some

kind of prominence compared with the environment.

For example, in the Battle of the Sexes game, in order to woo the woman (man), the man (woman) usually pays more attention to the woman (man)'s feelings in their interactions. As such, their strategy profile is more likely to be the Nash equilibrium (Opera, Opera). If the background of their decisions is that they have watched the opera last time and they pay attention to equality, then this time they will choose the Nash equilibrium (Basketball, Basketball). In addition, in real life, people usually communicate in advance. In the case of the Battle of the Sexes, it is far-fetched to construe the mixed strategy equilibrium as a strategic choice in interactions, because under the mixed strategy equilibrium, both players' expected payoffs are $2/3$, which are less than the payoffs under pure strategy equilibrium. If the players can negotiate in advance in the process of interaction and there is a strategy profile which is the consensus of both parties after the negotiation and is also a Nash equilibrium, then individuals will not unilaterally deviate from this outcome. If the previously negotiated strategy profile is not a Nash equilibrium, then this *ex ante* agreement may not be followed.

Several technical standards to eliminate some Nash equilibria have been introduced. One of them is the concept of trembling-hand perfect equilibrium proposed by Selten (1975), which is a refinement of Nash equilibrium. The main idea of this concept is that a Nash equilibrium is stable if it is preserved against small modifications that may have originated from individuals' minor error in action. The trembling-hand perfect equilibrium means that if the sequence of probability with which individuals make mistakes approaches zero, then the trembling-hand perfect equilibrium is the (convergence) limit of the equilibrium sequence in this process.

We can fully comprehend the trembling-hand perfect equilibrium through the concept of subjective beliefs. When a player's subjective beliefs in the judgment of other players' actions have a minor error and this error becomes infinitely small, the player's strategy is still the best response to rational expectations (or correct beliefs).

Given game $\Gamma_N = [N, \{\Delta S_i\}, \{u_i(\cdot)\}]$, we say it is a **perturbed game** $\Gamma_\varepsilon = [N, \{\Delta_\varepsilon(S_i)\}, \{u_i(\cdot)\}]$ if for each player i and each of his or her strate-

gic choices, there is a certain disturbance term $\varepsilon_i(s_i) \in (0, 1)$, such that

$$\sum_{s_i \in S_i} \varepsilon_i(s_i) < 1.$$

On this basis, the (mixed) strategy space of player i after perturbation is defined as $\Delta_\varepsilon(S_i) = \{\sigma_i : \sigma_i(s_i) \geq \varepsilon_i(s_i), \sum_{s_i \in S_i} \sigma_i(s_i) = 1\}$.

Definition 6.3.13 (Trembling-Hand Perfect Equilibrium) Nash equilibrium $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_n)$ for a game $\Gamma_N = [N, \{\Delta S_i\}, \{u_i(\cdot)\}]$ is a *trembling-hand perfect equilibrium*, if there is a perturbed game sequence $\{\Gamma_{\varepsilon^k}\}_{k=1}^\infty$ that converges to the original game $\Gamma_N = [N, \{\Delta S_i\}, \{u_i(\cdot)\}]$, and the corresponding Nash equilibrium sequence $\{\sigma^k\}_{k=1}^\infty$ of the perturbed game sequence converges to the Nash equilibrium of the original game. Here, convergence means that for each player i and his or her strategy $s_i \in S_i$, we have $\lim_{k \rightarrow \infty} \varepsilon_i^k(s_i) = 0$.

The following example shows how to use the trembling-hand perfect equilibrium to eliminate some Nash equilibria of a game.

		player 2	
		L	R
player 1	U	<u>1</u> , <u>1</u>	<u>0</u> , -3
	D	-3, <u>0</u>	<u>0</u> , <u>0</u>

Table 6.13: Trembling-Hand Perfect Equilibrium

Example 6.3.10 The game with two players 1 and 2 is described by Table 6.13.

In this game, there are two pure strategy Nash equilibria, (U, L) and (D, R) . Strategy D is a weakly dominated strategy for player 1, and strategy R is a weakly dominated strategy for player 2. Although (D, R) is a Nash equilibrium, it is not a trembling-hand perfect equilibrium. This is because if each player is likely to have a choice deviation, no matter how small the probability of this deviation is, as long as this probability is positive, choosing a weakly dominated strategy will not be a player's best response. As such, in a perturbed game, there is only one Nash equilibrium,

i.e., (U, L) , and (D, R) is not the limit of the Nash equilibrium sequence of perturbed games.

In this way, with the concept of the trembling-hand perfect equilibrium, we can eliminate certain strategic choices of some players. For example, a weakly dominated strategy cannot be a strategic choice for a trembling-hand perfect equilibrium.

For the characterization of the trembling-hand perfect equilibrium, Selten (1975) proved the following proposition.

Proposition 6.3.2 *A Nash equilibrium $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_n)$ of Game $\Gamma_N = [N, \{\Delta S_i\}, \{u_i(\cdot)\}]$ is a trembling-hand perfect equilibrium if and only if there is a sequence of completely mixed strategy profile $\{\sigma^k\}_{k=1}^\infty$ (i.e., each pure strategy is selected with a positive probability), such that $\lim_{k \rightarrow \infty} \sigma^k = \sigma$, and for every k and every player $i \in N$, σ^k_i is the best response to its opponents' strategy profile σ^k_{-i} .*

The proof of the proposition can be found in Selten (1975). Trembling-hand perfect equilibrium is only one way to refine Nash equilibria. There are many other way to refine Nash equilibria. For instance, **subgame perfect Nash equilibrium** to be discussed below is another typical way to refine Nash equilibrium in context of extensive-form game.

6.4 Dynamic Games with Complete Information

A dynamic game with complete information usually possesses a sequential structure. We may convert an extensive-form game to a normal-form game, and then solve for the equilibrium of a sequential game with the equilibrium concept of the static game. However, some undesirable equilibria may appear with this approach. In a dynamic game, since there is an order of decision-making, there will be a problem of credibility of “commitment”. A reasonable dynamic equilibrium needs to satisfy the requirement of “credible commitment” (“credible threat”), and thus we can refine equilibria. The “credible commitment” has new requirements for players' rationalities. It requires that in the process of decision-making,

players are rational in every possible decision-making environment (more precisely, every information set). This rationality is also called **sequential rationality**. In many situations, it is necessary to solve dynamic games through backward induction.

We first consider the issue of commitment through an example.

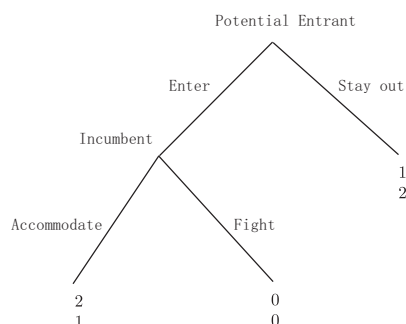


Figure 6.4: Extensive-Form of Market Entry Game

		Incumbent	
		Accommodate if Enter	Fight if Enter
Potential Entrant	Enter	<u>2</u> , <u>1</u>	0, 0
	Stay Out	1, <u>2</u>	<u>1</u> , <u>2</u>

Table 6.14: Strategic-Form of Market Entry Game

Example 6.4.1 (Market Entry Game) Suppose that there are two firms in a market, the Incumbent and the Potential Entrant. The Potential Entrant moves first to choose whether to enter the market, and then the Incumbent decides whether or not to launch a price war. The payoffs of their actions are shown in Figure 6.4. Table 6.14 is the normal-form representation of the game. The game has two Nash equilibria, i.e., (Enter, Accommodate if Enter) and (Stay out, Fight if Enter).

It is not difficult to see that the Incumbent has an incredible threat at Nash equilibrium (Stay Out, Fight if Enter). The reason is that the Potential Entrant would evaluate the gain or loss of entering the market: if the Potential Entrant chooses to enter, he knows that the rational Incumbent will choose to accommodate so that his payoff is 1; otherwise, it is 0, and

then the Potential Entrant's income is 2. If the Potential Entrant chooses to stay out, his payoff is only 1. Thus, the rational choice of the Potential Entrant is to enter. Therefore, only Nash equilibrium (Enter, Accommodate if Enter) is sequentially rational while Nash equilibrium (Stay Out, Fight if Enter) violates the principle of sequential rationality (i.e., in any possible environment, the decision-maker should make a rational decision).

In order to eliminate Nash equilibria that violate sequential rationality, backward induction is frequently used to solve for equilibrium of a dynamic game with perfect information. The equilibrium obtained through this method is called **subgame perfect Nash equilibrium** or simply termed **subgame perfect equilibrium**.

6.4.1 Subgame Perfect Nash Equilibrium

A subgame is a subset of an entire game, but not all subsets can be a subgame. If an entire game begins with a singleton information set, then the entire game is also a subgame. The subgame is defined below.

Definition 6.4.1 (Subgame) A *subgame* of an extensive-form game Γ_E is a subset of the game if it satisfies the following properties:

- (1) It begins with a singleton information set. Let x_0 be the initial decision node of the subgame. The subgame contains and only contains all successors starting from this decision node. If x belongs to the subgame starting from x_0 , and $x \neq x_0$, then $x \notin h(x_0)$, and there is a sequence y_1, \dots, y_n , such that $y_1 = x, y_2 = p(y_1), \dots, y_n = x_0 = p(y_{n-1})$, i.e., there are a series of immediately connected nodes connecting from x_0 to x .
- (2) If decision node x is in the subgame, then every decision node $x' \in h(x)$ is also in the subgame, i.e., the subgame does not divide any information set.

Example 6.4.2 (Continuation of Example 6.4.1) In the above Market Entry Game, there are two subgames. The original game is a subgame. In addition, the game described in Figure 6.5 is also a subgame of the game.

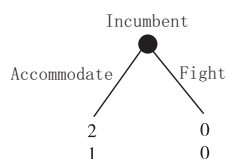


Figure 6.5: A Subgame of the Market Entry Game

None of the subsets of the game described in Figure 6.6 are subgames.

Example 6.4.3 (Game and Subgame) In the game described in Figure 6.6, the game's three subsets surrounded by the dashed line are not subgames. In Figure 6.6(a), the initial node of the subset is not a singleton information set; in Figure 6.6(b), the subset divides the information set of firm I ; in Figure 6.6(c), the initial node of the subset is not a singleton information set, and the subset divides the information set.

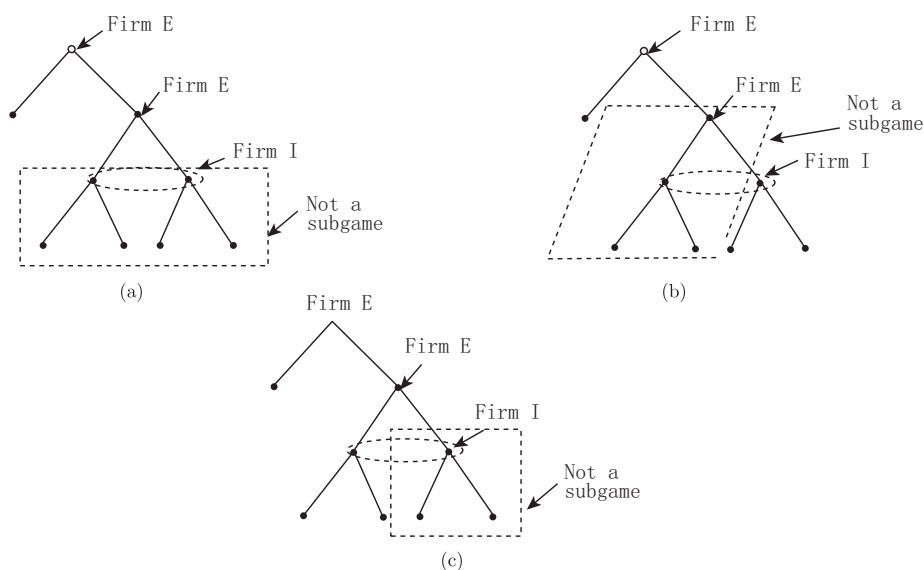


Figure 6.6: Non-Subgame Scenarios

6.4.2 Backward Induction

In an extensive-form dynamic game, since there is a sequence of players' decisions, an issue of the credibility of threats arises in their interaction

process.

Example 6.4.4 (Market Entry Game Continued) Suppose that there are two firms I and E , in which firm I is the incumbent, and firm E is the potential entrant. Firm E decides whether to enter the market. If firm I observes that firm E has decided to enter the market, then firm I may either fight or acquiesce. Their payoffs are shown in Figure 6.7. (Stay out, Fight if firm E enters) is a Nash equilibrium. However, there is a problem with this equilibrium. Once firm E has chosen to enter the market, it is irrational for firm I to choose to fight. In other words, firm I 's threat, "If firm E enters, I will choose to fight", is non-credible.

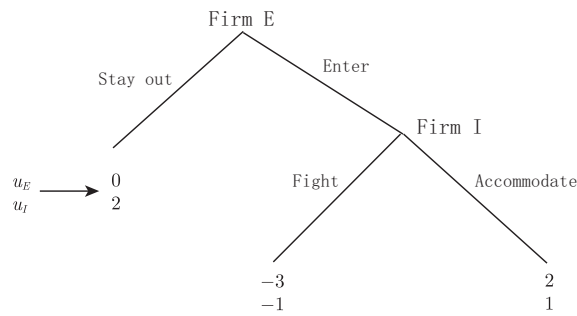


Figure 6.7: Incumbent's Non-credible Threat

A reasonable equilibrium concept of an extensive-form game is more demanding than Nash equilibrium. The equilibrium concept related to extensive-form games is subgame perfect Nash equilibrium. We have the following formal definition.

Definition 6.4.2 (Subgame Perfect Nash Equilibrium) In an extensive-form game with n players, a strategy profile is called *subgame perfect (Nash) equilibrium*, if it is a Nash equilibrium in each subgame.

In a subgame perfect Nash equilibrium, each player's strategy is rationalizable on every possible information set, and each player's choice on each information set is based on the subjective beliefs about the information set that meet the rational expectation assumption.

A subgame perfect equilibrium is a Nash equilibrium in each subgame, which implies that for a dynamic perfect game, players' decisions are rational on each information set, i.e., they satisfy the requirements of sequential rationality. If an extensive-form game is a perfect information game, i.e., each information set is a singleton, then backward induction can be used to solve for the subgame perfect Nash equilibrium of this game. Backward induction means that the solving process starts from the decision-making node at the bottom level, *reducing the subgames at the bottom level to equilibrium payoffs of these subgames*, and then advances recursively to the subgames at the upper level, *reducing the subgames at this level to equilibrium payoffs of these subgames*. This process continues until the very beginning of the game is reached.

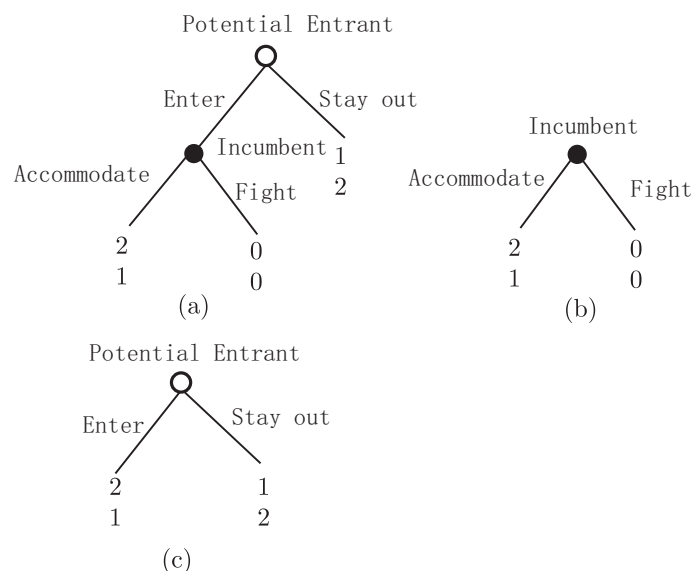


Figure 6.8: Market Entry Game(continuation)

Example 6.4.5 (Continuation of Example 6.4.4) Consider the previous market entry game. Diagram (a) in Figure 6.8 is the entire game. In the market entry game, there are two subgames: the original game (Diagram (a)), and the subgame starting from the incumbent's decision-making node (Diagram (b)). Backward induction starts from the subgame at the lowest level, i.e., from the game in Diagram (b). The Nash equilibrium of the game in

Diagram (b) is the incumbent choosing to acquiesce, and thus the equilibrium payoff of this subgame is $(2, 1)$, and then the entire game is reduced to the game in Diagram (c). At this time, the game has advanced to the topmost level, i.e., the subgame is replaced by its equilibrium payoff. The Nash equilibrium of the game in Diagram (c) is the Potential Entrant choosing to enter. Therefore, (Entry, Accommodate) is the subgame perfect Nash equilibrium of the entire market entry game.

In the following, we consider the subgame perfect Nash equilibrium of a dynamic game in which players make decisions alternately. As the shape of this dynamic game's extensive-form representation is similar to that of a centipede, it is called the Centipede Game. This game reveals that, although the total payoff increases after each cooperation, unfortunately, this happy ending is hard to achieve, i.e., no cooperation from the very beginning is a rational choice.

This conclusion is very counter-intuitive. In practice, although cooperation is difficult to last long, the willingness to cooperate is actually common in the short run. Because of this, the Centipede Game is considered the best example of what is known as the “backward induction paradox.”

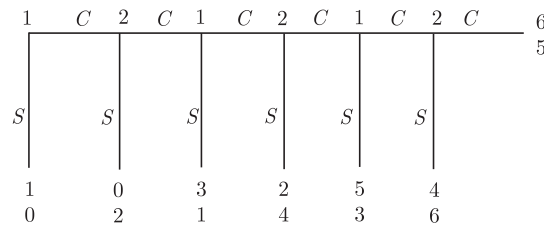


Figure 6.9: Centipede Game

Example 6.4.6 (Centipede Game) The centipede game is a dynamic game problem proposed by Rosenthal (1981). The extensive form game is very similar to a centipede, so it is called “centipede game” in which two players take turns choosing either to pass the pot to the other player (cooperation), or to take a slightly larger share of an increasing pot (stop cooperation). The payoffs are arranged so that if one passes the pot to one's opponent and the opponent takes the pot in the next round, one receives

slightly less than if one had taken the pot in this round. The interactions are described by Figure 6.9.

The extensive form representation of a six-stage centipede game ends after six rounds with the money being split. Passing the coins across the table is represented by a move of C (going across the row of the lattice) and pocketing the coins is a move of S (down the lattice). The numbers 1 and 2 along the top of the diagram show the alternating decision-maker between two players denoted here as 1 and 2, and the numbers at the bottom of each branch show the payoff for players 1 and 2, respectively. Player 1 moves first. After six rounds, the income is distributed.

We use the backward induction method to solve for the subgame perfect Nash equilibrium. For the lowest subgame in which player 2 makes her third decision, the Nash equilibrium is that player 2 chooses strategy S . Then, we advance recursively to the upper level subgame. The Nash equilibrium of this subgame is that player 1 chooses S , and player 2 chooses S . This process continues until the topmost level of the game is reached. The subgame perfect Nash equilibrium of the entire game is $(S, S, S; S, S, S)$, i.e., players 1 and 2 choose S in each period.

Therefore, like the Prisoner's Dilemma, this centipede game presents a conflict between self-interest and mutual benefit. If some enforcement or incentive mechanism could be imposed, both players would prefer that they both cooperate throughout the entire game. However, a player's self-interest or players' distrust interferes the cooperation and creates a situation where both do worse than if they had blindly cooperated.

The above extensive-form games are all perfect information games in which every information set is singleton. In a game with complete but imperfect information, we can use a more general backward induction method to solve for subgame perfect Nash equilibrium.

Below, we discuss by example how to employ the backward induction method to solve a dynamic game of complete but imperfect information.

Example 6.4.7 (Market Entry and Site Selection) There are two firms E and I . Firm E chooses whether to enter the market first. If firm E does not enter, the game ends; if firm E enters, in the second stage, firms E and I select

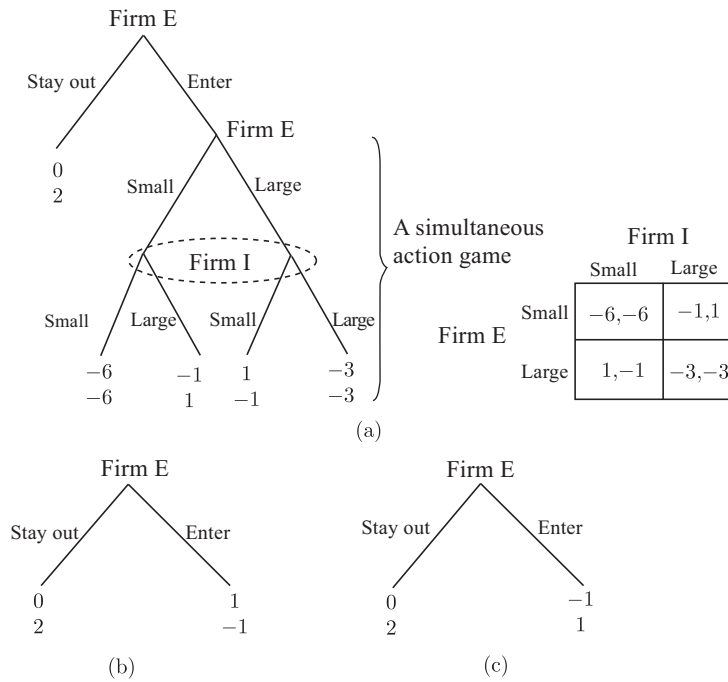


Figure 6.10: Market Entry and Site Selection

their sites at the same time. In this game, there is a non-singleton information set, so that it is not a perfect game. There are two subgames in this game. In addition to the original game, after firm E chooses to enter, the game in which two firms move simultaneously is also a subgame.

The subgame in which two firms move simultaneously can be described by Diagram (a) in Figure 6.10. This subgame has two Nash equilibria, i.e., (Large, Small) and (Small, Large), and the equilibrium payoffs are (1, -1) and (-1, 1), respectively, which can be used to reduce this simultaneous move game. Therefore, the backward induction of this step produces two possibilities, which are given in Diagrams (b) and (c), respectively. In the game in Diagram (b), the Nash equilibrium is firm E choosing to enter; whereas, in the game in Diagram (c), the Nash equilibrium is firm E choosing to stay out. As such, the entire game has two subgame perfect Nash equilibria ((Enter, Large after Entering), Small) and ((Stay Out, Small after entering), Large).

Now, we use the concept of subgame perfect Nash equilibrium to discuss a classic example of economics (which is also a common situation in real life), i.e., the bargaining game. There are many versions of the bargaining game, including the Nash bargaining game, the Rubinstein bargaining game, and bargaining games with finite or infinite periods. Here we discuss the Rubinstein bargaining game.

Example 6.4.8 (Rubinstein Bargaining Game, 1982) Suppose that there are two players who conduct a bargaining on how to split a total of 1 unit of an infinitely divisible property. Obviously, in this game, any (x_1, x_2) with $x_1 + x_2 = 1$ is a Nash equilibrium, and thus there are infinitely uncountable Nash equilibria, but the subgame perfect equilibrium is unique. Since backward induction is more suitable for bargaining games with finite periods, we first discuss the bargaining game with T periods.

In the $2k + 1$ period, player 1 proposes a distribution plan (in which the payoff received by any player is not allowed to be negative), and player 2 chooses whether to accept it; in the $2k$ period, player 2 proposes a distribution plan, and player 1 chooses whether to accept it. Once an agreement is reached in a certain period, i.e., player 1 (or player 2) chooses to accept the distribution plan proposed by his or her opponent, the game ends, and the distribution of the property is determined by the distribution plan. If the two players still have not reached an agreement in the T period, then the property is confiscated and the two players will receive nothing. Suppose that the time discount rates for both players are both δ . Let $(x_t, 1 - x_t)$ be a distribution plan proposed by the player who has the right to make a proposal during the t period, where x_t is the amount of property distributed to player 1 and $1 - x_t$ is the amount of property distributed to player 2.

When $T = 1$, player 1 has the right of proposal, and player 2 chooses whether or not to accept it. Obviously, as long as $1 - x_1 \geq 0$, player 2 will not choose to reject the proposal. Therefore, the Nash equilibrium of the game is that player 1 proposes a distribution plan of $(1, 0)$, and player 2 accepts it.

When $T = 2$, in the last period, player 2 has the proposal right. Similar to the logic in the previous case, in this period, player 2's proposal is

$(0, 1)$, and player 1 will also choose to accept it. In this plan, the payoff pair of these two players (calculated regarding discounted value) is $(0, \delta)$. Returning to the first period, player 1 has the right of proposal. As long as $1 - x_1 \geq \delta$, player 2 will accept player's 1 proposal, because if player 2 does not accept player 1's proposal, player 2's final payoff is only δ in the subgame in the last period. Therefore, in the equilibrium path of the game, player 1 proposes a distribution plan $(1 - \delta, \delta)$ and player 2 accepts it in the first period, and the game ends.

When $T = 3$, in the last period, player 1 has the proposal right. Similar to the logic in the previous cases, in this period, player 1's proposal is $(1, 0)$, and player 2 will also choose to accept it. Returning to the second period, player 2 has the right of proposal, and as long as $x_2 \geq \delta$, player 1 will accept player 2's proposal. Returning to the first period, player 1 has the right of proposal, and as long as $1 - x_1 \geq \delta(1 - \delta)$, player 2 will accept player 1's proposal. Therefore, in the equilibrium path of the game, player 1 proposes a distribution plan $(1 - \delta + \delta^2, \delta - \delta^2)$ and player 2 accepts it in the first period, and the game ends.

We find that in the case of $T = 1, 2, 3$, player 1 proposes a distribution plan $(\frac{1-(-\delta)^T}{1+\delta}, 1 - \frac{1-(-\delta)^T}{1+\delta})$ and player 2 accepts this plan in the first period, and then the game ends. As a consequence, we conjecture that for all T , we have: player 1 proposes a distribution plan $(\frac{1-(-\delta)^T}{1+\delta}, 1 - \frac{1-(-\delta)^T}{1+\delta})$ and player 2 accepts this plan in the first period, and then the game ends. This can be proven by mathematical induction.

Let $(x_t(T), y_t(T))$ denote the distribution plan of the t period of the bargaining game with the deadline of T period(s), which satisfies $x_t(T) + y_t(T) = 1$. First of all, when $T = 1$, the above conclusion holds. Assume that when $T \leq K$, the above conclusion is also true, i.e., the distribution plan $(x_1(K) = \frac{1-(-\delta)^K}{1+\delta}, y_1(K) = 1 - x_1(K))$ proposed by player 1 in the first period will be accepted by player 2.

Suppose now that $T = K + 1$. Consider the distribution plan in the second period $(x_2(K+1), y_2(K+1))$. The subgame starting from the second period is the same as the bargaining game with the deadline of K period(s) in which player 2 proposes a distribution plan first, and thus $y_2(K+1) = x_1(K) = \frac{1-(-\delta)^K}{1+\delta}$ and $(x_2(K+1) = 1 - x_1(K), y_2(K+1) = x_1(K))$ will be

accepted by player 1. Returning to the first period, $(x_1(K+1) = 1 - \delta y_2(K+1), y_1(K+1) = \delta y_2(K+1))$ will be accepted by player 2. Therefore, the distribution plan:

$$(x_1(K+1) = 1 - \delta \left(\frac{1 - (-\delta)^K}{1 + \delta} \right) = \frac{1 - (-\delta)^{K+1}}{1 + \delta}, y_1(K+1) = 1 - x_1(K+1))$$

proposed by player 1 in the first period will be accepted by player 2.

In this way, in the equilibrium path of the bargaining game with the deadline of T period(s), player 1 proposes a distribution plan:

$$\left(\frac{1 - (-\delta)^T}{1 + \delta}, 1 - \frac{1 - (-\delta)^T}{1 + \delta} \right)$$

and player 2 accepts this plan in the first period, and then the game ends.

When $\delta < 1$ and $T \rightarrow \infty$, the subgame perfect equilibrium becomes

$$\left(\frac{1}{1 + \delta}, \frac{\delta}{1 + \delta} \right),$$

In which player 1 gets the first-mover advantage.

6.5 Static Games with Incomplete Information

In many interactions, players do not know information about other players' payoffs or types. Taking auctions as an example, a bidder does not know other bidders' evaluations of the auction item. In a market competition, a firm often does not know how strong its competitors are, but this information will affect the interaction process and outcomes. In addition, even when player 1 knows player 2's information, player 2 may not know that player 1 knows player 2's information (e.g., when some of the decision-maker-related information is not common knowledge), and thus the Nash equilibrium concept cannot be applied to the analysis of strategic interactions with incomplete information.

Another development milestone in game theory is the analytical framework proposed by Harsanyi (1967, 1968) to investigate games under incomplete information. Harsanyi converted games of incomplete informa-

tion into games of complete information. The key to this is to convert a player's subjective judgment about all other players' interaction-related unknown information into random variables which describe other players' types. The variables of players' types are some exogenously-given random variables which can be described by Nature's actions. Nature's actions are based on type variables' prior probability distribution, which is common knowledge of all players of the game. Of course, different players have different signals for these type variables. These signals can also be utilized to revise posterior beliefs. In this way, incomplete information can be converted into complete but imperfect information by describing all unknown information and beliefs by type variables. The complete information defined here refers to a situation in which a player's information state can be defined by an information set, and this information state is common knowledge.

At the beginning of a game, Nature assigns a random variable to each player, which could take values of types for each player. Each player knows his or her own type, and does not know other players' types, but knows their prior distribution. The game after transforming from incomplete information to complete but imperfect information is called the **Bayesian game**. In the Bayesian game, belief is an essential concept, which is a player's subjective judgment of other players' types distribution. If a player obtains some new information, he or she will update beliefs about other players' types according to Bayes' rule. We will use the concept of Bayesian-Nash equilibrium to analyze the equilibrium of strategic interactions in the game.

6.5.1 Bayesian Game

The Bayesian game under incomplete information is now formally defined.

Definition 6.5.1 (Bayesian Game) A Bayesian game

$(\tilde{N}, (A_i)_{i \in N}, (T_i)_{i \in N}, p, (u_i(\cdot))_{i \in N})$ is characterized by the following components:

- (1) The set of players: $\tilde{N} = \{N, N_0\}$ is the set of players, where N_0 is Nature.

- (2) Player's types: t_i is the type of player i , $\mathbf{t} = (t_i)_{i \in N}$ is a profile of all players' types, T_i is the set of player i 's types and $T \equiv \prod_i T_i$ is the set of all profiles of players' types. Nature randomly selects all players' types, and each player knows his or her own type. The probability that Nature randomly selects all players' types is given exogenously. This probability distribution is p , a common prior distribution for all players, and is denoted by $p(\mathbf{t})$. Nature randomly selects the profile of types \mathbf{t} with the probability of $p(\mathbf{t})$. After player i knows his or her own type t_i , his or her posterior belief in the distribution of other players' types is $p(\mathbf{t}_{-i}|t_i) = \frac{p(t_i, \mathbf{t}_{-i})}{p(t_i)}$, where $p(t_i) \equiv \sum_{\mathbf{t}_{-i}} p(t_i, \mathbf{t}_{-i})$ is the probability of player i 's type and $\mathbf{t}_{-i} \equiv (t_1, \dots, t_{i-1}, t_{i+1}, \dots, t_n)$. In a Bayesian game, the type of player i is his or her private information that is not known to others than himself or herself.
- (3) Player's action set: A_i is the set of player i 's actions, and $A \equiv \prod_{i \in N} A_i$ is the set of action profiles of all players.
- (4) Payoffs: on $A \times T$, player i 's utility function is $u_i(\cdot)$.
- (5) Player i 's strategy: $s_i(\cdot) : T_i \rightarrow A_i$ is a complete plan formulated by player i for all possible types, i.e., $s_i(\cdot) \equiv \prod_{t_i \in T_i} s_i(t_i)$, where $s_i(t_i)$ is an action plan of player i when his or her own type is t_i . Player i 's pure strategy set is denoted as $S_i(\cdot) \equiv \prod_{t_i \in T_i} S_i(t_i)$, and the corresponding mixed strategy space is denoted as $\Delta S_i(\cdot) \equiv \prod_{t_i \in T_i} \Delta S_i(t_i)$.

In the following, we describe the Bayesian game with two examples.

Example 6.5.1 Consider a variant of the Prisoner's Dilemma in which the set of players is $N = \{1, 2\}$. Player 1 only has one type. Player 2 has two types, and the set of player 2's types is $T_2 = \{I, II\}$.

The common prior probability of player 2's type distribution is $p(I) = p(II) = 0.5$. Each player's action set is {Deny, Confess}. If Prisoner 2 is of type I , the payoffs of Prisoner 1 and Prisoner 2's interaction are represented

		Prisoner 2: Type <i>I</i>	
		Deny	Confess
Prisoner 1	Deny	-2, -2	-10, -1
	Confess	-1, -10	-5, -5

		Prisoner 2: Type <i>II</i>	
		Deny	Confess
Prisoner 1	Deny	-2, -2	-10, -7
	Confess	-1, -10	-5, -11

Table 6.15: Prisoner's Dilemma with Incomplete Information

by the first matrix in Table 6.15; If Prisoner 2 is of type *II*, the payoffs of Prisoner 1 and Prisoner 2's interaction are represented by the second matrix in Table 6.15.

An auction example is discussed below. In an auction, each bidder has some incomplete information about other bidders. Auctions come in numerous forms. Assume that the auction format employed here is the Second-Price Sealed-Bid Auction.

Example 6.5.2 (Second-Price Sealed-Bid Auction) Suppose that there are n bidders $\{1, 2, \dots, n\}$ participating in an antique auction. Player i 's value of the antique is v_i . Each player only knows his or her own value, but does not know other players' values. Each player's value is independent. A bidder's value of the antique is then the type of the bidder and obeys the same probability distribution $q_i(\cdot) : V \rightarrow (0, 1)$, where V is a set of all possible values. Therefore, each bidder's type set is V . Let b_i be the bid chosen by player i . The bidder with the highest bid gets the antique, but pays the second-highest bidding price. If there are multiple bidders at the highest price, they will get the antique with the same probability and pay their bid price once they win.

In this game, the set of players is $\tilde{N} = \{N, N_0\}$, where N_0 is Nature who determines bidders' types based on the prior probability distribution of other bidders' types. Player i 's action set is $A_i = \mathcal{R}_+$. The set of signals received by player i is $T_i = V$, i.e., all players know their own types. All

players have a common prior probability $p(\mathbf{t}) = \prod_{i \in N} q(t_i)$, where $\mathbf{t} = (t_i)_{i \in N}$.

A profile of players' bids is (b_1, \dots, b_n) . If $b_i > b_j$ for all $j \in N \setminus i$, player i 's payoff is

$$\Pi_i(b_i, \mathbf{b}_{-i}) = \begin{cases} v_i - \max_{j \neq i} b_j & \text{if } b_i > \max_{j \neq i} b_j \\ 0 & \text{if } b_i < \max_{j \neq i} b_j. \end{cases} \quad (6.5.3)$$

If $b_i = \max_{j \neq i} b_j$, the types shall be decided by lot, i.e., the object is randomly assigned by the same probability.

6.5.2 Bayesian-Nash Equilibrium

An equilibrium concept corresponding to the game of incomplete information is Bayesian-Nash equilibrium.

Definition 6.5.2 (Pure-Strategy Bayesian-Nash Equilibrium) A strategy profile $(s_i(t_i)_{t_i \in T_i})_{i \in N}$ is said to be a *pure-strategy Bayesian-Nash equilibrium* of game $(\tilde{N}, (A_i)_{i \in N}, (T_i)_{i \in N}, p, (u_i(\cdot))_{i \in N})$ if for any $i \in N$, $t_i \in T_i$, $s_i'(t_i) \in S(t_i)$, we have

$$\sum_{\mathbf{t}_{-i}} p(\mathbf{t}_{-i} | t_i) u_i(s_i(t_i), \mathbf{s}_{-i}(\mathbf{t}_{-i})) \geq \sum_{\mathbf{t}_{-i}} p(\mathbf{t}_{-i} | t_i) u_i(s_i'(t_i), \mathbf{s}_{-i}(\mathbf{t}_{-i})).$$

It is clear that every dominant strategy equilibrium is a Bayesian-Nash equilibrium, and the converse may not be true.

Definition 6.5.3 (Mixed-Strategy Bayesian-Nash Equilibrium) A strategy profile $(\sigma_i(t_i)_{t_i \in T_i})_{i \in N}$ is a *mixed-strategy Bayesian-Nash equilibrium* of game $(\tilde{N}, (A_i)_{i \in N}, (T_i)_{i \in N}, p, (u_i(\cdot))_{i \in N})$ if for any $i \in N$, $t_i \in T_i$, $\sigma_i'(t_i) \in \Delta S(t_i)$, we have

$$\sum_{\mathbf{t}_{-i}} p(\mathbf{t}_{-i} | t_i) u_i(\sigma_i(t_i), \boldsymbol{\sigma}_{-i}(\mathbf{t}_{-i})) \geq \sum_{\mathbf{t}_{-i}} p(\mathbf{t}_{-i} | t_i) u_i(\sigma_i'(t_i), \boldsymbol{\sigma}_{-i}(\mathbf{t}_{-i})).$$

For the Prisoner's Dilemma with incomplete information, its (pure-strategy) Bayesian-Nash equilibrium is: for player 1, regardless of the type of his opponent, choosing "Confess" is his dominant strategy. For player 2, if her

type is I , choosing “Confess” is her dominant strategy, and if her type is II , choosing “Deny” is her dominant strategy. Therefore, the Bayesian-Nash equilibrium of this game is $(s_1 = \text{Confess}; s_2(I) = \text{Confess}, s_2(II) = \text{Deny})$.

For the Second-Price Sealed-Bid Auction, for any player, choosing $b_i(v_i) = v_i$ is his or her weakly dominant strategy. Therefore, the Bayesian-Nash equilibrium of this game is $(b_i(t_i = v_i) = v_i)_{i \in N, t_i \in T_i}$.

In the above examples, there is a (weakly) dominant strategy for each type of player. In many interaction situations, there is no dominant equilibrium. The following example illustrates that the Bayesian-Nash equilibrium with dominant strategies may not exist.

Example 6.5.3 (First-Price Sealed-Bid Auction) There are bidders whose valuations are independent and follow uniform distributions on $[0, 1]$. The bidder with the higher bid wins the auction item at the bidding price. If they submit the same bidding price, each player obtains the item with equal probability and pays the bidding price. Let v_1 and v_2 be the types of bidders 1 and 2 whose strategies are $b_1(v_1)$ and $b_2(v_2)$, respectively.

We want to solve the symmetric Bayesian-Nash equilibrium, i.e., $b_1(v_1) = b(v_1)$ and $b_2(v_2) = b(v_2)$. We will verify that $b(v)$ is a strictly increasing function. The expected utility of bidder i with type v_i when he or she bids b_i is

$$EU_i(b_i, v_i) = (v_i - b_i) \text{prob}(b(v_j) < b_i) + \frac{1}{2}(v_i - b_i) \text{prob}(b(v_j) = b_i).$$

Since player j has a continuous distribution and the bid is a strictly increasing function of the type, we have $\text{prob}(b(v_j) = b_i) = 0$ and $\text{prob}(b(v_j) < b_i) = \text{prob}(v_j < b^{-1}(b_i)) \equiv \Phi(b_i) = \Phi(b_i)$. Then, $EU_i(b_i, v_i) = (v_i - b_i)\Phi(b_i)$. The first-order condition of maximization is $-\Phi(b(v_i)) + (v_i - b(v_i))\Phi'(b(v_i)) = 0$ or $v_i = 2\Phi(b(v_i))$. Therefore, $b(v_i) = \frac{v_i}{2}$. Obviously, $b_i(v_i) = b(v_i)$ is a strictly increasing function. Therefore, the Bayesian-Nash equilibrium of this First-Price Sealed-Bid Auction is $b_i(v_i) = \frac{v_i}{2}$.

It is sometimes difficult to understand mixed strategies, such as the previous Battle of the Sexes game, in which players randomly choose their

actions. The Bayesian game can provide a rationalized explanation for a mixed strategy. This kind of interpretation of a mixed strategy equilibrium was first proposed by Harsanyi (1974).

Example 6.5.4 (Mixed Strategy and Bayesian-Nash Equilibrium) Consider the Battle of the Sexes game with a mixed strategy in which the woman will choose opera with probability $2/3$ and the man will choose basketball with probability $2/3$. In real life, however, it is an extreme situation in which players know all of the information of other players. For more general and practical interactions, there will always be more or less incomplete information. Some very small incomplete information is introduced to the Battle of the Sexes game, as shown in Table 6.16. What will occur when the degree of incomplete information reaches a certain limit (i.e., complete information)?

Let x_1 and x_2 be the types of the woman and the man, respectively, and they both obey the uniform distribution on $[0, x]$. When $x \rightarrow 0$, the limit of this incomplete information game is a complete information game, i.e., returning to the previous example of Battle of the Sexes. We can use pure strategy Bayesian-Nash equilibrium of the incomplete information game to explain mixed strategy Nash equilibrium of the complete information game. Assume that the Bayesian-Nash equilibrium of this incomplete information game has the following properties: for the woman, as long as x_1 does not exceed a certain threshold $c < x$, she will choose opera; otherwise, she will choose basketball, because the size of x_1 represents the woman's love for basketball. Similarly, for the man, as long as x_2 does not exceed a certain threshold $d < x$, he will choose basketball; otherwise, he will choose opera.

		Male	
		Opera	Basketball
Female	Opera	$2, 1 + x_2$	$0, 0$
	Basketball	$0, 0$	$1 + x_1, 2$

Table 6.16: The Bayesian-Nash Equilibrium Interpretation of Mixed Strategy in the Battle of the Sexes

In this way, the woman expects that the man will choose basketball with a probability of $\frac{d}{x}$ and choose opera with a probability of $1 - \frac{d}{x}$. Similarly, the man expects that the woman will choose opera with a probability of $\frac{c}{x}$ and choose basketball with a probability of $1 - \frac{c}{x}$. For women of type x_1 , the expected utility for choosing opera is $2\frac{x-d}{x}$, and the expected utility for choosing basketball is $(1+x_1)\frac{d}{x}$. When $x_1 = c$, choosing opera or basketball makes no difference for the woman. Therefore, an equilibrium requires $(1+c)\frac{d}{x} = 2\frac{x-d}{x}$ or $(1+c)d = 2(x-d)$. In the same way, we can obtain $(1+d)c = 2(x-c)$. Solving the two equations, we obtain $c = d$ and $\frac{c}{x} = \frac{4}{\sqrt{9+8x+3}}$. Since $\lim_{x \rightarrow 0} \frac{4}{\sqrt{9+8x+3}} = \frac{2}{3}$, when $x \rightarrow 0$, the woman chooses opera and basketball with probabilities $2/3$ and $1/3$, respectively. Similarly, we can also get: when $x \rightarrow 0$, the man chooses basketball and opera with probabilities $2/3$ and $1/3$, respectively.

From the above example, it can be seen that mixed strategy equilibria in a game of complete information can be constructed by the limit of a sequence of pure strategy Bayesian-Nash equilibria for games of incomplete information. We can thus explain why individuals choose mixed strategies by introducing some incomplete information. In other words, a player's judgment of other players' mixed strategies may stem from a lack of understanding them.

6.6 Dynamic Games with Incomplete Information

Now, we discuss dynamic games of incomplete information. This type of game is relatively more realistic. As this type of game exhibits features of both dynamic and incomplete information, it has more subtle factors that affect individuals' interactions, one of which is belief. In a dynamic game of incomplete information, a player can obtain information of his or her opponents' types through their moves, i.e., the move of an opponent may contain some signal about the opponent's type. After obtaining signals, the player can update the subjective judgments of other players' types through Bayes' rules. **Weak perfect Bayesian equilibrium (weak PEB)** is a refinement of both Bayesian-Nash equilibrium and subgame perfect equi-

librium.¹ Since there are usually multiple weak perfect Bayesian equilibria, it is necessary to make further refinements, so the modifier “weak” was added. Imposing some constraints on beliefs, especially beliefs about out-of-equilibrium paths, will result in new equilibrium concepts.

Below, we first introduce the concept of belief system.

Definition 6.6.1 (Belief System) A belief system μ is a decision-maker's subjective judgment on a decision set based on an information set, and is a function defined on the set of decision-making nodes of an extensive-form game Γ_E : $\mu : X \rightarrow [0, 1]$ with $\sum_{y \in h(x)} \mu(y) = 1$ for any information set $h \in H$.

A belief system is actually a subjective judgment on opponents' types and their previous actions. Since players' actions depend on their types, a belief system on an information set is a subjective judgment on opponents' types. Opponents' previous actions may reveal their types. A belief system on an information set is usually a revision of posterior beliefs about opponents' types.

In a dynamic game of incomplete information, just like in a dynamic game of complete information, the desirability for players' rationality is sequential rationality. Sequential rationality refers to a situation in which on any information set, given his or her beliefs, a player will choose the behavior strategy that can bring the highest expected payoffs from all possible behavior strategies.

Definition 6.6.2 (Sequential Rationality) A strategy profile $\sigma = (\sigma_h)_{h \in H}$, $\sigma_h \in \Delta A(h)$ and a corresponding belief system μ are *sequentially rational* of an extensive-form game, if for every $h \in H$, we have

$$E_{\iota(h)}[u_{\iota(h)}(\sigma_h, \sigma_{-h})|h, \mu] \geq E_{\iota(h)}[u_{\iota(h)}(\sigma'_h, \sigma_{-h})|h, \mu], \forall \sigma'_h \in \Delta A(h),$$

where $\sigma_{-h} = (\sigma_{h'})_{h' \in H/h}$.

We first provide an example of strategy profile that is not sequentially rational.

¹When defining the signaling game in a subsequent subsection, we will impose further restriction so that we will introduce the solution concept of **perfect Bayesian equilibrium**, PBE).

Example 6.6.1 (A Behavior Strategy Profile That is Not Sequentially Rational)

Figure 6.11 depicts a market entry game with two firms. Although there is no uncertain information on players' types, there are multiple decision-making nodes on firm I 's information set. Firms' profile of behavior strategies can be discussed in the framework of games of incomplete information.

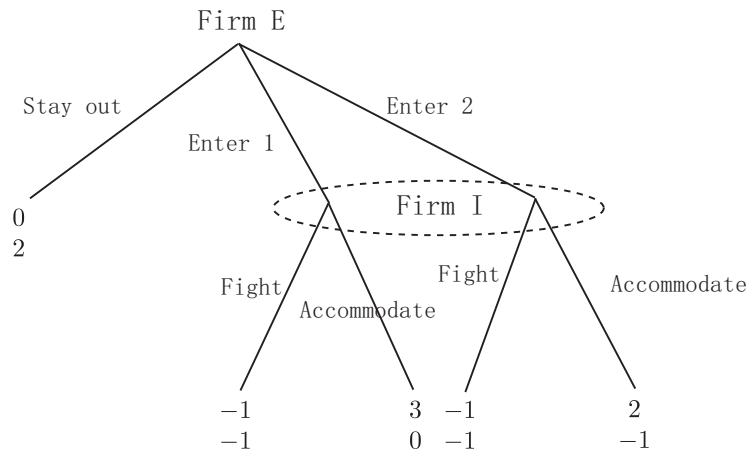


Figure 6.11: Market Entry Game

We first discuss firm I 's behavior strategy. In the behavior strategy profile (Stay Out, Fight), "Stay Out" is chosen on firm E 's information set, and "Fight" is chosen on firm I 's information set. Since each player has only one information set in this game, this behavior strategy profile is also a strategy profile of this game. Since there is only one subgame in this game, i.e., the original game, (Stay Out, Fight) is a subgame perfect Nash equilibrium. However, (Stay Out, Fight) does not satisfy sequential rationality, because on firm I 's information set, regardless of her belief system, "Accommodate" is always more favorable than "Fight".

In the above definition of sequential rationality, rationality relies on beliefs, and thus in an equilibrium concept, the consistency between a behavior strategy's sequential rationality and beliefs is required, which is different from previous equilibrium concepts. An equilibrium of a dynamic game with incomplete information constitutes a tuple of strategy and be-

liefs. Since players' strategy choices will affect decision-makers' information sets and their belief systems on these information sets, a player's belief system depends on players' strategies. Usually, the revision of a player's beliefs obeys Bayes' rule. If a certain information set h can be reached with a positive probability under players' behavior strategy profile σ , then by Bayes' rule, the belief system on information set h is:

$$\text{prob}(x|h, \sigma) = \frac{\text{prob}(x|\sigma)}{\text{prob}(h|\sigma)}. \quad (6.6.4)$$

However, under certain behavior strategy profiles, since not all possible information sets can be reached with a positive probability, the denominator in the above formula may be zero, which is a situation in which Bayes' rule does not apply. We usually refer to the information set which will be reached with zero probability as an *information set off the equilibrium path*. In the dynamic game of incomplete information, there are different equilibrium concepts corresponding to different requirements for the information sets off the equilibrium path. If an information set contains only one decision-making node, it is clear that the belief on this information set is giving probability 1 to this decision-making node. The solution concept of **weak perfect Bayesian equilibrium** given below does not impose any constraints on the beliefs on information sets off the equilibrium path, but rather imposes restrictions only on the information sets on an equilibrium path, requiring that beliefs are consistent with the equilibrium strategy's sequential rationality.

6.6.1 Weak Perfect Bayesian Equilibrium

An equilibrium concept of the dynamic game with incomplete information is weak perfect Bayesian equilibrium.

Definition 6.6.3 (Weak Perfect Bayesian Equilibrium) A behavior strategy profile and system of beliefs (σ, μ) is a *weak perfect Bayesian equilibrium* (weak PBE) of an extensive-form game Γ_E if the following conditions are met:

- (1) Given belief system μ , the behavior strategy profile σ is se-

quentially rational;

- (2) Belief system μ is derived from behavior strategy profile σ and initial beliefs (if they exist) through Bayes' rule. In other words, as long as information set h can be reached with a positive probability under the behavior strategy profile σ , i.e., $\text{prob}(h|\sigma) > 0$, then for all $x \in h$, the belief on information set h is $\mu(x) = \frac{\text{prob}(x|\sigma)}{\text{prob}(h|\sigma)}$.

If $\text{prob}(h|\sigma) = 0$, the concept of weak perfect Bayesian equilibrium imposes no constraints on the belief of information set h .

We now solve for the weak PBE of the previous game given by 6.11.

Example 6.6.2 In the Market Entry Game with firm E and firm I , the game has a unique weak PBE strategy profile (Enter, Accommodate). Indeed, their beliefs satisfy the following: the belief probability on firm E 's decision-making node is 1, since firm E 's information set contains only one decision-making node, and the belief probability on the left decision-making node of firm I 's information set is 0, and the belief probability on the right decision-making node of firm I 's information set is 1. This is attributable to the fact that “Accommodate” is a dominant strategy on firm I 's information set, i.e., irrespective of the beliefs on firm I 's information set, sequential rationality requires that firm I chooses “Accommodate”. If firm I chooses “Accommodate”, the optimal choice for firm E is “Enter”, which is also an equilibrium strategy under the requirement of sequential rationality. Under this strategy choice, the belief probability on the left decision-making node of firm I 's information set is 0, and the belief probability on the right decision-making node of firm I 's information set is 1.

However, the weak perfect Bayesian equilibrium may sometimes include some unreasonable equilibria because it does not impose any constraints on beliefs off the equilibrium path. In some cases, a weak perfect Bayesian equilibrium may not even be a subgame perfect equilibrium. Consider the following market entry game.

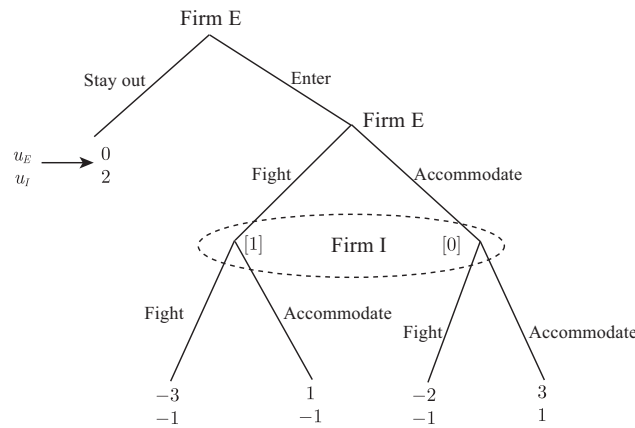


Figure 6.12: Weak perfect Bayesian equilibrium is not a subgame perfect equilibrium.

Example 6.6.3 As shown in Figure 6.12, firm E is the potential market entrant, and firm I is the incumbent. Firm E first chooses whether or not to enter. Once firm E enters, firm E and firm I choose whether to accommodate or fight simultaneously.

A weak perfect Bayesian equilibrium of this game is that firm E chooses to stay out and if firm E enters, then firm E chooses to accommodate and firm I chooses to fight. Indeed, firm I 's beliefs are: on firm I 's information set, the belief probability on the left decision-making node is 1, and the belief probability on the right decision-making node is 0. Thus, firm I 's information set is located off the equilibrium path, there are no constraints on beliefs on this information set. In addition, on the basis of these beliefs, firm I 's strategy choice "Fight" satisfies the requirement of sequential rationality. Given firm I 's strategy choice, firm E 's strategy choice "Accommodate" also satisfies the requirement of sequential rationality. The arrows in Figure 6.12 describe equilibrium strategies for firm E and firm I . The figures next to two decision-making nodes on firm I 's information set describe firm I 's beliefs.

However, this weak perfect Bayesian equilibrium is not a subgame perfect equilibrium, because in the subgame after firm E enters, the only Nash equilibrium is that both firm E and firm I choose "Accommodate". Indeed, the only subgame perfect equilibrium of this game is that firm E

chooses to enter, and once it enters, it chooses “Accommodate” and firm I chooses “Accommodate” after firm E enters.

This example shows that since there are no constraints on beliefs off the equilibrium path, a weak perfect Bayesian equilibrium may not be a subgame perfect equilibrium. Firm I 's beliefs off the equilibrium path do not match firm E 's strategy in the subgame.

The following example further illustrates that due to the lack of constraints on beliefs off the equilibrium path, these beliefs become unreasonable.

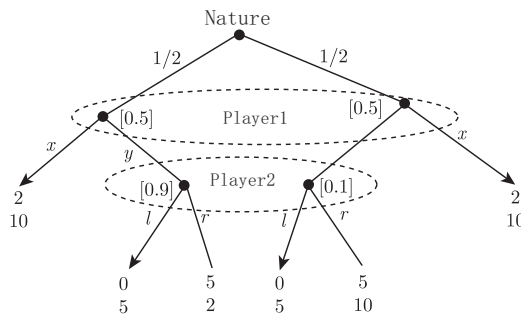


Figure 6.13: Beliefs off the equilibrium path

Example 6.6.4 In the game depicted in Figure 6.13, “Nature” randomly selects a decision-making node on player 1’s information set, and the probability with which any decision-making node on this information set is selected is 0.5. Player 1 does not know “Nature”’s choice; player 2 on her information set does not know “Nature”’s choice either, but knows that player 1’s choice is “ y ”. A weak perfect Bayesian equilibrium of this game is that player 1 chooses “ x ” and player 2 chooses “ l ”. Moreover, player 1’s beliefs on his information set are 0.5 for the left decision-making node and 0.5 for the right decision-making node, and player 2’s beliefs on her information set are 0.9 for the left decision-making node and 0.1 for the right decision-making node.

Obviously, given the beliefs of players 1 and 2, their strategies meet the requirement of sequential rationality, while player 1’s beliefs coincide with “Nature’s” selection probabilities. Player 2’s information set is off

the equilibrium path, and there are no constraints on her beliefs on player 1's information set. However, player 2's beliefs are not consistent with player 1's strategy and "Nature" 's choices. Since player 1 has the same probability on his two decision-making nodes, once player 1 has chosen y , player 2's beliefs on player's 1 information set should be 0.5 for both of her decision-making nodes, instead of 0.9 for the left decision-making node and 0.1 for the right decision-making node.

The above example shows that it is necessary to impose some rational constraints on beliefs off the equilibrium path; otherwise, a weak perfect Bayesian equilibrium may contain some unreasonable beliefs and strategy profiles. Below, we will discuss some other equilibrium concepts which impose certain reasonable constraints on beliefs off the equilibrium path.

6.6.2 Sequential Equilibrium

Due to the problem of weak perfect Bayesian equilibrium, a reasonable equilibrium concept needs to impose suitable constraints on information sets off the equilibrium path. Kreps and Wilson (1982a) proposed the concept of sequential equilibrium to constrain beliefs off the equilibrium path. In a Bayesian game, if a certain information set is reached with probability 0 in an equilibrium path, Bayes' rule cannot be used to restrict the beliefs on this information set. The concept of sequential equilibrium uses a method of **completely mixed strategy**, which means that the probability of reaching any information set is positive, so that beliefs on any information set can be constrained according to Bayes' rule.

Definition 6.6.4 (Sequential Equilibrium) A strategy profile and a belief system (σ, μ) constitute a *sequential equilibrium* of an extensive-form game Γ_E if the following conditions are satisfied:

- (1) The strategy profile σ is sequential rational given belief system μ .
- (2) There is a completely mixed strategy sequence $\{\sigma^k\}_{k=1}^{\infty}$, such that $\lim_{k \rightarrow \infty} \sigma^k = \sigma$ and $\lim_{k \rightarrow \infty} \mu^k = \mu$, where μ^k are the beliefs derived from strategy σ^k using Bayes' rule.

When verifying whether or not a strategy profile and a belief system constitute a sequential equilibrium, we only need to find a completely mixed disturbing sequence which approaches the strategy profile, and determine if the belief system sequence based on the completely mixed strategy sequence converges to the belief system.

The following example, as shown in Figure 6.14, discusses completely mixed strategies and the calculation of beliefs.

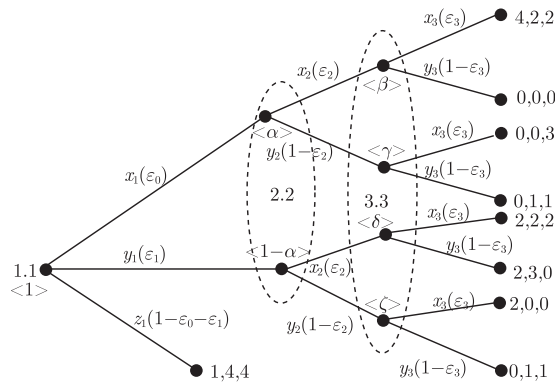


Figure 6.14: Completely Mixed Strategy and Its Beliefs

Example 6.6.5 (Beliefs of Completely Mixed Strategy Sequence) Suppose that strategy profile (z_1, y_2, y_3) is disturbed into a completely mixed strategy profile. Player 1 chooses strategy z_1 with probability $1 - \varepsilon_0 - \varepsilon_1$, chooses strategy y_1 with probability ε_1 and chooses strategy x_1 with probability ε_0 . When $\varepsilon_0 \rightarrow 0, \varepsilon_1 \rightarrow 0$, player 1's strategies converge to pure strategy z_1 . Similarly, player 2 chooses strategy y_2 with probability $1 - \varepsilon_2$ and chooses strategy x_2 with probability ε_2 ; and player 3 chooses strategy y_3 with probability $1 - \varepsilon_3$ and chooses strategy x_3 with probability ε_3 .

Under this completely mixed strategy profile, players' belief systems are as follows: since player 1's information set has only one decision-making node, the belief probability on this decision-making node is 1; player 2's information set has two decision-making nodes, and according to player 1's completely mixed strategy, the belief probability of player 2 on the above decision-making node is $\alpha = \frac{\varepsilon_0}{\varepsilon_0 + \varepsilon_1}$, and the belief probability of player 2 on the below decision-making node is $1 - \alpha = \frac{\varepsilon_1}{\varepsilon_0 + \varepsilon_1}$; the belief

probability of player 3 on the top decision-making node is $\beta = \frac{\varepsilon_0 \varepsilon_2}{\varepsilon_0 + \varepsilon_1} = \alpha \varepsilon_2$, the belief probability of player 3 on the second decision-making node above is $\gamma = \frac{\varepsilon_0(1-\varepsilon_2)}{\varepsilon_0 + \varepsilon_1} = \alpha(1 - \varepsilon_2)$, the belief probability of player 3 on the third decision-making node above is $\delta = \frac{\varepsilon_1 \varepsilon_2}{\varepsilon_0 + \varepsilon_1} = (1 - \alpha)\varepsilon_2$, and the belief probability of player 3 on the bottom decision-making node is $\zeta = \frac{\varepsilon_1(1-\varepsilon_2)}{\varepsilon_0 + \varepsilon_1} = (1 - \alpha)(1 - \varepsilon_2)$. When $\varepsilon_0, \varepsilon_1, \varepsilon_2$ and ε_3 approach to 0, β and δ both approach to 0, and $\gamma \rightarrow \alpha, \zeta \rightarrow 1 - \alpha$.

Strategy profile (z_1, y_2, y_3) , and player 2's belief probability $\alpha \in (0, 1)$ and player 3's belief probability $\gamma = \alpha, \zeta = 1 - \alpha, \beta = 0, \delta = 0$ constitute a weak perfect Bayesian equilibrium which is not a sequential equilibrium here.

This is because when $\varepsilon_0, \varepsilon_1, \varepsilon_2$ and ε_3 all approach to 0, completely mixed strategy profiles converge to (z_1, y_2, y_3) . The fact that player 3's choice of y_3 is sequentially rational requires that the expected payoff from choosing strategy x_3 is lower than that from choosing strategy y_3 when $\gamma = \alpha, \zeta = 1 - \alpha, \beta = 0, \delta = 0$ so that the belief system satisfies $3\alpha < 1$. The fact that player 2's choice of y_2 is sequentially rational requires that the belief system satisfies $3(1 - \alpha) < 1$. Obviously, the above two inequalities cannot be satisfied at the same time.

Now, we solve for the sequential equilibrium of this example. In Figure 6.14, we consider $\varepsilon_0, \varepsilon_1, \varepsilon_2$ and ε_3 to be real numbers that belong to $[0, 1]$, and $\varepsilon_0 + \varepsilon_1 \leq 1$ is satisfied. In the belief system described above, a sequential equilibrium requires that player 3 satisfies sequential rationality.

Player 3's sequential rationality means:

- If $\varepsilon_3 = 0$, i.e., for player 3, the expected payoff from choosing strategy x_3 is lower than that from choosing strategy y_3 , $\zeta > 2(\beta + \gamma + \delta) = 2(1 - \zeta)$, i.e., $(1 - \alpha)(1 - \varepsilon_2) > \frac{2}{3}$ is required;
- If $\varepsilon_3 = 1$, $(1 - \alpha)(1 - \varepsilon_2) < \frac{2}{3}$ is required;
- If $\varepsilon_3 \in (0, 1)$, $(1 - \alpha)(1 - \varepsilon_2) = \frac{2}{3}$ is required.

Player 2's sequential rationality means the following:

- If $\varepsilon_2 = 0$, $2\varepsilon_3 + 3(1 - \alpha)(1 - \varepsilon_3) < 1 - \varepsilon_3$ is required;
- If $\varepsilon_2 = 1$, $2\varepsilon_3 + 3(1 - \alpha)(1 - \varepsilon_3) > 1 - \varepsilon_3$ is required;

If $\varepsilon_2 \in (0, 1)$, $2\varepsilon_3 + 3(1 - \alpha)(1 - \varepsilon_3) = 1 - \varepsilon_3$ is required.

Beliefs need to be consistent. Suppose that $\varepsilon_3 = 0$ is player 3's strategy in a sequential equilibrium. Then, there is $(1 - \alpha)(1 - \varepsilon_2) > \frac{2}{3}$. However, $2\varepsilon_3 + 3(1 - \alpha)(1 - \varepsilon_3) = 3(1 - \alpha) > 2 > 1 - \varepsilon_3 = 1$ implies that $\varepsilon_2 = 1$, which contradicts $(1 - \alpha)(1 - \varepsilon_2) > \frac{2}{3}$. As such, $\varepsilon_3 = 0$ cannot be player 3's strategy in a sequential equilibrium. Assume that $\varepsilon_3 \in (0, 1)$, i.e., choosing a completely mixed strategy, is player 3's strategy in a sequential equilibrium. Then, there is $(1 - \alpha)(1 - \varepsilon_2) = \frac{2}{3}$, which further requires $(1 - \alpha) \geq \frac{2}{3}$. $(1 - \alpha) \geq \frac{2}{3}$ means $2\varepsilon_3 + 3(1 - \alpha)(1 - \varepsilon_3) > 1 - \varepsilon_3$, which further requires $\varepsilon_2 = 1$, contradicting $(1 - \alpha)(1 - \varepsilon_2) = \frac{2}{3}$. Therefore, if a sequential equilibrium exists, player 3's strategy must inevitably satisfy $\varepsilon_3 = 1$, and thus $(1 - \alpha)(1 - \varepsilon_2) < \frac{2}{3}$ must be satisfied. $\varepsilon_3 = 1$ means $2\varepsilon_3 + 3(1 - \alpha)(1 - \varepsilon_3) > 1 - \varepsilon_3$, and thus $\varepsilon_2 = 1$, which is compatible with $(1 - \alpha)(1 - \varepsilon_2) < \frac{2}{3}$. The above discussion implies that a sequential equilibrium requires $\varepsilon_3 = 1$ and $\varepsilon_2 = 1$. Further, player 1's sequential rationality requires that player 1 chooses strategy x_1 , i.e., $\varepsilon_0 = 1$. Therefore, the game has a unique sequential equilibrium, i.e., strategy profile (x_1, x_2, x_3) , and the belief system satisfies $\alpha = 1$, $\beta = 1$ and $\gamma = \delta = \zeta = 0$.

Below, we solve an example which is an extensive-form game with "Nature" 's actions. We solve for the sequential equilibrium of this game. The game representation is shown in Figure 6.15.

Example 6.6.6 (Solving for Sequential Equilibrium) In this game, "Nature" chooses the upper decision-making node on player 1's information set with probability 0.95. The number in the angled brackets on a decision-making node indicates the belief probability of this decision-making node, and the number in the parentheses below a branch indicates the probability with which the corresponding player chooses this pure strategy. A sequential equilibrium of this game necessitates that the beliefs on player 1's information set satisfy $\alpha = 0.95$. Information set 2.4 is the second information set for player 2, and according to the requirement of sequential rationality, player 2 chooses f_4 or $\zeta = 0$. Information set 1.3, which contains two decision-making nodes, is player 1's information set, which is the outcome of player

1 choosing g_1 . Player 1 cannot distinguish between these two decision-making nodes on information set 1.3. Assume that the probability with which player 2 chooses g_2 is γ . By Bayes' rule, the beliefs on information set 1.3 satisfy $\delta = \frac{0.95\beta\gamma}{0.95\beta\gamma+0.05\beta} = \frac{19\gamma}{19\gamma+1}$.

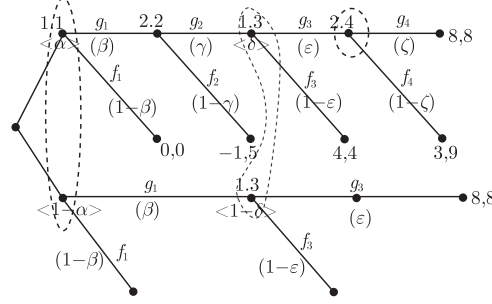


Figure 6.15: Solving for Sequential Equilibrium

Under this belief system, player 1's sequential rationality requires the following:

When $4 > 3\delta + 8(1 - \delta)$, player 1's behavior strategy on information set 1.3 is $\varepsilon = 0$;

When $4 < 3\delta + 8(1 - \delta)$, player 1's behavior strategy on information set 1.3 is $\varepsilon = 1$;

When $4 = 3\delta + 8(1 - \delta)$, player 1's behavior strategy on information set 1.3 is $\varepsilon \in (0, 1)$.

On information set 2.2, player 2's sequential rationality requires:

When $9\varepsilon + 4(1 - \varepsilon) > 5$, player 2's behavior strategy on information set 2.2 is $\gamma = 1$;

When $9\varepsilon + 4(1 - \varepsilon) < 5$, $\gamma = 0$;

When $9\varepsilon + 4(1 - \varepsilon) = 5$, $\gamma \in [0, 1]$.

On information set 1.1, player 1's sequential rationality requires the following:

When $0.95[3\gamma\varepsilon + 4\gamma(1 - \varepsilon) - (1 - \gamma)] + 0.05[8\varepsilon + 4(1 - \varepsilon)] > 0$, player 1's behavior strategy on information set 1.1 is $\beta = 1$;

When $0.95[3\gamma\varepsilon + 4\gamma(1 - \varepsilon) - (1 - \gamma)] + 0.05[8\varepsilon + 4(1 - \varepsilon)] < 0$,
 $\beta = 0$;

When $0.95[3\gamma\varepsilon + 4\gamma(1 - \varepsilon) - (1 - \gamma)] + 0.05[8\varepsilon + 4(1 - \varepsilon)] = 0$,
 $\beta \in [0, 1]$.

Below, we describe the solution process of sequential equilibrium.

Suppose that $\varepsilon = 1$ holds in a sequential equilibrium. $\varepsilon = 1$ implies that $4 < 3\delta + 8(1 - \delta)$, i.e., $\delta < 0.8$, or $\frac{19\gamma}{19\gamma+1} < 0.8$, or $\gamma < 4/19$. However, $\varepsilon = 1$ also means that $9\varepsilon + 4(1 - \varepsilon) = 9 > 5$, and $\gamma = 1$, which contradicts $\gamma < 4/19$.

Suppose that $\varepsilon = 0$ holds in a sequential equilibrium. $\varepsilon = 0$ means $4 > 3\delta + 8(1 - \delta)$, and $\gamma > 4/19$. However, $\varepsilon = 0$ also means that $9\varepsilon + 4(1 - \varepsilon) = 4 < 5$, and $\gamma = 0$, which contradicts $\gamma > 4/19$.

As a result, in a sequential equilibrium, there must be $\varepsilon \in (0, 1)$, which requires $4 = 3\delta + 8(1 - \delta)$ or $\gamma = 4/19$. $\gamma = 4/19$ means $9\varepsilon + 4(1 - \varepsilon) = 5$, and $\varepsilon = 0.2$.

When $\varepsilon = 0.2$ and $\gamma = 4/19$, $0.95[3\gamma\varepsilon + 4\gamma(1 - \varepsilon) - (1 - \gamma)] + 0.05[8\varepsilon + 4(1 - \varepsilon)] = 0.25 > 0$, which means that $\beta = 1$.

Therefore, the only sequential equilibrium of the entire game is as follows: the behavior strategy profile is $\beta = 1, \gamma = 4/19, \varepsilon = 0.2, \zeta = 0$; and the belief system is $\alpha = 0.95, \delta = 0.8$.

In a finite extensive-form game, a sequential equilibrium usually exists. Readers who are interested in the proof of this theorem can refer to the classical literature of Kreps and Wilson (1982).

6.6.3 Forward Induction

In some games, the rationalization of beliefs not only requires rational backward induction, but also rational forward induction. The following example (see Figure 6.16) reflects rational forward induction.

Example 6.6.7 In the game depicted in Figure 6.16, there are two (pure strategy) sequential equilibria. One is behavior strategy profile (a_1, y_1, y_2) , and the belief probability of the lower decision-making node on player 2's information set 2.2 is 1. The other is behavior strategy (b_1, x_1, x_2) , and the belief probability of the upper decision-making node on player 2's information set 2.2 is 1. However, the first sequential equilibrium does not satisfy the forward induction criterion. If player 2 enters information set 2.2, player 1 has not chosen behavior strategy a_1 on information set 1.0. If player 1 chooses a_1 , his payoff is 4. If player 1 is rational, the goal of having not chosen strategy a_1 on information set 1.0 is to obtain a higher payoff in the following subgame equilibrium. In consequence, the Nash equilibrium of the subgame starting from information set 1.1 is (x_1, x_2) .

If the Nash equilibrium of this subgame is (y_1, y_2) , player 1 only obtains a payoff of 3, which is not as good as choosing strategy a_1 on information set 1.0. In other words, the strategy (b_1, y_1) is player 1's strictly dominated strategy (relative to strategy a_1). If player 1 knows that player 2 will reason in this way, then once the subgame starting from information set 1.1 is entered, the Nash equilibrium will be (x_1, x_2) . This reasoning process is called forward induction. In the game shown in this example, only the sequential equilibrium (b_1, x_1, x_2) satisfies the forward induction criterion. In this sequential equilibrium, the belief probability of the upper decision-making node on player 2's information set 2.2 is 1.

However, the forward induction criterion may conflict with the backward induction criterion. The following example (see Figure 6.17) shows the possibility of such a conflict.

Example 6.6.8 In this example, by backward induction, there are two Nash equilibria (x_1, x_2) and (y_1, y_2) in the subgame that starts from information set 1.3, and their equilibrium payoffs are $(9, 0)$ and $(1, 8)$, respectively. On information set 2.2, if player 2 chooses a_2 , her payoff is only 7, and if player 2 chooses b_2 , he or she wants to obtain an equilibrium payoff of 8 in the subgame starting from the information set 1.3 (otherwise, he or she has chosen a_2). Therefore, the Nash equilibrium is (y_1, y_2) .

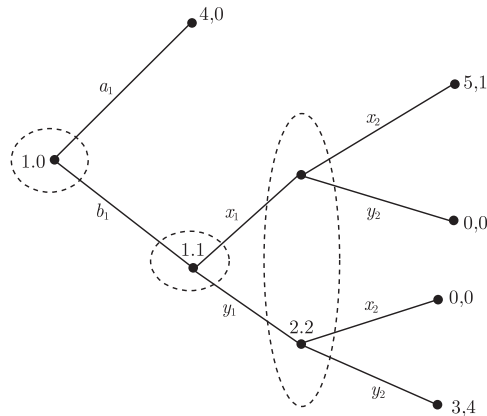


Figure 6.16: Forward Induction

However, reaching information set 1.3 indicates that player 1 has chosen b_1 on information set 1.1. If player 1 chooses a_1 on information set 1.1, his payoff is only 2. As a consequence, for player 1, the purpose of choosing b_1 is to ultimately obtain a payoff that is no less than 2. However, when the backward induction is combined with the previous forward induction, player 1's final equilibrium payoff is 1 if he or she chooses b_1 , which contradicts the forward induction here.

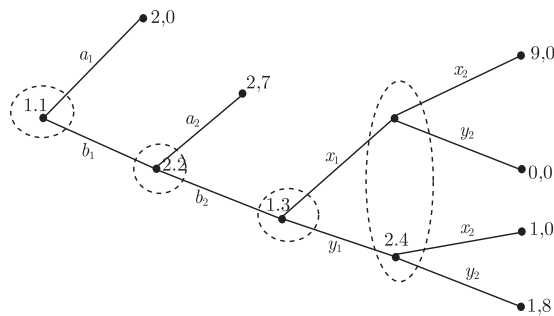


Figure 6.17: Conflict between Forward Induction and Backward Induction

Another objection to forward induction is that some irrational strategy disturbances may be misunderstood as purposefully rational actions. As in the previous example, player 1 originally intends to choose a_1 , but may accidentally choose b_1 . Therefore, player 2 may think that player 1 will

choose x_1 in the subsequent subgame.

The above extensive-form dynamic games do not consider players' incomplete information, i.e., we assume types of players are complete information. In dynamic games with incomplete information, an important kind of game is called the signaling game, in which players' types can be inferred through their actions. In this kind of game, there are many sequential equilibria which call for further refinements.

6.6.4 Signaling Game

Spence (1973) proposed a new idea when discussing the value of education. He found that an important function of education was to deliver signals. In the labor market, different workers have heterogeneous productivities. However, individual productivity is private information so that employers usually do not know or is costly to obtain. A simple and convenient way of judging individual productivity is through years of education and diplomas. Different levels of education may reflect different intrinsic productivity. From years of education and diplomas, employers can speculate on potential employees' types.

Generally, a signaling game can be described by a two-stage extensive-form game. Assume that there are two players 1 and 2, and player 1's type θ is his private information. The set of all possible types is denoted as Θ . The prior distribution of types is $p(\cdot) : \Theta \rightarrow [0, 1]$, which is common knowledge. Player 1's action in the first stage is represented by a_1 , and the set of all possible actions is denoted as A_1 . In the second stage, after observing player 1's action a_1 , player 2 chooses action a_2 . The set of all possible actions of player 2 is denoted as A_2 . When player 1's type θ becomes public information, these two players' payoffs are $u^1(a_1, a_2, \theta)$ and $u^2(a_1, a_2, \theta)$, respectively. Let $\alpha_1 \in \Delta A_1$ and $\alpha_2 \in \Delta A_2$ represent the mixed actions of players 1 and 2, respectively.

Player 1's strategy $\sigma_1(\cdot|\theta)$ describes the probability distribution on his action set A_1 when his type is θ ; player 2's strategy $\sigma_2(\cdot|a_1)$ describes the probability distribution on her action set A_2 after he or she observes player 1's action a_1 . Prior to taking her action, player 2 speculates that the prob-

ability with which player 1's type is θ is $\mu(\theta|a_1)$. The formation of this posterior belief depends on player 1's strategy a_1 and Bayes' rule.

The equilibrium concept adopted here is that of a weak perfect Bayesian equilibrium (PBE), but with an added condition that requires that player 2's beliefs following a_1 are consistent with the equilibrium strategies both on and off the equilibrium path. We refer a weak perfect Bayesian equilibrium satisfying this extra requirement as a perfect Bayesian equilibrium (PBE).

Definition 6.6.5 (Perfect Bayesian Equilibrium of Signaling Games) A *perfect Bayesian equilibrium* of a signaling game consists of a strategy profile $(\sigma_1^*(\cdot|\theta), \sigma_2^*(\cdot|a_1))$ and posterior beliefs $\mu(\cdot|a_1)$, satisfying:

- (1) Given a_1 , $\sigma_2^*(\cdot|a_1) \in \arg \max_{\alpha_2} \sum_{\theta \in \Theta} \mu(\theta|a_1) u^2(a_1, \alpha_2, \theta)$;
- (2) Given $\theta \in \Theta$, $\sigma_1^*(\cdot|\theta) \in \arg \max_{\alpha_1} u^1(\alpha_1, \sigma_2^*(\cdot|a_1), \theta)$;
- (3) $\mu(\theta|a_1) = \frac{p(\theta)\sigma_1^*(a_1|\theta)}{\sum_{\theta'} p(\theta')\sigma_1^*(a_1|\theta')}$, if $\sum_{\theta'} p(\theta')\sigma_1^*(a_1|\theta') > 0$; otherwise $\mu(\cdot|a_1)$ is an arbitrary probability distribution on Θ , if $\sum_{\theta'} p(\theta')\sigma_1^*(a_1|\theta') = 0$.

Although the definition of perfect Bayesian equilibrium in principle is consistent with the previous weak perfect Bayesian equilibrium, in the signaling game, a great correlation exists between the perfect Bayesian equilibrium and sequential equilibrium. Indeed, Fudenberg and Tirole (1991) proved that in a two-stage or two-type signaling game, they are equivalent. In the following, we will discuss the signaling game's equilibrium concept through an example.

Example 6.6.9 (Education Game) Suppose that there are two different types of individuals whose intrinsic productivities are θ_h and θ_l , respectively, where $\theta_h > \theta_l$. Productivity can be viewed as unit labor's output value, and the proportion of high-productivity individuals is λ (priori distribution).

The costs of education level e for different types of individuals are $C(e, \theta)$, which satisfy $C(0, \theta) = 0$, $C_e(e, \theta) > 0$, $C_{ee}(e, \theta) > 0$, $C(e, \theta_h) < C(e, \theta_l)$, and $C_e(e, \theta_h) < C_e(e, \theta_l)$.

Assume that in the labor market, due to competition, the wage paid by an employer equals to the expected labor productivity of the worker. In the first stage, a job seeker chooses his education level e . In the second stage, after observing the job seeker's education level e , an employer's posterior belief (the probability inferred by an employer that the job seeker's type is θ_h) is $\mu(\theta_h|e)$, and thereby the employer pays wage $w = \mu(\theta_h|e)\theta_h + (1 - \mu(\theta_h|e))\theta_l$. The employer chooses to provide a labor contract with wage w , and the job seeker chooses whether or not to accept it. If the labor contract is accepted, the payoff of the job seeker whose type is θ is $u(w, e|\theta) = w - C(e, \theta)$; if the labor contract is not accepted, it is assumed that the payoff of each type of job seeker is zero.

The equilibrium of such a signaling game is usually divided into two types. One is **separating equilibrium**, in which different types of individuals choose different actions in the first stage; the other is **pooling equilibrium**, in which all types of individuals choose the same action in the first stage.

Separating equilibrium: Let e_h and e_l be the education levels chosen by job seekers of type θ_h and θ_l , respectively, in the first stage and $e_h \neq e_l$. After observing the education level e , the employer provides the labor contract $w(e)$, which satisfies $w_h = w(e_h)$ and $w_l = w(e_l)$ in the separating equilibrium. Since $e_h \neq e_l$, the employer's beliefs in the separating equilibrium are $\mu(e_h) = 1$ and $\mu(e_l) = 0$. Thus, the wages provided by the employer after observing different education levels satisfy $w_h = \theta_h$ and $w_l = \theta_l$. At the same time, for job seekers of type θ_l , since their type will be known to the employer in the separating equilibrium, the rational choices satisfy $e_l = 0$. Since it is a separating equilibrium, $e_h \neq 0$. However, arbitrary $e_h > 0$ does not necessarily constitute a separating equilibrium. First, under e_h and wage $w(e_h) = \theta_h$, job seekers of type θ_l have no incentive to imitate the action of job seekers of type θ_h ; otherwise, $\mu(e_h) \neq 1$, which contradicts separating equilibrium. As a consequence, a separating equilibrium necessitates $u(\theta_h, e_h|\theta_l) < u(\theta_l, e_l|\theta_l)$, i.e., $\theta_h - C(e_h, \theta_l) < \theta_l$. Let \tilde{e} satisfy $\theta_h - C(\tilde{e}, \theta_l) = \theta_l$. Then, a separating equilibrium requires $e_h \geq \tilde{e}$. Secondly, under $e_l = 0$ and wage $w(e_l) = \theta_l$, job seekers of type θ_h have no incentive to imitate the action of job seekers of type θ_l ; otherwise, $\mu(e_l) \neq 0$,

which contradicts separating equilibrium. As such, a separating equilibrium requires $u(\theta_l, e_l|\theta_h) < u(\theta_h, e_h|\theta_h)$, i.e., $\theta_l < \theta_h - C(e_h, \theta_h)$. Let \bar{e} satisfy $\theta_l = \theta_h - C(\bar{e}, \theta_h)$. Obviously, $\bar{e} > \tilde{e}$. So, when $\bar{e} \geq e_h \geq \tilde{e}$, each type of job seeker has no incentive to imitate the action of the other type of job seeker. Therefore, the separating equilibria of this game are: $e_l = 0$, $e_h \in [\tilde{e}, \bar{e}]$, the beliefs of the employer are $\mu(e_h) = 1$ and $\mu(0) = 0$, and the strategies of the employer are $w(e_h) = \theta_h$, $w(0) = \theta_l$. In all of the above separating equilibria, $e_l = 0$, $e_h = \bar{e}$, $\mu(\bar{e}) = 1$, $\mu(0) = 0$, $w(e_h) = \theta_h$ and $w(0) = \theta_l$ is a Pareto optimal equilibrium, in the sense that no one can be better off without harming the other (a general definition of Pareto optimality will be given in Chapter 11).

Pooling equilibrium: In this type of equilibrium, different types of job seekers choose the same level of education, i.e., $e(\theta_h) = e(\theta_l) = e^*$. Since employers observe only one level of education in the equilibrium, the beliefs of employers are the same as the initial beliefs, i.e., $\mu(\theta_h|e^*) = \lambda$. The beliefs off the equilibrium path are $\mu(\theta_h|e \neq e^*) = 0$. At this point, employers pay $w_p = \lambda\theta_h + (1 - \lambda)\theta_l$. In order to form a pooling equilibrium, e^* is also faced with the constraint that no type of job seeker deviates from this choice. The fact that job seekers of type θ_l are willing to accept employers' labor contracts means that $u(w_p, e^*|\theta_l) = w_p - C(e^*, \theta_l) \geq 0$. Let \hat{e} satisfy $w_p - C(\hat{e}, \theta_l) = 0$. When $e^* > \hat{e}$, job seekers of type θ_l will reject employers' contracts, and thus $\mu(\theta_h|e^*) \neq \lambda$, which contradicts the pooling equilibrium. Consequently, this game's pooling equilibrium requires $e^* \leq \hat{e}$.

In the signaling games above, we obtain many (continuum) equilibria. These equilibria are all sequential equilibrium or perfect Bayesian equilibrium. Therefore, these equilibrium concepts need to be further refined. The refinement principle of “intuitive criterion” proposed by Cho and Kreps (1987) can reduce the above signaling game equilibria to only one after refinement. The intuitive criterion involves further constraints on beliefs, which is similar to the idea of eliminating dominated strategies. Below, we first introduce some related concepts.

Consider a more general signaling game. There are N players and one “Nature” player. First, “Nature” selects player 1's type, $\theta \in \Theta$, and only

player 1 knows his own type. Other players only know a priori probability distribution $p(\theta)$ about player 1's type, which is a common knowledge among players. Then, player 1 chooses his action $a_1 \in A_1$. After observing player 1's action, other players $i \in \{2, \dots, N\}$ choose their strategies $s_i \in S_i$ simultaneously. Define $\mathbf{S}_{-1} = S_2 \times \dots \times S_N$. After observing player 1's action a_1 , the posterior beliefs of other players $i \neq 1$ are $\mu(\theta|a_1)$. If player 1 chooses action a_1 and other players choose strategy $\mathbf{s}_{-1} = (s_2, \dots, s_N)$, player 1's utility is $u_1(a_1, \mathbf{s}_{-1}, \theta)$, and player $i \neq 1$'s utility is $u_i(a_1, \mathbf{s}_{-1}, \theta)$.

Below, we further examine the refinement of beliefs. If a strategy is a strictly dominated strategy for a certain type of player, after observing this strategy, assigning a positive probability to this type is clearly not a reasonable belief. The formal definition is as follows.

Definition 6.6.6 (Type Strictly Dominated Strategy) We say that an action a_1 is a *strictly dominated strategy* for type θ , if there is an action $a'_1 \in A_1$, such that:

$$\min_{\mathbf{s}'_{-1} \in \mathbf{S}_{-1}} u_1(a'_1, \mathbf{s}'_{-1}, \theta) > \max_{\mathbf{s}_{-1} \in \mathbf{S}_{-1}} u_1(a_1, \mathbf{s}_{-1}, \theta), \quad (6.6.5)$$

and the updating of beliefs $\mu(\theta|a_1) = 0$.

Define

$$\Theta(a_1) = \{\theta : \nexists a'_1 \in A_1 \text{ makes the above strict inequality (6.6.5) true}\},$$

i.e., $\Theta(a_1)$ presents that under types $\theta \in \Theta(a_1)$, a_1 is not a strictly dominated strategy.

In a perfect Bayesian game, reasonable beliefs need to satisfy: if $\mu(\theta|a_1) > 0$, then $\theta \in \Theta(a_1)$.

Since we must also take into account all players' strategies in equilibrium, we need an equilibrium related belief system for eliminating dominated strategies. Let $\mathbf{S}_{-1}^*(\Theta, a_1) \equiv S_2^*(\Theta, a_1) \times \dots \times S_N^*(\Theta, a_1) \subseteq \mathbf{S}_{-1}$ be all possible equilibrium responses of other players $i \neq 1$ to a given belief $\mu(\theta|a_1)$ after observing player 1's action a_1 , i.e., if $s_i^* \in S_i^*(\Theta, a_1)$, then $s_i^* \in \arg \max_{s_i} u_i(a_1, s_i, \theta)$.

Applying the above criterion, we have the following definition.

Definition 6.6.7 (Belief System after Eliminating Dominated Strategy) Let action $a'_1 \in A_1$ be a strictly dominated strategy for type θ under $S_{-1}^*(\Theta, a_1)$, i.e.,

$$\min_{s'_{-1} \in S_{-1}^*(\Theta, a'_1)} u_1(a'_1, s'_{-1}, \theta) > \max_{s_{-1} \in S_{-1}^*(\Theta, a_1)} u_1(a_1, s_{-1}, \theta). \quad (6.6.6)$$

The **belief system after eliminating dominated strategy** $\Theta^*(a_1)$ is defined as: for any $a_1 \in A_1$,

$$\Theta^*(a_1) = \{\theta : \nexists a'_1 \in A_1 \text{ makes the above strict inequality (6.6.6) true}\}.$$

Using the constraints imposed on beliefs above, we can further refine the education game's separating equilibria. When $e > \tilde{e}$, $\theta_h - C(\tilde{e}, \theta_l) < \theta_l$. We have $\theta_h - C(\tilde{e}, \theta_l) \geq w(\tilde{e}) - C(\tilde{e}, \theta_l)$, where $w(\tilde{e}) = \mu(\tilde{e})\theta_h + (1 - \mu(\tilde{e}))\theta_l$ is the employer's equilibrium response under given belief $\mu(\tilde{e})$. We also have $\theta_l \leq w(e_l = 0) - C(0, \theta_l)$, where $w(0) = \mu(0)\theta_h + (1 - \mu(0))\theta_l$ is the employer's equilibrium response under given belief $\mu(0)$. Therefore, when $e > \tilde{e}$, $\mu(e) = 1$. Based on these beliefs, separating equilibria, including $e_h > \tilde{e}$, can be refined. In this way, only $e_l = 0$, $e_h = \tilde{e}$, $\mu(\tilde{e}) = 1$, $\mu(0) = 0$, $w(e_h) = \theta_h$, $w(0) = \theta_l$ satisfies the above belief constraint in all separating equilibria.

In addition, in the pooling equilibrium, if $u(w_p, e^*|\theta_h) = w_p - C(e^*, \theta_h) < \theta_h - C(\tilde{e}, \theta_h)$ is established, such pooling equilibria can also be similarly refined.

In the following, we introduce two additional criteria to further strengthen the constraints imposed on beliefs.

6.6.5 Dominated Strategy in Equilibrium

Assume that at a perfect Bayesian equilibrium $((a_1^*(\theta))_{\theta \in \Theta}, s_{-1}^*(a_1), \mu(\theta|a_1))$, the utility of the type θ player is $u_1^*(\theta) \equiv u_1(a_1^*(\theta), s_{-1}^*(a_1^*), \theta)$.

Definition 6.6.8 (Dominated Strategy in Equilibrium) Action a_1 is called a *dominated strategy in equilibrium* for the player of type θ if

$$u_1^*(\theta) > \max_{s_{-1} \in S_{-1}^*(\Theta, a_1)} u_1(a_1, s_{-1}, \theta). \quad (6.6.7)$$

If the above inequality holds, then $\mu(\theta|a_1) = 0$.

Define

$$\Theta^{**}(a_1) = \{\theta : \nexists a_1 \in A_1 \text{ makes the above strict inequality (6.6.7) true}\}.$$

A perfect Bayesian equilibrium needs to have reasonable beliefs, and thus when $\mu(\theta|a_1) > 0$, we must have $\theta \in \Theta^{**}(a_1)$. Since

$$u_1(\theta) \equiv u_1(a_1^*(\theta), s_{-1}^*(a_1^*), \theta) > \min_{s'_{-1} \in \mathcal{S}_{-1}^*(\Theta, a_1')} u_1(a_1^*(\theta), s'_{-1}, \theta),$$

the dominated strategy in equilibrium imposes stronger constraints on beliefs than previous belief constraints based on eliminating dominated strategies.

Applying the dominated strategy in equilibrium to eliminate irrational beliefs, we can eliminate all pooling equilibria of the education game.

$$(e(\theta_h) = e(\theta_l) = e^* < \hat{e}, w(e^*) = w_p = \lambda\theta_h + (1-\lambda)\theta_l, \mu(\theta_h|e^*) = \lambda, \mu(\theta_h|e^*) = 0)$$

is a pooling equilibrium.

Let e' satisfy $\theta_h - C(e', \theta_l) = w_p - C(e^*, \theta_l)$ and e'' satisfy $\theta_h - C(e'', \theta_h) = w_p - C(e^*, \theta_h)$. Obviously, we have $e'' > e'$. When $e \in (e', e'')$, $\theta_h - C(e, \theta_l) < w_p - C(e^*, \theta_l)$ and $\theta_h - C(e, \theta_h) > w_p - C(e^*, \theta_h)$. In other words, when the employer observes $e \in (e', e'')$, job seekers of type θ_l prefer the payoff of pooling equilibrium instead of choosing e to obtain the maximum possible payoff, while job seekers of type θ_h are just the opposite. According to the constraints of the dominated strategy in equilibrium on beliefs, when $e \in (e', e'')$, the employer's posterior belief is $\mu(e) = 1$. Therefore, job seekers of type θ_h have an incentive to choose e , and the pooling equilibrium does not satisfy the belief constraints imposed by the dominated strategy in equilibrium.

6.6.6 Intuitive Criterion

Based on the above constraints on beliefs, Cho and Kreps (1987) proposed another refinement criterion for reducing the set of equilibria, i.e., **intuitive**

criterion.

Definition 6.6.9 (Intuitive Criterion) A perfect Bayesian equilibrium $((a_1^*(\theta))_{\theta \in \Theta}, s_{-1}^*(a_1), \mu(\theta|a_1))$ violates the *intuitive criterion* if there is a type $\theta \in \Theta$ and an action $a_1 \in A_1$, such that

$$u_1^*(\theta) < \min_{s_{-1} \in S_{-1}^*(\Theta^{**}(a_1), a_1)} u_1(a_1, s_{-1}, \theta).$$

According to the above discussion of the education game, only the Pareto optimal separating equilibrium can pass the intuitive criterion among all perfect Bayesian equilibria. In the following, we refer to the example in Cho and Kreps (1987) (see Figure 6.18) to explore how to employ the intuitive criterion to refine perfect Bayesian equilibria.

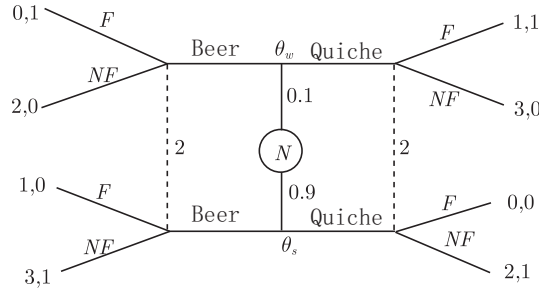


Figure 6.18: An Example of Intuitive Criterion

Example 6.6.10 In the game depicted in Figure 6.18, “Nature” chooses the type of player 1. θ_w denotes the type of “weak”, and θ_s denotes the type of “strong”. The initial probability of the “weak” type is 0.1. Player 1 chooses breakfast between “Beer” and “Quiche”. After player 2 observes player 1’s choice, he or she chooses an action from “Fight” (F) and “Not Fight” (NF). If the “weak” type is encountered, for player 2, the payoff of choosing F is greater than that of choosing NF ; if the “strong” type is encountered, for player 2, the payoff of choosing NF is greater than that of choosing F . Regardless of the type, player 1 does not hope that player 2 chooses F .

First, we can verify that there is no “separating equilibrium” in this

game. This game has the following two classes of perfect Bayesian equilibrium or sequential equilibrium.

The first class: both types of player 1 choose “Beer”, and player 2 chooses NF if he or she observes that player 1 has chosen “Beer” and chooses F if he or she observes that player 1 has chosen “Quiche”, and $\mu(\theta_w|\text{Beer}) = 0.9$.

The second class: both types of player 1 choose “Quiche”, and player 2 chooses F if he or she observes that player 1 has chosen “Beer” and chooses NF if he or she observes that player 1 has chosen “Quiche”, and $\mu(\theta_w|\text{Quiche}) = 0.9$.

We find that the second class of perfect Bayesian equilibrium does not satisfy the “intuitive criterion”. If player 2 observes that player 1 has chosen “Beer”, he or she should be able to infer that player 1 is “strong”. This is due to the fact that if player 1 is “weak”, his payoff is 3 in Bayesian equilibrium, which is the highest payoff of all possible outcomes in the game, and thus the “weak” type has no incentive to choose “Beer”. However, for the “strong” type, choosing “Beer” can make player 2 choose NF because she believes that “Beer” reveals that player 1 is “strong”. In this case, the “strong” type has a higher payoff.

In a more rigorous way, since $u_1^*(\theta_w) > \max_{s_2} u_1(\text{Beer}, s_2, \theta_w)$ and $\mu(\theta_w|\text{Beer}) = 0$, $\Theta^{**} = \theta_s$. When $\theta = \theta_s$, we have

$$u_1^*(\theta_s) = 2 < 3 = \min_{s_2 \in S_2^*(\Theta^{**}(\text{Beer}, \text{Beer}))} u_1(\text{Beer}, s_2, \theta_s).$$

It can be verified that the first class of perfect Bayesian equilibrium does not violate the “intuitive criterion”.

There are other criteria for refining the equilibrium of dynamic games of incomplete information, such as the “divinity” and the “universal divinity” proposed by Banks and Sobel (1987), and the concept of “stable equilibrium” proposed by Kohlberg and Mertens (1986).

Above, we have discussed various equilibrium concepts. In the following, we discuss the existence of equilibrium.

6.7 Existence of Nash Equilibrium

In using game theory to examine an interaction process, the most basic and important premise is that the game has an equilibrium solution. In the non-cooperative game, Nash equilibrium is a crucial concept. Nash (1951) proved the existence theorem of Nash equilibrium.

Below, we briefly discuss some existence theorems of game equilibrium.

6.7.1 Existence of Nash Equilibrium in Continuous Games

Theorem 6.7.1 (Existence Theorem of Pure Strategy Nash Equilibrium) *For a normal-form game $\Gamma = (N, (A_i)_{i \in N}, (u_i)_{i \in N})$, if for each player $i \in N$, A_i is a nonempty compact convex subset in Euclidean space, u_i is continuous on $A = \prod_{i \in N} A_i$ and quasiconcave on A_i , then there is a pure strategy Nash equilibrium in the game.*

PROOF. For any $\mathbf{x}_{-i} = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_N)$, define

$$BR_i(\mathbf{x}_{-i}) = \{x_i \in A_i : u_i(x_i, \mathbf{x}_{-i}) \geq u_i(x'_i, \mathbf{x}_{-i}), \quad \forall x'_i \in A_i\},$$

i.e., $BR_i(\mathbf{x}_{-i})$ is a set of best responses to other players' strategy \mathbf{x}_{-i} . Define $BR(\mathbf{x}) = \times_{i \in N} BR_i(\mathbf{x}_{-i})$. Then, $BR : A \rightarrow 2^A$ is a multi-valued mapping. Since for any $i \in N$, u_i is continuous and quasiconcave on A_i , then the correspondence BR_i is convex-valued and has a closed graph². Applying the Kakutani fixed point theorem (see Theorem 2.6.19), there is an \mathbf{x}^* , such that $\mathbf{x}^* \in BR(\mathbf{x}^*)$. \mathbf{x}^* is the pure strategy Nash equilibrium of the game. \square

Since the utility function is linear on mixed strategy space ΔA_i , it is quasiconcave. We immediately have the following corollary.

Corollary 6.7.1 (Existence Theorem of Mixed Strategy Nash Equilibrium) *For a normal-form game $\Gamma = (N, (A_i)_{i \in N}, (u_i)_{i \in N})$, if for each player $i \in N$, mixed strategy space ΔA_i is a nonempty compact convex subset in Euclidean space, u_i is continuous, then there is a mixed strategy Nash equilibrium in the game.*

²A mapping $f : X \rightarrow X$ has a closed graph, if for all sequences $\{x_n\}$ and $\{y_n\}$, where $x_n \in X$, $y_n \in f(x_n)$, $x_n \rightarrow x$ and $y_n \rightarrow y$, then $y \in f(x)$.

However, in reality, many games do not satisfy some of the above assumptions. For example, in the first-price sealed-bid auction, if two bidders bid the highest price at the same time, these two bidders obtain the auction item with the same probability. If one of these two bidders increases his or her price slightly, his or her utility level will experience a large leap. As a consequence, the utility function is not continuous at this point. The classical Bertrand (1883) price war game also has discontinuous payoff functions.

If some of the above conditions are not satisfied, does it mean that no equilibrium exists? In the literature, there are intensive discussions on the existence of Nash equilibrium after appropriate relaxation of continuity and quasiconcavity, such as Dasgupta and Maskin (1986), Baye, Tian, and Zhou (1993), Reny (1999), and Tian (2015). Below, we introduce some characterization results on the existence of Nash equilibrium given by Baye, Tian and Zhou (1993) and Tian (2015).

6.7.2 Existence of Nash Equilibrium in Discontinuous Games

Consider a normal-form game $(\Gamma = (N, (X_i)_{i \in N}, (u_i)_{i \in N}))$, $X = \prod_i X_i$. We first define upsetting a binary relation \succ .

Definition 6.7.1 For any $x, y \in X$, define *upsetting* the binary relation \succ as: $y \succ x$ if and only if there is $i \in N$, such that $u_i(y_i, x_{-i}) > u_i(x_i, x_{-i})$.

Obviously, if a strategy profile is a Nash equilibrium, no one will upset his or her strategy.

Define $U(y, x) = \sum_{i \in N} u_i(y_i, x_{-i})$, which represents the sum of utilities that each player uses strategy y_i to upset strategy profile x . For any $(x, y) \in X \times X$, based on the summation of all individual utilities, we define a similar upsetting binary relation \succ , i.e., $y \succ x$ if and only if $U(y, x) > U(x, x)$. Obviously, if x is a Nash equilibrium, then there is no $y \in X$, such that $y \succ x$.

We introduced the diagonal transfer continuity of function $U : X \times X \rightarrow \mathcal{R}$ with respect to y in Chapter 2, and we now define the diagonal transfer continuity with respect to $\Gamma = (N, (X_i)_{i \in N}, (u_i)_{i \in N})$.

Definition 6.7.2 A game $\Gamma = (N, (X_i)_{i \in N}, (u_i)_{i \in N})$ is *diagonally transfer continuous*, if the function $U : X \times X \rightarrow \mathcal{R}$ is diagonally transfer continuous with respect to \mathbf{y} , i.e., for any $\mathbf{x}, \mathbf{y} \in X$, once $U(\mathbf{y}, \mathbf{x}) > U(\mathbf{x}, \mathbf{x})$, then there is another strategy profile $\mathbf{z} \in X$ and a neighborhood of \mathbf{x} , $V_x \subseteq X$, such that $U(\mathbf{z}, V_x) > U(V_x, V_x)$, i.e., for any $\mathbf{x}' \in V_x$, we have $U(\mathbf{z}, \mathbf{x}') > U(\mathbf{x}', \mathbf{x}')$.

Definition 6.7.3 (Diagonally Transfer Quasiconcavity) A function $U(\mathbf{x}, \mathbf{y}) : X \times X \rightarrow \mathcal{R}$ is *diagonally transfer quasiconcave with respect to \mathbf{x}* , if for any finite subset $X^m = \{\mathbf{x}^1, \dots, \mathbf{x}^m\} \subseteq A$, there is a corresponding finite subset $Y^m = \{\mathbf{y}^1, \dots, \mathbf{y}^m\} \subseteq C$, such that for any subset $\{\mathbf{y}^{k^1}, \mathbf{y}^{k^2}, \dots, \mathbf{y}^{k^s}\} \subseteq Y^m$, where $1 \leq s \leq m$, and any $\mathbf{y}^{k^0} \in \text{co}\{\mathbf{y}^{k^1}, \mathbf{y}^{k^2}, \dots, \mathbf{y}^{k^s}\}$, we have

$$\min_{1 \leq l \leq s} U(\mathbf{x}^{k^l}, \mathbf{y}^{k^0}) \leq U(\mathbf{y}^{k^0}, \mathbf{y}^{k^0}). \quad (6.7.8)$$

Similarly, a game $\Gamma = (N, (X_i)_{i \in N}, (u_i)_{i \in N})$ is *diagonally transfer quasiconcave*, if the function $U : X \times X \rightarrow \mathcal{R}$ is diagonally transfer quasiconcave with respect to \mathbf{x} .

Remark 6.7.1 Diagonal transfer quasiconcavity of U is a weak version of quasiconcavity. For example, if U is quasiconcave or diagonally quasiconcave with respect to \mathbf{x} , then it is diagonally transfer quasiconcave with respect to \mathbf{x} (Let $\mathbf{y}^k = \mathbf{x}^k$).³

Remark 6.7.2 Let $G(\mathbf{x}) = \{\mathbf{y} \in C : U(\mathbf{x}, \mathbf{y}) \leq U(\mathbf{y}, \mathbf{y})\}$. It is easy to verify that U is diagonally transfer quasiconcave with respect to \mathbf{x} if and only if the corresponding $G : A \rightarrow 2^C$ is transfer FS-convex (see Definition 3.4.4).

In fact, the following theorem proves that diagonal transfer quasi-concavity is a necessary condition for the existence of Nash equilibrium, and it is also a sufficient condition under diagonal transfer continuity.

Theorem 6.7.2 (Baye, Tian, and Zhou (1993)) Suppose that game $\Gamma = (N, (X_i)_{i \in N}, (u_i)_{i \in N})$ satisfies diagonal transfer continuity. Γ has a pure strategy Nash equilibrium if and only if it is diagonally transfer quasiconcave.

³ A function $U : Z \times Z \rightarrow \mathcal{R}$ is diagonally quasiconcave with respect to \mathbf{x} , if for any finite subset X^m of Z and $\mathbf{x}^0 \in \text{co } X^m$ we have $\min_k U(\mathbf{x}^k, \mathbf{x}^0) \leq U(\mathbf{x}^0, \mathbf{x}^0)$.

PROOF. Necessity: Suppose that game Γ has a pure strategy Nash equilibrium $\mathbf{y}^* \in X$. We need to prove that U is diagonally transfer quasiconcave with respect to \mathbf{x} . For any finite subset $X^m = \{\mathbf{x}^1, \dots, \mathbf{x}^m\} \subseteq X$, let the corresponding finite subset be $Y^m = \{\mathbf{y}^1, \dots, \mathbf{y}^m\} = \{\mathbf{y}^*\}$. So, for any $\{\mathbf{y}^{k^1}, \mathbf{y}^{k^2}, \dots, \mathbf{y}^{k^s}\} \subseteq Y^m = \{\mathbf{y}^*\}$, where $1 \leq s \leq m$ and any $\mathbf{y}^{k^0} \in co\{\mathbf{y}^{k^1}, \mathbf{y}^{k^2}, \dots, \mathbf{y}^{k^s}\} = \{\mathbf{y}^*\}$, we have

$$\min_{1 \leq l \leq s} [U(\mathbf{x}^{k^l}, \mathbf{y}^{k^0})U(\mathbf{y}^{k^0}, \mathbf{y}^{k^0})] \leq [U(\mathbf{x}^{k^l}, \mathbf{y}^*)U(\mathbf{y}^*, \mathbf{y}^*)] = \sum_{i \in I} [u_i(\mathbf{x}_i^{k^l}, \mathbf{y}_i^*)u_i(\mathbf{y}^*)] \leq 0.$$

Therefore, U is diagonally transfer quasiconcave with respect to \mathbf{x} .

Sufficiency: For each $\mathbf{x} \in Z$, let $G(\mathbf{x}) = \{\mathbf{y} \in X : U(\mathbf{x}, \mathbf{y}) \leq U(\mathbf{y}, \mathbf{y})\}$. It is easy to verify that U is diagonally transfer continuous with respect to \mathbf{x} if and only if $G : X \rightarrow 2^X$ is transfer closed-valued (see Chapter 2 for its definition). Moreover, U is diagonally transfer quasiconcave with respect to \mathbf{x} if and only if the corresponding $G : A \rightarrow 2^C$ is transfer FS-convex. From Lemma 3.4.2, we thus know that $\bigcap_{\mathbf{x} \in Z} G(\mathbf{x}) = \bigcap_{\mathbf{x} \in Z} cl_Z G(\mathbf{x}) \neq \emptyset$. Therefore, there is $\mathbf{y}^* \in X$, such that $U(\mathbf{x}, \mathbf{y}^*) \leq U(\mathbf{y}^*, \mathbf{y}^*)$ holds for all $\mathbf{x} \in X$. Let $\mathbf{x} = (\mathbf{x}_i, \mathbf{y}_{-i}^*)$, then

$$U(\mathbf{x}, \mathbf{y}^*) - U(\mathbf{y}^*, \mathbf{y}^*) = [u_i(\mathbf{x}_i, \mathbf{y}_{-i}^*) - u_i(\mathbf{y}^*)] \leq 0 \quad (6.7.9)$$

holds for all $\mathbf{x}_i \in X_i$. Therefore, \mathbf{y}^* is a pure strategy Nash equilibrium of Γ . \square

Tian (2015) further provided the sufficient and necessary topological conditions for the existence of pure strategy Nash equilibrium in any normal-form game. In a general game, the number of players can be finite or infinite; strategy spaces are arbitrary, which can be discrete or continuous and can be non-compact or non-convex; and players' utility functions can be discontinuous or non-quasiconcave on strategy spaces. The way of proof shown in Tian (2015) does not use any form of a fixed point theorem as usual, but is based on a more basic mathematical result (Borel-Lebesgue covering theorem). The following only discusses situations in which utility functions exist. For general situations, please refer to Tian (2015).

Definition 6.7.4 A game $\Gamma = (N, (X_i)_{i \in N}, (u_i)_{i \in N})$ is said to be *recursively*

diagonal transfer continuous if, for any $x, y \in X$ satisfying $y \succ x$, there exists a strategy profile $y^0 \in X$ (possibly $y^0 = x$) and a neighborhood $V_x \subseteq X$, such that for any $z \in X$ that recursively upsets y^0 ,⁴ there exists $U(z, V_x) > U(y^0, V_x)$.

We can similarly define m -recursively diagonal transfer continuity. A game $G = (X_i, u_i)_{i \in I}$ is said to be m -recursively diagonal transfer continuous if and only if “recursively upsets y^0 ” in the above definition is replaced by “ m -recursively upsets y^0 ”.

Based on the introduction of these concepts, Tian (2015) gave the necessary and sufficient conditions for the existence of pure strategy Nash equilibrium.

Theorem 6.7.3 (Necessary Conditions for the Existence of Nash Equilibrium)

If a game $\Gamma = (N, (X_i)_{i \in N}, (u_i)_{i \in N})$ has a pure strategy Nash equilibrium, then the game must satisfy recursive diagonal transfer continuity.

PROOF. First, note that, if $x^* \in X$ is a pure strategy Nash equilibrium of a game G , we must have $U(y, x^*) \leq U(x^*, x^*)$ for all $y \in X$, which is obtained by summing up $u_i(y_i, x_{-i}^*) \leq u_i(x_i^*, x_{-i}^*) \quad \forall y_i \in X_i$ for all players.

If for any $x, y \in X$, there is $U(y, x) > U(x, x)$. Let $y^0 = x^*$ and V_x be a neighbourhood of strategy profile x . Since $U(y, x^*) \leq U(x^*, x^*)$, it is impossible to find a strategy profile y^1 , such that $U(y^1, y^0) > U(y^0, y^0)$, and of course, it is impossible to find a finite strategy profile chain, $\{y^1, y^2, \dots, y^m\}$, such that $U(y^{i+1}, y^i) > U(y^i, y^i)$, $i = 1, \dots, m-1$. This means that the game satisfies recursive diagonal transfer continuity. \square

Theorem 6.7.4 (Sufficient Conditions for the Existence of Nash Equilibrium)

Suppose that the strategy profile space X of game $\Gamma = (N, (X_i)_{i \in N}, (u_i)_{i \in N})$ is compact. If the game satisfies recursive diagonal transfer continuity on X , then there is a pure strategy Nash equilibrium.

PROOF. First, note that if there is $U(y, x^*) \leq U(x^*, x^*)$ for all $y \in X$, then $x^* \in X$ must be a Nash equilibrium of the game. We can let $y = (y_i, x_{-i}^*)$, $U(y, x^*) \leq U(x^*, x^*)$ means $u_i(y_i, x_{-i}^*) \leq u_i(x_i^*, x_{-i}^*)$.

⁴A strategy profile $y^0 \in X$ is said to be *recursively upset* by $z \in X$ if there exists a finite set of deviation strategy profiles $\{y^1, y^2, \dots, y^{m-1}, z\}$, such that $U(y^1, y^0) > U(y^0, y^0)$, $U(y^2, y^1) > U(y^1, y^1)$, \dots , $U(z, y^{m-1}) > U(y^{m-1}, y^{m-1})$.

Suppose, by way of contradiction, that there is no pure strategy Nash equilibrium. Then, for each $x \in X$, there exists $y \in X$, such that $U(y, x) > U(x, x)$, and thus by recursive diagonal transfer continuity, for each $x \in X$, there is y^0 and a neighborhood of x , V_x , such that for any z that recursively upsets y^0 , we have $U(z, V_x) > U(V_x, V_x)$. Since there is no equilibrium by the contrapositive hypothesis, y^0 is not an equilibrium and thus, by recursive diagonal transfer continuity, such a sequence of recursive securing strategy profiles $\{y^0, \dots, y^{m-1}, y^m = z\}$ exist for some $m \geq 1$, such that $U(y^{i+1}, y^i) > U(y^i, y^i), i = 0, \dots, m-1$.

Since X is compact and $X \subseteq \bigcup_{x \in X} V_x$, there are finite strategies $\{x^1, \dots, x^L\}$, such that $X \subseteq \bigcup_{i=1}^L V_{x^i}$. For each such x^i , there is a corresponding y^{0i} , so that $U(z^i, V_{x^i}) > u(x^i, V_{x^i})$ whenever y^{0i} is recursively upset by z^i .

Since there is no equilibrium, then for each such y^{0i} , there must be $z^i \in X$, such that $U(z^i, y^{0i}) > u(y^{0i}, y^{0i})$, and then, by 1-recursive diagonal transfer continuity, we have $U(z^i, V_{x^i}) > U(V_{x^i}, V_{x^i})$. For strategy profile $\{z^1, \dots, z^L\}$, we must have $z^i \notin V_{x^i}$; otherwise, by $U(z^i, V_{x^i}) > U(V_{x^i}, V_{x^i})$, we have $U(z^i, z^i) > U(z^i, z^i)$, which is a contradiction. As such, we must have $z^1 \notin V_{x^1}$. We assume that $z^1 \in V_{x^2}$, which does not lose generality.

Since $U(z^2, z^1) > u(z^1, z^1)$ and $U(z^1, y^{01}) > u(y^{01}, y^{01})$, by 2-recursive diagonal continuity, we have $U(z^2, V_{x^1}) > U(V_{x^1}, V_{x^1})$. Similarly, since $U(z^2, V_{x^2}) > U(V_{x^2}, V_{x^2})$, $U(z^2, V_{x^1} \cup V_{x^2}) > U(V_{x^1} \cup V_{x^2}, V_{x^1} \cup V_{x^2})$, from which $z^2 \notin (V_{x^1} \cup V_{x^2})$ is obtained. With this recursive process, for $k = 3, \dots, L$, we can show that $z^k \notin V_{x^1} \cup V_{x^2} \cup \dots \cup V_{x^k}$. When $k = L$, we can obtain that $z^L \notin V_{x^1} \cup V_{x^2} \cup \dots \cup V_{x^L}$, which contradicts $X \subseteq \bigcup_{i=1}^L V_{x^i}$ and $z^L \in X$. Therefore, the game must have a pure strategy Nash equilibrium. \square

In the following, we define a relatively stronger concept based on recursive diagonal transfer continuity, and thus that we can identify the necessary and sufficient conditions for the existence of (pure strategy) Nash equilibrium in any game.

Definition 6.7.5 Let $B \subseteq X$. A game $\Gamma = (N, (X_i)_{i \in N}, (u_i)_{i \in N})$ is said to satisfy *recursive diagonal transfer continuity relative to B* on X , if x is not a Nash equilibrium, and then there exists a strategy profile $y^0 \in B$ (pos-

sibly $\mathbf{y}^0 = \mathbf{x}$) and a neighborhood of strategy profile \mathbf{x} , $V_{\mathbf{x}}$, such that: (1) \mathbf{y}^0 is upset by a strategy on B . (2) If for any finite strategy profile chain $\{\mathbf{y}^1, \dots, \mathbf{y}^m = \mathbf{z}\}$ with $U(\mathbf{y}^{i+1}, \mathbf{y}^i) > U(\mathbf{y}^i, \mathbf{y}^i)$, $i = 0, \dots, m-1$, we have $U(\mathbf{z}, V_{\mathbf{x}}) > U(V_{\mathbf{x}}, V_{\mathbf{x}})$.

Theorem 6.7.5 (Full Characterization for the Existence of Nash Equilibrium)

A game $\Gamma = (N, (X_i)_{i \in N}, (u_i)_{i \in N})$ has a pure strategy Nash equilibrium if and only if there is a compact set $B \subseteq X$, such that the game satisfies recursive diagonal transfer continuity relative to B on X .

PROOF. Since the sufficiency proof of the theorem is similar to the above, it is approximately given here. We first prove that the game has a Nash equilibrium \mathbf{x}^* in strategy space B . Suppose that this is not the case. Since game G satisfies recursive diagonal transfer continuity with respect to B on X , for every $\mathbf{x} \in B$, there exists $\mathbf{y}^0 \in B$ and neighborhood $V_{\mathbf{x}}$, such that for any finite subsequence $\{\mathbf{y}^1, \dots, \mathbf{y}^m\} \subseteq B$ satisfying $\mathbf{y}^m = \mathbf{z}$ and $U(\mathbf{z}, \mathbf{y}^{m-1}) > U(\mathbf{y}^{m-1}, \mathbf{y}^{m-1})$, $U(\mathbf{y}^{m-1}, \mathbf{y}^{m-2}) > U(\mathbf{y}^{m-2}, \mathbf{y}^{m-2})$, \dots , $U(\mathbf{y}^1, \mathbf{y}^0) > U(\mathbf{y}^0, \mathbf{y}^0)$, we have $U(\mathbf{z}, V_{\mathbf{x}}) > U(V_{\mathbf{x}}, V_{\mathbf{x}})$. By the assumption, since there is no equilibrium in B , \mathbf{y}^0 is not an equilibrium in B . Therefore, by recursive diagonal transition continuity with respect to B on X , for some $m \geq 1$, there exists such a recursive sequence $\{\mathbf{y}^1, \dots, \mathbf{y}^{m-1}, \mathbf{z}\}$.

Since B is compact and $B \subseteq \bigcup_{\mathbf{x} \in X} V_{\mathbf{x}}$, there is a finite set $\{\mathbf{x}^1, \dots, \mathbf{x}^L\} \subseteq B$, such that $B \subseteq \bigcup_{i=1}^L V_{\mathbf{x}^i}$. For each such \mathbf{x}^i , there is a corresponding initial deviation \mathbf{y}^{0i} , such that as long as \mathbf{y}^{0i} is recursively upset by \mathbf{z}^i through finite strategies $\{\mathbf{y}^{1i}, \dots, \mathbf{y}^{mi}\} \subseteq B$ and $\mathbf{y}^{mi} = \mathbf{z}^i$, we have $U(\mathbf{z}^i, V_{\mathbf{x}^i}) > U(V_{\mathbf{x}^i}, V_{\mathbf{x}^i})$. By the same reasoning as in the previous theorem's proof, for all $k = 1, 2, \dots, L$, \mathbf{z}^k is not in the union of $V_{\mathbf{x}^1}, V_{\mathbf{x}^2}, \dots, V_{\mathbf{x}^k}$. Especially for $k = L$, we have $\mathbf{z}^L \notin V_{\mathbf{x}^1} \cup V_{\mathbf{x}^2} \cup \dots \cup V_{\mathbf{x}^L}$ and thus $\mathbf{z}^L \notin B \subseteq \bigcup_{i=1}^L V_{\mathbf{x}^i}$, which is a contradiction. As a consequence, the game has a pure strategy Nash equilibrium \mathbf{x}^* on B .

We now prove that \mathbf{x}^* must also be a pure strategy Nash equilibrium on X . Suppose that \mathbf{x}^* was not a pure strategy Nash equilibrium on X . Then, \mathbf{x}^* will be upset by a strategy in $X \setminus B$, and thus it is upset by a strategy in B , which means that \mathbf{x}^* is not a Nash equilibrium on B , which is a contradiction.

The proof of necessity is the same, since its proof does not rely on the compactness of the set. \square

In Tian (2015), the sufficient and necessary conditions for the existence of mixed strategy equilibrium and for the existence of equilibrium under general preferences are also discussed.

6.8 Biographies

6.8.1 John Forbes Nash Jr.

John Forbes Nash Jr. (1928-2015) was an American mathematician who made fundamental contributions to game theory. He was once a C. L. E. Moore instructor at MIT, and later became a professor of mathematics at Princeton University. He mainly studied game theory, differential geometry, and partial differential equations. He won the 1994 Nobel Memorial Prize in Economic Sciences for his pioneering analysis of equilibria in the theory of non-cooperative games, and the concept of Nash equilibrium proposed at the age of 22 that has had a profound influence on game theory and economics.

He received his Ph.D. from Princeton University in 1950. His doctoral dissertation, entitled “Non-Cooperative Games”, comprised only 27 pages. In 1950 and 1951, his two important papers on non-cooperative games theory completely transformed people’s views on competition and the market. He defined the non-cooperative game and its equilibrium solution, and proved the existence of equilibrium solution (i.e., the well-known Nash equilibrium). His research shows that making decisions based on maximizing individual interests does not necessarily maximize collective interests. In other words, the maximization of individual interests and the maximization of collective interests may conflict, thus revealing the intrinsic link between game equilibrium and economic equilibrium. Nash’s research laid the foundation for the modern non-cooperative game theory, and later game theory research was basically carried out along this main line. However, Nash’s genius discovery was flatly denied by von Neumann, and he had previously received an indifferent reception from Einstein.

Nash's instinct of challenging authority made him adhere to his own ideas, and led to his great accomplishments. He had a serious mental illness for more than 30 years, and the Oscar-winning film *A Beautiful Mind* adapted from the biography of the same name in 2001 is about his life experience.

The minimax theorem proposed by von Neumann in 1928 and the equilibrium theorem proposed by Nash in 1950 formed the cornerstone of game theory. The former primarily considers the zero-sum game, while the latter considers the more general non-zero-sum game. By extending this theory to games involving various cooperation and competition, Nash successfully opened the door to application of game theory in economics, political science, sociology, and even evolutionary biology. In 1958, Nash was identified by *Fortune* (magazine) as the most outstanding figure among genius mathematicians of the modern generation for his excellent work in mathematics. In 1994, he and John C. Harsanyi and Reinhard Selten jointly won the Nobel Memorial Prize in Economic Sciences. In 1999, he was awarded the Leroy P. Steele Prize by the Mathematical Association of America.

Although the Nobel Memorial Prize in Economic Sciences, the highest prize in economics, brought a new life to Nash, and his physical and mental condition seemed to be improving in the 21 years since winning the prize, his life ended tragically. Nash won the Abel Prize in 2015, which is awarded by the Norwegian Royal Family and rewards scientists who have made outstanding contributions in the field of mathematics. On their way back to the United States after attending the Norwegian Royal Family Award Ceremony, Nash and his wife Alicia died in an automobile accident on the New Jersey Turnpike on May 23, 2015.

6.8.2 John C. Harsanyi

John C. Harsanyi (1920-2000) was one of the pioneers in developing game theory as a tool of economic analysis. He made fundamental contributions to games of incomplete information, i.e., Bayesian games. Other important contributions included the application of game theory, and economic reasoning in political and moral philosophy (specifically utilitarian ethics). His contributions made him a co-recipient of the 1994 Nobel Memorial

Prize in Economic Sciences, together with John Nash and Reinhard Selten.

Harsanyi was born into a Jewish family in Budapest, Hungary. He followed his parents' wishes and studied pharmacology at the University of Budapest. In early 1944, he received a master's degree in pharmacy. In March 1944, German troops occupied Hungary. Harsanyi was compelled to join a forced labor unit from May to November. In November of the same year, the Nazi authorities decided to deport his unit at Budapest to a concentration camp in Austria. Harsanyi was fortunate enough to escape from the Budapest railway station, just before the train left for Austria. After the end of the war, in 1946, Harsanyi returned to the University of Budapest to obtain his Ph.D. in philosophy with minors in sociology and psychology. As he had credit for his previous studies in pharmacy, Harsanyi received a Ph.D. in philosophy in June 1947, after only one more year of course work and after writing a dissertation in philosophy. From September 1947 to June 1948, Harsanyi was an assistant professor at the University Institute of Sociology, where he met Anne Klauber, his future wife. In June 1948, he was forced to resign from the Institute because of openly expressing his anti-Marxist opinions. In April 1950, Harsanyi and Anne decided to leave Hungary. They crossed the Hungarian border over a marshy terrain, which was less guarded, and fled to Austria before they reached Sydney, Australia.

While working in a factory during the day in Sydney, Harsanyi took economics courses at the University of Sydney in the evening and obtained a master's degree in economics in 1953. During his study in Sydney, he began publishing papers in economic journals (including the *Journal of Political Economy* and the *Review of Economic Studies*). The degree allowed him to secure a teaching position in 1954 at the University of Queensland in Brisbane. Harsanyi wrote a paper on game theory under the guidance of Kenneth J. Arrow (1921-2017) at Stanford University in 1958 and received his second doctoral degree (Ph.D. in economics) in 1959. He was appointed Professor of Economics at Wayne State University in Michigan between 1961 and 1963. In 1964, he moved to the University of California at Berkeley and remained there until his retirement in 1990.

The second half of the 1960s witnessed the most important achieve-

ments in Harsanyi's academic career. In 1967 and 1968, Harsanyi published a three-part paper, entitled "Games with Incomplete Information Played by 'Bayesian' Players". The paper studied games with incomplete information that game theory at that time could not effectively discuss. He proposed a method to convert a game with incomplete information into one with complete but imperfect information, in order to make it accessible to game-theoretic analysis. Currently, this method is called the "Harsanyi Transformation" and is the standard method of analysing games with incomplete information. Due to Harsanyi's paper, the difficulty in analysing the incomplete information was solved, and the incomplete information game was incorporated into the analytical framework of game theory, which markedly expanded the analysis and application scope of game theory, and thus constituted a milestone achievement in the development of game theory. It was because of this contribution that Harsanyi received the Nobel Memorial Prize in Economic Sciences. In addition to his outstanding achievements in the study of game theory, Harsanyi also obtained important results in welfare economics and economic philosophy. From the early 1950s to the 1990s, Harsanyi published a series of articles in these two fields, which further established his position in the economics profession. John Harsanyi passed away on August 9, 2000, from a heart attack, in Berkeley, California.

6.9 Exercises

Exercise 6.1 Consider a normal-form game. Prove that if only one strategy profile survives the iterated elimination of non-best response strategies, then it is the unique Nash equilibrium.

Exercise 6.2 Consider the following game: There are 20 students in a class. Each of them chooses an integer between 1 and 100. Students who choose the number closest to $1/2$ of the class average will equally divide 100 dollars.

1. Find strictly dominated strategies for each student.
2. Find the Nash equilibrium of the game by iterated elimination of strictly dominated strategies.
3. Assume that the winning rule is changed to that students who choose the number closest to 2 times the class average will equally divide 100 dollars. Find all Nash equilibria of the game.

Exercise 6.3 There are n herdsmen in a public grassland. Herdsman i can choose to herd g_i sheep on the public grassland. The cost per head is $c > 0$. One year later, each herdsman can sell his sheep at the market price $v(G)$, where $G = \sum_{i=1}^n g_i$. Assume that $v(G)$ is twice continuously differentiable, and that its second derivative is less than zero.

1. Solve for the socially optimal amount of sheep.
2. Solve for the amount of sheep held by each herdsman in the Nash equilibrium.

Exercise 6.4 Consider a normal-form game $(N, (S_i)_{i \in N}, (u_i)_{i \in N})$, where $N = \{1, 2, \dots, n\}$. We say that the game is symmetric if it satisfies the following conditions: (1) For any players i and j , $S_i = S_j$; (2) if $s_{-i} = s_{-j}$ and $s_i = s_j$, then $u(s_i, s_{-i}) = u(s_j, s_{-j})$. Suppose that for any player i and j , $\sigma_i = \sigma_j$, strategy $\sigma = (\sigma_1, \dots, \sigma_n)$ is symmetric. The symmetric Nash equilibrium refers to the Nash equilibrium whose strategy is symmetric.

1. Determine whether any finite symmetric game has a symmetric Nash equilibrium.
2. Prove that not all Nash equilibrium strategies are symmetric.

Exercise 6.5 Find all Nash equilibria of the following game (including pure-strategy Nash equilibrium and mixed-strategy Nash equilibrium):

		Player 2	
		L	R
Player 1	T	0, 0	0, -2
	M	3, 0	-1, 0
	B	1, 0	-1, 1

Exercise 6.6 Two individuals have to decide how to allocate 100 thousand dollars. They use the following allocation rule: Each decision-maker reports a positive integer less than 100 thousand. If the sum of the numbers reported by these two individuals does not exceed 100 thousand, then the amount of money each decision-maker receives is his or her own number (the extra money is discarded). If the sum of the numbers reported by these two individuals is greater than 100 thousand and their numbers are different, then the decision-maker who reports the smaller number receives the amount of money that he or she reports, and the other decision-maker gets what remains of the 100 thousand dollars. If the sum of the two numbers is greater than 100 thousand and the two numbers are the same, then each decision-maker receives 50 thousand dollars.

1. Find all pure strategy Nash equilibria of the game.
2. Find all mixed strategy Nash equilibria of the game.

Exercise 6.7 Consider the following strategic-form game:

		Player 2		
		L	C	R
Player 1	T	1, -3	-3, 1	0, 0
	M	-3, 1	1, -3	0, 0
	B	0, 0	0, 0	2, 2

1. Find the pure strategy Nash equilibrium in this game.
2. Prove that there is no additional mixed strategy Nash equilibrium in this game.

Exercise 6.8 Consider the following simultaneous-move game:

		Player 2	
		L	R
Player 1	T	a_1, a_2	b_1, b_2
	B	c_1, c_2	d_1, d_2

1. Under what conditions is there a Nash equilibrium such that both of the two players choose completely mixed strategies?
2. Under what conditions is the Nash equilibrium in question (1) the unique Nash equilibrium of the game?
3. Find the Nash equilibrium in question (2) (where all players' payoffs are constant).

Exercise 6.9 Two players compete for one item, and each player simultaneously chooses a time node of giving up. If one of the two players first gives up, the other player will receive the item; if both players give up at the same time, then both players get the item with the same probability. Let time be a continuous variable that starts at 0 and tends to infinity. Assume that if player i receives the item, his or her payoff is v_i . For each unit of time passed, each player needs to pay one unit cost. Let t_1 and t_2 represent the time node of giving up chosen by these two players.

1. Write down the normal-form representation of the above game.

2. Write down player i 's best response function (or correspondence).
3. Suppose $v_1 > v_2$. Find the best response curves for these two players.
4. Find the Nash equilibrium.

Exercise 6.10 Prove that in game $G = (I; \{S_i, u_i\}_{i=1}^n)$, if $s^* = (s_1^*, \dots, s_n^*)$ is the only strategy profile that survives the iterated elimination of strictly dominated strategies, then s^* is the unique Nash equilibrium of the game.

Exercise 6.11 In the second-price sealed-bid auction with complete information (i.e., each player knows other players' true valuations, and the highest bidder obtains the auction item at the second-highest bid price), the true valuations of players $i = 1, 2, \dots, n$ are $v_1 > v_2 > \dots > v_n$, respectively. Find a Nash equilibrium in which the player with the highest valuation does not obtain the auction item.

Exercise 6.12 A game in which the sum of two players' payoffs is zero is called a zero-sum game. When player 1 chooses strategy $a_1 \in A_1$ and player 2 chooses strategy $a_2 \in A_2$, player 1's payoff is $u(a_1, a_2)$. In a zero-sum game, player 2's payoff is $-u(a_1, a_2)$. A_1 and A_2 are strategy spaces for players 1 and 2, respectively.

1. Prove the minimax theorem, i.e., prove that the following formula holds:

$$\max_x \min_y u(x, y) = u(x^*, y^*) = \min_y \max_x u(x, y).$$

2. Prove that if (m_1, m_2) and (m_1^*, m_2^*) are Nash equilibria, then (m_1, m_2^*) and (m_1^*, m_2) also are Nash equilibria.
3. Prove that player 1's payoff is always zero in Nash equilibrium in a symmetric zero-sum game.

Exercise 6.13 Consider the following simultaneous-move game:

		Player B	
		L	R
Player A	T	x, x	$0, 0$
	B	$0, 0$	x, x

Player A knows the exact value of x , and player B only knows that the probability of x being 5 or 10 is 0.5.

1. Describe the above game of incomplete information.
2. Find all pure strategy and mixed strategy Bayesian-Nash equilibria of the above game.
3. Now, suppose that, after observing the value of x , player A can choose to play a simultaneous-move game with player B , or pay a cost of 2 to play a sequential-move game and move first. For certain values of x , player A will choose to pay a cost of 2 and move first. Find a Bayesian-Nash equilibrium for this dynamic game. Is this Bayesian-Nash equilibrium a sequential equilibrium? If yes, why?

Exercise 6.14 Consider the following game with two players. There are 21 coins on the table. Player 1 and Player 2 take turns to take away 1 to 3 coins. The last player to take away the coin on the table loses the game. Specifically, player 1 can choose to take away 1, 2 or 3 coins, and then player 2 chooses the number of coins to take away, and they will take turns until the last player to take away the coin on the table loses.

1. Use backward induction to solve this game.
2. What is the number of coins on the table that can make player 2 always be the loser in equilibrium?

Exercise 6.15 Two players play the following game: In the first stage, player 1 makes a choice between actions A and B ; in the second stage, player 2 makes a choice between actions C and D after observing player 1's choice; in the third stage, player 1 makes a choice between actions a and b after not observing player 2's choice.

1. Give the extensive-form representation of the game.
2. Is the game of perfect information or imperfect information? Why?
3. Give the strategy set for each player.

Exercise 6.16 In a wild grassland, there are n hungry lions that have not eaten food for a long period of time. One of the lions has fallen into a coma and is defenseless due to an illness. These lions have a strict hierarchy. Only the highest-ranked lion can eat the sick lion. However, the highest-ranked lion may become sick and slip into a coma once he has eaten the sick lion, and may subsequently be eaten by the second highest-ranked lion. The preferences of these lions are as follows: “eat a sick lion and not be eaten” \succ , “not be eaten” \succ , and “eaten by another lion (they don’t care whether or not they are sick)” . Use backward induction to find the subgame perfect equilibrium of this game.

Exercise 6.17 There is a duel among three musketeers, each with a pistol. In each round, they aim at the target and fire at the same time, and the entire process continues until at most one person survives. It is known that the hit rate of A is 1, the hit rate of B is 0.8, and the hit rate of C is 0.6.

1. In the first round of the duel, who should be the shooting target of A , B , and C , respectively?
2. What are the survival rates of A , B , and C ?
3. The dueling rules are now amended to the following: In each round, C fires first, then B fires, and A fires last. In the first round of the duel, who should be the shooting target for A , B , and C , respectively? What is the final survival rate of each of the three musketeers?

Exercise 6.18 Consider the following pirate game: 10 pirates consider how to allocate 100 gems, and they alternately propose a distribution plan in order of 1 to 10, and the order is determined by a lottery. The rules of the game are as follows: Gems can only be allocated in integer quantities. Pirate 1 first proposes an allocation plan. If half or more of the pirates vote to accept the plan, the plan will be implemented, and the game ends; otherwise, pirate 1 must leave the game, and the other 9 pirates continue the game. Next, pirate 2 proposes an allocation plan. The rules of the game are the same as previously. Suppose that all pirates must choose to reject an allocation plan when they are indifferent between accepting and rejecting

this allocation plan. Solve for the subgame perfect Nash equilibrium of this game.

Exercise 6.19 Two players A and B consider how to allocate two cakes X and Y . Each cake is 1 unit in size. The utility function of player A is $u(x, y) = x + \lambda y$, where (x, y) is his share, i.e., x is obtained from cake X and y is obtained from cake Y ; the utility function of player B is $v(x, y) = x + \delta y$, where (x, y) is her share. Suppose that $\delta > \lambda > 0$. The mechanism for allocating cakes is as follows: First, each cake is divided into two pieces by A , i.e., X is divided into x and $1 - x$ and Y is divided into y and $1 - y$, and the divided cakes are merged into two groups: (x, y) and $(1 - x, 1 - y)$. Then, B chooses one group first, and A gets the other group.

1. Solve for the subgame perfect Nash Equilibrium of this game with backward induction.
2. If the roles of A and B are reversed, i.e., the cakes are divided and grouped by B , and A chooses one group first. What is the outcome?
3. Can this “distributor chooses last” allocation mechanism result in a fair distribution?
4. The distribution mechanism is now changed to: First, A divides the cake X into two pieces, B chooses one piece, and A receives the other piece; then, B cuts the cake Y into two pieces, A chooses one of them and B gets the other one. Find the subgame perfect Nash equilibrium under this mechanism. Compared with the original mechanism, which is more efficient?

Exercise 6.20 Consider the following dynamic game (arms race) with two players (two countries that are competing with each other). In each period $t = 0, 1, 2, \dots$, each player can choose to participate in or withdraw from the competition. The cost of participating in the competition in each period is 1. If both players choose to participate in the competition in a certain period, then the returns of both players are 0 for the current period, and they enter the game in the next period; if one player chooses to participate

in the competition and the other player chooses to withdraw from the competition in a certain period, then the player who chooses to participate will receive v for the current period and the player who chooses to withdraw will receive 0 for the current period, and the game will end (i.e., there will be no game in the subsequent period).

1. Prove that (always participate, always withdraw) is a subgame perfect Nash equilibrium.
2. Find a p , such that (always participate with probability p , always withdraw with probability p) is a subgame perfect Nash equilibrium.

Exercise 6.21 Consider a bargaining game with three players. During periods $t = 1, 4, 7, \dots$, the first player can propose an allocation plan (x_1, x_2, x_3) , where $x_i \geq 0$ and $x_1 + x_2 + x_3 \leq 1$, and other players can choose whether or not to accept the allocation plan. During periods $t = 2, 5, 8, \dots$, the second player can propose an allocation plan. During periods $t = 3, 6, 9, \dots$, the third player can propose an allocation plan. If all players during a certain period accept the allocation plan, the allocation plan will be implemented; if there is one player in a certain period who rejects the allocation plan, these three players will perform the next round of distribution. The discount rate of each player per period is δ .

1. Prove that $(1/(1 + \delta + \delta^2), \delta/(1 + \delta + \delta^2), \delta^2/(1 + \delta + \delta^2))$ is a subgame perfect Nash equilibrium.
2. Prove that the above equilibrium is unique.

Exercise 6.22 Prove the following theorem on mixed strategies and behavior strategies.

1. In a finite extensive-form game that satisfies perfect recall, any mixed strategy has an outcome-equivalent behavior strategy.
2. Show by an example: in a situation in which perfect recall fails, a mixed strategy and a behavior strategy are not necessarily outcome-equivalent.

Exercise 6.23 Suppose that two investors decide whether to invest in a certain firm, and their returns on investment can be represented in the following payoff matrix, where θ is the firm's operating cost.

		Investor 2	
		Invest	Don't Invest
Investor 1	Invest	θ, θ	$\theta - 1, 0$
	Don't Invest	$0, \theta - 1$	$0, 0$

1. If the investors know the operating cost θ . Find all Nash equilibria.
2. If the investors do not know the operating cost θ , investor i can observe a signal $x_i = \theta + \varepsilon_i$ on the operating cost, where $\varepsilon_i \sim N(0, \sigma^2)$. Assume that the belief of investor i prior to observing the cost signal is that θ obeys a uniform distribution in \mathcal{R} . The belief of investor i after observing the cost signal becomes $\theta|x \sim N(x, \sigma^2)$. Find the unique Bayesian-Nash equilibrium of this game.

Exercise 6.24 Two households decide whether to maintain a shared facility simultaneously. If one of these two households maintains, then each household will get 1 unit gain; if there is no maintenance, then there is no gain for the two households. The maintenance costs for these two households are c_1 and c_2 , respectively.

1. Suppose that c_1 and c_2 are 0.1 and 0.5, respectively. What is the Nash equilibrium of this game? What will be the outcome of this game?
2. Assume that c_1 and c_2 are random variables that independently obey a uniform distribution on $[0, 1]$, and the true cost of each household is only known to itself. What is the Bayesian-Nash equilibrium of this game?

Exercise 6.25 Two hostile armies want to occupy an island. Each army can decide whether or not to attack. The probability that the army is strong or weak is one-half (the strength of each army is independent of each other). Each army knows its own strength. If the island is occupied by an army,

this army gains M . If one army attacks the island and the other army does not attack the island, the attacker will occupy the island. If two armies choose to attack at the same time, then the stronger army will occupy the island; if the two armies have the same strength, then no army will occupy the island. Each attacker needs to pay a certain cost: the cost for a strong army is s , and the cost for a weak army is w . The army that does not attack the island will not suffer any cost. Assume $M > w > s$ and $w > M/2 > s$. Find the pure strategy Bayesian-Nash equilibrium of this game.

Exercise 6.26 Consider the following bilateral auction. Both the buyer and the seller quote a price simultaneously. If the seller's price p_s is greater than or equal to the buyer's price p_b , they trade at the price of $p = (p_s + p_b)/2$; however, if p_s is less than p_b , no transaction occurs. The value of the auction item for the buyer is v_b , and for the seller it is v_s . The value is private information for each party, and independently obeys a uniform distribution on $[0, 1]$. If the buyer gets the item at a price of p , then the buyer's payoff is $v_b - p$; if no transaction occurs, the buyer's utility is zero. If the seller sells the item at a price of p , the seller's payoff is $p - v_s$; if no transaction occurs, the seller's utility is zero.

1. Find out the objective functions for the buyer and the seller.
2. Suppose that the equilibrium strategy of the buyer or the seller is a linear function of the value of the item. Show their equilibrium strategies.

Exercise 6.27 Consider a game with two players. Player 1 chooses from three strategies: U , V , and W . Player 2 chooses from two strategies: L and R . Player 2 only knows whether player 1 has chosen U while making her decision and does not know any other information. In the case in which player 1 has chosen U , if player 2 then chooses L , and their payoffs will be $(0, 2)$, where 0 is player 1's payoff and 2 is player 2's payoff. If player 2 then chooses R , their payoffs will be $(2, 0)$. Similarly, if these two players choose V and L one after another, their payoffs will be $(-1, -1)$; if these two players choose V and R one after another, their payoffs will be $(3, 0)$. If these two players choose W and L one after another, their payoffs will

be $(-1, -1)$; if these two players choose W and R one after another, their payoffs will be $(2, 1)$.

1. Represent this game in extensive-form.
2. Find all pure strategy weak perfect Bayesian equilibria of this game.
3. Find out all pure strategy sequential equilibria of this game and compare them with the results in question 2.

Exercise 6.28 Prove the following theorem on the trembling hand perfect Nash equilibrium:

1. In a two-player finite strategy game, a strategy profile is a trembling hand perfect Nash equilibrium if and only if it is a Nash equilibrium, and no strategy is weakly dominated.
2. There is a trembling hand perfect Nash equilibrium in any finite strategy game.
3. In a finite extensive-form game that satisfies perfect recall, there is a belief system μ for each trembling hand perfect Nash equilibrium σ , such that (σ, μ) is a sequential equilibrium of this game.

Exercise 6.29 For a firm recruiting a worker, the probability of recruiting a high-ability worker is 0.8, while the probability of recruiting a low-ability worker is 0.2. Workers can choose whether or not to attend training. The firm does not know workers' abilities, but can observe whether they have been trained. The firm can appoint the recruited worker as either a manager or an employee. If a trained high-ability worker is hired as a manager, the payoffs of the worker and the firm are $(4, 2)$, and if a trained high-ability worker is hired as an employee, the corresponding payoffs are $(2, 1)$. If an untrained high-ability worker is hired as a manager, the payoffs of the worker and the firm are $(3, 2)$, and if an untrained high-ability worker is hired as an employee, the corresponding payoffs are $(1, 1)$. If a trained low-ability worker is hired as a manager, the payoffs of the worker and the firm are $(3, 1)$, and if a trained low-ability worker is hired as an employee, the corresponding payoffs are $(1, 2)$. If an untrained low-ability worker is

hired as a manager, the payoffs of the worker and the firm are $(4, 1)$, and if an untrained low-ability worker is hired as an employee, the corresponding payoffs are $(2, 2)$.

1. Represent this game with extensive-form.
2. Find out all pure-strategy sequential equilibria of this game.
3. In this game, are all pure-strategy sequential equilibria reasonable? Provide an explanation for your answer.

Exercise 6.30 An investor needs to decide how much to invest in an electronics manufacturer, but he or she does not know this manufacturer's true profitability θ . The investor can consult an audit company. Suppose that the audit company knows this manufacturer's true profitability θ and can send the investor a report m on the profitability θ . However, the objectives of the audit company and the investor are not consistent, and the audit company does not necessarily report its true assessment result. Suppose that the investor knows in advance that this manufacturer's true profitability $\theta \in [0, 1]$ and that its probability density function is f . The investor's investment is $a \in [0, 1]$. The payoff functions of the audit company and the investor are $u^S(a, \theta, b)$ and $u^R(a, \theta)$, respectively, where b represents the difference in preferences between the audit company and the investor. Assume that each player's payoff function is continuous and twice differentiable, and $\frac{\partial^2 u^i}{\partial a^2} < 0$ and $\frac{\partial^2 u^i}{\partial a \partial \theta} > 0$, where $i = R, S$. Therefore, there is a unique optimal investment amount $y^R(\theta)$ ($y^S(\theta)$) that maximizes the payoff of the investor (the audit company).

1. Prove that there is a pooling equilibrium, in which regardless of what m the audit company reports, the investment amount of the investor remains consistent.
2. Prove that if for any θ , $y^R(\theta) \neq y^S(\theta, b)$, then the number of equilibrium actions is finite. Hint: It can be proven that for any two different equilibrium actions $a^1 < a^2$, there is $\varepsilon > 0$, such that $a^1 - a^2 \geq \varepsilon$.
3. Prove that the state space is divided into finite intervals in each equilibrium, and in the same interval, the audit company will adopt the

same strategy that leads to the same amount of investment.

4. Prove that for any θ , such that $y^R(\theta) \neq y^S(\theta, b)$, there is a positive integer $N(b)$, such that for any positive integer k between 1 and $N(b)$, there is a corresponding equilibrium which divides the state space into k intervals.

Exercise 6.31 Consider the following dynamic game of incomplete information: Nature chooses Game 1 or Game 2 first, and the probability of choosing Game 1 is 0.6, and the probability of choosing Game 2 is 0.4.

Game 1 is as follows:

		Player 2	
		<i>C</i>	<i>D</i>
Player 1	<i>C</i>	1, 2	2, 1
	<i>D</i>	2, 0	0, 2

Game 2 is as follows:

		Player 2	
		<i>C</i>	<i>D</i>
Player 1	<i>C</i>	2, 1	1, 2
	<i>D</i>	0, 2	4, 0

After observing which game Nature has chosen, player 1 chooses between actions *C* and *D*. Player 2 cannot observe Nature's choice, and after observing player 1's action, he or she chooses between actions *C* or *D*.

1. Prove that there is no separating equilibrium in this game.
2. Solve for the pooling equilibrium of this game.
3. If player 1 cannot observe Nature's choice, can he or she receive a higher payoff?

Exercise 6.32 Consider the following conspicuous consumption model: player *A*'s wealth level is either H or L , where $H > L > 0$. Suppose that *A* knows his own wealth level precisely, while nobody else knows it, and *A*

wants others to think that his wealth level is high because it will make him more satisfied. If other players think that the probability of A 's wealth level being H is q , then the payoff of A is q .

Assume that at the beginning of a period, other players think that the prior probability of A 's wealth level being H is p . Now, A can choose to purchase some expensive-looking goods as a signal of being richer. Let c be the conspicuous consumption on the expensive-looking goods, which does not in itself bring any benefit to A . Assume that the cost of c units of conspicuous consumption is c/w , where w is the actual wealth level of A , i.e., w is equal to H or L .

Assume that other players can observe c and update their posterior probability of A having a wealth level of H based on c and that the total payoff function of A is $q - c/w$, where q is the posterior probability of A having a wealth level of H and w is the actual wealth level of A .

1. Under what conditions does the game have a separating equilibrium?
2. Under what conditions does the game have a pooling equilibrium?
3. Which equilibria above satisfy the intuitive criterion?

Exercise 6.33 Consider the following education investment signaling model: Each employee has two possible types: $\theta \in \{\theta_H, \theta_L\}$, where $\theta_H > \theta_L$. Given $i \in \{H, L\}$, the prior probability of this employee being type θ_i is β_i . The reservation utility for each employee is $\bar{u} = 0$. Employees of type θ can produce θ for a firm. The firm is willing to hire an employee at a wage of w if and only if the employee's expected productivity can at least offset the wage. Employees of type θ can receive e years of education at cost $c(e, \theta) = \frac{e}{\theta}$. The education investment cost function $c(e, \theta)$ satisfies the single-crossing property with respect to (e, θ) , i.e., if $e > e'$, then $c(e, \theta_L) - c(e', \theta_L) > c(e, \theta_H) - c(e', \theta_H)$. Given wage w and education level e , each type θ employee has a payoff function $u(w, e|\theta) = w - c(e, \theta)$. Consider the following sequential actions:

- An employee observes his own type, which is his or her private information;

- The employee chooses his education investment level;
- The firm observes the education level of the employee, but cannot observe the employee type;
- The employee requests a raise in wage from the firm;
- The firm either rejects the request or accepts the request and employs the employee at the wage level.

It is assumed that educational investment can promote a low-type employee to a high-type employee. Specifically, suppose that the probability that a type θ_L employee becomes type θ_H after investing in e years' education is $p(e)$, which satisfies the following properties: $p(0) = 0$, $\lim_{e \rightarrow \infty} p(e) = 1$, $p' > 0$, $p'(0) = \infty$, $\lim_{e \rightarrow \infty} p'(e) = 0$ and $p'' < 0$. Once an employee of type θ_L has invested in education and converts to type θ_H , he can observe his own type conversion prior to entering the labor market. First, assume that there is no asymmetric information between the firm and its employees. Answer the following questions:

1. Write down the optimization problem that can solve for the first best (the information is complete) wage and the first best education investment level.
2. Solve this problem. Prove that the type θ_H employee's investment level in education is 0, and the type θ_L employee's investment level in education is strictly positive.
3. Suppose that education investment now becomes more effective, i.e., the probability that a type θ_L employee will become type θ_H after investing in e years' education is $q(e)$, where $q(e) > p(e)$ for any $e > 0$. What is the first best investment level in education at this time? What is the payoff of type θ_L employee at this time? Provide an intuitive explanation for your answers.

Exercise 6.34 Based on the previous exercise, suppose that there is asymmetric information between the firm and its employees and that the solution concept of pure strategy perfect Bayesian equilibrium is adopted. Con-

sider separating equilibrium that satisfies $e_H \neq e_L$. Answer the following questions:

1. In any separating equilibrium, type θ_L employees choose education level $e_L = e_{FB}$, where FB represents first best.
2. Characterize the investment level in education of type θ_H employees, which satisfies $e_H \neq e_{FB}$.
3. Explain whether or not separating equilibrium always exists.
4. Suppose that education investment now becomes more efficient, i.e., the probability that a type θ_L employee will become type θ_H after investing in e years' education is $q(e)$, where $q(e) > p(e)$ for any $e > 0$. Will type θ_H employees change the education investment level in the above separating equilibrium? Explain your conclusion.

Next, consider pooling equilibrium which satisfies $e_H = e_L = e^*$. Answer the following questions:

1. Given the investment level in education e^* under a pooling equilibrium, what is an employee's wage?
2. Characterize the pooling equilibrium e^* .
3. Does the pooling equilibrium $e^* = 0$ always exist? Does the pooling equilibrium $e^* = e_{FB}$ always exist?
4. Suppose that educational investment now becomes more efficient, i.e., the probability that a type θ_L employee will become type θ_H after investing in e years' education is $q(e)$, where $q(e) > p(e)$ for any $e > 0$. Characterize the impact of this change on the pooling equilibrium e^* and provide your explanation.
5. Given the efficient educational investment, is there a pooling equilibrium that satisfies the intuitive criterion?

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Chapter 7

Repeated Games and Reputation Mechanism

7.1 Introduction

In game theory, the dynamic game refers to a game in which players move either sequentially or repeatedly. The previous chapter mainly discusses the sequential game, in which one player chooses his or her action before the others choose theirs. Importantly, the later players must have some information about the former player's choice; otherwise, the difference in time would have no strategic effect. Sequential games are thus governed by the time axis, and represented in the form of decision trees, but the structure of different sub decision trees (subgames) may be different. Although the repeated game also exhibits a dynamic structure, the difference is that the structure of its sub-decision tree (called the stage game) is the same, i.e., the same game structure is repeated many times. Therefore, we discuss repeated games in this separate chapter. The theory of repeated games provides a central underpinning for understanding social, political, and economic institutions, both formal and informal.

A key factor in understanding institutions and other long-term relationships is the role of shared expectations of behavioral norms such as cultural beliefs, as well as the role of sanctions in ensuring compliance with the “rules” . The so-called repeated game is *an extensive form game that consists*

of a number of repetitions of some base game (called a *stage game*), i.e., it refers to a strategic situation in which all or some of the participants interact repeatedly. Repeated games capture the idea that a player will have to take into account the impact of his or her current action on the future actions of other players; this impact is sometimes called his or her reputation. We can find many examples of repeated games: neighbours in the countryside often interact repeatedly in certain activities, such as sowing and harvesting, social events (weddings, funerals, etc.), and borrowing. When making policies, including fiscal and monetary policies, governments may face similar environments repeatedly; furthermore, policy choices of governments may be constrained by possible future interactions. In addition, firms within the same industry may face repeated games of competition and cooperation.

Compared with one-shot interaction, these repeated interactions have quite dissimilar impacts on individuals' behaviors and incentives. For instance, discussions about increased urban crime rates after the opening-up of China often focus on the reason of migration. Some people behave appropriately in places where they have been living for a long period of time, while they may misbehave when visiting a location for a short period of time. This is not due to a change of preferences, but rather attributable to changed constraints and environments that face them. In a relatively stable environment, information is sufficient that punishment or encouragement is relatively direct and effective. However, for a temporary stay, neither punishment nor encouragement works well. The common logic behind these phenomena is what the theory of repeated games is going to reveal.

The idea of the repeated game has been applied broadly in reality, with the reputation mechanism being one of the most important applications. Since individuals usually do not know much about the quality of goods that they consume, how do they make rational decisions about them? Since different companies have varied reputations (e.g., their product brands), these reputations will affect behavioral decisions regarding the quality of products. For example, individuals may encounter two kinds of restaurants. One is proximal to a railway station, and the other is near residential areas. In general, people only go to the former restaurant occasionally while visiting the latter frequently. Indeed, the quality of food of the former is likely

to be inferior to that of the latter. With the idea of repeated games (e.g., Folk Theorem), repeated interactions will make the reputation mechanism more effective. Then, it is not difficult to understand the above phenomenon. The Folk Theorem actually concerns the fact that virtually any outcome of an infinitely repeated game becomes a subgame perfect Nash equilibrium. Expressed more simply, the theorem suggests that anything that is feasible and individually rational is possible. This result was termed the Folk Theorem because it was widely known among game theorists in the 1950s, even though no one had published it.

In this chapter, we will focus on how multiple interactions change individuals' incentives. The theoretical result shows the reason why the modern market economy can better solve the problem of honesty. Specifically, honesty is not only a traditional virtue and a positive social climate, but also an incentive mechanism, and the long-term outcome of the law and incentive mechanism such as market mechanism such that individuals who are dishonest will be punished. According to the Folk Theorem, any established social climate constitutes a social equilibrium. Indeed, once the social norm or convention is established, as long as the discount factor is sufficiently large (i.e., the future penalty after deviation is sufficiently large), no one has the incentive to deviate unilaterally. If the process of strategic choices can be divided into successive steps, rational behaviors would consider the influences of all individuals' initial behaviors on subsequent choices and final outcomes. The key issue here is credibility, i.e., once deviation occurs, whether or not the punishment is actually executed. In fact, this is precisely the key to establish credibility in a realistic society.

As will be revealed in this chapter (Section 7.5.3), in a society replete with deceit and dishonesty, even if the discount factor equals one, if a person chooses to be honest, his or her interest will be harmed. As a result, deceiving each other is a Nash equilibrium. However, in a society in which most people are honest, cheating is subject to legal sanctions and public condemnation (the discount factor is greater than some lower bound), and being honest would constitute a Nash equilibrium. If the discount factor in the society is not particularly large, the larger is the proportion of "bad people" in the society, the lower is the effectiveness of social norms, since

it increases with the decrease of the proportion of good people. This is the profound concept contained in the Folk Theorem that tells us that if agents (usually referring to firms in markets) live long enough, they will be in an infinitely repeated game, rather than in a static one. When most of them are patient enough and far-sighted enough to be honest (discount factor is close to 1), keeping honest is a Nash equilibrium. As such, both dishonesty and honesty could be Nash equilibria. The determinant factor is which one is in the majority (characterized by discount factor). As long as a mechanism is designed properly, such that the punishment of deviations is credible and the cost is sufficiently large to offset the extra benefit of deviating behaviors, no one has the incentive to deviate unilaterally.

The repeated game is a profile of stage games that repeat many times. Depending on the horizon, it can be either a finitely repeated game or an infinitely repeated game, which is also called a supergame. In addition, based on the information distribution structure, it can also be a repeated game with perfect monitoring, a repeated game with imperfect public monitoring, or a repeated game with private monitoring, resulting in conclusions that may be quite different. We will mainly focus on the discussion of the first two kinds of repeated games.

7.2 Examples of Repeated Games

Repeated games can clearly describe not only the short-sighted incentives for which agents do not follow the rules, but also the incentive measures to prevent such opportunistic behaviors through appropriate norms, rewards and punishments for future behaviors.

Firstly, we illustrate this point through examples of long-term relationship and opportunistic behavior. These examples can assist us to understand the basic idea behind repeated games.

Example 7.2.1 Table 7.1 describes the incentive problem in a team. Consider a team with two members. If both players work hard, each receives a payoff of 3; if one is lazy and the other works diligently, the lazy worker receives a payoff of 4 and the other receives a payoff of -1; if both are lazy,

each receives a payoff of 1. From the perspective of team welfare, both should choose to work hard; however, from the perspective of individual welfare, both will choose to be lazy because being lazy is a dominant strategy for both, and it is the unique equilibrium.

Now, suppose that they will interact infinitely often, with the payoff at the end of each period given by Table 7.1. At the beginning of each period, they can observe all previous choices. Let the discount factor be $0 < \delta < 1$. In such a repeated game, a player cares about the utility of all periods. As is standard, the (normalized) utility of player i in the repeated game is

$$U_i = (1 - \delta) \sum_{t=0}^{\infty} \delta^t u_{it},$$

where u_{it} is the payoff obtained in period t . Note that if $u_{it} = u_i$ for all t , then $U_i = u_i$.

		player 2	
		E	S
player 1	E	3, 3	-1, 4
	S	4, -1	1, 1

Table 7.1: Dilemma of Work Incentives

Is the collectively rational payoff (team welfare) a Nash equilibrium payoff of an infinitely repeated game? Since the history of previous actions can be observed, the choice of a player in a certain stage game depends on the past history of other players. Consider a “**grim strategy profile**”: each player chooses to work hard at the beginning, but as soon as one player defects, both defect forever. If we regard “work hard” and “be lazy” as “cooperation” and “punishment”, respectively, “grim strategy” means that once some player has deviated from “cooperation”, both will choose to “punish the other”. In other words, players do not forgive “treachery”. Because of this, such a strategy is also called **trigger strategy**: any participant’s non-cooperation triggers the other party to never cooperate.

We shall show that when $\delta \geq \frac{1}{3}$, the grim strategy profile is a subgame perfect Nash equilibrium, which is a Nash equilibrium that satisfies se-

quential rationality in any subgame, i.e., provided one player has chosen to “be lazy” in some previous round, the opponent will choose to “be lazy” in every future round; then, the optimal choice for the player who triggered the punishment is also to “be lazy”. To see this, consider a stage game starting in period t where both players chose to “work hard” in all previous periods. Under the “grim strategy” profile, player i chooses to “work hard” and earns the utility

$$(1 - \delta) \left[\sum_{s=0}^{t-1} \delta^s 3 + \delta^t 3 + \sum_{s \geq t+1} \delta^s 3 \right] = 3.$$

If player i chooses to “be lazy” once and follow the “grim strategy” thereafter, his or her utility is

$$(1 - \delta) \left[\sum_{s=0}^{t-1} \delta^s 3 + \delta^t 4 + \sum_{s \geq t+1} \delta^s 1 \right] = 3(1 - \delta^t) + 4(1 - \delta)\delta^t + \delta^{t+1}.$$

When $\delta \geq \frac{1}{3}$, it can be shown that

$$3(1 - \delta^t) + 4(1 - \delta)\delta^t + \delta^{t+1} \leq 3 = 3(1 - \delta^t) + 3(1 - \delta)\delta^t + 3\delta^{t+1}.$$

Therefore, given that the opponent chooses the grim strategy, in any subgame player i will choose to follow the grim strategy, as well.

In this infinitely repeated game, why are players willing to give up the best choice in the short run, “be lazy”, and instead “work hard”? The above reasoning process of “grim strategy” shows that each player will weigh the short-run benefits of choosing to “be lazy” and the long-run returns of choosing to “work hard”. When each player’s extra return, 2δ , from cooperation is greater than the short-run extra return, $(1 - \delta)$, from non-cooperation, they both resist opportunistic behavior, i.e., not to “be lazy”. As the discount factor δ increases, players place more weight on long-run returns.

In this infinitely repeated game, there are multiple subgame-perfect Nash equilibria. One such equilibrium involves each player choosing to “be lazy” in every period; this gives $(1, 1)$ as the equilibrium payoff. In

fact, $(t + 3(1 - t), t + 3(1 - t))$, for all $t \in [0, 1]$, are all payoffs of some Nash equilibria in this infinitely repeated game. This conclusion is called the “Folk Theorem” .

Now, instead of infinitely many periods, we allow the players to interact for a finite horizon $T < \infty$. Then, under grim strategies, for any $\delta \leq 1$, the unique subgame perfect Nash equilibrium is: “be lazy” in each period. This is because, in the last period T , players choose to “be lazy” as there are no future returns; in period $T - 1$, as the choices in this period will not influence their choices in the next period, both players will choose to “be lazy” , as well. By backward induction, since the choices in any period have no effect on their behavior in future periods, they always choose their short-run optimum, i.e., to “be lazy” .

However, this conclusion depends on the uniqueness of Nash equilibrium in the one-stage game. This is true when there are multiple equilibria in a stage game. The finiteness of the horizon does not fully determine players’ behavior in a repeated interaction. The key determinant is how behavior in the current period affects future interaction. This is the case when a stage game has more than one equilibrium. We now explain this idea with a two-stage game.

Example 7.2.2 Suppose that there are two players, denoted $\{1, 2\}$. Each has three choices: $\{L, M, R\}$. The payoffs are shown in Table 7.2. The game is repeated twice, using the discount factor 1 for simplicity (analysis of the general situation with $\delta \leq 1$ is similar).

		player 2		
		L_2	M_2	R_2
player 1	L_1	1, 1	5, 0	0, 0
	M_1	0, 5	4, 4	0, 0
	R_1	0, 0	0, 0	3, 3

Table 7.2: Two-Stage Repeated Game

The single-stage game has two Nash equilibria, (L_1, L_2) and (R_1, R_2) , with payoffs $(1, 1)$ and $(3, 3)$, respectively. However, compared with the two Nash equilibria, the action profile (M_1, M_2) is better off for both the play-

ers in terms of team welfare. The two-stage game has more than one subgame perfect Nash equilibrium. For example, playing any of the above Nash equilibria in each stage forms a subgame perfect Nash equilibrium. In addition, there is another subgame perfect Nash equilibrium: in stage 1, players choose (M_1, M_2) ; if the choice in stage 1 is (M_1, M_2) , in stage 2 the players choose (R_1, R_2) ; otherwise, they choose (L_1, L_2) .

We now show that the two stage strategy profile $\{(M_1, M_2), (R_1, R_2)\}$ constitutes a subgame perfect Nash equilibrium. First, in stage 2, this constitutes a Nash equilibrium in any subgame. Next, we consider that in stage 1, given that the opponent chooses M_j , if player i chooses M_i , the total payoff of two periods for him or her is $4 + 3 = 7$. If player i chooses L_i , the payoff for him or her is $5 + 1 = 6$; if he or she chooses R_i , the payoff is $0 + 1 = 1$. Thus, M_i is the best response of player i in stage 1, as desired.

Why do players choose individually irrational, but collectively rational behavior in the first stage (relative to the single stage game)? The key reason is that the choice in stage 1 will influence payoffs thereafter. In other words, when players make decisions, they are weighing short-run and long-run returns. When the latter is larger, each player will choose cooperation as an optimal choice in the long run. This is exactly the logic of repeated games.

In the above repeated game, a “cooperation” mechanism is used to punish deviations. However, punishment or other mechanisms that encourage cooperative behavior have different levels of efficiency in different situations. In the above example, players can observe all past choices. If they cannot, can the outcomes that they observe assist them to infer past behavior? If not, the punishment scheme may not effectively prevent deviations. In addition, the punishment scheme itself needs to satisfy some conditions. Draconian punishment schemes may run counter to the rationality of those who carry them out; thus, it is important to construct appropriate punishment schemes. Indeed, punishment schemes may also involve cooperation among players. To this end, it may be necessary to encourage the performers. Furthermore, cooperation during a punishment process may involve information issues, as well. The degree of punishment, or when the

punishment ends, is also an important issue. There are different solutions for different types of interaction. These will all be discussed and answered in this chapter.

Next, we start with the simplest repeated game with perfect monitoring, in which players can observe previous actions. Subsequently, we will discuss repeated games with imperfect public monitoring, in which players can observe the outcome of a public behavior instead of previous actions. We then discuss the repeated game with private monitoring, in which different players observe different outcomes. Finally, we examine the economic logic of the reputation mechanism. The repeated game is an important branch in the development of game theory. This literature also proposes and solves new problems. The most comprehensive survey of this literature is given in Mailath and Samuelson (2006), to which many discussions in this chapter refer.

7.3 Repeated Games with Perfect Monitoring

This section first sets up the basic structure and concepts of the repeated game model with perfect monitoring (i.e., observable previous actions), and then focuses on providing important techniques and tools for proving the Folk Theorem in the next section and its extensions in more general environments in the consequent sections.

A repeated game consists of a stage or constituent game. Generally, in a repeated game, the stage game is a static game with simultaneous actions (in some repeated games, the stage game may also be in extensive form). Let Γ^t be the stage game in period t . The set of players in period t is N^t , the set of actions of player $i \in N^t$ is (A_i^t) , and the set of mixed strategies is ΔA_i^t , which is the probability distribution on the action set of period t , and the utility/payoff function is $(u_i(a^t))_{i \in N^t}$, with $a^t \in A^t \equiv A_1^t \times \cdots \times A_{N^t}^t$ being the action profile in period t .

Let $h^t = (a^0, a^1, \dots, a^{t-1})$ be the previous action history at period t , where $a^0 \in H^0$ is the initial action history. The set of action histories at period t is denoted by H^t . All possible action histories are contained in $H \equiv \bigcup_{t=0}^{\infty} H^t$. With perfect information, a player can observe all previous actions of all players. In period t , the behavioral strategy of player i is denoted as $\sigma_i^t : H^t \rightarrow \Delta A_i^t$. The (mixed) strategy of player i in the repeated game is denoted by $\sigma_i = (\sigma_i^t)_{t \in \{1, 2, \dots, \infty\}}$.

For any action history h^t , we define a **continuation game** for an (infinitely) repeated game. From the beginning of action history h^t at period t , for any strategy profile σ , the continuation strategy of player i given h^t is denoted as $\sigma_{i|h^t}$, which is defined as

$$\sigma_{i|h^t}(h^\tau) = \sigma_i(h^t h^\tau), \forall h^\tau \in H,$$

where $h^t h^\tau$ is the profile of action histories that start with h^t . The continuation game generated from a given action history is a subgame of the whole repeated game.

If the repetition period is finite, the repeated game $\Gamma_R = (\Gamma^t)_{t \in T}$ is

called **finitely repeated game**; otherwise, it is called **infinitely repeated game**. The simplest infinitely repeated game is that the game in each stage is the same, i.e., we have $N^t = N$ and $A_i^t = A_i$. In the following, we mainly discuss this kind of repeated game.

As the game has multiple periods, the utility of a player is, in general, defined as the sum of intertemporal discounted utilities, and the discount factor δ is the same for all players. Of course, in some cases, such as bargaining, different players may have different discount factors.

Denote the strategy profile for players by $\sigma = (\sigma_i)_{i \in N} = (\sigma_i^t)_{t=1}^\infty$, where $\sigma_i = (\sigma_i^t)_{t=1}^\infty$ is the strategy of player i and $\sigma^t = (\sigma_i^t)_{i \in N}$ is the strategy profile of all players in period t .

Given the strategy profile $\sigma = (\sigma_i)_{i \in N}$, the payoff for player i is

$$U_i(\sigma) = (1 - \delta) \sum_{t=0}^{\infty} \delta^t u_i(\sigma^t).$$

When defined in this way, the domain of this utility is the same as that of utility in the stage game. It is worth noting that U_i is the payoff of player i in the whole repeated game, while u_i is the payoff in the stage game.

Then, under what conditions are there equilibria of the repeated game? What are their features and range? These are questions that we will answer in the remainder of this chapter.

7.3.1 Feasible and Individually Rational Payoffs

We first give some basic concepts of the stage game and equilibrium solution concept of a repeated game.

Define $F \equiv \{v \in \mathcal{R}^n : \exists a \in A, \text{ s.t. } v = u(a)\}$ as the payoff set of pure strategy profiles of the stage game, and define $F^+ \equiv \text{co}F$ as the convex hull of the set F .

Definition 7.3.1 (Feasible Payoffs) A profile of payoffs is *feasible* in the stage game Γ^t if $v \in F^+$.

Unfeasible payoffs cannot be outcomes of the game.

Towards finding a lower bound on the payoffs from pure-strategy Nash equilibria, we define the following concept.

Definition 7.3.2 (Minmax Payoffs) In the stage game Γ^t , player i 's pure strategy minmax payoff \underline{v}_i^p is:

$$\underline{v}_i^p \equiv \min_{\mathbf{a}_{-i} \in A_{-i}} \max_{a_i \in A_i} u_i(a_i, \mathbf{a}_{-i}),$$

i.e., it is the lowest payoff that the player can obtain regardless of all the other players' choices.

In other words, it is the minimum of player i 's best response function over other players' strategies.

When other players can employ mixed strategies in the stage game, the **mixed strategy minmax payoff** is defined as

$$\underline{v}_i \equiv \min_{\sigma_{-i} \in \times_{j \neq i} \Delta A_j} \max_{a_i \in A_i} u_i(a_i, \mathbf{a}_{-i}).$$

In the stage game, player i will never receive a payoff lower than the minmax payoff.

Definition 7.3.3 (Individually Rational Payoffs) A pure strategy profile is *individually rational* in the stage game Γ^t if the pure strategy payoff of every player is not less than its pure strategy minmax payoff, i.e., $v_i \geq \underline{v}_i^p$ for all $i \in N$. Similarly, a mixed strategy profile is *individually rational* in the stage game Γ^t if $v_i \geq \underline{v}_i$ for all $i \in N$.

In other words, a payoff profile is individually rational if it gives each player at least his or her guarantee.

The set of pure-strategy individually rational payoffs of the form $\mathbf{v} = (v_i)_{i \in N}$ is defined as

$$F^p \equiv \{\mathbf{v} : v_i \geq \underline{v}_i^p \mid \mathbf{v} \in F^+\}.$$

Similarly, the payoff of mixed strategy individually rational payoffs is defined as

$$F^* \equiv \{\mathbf{v} : v_i \geq \underline{v}_i \mid \mathbf{v} \in F^+\}.$$

The feasibility and individual rationality of payoff profiles are very important requirements. In the next section, we will prove that any feasible

and individually rational payoff profile v is an equilibrium payoff profile of a subgame perfect equilibrium.

Here are some examples of calculating the minmax payoff of players.

Example 7.3.1 Consider the following matching pennies game.

In the matching pennies game in Table 7.3, the minmax payoff of pure strategy for player 1 (or player 2) is $\underline{v}_1^p = \underline{v}_2^p = 1$. The mixed strategy minmax payoff is $\underline{v}_1 = \underline{v}_2 = 0$ since optimal mixed strategy of player 1 (or player 2) is to choose heads or tails with probability 0.5.

		player 2	
		Head	Tail
player 1	Head	1, -1	-1, 1
	Tail	-1, 1	1, -1

Table 7.3: The Minmax Payoff of Matching Pennies Game

Next, we solve for the set of feasible and individually rational payoffs in the following example.

		player 2	
		Left	Right
player 1	Up	-2, 2	1, -2
	Middle	1, -2	-2, 2
	Down	0, 1	0, 1

Table 7.4: Example of Feasible Payoff

Example 7.3.2 Consider the game given by Table 7.4. It can be shown that this game does not have a pure strategy Nash equilibrium, but there is a mixed strategy Nash equilibrium at which two players choose (Up, Middle) and (Left, Right) with probability 0.5, respectively, and then the mixed strategy minmax payoff is 0. Thus, the set of feasible and individually rational payoffs is the shaded area in Figure 7.1.

During the interaction, players may use some public correlation devices to coordinate behavior among players. For example, in the 1950s, during

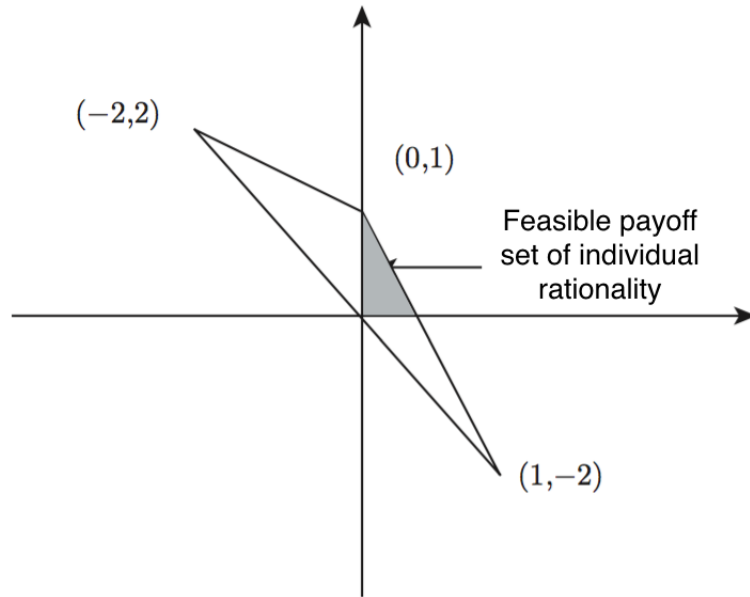


Figure 7.1: Feasible individual rational payoff

the collusion of bids for electrical equipment, bidders used the phase of the moon to coordinate their bids. Let W be the set of public correlation devices in which $w \in W$ is one of its states that can be observed by all players, and p be a probability distribution over W . The behavior strategy of each player can conditional on the state of the public correlation device, i.e., $\sigma_i(\cdot) : W \rightarrow \Delta A_i$. With public correlation devices, the set of payoffs will be extended.

Example 7.3.3 Consider the game in Table 7.5. Let $W = \{w_1, w_2\}$, and the probability of each state be 0.5. When w_1 appears, the choices of players 1 and 2 are (Up, Right); when w_2 appears, the choices of players 1 and 2 are (Down, Left). Thus, the expected payoffs obtained under the public correlation device are (3,3). However, without such a device, they cannot achieve these payoffs.

Assuming that public correlation devices exist, the set of feasible pay-

		player 2	
		Left	Right
player 1	Up	2, 2	1, 5
	Down	5, 1	0, 0

Table 7.5: Example of public correlation devices

offs in the repeated game is:

$$V^* = F^+ \equiv \left\{ \sum_{a \in A} \lambda(a) u(a) \mid \exists \lambda(\cdot), \lambda(a) \in [0, 1], \sum_{a \in A} \lambda(a) = 1 \right\},$$

which is the convex hull of F

After introducing public correlation devices, the set of feasible and individually rational payoffs can be defined as $FV^* = \{v \mid v \in V^*, v_i \geq \underline{v}_i\}$. If the weak inequality is replaced by a strict inequality, it becomes the set of feasible and strictly individually rational payoffs.

Next, we discuss what equilibrium concept is appropriate for repeated games. We first discuss Nash equilibrium.

Definition 7.3.4 (Nash Equilibrium of Repeated Games) A strategy profile σ is a *Nash equilibrium of a repeated game*, if for any $i \in N$ and any σ'_i , we have $U_i(\sigma) \geq U_i(\sigma'_i, \sigma_{-i})$.

Since a repeated game with perfect monitoring is a dynamic game with complete information, it is natural to use subgame perfect Nash equilibrium.

Definition 7.3.5 (Subgame Perfect Nash Equilibrium of Repeated Games) A strategy profile σ is a *subgame perfect Nash equilibrium of a repeated game*, if for any history $h^t \in H$, $\sigma|_{h^t}$ is the Nash equilibrium of the repeated game that starts from the history h^t .

An infinitely repeated game has infinitely many histories and subgames. As a consequence, it is difficult to verify whether a strategy profile is subgame perfect Nash equilibrium. As such, we will provide several key techniques and tools, which will be discussed in the rest of this section.

First of all, we introduce a criterion called the one-shot deviation principle, which can be used to identify whether a strategy profile is a subgame perfect Nash equilibrium. The idea comes from dynamic programming.

7.3.2 One-Shot Deviation Principle

For player i , a **one-shot deviation** from σ_i is any strategy $\hat{\sigma}_i \neq \sigma_i$ that agrees with σ_i at all histories but one: there exists a unique history $\tilde{h}^t \in H$ such that $\hat{\sigma}_i(\tilde{h}^t) \neq \sigma_i(\tilde{h}^t)$ while $\hat{\sigma}_i(h^s) = \sigma_i(h^s)$ for all other histories $h^s \neq \tilde{h}^t$.

Definition 7.3.6 (Profitable One-Shot Deviation) Given the strategy profile σ_{-i} of other players, one-shot deviation $\hat{\sigma}_i$ of strategy σ_i is *profitable*, if there exists a unique history $\tilde{h}^t \in H$ with $\hat{\sigma}_i(\tilde{h}^t) \neq \sigma_i(\tilde{h}^t)$, such that $U_i(\hat{\sigma}_i |_{\tilde{h}^t}, \sigma_{-i} |_{\tilde{h}^t}) > U_i(\sigma_i |_{\tilde{h}^t}, \sigma_{-i} |_{\tilde{h}^t})$.

Next, we discuss the relationship between the one-shot deviation principle and subgame perfect Nash equilibrium.

Proposition 7.3.1 (One-Shot Deviation Principle) *A strategy profile is a subgame perfect Nash equilibrium of a repeated game if and only if no player has a profitable one-shot deviation.*

The importance of this principle is that we do not need to consider all possible deviations when solving for the subgame perfect Nash equilibrium of a repeated game. For instance, to check if the strategy profile σ_i is subgame perfect, we do not need to consider strategies of player i that deviate in period t , again in $t' > t$, etc.

PROOF. Here, we prove the one-shot deviation for the case of pure strategies without public correlation devices. When mixed strategies and/or public correlation devices are allowed, proofs are similar, but require more technical details. Obviously, if a strategy profile is a subgame perfect Nash equilibrium, for each player there are no better strategies, which includes one-shot deviation strategies. Thus, the necessary condition is immediate.

Now, we prove sufficiency by way of contradiction. If a strategy profile is not a subgame perfect Nash equilibrium, there must exist a profitable one-shot deviation strategy.

Suppose that the strategy profile σ is not the subgame perfect Nash equilibrium of a repeated game. Then, there exists at least one history, e.g., \tilde{h}^t , such that for player i , there exists a strategy $\tilde{\sigma}_i \neq \sigma_i$ leading to

$$U_i(\tilde{\sigma}_i |_{\tilde{h}^t}, \sigma_{-i} |_{\tilde{h}^t}) > U_i(\sigma |_{\tilde{h}^t}).$$

Define $\varepsilon = U_i(\tilde{\sigma}_i |_{\tilde{h}^t}, \sigma_{-i} |_{\tilde{h}^t}) - U_i(\sigma |_{\tilde{h}^t}) > 0$. Let $M = \max_{\mathbf{a}} u_i(\mathbf{a})$ and $m = \min_{\mathbf{a}} u_i(\mathbf{a})$ be the highest and the lowest payoffs that player i can obtain in a stage game. As $\delta < 1$, there exists $T > t$, such that $\delta^T(M - m) < \frac{\varepsilon}{2}$.

Now, consider a strategy $\hat{\sigma}_i$ that satisfies: if $\tau < T$, for any $h^\tau \in H$, $\hat{\sigma}_i(h^\tau) = \tilde{\sigma}_i(h^\tau)$ always holds; if $\tau \geq T$, for any $h^\tau \in H$, $\hat{\sigma}_i(h^\tau) = \sigma_i |_{\tilde{h}^t}(h^\tau)$ always holds, where $\sigma_i |_{\tilde{h}^t}(h^\tau)$ is the behavioral strategy that conditions on h^τ that includes \tilde{h}^t .

Obviously, if $U_i(\tilde{\sigma}_i |_{\tilde{h}^t}, \sigma_{-i} |_{\tilde{h}^t}) - U_i(\hat{\sigma}_i |_{\tilde{h}^t}, \sigma_{-i} |_{\tilde{h}^t}) < \varepsilon/2$, then $U_i(\hat{\sigma}_i |_{\tilde{h}^t}, \sigma_{-i} |_{\tilde{h}^t}) - U_i(\sigma |_{\tilde{h}^t}) > \varepsilon/2$. This implies that for σ_i , if a profitable deviation $\tilde{\sigma}_i$ exists, there must exist a profitable deviation $\hat{\sigma}_i$, which deviates at most T periods. Let $\hat{h}^{T-1} = (\hat{\mathbf{a}}^0, \dots, \hat{\mathbf{a}}^{T-2})$ be a history of $T-1$ periods which is generated from $(\hat{\sigma}_i, \sigma_{-i} |_{\tilde{h}^t})$.

If $U_i(\hat{\sigma}_i |_{\tilde{h}^t \hat{h}^{T-1}}, \sigma_{-i} |_{\tilde{h}^t \hat{h}^{T-1}}) > U_i(\sigma |_{\tilde{h}^t \hat{h}^{T-1}})$, a favorable one-shot deviation strategy is obtained at $\tilde{h}^t \hat{h}^{T-1}$. (Note that in period T and after that $\hat{\sigma}_i |_{\tilde{h}^{T-1}}$ is consistent with σ_i). If $U_i(\hat{\sigma}_i |_{\tilde{h}^{T-1}}, \sigma_{-i} |_{\tilde{h}^t \hat{h}^{T-1}}) \leq U_i(\sigma_i |_{\tilde{h}^t \hat{h}^{T-1}}, \sigma_{-i} |_{\tilde{h}^t \hat{h}^{T-1}})$, construct a new strategy $\bar{\sigma}_i$, which is defined as:

$$\bar{\sigma}_i(h^\tau) = \begin{cases} \hat{\sigma}_i(h^\tau), & \text{if } \tau < T-1; \\ \sigma_i |_{\tilde{h}^t}(h^\tau), & \text{if } \tau \geq T-1. \end{cases}$$

It can be shown that

$$\begin{aligned} U_i(\hat{\sigma}_i |_{\hat{h}^{T-2}}, \sigma_{-i} |_{\tilde{h}^t \hat{h}^{T-2}}) &= (1 - \delta)u_i(\hat{\mathbf{a}}^{T-1}) + \delta U_i(\hat{\sigma}_i |_{\hat{h}^{T-1}}, \sigma_{-i} |_{\tilde{h}^t \hat{h}^{T-1}}) \\ &\leq (1 - \delta)u_i(\hat{\mathbf{a}}^{T-1}) + \delta U_i(\sigma_i |_{\tilde{h}^t \hat{h}^{T-1}}, \sigma_{-i} |_{\tilde{h}^t \hat{h}^{T-1}}) \\ &= U_i(\bar{\sigma}_i |_{\hat{h}^{T-1}}, \sigma_{-i} |_{\tilde{h}^t \hat{h}^{T-2}}). \end{aligned}$$

In this way, $U_i(\bar{\sigma}_i | \tilde{h}^t, \sigma_{-i} | \tilde{h}^t) \geq U_i(\hat{\sigma}_i | \tilde{h}^t, \sigma_{-i} | \tilde{h}^t)$. Therefore, for $\sigma_i | \tilde{h}^t$, we obtain a profitable deviation $\bar{\sigma}_i$ that deviates during at most the first $T - 1$ periods conditioning on \tilde{h}^t . By using backward induction, we can obtain a profitable one-shot deviation strategy. \square

Example 7.3.4 (Prisoner's Dilemma Continued) Consider the problem of cooperation during work, where each player wants to be a free-rider. The payoff matrix in a stage game for two players is represented in Table 7.6.

		player 2	
		<i>E</i>	<i>S</i>
player 1	<i>E</i>	3, 3	-1, 4
	<i>S</i>	4, -1	1, 1

Table 7.6: Incentives in the Prisoner's Dilemma

Suppose that players choose the “**tit for tat**” strategy in which tit-for-tat strategies at $t + 1$ only depends on what is played at t not any previous play, i.e., a player chooses to work in the initial stage and thereafter copy the opponent's behavior in the previous period. In order to ensure that a player does not have a profitable one-shot deviation strategy, for player 1, we only need to examine whether the payoff of choosing to be “lazy” (*S*) in period 1 and then going back to “tit for tat” is higher than that of choosing to “work” (*E*) in period 1. The deviation beginning at period t is similar to what we have described.

If each player chooses the “tit for tat” strategy, the expected equilibrium payoff is 3; if player 2 chooses the “tit for tat” strategy while player 1 chooses deviation only in period 1, the outcomes of such actions in different periods are (“lazy” , “work”), (“work” , “lazy”), (“lazy” , “work”), and (“work” , “lazy”), \dots . The expected payoff for player 1 is

$$\begin{aligned}
 U_1 &= 4 - \delta + 4\delta^2 - \delta^3 + \dots \\
 &= (1 - \delta) \left[\sum_{t=0}^{\infty} 4\delta^{2t} - \sum_{t=0}^{\infty} \delta^{2t} \right] \\
 &= \frac{4 - \delta}{1 + \delta}.
 \end{aligned}$$

Thus, when $\delta \geq \frac{1}{4}$ there is no profitable one-shot deviation strategy for both players.

However, when $\delta = 1/4$, the most profitable deviation strategy for “tit for tat” is choosing to be “lazy” in every period, rather than one-shot deviation. In this situation of history, if the opponent chooses the “tit for tat” strategy, a player’s expected payoff of choosing to be “lazy” in every period is $4(1 - \delta) + \delta$. If $\delta \geq \frac{1}{3}$, then $4(1 - \delta) + \delta \leq 3$ holds. Then, when $\delta \in [1/4, 1/3)$, the “tit for tat” strategy is not profitable one-shot deviation, but profitable multiple-deviation strategy. Therefore, the “tit for tat” strategy is not a Nash equilibrium strategy. Consequently, there is no subgame perfect Nash equilibrium for $\delta \in [1/4, 1/3)$.

Here we have only considered three types of histories of actions: no deviation, one-shot deviation, and multiple deviation. In fact, by introducing the technique of automata for simplifying examination of all histories, we can show that “tit for tat” strategy profile is not a subgame perfect Nash equilibrium for any $\delta \geq (0, 1)$.

7.3.3 Automaton Representation of Strategy Profiles

Although the one-shot deviation principle greatly simplifies the test of subgame perfect Nash equilibrium, many histories need to be examined to check whether a profitable one-shot deviation exists. A further simplification is to divide histories into equivalence classes, such that all histories in an equivalence class produce the same continuation strategy. If we describe an equivalence class as a state, we can describe strategies in different equivalence classes in a manner that is similar to an automaton. The use of automata in repeated games was pioneered by Aumann (1981). Rubinstein (1986), Abreu and Rubinstein (1988), and Osborne and Rubinstein (1994) describe the problem of strategic choice in repeated games using automata.

An **automaton** consists of $(\Omega, \omega^0, f(\cdot), \tau(\cdot))$, where Ω represents all possible states (all possible equivalence classes of histories), and $\omega^0 \in \Omega$ is the initial state. The function $f : \Omega \rightarrow \Pi_i \Delta(A_i)$ describes the mapping from states to behavioral action profiles, and $f^w(a)$ denotes the probability of choosing profile a at state w , which satisfies $\sum_{a \in A} f^w(a) = 1$. The state

transition function $\tau : \Omega \times A \rightarrow \Omega$ characterizes how we transition from the current state and the current action to the next period's state.

If the outcome of $f(\cdot)$ is a pure strategy profile, the unique history generated by the automaton $(\Omega, \omega^0, f(\cdot), \tau(\cdot))$ is $(\mathbf{a}^0, \mathbf{a}^1, \dots)$, where

$$\mathbf{a}^0 = f(\omega^0), \mathbf{a}^1 = f(\tau(\omega^0, \mathbf{a}^0)), \mathbf{a}^2 = f(\tau(\tau(\omega^0, \mathbf{a}^0), \mathbf{a}^1)), \dots$$

Thus, we can redefine a new transition function $\tau : \Omega \times H/\{\emptyset\} \rightarrow \Omega$, i.e., $\tau(w, h^t) = \tau(\tau(w, h^{t-1}), \mathbf{a}^{t-1})$. Based on this, we can form a one-to-one correspondence between the strategy profile and the automaton: $\sigma(h^t) = f(\tau(\omega^0, h^t))$. Meanwhile, $f(h^t) = \sigma(h^t)$ and $h^{t+1} \equiv (h^t, \mathbf{a}^t) = \tau(h^t, \mathbf{a}^t)$.

The automaton can divide the entire history H into equivalence classes and generate the same strategy within an equivalence class. The set of states under automaton is usually a finite set. Under automaton representation, for the strategy $\sigma|_{h^t}$ after history h^t , each state forms a specific continuation strategy.

The automaton of a player can be defined as $(\Omega_i, \omega_i^0, f_i, \tau_i)$, and it is interchangeable with individual strategy σ_i .

Example 7.3.5 (Automaton Representation of Grim Strategy) Consider the previous example of grim strategy that implies that a player chooses to work hard (E) first. If players always choose to work hard (E) prior to period t , they choose to work hard in this period, as well. If some player has chosen to be lazy (S), he or she chooses to be lazy from then on.

In the static (one-shot) play, if each player plays S , it will result in SS outcome (action); if player 1 plays S and player 2 plays E , it will result in SE outcome; other outcomes can be similarly denoted. Also, since each equivalence class produces the same continuation strategy, the state set for the grim strategy only contains two element $\{EE, SS\}$.

Then the automaton to represent the grim strategy can be expressed as

$$\Omega = \{w_{EE}, w_{SS}\}, f(w_{EE}) = EE, f(w_{SS}) = SS,$$

and

$$\tau(w, a) = \begin{cases} w_{EE}, & \text{if } w = w_{EE}, a = EE, \\ w_{SS}, & \text{otherwise.} \end{cases}$$

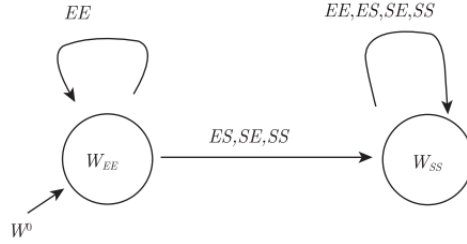


Figure 7.2: Automaton Representation of Grim Strategy

The state transition function is represented by Figure 7.2.

Combining the one-shot deviation principle with the automaton representation, when verifying whether or not a Nash equilibrium is subgame perfect, we only need to make sure that in each state $w \in \Omega$, the strategy profile generated by the automaton $(\Omega, \omega^0, f(\cdot), \tau(\cdot))$ is a Nash equilibrium.

In the case of incentives in the Prisoner's Dilemma, it is easy to prove that when $\delta \geq 1/3$, the grim strategy profile is a subgame perfect Nash equilibrium.

7.3.4 Credible Continuation Promises

In order to analyse repeated games using automata, we first characterize the set of equivalence classes of states. At every stage, a player needs to consider not only his or her payoff in the current period, but also the impact of his or her decision on future states. Abreu, Pearce, and Stacchetti (1986, 1990) proposed a method to describe the state. Specifically, they suggested using the continuation (expected) discounted value to describe the state; thus, the state determines not only a player's incentives in the stage game, but also his or her payoff from the continuation game. The idea of this approach comes from the dynamic programming method that transforms the dynamic optimization problem into Bellman equation. It writes

the “value” of a decision problem at a certain point in time in terms of the payoff from some initial choices and the “value” of the remaining decision problem that results from those initial choices. This breaks a dynamic optimization problem into a sequence of simpler subproblems, as the Bellman’s “principle of optimality” prescribes. So is here: the decision problem of the dynamic game is transformed into a series of (correlated) static decision problems, i.e., establishing a recursive structure to analyze repeated interactions among players. The approach has become a standard approach to solving repeated games. Next, we discuss the logic behind this approach.

Fix an automaton $(\Omega, \omega^0, f(\cdot), \tau(\cdot))$, which starts from ω^0 . The set of all reachable states is Ω , with generic element $w \in \Omega$. For player i , let $V_i(w)$ be the continuation (expected) discounted value starting from state w . In other words, $V_i(w)$ is player i ’s expected discounted value that is generated by the strategy sequence that comes from the automaton under state w .

Stating that the strategy generated by $(\Omega, \omega^0, f(\cdot), \tau(\cdot))$ is a subgame perfect Nash equilibrium means that, given that other players follow the automaton, the sum of the current payoff and the expected discounted value of the strategy starting from any state is the highest for a player when the player follows the recommendations of the automaton. In other words, no one will deviate unilaterally.

The probability of choosing the action profile \mathbf{a} in state w is $f^w(\mathbf{a})$. Based on the action profile \mathbf{a} and the current state w , the transition function selects the new state $\tau(w, \mathbf{a})$. In a subgame perfect Nash equilibrium, player i receives the value $V_i(\tau(w, \mathbf{a}))$ from $\tau(w, \mathbf{a})$. Thus, $V_i(w)$ can be written as

$$V_i(w) = (1 - \delta) \sum_{\mathbf{a} \in A} u_i(\mathbf{a}) f^w(\mathbf{a}) + \delta \sum_{\mathbf{a} \in A} V_i(\tau(w, \mathbf{a})) f^w(\mathbf{a}). \quad (7.3.1)$$

Since $V_i(w)$ is an optimal value, it ought to remind you that this is the Bellman’s equation for dynamic programming.

This optimal value is credible in a subgame perfect Nash equilibrium since, given any $a'_i \in \text{supp}(f_i(\omega)) \equiv \{a_i \mid f^w(\mathbf{a}) > 0\}$, for any $\hat{a}_i \in A_i$, we

have

$$\begin{aligned}
V_i(\omega) &= (1 - \delta) \sum_{\mathbf{a}_{-i} \in A_{-i}} u_i(a'_i, \mathbf{a}_{-i}) f^w(a'_i, \mathbf{a}_{-i}) \\
&\quad + \delta \sum_{\mathbf{a}_{-i} \in A_{-i}} V_i(\tau(w, (a'_i, \mathbf{a}_{-i}))) f^w(a'_i, \mathbf{a}_{-i}) \\
&\geq (1 - \delta) \sum_{\mathbf{a}_{-i} \in A_{-i}} u_i(\hat{a}_i, \mathbf{a}_{-i}) f^w(\hat{a}_i, \mathbf{a}_{-i}) \\
&\quad + \delta \sum_{\mathbf{a}_{-i} \in A_{-i}} V_i(\tau(w, (\hat{a}_i, \mathbf{a}_{-i}))) f^w(\hat{a}_i, \mathbf{a}_{-i}).
\end{aligned}$$

We call $V_i(w)$ that satisfies the above inequality as **credible continuation promises** of player i . On the basis of credible continuation promises, we can re-characterize (under automaton representation) the subgame perfect Nash equilibria of repeated games.

Proposition 7.3.2 *The strategy σ described by automaton $(\Omega, \omega^0, f(\cdot), \tau(\cdot))$ is a subgame perfect Nash equilibrium if and only if for any state $w \in \Omega$ that can be reached from ω^0 , $f(w)$ is a Nash equilibrium of game $G = (N, A_i, U_i(\cdot) = g_i^w(\cdot))_{i \in N}$, where $g_i^w(\cdot) = (1 - \delta)u_i(\mathbf{a}) + \delta V_i(\tau(w, \mathbf{a}))$.*

PROOF. We prove this conclusion only in the case of pure strategies.

Sufficiency: Let strategy σ be generated by an automaton $(\Omega, \omega^0, f(\cdot), \tau(\cdot))$. If there is no profitable one-shot deviation, σ is a subgame perfect Nash equilibrium. We prove the absence of such a deviation by means of contradiction. Suppose that there exists a profitable one-shot deviation $\hat{\sigma}$. In other words, there exists a history \hat{h}^t , such that $\hat{\sigma}_i$ is a profitable one-shot deviation for player i . Let $\hat{w} = \tau(\omega^0, \hat{h}^t)$, $\hat{a}_i = \hat{\sigma}_i(\hat{h}^t) \neq \sigma_i(\hat{h}^t) = f(\hat{w}) = a_i$. Since $\hat{\sigma}_i$ is a profitable one-shot deviation for player i , then

$$\begin{aligned}
U_i(\hat{\sigma}_i|_{\hat{h}^t}, \sigma_{-i}|_{\hat{h}^t}) &= g_i^{\hat{w}}(\hat{a}_i, \mathbf{a}_{-i}) = (1 - \delta)u_i(\hat{a}_i, \mathbf{a}_{-i}) + \delta V(\tau(\hat{w}, (a_i, \mathbf{a}_{-i}))) \\
&> (1 - \delta)u_i(a_i, \mathbf{a}_{-i}) + \delta V(\hat{w}, (a_i, \mathbf{a}_{-i})) \\
&= g_i^{\hat{w}}(a_i, \mathbf{a}_{-i}) = U_i(\sigma_i|_{\hat{h}^t}, \sigma_{-i}|_{\hat{h}^t}),
\end{aligned}$$

contradicting the fact that $f(w)$ is a Nash equilibrium of G . Therefore, there is no profitable one-shot deviation, and σ is a subgame perfect Nash equilibrium.

Necessity: If σ is a subgame perfect Nash equilibrium, then no player has a profitable one-shot deviation. Furthermore, there does not exist \hat{h}^t , such that $V_i(\hat{w}) = U_i(\sigma_i|_{\hat{h}^t}, \sigma_{-i}|_{\hat{h}^t}) < U_i(\hat{\sigma}_i|_{\hat{h}^t}, \sigma_{-i}|_{\hat{h}^t})$ for player i and $\hat{\sigma}_i$, which means that for any $\hat{\sigma}_i$ and $\hat{a}_i = \hat{\sigma}_i(\hat{h}^t) = f(\hat{w}) = a_i$, we always have

$$(1 - \delta)u_i(\hat{a}_i, \mathbf{a}_{-i}) + \delta V(\tau(\hat{w}, (\hat{a}_i, \mathbf{a}_{-i}))) \leq (1 - \delta)u_i(a_i, \mathbf{a}_{-i}) + \delta V(\tau(\hat{w}, (a_i, \mathbf{a}_{-i}))).$$

Therefore, $f(w)$ is a Nash equilibrium of G for $g_i^w = (1 - \delta)u_i(\mathbf{a}) + \delta V_i(\tau(w, \mathbf{a}))$.

□

Example 7.3.6 (Incentives in the Prisoner's Dilemma (continued)) Consider the two automaton representations for playing the repeated game based on the stage game in Table 7.6. The first is the “tit for tat” strategy profile, and the second is the grim strategy profile. The automaton representation of the “tit for tat” strategy profile is: $\Omega = \{w_{EE}, w_{SE}, w_{ES}, w_{SS}\}$, $w^0 = w_{EE}$, $f(w_{a_1 a_2}) = a_1 a_2$, $\tau(w_{a_1 a_2}, a'_1 a'_2) = w_{a'_2 a'_1}$.

If the “tit for tat” strategy profile is a subgame perfect Nash equilibrium, we have:

$$V_1(w_{EE}) = (1 - \delta)3 + \delta V_1(w_{EE}) \geq (1 - \delta)4 + \delta V_1(w_{ES}) \quad (7.3.2)$$

$$V_1(w_{SE}) = (1 - \delta)4 + \delta V_1(w_{ES}) \geq (1 - \delta)3 + \delta V_1(w_{EE}) \quad (7.3.3)$$

$$V_1(w_{ES}) = (1 - \delta)(-1) + \delta V_1(w_{SE}) \geq (1 - \delta)1 + \delta V_1(w_{SS}) \quad (7.3.4)$$

$$V_1(w_{SS}) = (1 - \delta)1 + \delta V_1(w_{SS}) \geq (1 - \delta)(-1) + \delta V_1(w_{SE}). \quad (7.3.5)$$

Then,

$$V_1(w_{EE}) = 3, V_1(w_{SS}) = 1, V_1(w_{SE}) = \frac{4 - \delta}{1 + \delta}, V_1(w_{ES}) = \frac{4\delta - 1}{1 + \delta}.$$

Inequality (7.3.2) implies that $\delta \geq 1/4$; inequality (7.3.3) implies that $\delta \leq 1/4$; inequality (7.3.4) implies that $\delta \geq 2/3$; obviously, there does not exist any $\delta \in (0, 1)$ that simultaneously satisfies inequalities (7.3.2), (7.3.3) and (7.3.4). Therefore, the “tit for tat” strategy profile is not a subgame perfect Nash equilibrium.

For the grim strategy, the automaton has been described earlier. Now,

we study when it will be a subgame perfect Nash equilibrium. As

$$V_1(w_{EE}) = (1 - \delta)3 + \delta V_1(w_{EE}) \geq (1 - \delta)4 + \delta V_1(w_{SS}), \quad (7.3.6)$$

$$V_1(w_{SS}) = (1 - \delta)1 + \delta V_1(w_{SS}) \geq (1 - \delta)(-1) + \delta V_1(w_{SS}), \quad (7.3.7)$$

we obtain $V_1(w_{EE}) = 3$ and $V_1(w_{SS}) = 1$. Therefore, inequality (7.3.7) holds naturally. Moreover, if inequality (7.3.6) holds, $\delta \geq 1/3$. In other words, when $\delta \geq 1/3$, the grim strategy is a subgame perfect Nash equilibrium. This result is consistent with the previous one.

To solve for equilibria of repeated games, Abreu, Pearce, and Stacchetti (1986, 1990) proposed the concept of the self-generating set of payoffs. Again, for simplicity, we only discuss the case of pure strategies below.

7.3.5 Self-Generation of Repeated Games

First, let E^p be the set of payoffs of all subgame perfect Nash equilibria. For every $v \in E^p \subseteq \mathcal{R}^n$, let σ^v be a subgame perfect Nash equilibrium that results in payoff v . This profile σ^v generates a current action profile \mathbf{a}^* and a continuation promise function $\gamma : A \rightarrow E^p$, such that $v_i = (1 - \delta)u_i(\mathbf{a}^*) + \delta\gamma_i(\mathbf{a}^*)$. Since σ^v is a subgame perfect Nash equilibrium, it satisfies:

$$v_i = (1 - \delta)u_i(\mathbf{a}^*) + \delta\gamma_i(\mathbf{a}^*) \geq (1 - \delta)u_i(a_i, \mathbf{a}_{-i}^*) + \delta\gamma_i(a_i, \mathbf{a}_{-i}^*).$$

As $\gamma(\cdot) \in E^p$, it is the value of a subgame perfect Nash equilibrium, as well. After generating the action \mathbf{a}^* , σ^v also generates a continuation subgame perfect Nash equilibrium strategy $\sigma^{\gamma(\mathbf{a}^*)} \equiv \sigma^v|_{h^0, \mathbf{a}^*}$. The equilibrium value generated by $\sigma^{\gamma(\mathbf{a}^*)}$ is $\gamma(\mathbf{a}^*)$. Then, the process continues recursively. This approach ingeniously divides an optimization problem with infinitely many periods into a series of one-stage optimization problems. What connects these problems is a sequence of subgame perfect Nash equilibrium payoffs. The discounted value of each continuation subgame equilibrium payoff is called a state. The equilibrium strategy, which is generated by the state, transitions the previous state to the subsequent one. This approach involves some basic concepts, which we now introduce.

Definition 7.3.7 (Enforceable Action Profiles) In subset $W \subseteq F^*$ of a feasible payoff set, an action profile \mathbf{a}^* is *enforceable*, if there is a credible continuation promise function $\gamma : A \rightarrow E^p$, such that for any player $i \in N$ and for any $a_i \in A_i$, we have

$$(1 - \delta)u_i(\mathbf{a}^*) + \delta\gamma_i(\mathbf{a}^*) \geq (1 - \delta)u_i(a_i, \mathbf{a}_{-i}^*) + \delta\gamma_i(a_i, \mathbf{a}_{-i}^*).$$

This definition depicts the incentive of continuation equilibrium discounted payoff (or state) for a player to choose (equilibrium) strategy. Next, we will propose several related concepts of equilibrium payoff.

Definition 7.3.8 (Action Decomposable Payoffs) A feasible payoff $\mathbf{v} \in F^*$ in $W \subseteq F^*$ is *action decomposable*, if there is an enforceable action profile \mathbf{a}^* on $W \subseteq F^*$, i.e.:

$$v_i = (1 - \delta)u_i(\mathbf{a}^*) + \delta\gamma_i(\mathbf{a}^*) \geq (1 - \delta)u_i(a_i, \mathbf{a}_{-i}^*) + \delta\gamma_i(a_i, \mathbf{a}_{-i}^*),$$

in which $\gamma(\mathbf{a}^*)$ is a credible continuation promise function of enforceable action profile \mathbf{a}^* . Here, the payoff \mathbf{v} is decomposed by \mathbf{a}^* on F^* .

Definition 7.3.9 (Self-Generating Set of Payoffs) A set $W \subseteq F^*$ is *pure-strategy self-generating* if every $\mathbf{v} \in W$ is action decomposable.

The notion of a self-generating set of payoffs is closely related to the payoff set of subgame perfect Nash equilibria. The following proposition characterizes their relationship.

Proposition 7.3.3 *The pure-strategy self-generating set of payoffs $W \subseteq F^*$ is the subset of pure strategy subgame perfect Nash equilibrium payoffs, i.e., $W \subseteq E^p$.*

PROOF. Let payoff set $\Omega = W \subseteq F^*$ be the state set of the automaton. W is a pure-strategy self-generating set of payoffs, which implies that: for arbitrary $\mathbf{v} \in W$, there always exists a corresponding pure action profile $\mathbf{a}(\mathbf{v})$ and continuation promises $\gamma^{\mathbf{v}} : A \rightarrow W$. Consider the following automaton set $\{(W, \mathbf{v}, f, \tau) : \mathbf{v} \in W\}$ that satisfies: for all $\mathbf{v} \in W$, $f(\mathbf{v}') = \mathbf{a}(\mathbf{v}')$; and for all $\mathbf{v} \in W, \mathbf{a} \in A, \tau(\mathbf{v}', \mathbf{a}) = \gamma^{\mathbf{v}'}(\mathbf{a})$.

We show that for any $\mathbf{v} \in W$, the automaton $\{W, \omega^0 = \mathbf{v}, f, \tau\}$ describes a subgame perfect Nash equilibrium with payoff \mathbf{v} . For every state $v_i = V_i(\mathbf{v})$, $V_i(\mathbf{v})$ is the equilibrium payoff of player i under state \mathbf{v} . Since each $\mathbf{v} \in W$ is decomposable, let $\mathbf{v}^0 = \mathbf{v}$, and thus a sequence of payoff-action profiles can be generated as follows:

$$\mathbf{v}^0 = \mathbf{v}, \mathbf{a}^0 = f(\mathbf{v}^0), \mathbf{v}^k = \tau(\mathbf{v}^{k-1}, \mathbf{a}^{k-1}) = \gamma^{\mathbf{v}^{k-1}}(\mathbf{a}^{k-1}), \mathbf{a}^k = f(\mathbf{v}^k).$$

Thus,

$$\begin{aligned} v_i &= (1 - \delta)u_i(\mathbf{a}^0) + \delta v_i^1 = (1 - \delta)u_i(\mathbf{a}^0) + \delta((1 - \delta)u_i(\mathbf{a}^1) + \delta v_i^2) \\ &= \dots = (1 - \delta) \sum_{s=0}^{t-1} \delta^s u_i(\mathbf{a}^s) + \delta^t v_i^t. \end{aligned}$$

As v_i^t is bounded, when $t \rightarrow \infty$, $v_i = (1 - \delta) \sum_{s=0}^{\infty} \delta^s u_i(\mathbf{a}^s)$, we have $v_i = V_i(\mathbf{v})$. \square

The above proposition can be extended to a corollary: the set of payoffs of pure strategy subgame perfect Nash equilibria, E^p , is the largest pure-strategy self-generating set. Furthermore, Abreu, Pearce, and Stacchetti (1990) proved that E^p is a compact set.

Example 7.3.7 (Incentives in the Prisoner's Dilemma (continued)) Returning to the Prisoner's Dilemma, we analyse under what conditions both players working hard in each period is a subgame perfect Nash equilibrium. Therefore, the problem is under which conditions a self-generating set of payoffs, W , contains the payoff (3,3). Furthermore, if such a $W \subseteq F^*$ exists, action profile (E, E) in W is enforceable. This implies that:

$$\begin{aligned} (1 - \delta)3 + \delta\gamma_1(E, E) &\geq (1 - \delta)4 + \delta\gamma_1(S, E), \\ (1 - \delta)3 + \delta\gamma_2(E, E) &\geq (1 - \delta)4 + \delta\gamma_2(E, S). \end{aligned}$$

Moreover, $\gamma(E, E) \geq \underline{v}_i = 1, \gamma_1(S, E) \geq \underline{v}_i = 1, \gamma_2(E, S) \geq \underline{v}_i = 1$. In addition, (S, S) is a Nash equilibrium of the stage game, which makes $\gamma(S, S) = 1$ belong to E^p definitely. When $\gamma_1(S, E) = \gamma_2(E, S) = 1$ and $\gamma_1(E, E) = \gamma_2(E, E) = 3$, the above two inequality constraints are the weakest. Thus, the condition to guarantee the two inequalities is $\delta \geq 1/3$,

and $W = \{(1, 1), (3, 3)\}$ is a self-generating payoff set. In this situation, working hard in every period is a subgame perfect Nash equilibrium for both players.

Using the above method, we can solve for the equilibrium payoffs in repeated games. However, empirical research is more concerned with analysing interactive behavior in repeated games. Dynamic inconsistency (or time inconsistency) is a difficulty usually faced by governments in the process of policy-making. Much research focuses on this topic, such as the dynamic inconsistency of monetary policy by Kydland and Prescott (1977). For such a problem, people will think of the credibility of the government. For example, the government can fix such decisions in the form of a rule that cannot be changed at will, or give the relevant decision-making power to certain people or groups with specific preferences. The idea of a repeated game can also deal with other problems conveniently. Consider the following example of public goods provision (see Samuelson, 2006).

Example 7.3.8 Consider an infinitely repeated game with two types of players: the first type of player is a government, while the second type of player comprises consumers (a continuum of measure 1). For simplicity, assume that consumers are homogeneous. Each consumer is negligible relative to society, which means that his behavior has little influence on the society, and he will choose an action to maximize his short-run benefit. Assume that in every period, a consumer has 1 unit of endowment that he can consume or invest. Assume that the return on investment is $R > 1$, and that the consumption amount is c ; then, the benefit from investment is $R(1 - c)$. The government's tax rate on investment is t , and then her revenue is $tR(1 - c)$. Assume that all of the revenue is used for provision of public goods. Each unit of revenue can provide γ units of public goods, and let $R - 1 \leq \gamma \leq R$. For simplicity, suppose that there is no savings across periods, and the problem of dynamic inconsistency occurs within each period.

The utility function of a consumer is:

$$c + (1 - t)R(1 - c) + 2\sqrt{G},$$

in which G is the public good provided by the government.

Since everyone is negligible to society, everyone's contribution to the financing of the good is also trivial. If the amount of public good provided by the government is G , the optimal decision for a consumer is: if $t < \frac{R-1}{R}$, there is $c = 0$; otherwise, $c = 1$.

The goal of the government is to maximize social welfare, which in this case is the total utility of all consumers. Therefore, the government's decision is to choose the tax rate t to maximize social welfare:

$$c + (1 - t)R(1 - c) + 2\sqrt{\gamma t R(1 - c)}.$$

Then, combining with consumers' decisions, the government's choice of tax rate is: $t(c) = \frac{\gamma}{R(1-c)}$.

The best response functions for consumers and the government are shown in Figure 7.3. Since $R > 1$, the most efficient consumption arrangement is $c = 0$, i.e., all endowments are utilized for investment. In this situation, tax revenue is $t = \frac{\gamma}{R}$, which is point B in Figure 7.3. From Figure 7.3, we can find that there is a unique equilibrium $c = 1, t = 1$ in this (stage) game, which is point A . This equilibrium does not optimize the goal of consumers or the government. This conclusion is somewhat surprising since, in this model, both the consumers and the government have identical goals. However, in equilibrium, they choose a strategy that is unfavourable for both of them.

Since each individual has very little influence on society, he or she will not consider the consequences of his or her actions, and each will choose the strategy to optimize his or her short-run benefit. Considering the behavioral constraints of consumers (i.e., consumers always make a best response to the government's decision), the optimal strategy profile in this economy is point C in Figure 7.3. The highest tax rate which is compatible with the best consumption decision is $\frac{R-1}{R}$.

Let \underline{v}_1 be the utility of the government (this is also the consumers' utility) in this profile. If the government can make a prior commitment to the tax rate, it can obtain revenue \underline{v}_1 . If the government cannot promise in advance, it will be faced with a problem of policy commitment. However,

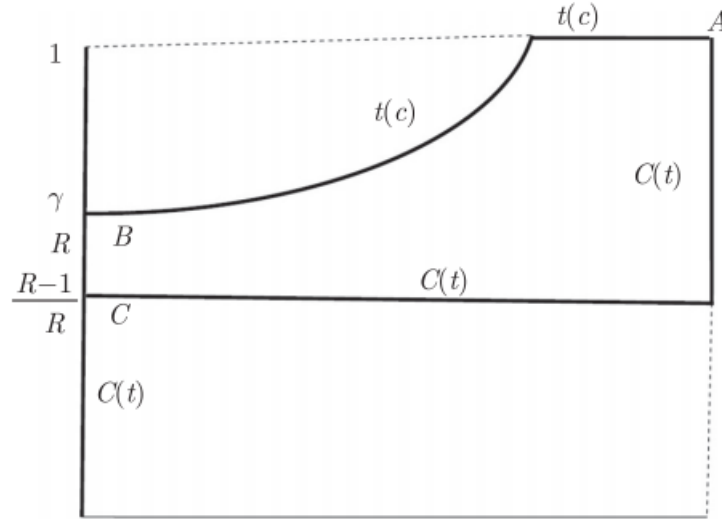


Figure 7.3: The Best Reaction Functions for Consumers and the Government

in a repeated game, the government can solve the problem of commitment even if it cannot make a commitment to the tax rate in advance. If the government is sufficiently concerned about its future benefits, once the tax rate that is chosen by the government is not $\frac{R-1}{R}$, the future interaction will revert to the single-stage Nash equilibrium permanently, i.e., $c = 1$ and $t = 1$. This can ensure that the government chooses an efficient tax rate for society.

Therefore, using the idea of repeated games, we can conclude that there exists a lower bound of time discount factor $\underline{\delta}$, such that when the time discount factor of the government satisfies $\delta \in [\underline{\delta}, 1]$, there is a subgame perfect Nash equilibrium in which the choices for the government and consumers are $t = \frac{R-1}{R}$ and $c = 1$, respectively, which is given by point C in Figure 7.3.

7.4 Folk Theorems with Perfect Monitoring

In the above examples of the repeated game, there often exist multiple equilibria. Indeed, there are infinitely many feasible and (strictly individually rational) payoffs in repeated games. Folk Theorems show that all of them

are subgame perfect Nash equilibrium payoffs.

Discounted subgame perfect Folk Theorems were first studied in Friedman (1971) and then comprehensively in Fudenberg and Maskin (1986), which claim that for any feasible and individually rational payoff profile v , if players are sufficiently patient (i.e., δ is large enough), then there is a subgame perfect equilibrium with payoff v . Consequently, the theorem implies that efficient payoffs are consistent with equilibrium (then collective rationality is consistent with individual rationality), so are many other payoffs and associated behaviors. Moreover, multiple equilibria may be consistent with the same payoff.

We first introduce the simplest Folk Theorem, attributable to Friedman (1971).

Theorem 7.4.1 (Folk Theorem under Nash Threats (Friedman, 1971)) *Let a^* be the Nash equilibrium of a stage game, and its equilibrium payoff profile be e . Then, for arbitrary $v' \in \{v | v_i > e_i, v \in F^p\}$, there is $\underline{\delta}$, such that for any $\delta \in [\underline{\delta}, 1)$, there exists a subgame perfect Nash equilibrium of the repeated game having payoff v' .*

PROOF. Suppose that there is a pure action profile a , such that $u_i(a) = v'_i$. Consider the following strategy profile: in period 0, for any player $i \in N$, his or her choice is a_i . If players' action profile is a in all previous stages, then any player $i \in N$ still chooses a_i ; otherwise, player $i \in N$ chooses a_i^* .

When $\delta \rightarrow 1$, this strategy profile is a Nash equilibrium since $(1 - \delta) \max_{a'_i} u_i(a'_i, a_{-i}) + \delta e_i < v'_i$. This is because, there exists a $\underline{\delta}$, such that for $\delta \in [\underline{\delta}, 1)$, this strategy profile is a Nash equilibrium. As for any subgame off the equilibrium path, a^* is always a Nash equilibrium, and this equilibrium is naturally a subgame perfect Nash equilibrium.

If there is no such pure strategy profile a , such that $u_i(a) = v'_i$, a public correlation device can be introduced. Let W be the set of public states, i.e., everyone observes it. Let p be a probability distribution on W , such that $\sum_{w \in W} u_i(a_i(w)) p(w) = v'_i$. Then, the pure action profile above, a , can be replaced by the action profile $a(w)_{w \in W}$ using the public correlation device. At this time, the strategy profile is: at stage 0, for any player $i \in N$, the choice is $a_i(w)$. If in all previous stages, players' strategy profile is $a(w)$,

then for any player $i \in N$, he or she still chooses $a_i(w)$; otherwise, each player $i \in N$ chooses a_i^* .

When $\delta \rightarrow 1$, this strategy profile is a Nash equilibrium since $(1 - \delta) \max_{a'_i \in A_i} u_i(a'_i, \mathbf{a}_{-i}(w)) + \delta e_i < v'_i$. Meanwhile, for any subgame off the equilibrium path, \mathbf{a}^* is a Nash equilibrium, and thus this equilibrium is naturally a subgame perfect Nash equilibrium. \square

For a more general Folk Theorem, it is only required that each player's payoff is greater than the minmax payoff, and is not necessarily greater than the equilibrium payoff of the stage game. Next, we prove this Folk Theorem for two players first, and then prove it in a more general situation.

Theorem 7.4.2 *Suppose $n = 2$. For any feasible and individually rational profile $v \in F^p$, there always exists $\underline{\delta}$, such that for any $\delta \in [\underline{\delta}, 1)$, there is a subgame perfect Nash equilibrium of repeated game with payoff v .*

PROOF. Suppose that there is a pure action profile $\tilde{\mathbf{a}}$, such that $u_i(\tilde{\mathbf{a}}) = v$. Let $M = \max_{\mathbf{a} \in A} u_i(\mathbf{a}) < \infty$. Define a mutually punishing strategy $\hat{\mathbf{a}} = (a_i^j, a_j^i), i \neq j$ by

$$u_i(\hat{a}_i^i, \hat{\mathbf{a}}_{-i}^i) = \min_{\mathbf{a}_{-i}} \max_{a_i} u_i(a_i, \mathbf{a}_{-i}) = \underline{v}_i^p,$$

which is the minmax strategy for both players. Note that $u_i(\hat{\mathbf{a}}) \leq \underline{v}_i^p$. Consider the following strategy: players choose $\mathbf{a}(0) = \tilde{\mathbf{a}}$. If in previous stages both players choose $\mathbf{a}(0)$, they choose $\mathbf{a}(0)$ in this stage, as well; however, if in the last stage a player has deviated from $\mathbf{a}(0)$, a punishment process that lasts L periods begins at this stage. During the punishment process, players choose $\hat{\mathbf{a}}$.

Meanwhile, if some player deviates from $\hat{\mathbf{a}}$, from the next stage onwards a new punishment process that lasts L periods starts again. If during the punishment process both players choose $\hat{\mathbf{a}}$, it returns to the initial choice $\mathbf{a}(0)$ when the punishment process is over. This strategy can be expressed as the following automaton.

The state set of the automaton is $\Omega = \{w(l) : l = 0, \dots, L\}$, and the

initial state is $\omega^0 = w(0)$. The strategy action function is

$$f(w(l)) = \begin{cases} \tilde{\mathbf{a}}, & \text{if } l = 0, \\ \hat{\mathbf{a}}, & \text{if } l = 1, 2, \dots, L. \end{cases}$$

The state transition function is

$$\tau(w(l), \mathbf{a}) = \begin{cases} w(0), & \text{if } l = 0, \mathbf{a} = \tilde{\mathbf{a}} \text{ or } l = L, \mathbf{a} = \hat{\mathbf{a}}, \\ w(l+1), & \text{if } 0 < l < L, \mathbf{a} = \hat{\mathbf{a}}, \\ w(1), & \text{otherwise,} \end{cases}$$

where $\hat{\mathbf{a}} = (\hat{a}_1^2, \hat{a}_2^1)$.

Let L such that

$$L \min_i (u_i(\tilde{\mathbf{a}}) - u_i(\hat{\mathbf{a}})) > M - \min_i u_i(\tilde{\mathbf{a}}). \quad (7.4.8)$$

When δ is sufficiently large (i.e., $\delta \rightarrow 1$), we have

$$u_i(\tilde{\mathbf{a}}) \geq (1 - \delta)M + \delta v_i^*, \quad (7.4.9)$$

where $v_i^* = (1 - \delta^L)u_i(\hat{\mathbf{a}}) + \delta^L u_i(\tilde{\mathbf{a}})$. Substituting v_i^* into inequality (7.4.9), we have:

$$(1 - \delta^{L+1})u_i(\tilde{\mathbf{a}}) \geq (1 - \delta)M + \delta(1 - \delta^L)u_i(\hat{\mathbf{a}}).$$

Dividing both sides of the inequality by $(1 - \delta)$, we have

$$\sum_{t=0}^L \delta^t u_i(\tilde{\mathbf{a}}) \geq M + \sum_{t=0}^{L-1} \delta^t u_i(\hat{\mathbf{a}}). \quad (7.4.10)$$

Obviously, when (7.4.9) holds, (7.4.10) holds, as well. In other words, there exists a $\underline{\delta}$, such that for $\delta \in [\underline{\delta}, 1)$, (7.4.10) holds. Consequently, the strategy generated by this automaton is a Nash equilibrium. If the deviation is profitable during the punishment stage off the equilibrium path, it will also be profitable under $w(1)$. This is because in the punishment stage, deviation from the first stage is more profitable than deviation in later stages. If there is no deviation under $w(1)$, the payoff is v_i^* ; if there is a deviation

tion, the player obtains a payoff in the current stage which is no more than $\underline{v}_i^p < v_i^*$ (since the other player chooses the minmax action) and a continuation payoff v_i^* . Consequently, deviation cannot increase his or her payoff. Therefore, the above strategy is a subgame perfect Nash equilibrium.

If there does not exist such a pure action profile $\tilde{\mathbf{a}}$, such that $u_i(\tilde{\mathbf{a}}) = v$, by the payoff structure, there exists a mixed action α that assigns probability $\alpha(\mathbf{a})$ to the pure action profile \mathbf{a} , such that $\sum_{\mathbf{a} \in A} u_i(\mathbf{a})\alpha(\mathbf{a}) = v_i$ for all i . Then, we can use a public correlation device to describe the strategy using an automaton (Ω, μ^0, f, τ) , where $\Omega = \{w^{\mathbf{a}}\}_{\mathbf{a} \in A} \cup \{w(l), l = 1, \dots, L\}$, μ^0 is the probability distribution induced by α on $\{w^{\mathbf{a}} : \mathbf{a} \in A\}$,

$$f(w) = \begin{cases} \mathbf{a}, & \text{if } w = w^{\mathbf{a}}, \\ \hat{\mathbf{a}}, & \text{if } w = w(l) \text{ for } l = 1, \dots, L; \end{cases}$$

and

$$\tau(w, \mathbf{a}') = \begin{cases} \alpha, & \text{if } w = w^{\mathbf{a}}, \mathbf{a}' = \mathbf{a} \text{ or } w = w(L), \mathbf{a}' = \hat{\mathbf{a}}, \\ w(l+1), & \text{if } w = w(l) \text{ for } 0 < l < L, \mathbf{a}' = \hat{\mathbf{a}} \\ w(1), & \text{otherwise.} \end{cases}$$

After replacing $w(0)$ by $\{w^{\mathbf{a}}\}_{\mathbf{a} \in A}$, and the initial state w^0 by μ^0 , the remainder of the proof is similar to the previous one and is thus omitted. \square

The intuitive meaning of the Folk Theorem is that if a player is sufficiently patient, the increase in payoff from a one-shot deviation in any stage game will be offset by the loss in payoff from the punishment stage in the future. In other words, the penalty concerning future payoff exceeds what a player can currently obtain by a deviation. Therefore, players have no incentive to deviate from the equilibrium.

In the previous proof, the punishment that is employed to guarantee no deviation from the equilibrium path applies to all players. However, in repeated games with more than two players, the punishment is aimed at the player who deviated most recently. To describe this idea, we now state a more general Folk Theorem based on the classic paper by Fudenberg and Maskin (1986).

Theorem 7.4.3 (Folk Theorem (Fudenberg and Maskin, 1986)) *Suppose that the dimension of feasible and individually rational payoff set FV^* equals the number of players (which is also called full dimensionality). Then, given $v \in FV^*$, there exists a $\underline{\delta}$, such that for arbitrary $\delta \in [\underline{\delta}, 1)$, there is a subgame perfect Nash equilibrium with payoff v .*

In the proof of Fudenberg and Maskin (1986), three stages were introduced. The first stage is equilibrium path; the second is the punishment stage, in which the player who deviated most recently is punished; and the last stage is the one in which players who carried out the punishments required by the equilibrium are rewarded.

PROOF. Here, we only consider the case of pure strategies both on and off the equilibrium path. In particular, we assume that there exists \mathbf{a} , such that the payoff $u(\mathbf{a}) = v$ holds. The case of mixed strategies and the case in which a public correlation device exists can be analyzed similarly; we refer the reader to Fudenberg and Maskin (1986) for details.

Their proof imposes a condition about feasible and individually rational payoffs, i.e., a full dimensionality requirement is imposed on the feasible and individually rational payoff set. This means that for arbitrary $v \in FV^*$, when $\underline{v}_i < v'_i < v_i$ for any $i \in N$, we can find $\varepsilon > 0$, such that the payoff $v'(i) = (v'_1 + \varepsilon, \dots, v'_{i-1} + \varepsilon, v'_i, v'_{i+1} + \varepsilon, \dots, v'_n + \varepsilon)$ is in FV^* for each i . Note that if v is on the lower boundary of the feasible and individually rational set, we can construct the points differently following Abreu, Dutta and Smith (1994).

To avoid introducing a public correlation device, we assume that for any $i \in N$, there is a pure action profile $\mathbf{a}(i)$, such that $u(\mathbf{a}(i)) = v'(i)$. Let $w_i^j = u_i(m^j)$, where m^j is a minmax strategy profile for player j : it gives j the minmax payoff when j is making a best response: $\max_{a_j} u_j(m_{-j}^j, a_j) = u_j(m^j) = \underline{v}_j$. Furthermore, assume that for each j , the action profile m^j is pure. Finally, pick a natural number k satisfying

$$k > \frac{\max_{\mathbf{a} \in A} u_i(\mathbf{a}) - v'_i}{v'_i - \underline{v}_i} \quad \forall i.$$

Such a number k exists because, by construction, both the numerator and

denominator of the fraction above are strictly positive.

Consider the following strategy profile. Interaction starts at stage I: in stage I, players choose \mathbf{a} , such that $u(\mathbf{a}) = v$. If only player j deviates from the strategy profile, the interaction enters stage II_j ; whereas, in other situations (e.g., more than one player deviates, or no player deviates) the interaction remains in stage I.

Stage II_j : players choose m^j . If no one deviates or more than one player deviates from m^j , stage II_j lasts for k periods, followed by stage III_j . In stage II_j , if only one player $i \in N$ deviates from m^j , the interaction enters II_i . Note that the construction of stage II_j makes sense only if m^j is a pure strategy. We will discuss the case of a mixed strategy below. Stage II_j can be regarded as the punishing stage for player j .

Stage III_j : players choose strategy profile $\mathbf{a}(j)$ all the time unless some player deviates from $\mathbf{a}(j)$ unilaterally. If there is some player $i \in N$ that deviates from this strategy, it enters stage II_i starting from the next period.

Next, we prove that this strategy profile is a subgame perfect Nash equilibrium. We only need to prove that in each subgame, no one deviates unilaterally.

In stage I, if player i deviates, the payoff that the player can obtain is no more than

$$(1 - \delta) \max_{\mathbf{a} \in A} u_i(\mathbf{a}) + \delta(1 - \delta^k) \underline{v}_i + \delta^{k+1} v'_i.$$

Obviously, when $\delta \rightarrow 1$, the best deviation payoff above converges to v'_i , which is strictly less than the payoff v_i of following the equilibrium path. Therefore, no player i has a profitable one-shot deviation in this stage.

In stage III_j , by following the strategy constructed above, player $i \neq j$ receives a payoff $v'_i + \varepsilon$. If the player deviates from it, the payoff is at most $(1 - \delta) \max_{\mathbf{a} \in A} u_i(\mathbf{a}) + \delta(1 - \delta^k) \underline{v}_i + \delta^{k+1} v'_i$. Obviously, when $\delta \rightarrow 1$, this converges to v'_i , which is strictly less than the payoff $v'_i + \varepsilon$. Therefore, player i does not have a profitable one-shot deviation in this stage. In stage III_i , to ensure that there is no profitable one-shot deviation for player i , it suffices

to show that

$$(1 - \delta) \max_{\mathbf{a} \in A} u_i(\mathbf{a}) + \delta(1 - \delta^k) \underline{v}_i + \delta^{k+1} v'_i < v'_i.$$

Rearranging and dividing by the factor $(1 - \delta)$, this inequality becomes

$$\max_{\mathbf{a} \in A} u_i(\mathbf{a}) + \delta \left(\sum_{\tau=0}^{k-1} \delta^\tau \right) \underline{v}_i < \left(\sum_{\tau=0}^{k-1} \delta^\tau \right) v'_i.$$

As $\delta \rightarrow 1$, this last inequality clearly reduces to the same inequality that was used to choose k .

In stage Π_j , consider player $i \neq j$. When $k^t \leq k$ periods of this stage are left, the payoff is $(1 - \delta^{k^t}) w_i^j + \delta^{k^t} (v'_i + \varepsilon)$. If deviating unilaterally, the player receives his or her minmax payoff for the next k periods, and then the game enters stage III_i ; thus, the payoff from deviation is no more than $(1 - \delta) \max_{\mathbf{a} \in A} u_i(\mathbf{a}) + \delta(1 - \delta^k) \underline{v}_i + \delta^{k+1} v'_i$. When $\delta \rightarrow 1$, this payoff converges to v'_i , while the payoff from not deviating converges to the strictly higher value $v'_i + \varepsilon$. In stage Π_i , consider player i . When $k^t \leq k$ periods are still left, if the player follows the equilibrium strategy, the payoff is $q_i(k^t) = (1 - \delta^{k^t}) \underline{v}_i + \delta^{k^t} v'_i$; otherwise, the payoff is $q_i(k) = (1 - \delta^k) \underline{v}_i + \delta^k v'_i$. Obviously, $v'_i > \underline{v}_i$ and $k^t \leq k$ imply that $q_i(k^t) \geq q_i(k)$, i.e., it is not profitable to restart one's own punishment.

As a consequence, no player i has a profitable one-shot deviation from the constructed three-phase strategy profile, which is therefore a subgame perfect Nash equilibrium. \square

For mixed strategies, the discussion is more complex, and details can be found in Fudenberg and Maskin (1986).

A popular explanation of the Folk Theorem is that as long as individuals are sufficiently patient, an arbitrary outcome in which the utility of each player exceeds the individually rational level can be obtained in a Nash equilibrium. This result with an infinite number of equilibria is considered as being negative for repeated games because if a theorem contains all possible outcomes, it is essentially meaningless. However, this constitutes a misunderstanding. Folk Theorems provide a highly profound explanation

for the crucial importance of a long-term institutional environment, such as culture and social norms, so that different (good or bad) environments will lead to different (good or bad) outcomes.

Firstly, research on long-run relationships can assist us to understand opportunistic behavior among individuals' and how institutions respond to them. In different institutional environments, including different cultures, habits, social norms, etc. (the degree is determined by discounting factors), individuals behave differently in long-run relationships. If a theory can ignore the institutional environment and obtain an explicit and a unique result, it is not necessarily a good theory since different institutional details tend to determine different equilibrium outcomes of long-run interactions. Secondly, regarding the Folk Theorem as a reference system, many theorists focus on conditions that invalidate this theorem. If individuals are not patient enough, or the identification of opportunism is not very accurate, or some players only interact in the short run, how would these changed conditions affect individuals' behavior in the long run? We discuss these factors in the following section.

7.5 Some Variations of Repeated Games

In the above discussion of repeated games, we assumed that players and action sets are the same in every period. Now, we introduce three examples to examine three variations—the case with short-run players, the case with entry and exit of players, and the case of social norms that constrain the behaviour of players.

7.5.1 Long-Run Players and Short-Run Players

In some multi-period interactions, players in different stage games may be different. For example, a seller meets different customers in different periods. In such a game, players can be divided into two categories: long-run players and short-run players. A short-run player only takes part in one round of play, and thus his goal is to maximize short-run payoff. However, a long-run player's goal is the aggregate payoff over periods, and thus she

aims at long-run payoff maximization.

Let $i \in \{1, 2, \dots, L\}$ be a long-run player, and $j \in \{L+1, L+2, \dots, n\}$ be a short-run player.

Let $B : \prod_{i=1}^L \Delta A_i \rightarrow \prod_{j=L+1}^n \Delta A_j$ be the best response of short-run players to the (mixed) strategy of long-run players. Based on this, the minmax payoff \underline{v}_i of long-run player i needs to be redefined as:

$$\underline{v}_i = \min_{\alpha \in \text{graph}(B)} \max_{a_i} u_i(a_i, \alpha_{-i}),$$

where $\text{graph}(B) \subseteq \prod_{i=1}^n \Delta A_i$ is the corresponding graph of B , which satisfies the following: for $j > L$, $\alpha_j(\alpha_{-j}) = \arg\max_{a_j} u_j(a_j, \alpha_{-j})$. However, in the situation with short-run players, there exists an upper bound in the repeated game that constrains the feasible and individually rational payoffs of long-run players. Define

$$\bar{v}_i = \max_{\alpha \in \text{graph}(B)} \min_{a_i \in \text{support}(\alpha_i)} u_i(a_i, \alpha_{-i}),$$

in which $\text{support}(\alpha_i)$ denotes that those pure actions a_i have strictly positive probability in the mixed action α_i of player i .

Example 7.5.1 (Short-Run Players and Long-Run Players) Consider a repeated game with short-run players. Player 1 (row) comprises a single long-run player, and player 2 (column) comprises a sequence of short-run players, each of whom interacts with the long-run player in only one period. In other words, the discount factor of player 2 is 0. The two-player stage game is represented in Table 7.7.

$$\begin{aligned} \text{graph}(B) = \{(\alpha_1^T, L) : \alpha_1^T \geq 2/3\} \cup \{(\alpha_1^T, R) : 1/3 \leq \alpha_1^T \leq 2/3\} \\ \cup \{(\alpha_1^T, C) : \alpha_1^T \leq 1/3\}, \end{aligned}$$

in which α_1^T is the probability with which the long-run player 1 chooses T . We obtain $\underline{v}_1 = 1$ and $\bar{v}_1 = 6$.

In this game, the minmax payoff of the long-run player is 1, which equals her payoff from the Nash equilibrium (T, L) of the stage game; meanwhile, the upper bound of payoff 6 is equal to the payoff of player 1 when

		Player 2		
		L	C	R
Player 1	T	1, 3	0, 0	6, 2
	B	0, 0	2, 3	6, 2

Table 7.7: Interaction between long-run player and short-run player

the (mixed) Nash equilibrium $(0.5, R)$ is played in the stage game. Therefore, in this example, for player 1 the payoff is $v_1 \in (1, 6)$, and we can construct a public correlation device. For example, let the space W be $\{w_1, w_2\}$, and let $p = \text{prob}(w = w_1)$ and $1 - p = \text{prob}(w = w_2)$, such that $p + 6(1 - p) = v_1$. We construct the following strategy profile: under w_1 , players choose (T, L) ; under w_2 , they choose $(0.5, R)$. If player 2 observed that player 1 deviated from this strategy in previous stages (by not choosing T after w_1), they play (T, L) henceforth. For player 1, there is a lower bound of time discount factor $\underline{\delta}$, such that when $\delta \in [\underline{\delta}, 1)$, the above strategy is a subgame perfect Nash equilibrium where the payoff of player 1 is v_1 .

In a more general situation, Fudenberg, Kreps and Maskin (1990) and Fudenberg and Levine (1994) proved the Folk Theorem for games with short-run and long-run players. Let the dimension of the payoff space for the long-run players equal the number of long-run players L . If payoff $v = (v_1, \dots, v_L)$ of the long-run players satisfies $\underline{v}_i < v_i < \bar{v}_i$, $i \in \{1, 2, \dots, L\}$, then there exists a lower bound of the time discount factor, $\underline{\delta}$, such that for any $\delta \in [\underline{\delta}, 1)$, there is a subgame perfect Nash equilibrium of the repeated game in which the payoff of the long-run players is v .

7.5.2 Overlapping Generations Games

Some interaction involves players entering or exiting the game, and there is some time limit for everyone to interact with others. In this situation, no player interacts with others forever, and different types of players face different periods of interaction. In reality, such examples are very common, especially in organizations. In fact, most members in an organization will face retirement, and new ones will join. For different members, the

time that they stay in the organization is different, and their career expectations are also dissimilar. This situation is called a **repeated game with overlapping generations of finite-lived players**. Next, we use an example (Cremer, 1986) to explore interactions and incentives of such individuals.

Consider an organization in which every one stays for T years (which can be regarded as age at retirement). For simplicity, assume that in this organization, the measure of individuals with different ages is 1. In every period, there is 1 member (length of service is T) retiring and a new member joining (length of service is 1). Every member who stays in the organization for the next period increases his or her length of service by 1. Consider the cooperative interactions between members. Assume that everyone can choose to work hard or to be lazy, and the individual cost of working hard is 1. We assume that the output of the organization is determined by the number of members who choose to work hard. At the same time, each member gets the same proportion of output, i.e., there is a possibility of free-riding.

Assume that, except player i , the number of the players who choose to work hard is k . Let s be the output efficiency of working hard and assume $1 < s < T$. If player i chooses to work hard, his or her utility is $\frac{s(k+1)}{T} - 1$; otherwise, it is $\frac{sk}{T}$. Obviously, if the interaction lasts for only one period, all rational players choose to be lazy. However, the outcome is completely different in a repeated game. For simplicity, assume that the discount factor is $\delta = 1$. Next, we consider the incentives of members in the organization.

Obviously, the player who has length of service T stays in the organization only for the last period. Therefore, he or she has no incentive to work hard. Consider the following strategy profile for organization (players): the players that have length of service T choose to be lazy; if no one whose length of service is not T has ever chosen to be lazy, then these players choose to work hard; if someone whose length of service is not T has ever chosen to be lazy, then all players choose to be lazy. In the following, we prove that this strategy is a subgame perfect Nash equilibrium.

Firstly, for players whose length of service is T , to be lazy is a dominant strategy. Then, consider the incentive of players who have length of service $T - 1$. Assume that all other players follow the above strategy profile. If

only one player chooses to be lazy, his or her payoff is $\frac{s(T-2)}{T}$ in the current period and 0 in the next period. The total payoff is $\frac{s(T-2)}{T}$; if the player chooses to work hard, his or her payoff is $\frac{s(T-1)}{T} - 1$ in this period and $\frac{s(T-1)}{T}$ in the next period. The total payoff $2\frac{s(T-1)}{T} - 1 > \frac{s(T-2)}{T}$ since $s > 1$. As a consequence, for a player with length of service $T - 1$, there is no incentive to deviate unilaterally.

We consider the player whose length of service is $T - k$, in which $k \in 1, 2, \dots, T - 1$. If the player deviates from this strategy, his or her payoff is $\frac{s(T-2)}{T}$ in current period and 0 for the next period. His or her total payoff is $\frac{s(T-2)}{T}$. If this player follows the strategy, his or her payoff is $k\frac{s(T-1)}{T} - (k - 1) > \frac{s(T-2)}{T}$. Thus, the player with length of service $T - k$ does not deviate from the strategy profile, as well. In addition, off the equilibrium path where some players whose length of service is not T choose to be lazy, the strategy profile is that everyone chooses to be lazy forever afterwards. This is exactly a Nash equilibrium in the stage game from which no player will deviate unilaterally. Therefore, the above strategy profile is a subgame perfect Nash equilibrium.

Of course, it is not necessary to restrict attention to $\delta = 1$. In the above inference, we can find a lower bound $\underline{\delta}$ for time discount factor, such that when $\delta \in (\underline{\delta}, 1]$, the above strategy profile is still a subgame perfect Nash equilibrium.

7.5.3 Community Constraints and Social Norms

In many repeated games, players interact randomly. For example, people encounter different opponents at different times when they purchase something. As a result, punishment cannot be implemented by the participant who loses from a deviation, but by other participants. However, in general, punishing someone is costly. Therefore, other mechanisms are needed to constrain the punishment process. Here, we focus on discussing the constraining mechanism of social norms.

Assume that society consists of an even number M of players. In each period, each player randomly interacts with one of the other players by choosing “cooperation” or “non-cooperation”, and the payoff of the

stage game is given in Table 7.8. If M is sufficiently large, the probability that anyone encounters his or her previous opponent is quite small. How can we stimulate individuals to cooperate with others in such a situation? Social norms are a general way to achieve this. Social norms consist of two elements: a renewal function of individual social labels, and strategies dependent on the label. A renewal function of individual social labels is a transition function for labels. When the labels of player i and his or her opponent are x and z , respectively, and player i chooses a_i , the (updated) social label in the next period is $\tau_i(x, z, a_i)$. A social label dependent strategy $\sigma_i(x, z)$ denotes the strategy of player i when the social labels of player i and his or her opponent are x, z , respectively.

		player 2	
		C	D
player 1	C	4, 4	0, 5
	D	5, 0	1, 1

Table 7.8: Social Norm

Consider the social norm below, with the set of social labels being $\{G, B\}$:

$$\tau_i(x, z, a_i) = \begin{cases} G, & \text{if } (x, z, a_i) = (G, G, C) \text{ or } (G, B, D), \\ B, & \text{otherwise.} \end{cases}$$

$$\sigma_i(x, z) = \begin{cases} C, & \text{if } x = z = G, \\ D, & \text{otherwise.} \end{cases}$$

In the above, the definitions of social label and social label dependent strategy are quite intuitive: we can regard individuals who have social label G as a “good person” and those who have B as a “bad person”. If a person with the label “good person” faces another one with the label “good person”, choosing cooperation (C) maintains his or her social label as a “good person”; if his or her rival is a “bad person”, choosing noncooperation (D) keeps him or her a “good person”; in any other case, his or her social label becomes a “bad person”. In other words, the social norm requires that to be a good person one should cooperate if one encounters another good person, but not if one encounters a bad one; otherwise,

under social norms, the social label for this person is a “bad person”. For a bad person, the social norm always regards him or her as a bad person, which is similar to the grim strategy in that there is no forgiveness. We shall show that when $\delta \rightarrow 1$, no matter how large M is, the social norm described above is a subgame perfect Nash equilibrium.

Assume that at the initial state the social label of every one is G . First, we prove that on the equilibrium path, no player deviates unilaterally. If a player deviates, the discounted payoff is $5(1 - \delta) + \delta$; if he or she follows the equilibrium path, his or her discounted payoff is 4. As long as $\delta > 1/4$, no player deviates unilaterally.

Next, consider a situation off the equilibrium path. Assume that the proportion of individuals with social labels G and B are $\alpha > 0$ and $1 - \alpha$, respectively. Let $V(G)$ and $V(B)$ denote the equilibrium utilities of players with social label G and B , respectively, prior to knowing the type of the current opponent. For the individual with social label B , since (D, D) is the Nash equilibrium in a stage game, his or her optimal choice is D given that all others follow the above social norms.

For the individual with social label G , when encountering an opponent with social label B , following the social norm gives the expected payoff $(1 - \delta) + \delta V(G)$; if the individual does not follow the social norm, his or her expected payoff is $0 + \delta V(B)$. When encountering an opponent with social label G , if the individual follows the social norm, the expected payoff is $4(1 - \delta) + \delta V(G)$; whereas, the expected payoff is $5(1 - \delta) + \delta V(B)$ if he or she deviates from the norm. Clearly, $V(B) = 1$. Therefore, we have $V(G) = (1 - \delta)[\alpha 4 + (1 - \alpha)] + \delta V(G)$, which yields $V(G) = 1 + 3\alpha > V(B)$.

Therefore, off the equilibrium path, an individual with label G will follow social norms when the individual meets an opponent with label B , since $(1 - \delta) + \delta V(G) > 0 + \delta V(B)$. When the individual's opponent has social label G , he or she strictly prefers to follow the social norm if

$$4(1 - \delta) + \delta V(G) = 4(1 - \delta) + \delta(1 + 3\alpha) > 5(1 - \delta) + \delta V(B) = 5 - 4\delta,$$

which is equivalent to $\delta > \underline{\delta} = \frac{1}{1+3\alpha}$. Thus, as long as $\alpha > 0$, no matter what the distribution of good persons and bad persons is initially, the above

norm is always a subgame Nash equilibrium as $\delta \rightarrow 1$.

In the above proof, we find that when the discount factor δ of a player is not very large, the greater is the proportion of “bad people” in society, the lower will be the efficiency of social norms; especially, in a society replete with deceit and dishonesty, even if the discount factor is close to one, if a person chooses to be honest, his or her interest will be harmed. This is because the lower bound of the discount factor that guarantees upholding the social norm is $\underline{\delta} \equiv \frac{1}{1+3\alpha}$, and this is increasing with a smaller α . When $\alpha < \frac{1-\delta}{3\delta}$, the above social norm is no longer a Nash equilibrium, i.e., the social norm will collapse. As such, both dishonesty and honesty can be Nash equilibria. The determinant factor is which equilibrium is in the majority. In the development of a society, there are various kinds of traps. In addition to resource endowment, social culture, which takes trust as a key element, has become a crucial constraining factor.

The long-run interaction of players discussed in this section is based on the extreme assumption that the previous behaviors of players will be accurately identified by themselves and their opponents. Correspondingly, in order to maintain cooperation among players, an important mechanism is punishment. In general, we can obtain the following conclusion: the more severe is the punishment for deviating from cooperative behavior, the easier it is for players to maintain cooperation. Some extreme punishment approaches, such as the grim strategy, play an important role in this kind of repeated game. However, once we relax the precise observation of players' behavior histories, many conclusions may need to be revised. For example, in the presence of observation errors, extreme punishment often destroys cooperation, and if minor disturbances occur, long-run cooperation between players will vanish. The example of imperfect public monitoring to be discussed in the next section illustrates this point. Under the punishment mode of grim strategy, all individuals choose to be lazy, while under the relatively loose punishment mode, they choose to work hard. Therefore, when maintaining long-term cooperative relations, it is critical that the mechanism offers forgiveness (to some extent) in addition to punishing opportunistic behavior. In many experiments, researchers also found that “tit for tat” is usually a more efficient way to maintain cooperation than

strategies such as the grim strategy.

7.6 Repeated Games with Imperfect Public Monitoring

The study of repeated games above assumes that players can observe all previous actions, which is a very strong assumption in real life. For many repeated interactions, individuals are unable to observe previous actions, but can only observe certain outcomes, the distribution of which depends on individuals' actions. In order to support the cooperation outcome, players can punish actions highly related to noncooperation outcomes, and thus indirectly punish opportunistic behavior, although they cannot directly impose punishments on players who deviate.

Repeated interactions that cannot be accurately observed can be further divided into two categories, according to the outcomes that players observe. One is that the outcomes observed by the players are public. For example, when firms cannot directly observe their opponents' previous pricing behaviors, it is possible to observe the aggregate demand in the market (e.g., industry reports issued by trade associations), and the size of aggregate demand depends on the pricing behaviors. The other is that the outcomes observed by players are private. For example, firms cannot observe the overall market demand, but can observe their own demands. This section focuses on the situation of publicly observable outcomes.

7.6.1 Basics of Repeated Games with Imperfect Public Monitoring

Compared to the model with perfect public monitoring discussed above, the model of repeated games with imperfect public monitoring differs primarily in the players' knowledge of histories.

First, we discuss the structure of repeated games with imperfect public monitoring, in which the outcomes are publicly observable.

In the stage game, each player $i \in N \equiv \{1, 2, \dots, n\}$ chooses a_i from action set A_i simultaneously. Each action $a \in A \equiv \prod_i A_i$ generates a proba-

bility distribution on the outcome set Y , $\pi_y(a)$, $y \in Y$, which represents the probability of outcome y conditional on the action a . The realized payoff of player i is $r_i(a_i, y)$, which means that the payoff does not directly depend on the actions of other players. Under action a , the expected utility of player i in the stage game is $u_i(a) = \sum_y \pi_y(a) r_i(a_i, y)$.

In repeated games, the common information at the beginning of stage t is $h^t = (y^0, y^1, \dots, y^{t-1})$, because no player's action is known to others, and each player has different information; specifically, player i knows $z_i^t = (a_i^0, a_i^1, \dots, a_i^{t-1})$, all previous own actions, at the stage t . In the stage t , the information of the player i is $h_i^t = (h^t, z_i^t)$, the set of all possible information is H_i^t , and the behavior strategy of players in the stage t is $\sigma_i^t(\cdot) : H_i^t \rightarrow \Delta A_i$.

A repeated game with perfect monitoring constitutes a special case of repeated game with imperfect monitoring, in which the common information set is $h^t = (a^0, a^1, \dots, a^{t-1})$, i.e., the publicly observable outcomes of previous actions of all players.

In repeated games with perfect monitoring, we employ the idea of dynamic programming, which means that the whole dynamic game is transformed into a series of (correlated) static decision problems, i.e., establishing a recursive structure to analyze repeated interactions among players. Similar recursive structures can be established in repeated games under imperfect monitoring. However, one of the important factors in the recursive structure is the existence of some factors that can coordinate the interaction of players. Therefore, the equilibrium solution of repeated games with imperfect monitoring is concentrated in public strategy equilibrium, i.e., the player's strategies rely only on public outcomes. Through public outcomes, the whole game can be converted into a series of (correlated) static decisions. In terms of private strategy, we will use some examples to illustrate the differences between them.

Definition 7.6.1 (Public Strategy) Strategy σ_i is a *public strategy*, if at any stage t , for any common history h^t , and any two different private histories z_i^t and \tilde{z}_i^t , we have $\sigma_i(h^t, z_i^t) = \sigma_i(h^t, \tilde{z}_i^t)$.

For public strategy, we have the following conclusion: if all players other than i choose a public strategy, player i can best respond with a public

strategy. The logic behind this conclusion is simple: the rational behaviour of player i depends on his or her belief of the behavior of other players, while the behavior of other players depends only on public outcomes; thus, player i 's belief in the behavior of other players is independent of the private information observed by other players (i.e., their previous actions). Therefore, the player has a best response on the basis of public outcomes only.

Although not all pure strategies are public strategies, the choice of public strategies has a certain degree of universality. This is because for any pure strategy equilibrium, there is an equivalent public strategy equilibrium. Two strategies σ_i and $\hat{\sigma}_i$ are (outcome) equivalent for player i means that for the strategies of other players σ_{-i} , the distributions of common outcomes induced by action profiles (σ_i, σ_{-i}) and $(\hat{\sigma}_i, \sigma_{-i})$ are the same. The following lemma reveals this idea.

Lemma 7.6.1 *In the game with imperfect public monitoring, each pure strategy has an (outcome) equivalent pure public strategy.*

PROOF. Let σ_i be the pure strategy of player i , and $a_i^0 = \sigma_i(\emptyset)$ be the action in the first stage, where \emptyset is the empty set. In the second stage, given the public outcome y^0 , player i chooses $a_i^1(y^0) = \sigma_i(y^0, a_i^0) = \sigma_i(y^0, \sigma_i(\emptyset))$. The rest can be done in the same manner. In stage t , public history is $h^t = (y^0, y^1, \dots, y^{t-1})$, and the history of player i is

$$a_i^t(h^t) \equiv \sigma_i(h^t, a_i^0, a_i^1(y^0), \dots, a_i^{t-1}(h^{t-1})).$$

Therefore, for any infinite public outcome h , the pure strategy gives a probability of 1 to $(a^0, a^1(y^0), \dots, a^t(h^t), \dots)$. Therefore, pure strategy σ_i is outcome equivalent to a public strategy of player i (choosing $a_i^t(h^t)$ in stage t). \square

The logic behind this lemma is as follows: in a pure strategy equilibrium, each player accurately anticipates the actions of other players. Player i chooses a_i^0 in stage 1, and chooses $\sigma_i^1(a_i^0, y^0)$ in stage 2. Since the action a_i^0 in stage 1 was previously determined, the dependence of the behavior

strategy in stage 2 on a_i^0 becomes redundant. We can replace the behavioral strategy σ_i^1 of player i with the public strategy $\hat{\sigma}_i^1(y^0) = \sigma_i^1(a_i^0, y^0)$, and then the public strategy of the other stage is constructed in a similar way.

On the basis of public outcomes, we can give concepts and tools similar to those in repeated games with perfect monitoring. First, under imperfect public monitoring, an automaton consists of the following elements: state set Ω , initial state ω^0 , behavioral action function $f : \Omega \rightarrow \prod_i \Delta(A_i)$, and state transition function $\tau : \Omega \times Y \rightarrow \Omega$.

The second is the concept of equilibrium. When all individuals adopt public strategies, given public history h^t , they agree on the distribution of future actions and outcomes (common knowledge). Similar to repeated games with perfect monitoring, we can define the continuation payoffs of a given public history, the public strategy associated with the continuation payoffs, and discuss what kind of public strategy profile is a Nash Equilibrium after stage t . Since there is such a structure at every possible stage, the equilibrium discussed here is perfect public equilibrium.

Definition 7.6.2 (Perfect Public Equilibrium) A strategy $\sigma = (\sigma_i)_{i \in N}$ is a *perfect public equilibrium*, if it satisfies the following two conditions:

- (1) Each σ_i is a public strategy.
- (2) The strategy profile is a Nash equilibrium of the continuation game starting at any public history h^t .

Under imperfect public monitoring, subgame perfect equilibrium would not be restrictive. Starting from stage 2 of the game, the information set has no singleton information sets, and there is only one subgame in the repeated game. When players only adopt public strategies, the private information of the player's own actions will not have a direct impact on the strategy. Consequently, the perfect public strategy is an extension of subgame perfect equilibrium for repeated games with imperfect public monitoring.

There is also a similar profitable one-shot deviation for perfect public equilibrium. The one-shot deviation strategy means that for player i 's public strategy σ_i and another strategy $\hat{\sigma}_i \neq \sigma_i$, we have a unique public history $\tilde{h}^t \in Y^t$ with $\hat{\sigma}_i(\tilde{h}^t) \neq \sigma_i(\tilde{h}^t)$, and for other history $\tilde{h}^\tau \neq \tilde{h}^t$, we have

$\hat{\sigma}_i(\tilde{h}^\tau) = \sigma_i(\tilde{h}^\tau)$. A profitable one-shot deviation refers to the fact that under such a deviation, the discounted payoff of player i is higher. The following proposition shows the relationship between perfect public equilibrium with imperfect monitoring and profitable one-shot deviation.

Proposition 7.6.1 *A public strategy σ is a perfect public equilibrium if and only if there is no profitable one-shot deviation, i.e., for all public histories $h^t \in Y^t$, $\sigma(h^t)$ is a Nash equilibrium for the following normal-form game. The payoff for this game is:*

$$g_i(\mathbf{a}) = (1 - \delta)u_i(\mathbf{a}) + \delta \sum_{y \in Y} U_i(\sigma|_{h^t, y})\pi_y(\mathbf{a}),$$

where $U_i(\sigma|_{h^t, y})$ is the (expected) discounted value of player i under public history h^t and public strategy σ with outcome y .

The proof of this proposition is similar to the previous one, and thus omitted. By adopting the automaton representation for repeated games, the following equivalent formulation of a perfect public equilibrium can be obtained (the proof is similar to the previous one with perfect monitoring, and thus omitted).

Proposition 7.6.2 *For the automaton $(\Omega, \omega^0, f, \tau)$, let $V_i(w)$ is the total discounted payoff of player i from state w . A public strategy σ is a perfect public equilibrium if and only if for any state $w \in \Omega$ that can be reached from the initial state ω^0 , $f(w)$ is an equilibrium of the following normal-form game with payoff:*

$$g_i(\mathbf{a}) = (1 - \delta)u_i(\mathbf{a}) + \delta \sum_{y \in Y} V_i(\tau(w, y))\pi_y(\mathbf{a}).$$

7.6.2 The Recursive Method of Repeated Games with Imperfect Monitoring

Public strategies can be analysed with dynamic programming techniques developed under perfect public monitoring, i.e., the concepts of enforcement and self-generation can be used to construct a recursive structure for repeated games with imperfect monitoring.

First, we introduce enforceable actions and payoffs with imperfect monitoring.

Definition 7.6.3 (Enforcement under Public Outcomes) A profile of (mixed) actions and payoffs (α^*, v) is *enforceable* under a discount factor of δ and on a feasible payoff set $W \in \mathcal{R}^n$, where $\alpha^* \in \prod_i \Delta(A_i)$, if there are continuation promises established on the public outcomes $\gamma : Y \rightarrow W$, such that for any player $i \in N$ and any $\alpha_i \in \Delta(A_i)$, we have

$$v_i = (1 - \delta)u^i(\alpha^*) + \delta \sum_y \pi_y(\alpha^*)\gamma_i(y),$$

$$(1 - \delta)u^i(\alpha^*) + \delta \sum_y \pi_y(\alpha^*)\gamma_i(y) \geq (1 - \delta)u^i(\alpha_i, \alpha_{-i}^*) + \delta \sum_y \pi_y(\alpha_i, \alpha_{-i}^*)\gamma_i(y).$$

We call α^* that satisfies the above conditions as enforceable on payoff set W , and v as being self-generated from (W, δ) . All payoffs that can be generated from (W, δ) are denoted by $B(W, \delta)$.

The following defines the self-generating set of payoffs.

Definition 7.6.4 (Self-Generating Set of Payoffs) A set $W \subseteq \mathcal{R}^n$ is *self-generating* at discount factor δ if $W \subseteq B(W, \delta)$.

Let $E^{PPE}(\delta)$ be the payoff set of all perfect public equilibrium payoffs at the discount factor δ . Abreu, Pearce, and Stachetti (1986, 1990) proved the following theorem.

Theorem 7.6.1 *If W is a bounded and self-generating set, then $W \subseteq B(W, \delta) \subseteq E^{PPE}(\delta)$.*

The theorem is similar to the one under perfect monitoring. More generally, for the case of mixed strategy, see the proof of Proposition 7.3.1. in Mailath and Samuelson (2006).

PROOF. Let W be self-generating, and fix $v \in B(W, \delta)$. By definition of $B(W, \delta)$, we can find an action profile α and a mapping $\gamma : Y \rightarrow W$ that leads to payoff profile v . Suppose that the strategy in period 0 is $\sigma^0 = \alpha^0(v)$, and for each outcome y^0 in stage 0, $v^1 = \gamma^0(y^0) \in W$. Then, we have $v = (1 - \delta)u(\alpha^0) + \delta \sum_{y \in Y} \pi_y(\alpha^0)\gamma^0(y)$. In addition, since $v^1 \in W$ and W is a

self-generating set, we have $v^1 \in B(W, \delta)$; thus, v^1 is also decomposable by an action profile $\alpha(v^1)$ and a continuation promise $\gamma^1(y^1)$. Let the strategy be the following in period 1: $\sigma^1(y^0) = \alpha^1(\gamma^0(y^0))$. Furthermore, for any sequence y^0, y^1 , let $v^2 = \gamma^1(\gamma^0(y^0))(y^1)$. Continuing the above construction, we can obtain a public strategy profile. For any t , the payoff of this public strategy profile can be written as the discounted sum of t action profiles and a continuation payoff profile; since this continuation payoff is discounted and in a bounded set W , the discounted sum of the actions converges to v .

Finally, it is necessary to check that there is no incentive to unilaterally deviate from any stage of this public strategy profile. In essence, this follows from the incentive compatibility conditions imposed by the notion of enforcement (and thus by self-generation) and the one-shot deviation principle mentioned earlier. Therefore, the constructed strategy profile is a perfect public equilibrium. \square

Abreu, Pearce, and Stacchetti (1990) also proved that the set of all perfect public equilibrium payoffs is self-generating:

$$E^{PPE, \delta} = B(E^{PPE}, \delta);$$

they also showed that it is the largest set with this property.

We now present some examples to explore incentives and penalties in repeated games with imperfect public monitoring.

Example 7.6.1 There are two players who choose to work hard (also known as co-operation, E) or be lazy (S). Their actions are not observed by the other player, but their actions affect the distribution of the observed public outcomes. Suppose that there are two public outcomes \bar{y} and \underline{y} representing high or low output, respectively. The relationship between actions and public outcomes is as follows:

$$\pi_{\bar{y}}(a) = \begin{cases} p, & \text{if } a = EE, \\ q, & \text{if } a = SE \text{ or } a = ES, \\ r, & \text{if } a = SS, \end{cases}$$

where $1 > p > q > 0, 1 > p > r > 0$. Obviously, high outcome \bar{y} is

more likely under cooperation between both parties, while low outcome \underline{y} is more likely to be the outcome of one or both parties being lazy.

We can show that the parameters of the game can be specified to yield ex-ante payoffs given by Table 7.9.

		player 2	
		C	D
player 1	C	2, 2	-1, 3
	D	3, -1	0, 0

Table 7.9: The Stage Game of the Prisoner's Dilemma

Here, we consider different types of punishment: one is the grim strategy, in which the occurrence of low output \underline{y} triggers a permanent punishment stage for both players; the other is a relatively tolerant punishment mode, in which the two sides return to cooperation if the outcome of the previous period was the high output.

First, consider the grim punishment mode, i.e., choosing cooperation at the beginning (i.e., E). Once there is a state of low output, all players will not choose to cooperate (i.e., S). The strategies of the two players can be described by an automaton with two states (see Figure 7.4).

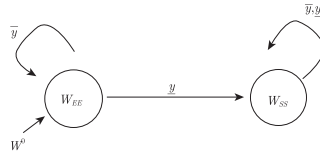


Figure 7.4: Automaton Representation of the Grim Strategy

The state space is $\Omega = \{w_{EE}, w_{SS}\}$, the initial state is w_{EE} , the automaton action function is $f(w_{EE}) = EE$, $f(w_{SS}) = SS$, and the state transition function is

$$\tau(w, y) = \begin{cases} w_{EE}, & \text{if } w = w_{EE}, y = \bar{y}, \\ w_{SS}, & \text{otherwise.} \end{cases}$$

A value function is assigned to the sum of expected discounted payoffs

of players in different states:

$$\begin{aligned} V_i(w_{EE}) &= (1 - \delta)2 + \delta\{pV_i(w_{EE}) + (1 - p)V_i(w_{SS})\}, \\ V_i(w_{SS}) &= (1 - \delta)0 + \delta V_i(w_{SS}). \end{aligned}$$

We then obtain $V_i(w_{EE}) = \frac{2(1-\delta)}{1-\delta p}$, $V_i(w_{SS}) = 0$ from the above two equations. If the above grim strategy is a Nash equilibrium, the incentive compatibility conditions (i.e., players have incentives to follow the automaton) need to hold:

$$\begin{aligned} V_i(w_{EE}) &\geq (1 - \delta)3 + \delta\{qV_i(w_{EE}) + (1 - q)V_i(w_{SS})\}, \\ V_i(w_{SS}) &\geq (1 - \delta)(-1) + \delta V_i(w_{SS}). \end{aligned}$$

To make the above two inequalities be satisfied, it is necessary to satisfy the condition $3p - 2q \geq \frac{1}{\delta}$. If $3p - 2q \geq \frac{1}{\delta}$, then $\{(\frac{2(1-\delta)}{1-\delta p}, \frac{2(1-\delta)}{1-\delta p}), (0, 0)\}$ is a self-generating set of payoffs. Since the state w_{SS} is an attractor (i.e., once we enter this state, there is no way to leave), the state w_{EE} will eventually be attracted to w_{SS} . In the grim strategy, everyone will choose to be lazy eventually. In this way, we find that when players are sufficiently patient, i.e., $\delta \rightarrow 1$, we have $V_i(w_{EE}) = \frac{2(1-\delta)}{1-\delta p} \rightarrow V_i(w_{SS}) = 0$. In other words, under the grim strategy, the player's payoff of (2, 2) per period cannot be the payoff of this repeated game.

The following considers another mode of interaction, i.e., the relatively tolerant punishment mode (also known as the punishment mode under one-term memory). If the previous state has a high outcome, then this period returns to the cooperation state; otherwise, it enters the punishment period.

The above strategy is described by an automaton (see Figure 7.5).

State space is $\Omega = \{w_{EE}, w_{SS}\}$, the initial state is w_{EE} , and the payoff function is:

$$f(w_{EE}) = EE, f(w_{SS}) = SS.$$

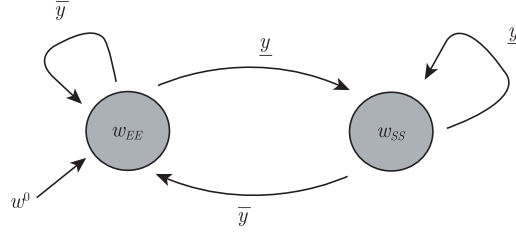


Figure 7.5: Automaton with One-term Memory Punishment

The state transition function is:

$$\tau(w, y) = \begin{cases} w_{EE}, & \text{if } y = \bar{y}, \\ w_{SS}, & \text{if } y = \underline{y}. \end{cases}$$

Similarly, if $\{(V_1(w_{EE}), V_2(w_{EE})), (V_1(w_{SS}), V_2(w_{SS}))\}$ is the self-generating set of payoffs from the above tolerant strategy, then it satisfies:

$$\begin{aligned} V_i(w_{EE}) &= (1 - \delta)2 + \delta[pV_i(w_{EE}) + (1 - p)V_i(w_{SS})] \\ &\geq (1 - \delta)3 + \delta[qV_i(w_{EE}) + (1 - q)V_i(w_{SS})], \end{aligned} \quad (7.6.11)$$

$$\begin{aligned} V_i(w_{SS}) &= (1 - \delta)0 + \delta[rV_i(w_{EE}) + (1 - r)V_i(w_{SS})] \\ &\geq (1 - \delta)(-1) + \delta[qV_i(w_{EE}) + (1 - q)V_i(w_{SS})]. \end{aligned} \quad (7.6.12)$$

we then have

$$V_i(w_{EE}) = \frac{2(1 - \delta(1 - r))}{1 - \delta(p - r)}, \quad V_i(w_{SS}) = \frac{2\delta r}{1 - \delta(p - r)}.$$

Meanwhile, inequality (7.6.11) means $\delta \geq \frac{1}{3p-2q-r}$, and inequality (7.6.12) means $\delta \leq \frac{1}{p+2q-3r}$. To make the above two inequalities hold simultaneously, we need $p - q \geq q - r$. In addition, the above two inequalities have some mutual restriction, satisfying the incentive compatibility of state w_{EE} , requiring the players to have enough patience (high discount factor); however, the incentive compatibility of state w_{EE} requires that the patience cannot exceed a certain degree. Moreover, $p - q \geq q - r$ means that the signal (payoff) requires a higher degree of accuracy in reflecting effort.

In the tolerant punishment, unlike the grim punishment pattern, the player will not eventually enter the mutual punishment state, and the value of maintaining the cooperation exceeds that of the grim mode, namely

$$\frac{2(1 - \delta(1 - r))}{1 - \delta(p - r)} > \frac{2(1 - \delta)}{1 - \delta p}.$$

Then, the conclusion in this example is different from the one in repeated games with perfect monitoring: under the grim strategy punishment mode, all individuals choose to be lazy; whereas, under the relatively loose punishment mode, they choose to work hard.

Through the above discussion, we find that under the above two punishment modes, the highest possible ex-ante expected payoff of the perfect public equilibrium is less than the ex-ante expected payoff of the cooperation between two parties. Under the punishment model of one-term memory, i.e., $\frac{2(1 - \delta(1 - r))}{1 - \delta(p - r)} < 2$, which means that in repeated games with imperfect monitoring, the Folk Theorem may fail in some cases.

7.6.3 Potential Efficiency Loss in Repeated Games with Imperfect Monitoring

The following example reveals the potential inefficiency in repeated games.

Example 7.6.2 The following is a discussion of incentive issues among players in Example 7.6.1. Here, we focus on the strongly symmetric pure strategy equilibrium (i.e., in each possible history, the player chooses the same action), and discuss the maximum possible discounted payoff that can be supported by the player's interaction in a symmetric case (or the most efficient symmetric pure strategy equilibrium payoff). Suppose that a public correlation device can be used, and thus we can consider automata of the sort shown in Figure 7.6.

This can be represented as an automaton: $\Omega = \{w_{EE}, w_{ss}\}$, $f(w_{EE}) =$

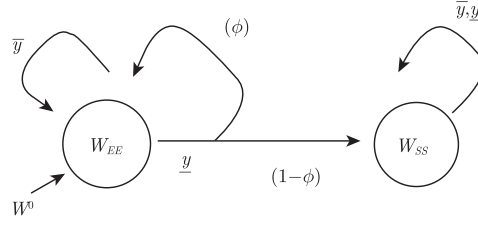


Figure 7.6: Automaton Representation in Symmetric Strategies

$EE, f(w_{SS}) = SS, \tau : \Omega \times Y \rightarrow \Omega$:

$$\tau_{w_{EE}}(w, y) = \begin{cases} 1, & \text{if } w = w_{EE}, y = \bar{y}, \\ \phi, & \text{if } w = w_{EE}, y = \underline{y}, \\ 0, & \text{if } w = w_{SS}, \end{cases}$$

where $\tau_{w_{EE}}(w, y)$ is the probability of state w_{EE} in the next stage. In the above state transition function, increasing ϕ reduces the difference between the observable signal and the payoff function; however, if ϕ is too high, it will impede players' incentive to work hard. The following discusses how automaton should be chosen to obtain the maximum possible payoff. If $\phi = 0$, it corresponds to the previous grim strategy.

The following focuses on the value $V(w_{EE})$ for players' cooperation. From

$$\begin{aligned} V(w_{EE}) &= (1 - \delta)2 + \delta[pV(w_{EE}) + (1 - p)(\phi V(w_{EE}) + (1 - \phi)V(w_{SS}))] \\ &\geq (1 - \delta)3 + \delta[qV(w_{EE}) + (1 - q)(\phi V(w_{EE}) + (1 - \phi)V(w_{SS}))], \end{aligned}$$

and

$$\begin{aligned} V(w_{SS}) &= (1 - \delta)0 + \delta V(w_{SS}) \\ &\geq (1 - \delta)(-1) + \delta V(w_{SS}), \end{aligned}$$

we have that $V(w_{SS}) = 0$, $V(w_{EE}) = \frac{2(1-\delta)}{1-\delta(p+(1-p)\phi)}$, the incentive compatibility conditions are satisfied, and the larger is the ϕ , the greater is the $V(w_{EE})$.

To achieve the above automaton, it is necessary to have:

$$\delta(1 - \phi)(p - q)V(w_{EE}) \geq 1 - \delta.$$

The maximum possible ϕ to satisfy the above inequality is $\phi = \frac{\delta(3p-2q)-1}{\delta(3p-2q-1)}$. Substituting the above solution into $V(w_{EE})$, we obtain the maximum possible payoff: $V(w_{EE}) = 2 - \frac{1-p}{p-q} < 2$. The maximum possible symmetric equilibrium payoffs are not dependent on the discount factor δ , and it is strictly less than the most efficient symmetric equilibrium payoff under perfect monitoring, i.e., (2, 2).

The efficiency losses in a repeated game are due to the fact that some strategies need to be implemented by mutual punishment under certain conditions (observed public outcomes), which reduces the players' expected discounted payoffs. To avoid the decrease in payoffs due to punishment for all players, there is a need for more accurate signals that allow speculation on individual behavior, thus linking the state (observed public outcomes) more directly to individual behavior. In other words, sufficient information is needed to identify and punish opportunistic players, while avoiding accidental injury to "innocent" players.

Fudenberg, Levine and Maskin (1994) systematically discussed the conditions for Folk Theorem in repeated games with imperfect monitoring, i.e., infer players' actions probabilistically from public outcomes. For different public outcomes, the appropriate punishments or rewards are applied to different players to support some of the interactive patterns to a greater extent. As the proofs of such theorems are relatively complex, the discussion is omitted here, and readers are referred to their original work.

7.6.4 Private Strategy

The cases discussed above are based on public strategies of players, because any pure strategy has an (outcome) equivalent public strategy. Does this then mean that the private strategy (i.e., the strategic choices of players are based not only on the history of the public outcomes, but also on the history of one's own actions) irrelevant? The answer is negative. The

following is an example from Mailath and Samuelson (2006) that discusses the difference between private strategies and public strategies.

Example 7.6.3 Consider a two-stage repeated game. Based on the decomposition of repeated game payoffs, the infinitely repeated game can be reduced to a two-stage game, in which the second stage payoff can be viewed as a continuation discounted payoff on the basis of public outcomes (see Table 7.10).

		player 2				player 2	
		<i>E</i>	<i>S</i>			<i>R</i>	<i>P</i>
player 1	<i>E</i>	2, 2	−1, 3	player 1	<i>R</i>	$\frac{8}{5}, \frac{8}{5}$	$0, \frac{8}{5}$
	<i>S</i>	3, −1	0, 0		<i>P</i>	$\frac{8}{5}, 0$	0, 0

Table 7.10: Left table: payoff in the first stage; Right table: payoff in the second stage

In the second stage, *R* can be understood as a reward behavior, while *P* is a punishment behavior. Let the discount factors of both players be $\delta = \frac{25}{27}$. The set of public outcomes is $Y = \{\underline{y}, \bar{y}\}$. Actions affect the distribution of public outcomes according to the following:

$$\pi_{\bar{y}}(a) = \begin{cases} p = \frac{9}{10}, & \text{if } a = EE; \\ q = \frac{4}{5}, & \text{if } a = SE \text{ or } a = ES; \\ r = \frac{1}{5}, & \text{if } a = SS. \end{cases}$$

The following considers the choices of different strategies by players.

First, consider the case of pure strategies without public correlation device. Without loss of generality, we consider public strategy under pure strategy, and we focus on pure public strategy equilibrium. Symmetric equilibrium payoffs are: in the first and second stages, the stage payoffs of two players are (2, 2) and $(\frac{8}{5}, \frac{8}{5})$. If two players choose strategy (*E*, *E*) in the first stage, and the common outcome in the second stage is \bar{y} , then they choose the strategy (*R*, *R*); otherwise, they choose (*P*, *P*). Since in the second stage, the strategic choice given above under all possible public outcomes is a Nash equilibrium, we only need to consider the first stage. Given

each other's strategy, if player i chooses the above strategy, the payoff is:

$$2(1 - \delta) + \delta p \frac{8}{5} = \frac{40}{27},$$

and if he or she changes the strategy, the payoff is:

$$3(1 - \delta) + \delta q \frac{8}{5} = \frac{38}{27}.$$

Obviously, the strategic choice above is an equilibrium.

Second, consider the pure strategy with public correlation devices. Similar to the examples discussed earlier, a good symmetric equilibrium is: choosing (E, E) in the first stage. If the public outcome of the first stage is \bar{y} , then choose strategy (R, R) in the second stage. If the public outcome of the first stage is \underline{y} , then choose strategy (R, R) with probability ϕ , and choose strategy (P, P) with probability $1 - \phi$ in the second stage. While satisfying the incentive compatibility (i.e., players have incentives to choose E in the first stage), it is possible to improve ϕ . The conditions for the first stage of incentive compatibility are:

$$2(1 - \delta) + \delta \frac{8}{5} [p + (1 - p)\phi] \geq 3(1 - \delta) + \delta \frac{8}{5} [q + (1 - q)\phi],$$

The highest ϕ that satisfies the above inequality is seen to be $\phi = \frac{1}{2}$. As a consequence, with a public correlation device, the best symmetric equilibrium payoff is

$$2(1 - \delta) + \delta \frac{8}{5} \left[p + (1 - p) \frac{1}{2} \right] = \frac{42}{27}.$$

Again, consider a mixed public strategy (with public correlation devices). In this example, comparing with the situation in which player i chooses to “work hard”, if the player is “lazy” (S), public action outcomes better reflect player j 's action. This is because when player i is “lazy”, if player j is also “lazy,” then the probability of $r = \frac{1}{5}$ in the public outcome distribution is \bar{y} ; if player j chooses to “work hard”, the probability of $q = \frac{4}{5}$ is \bar{y} , and the difference is $q - r = \frac{3}{5}$. When player i chooses to work hard, if player j is “lazy”, the probability of \bar{y} is $q = \frac{4}{5}$; if player j chooses to “work hard”, the probability of \bar{y} is $p = \frac{9}{10}$. Fur-

thermore, the difference is only $p - q = \frac{1}{10}$. Therefore, choosing a mixed strategy in the first stage will make it easier to solve incentive compatibility issues.

Consider a symmetric mixed strategy: in the first stage, the probability of choosing E is α ; if the public outcome of the first stage is \bar{y} , choose strategy (R, R) in the second stage. If the public outcome of the first stage is \underline{y} , in the second stage choose strategy (R, R) with probability ϕ , and strategy (P, P) with probability $1 - \phi$. The condition for the above strategies to be an equilibrium is:

$$\begin{aligned} & \alpha \left\{ 2(1 - \delta) + \delta \frac{8}{5} [p + (1 - p)\phi] \right\} + (1 - \alpha) \left\{ (-1)(1 - \delta) + \delta \frac{8}{5} [q + (1 - q)\phi] \right\} \\ & \geq \alpha \left\{ 3(1 - \delta) + \delta \frac{8}{5} [q + (1 - q)\phi] \right\} + (1 - \alpha) \left\{ (0)(1 - \delta) + \delta \frac{8}{5} [r + (1 - r)\phi] \right\}. \end{aligned}$$

The highest outcome is $\phi(\alpha) = \frac{11-10\alpha}{12-10\alpha} > \frac{1}{2}$. When $\alpha \in (0, 1)$, it will decrease as α increases. Substituting this into the above equation, the expected payoff is $\frac{224-152\alpha-30\alpha^2}{27(6-5\alpha)}$, which is maximum at $\alpha \approx 0.969$. This maximum payoff approximately equals to $1.5566 > \frac{42}{27}$.

Finally, consider private (mixed) strategies. The public outcome better reflects the action of player j when player i is “lazy” (S). Therefore, it is a natural consideration to punish player j under the outcome \underline{y} (which makes it more likely that player j chose to be “lazy”) when player i is “lazy” . This strategy obviously takes advantage of the private strategy of player i to solve the incentive problem for j . Due to symmetry, similar considerations apply when i and j are swapped.

Consider the following symmetric private strategy. Suppose that player i chooses a mixed strategy in the first stage. If the outcome is \bar{y} in the first stage, then the player chooses R in the second stage; if the outcome is \underline{y} and the player chose to be “lazy” in the first stage, he or she will choose P with a strict positive probability in the second stage; if the player chooses to “work hard” in the first stage but the outcome is \underline{y} , he or she will still choose R in the second stage.

Let α be the probability that player i chooses E in the first stage, and ξ be the probability that player i chooses R in the second stage if he or she

observes the outcome \underline{y} and chose to be “lazy” in the first stage. Figure 7.7 describes the private strategy of each player by the automaton below.

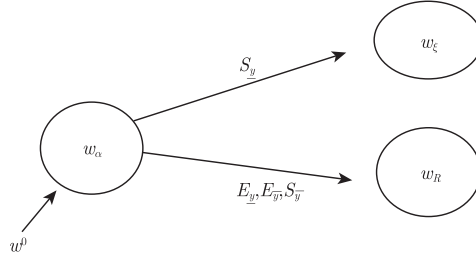


Figure 7.7: An Automaton Representation of Private Strategies

Getting player i to mix ($\alpha \in (0, 1)$) in the first stage requires that the player is indifferent regarding the two actions:

$$\begin{aligned} & \alpha \left\{ 2(1 - \delta) + \delta \frac{8}{5} \right\} + (1 - \alpha) \left\{ (-1)(1 - \delta) + \delta \frac{8}{5} [q + (1 - q)\xi] \right\} \\ &= \alpha \left\{ 3(1 - \delta) + \delta \frac{8}{5} \right\} + (1 - \alpha) \left\{ (0)(1 - \delta) + \delta \frac{8}{5} [r + (1 - r)\xi] \right\}. \end{aligned}$$

Substituting $\delta = \frac{25}{27}$, $q = \frac{4}{5}$, $r = \frac{1}{5}$ into the above equality, we obtain $\xi(\alpha) = \frac{11-12\alpha}{12-12\alpha}$. Substituting this, it can be checked that a player's expected payoff is $\frac{2}{9}(\alpha + \frac{56}{9})$. To maximize this, we maximize α subject to $\xi(\alpha) \in [0, 1]$; this requires $\alpha = \frac{11}{12}$, $\xi = 0$, which gives a maximum expected utility of approximately 1.5864.

Since $1.5864 > 1.5566 > \frac{42}{27} > \frac{40}{27}$, in the above four symmetric equilibrium cases, the private strategy can support a higher degree of cooperation than a mixed strategy with a public correlation device. The pure strategy with a public correlation device, when compared to the pure strategy, can support a higher degree of cooperation.

There are other types of repeated games, such as repeated games with private monitoring, i.e., players can only observe their own signals, and there is no public signal. It is more difficult to establish the previous recursive structure under this type of repeated game. However, for repeated games with private monitoring, Ely, Horner, and Olszewski (2005) proposed a new concept of equilibrium with stronger constraints - “belief-

free equilibrium” – to reconstruct a recursive structure. Mailath and Samuelson (2006) systematically discuss this type of repeated games, and their monograph is considered to constitute an encyclopedia of repeated games and reputation mechanisms to be discussed below.

7.7 Reputation Mechanism

Below is a practical application related to repeated games, i.e., how players develop a reputation for a particular behavioral pattern in multiple-period interactions. In contrast to the previous repeated games, the reputation mechanism has privately-informed players in long-term interactions. Some players will use specific actions to influence their opponents’ beliefs. The intuition behind it is that if a player always takes the same action, his or her opponents would expect the player to take similar actions in the future. An important question is under what conditions can the players develop and maintain the reputation that they want to achieve. For example, companies attempt to build a reputation for product quality; and policy-makers hope to build a credible reputation for a policy. In this section, the basic principle of reputation mechanism is discussed mainly through an example.

The Chain Store Paradox

A long-lived company has chain stores in multiple markets that are independent. There is a sequence of short-run players, each of whom is a potential entrant in a different market. Each short-run player observes all previous actions. Each potential entrant decides whether or not to enter the market. If he does not enter, then the incumbent in the market is a monopolist with payoff $a > 0$. If the potential entrant enters, the incumbent decides whether to fight or accommodate to the entry. If the incumbent fights, her payoff is -1 ; if not, her payoff is 0 . The incumbent’s goal is to maximize the sum of expected discounted payoffs with $\delta < 1$ as the discount factor.

Suppose that there are two types of potential entrants: with probability $q \in (0, 1)$ the short-run player is tough and will enter in any case, while with the remaining probability she is weak and receives a payoff of 0 if

she does not enter (so this type will perform a cost-benefit analysis when deciding whether or not to enter). If the potential entrant enters and the incumbent fights back, then the entrant's payoff is -1 ; if the incumbent accommodates and accepts the entry, then the entrant's payoff is $b > 0$. The type of the potential entrants is private information, while the types of different potential entrants are independent.

This is a finitely repeated game, in which the number of repetitions equals the number of markets. Selten (1978) noted that from a theoretical point of view, there exists a unique sequential equilibrium in this finitely repeated game, in which potential entrants enter in each period, and the incumbent chooses to accommodate to each entry. However, this equilibrium seems counter-intuitive given that there are multiple markets, and the incumbent can fight entrants in certain markets and create a tough image that discourages entry into other markets. Selten referred to this contrast between the theoretical prediction and the intuitive view as the **chain store paradox**.

Market entry is a crucial issue in industrial organizations because it influences the competition and efficiency of a market. Much work has been carried out to solve the Chain Store Paradox. Kreps and Wilson (1982b), Milgrom and Roberts (1982), and Kreps, Wilson, Milgrom and Roberts (1982) solved this paradox by introducing incomplete information.

Now, suppose that the incumbent also has private information. Specifically, the long-run incumbent has a probability p^0 of being a irrational or hard type who will always fight an entrant; she has a probability of $1 - p^0$ to be rational, when her payoff equals that of the incumbent described earlier. The following argument demonstrates that even if p^0 is small, as long as the duration of the repetition is sufficiently long, the incumbent will always maintain a hard-line attitude toward entry, i.e., the incumbent will establish a reputation for being irrational when it comes to responding to entry.

We now discuss the mechanism of reputation. If there is only one period, as long as the potential entrant enters, the rational type incumbent will choose to acquiesce, and the tough type incumbent will choose to fight. If $(1 - p^0)b - p^0 < 0$, or $p^0 > \bar{p} \equiv \frac{b}{b+1}$, a weak potential entrant will choose to

not enter; otherwise, he will choose to enter.

Consider two periods of interaction now. Potential entrants 1 and 2 make decisions for markets 1 and 2, respectively. Player 1 firstly faces the entry choice, then player 2 chooses after seeing the outcome of market 1. The following focuses on the behaviour of the rational type incumbent and the weak type of potential entrants.

In period 1, if the incumbent chooses to acquiesce when faced with market entry, the incumbent is revealed as the rational type. Therefore, in period 2, the potential entrant will definitely choose to enter market 2. The following focuses on the consideration of the rational type incumbent. If faced with entry in the first period, the cost of choosing to fight is 1. However, the benefit is to build the reputation of the hard type to discourage potential weak entrants. At this time, his or her revenue is at most $\delta(1-q)a$. In this way, when $q > \bar{q} \equiv \frac{a\delta-1}{a\delta}$, the incumbent will not choose to fight.

When $q \leq \bar{q}$, if the fight in market 1 can make weak potential entrant 2 quit, then reputation can produce value. However, player 2's decision-making depends on his belief in the type incumbent.

Consider the case of $p^0 > \bar{p}$. Since the tough type incumbent always chooses to fight, the probability of the rational type incumbent choosing to fight is not more than that of the tough type. After observing the fight in market 1, the potential entrants in market 2 believe that the probability of the tough type will not be lower than p^0 . At this time, the weak type player 2 will not choose to enter; this, in turn, means that the rational type incumbent will establish reputation through fighting in market 1. In this way, potential weak entrants will choose to not enter market 1.

Consider the case of $p^0 \leq \bar{p}$. We prove that the incumbent will not choose a pure strategy. First, consider the pooling equilibrium and show that the rational type incumbent will not choose to fight entry into market 1. Under pooling, if there is a fight in market 1, the posterior belief of the potential entrant of market 2 is the same as the initial belief. Therefore, the weak entrant 2 will also choose to enter, which means that the rational type incumbent should definitely not choose to fight in market 1. Second, consider the separating equilibrium, in which the rational type incumbent will certainly choose to accommodate in market 1. If instead she deviates

from this strategy and fights in market 1, the potential entrant in market 2 will think that the incumbent is tough and the potential entrant will choose to not enter. The rational type incumbent then has an incentive to fight entry in the market.

Therefore, the only possible equilibrium is the quasi-separation equilibrium, and thus the rational type incumbent chooses a mixed strategy to fight in market 1. Let $\beta < 1$ be the probability that rational type incumbent chooses to fight. Because the rational type incumbent adopts a strictly mixed strategy, choosing to fight and choosing to accommodate must bring the same benefits, which means that weak entrants must also choose mixed strategies. After observing the fight in the market, the posterior belief of the potential entrant in market 2 satisfies

$$\text{prob}(\text{tough}|\text{fight}) = \frac{p^0}{p^0 + (1 - p^0)\beta} = \bar{p} = \frac{b}{b + 1},$$

and thus it follows that $\beta = \frac{p^0}{(1 - p^0)b}$. In market 1, the probability of the potential entrant facing a fight with the incumbent is

$$p^0 + (1 - p^0) \frac{p^0}{(1 - p^0)b} = \frac{p^0(1 + b)}{b}.$$

If $p^0 > (\frac{b}{b+1})^2 = \bar{p}^2$, weak potential entrants in market 1 will not choose to enter, the ex-ante expected payoff of the rational type incumbent is greater than zero; if $p^0 < \bar{p}^2$, weak potential entrants in market 1 will inevitably enter.

The following continues to discuss the presence of chain stores in three markets. If $p^0 > \bar{p}^2$, the rational type incumbent facing an entrant in market 1 will definitely choose to fight, while the weak potential entrants of market 1 will not choose to enter. If $p^0 \in (\bar{p}^3, \bar{p}^2)$, the rational type incumbent chooses a mixed strategy when facing entry in market 1, while the weak potential entrants in market 1 will not choose to enter. If $p^0 < \bar{p}^3$, the weak potential entrants in market 1 will definitely enter.

More generally, the incumbent has N chain stores. If $p^0 > \bar{p}^k, k < N$, then the weak potential entrants from market 1 to market $N - k$ will not enter, and the incumbent will certainly build strong reputation in these mar-

kets by choosing to fight should there be entry. In addition, note that as $N \rightarrow \infty$, we obtain that $\bar{p}^N \rightarrow 0$, and thus $p^0 > \bar{p}^N$ holds for sufficiently large N . This means that the first potential entrant will not enter, and if there is entry the rational type incumbent has a strong motivation to fight it to build a strong reputation. Therefore, as long as there is incomplete information, even if the degree of incomplete information is small, long-term players can build their reputations through certain behavior.

In different contexts, reputation operates differently and performs dissimilar roles. Fudenberg and Tirole (1991) and Mailath and Samuelson (2006) provide systematic introductions to the rich literature in this field; the latter also reviews some recent literature on the reputation mechanism.

7.8 Biographies

7.8.1 John Richard Hicks

John Richard Hicks (1904—1989), one of the founders of the general equilibrium theory, made great contributions to microeconomics, macroeconomics, economic methodology and economic history, and shared the Nobel Memorial Prize in Economic Sciences with Kenneth Joseph Arrow in 1972.

He was educated at Clifton College (1917-1922), and later obtained a scholarship to study mathematics at Balliol College, Oxford (1922-1926). In 1923, he moved to Philosophy, Politics and Economics, the “new school” that was just being started at Oxford, and graduated with second-class honors. During 1925 and 1926, he studied labor economics under Cole’s guidance. From 1926 to 1935, Hicks worked as an assistant and later as a lecturer at the London School of Economics, during which time he received a Ph.D. degree from the University of London in 1932. In the same year, he published *The Theory of Wages*. During the time in the London School of Economics and Political Science, Hicks learned a great deal about economics, and gradually developed from a beginner to an accomplished economist who published a series of important academic papers, including *A Reconsideration of the Theory of Value* (1934) with Roy Allen and *A Suggestion for*

Simplifying the Theory of Money (1935). In the summer of 1935, Hicks left the London School of Economics, and became a researcher and lecturer at Gonville and Caius College of the University of Cambridge until 1938. During this period, his main achievement at the University of Cambridge was the book entitled *Value and Capital*. In addition, he wrote two influential book reviews of Keynes's *The General Theory of Employment, Interest, and Money*, in which the article *Mr. Keynes and the "Classics" – A Suggested Interpretation* had far-reaching effects.

Hicks is one of the founders of the general equilibrium theory in microeconomics. The general equilibrium theory was originally characterized by normative analysis, but in his most famous work, *Value and Capital* published in 1939, he abandoned this tradition and gave a powerful empirical implication to this theory. He put forward a complete economic equilibrium model with markets for commodities, factors of production, credit, and money. There are many innovations in the model, including perfecting the original consumption and production theory, clarifying the conditions of the stability of the market, extending the application scope of the static analysis, considering the dynamic analysis, and adopting the capital theory based on the assumption of profit maximization. The well-known Hicksian demand in microeconomics describes the minimum consumption expenditure under a given level of utility, which, together with the traditional Marshallian demand, is one of the two optimal solutions to a dual problem. Deeply rooted in consumer theory and producer theory, the Hicks model provides much greater possibilities for performing comparative static analyses than previous models in this field, and thus many important economic theorems were established using this model. His model is also an important link between the general equilibrium theory and the prevailing business cycle theory.

The most notable contributions of Hicks to welfare economics were the standard analysis of comparing different economic conditions and the revision of the concept of consumer surplus. He proposed the Kaldor-Hicks efficiency test, which is another well-known criterion for comparing different public policies and economic states besides the efficiency criterion of Pareto improvement. He also perfected the theory of marginal utility ex-

plained by the ordinal utility theory and indifference curve, and developed the general equilibrium theory. He systematically studied and clarified the general equilibrium theory based on the theories of Walrasian, Pareto, etc. In his theoretical system, general equilibrium was divided into static and dynamic general equilibrium, and its contribution was primarily to establish the dynamic general equilibrium theory, based on previous theory. He proposed the *IS-LM* model and used it to analyze the simultaneous equilibrium of goods market and money market, the simultaneous determination of national income and interest rate, as well as the interrelationships between them. This model combines the general equilibrium analysis of neoclassical economics with the Keynesian theory of national income determination, and becomes the theoretical hallmark model of modern Keynesian macroeconomics.

Hicks used the accelerator multiplier interaction for building up a new theory of business cycle. The theory holds that the increase of output and income will lead to acceleration of investment through the acceleration effect. Moreover, due to the multiplier effect, the growth of investment causes output and income to increase accordingly by a magnified amount, and thus production capacity expands rapidly. When expansion reaches the limit of the cycle, it shifts to economic contraction. During the contraction, due to the role of acceleration, the decline in investment will lead to output and income decline in a certain proportion, and this decline is limited by the lower bound of the cycle. The economy starts to rebound again when it reaches the bottom of the cycle. Hicks discovered a regular cyclical fluctuation of 7-10 years, based on a study of the economic history of the past one and a half century.

7.8.2 Thomas Schelling

Thomas C. Schelling (1921-2016) was an American economist, an expert on foreign affairs, national security, nuclear strategy and arms control, and one of the founders of the theory of limited war. He was born in California in April 14, 1921 and received a Ph.D. degree in economics from Harvard University in 1948. He won the Frank E. Seidman Distinguished Award in

Political Economy in 1977 and the Nobel Prize in Economics for “having enhanced our understanding of conflict and cooperation through game-theory analysis” in 2005.

Unlike conventional game theory, which has traditionally used mathematics extensively, Schelling’s main research field is called “non-mathematical game theory”. Schelling and Aumann further developed the non-cooperative game theory and began to deal with some major problems in the field of sociology. They came from different perspectives—Aumann mainly from the perspective of mathematics while Schelling primarily from the perspective of economics—and both thought that it was possible to reconstruct the analytical paradigm of human interaction using game theory. More importantly, Schelling pointed out that many social interactions with which people are familiar can be understood from the perspective of non-cooperative games; Aumann also found that some long-term social interactions can be analyzed deeply with formal non-cooperative game theory.

Schelling’s game theory was based on the breakthrough of the analytical method of neoclassical economic theory, different from the mainstream game theory in research method and focus, thereby improving, enriching and developing the modern game theory. In his classic book, *The Strategy of Conflict*, Schelling first defined and clarified concepts, such as deterrence, credible commitment, strategic mobility, etc., began to study social science issues using a unified analytical framework of game theory, and made a detailed analysis of the bargaining and conflict management theory. Bargaining theory is the primary contribution of Schelling’s early-period research. Although he did not deliberately set out to establish a formal model, many of his views were later shaped by the new development of game theory. The concepts that he defined are also the most basic in game theory, e.g., the non-credible threat of perfect equilibrium.

His fruitful work contributed to the new development of game theory and accelerated the application of game theory in the field of social science. In particular, his research on strategic commitments explains many phenomena (e.g., the firm’s competitive strategy, and the mandate of political decision-making). In 1988, the American Economic Association gave him the “Distinguished Fellow Award”, and stated “Schelling’s theory

about social relations and his application of the theory are derived from his fruitful integration of theory with practice. He has an unusual talent, which enabled him to capture the nature of the social and economic situations in which the participants share the same or different interests, and to vividly describe the nature.” The Nobel committee evaluated him as follows, ” Schelling, a self-described ‘errant economist’, has been proven to be a very distinguished and pioneering explorer.”

7.9 Exercises

Exercise 7.1 Consider a three-player symmetric infinitely repeated game: the discount factor is δ , and the stage game is $(1, 2, 3, A_i, u_i)$, where $A_i = [0, 1]$. For any $(a_1, a_2, a_3) \in A_1 \times A_2 \times A_3$, we have

$$u_i(a_1, a_2, a_3) = a_1 a_2 a_3 + (1 - a_1)(1 - a_2)(1 - a_3).$$

1. Find the set of feasible payoffs of the stage game.
2. Prove: for any discount factor $\delta \in (0, 1)$, the payoff of any player in a subgame perfect Nash equilibrium (SPNE) in the repeated game is at least $1/4$.

Exercise 7.2 Consider the following game:

		buyer	
		<i>B</i>	<i>D</i>
seller	<i>H</i>	1, 1	-1, 2
	<i>L</i>	2, -1	0, 0

The seller can choose to work hard (*H*) or to be lazy (*L*), and the buyer can choose to purchase (*B*) or not (*D*).

1. Find the set of feasible payoffs and the set of individually rational payoffs.
2. Suppose that the game is repeated infinitely, and the discount factor for both players is δ . Find a SPNE such that, for some range of δ , the path of the repeated game is $(H, B)^\infty$. Solve for the range of δ .

3. Now, suppose that in the stage game, the buyer could observe the effort of the seller prior to making a purchasing decision. Then, solve question 2.

Exercise 7.3 There is a static game with complete information between player 1 and player 2, and the strategy space for both is $\{A, B\}$. If both choose A , then each obtains α ; if both choose B , then each obtains β ; if they choose A and B , respectively, then the player with action A receives γ , and the player with action B receives λ .

1. Give the normal form of the above game.
2. If it is a Prisoner's Dilemma game, give the range of parameter.
3. With the answer from question 2, suppose that the game is repeated infinitely, and the discount factor is δ . Prove that if $\delta \geq \bar{\delta}$, then the grim strategy can resolve the Prisoner's Dilemma. Solve for $\bar{\delta}$.

Exercise 7.4 Consider a two-player, two-stage repeated game, and the stage game is the following: Prior to stage two, both players could observe

		player 2		
		d	e	f
player 1	a	3, 1	0, 0	5, 0
	b	2, 1	1, 2	3, 1
	c	1, 2	0, 1	4, 4

the outcomes of stage one, and there is no discount factor. Is there an SPNE, such that the payoff in stage one is $(4, 4)$? If yes, give the corresponding strategy. If no, explain why.

Exercise 7.5 Consider the following infinitely repeated game with the stage game:

Solve for the range of discount factor δ , such that the following strategy is an SPNE.

State 1: first choose (B, R) ; if no one deviates, then continue to choose (B, R) ; otherwise, change to state 2;

State 2: choose (T, L) ; if no one deviates, then continue to choose (T, L) ; otherwise, change to state 1.

		player 2		
		L	C	R
player 1	T	1, 1	-1, -8	6, -8
	M	-8, 1	-4, -4	1, -4
	B	-8, 6	-4, 1	3, 3

Exercise 7.6 Consider the infinitely repeated Prisoner's Dilemma game with the stage game:

		player 2	
		candor(C)	disavow(D)
player 1	candor (C)	1, 1	-1, 2
	disavow(D)	2, -1	0, 0

Let the discount factor be δ .

1. Prove: if $\delta \geq 0.5$, then there is an SPNE, such that the strategy profile in every stage is (C, C) . Find the complete strategies of the two players in this equilibrium.
2. Prove the following Folk Theorem: if δ is sufficiently close to 1, then any feasible and individually rational payoff is subgame perfect Nash equilibrium payoff.

Exercise 7.7 Consider the following infinitely repeated Prisoner's Dilemma with the discount factor δ and the stage game:

		player 2	
		candor(C)	disavow(D)
player 1	candor (C)	3, 2	-1, 4
	disavow(D)	6, -1	0, 0

1. Solve for the minimum SPNE payoff for players, and prove that there is no lower equilibrium payoff.
2. Solve for the minimum discount factor δ^* in SPNE where cooperation can be realized.

3. If the discount factor is lower than δ^* , is there an SPNE, such that the payoff is bigger than $(0, 0)$?

Exercise 7.8 Consider the following infinitely repeated Prisoner's Dilemma game with the stage game:

		player 2	
		candor(C)	disavow(D)
player 1	candor (C)	1, 2	-1, 3
	disavow(D)	2, -4	0, 0

Let the discount factor be δ .

1. Prove: if $\delta < 0.5$, in pure strategy SPNE, the maximum average discounted payoff of players is 0; if $\delta = 0.5$, then there is a pure strategy SPNE, such that the average discounted payoff of players is 1.
2. Now suppose that a player plays the above game with N players, and the payoff in one period is the sum of payoff in all N games. Then, will the result in question 1 change? Explain your answer.

Exercise 7.9 Consider the following infinitely repeated Prisoner's Dilemma, where $2a > b + c$, and the discount factor is δ :

		player 2	
		candor(C)	disavow(D)
player 1	candor(C)	a, a	b, c
	disavow(D)	c, b	0, 0

1. If the grim strategy is an SPNE, then what is the minimum discount factor?
2. Prove that if $\delta = 1$, then the "tit-for-tat" strategy is not an SPNE.

Exercise 7.10 In the Cournot model of n firms, the inverse demand function is $p = 1 - 2q$, and the marginal cost and fixed cost of all firms are zero. Consider the infinitely repeated game of this stage game:

1. Solve for the minimum δ such that a firm can maintain monopoly output through the grim strategy in SPNE.
2. If δ is too small to maintain monopoly output through the grim strategy, then solve for the SPNE, in which the grim strategy leads to profit maximization.

Exercise 7.11 Consider a game between a firm and a labor union. The union determines the wage level, and the firm decides the number of employees. The union's utility function is $u(w, l)$, where w is wage, l is employment, and $u(w, l)$ is an increasing function of w and l . The firm's profit function is $\pi(w, l) = R(l) - wl$, where $R(l)$ is the enterprise's revenue, which is an increasing and concave function. The time sequence of the game is: the union first gives the salary level; the firm observes and accepts the salary level, and then chooses the number of employees. u^* and π^* represent the union's utility and the firm's profit, respectively, by backward induction of a one-shot game. Consider another wage-employment combination (w, l) and the corresponding utility-profit combination (u, π) , where the discount factor for both is δ . Solve for the condition that (w, l) satisfy: (u, π) is Pareto superior to (u^*, π^*) ; and (u, π) is the outcome of a subgame perfect Nash equilibrium for an infinitely repeated game, where as long as any deviation occurs, it permanently shifts to (u^*, π^*) .

Exercise 7.12 (Shapiro and Stiglitz, 1984) Consider the following stage game: in the first stage, the firm sets wage level w for a worker; in the second stage, the worker chooses to accept or reject the wage. If rejected, the worker chooses a self-employment salary of w_0 ; if accepted, the worker chooses to work hard or to be lazy. Hard work will result in a negative utility of f , while laziness does not have a negative effect. The firm cannot observe the worker's effort, but the firm and the worker can observe the level of the worker's output, with a low output of 0 and a high output of $y > 0$. The worker who works hard can inevitably achieve high output, while the lazy worker gets high output with probability p and low output with probability $1 - p$.

Suppose $y - f > w_0 > py$, and consider the following combination of strategies in an infinitely repeated game: the firm's strategy is to set

the wage level w^* for the first stage, and if at each subsequent stage, the outcome of the game is (w^*, y) , then continue the wage level w^* ; otherwise, change the wage level to be $w = 0$; the worker's strategy is if $w \geq 0$, then accept the firm's wage (or choose self-employment), and if the outcome in each stage of the game is (w^*, y) , then work hard (otherwise choose to be lazy). Solve for the conditions under which the above strategies form a subgame perfect Nash equilibrium.

Exercise 7.13 Consider the following infinitely repeated game of two players with the discount factor δ and the stage game:

		player 2	
		candor(C)	disavow(D)
player 1	candor (C)	2, 3	1, 6
	disavow(D)	0, 1	0, 1

Prove that if δ satisfies a certain condition, then $(C, C), (C, C), \dots$ is not the equilibrium path of an SPNE, and solve for the range of δ in this case.

Exercise 7.14 Prove that in a repeated game, if there is an SPNE for each player where the equilibrium payoff is the player's minmax payoff, then every Nash equilibrium payoff is an SPNE payoff.

Exercise 7.15 Prove that if $(w^k)_{k=1}^\infty$ is a sequence of the SPNE payoff profile of an infinitely repeated game with discount factor δ , and it converges to w^* , then w^* is an SPNE payoff profile for this repeated game.

Exercise 7.16 In a symmetric finitely repeated game, suppose that there is a symmetric minmax strategy profile m^* , where the pure strategy m satisfies $\max_{a_i} g(a_i, m_{-i}^*) \leq \underline{v}$. Prove that if public randomization is possible, then for sufficiently large discount factors, the minimum strongly symmetric equilibrium payoff e can be obtained by the following strategy: in state A , players choose m^* ; if players do not deviate in state A , then change to state B ; if players deviate, then remain in state A with probability 1. In state B , the game is played by strategies with maximum equilibrium payoff.

Exercise 7.17 Consider a repeated game between a long-term player 1 and other players in an infinite sequence. Each player in the sequence only exists for one period and knows the previous actions of player 1. Player 1 evaluates the payoff sequence by limit in mean, and any of other players only considers the payoff for the period in which he or she is present.

1. If two players in each period perform the Prisoner's Dilemma game as follows, solve for the set of subgame perfect Nash equilibria of the game.

		player 2	
		candor (C)	disavow (D)
player 1	candor (C)	3, 3	0, 4
	disavow (D)	4, 0	1, 1

2. If the payoff of (C, D) is changed to 0 for each player in each period, prove that for each $x \in [1, 3]$, there is a subgame perfect Nash equilibrium, in which the average payoff for player 1 is x .

Exercise 7.18 Consider the following game:

		player 2		
		a	b	c
player 1	a	4, 4	0, 5	0, 0
	b	5, 0	2, 2	0, 0
	c	0, 0	0, 0	3, 3

1. Suppose that the game proceeds in two periods, and the discount factor is δ . Find a subgame perfect Nash equilibrium, such that for a range of δ , the first stage can achieve the payoff profile of (a, a) . Solve for the range of δ .
2. Suppose that the game repeats $T > 2$ periods, and the discount factor is δ . Find a subgame perfect Nash equilibrium, such that for some range of δ , the payoff profile of (a, a) can be achieved in the previous $T - 1$ stage. Find the range of δ and equilibrium payoffs for each player.

3. Suppose that the game repeats infinite periods, and the discount factor is δ . Is there a subgame perfect Nash equilibrium where for some δ , every $T - 1$ stage can achieve the payoff profile of (a, a) ? If it exists, find the range of δ and the equilibrium payoff for each player; if it does not exist, provide the corresponding proof.

Exercise 7.19 Workers will face off season and peak seasons each year. In the off season, they will receive wage w_* . During the peak season, they will receive wage w^* , and $w^* > w_*$. Workers cannot save, and their utility function u is a strictly concave function defined in each period of consumption. The discount factor between different periods is δ . Each season is a period.

1. Suppose that the wage cannot be used for saving or borrowing. Write the worker's lifelong utility.
2. Suppose that an employer provides a contract (x_*, x^*) to the worker, where x_* and x^* are the worker's wage levels in the off season and the peak season, respectively. Set the utility of the employer as a linear function and assume that the employer keeps its promise to perform the contract, while the worker can choose to break the contract. Once the worker defaults, the employer can continue or cease providing the contract. As a result, the wages of the worker are still determined by w_* and w^* .
 - (a) The relationship between workers and employers is represented by a repeated game. Is the payoff function of each period continuous? Explain your answer.
 - (b) Suppose that the employer provides the contract in the off season. What are the two constraints that need to be met, such that the workers accept and perform the contract?
 - (c) Prove that $\delta^2 u'(w_*) > u'(w^*)$ is the sufficient and necessary condition for the existence of an incentive compatible contract in which both parties can make a profit.

Exercise 7.20 Consider the following stage game:

		player 2	
		<i>C</i>	<i>D</i>
player 1	<i>C</i>	5, 5	3, 6
	<i>D</i>	6, 3	4, 4

This stage game is repeated infinitely, and two players in each period fully know the history of the previous stages. Both have a discount factor $0 < \delta < 1$. Answer the following questions:

1. Consider the following trigger strategy: both parties choose *C* in stage 0. In any subsequent stage game, if the outcome of each previous stage is (C, C) , then both parties continue to choose *C*; otherwise, they choose *D*. Find the range of δ , where the trigger strategy is the subgame perfect Nash equilibrium.
2. Consider another trigger strategy: in the 0-stage game, player 1 chooses *C*, and player 2 chooses *D*. In any subsequent game, if the history of outcomes is the following sequence, i.e., $(C, D), (D, C), (C, D), (D, C), (C, D), (D, C), \dots$, then they continue to follow the sequence (i.e., player 1 chooses *C* in even periods, and chooses *D* in odd periods; player 2 chooses *C* in odd periods, and *D* in even periods); otherwise, they choose *D*. Find the range of δ , where this trigger strategy is a subgame perfect Nash equilibrium.
3. According to the Folk Theorem, as long as δ is sufficiently close to 1, the range of payoff profiles can be achieved by reasonably choosing trigger strategies in an infinitely repeated game. Illustrate this using figures.

Exercise 7.21 Consider the two-player game:

		player 2	
		<i>L</i>	<i>R</i>
player 1	<i>T</i>	2, 3	1, 5
	<i>B</i>	0, 1	0, 1

1. Find the Nash equilibrium of the game.

2. Suppose that this game repeats infinitely. According to the Folk Theorem, what is the limit of average payoffs that can be achieved in subgame perfect Nash equilibrium? Does it depend on the choice of the common discount factor δ ?
3. Suppose that this game repeats infinitely. According to the Folk Theorem, what is the limit of average payoffs that can be realized in Nash equilibrium?

Exercise 7.22 Consider the two-player game:

		player 2	
		L	R
player 1	T	2, 1	0, 0
	B	0, 0	1, 2

1. Suppose that this game repeats infinitely. According to the Folk Theorem, what is the limit of average payoffs that can be achieved in subgame perfect Nash equilibrium?
2. Give a subgame perfect Nash equilibrium that can achieve the limit of average payoff, $(\frac{3}{2}, \frac{3}{2})$, and give the requirement for common discount factor δ .
3. Suppose that this game repeats infinitely. According to the Folk Theorem, what is the limit of average payoffs that can be realized in Nash equilibrium?

Exercise 7.23 Consider the two-player game:

		player 2	
		C	D
player 1	C	3, 3	$k, 1$
	D	1, k	2, 2

Suppose that the new round of probability that the stage game continues is p , and this probability is independent of the number of repetitions of the stage game.

1. If $k = 4$, under what conditions does a “tit-for-tat” strategy form a Nash equilibrium?
2. Consider the definition of a “tit-for-tat” strategy. They choose cooperation in the first round. After that, they adopt the action of the opponent in the previous round (betrayal or cooperation). Prove that if k is sufficiently large, the alternate strategy defined below is better than the “tit-for-tat” strategy. At this time, does the alternate strategy constitute Nash equilibrium? If not, what strategy is a Nash equilibrium?

The **alternate strategy** is defined as follows. Cooperation in the first round. Since then,

if both cooperated with each other last time, then you will defect this time;

if you defected the last time but the other party chose cooperation, then you will cooperate this time;

if you cooperated last time but the other side defected, then you will defect this time;

if both sides defected last time, then you will cooperate this time.

Exercise 7.24 Consider the two-player game:

		player 2	
		C	D
player 1	C	3, 3	1, 4
	D	4, 1	2, 2

1. Suppose that the two players know in advance that the stage game will only repeat for 3 rounds. What is the SPNE of this repeated game?
2. Let this game repeat infinitely. According to the Folk Theorem, what is the limit of average payoffs that can be achieved in SPNE?

Exercise 7.25 Consider the two-player game:

		player 2	
		<i>C</i>	<i>D</i>
player 1	<i>C</i>	3, 3	1, 4
	<i>D</i>	4, 1	2, 2

Describe the following strategies using the automaton representation of strategies, and use Proposition 7.3.2 to verify whether it is a subgame perfect Nash equilibrium.

1. Trigger strategy: both parties first choose to cooperate (*C*). At period t , if the previous participants all chose to work hard (*C*), then they choose to work hard; if anyone had chosen to defect (*D*), then they choose to defect from this time forward.
2. “Tit-for-tat” strategy: both sides choose to cooperate (*C*) in the first round. At period t , both choose the opponent’s action in the previous round (defect or cooperation).
3. Alternate strategy (defined in Exercise 7.23).

Exercise 7.26 (Product Choice Game) Player 1 is the manufacturer, and player 2 is the consumer. The manufacturer can choose either high effort (*H*) or low effort (*L*). The consumer can choose from two products, high-end products (*h*) or low-end products (*l*). The payoff matrix of the game is shown in the table:

		player 2	
		<i>h</i>	<i>l</i>
player 1	<i>H</i>	2, 3	0, 3
	<i>L</i>	4, 0	1, 1

1. If the game is played for a finite period, what is the subgame perfect Nash equilibrium?
2. If the game is played for 3 rounds, is there another Nash equilibrium besides the subgame perfect Nash equilibrium derived in question 1? If yes, give one.

3. Suppose that this game repeats infinitely. According to the Folk Theorem, what is the limit of average payoffs that can be achieved in subgame perfect equilibrium?
4. Suppose that this game repeats infinitely. According to the Folk Theorem, what is the limit of average payoffs that can be realized in Nash equilibrium?

Exercise 7.27 (Product Choice Game) There are two types of manufacturers. The probability of being effort type is p , who can only use strategy H ; the other type of manufacturers is called common type, and the probability of being such type is $1 - p$. The discount factor for both types is δ . The payoff matrixes are as follows:

Common type:

		player 2	
		h	l
player 1	H	2, 3	0, 2
	L	3, 0	1, 1

Effort type:

		player 2	
		h	l
player 1	H	2, 3	0, 2

1. For the common type of manufacturers, if the type is ex-ante public knowledge, what is the strategy of a subgame perfect Nash equilibrium in a finitely-repeated game?
2. Suppose that the type is private information, and the game repeats for 2 periods. Under what conditions can manufacturers of the common type benefit from the existence of effort type? What is the specific strategy? (Hint: the common type can maintain the confidentiality of their type by adopting strategy H in the first period).
3. Suppose that the type is private information, and the game repeats for n periods ($n \geq 3$). What is the equilibrium strategy?

4. If the game repeats infinitely, what is the equilibrium?

Exercise 7.28 (Reputation Mechanism) Consider the chain store model given in this chapter. Suppose that the incumbent has chain stores in 3 markets. Answer the following questions:

1. For market 2, suppose that the rational type incumbent has not revealed her own type in market 1 (fight when facing an entrant). Discuss her strategies for different ranges of q and p_0 . (Hint: it is the same as the case of 2 markets).
2. For market 2, suppose that the rational type incumbent has already revealed her own type in market 1 (did not fight when faced with entry). Then, obviously her subsequent strategy is not to fight for any entry, while tough or weak type potential entrants will choose to enter. Based on this and the conclusion in question 1, use backward induction to discuss the strategy of the rational type incumbent for different ranges of q and p_0 in the market 1. Provide the reasoning process.
3. Give equilibrium strategies for this game, i.e., provide strategies for the rational type incumbent and weak potential entrants in these markets.

Exercise 7.29 (Reputation Mechanism) Consider the general case of the chain store model in this chapter. Suppose that the incumbent has chain stores in N markets, and potential entrants are tough with probabilities q_1, q_2, \dots, q_N , respectively. If the incumbent does not fight, then the benefits of entry are b_1, b_2, \dots, b_N , respectively. The incumbent is rational with probability $1 - p_0$.

1. First, suppose that $q_i = q$ and $b_i = b$ for any $i \in \{1, 2, \dots, N\}$. Use backward induction to find a market equilibrium strategy.
2. Is it true that no matter how small the p_0 is, as long as N is sufficiently large, then the rational type incumbent will always choose to fight in the first few markets to establish a tough reputation? Is there

such a conclusion when q_i and b_i are not exactly equal across these N markets?

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Chapter 8

Cooperative Game Theory

8.1 Introduction

This chapter discusses cooperative games, also known as coalitional games. The so-called coalition is a nonempty subset of players. The basic elements of the non-cooperative game discussed in the previous two chapters are based on each player's actions and preferences for possible outcomes. In cooperative games, the basic elements are coalitions consisting of players and the coalitional actions that they take. Although the actions are chosen by the coalition, it is also based on individual preferences. Similar to the equilibrium solution of the non-cooperative game, the equilibrium of a cooperative game must also satisfy stability, i.e., the stability of stable coalition and outcome.

Compared with non-cooperative games, cooperative games pay more attention to group choices rather than individual choices, and meanwhile ignore details of the interaction within a group. These two types of interactions reflect different strategic considerations. Game theory increasingly looks at the connection between the two, such as providing non-cooperative game foundations for solutions to cooperative games.

8.2 The Core

The core is a basic solution concept to coalitional games. It reflects that there is no other coalition that can bring better outcomes to the players in the coalition, and thus such an outcome is stable. Judging by whether or not payoffs can be transferable among members of the coalition, coalitional games can be divided into coalitional games with transferable payoff and coalitional games without transferable payoff. Our discussion mainly focuses on the former in the current chapter. We will discuss coalitional games without transferable payoff somewhat in detail when examining the core property of competitive equilibrium in Chapter 12.

8.2.1 Coalitional Game with Transferable Payoff

The coalitional game with transferable payoff concerns the payoff obtained by a group of players, and the payoff is allocated among the members without restriction.

Definition 8.2.1 (Coalitional Game with Transferable Payoff) A *coalitional game with transferable payoff* or simply a *coalitional game* consists of the following two elements: a set of players N , and a value $v(S)$ assigned to each coalition (i.e., a non-empty subset of N , $S \subseteq N$). $v(S)$ that is called *characteristic function* can be viewed as the total payoff of the coalition S to be assigned among its members.

A coalitional game with transferable payoff is denoted by $\langle N, v \rangle$. The payoff that a coalition can obtain is usually dependent on the actions of other players, and thus $v(S)$ can be interpreted as the highest payoff that coalition S receives independently of other coalitions $N \setminus S$. We can utilize the strategic form of a non-cooperative game $\Gamma = (N, (C_i)_{i \in N}, (u_i)_{i \in N})$ to discuss the payoff of a coalition in the coalitional game, where C_i is the choice space of player i , and u_i is the utility function of player i . Von Neumann and Morgenstern (1944) defined the characteristic function as:

$$v(S) = \min_{\sigma_{N \setminus S} \in \Delta(C_{N \setminus S})} \max_{\sigma_S \in \Delta(C_S)} \sum_{i \in S} u_i(\sigma_S, \sigma_{N \setminus S}).$$

We usually assume $v(\emptyset) = 0$, where \emptyset denotes the empty set. The definition of coalitional payoff is based on a pessimistic judgment of actions of players outside of the coalition, i.e., they will choose the most unfavorable actions against coalition members. There are also other ways to define coalitional payoff. Different judgment criteria of players' actions outside of the coalition will lead to dissimilar definitions.

Myerson (1991) proposes using defensive-equilibrium representation to define the group's payoff in a coalitional game. The entire group is then divided into the coalition and its complement. Their respective values are equivalent to Nash equilibrium payoffs. Formally, define

$$\begin{aligned}\bar{\sigma}_S &= \operatorname{argmax}_{\sigma_S \in \Delta(C_S)} \sum_{i \in S} u_i(\sigma_S, \bar{\sigma}_{N \setminus S}); \\ \bar{\sigma}_{N \setminus S} &= \operatorname{argmax}_{\sigma_{N \setminus S} \in \Delta(C_{N \setminus S})} \sum_{j \in N \setminus S} u_j(\bar{\sigma}_S, \sigma_{N \setminus S}); \\ v(S) &= \sum_{i \in S} u_i(\bar{\sigma}_S, \bar{\sigma}_{N \setminus S}); \\ v(N \setminus S) &= \sum_{j \in N \setminus S} u_j(\bar{\sigma}_S, \bar{\sigma}_{N \setminus S}).\end{aligned}$$

In addition, Harsanyi (1963) proposed to define characteristic functions by Nash's rational-threats criterion. Here, the payoff to the coalitional group is similar to the idea of coalitional group in Myerson (1991), except that the goal of each group is to maximize the difference between the payoffs of its own and its opponent. Formally, we define

$$\begin{aligned}\bar{\sigma}_S &= \operatorname{argmax}_{\sigma_S \in \Delta(C_S)} \left(\sum_{i \in S} u_i(\sigma_S, \bar{\sigma}_{N \setminus S}) - \sum_{j \in N \setminus S} u_j(\sigma_S, \bar{\sigma}_{N \setminus S}) \right); \\ \bar{\sigma}_{N \setminus S} &= \operatorname{argmax}_{\sigma_{N \setminus S} \in \Delta(C_{N \setminus S})} \left(\sum_{j \in N \setminus S} u_j(\bar{\sigma}_S, \sigma_{N \setminus S}) - \sum_{i \in S} u_i(\bar{\sigma}_S, \sigma_{N \setminus S}) \right); \\ v(S) &= \sum_{i \in S} u_i(\bar{\sigma}_S, \bar{\sigma}_{N \setminus S}); \\ v(N \setminus S) &= \sum_{j \in N \setminus S} u_j(\bar{\sigma}_S, \bar{\sigma}_{N \setminus S}).\end{aligned}$$

We employ an example to illustrate the characteristic function in different meanings.

Example 8.2.1 (Characteristic Function) Consider a three-person coalition game with transferable payoff. Suppose that each player has two actions, $C_i = \{a_i, b_i\}, i \in \{1, 2, 3\}$, which represent “generosity” and “selfishness”, respectively. The payoff matrix for their actions is shown in Figure 8.1.

C_1	$C_2 \times C_3$			
	a_2, a_3	b_2, a_3	a_2, b_3	b_2, b_3
a_1	(4,4,4)	(2,5,2)	(2,2,5)	(0,3,3)
b_1	(5,2,2)	(3,3,0)	(3,0,3)	(1,1,1)

Figure 8.1: Characteristic Function

Under the definition of characteristic function by von Neumann and Morgenstern (1944), we will have:

$$v(\{1, 2, 3\}) = 12, v(\{1, 2\}) = v(\{1, 3\}) = v(\{2, 3\}) = 4, v(\{1\}) = v(\{2\}) = v(\{3\}) = 1.$$

Under the definition of defensive-equilibrium by Myerson (1991), the characteristic function is then:

$$v(\{1, 2, 3\}) = 12, v(\{1, 2\}) = v(\{1, 3\}) = v(\{2, 3\}) = 4, v(\{1\}) = v(\{2\}) = v(\{3\}) = 5.$$

In the sense of rational-threats of Harsanyi (1963), the characteristic

function is:

$$v(\{1, 2, 3\}) = 12, \quad v(\{1, 2\}) = v(\{1, 3\}) = v(\{2, 3\}) = 2, \quad v(\{1\}) = v(\{2\}) = v(\{3\}) = 1.$$

Thrall and Lucas (1963) extended the characteristic function and proposed the concept of partition function, which can deal with the externality between coalitions in a more general framework. The coalitional games discussed later are based on the condition that each coalition has a corresponding characteristic function, thereby placing the focus on what kind of coalition the player will choose. We assume that the coalitional games satisfy the cohesive condition.

Definition 8.2.2 (Cohesive condition) A coalitional game with transferable payoff is said to be *cohesive*, if for each partition S_1, \dots, S_K of the set of all players N , we have $v(N) \geq \sum_{k=1}^K v(S_k)$.

The cohesive condition means that the coalition consisting of all players is optimal.

A stronger condition is the superadditive condition.

Definition 8.2.3 (Superadditive Condition) We say that the characteristic function is *superadditive*, if for any two disjoint subsets S and T , i.e., $S \cap T = \emptyset$, of the set of players N , we have $v(S \cup T) \geq v(S) + v(T)$.

Superadditivity means that if coalitions S and T act together, they can do at least as good as when they act separately.

We next discuss the solution concept of the coalitional game with transferable payoff. The idea is similar to the Nash equilibrium of a non-cooperative game: for a certain outcome, if there is no deviation for improvement, then the outcome is stable. Core is a fundamental equilibrium concept in cooperative games. Core (payoff allocation for all players) means that no coalition can increase the payoffs of its members. In the coalition with transferable payoff, since free transfers can be made among members, a stable condition is that the sum of payoffs obtained by any member in the coalition cannot exceed the sum of payoffs corresponding to the core. Then, we have the concept of feasible payoff allocation below.

Definition 8.2.4 (Feasible Payoff Allocation) Let $\langle N, v \rangle$ be a coalitional game with transferable payoff. For any payoff allocation $(x_i)_{i \in N}$ and any coalition S , define $x(S) = \sum_{i \in S} x_i$. We say that $(x_i)_{i \in S}$ is an S -feasible allocation if $x(S) = v(S)$. We say that $(x_i)_{i \in N}$ is a feasible (payoff) allocation when $S = N$.

Definition 8.2.5 (Core) We say that a feasible allocation $(x_i)_{i \in N}$ is in the core of a coalitional game with transferable payoff, if there exists no coalition S and an S -feasible allocation $(y_i)_{i \in S}$, such that $y_i > x_i$ for any $i \in S$.

Thus, an payoff allocation $(x_i)_{i \in N}$ is in the core of $\langle N, v \rangle$ if and only if $\sum_{i \in S} x_i = v(N)$ and $x(S) \geq v(S)$ for all $S \subseteq N$. We say a coalition S can **improve on** an payoff allocation x if the participants in S can obtain a S -feasible payoff allocation $(y_i)_{i \in S}$ such that $y_i > x_i, i \in S$. Then, if x is in the core, there is no such an improvement.

Remark 8.2.1 In a strict sense, the core defined above should be a weak core. A strong core means that there exist no subset S of N and an S -feasible payoff allocation $(y_i)_{i \in S}$, such that $y_i \geq x_i$ for any $i \in S$ and $y_j > x_j$ for at least one $j \in S$. This is similar to the difference between strong Pareto efficiency and weak Pareto efficiency (see Chapter 11). Obviously, a strong core implies a weak core, but the opposite may not be true. However, under continuous transfers, the concepts of weak core and strong core are equivalent. The transferable payoffs discussed in this chapter are mostly payoffs that can be transferred in a continuous manner, and thus a weak core implies a strong core.

The following example discusses the core of the coalitional game under different rules.

Example 8.2.2 (Coalitional game with collective allocation) There are three players, and a total of 300 units of resources that are available for allocation. Suppose that there are three different allocation rules. Rule 1: the allocation plan must win consent from all of the three players; otherwise, no one will receive any resource. Rule 2: the allocation plan can be passed with majority consent. Rule 3: if all players agree upon the allocation plan, then all of the resources can be allocated; if only two players agree upon the allocation

plan, the resources available for allocation are $2/3$ of the total resources; if only one player agrees upon the allocation plan, then no resource is available for allocation.

Under Rule 1, the coalitional game $\langle N, v \rangle_1$ can be described as $N = \{1, 2, 3\}$, $v(N) = 300$, and if $S \neq N$, then $v(S) = 0$. By the definition of core, every feasible payoff allocation is in the core. This is because for any feasible payoff allocation $(x_i)_{i \in N}$, we have $x_1 + x_2 + x_3 = 0$, and there exists no other feasible payoff allocation $(y_i)_{i \in N}$, such that $y_i > x_i$ for any i .

Under Rule 2, the coalitional game $\langle N, v \rangle_2$ can be described as $N = \{1, 2, 3\}$, and $v(N) = 300$ when $S \subseteq N$ and $|S| \geq 2$ (here the function $|\cdot|$ represents the number counting function); $v(S) = 0$ when $S \subseteq N$ and $|S| = 1$. If a feasible payoff allocation $(x_i)_{i \in N}$ is in the core, then there must exist i , such that $x_i > 0$. However, at this time, there exists a coalition $S = N/i$ satisfying $|S| = 2$ and $x(S) < 300 = v(S)$, and thus $(x_i)_{i \in N}$ cannot be in the core. Consequently, the core is an empty set in this coalitional game.

Under Rule 3, the coalitional game $\langle N, v \rangle_3$ can be described as: when $S = N = \{1, 2, 3\}$; $v(N) = 300$, and $v(S) = 200$ when $S \subseteq N$ and $|S| = 2$; $v(S) = 0$ when $S \subseteq N$ and $|S| = 1$. In this game, $(x_i)_{i \in N} = (100, 100, 100)$ is a unique allocation in the core. The reason for this is that if there exists an i , such that $x_i > 100$, then there must exist a coalition $S = N/i$ satisfying $|S| = 2$ and $x(S) < 200 = v(S)$.

Example 8.2.3 (Transactions in a Market with Indivisible Commodities)

In a market with an indivisible commodity, the set of consumers is denoted by B , and the set of sellers is denoted by L . Each seller has one unit of indivisible commodity. Each consumer can purchase one unit of commodity at most. The reserve prices of the commodity for consumers and sellers are 1 and 0, respectively. For a coalition $S \subseteq B \cup L$, its characteristic function is $v(S) = \min\{|S \cap B|, |S \cap L|\}$. In this game, the payoff allocations of consumers and sellers are denoted by x_b and x_l , respectively. We can verify that: when $|B| > |L|$, only one allocation is in the core, which is given by $(x_i)_{i \in N}$, where $N = B \cup L$, satisfying $x_i = x_b = 0$, $i \in B$; $x_i = x_l = 1$, $i \in L$. When $|B| = |L|$, the set of allocations in the core features

$x_i = x_b = \alpha$, $i \in B$; and $x_i = x_l = 1 - \alpha$, $i \in L$, $\alpha \in [0, 1]$.

In the examples above, the core is not always nonempty. Next, we discuss the conditions for the existence of nonempty cores.

8.2.2 The Existence Theorem on Nonempty Cores

According to the definition of core, if a feasible allocation is in a core, it needs to satisfy a series of inequalities. First, we introduce some related concepts.

The set of all coalitions is denoted by $C = \{S | S \neq \emptyset, S \subseteq N\}$. $1_S \in \mathcal{R}^N$ is called the characteristic vector of coalition S , satisfying

$$(1_S)_i = \begin{cases} 1, & i \in S; \\ 0, & \text{otherwise.} \end{cases}$$

Definition 8.2.6 (Balanced Collection of Weights) $(\lambda_S)_{S \in C}$, $\lambda_S \in [0, 1]$, is called a *balanced collection of weights* if $\sum_{S \in C} \lambda_S 1_S = 1$.

Example 8.2.4 The set of players is $\{1, 2, 3, 4\}$. If $|S| = 3$, $\lambda_S = 1/3$; if $|S| \neq 3$, $\lambda_S = 0$. Then, $(\lambda_S)_{S \in C}$ is a balanced collection of weights. In addition, if $|S| = 1$, $\lambda_S = 1$; if $|S| \neq 1$, $\lambda_S = 0$. Then, such defined $(\lambda_S)_{S \in C}$ is also a balanced collection of weights.

To interpret the balanced collection of weights, we can consider the players' time allocation. Suppose that the total time of player i is 1 unit. His or her time is allocated among all coalitions that include him or her, and the total amount is feasible: $\sum_{S \in C} (1_S)_i \lambda_S = 1$.

Definition 8.2.7 (Balanced Game) A game $\langle N, v \rangle$ is said to be *balanced*, if for each balanced collection of weights $(\lambda_S)_{S \in C}$, we have

$$\sum_{S \in C} \lambda_S v(S) \leq v(N).$$

We can comprehend the balanced game as the allocation of all feasible time of the player, in which, taking time allocation as weights, the sum of payoffs received by the player in all coalitions is less than what he or

she will receive in the biggest coalition that includes all players. Bondereva (1963) and Shapley¹ (1967) characterized the relationship between balanced game and nonempty core based on linear programming and duality theorem.

Theorem 8.2.1 (Bondereva-Shapley Theorem) *A sufficient and necessary condition for the existence of nonempty core in a coalitional game with transferable payoff is that the game is balanced.*

PROOF. Necessity: Let $(x_i)_{i \in N}$ be a payoff allocation in the core, while $(\lambda_S)_{S \in C}$ is one of its balanced collections of weights. Then, $\sum_{S \in C} \lambda_S v(S) \leq \sum_{S \in C} \lambda_S x(S) = \sum_{i \in N} x_i \sum_{i \in S} \lambda_S = \sum_{i \in N} x_i = v(N)$.

The inequality is attributed to the definition of the core; the first equal sign is attributed to different orders of summation of the equivalence; the second equal sign comes from the definition of balanced weights; the last equal sign comes from the definition of feasible payoff allocation.

Sufficiency: $\langle N, v \rangle$ is balanced, and thus there exists no balanced weights $(\lambda_S)_{S \in C}$ satisfying $\sum_{S \in C} \lambda_S v(S) > v(N)$. Therefore, the convex set $\{(1_N, v(N) + \varepsilon) : \varepsilon > 0\}$ and the convex cone $\{y \in \mathcal{R}^{N+1} : y = \sum_{S \in C} \lambda_S (v(S) + 1_S), \forall S \in C, \lambda_S \geq 0\}$ are disjoint. Using the hyperplane separation theorem, there exists a non-zero vector $(a_N, a) \in \mathcal{R}^{N+1}$, such that for any $y, \varepsilon > 0$, we have $(a_N, a)y \geq 0 > (a_N, a)(1_N, v(N) + \varepsilon)$. Since $(1_N, v(N))$ is in the convex cone, this inequality implies that $a < 0$. We construct $x = a_N/(-a)$. In addition, since for any $S \in C, (1_S, v(S))$ belongs to the above convex cone, then from the above inequality, we have $(a_N, a)(1_S, v(S)) = a(-x1_S + v(S)) = a(-x(S) + v(S)) \geq 0$, and thus $x(S) \geq v(S)$. Since for any $\varepsilon > 0$, we have $(a_N, a)(1_N, v(N) + \varepsilon) < 0$ and $(a_N, a)(1_N, v(N)) = a(-x1_N + v(N)) = a(-x(N) + v(N)) \geq 0$. Then, we have $x(N) = v(N)$, and thus the x constructed above is a payoff allocation in the core. \square

In the following, we discuss why some cores exist and some cores may be empty in the previous coalitional game with collective allocation.

Example 8.2.5 (Coalitional Game with Collective Allocation) It is clear under rule 1 that the coalitional game described by $v(S)_{S \in C}$ is a balanced

¹See the biography of Lloyd S. Shapley(1923—2016) in Section 22.5.1.

game, because when $S = N$, $v(S) = 300$; when $S \neq N$, $v(S) = 0$. Therefore, for any $i \in N$, $\sum_{i \in S} \lambda_S 1_S = 1$, we have $\sum_{S \in C} \lambda_S v(S) \leq v(N)$.

Under rule 2, consider the following balanced collection of weights. If $|S| = 2$, then $\lambda_S = \frac{1}{2}$; otherwise, $\lambda_S = 0$. However, $\sum_{S \in C} \lambda_S v(S) = 450 > 300 = v(N)$, and thus the coalitional game under rule 2 is not a balanced game.

Under rule 3, when $|S| = 2$, $v(S) = 200$, $v(N) = 300$; when $|S| = 1$, $v(S) = 0$. At this time, for any balanced collection of weights $(\lambda_S)_{S \in C}$, since it is a balanced collection of weights, it satisfies:

$$\lambda_{\{1,2\}} + \lambda_{\{1,3\}} + \lambda_{\{1,2,3\}} \leq 1,$$

$$\lambda_{\{1,2\}} + \lambda_{\{2,3\}} + \lambda_{\{1,2,3\}} \leq 1,$$

$$\lambda_{\{1,3\}} + \lambda_{\{2,3\}} + \lambda_{\{1,2,3\}} \leq 1,$$

so we have $\lambda_{\{1,2\}} + \lambda_{\{2,3\}} + \lambda_{\{1,3\}} \leq 3 \frac{1 - \lambda_{\{1,2,3\}}}{2}$. $\sum_{S \in C} \lambda_S v(S) = \lambda_{\{1,2,3\}} 300 + (\lambda_{\{1,2\}} + \lambda_{\{2,3\}} + \lambda_{\{1,3\}}) 200 \leq 300 = v(N)$.

To further understand the existence theorem on the core, we now discuss the issue based on linear programming and duality theorem, as in Bondereva (1963) and Shapley (1967).

Consider the following problem: what is the minimum utility transfer required under the constraint that no coalition can improve its members' payoffs? This problem can be expressed as the following linear programming:

$$\begin{aligned} & \min_{x \in \mathcal{R}^N} \sum_{i \in N} x_i \\ \text{s.t.} \quad & \sum_{i \in S} x_i \geq v(S), \forall S \subseteq N. \end{aligned}$$

The duality problem of the above linear programming is:

$$\begin{aligned} & \max_{\lambda \in \mathcal{R}_+^C} \sum_{S \in C} \lambda_S v_S \\ \text{s.t.} \quad & \sum_{S \ni i} \lambda_S = 1, \forall i \in N. \end{aligned}$$

According to the duality theorem of linear programming, if these two

problems have solutions, then they are the same.

8.2.3 Coalitional Game without Transferable Payoff

For a coalitional game without transferable payoff, allocations among its members are not arbitrary. In other words, within each coalition, given its total payoff, not all possible allocations can be implemented in the coalition. As such, for the characteristic function of the coalition, instead of giving a certain value $v(S)$, it gives a set of allocations $v(S)$. We can consider the coalitional game with transferable payoff as one special case, in which $v(S) \equiv \{x \in \mathcal{R}^N \mid \sum_{i \in S} x_i = v(S), x_j = 0, \forall j \in N \setminus S\}$.

A coalitional game without transferable payoff usually includes the following components: the set of players N ; the set of allocations X ; a set $v(S) \subseteq X$ given for any nonempty subset S of N , which can be understood as possible allocations under coalition S ; and the preference relation \succ_i of each player on X .

Accordingly, the core of the coalitional game without transferable payoff $\langle N, X, v(\cdot), \succ_i \rangle$ can be defined as: for all $x \in V(N)$, there exists no coalition $S \subseteq N$ and a feasible allocation y , such that $y_i \succ_i x_i, \forall i \in S$. Scarf (1967) provided the condition for the existence of nonempty cores of the coalitional game without transferable payoff.

In the general equilibrium theory to be discussed in Part IV, the market exchange can be regarded to some extent as the formation of coalitions among players, i.e., the transactions among them in a coalition without transferable payoff. Relevant content will be discussed in depth in Chapter 12.

8.3 Application of the Core: Market Design

In the following, we consider the application of the concept of core and its importance, especially the application of matching theory that will be highlighted in the last part of the textbook. We first consider the exchange of goods (or resources), including the exchange of homogeneous goods and that of heterogeneous goods. The discussion here deals primarily with the

transaction of a single indivisible commodity. The transaction of multiple types of (divisible) commodities are discussed in more detail in the general equilibrium theory in Part IV. We then provide a brief introduction to the problem of matching. Matching theory has numerous applications, including matchings in the marriage market, labor market, etc. Relevant discussions largely concern the problem of equity and efficiency of educational opportunities, especially the application in the reform of the school admission approach. These examples are adopted from Osborne (2004) and Peter (2008). A detailed discussion of the basic results of matching theory and its applications will be presented in the last chapter of the textbook.

8.3.1 Transaction of Homogeneous Goods

Suppose that there are some homogeneous and indivisible goods, such as horses of the same type. Different individuals have different values or reserve prices or willingness to pay for the horses. In addition, in this economy, some individuals have horses, while others do not. We denote the group of individuals who own horses (owners) as L , $|L| = L$, and those without horses (non-owners) as B , $|B| = M$. To simplify the discussion, everyone has at most one horse. At the same time, each individual $i \in N \equiv L \cup B$ has a value v_i for having the first horse and no extra value for having more horses, which means that the demand is at most the unit demand. We rank the values of non-owners for horses from the top to the lowest, β_1, \dots, β_M , and the values of owners for horses from the lowest to the top, $\sigma_1, \dots, \sigma_L$. We denote $k^* = \max\{k | \beta_k > \sigma_k\}$. When $k \leq k^*$, $\beta_k > \sigma_k$, the top k^* highest values of non-owners for horses are higher than the top k^* lowest values by owners. During the transaction, the horses are transferred from the owners to the non-owners. In this way, both parties can benefit when the transaction occurs between high-value non-owners and low-value owners.

We denote the group of individuals who have sold horses (the seller group) as $L^* \subseteq L$, and the group of individuals who do not own horses initially but have purchased horses now as $B^* \subseteq B$. Assume that during the entire process of transaction, $r_i, i \in L$ is the income of the horse seller i ,

and $p_j, j \in B$ is the payment of the horse buyer j . The payoff allocation of players corresponding to this transaction outcome is

$$\mathbf{x} = (\max\{\beta_j - p_j, 0\}, \max\{\sigma_i, r_i\})_{i \in L, j \in B}.$$

We now discuss below what conditions \mathbf{x} should be satisfied to become the core of this transaction.

First, for \mathbf{x} , we must have $p_j = 0$ for $j \in B \setminus B^*$. In other words, for the individual who does not participate in the transaction, his or her payment or income is zero. Obviously, for $j \in B \setminus B^*$, if $p_j > 0$, this means that even though agent j does not participate in the transaction, he still needs to make extra payment p_j . Obviously, this outcome will be improved by the coalition of the economic agent j alone because he does not need to make the extra payment in this way.

If $p_j < 0$, this means that other economic agents need to give agent j an extra positive payment $-p_j > 0$. Obviously, this outcome will also be improved by the coalition of other players that excludes j , because the coalition of other players that excludes j has the same amount of horses and positive revenue $-p_j > 0$ relative to this outcome. They can evenly allocate this amount of money to each player in the coalition, so that the payoff of each member of the coalition can be improved. As a consequence, the only possible outcome is $p_j = 0$. Similarly, we can also determine the income of the owner who is not involved in the transaction (i.e., $i \in L \setminus L^*$) to be $r_i = 0$.

Secondly, for the owners or non-owners who participate in the transaction, the income of each seller and the payment of each buyer must be the same, i.e., $r_i = p_j$ for any $i \in L^*, j \in B^*$. Indeed, this is true. If there is a set (i, j) , such that $r_i < p_j$, then players i and j can form a coalition $\{i, j\}$, who can have the same amount of horses relative to outcome \mathbf{x} , but increased benefits of $p_j - r_i > 0$. This additional increase can be evenly allocated among them, so that the coalition improves the outcome \mathbf{x} . As a result, we must have $r_i \geq p_j$.

Because buyers and sellers of horses are equal in number, the total amount paid to purchase horses must be the same as the total amount of

income from selling horses (because the buying and selling process constitutes a closed system). In other words, $\sum_{i \in L^*} r_i = \sum_{j \in B^*} p_j$, and thus we must have $r_i = p_j = p^*$.

Next, we discuss the value range of p^* . We want to verify that x , which satisfies $k^* = |L^*| = |B^*|$ and $p^* \in [\max\{\beta_{k^*+1}, \sigma_{k^*}\}, \min\{\beta_{k^*}, \sigma_{k^*+1}\}]$ is in the core (See Figure 8.2).

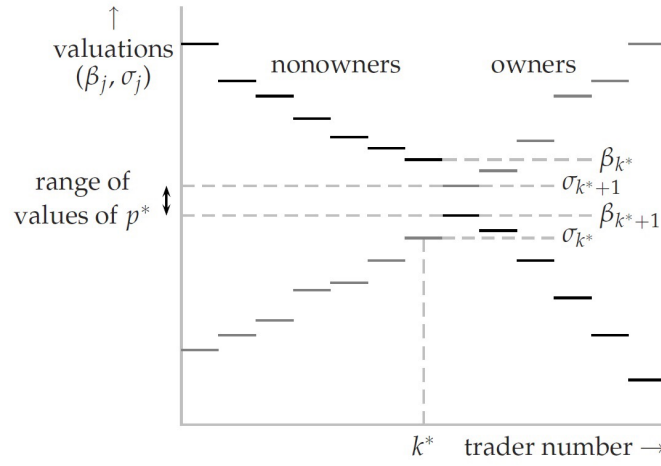


Figure 8.2: The Core of Market Transactions

In a market transaction, for non-owners, if their valuation $\beta_k \geq p^*$, they will participate in the transaction; and for owners, if their valuation $\sigma_k \leq p^*$, they will participate in the transaction. A transaction that maximizes the overall benefit will bring about all profitable transactions. In the previous setting, there are k^* non-owners who have higher values than k^* owners. Therefore, in all transactions of benefit maximization, there are k^* buyers and k^* sellers. In other words, the top k^* non-owners who have the highest values constitute the buyer group, and the top k^* owners who have the lowest values form the seller group. In order to prevent non-owners who have the $(k^* + 1)$ th highest value or below from joining the buyer group, we have $p^* \geq \beta_{k^*+1}$; at the same time, in order to prevent owners who have the $(k^* + 1)$ th lowest value and above from joining the seller group, we have $p^* \leq \sigma_{k^*+1}$. Consequently, the transaction price that maximizes the overall benefits must have $p^* \in [\max\{\beta_{k^*+1}, \sigma_{k^*}\}, \min\{\beta_{k^*}, \sigma_{k^*+1}\}]$. The

outcome of this transaction is:

$$\begin{aligned} x_i &= \max\{v_i, p^*\}, \quad i \in L, \\ x_j &= \max\{v_j, p^*\} - p^* \geq 0, \quad j \in B. \end{aligned}$$

To verify that the outcome x satisfying the above conditions is in the core, it is necessary to show that there are no coalitions whose members' benefits can be improved. For any coalition, the optimal arrangement for members is to allocate the horses to members with the highest valuations and transfer payoffs between the corresponding members, so that each member's payoff can be improved. For coalition S , we denote l as the number of owners in S and b as the number of non-owners in S . Let S^* be the top l members who have the highest values for horses in S , and thus $|S^*| = l$ and $|S \setminus S^*| = b$. When the coalition optimally allocates the horses, the total benefit of the coalition S is $v(S) = \sum_{i \in S^*} v_i$.

For the initial x ,

$$\begin{aligned} x(S) &= \sum_{i \in S} \max\{v_i, p^*\} - bp^* \\ &= \sum_{i \in S^*} \max\{v_i, p^*\} + \sum_{i \in S \setminus S^*} \max\{v_i, p^*\} - bp^* \\ &\geq \sum_{i \in S^*} v_i = v(S). \end{aligned}$$

Since the above coalition S is arbitrary, x is in the core.

8.3.2 Matching of Heterogeneous Goods

In the following, we discuss the exchange of indivisible items, such as the allocation problems of houses and offices. These problems are called the housing market problem in matching theory. For a more formal and rigorous discussion, see Section 22.3.1.

Now, we consider a group of individuals, each of whom owns a house. The houses are different. The values of the houses are also different to different players. If we do not consider the monetary factor, i.e., there are no transfers, then what would be a stable allocation that can maximize the wel-

fare of the individuals? A stable allocation means that there is no coalition that can improve the situation of its members through exchanges within the coalition. If an allocation does not maximize individuals' welfare, it is possible to improve their welfare by forming coalitions to obtain new allocations. The concept of core happens to possess such property.

In the previous section, we already have the existence theorem of core. However, in reality, what we need more is to determine how to find the specific allocations in the core. In the exchange of indivisible heterogeneous items where money is not the medium of exchange, there is an algorithm that can be utilized to find a core allocation in finite steps. This method is called the top trading cycle mechanism. It first appeared in Shapley and Scarf (1974),² but they gave credit for it to David Gale.

The top trading cycle mechanism can be described as follows. In step 1, everyone ranks all goods in order; everyone's most preferred good is owned by someone in this group, and everyone's most preferred good is different from others' (thus constituting strict orderings of goods). Let every one point to the owner of his or her most preferred good. Since there are only finitely many participants, there will be cycle(s) which are called top trading cycles. Note that a participant pointing to him or herself also constitutes a cycle. Let participants in cycles exchange and remove them. In step 2, with the remaining participants and goods, rank participants' preference for the goods and search for another top cycle. Subsequently, in each step, participants and goods in previous cycles are removed, until all of the participants and items have participated in top trading cycles (of different steps).

We now argue that there exist top trading cycles for all exchanges that involve a finite number of participants and goods. Let $N = \{1, \dots, n\}$ denote the set of players; for player i , the initial endowment owned by him or her is denoted by h_i ; the set of all initial endowments is H . In order to simplify the discussion, it is assumed that participant i 's preference for the set of goods is strict, i.e., player i is not indifferent between any two items, and is denoted by \succ_i . In this way, we rule out the possibility of a tie. In the

²For the biographies of Lloyd S. Shapley (1923-2016) and Herbert Scarf (1930-2015), see Sections 22.5.1 and 12.5.2, respectively.

case of dealing with indifferent preference, more sophisticated techniques are required. Participant i can rank the items in set H in order of preference from the highest to the lowest. If $|\{h' \in H | h' \succ_i h\}| = k - 1$, which means for player i , only $k - 1$ goods are preferred to h in the set of goods, then participant i is ranked k in h , denoted by $h = R_i(k)$.

Definition 8.3.1 (Top Trading Cycle) We say that $\{i_1, \dots, i_K\}$ constitutes a K -loop top trading cycle, if for any $k < K$, we have $h_{i_{k+1}} = R_{i_k}(1)$ and $h_{i_1} = R_{i_K}(1)$.

In the following, we show that if every agent has only one good, then there must be a top trading cycle.

First, consider $N = 2$. If $i \in \{1, 2\}$ and $h_i = R_i(1)$. Obviously, $\{i\}$ is a top trading cycle; otherwise, $h_1 = R_2(1)$ and $h_2 = R_1(1)$ must hold, and thus $\{1, 2\}$ is a top cycle.

Next, consider $N = 3$. If $i \in \{1, 2\}$ and $h_i = R_i(1)$, obviously, $\{i\}$ is a top cycle; otherwise, for player 1, we have $h_2 = R_1(1)$ or $h_3 = R_1(1)$. When $h_2 = R_1(1)$, consider the preference list of player 2. If $h_1 = R_2(1)$, then $\{1, 2\}$ is a top cycle; if $h_3 = R_2(1)$, consider the preference list of player 3. If $h_2 = R_3(1)$, then $\{2, 3\}$ is a top cycle; if $h_1 = R_3(1)$, then $\{1, 2, 3\}$ is a top cycle. When $h_3 = R_1(1)$, we will obtain similar results. Therefore, when $N = 3$, a top cycle exists, as well.

Mathematical induction shows that within finitely many individuals, if each agent owns only one good, then there must be a top trading cycle. Of course, we can also relax the assumption of each agent having only one good and let agents have different numbers of items.

We next examine why the outcome of such a top trading cycle mechanism is a core allocation. As we know, if an outcome is a core allocation, then there exists no improvable coalition. During the operation of the top trading cycle mechanism, it is impossible for any individual in top trading cycles of the first step to improve his or her welfare through any other allocation. As a consequence, coalitions that are likely to improve welfare must not include participants in top trading cycles of the first step. Second, for the participants in top trading cycles of the second step, no allocation can possibly improve their welfare by reallocation in the coalition without

the participation of the individuals in top trading cycles of the first step. Therefore, if all of the individuals in top trading cycles of the first step do not participate in a certain coalition, the individuals in top trading cycles of the second step will also not participate in the coalition. By analogy, if participants in top trading cycles of previous steps do not participate in a coalition, this coalition will not improve the welfare of participants in top trading cycles of the current step. As a result, there is no coalition that can improve the welfare of its members.

In the following, we discuss the top trading cycle mechanism through an example.

Example 8.3.1 (Exchange of Houses) Consider a group of four members. The house owned by player i is denoted by h_i . The values of houses by each player are shown in Figure 8.3.

Player 1	Player 2	Player 3	Player 4
h_2	h_1	h_1	h_3
—	—	h_2	h_2
—	—	h_4	h_4
—	—	h_3	—

Figure 8.3: The Top Trading Cycle of the First Step

Each player's ordering of houses is from top to bottom, and the horizontal lines “—” appearing in the figure indicate that these parts of orderings can be arbitrary, which is not our concern here.

In the first step, $\{1, 2\}$ constitutes a top trading cycle, while player 1 and player 2 make exchanges. In the second step, player 1 and player 2

have been removed, and player 3 and player 4 remain. Their preferences are shown in Figure 8.4.

Player 3	Player 4
h_4	h_3
h_3	h_4

Figure 8.4: The Top Trading Cycle of the Second Step

In the second step, $\{3, 4\}$ constitutes a top trading cycle, and thus player 3 and player 4 exchange. The outcome of the whole exchange is that player 1 owns the house of player 2, player 2 owns the house of player 1, player 3 owns the house of player 4, player 4 owns the house of player 3, and there does not exist any improved coalition for this allocation outcome.

For a more detailed discussion of the top trading cycle, see one-sided matching in Chapter 22, where we present a systematic discussion of the matching mechanism of players and indivisible goods, as well as the efficiency and incentive characteristics of different mechanisms. In addition, the one-sided matching mechanism has a broad range of applications in school admissions (Abdulkadiroglu and Sonmez, 2003) and organ transplantation (Roth, Sonmez and Unver, 2004). These problems are discussed in depth in Chapter 22.

8.3.3 Two-sided Matching: Marriage Market

Gale and Shapley (1962) published a paper in *The American Mathematical Monthly* that discussed the matching problem in the marriage market, and opened up an entirely novel field of research, i.e., matching among different groups. This mechanism has a very wide range of applications, such as

matching between companies and laborers in the labor market, matching between hospitals and interns, matching between universities and scholars in the realm of education, matching between donees and donors in the field of organ donation, etc. Roth and Sotomayor (1992) and Abdulkadiroglu and Sonmez (2013) made comprehensive reviews of relevant literature in two different periods of time.

Here, we introduce one of the simplest matching problems, i.e., one-to-one matching, such as the matching of men and women in the marriage market (e.g., “Blind Dating” shows frequently seen on television).

We assume that there are two groups that correspond to two sets of agents: $M = \{m_1, \dots, m_n\}$ and $W = \{w_1, \dots, w_k\}$, one for men and the other for women. The preference of member i is defined on the opposite set of agents and him or herself. To simplify the discussion, assume that the preference is strict, i.e., there exist no agents that are indifferent and denoted by \succ_i .

Definition 8.3.2 (Matching) A mapping $\mu : M \cup W \rightarrow M \cup W$ is called a *matching*, if

- (1) for all $i \in M$, $\mu(i) \in W \cup \{i\}$,
- (2) for all $j \in W$, we $\mu(j) \in M \cup \{j\}$,
- (3) $\mu(i) = j$ implies that $\mu(j) = i$.

We can understand that, in the matching of marriage, the matching of a man is either a woman, or himself (which can be understood as being single), and the matching of a woman is similar. We say that for a man $m \in M$, a woman w is **unacceptable** if $m \succ_m w$. We discuss below what kind of matching μ is stable.

Definition 8.3.3 (Stable Matching) A matching μ is *stable* if it satisfies the following conditions:

- (1) there exists no pair (m, w) with $m \in M$ and $w \in W$, such that $w \succ_m \mu(m)$ and $m \succ_w \mu(w)$;
- (2) for $i \in M \cup W$, if $\mu(i) \neq i$, then $\mu(i) \succ_i i$.

Stable matching means that if the mate of an agent is not him or herself, then the mate is surely acceptable to him or her; at the same time, there are no two agents of the opposite groups who would both rather have each other than their current matching mates. The stability of matching is consistent with the concept of the core. First, if one's matching is unacceptable, then according to the definition of the core, the coalition of one's own can improve his or her benefit. Second, for the matching problem, there are only two types of meaningful coalitions: one is a coalition of one agent, and the other is a coalition of a man and a woman.

In the matching, we rule out the possibility of polygamy, polyandry, or group marriages (shared husbands and wives), i.e., a coalition of multiple men and multiple women. Consequently, in a coalition formed by a man and a woman in the matching problem, stable matching is consistent with the core. However, the pertinent question is, how can we find the stable matching? Gale and Shapley (1962) proposed a deferred acceptance algorithm. The deferred acceptance algorithm introduced here is from Roth (2010).

There are two steps for each stage. We start with the first stage.

In the first step, each agent in the proposing group (e.g., a male agent) proposes to his most preferred choice (e.g., a female agent) in the proposed group (if there is anyone acceptable; otherwise, no proposal is made).

In the second step, each agent in the proposed group first removes the proposals of those unacceptable agents. If there are any remaining ones, choose the most preferred and reject the rest.

At stage k : an agent in the proposing group who was rejected at stage $k - 1$ can propose to his or her most preferred agents among the acceptable ones who have not yet rejected him or her. If no acceptable choice remains, he or she makes no proposal. Each agent of the proposed group chooses the most preferred agent after comparison between the retained proposal in the last stage and the new proposals (if any) received at the current stage and reject the rest.

Stop stage: no new proposal occurs. In this stage, the agent in the proposed group is matched to the agent who he or she has retained. If an agent of the proposing group does not receive any acceptance, or an agent of the proposed group does not receive any offer, he or she is matched with him or herself.

Gale and Shapley (1962) proved that there always exists a stable matching in the marriage market. We will discuss in detail the logic behind this in Chapter 22. Next, we use an example to understand the operation of the deferred acceptance algorithm.

Example 8.3.2 (Men Proposing Deferred Acceptance Algorithm) Consider marriage matching between three men and three women. Each agent's preference for agents in the opposite side is as follows (any unacceptable man/woman is removed from the ordered list of preference):

$$\begin{aligned} p(m_1) &= w_2, w_1, w_3; & p(w_1) &= m_1, m_2, m_3; \\ p(m_2) &= w_1, w_2, w_3; & p(w_2) &= m_3, m_1, m_2; \\ p(m_3) &= w_1, w_2, w_3; & p(w_3) &= m_1, m_2, m_3. \end{aligned}$$

The procedure for the deferred acceptance algorithm is shown in the table below, in which the underlined proposals are being held without commitment.

Stage	w_1	w_2	w_3
1	<u>m_2, m_3</u>	<u>m_1</u>	
2		$m_1, \underline{m_3}$	
3	<u>m_1, m_2</u>		
4		$m_2, \underline{m_3}$	
5			<u>m_2</u>

The marriage matching outcome between men and women is:

$$\mu_M^{DA} = \begin{pmatrix} w_1 & w_2 & w_3 \\ m_1 & m_3 & m_2 \end{pmatrix}.$$

It is easy to verify that this matching outcome satisfies the stability condition.

Of course, with different groups as the proposing group, the matching outcome of the deferred acceptance algorithm may be dissimilar. For instance, the matching outcome with male (or female) as the proposing group is the best matching outcome for the male (or female) group in all possible stable matchings. Simultaneously, the set of stable matchings coincides with the set of cores. In addition, stable matching is not strategy-proof (by manipulatively reporting preference information) (Roth, 1982). These conclusions are discussed in detail in Chapter 22.

Environmental changes or information asymmetry will cause the rupture of some initial stable matchings, and thus matching can also be highly dynamic. If employment is viewed as matching between firms and workers, while unemployment is regarded as the rupture of matching, then we can use the matching method to study employment and unemployment.

In many matching problems, there exists a medium of exchange, such as money. Moreover, the matching process may be accompanied by some contracts, such as labor contracts in the matching between workers and firms that provide terms, including wages, duties, etc. Kelso and Crawford (1982) introduced contracts in the analytical framework of matching, and this analytical framework is combined with the auction mechanism (Hatfield and Milgrom, 2005). For a more in-depth discussion of these issues, see Chapter 22.

We next discuss the concept of other stable allocations in the cooperative game.

8.4 Stable Set, Bargaining Set, and Shapley Value

The stability of the core comes from blocking any deviations. However, the deviation itself may be unstable, and the deviation can lead to new deviations. The initial deviation may also bring worse outcomes to the deviant. Therefore, the concept of stability needs a reasonable justification. In the following, we will further explore stability in the cooperative game by con-

straining the deviation. We will introduce some related concepts based on different constraints, including stable set, bargaining set, and the Shapley value. Here, we focus on the type of coalitional game with transferable payoff. Furthermore, the bargaining mechanism plays an important role in the process of formation of coalition. Ray (2006) employed the method of (cooperative and non-cooperative) game theory to discuss in depth the mechanism of coalition formation.

8.4.1 Stable Set

Von Neumann and Morgenstein (1944) proposed the concept of the stable set. This concept is related to the negotiation process. Assume that a feasible allocation disappoints some players who, therefore, may propose a blocking that is favorable for them. If their blocking is inherently unstable, there will be subsequent reactions. If the final stable outcome is not as good as the previous allocation, then it is not a meaningful deviation to the coalition. In other words, what a coalition S puts forward should be a credible blocking. The so-called credibility means that the blocking is stable, i.e., no new blocking or possibly chain of blockings will be triggered, such that eventually some members of the coalition are worse off than previously.

In this way, a set of stable outcomes need to satisfy two conditions: first, for each unstable outcome, there is a coalition that can present a credible blocking; second, for any stable outcome, there is no other credible blocking. Therefore, the solution concept is discussed in terms of sets. By the concept of stable set, we can divide the feasible outcomes into a stable set and an unstable set.

Definition 8.4.1 (Blocking of Coalition) An allocation x is called a *blocking* of coalition S against allocation y if for any $i \in S$, we have $x_i \succ_i y_i$ and $x(S) \leq v(S)$, denoted by $x \succ_S y$.

Definition 8.4.2 (Stable Set) We say that a subset Y of the feasible allocation set X of the coalitional game with transferable payoff $\langle N, v \rangle$ is a *stable set*, if it satisfies the following two conditions:

- (1) (Internal Stability) No allocation $y \in Y$ is blocked by any allocation $z \in Y$.
- (2) (External Instability) Every $z \in X/Y$ is blocked by some allocation $y \in Y$.

Von Neumann and Morgenstein explained that each stable set constitutes a behavioral pattern, and the allocations of different stable sets correspond to different behavioral patterns.

The following proposition summarizes the relationship between the core and the stable set, and the property of the stable set.

Proposition 8.4.1 *The core is a subset of every stable set; no stable set is a proper subset of another stable set; if the core forms a stable set, then it is the only stable set.*

In the following example, the core is empty, but there are multiple stable sets.

Example 8.4.1 (Continued from Example 8.2.2) There are three players, and a total of 300 units of resources that are available for allocation. The rule is: the allocation plan can be passed with majority consent. The expression of their coalition: for coalitional game $\langle N, v \rangle$, $N = \{1, 2, 3\}$, if $S \subseteq N$ and $|S| \geq 2$, then we have $v(S) = 300$ (here the function $|\cdot|$ represents the number counting function); if $S \subseteq N$ and $|S| = 1$, then we have $v(S) = 0$.

$Y_1 = \{(150, 150, 0), (150, 0, 150), (0, 150, 150)\}$ is a stable set. For this stable set, its behavioral pattern is that the coalition members allocate the coalition income evenly. Firstly, for each element of Y_1 , there exists no deviation in Y_1 . For any $z \in X/Y_1$, where $X = \{(x_1, x_2, x_3) | x_i \geq 0, x_1 + x_2 + x_3 = 300\}$, obviously there exists $i \neq j$, $z_i < 150$, $z_j < 150$ and a coalition $S = \{i, j\}$, such that in Y_1 , $(y_i = y_j = 150, y_k = 0)$ is a blocking in Y_1 to z .

$Y_{k,c} = \{(y_i)_{i \in N} | y_k = c \in [0, 300], \forall i \neq k, y_i \geq 0, \sum_{i \in N} y_i = 300\}$ is also a stable set. Its behavioral pattern is that the player k gets a fixed value c . Therefore, there are infinite stable sets in this coalitional game.

Next, we verify that $Y_{k,c}$ is a stable strategy. Firstly, the outcome $z_k > c$ will be improved with an outcome y in $Y_{k,c}$ by the coalition $\{i, j\}$, $i \neq k, j \neq$

k formed by other players, for example, $\mathbf{y} = (y_k = c, y_i = z_i + \frac{z_k - c}{2}, y_j = z_j + \frac{z_k - c}{2})$. If $z_k < c$, then there always exists an $i \neq k$, such that $z_i > 0$, and thus $\{k, j\}, j \neq k, j \neq i$ can be improved with any outcome \mathbf{y} in $Y_{k,c}$, for example, $\mathbf{y} = (y_k = c, y_i = 0, y_j = 300 - c)$. In addition, for any outcomes in the set $Y_{k,c}$, there does not possibly exist any coalition, such that the choice of other outcomes within $Y_{k,c}$ will improve the situation of the coalition members.

8.4.2 Bargaining Set, Kernel, and Nucleolus

We can also utilize the bargaining method to restrict blocking in the negotiation process. Under this method, the deviation chain generated by each blocking is divided into two steps. The stability condition means that for each blocking, there is a counter-blocking constraint. The idea here is that, if a certain set is stable, then there exists no stable blocking. Stability here is reflected in the way that if a player raises a blocking against other players and the blocking increases the player's payoff, then players suffering a loss because of this blocking can always put forward a counter-blocking against the previous objector. Therefore, the previous blocking is not a stable blocking, because the previous objector's payoff will eventually become worse.

In the following, we discuss the two stages of three types of blocking and counter-blocking coalitions. Different types correspond to the solution concepts of different stable outcomes.

The first type of blocking and counter-blocking:

Blocking: We say that a 2-tuple (\mathbf{y}, S) is a **blocking** to \mathbf{x} that player i proposes against player j , if: $i \in S, j \notin S$, \mathbf{y} is a feasible payoff allocation for S that improves the payoff of each coalition member, i.e., $\forall k \in S, y_k > x_k$.

Counter-blocking: We say that a 2-tuple (\mathbf{z}, T) is a **counter-blocking** that player j proposes against player i 's blocking (\mathbf{y}, S) , if $j \in T, i \notin T$, and \mathbf{z} is a feasible payoff allocation for T , such that $\forall k \in T \setminus S, z_k \geq x_k$, and $\forall l \in T \cap S, z_l \geq y_l$. In other words, for members who are in the counter-blocking

coalition T but not in blocking coalition S , their payoffs under counter-blocking will not be worse than under the initial allocation. For members who are in both blocking and counter-blocking coalitions, their payoffs will not be worse than the allocation under blocking. These are the necessary conditions for the members of T to join the coalition.

The second type of blocking and counter-blocking:

Let $e(S, x) = v(S) - x(S)$, which we call the **excess** of coalition S relative to allocation x . When $e(S, x) < 0$, it reflects the gain of coalition S , and when $e(S, x) > 0$, it reflects the sacrifice made by the coalition.

Blocking: We say that a coalition S is a blocking to x that player i proposes against player j , if $i \in S$, $j \notin S$, and $x_j > v(j)$. This condition reflects the fact that the player j gets more resources $x_j > v(j)$ in allocation x than in his or her own coalition, and the coalition S excludes j to reduce his or her benefit.

Counter-blocking: We say that a coalition T is a counter-blocking that player j proposes against player i 's blocking S , if $i \notin T$, $j \in T$, and $e(T, x) > e(S, x)$. This condition reflects that player j can find a counter-blocking coalition T that includes him or herself but does not include player i , such that the counter-blocking coalition which supports the allocation x can get less or sacrifice more.

The third type of blocking and counter-blocking:

Blocking: We say that a 2-tuple (y, S) is a blocking against feasible allocation x , if $e(S, x) > e(S, y)$, i.e., $y(S) > x(S)$. This means that the blocking coalition S obtains more in the blocking allocation y than in the initial allocation x .

Counter-blocking: We say that a coalition T is a counter-blocking against blocking (y, S) , if $e(T, y) > e(T, x)$, i.e., $x(T) > y(T)$, and $e(T, y) > e(S, x)$. This means that the counter-blocking coalition T can get more in allocation x than the

initial allocation y , while the counter-blocking coalition T sacrifices more to support the blocking allocation y than the blocking coalition S does in the initial allocation x .

The above two conditions are for the blocking coalition that deviates from the stable allocation, but not for any specific member of it.

These three types of different blocking and counter-blocking coalitions can form three associated stable solution concepts.

Definition 8.4.3 (Bargaining Set) We say that a set of coalitional game with transferable payoff is a *bargaining set*, if: (1) its element x is a feasible allocation; and (2) under the first type of blocking and counter-blocking, for a blocking (y, S) of any player i to x against another player j , there always exists a counter-blocking of player j against player i 's blocking (y, S) .

Definition 8.4.4 (Kernel) We say that a coalitional game with transferable payoff is a *kernel*, if: (1) its element x is a feasible allocation; and (2) under the second type of blocking and counter-blocking, for a blocking S of any player i to x against another player j , there always exists a counter-blocking of player j against player i 's blocking (y, S) .

For any two players i and j and any allocation x , define $s_{i,j}(x) = \max_{S \in C} \{e(S, x) : i \in S, j \notin S\}$, where $C = \{S : S \neq \emptyset, S \subseteq N\}$, which is the maximum surplus of coalition S including i , but not including j . The following definition of the kernel is in accordance with the above. If x is a feasible kernel element of N , then for any pair of players i and j , either $s_{j,i}(x) \geq s_{i,j}(x)$, or it satisfies $x_j = v(\{j\})$ for all $j \in N$.

The kernel models a stable arrangement of the group, making each member have the following collective logic about the allocation x in it: if player i proposes a blocking to the allocation x and establishes a coalition S including him or herself, then the coalition excludes player j who receives more benefits under the initial allocation than the payoff of his or her own. The blocking was raised because player i was dissatisfied with the payoff from the previous allocation x . Player j could raise a counter-blocking

and create a coalition T that includes him or herself (player j), but does not include the blocker (player i). This coalition sacrifices more or gains less than the blocking coalition S under the allocation x . In other words, if the initial allocation x is blocked, players can obtain more than in the previous blocking coalition by forming the counter-blocking coalition T .

Definition 8.4.5 (Nucleolus) We say that a coalitional game with transferable payoff is a *nucleolus* if: (1) its element x is a feasible allocation; and (2) under the third type of blocking and counter-blocking, for a blocking (S, y) to x , there always exists a counter-blocking to block (S, y) .

The idea of the nucleolus is closely related to the idea of the kernel. If the coalition S is not satisfied with allocation x , measured by $e(S, x)$, the coalition believes that it contributes too much. In the kernel, blocking is raised by one of the players; whereas, in the nucleolus, it is raised by the coalition. A blocking (S, y) can be interpreted as such opinion of the coalition: “In allocation x , our contribution is too big, and thus we propose a relatively less contributing allocation y .” The nucleolus depicts that one of the other coalitions, T , can raise such a counter-blocking: “Your opinion is not justified because we have contributed more in allocation y than in allocation x , and our contribution in allocation y is more than what you (i.e., the coalition S) contribute in allocation x .” Therefore, the idea of the nucleolus reflects the concept of equity, i.e., how much contribution players should make in the group.

In the following, we discuss the allocations (sets) corresponding to these three concepts through some examples.

Example 8.4.2 (Continued from Example 8.2.2 (2)) There are three players, and a total of 300 units of resources that are available for allocation. For coalitional game $\langle N, v \rangle$, $N = \{1, 2, 3\}$, if $S \subseteq N$ and $|S| \geq 2$, then we have $v(S) = 300$; if $S \subseteq N$ and $|S| = 1$, then we have $v(S) = 0$.

First, we solve for the bargaining set of the coalitional game. In this example, the set is a singleton whose allocation is $(x_i = 100, \forall i \in N)$. To see this, let feasible allocation x be in the bargaining set. If (y, S) is a blocking of player i to x against player j , then he or she will propose

$S = \{i, h\}, i \neq h \neq j$, satisfying: $y_h > x_h, y_i > x_i$, and $y_h \leq 300 - y_i$, getting $y_h \leq 300 - x_i$, and thus player i and player h receive more benefits in the blocking coalition. Since x is in the bargaining set, player j can always propose a counter-blocking (z, T) . Let $T = \{j, h\}$, and z satisfies: $z_h \geq y_h, z_j \geq x_j$, and $z_h \leq 300 - z_j$. Therefore, we have: $z_h \leq 300 - x_j$. Thus, if x is a bargaining set, then as long as $y_h < 300 - x_i$, we will have $y_h \leq 300 - x_j$; otherwise, $z_h \geq y_h > 300 - x_j$. Therefore, $300 - x_i \leq 300 - x_j$, i.e., $x_j \leq x_i$. Since i and j are arbitrary, we have $x_i = x_j = x_h$ and $x_i + x_j + x_h = 300$. Consequently, the only bargaining set is a singleton, i.e., the allocation $(x_i = 100, \forall i \in N)$.

Suppose that the feasible allocation x is the kernel. Then, it can always be ranked as $x_i \geq x_j \geq x_h, i \geq j \geq h$. We now verify that if one of the above inequalities is a strict inequality, then the allocation is not possible in the kernel. If at least one of the above is a strict inequality, then $x_i > x_j$, and $x_i > 100 > 0 = v(i)$. For the blocking of player j to x against player i , in coalition $S = \{h, j\}, h \neq i$, we have $s_{ji}(x) = e(S, x) = 300 - x_j - x_h$, and thus there exists no counter-blocking $T = \{i, h\}, h \neq j$ of i to the blocking of j because $s_{ij}(x) = e(T, x) = 300 - x_i - x_h \leq S_{ji}(x)$. Therefore, the allocation in the kernel must satisfy $x_i = 100, \forall i \in N$.

For nucleolus: consider feasible allocation $x_i \geq x_h \geq x_j$. If at least one is a strict inequality, then $x_i > x_j$ and $x_i > 100$. Consider a blocking to $x, S = \{j, h\}$, and $y = (100, 100, 100)$. Since $e(S, y) = 300 - 200 = 100 < 300 - x_j - x_h = x_i = e(S, x)$, there exists no counter-blocking, i.e., coalition T , such that $e(T, y) > e(T, x)$ and $e(T, y) > e(S, x)$.

$|T| \neq 3$; otherwise, $e(T, y) = e(T, x) = 0$. Meanwhile, $|T| \neq 1$; otherwise, $e(T, y) \leq 0$. If $|T| = 2$, then $e(T, y) = 300 - 200 < e(S, x)$. Therefore, $x_i > x_h$ cannot be in the nucleolus. It is easy to verify that $y = (100, 100, 100)$ is in the nucleolus.

Note that the sets of solutions for these three concepts are not necessarily identical. The following example shows that the kernel is a proper subset of the bargaining set.

Example 8.4.3 (Simple Game) A simple game refers to a coalitional game with transferable payoff whose characteristic value is either 1 or 0. Consid-

er the following simple game consisting of four players, $N = \{1, 2, 3, 4\}$, if and only if $S = \{2, 3, 4\}$, or $\{1, i\} \subseteq S$, for any $i \in \{2, 3, 4\}$. In this game, we can verify that no core exists. Player 1 is in a stronger position relative to other players; other than him or her, every player is in equal position. Do the allocations under the three solution concepts (i.e., bargaining set, kernel, and nucleolus) discussed above also reflect their positions?

Firstly, we discuss the bargaining set. If x is an element in it, we must have $x_2 = x_3 = x_4$; otherwise, for $i, j \in \{2, 3, 4\}$, we have $x_i < x_j$. Then, player i can propose a blocking against j , i.e., $(T = \{1, i\}, y)$, satisfying $y_1 = 1 - y_i$, $y_i = x_i + \frac{x_j - x_i}{2}$, and player j does not have a corresponding counter-blocking, and thus we must have $x_2 = x_3 = x_4 = \alpha$. In addition, α also has upper and lower bounds. If α is high, player 1 will have a credible blocking. For example, the player may raise a blocking $(y, S = \{1, 3\})$ against player 2, $y_1 > 1 - 3\alpha$, $y_3 = 1 - y_1 < 3\alpha$, and $y_j > \alpha$. If player 2 cannot raise a counter-blocking $(z, T = \{2, 3, 4\})$, $z_2 \geq \alpha$, $z_3 \geq y_3$, and $z_4 \geq \alpha$, then it must satisfy $\alpha + 3\alpha + \alpha > 1$, i.e., $\alpha > \frac{1}{5}$. Meanwhile, α cannot be too low; otherwise, player 2 can propose a blocking against player 1, i.e., $(y, S = \{2, 3, 4\})$, $y_2 > \alpha$. As such, $j, k \in \{3, 4\}$, and $y_j \leq y_k$, and thus we must have $y_j < \frac{1-\alpha}{2}$. If player 1 cannot propose a counter-blocking $(z, T = \{1, j\})$, then it satisfies $z_1 \geq 1 - 3\alpha$ and $z_j \geq y_j$, and we must have $1 - 3\alpha + \frac{1-\alpha}{2} > 1$, i.e., $\alpha < \frac{1}{7}$. As a consequence, the bargaining set is $\{(1 - 3\alpha, \alpha, \alpha, \alpha) : \frac{1}{7} \leq \alpha \leq \frac{1}{5}\}$.

Now, we discuss the set of the kernel. Firstly, if allocation x is in the kernel, we must have $x_2 = x_3 = x_4$; otherwise, without loss of generality, suppose that $x_2 \geq x_3 \geq x_4$ and $x_2 > x_4$. Thus, $x_2 > 0 = v(\{2\})$, player 4 can propose a blocking against player 2, i.e., $s_{4,2}(x) = e(\{1, 4\}, x) = 1 - x_4$, such that player 2 cannot propose a credible counter-blocking. This is because, for $s_{2,4}(x) = e(\{1, 2\}, x) = 1 - x_2 < s_{4,2}(x)$, we must have $x_2 = x_3 = x_4 = \alpha$ and $x_1 = 1 - 3\alpha$. Secondly, if allocation x is in the kernel, then we have $x_1 = \frac{2}{5}$ and $x_2 = x_3 = x_4 = \frac{1}{5}$. This is because when allocation x is in the kernel, if $x_2 > v(\{2\}) = 0$, i.e., $\alpha > 0$, for $s_{1,2}(x) = e(\{1, 3\}, x) = 2\alpha$ and $s_{2,1}(x) = e(\{2, 3, 4\}, x) = 1 - 3\alpha$, we have $s_{2,1}(x) \geq s_{1,2}(x)$, i.e., $\alpha \leq \frac{1}{5}$; if $x_1 > v(\{2\}) = 0$, i.e., $\alpha < \frac{1}{3}$, then we have $s_{2,1}(x) \leq s_{1,2}(x)$, i.e., $\alpha \geq \frac{1}{5}$. Therefore, $\alpha = \frac{1}{5}$.

Osborne and Rubinstein (1994) depicted the relationship between the above three concepts in the following way: the kernel is a subset of the bargaining set, and the nucleolus is a subset of the kernel. They also demonstrated that in any cohesive coalitional game with transferable payoff, the nucleolus is always nonempty, and it consists of a single point. For a more detailed discussion, one can refer to their book.

8.4.3 Shapley Value

The previous solution concepts are all based on a single game, while the Shapley value discussed below is based on a series of games. In all games, the marginal contributions of a player to the coalition and the payoff that its average value assigns to the player reflect some kind of equity, which means that one's revenue is associated with his or her contribution. The Shapley value is widely adopted and has good properties.

Let $\langle N, v \rangle$ be a coalitional game with transferable payoff. We call $\langle S, v^S \rangle$ a subgame of the coalitional game, where $S \subseteq N$, if for any $T \subseteq S$, we always have $v(T) = v^S(T)$. Let ψ be a value that is a feasible payoff allocation to the coalitional game with transferable payoff. For $i \in S$, $\psi_i(S, v^S)$ characterizes the payoff of player i in the subgame $\langle S, v^S \rangle$.

The blocking of player i to j is against the feasible payoff allocation x of the whole coalitional game $\langle N, v \rangle$.

There are two types of blocking:

The first type: player i requests more benefits; otherwise, the player will leave the initial game, making player j 's payoff decrease from x_j to $\psi_j(N/\{i\}, v^{N/\{i\}})$.

The second type: player i requests more benefits; otherwise, the player will join the others to exclude player j . During this process, player i 's payoff increases from x_i to $\psi_i(N/\{j\}, v^{N/\{j\}})$.

For the two types of blocking of player i , player j has two corresponding types of counter-blocking:

The first type of counter-blocking: although the payoff of player j decreases after blocker i 's leaving, player j 's leaving will

lead to a greater decrease in the payoff of blocker i , i.e., $x_i - \psi_i(N/\{j\}, v^{N/\{j\}}) \geq x_j - \psi_j(N/\{i\}, v^{N/\{i\}})$.

The second type of counter-blocking: player j proposes that if he or she joins other players to exclude player i , the increase in the payoff is greater than that when i excludes player j , i.e., $\psi_j(N/\{i\}, v^{N/\{i\}}) - x_j \geq \psi_i(N/\{j\}, v^{N/\{j\}}) - x_i$.

Under the solution concept of the previous coalitional game, a stable solution means that each blocking corresponds to a counter-blocking. Thus, if for any group of players, $i, j \in N, i \neq j$, we always have $\psi_j(N/\{i\}, v^{N/\{i\}}) - x_j = \psi_i(N/\{j\}, v^{N/\{j\}}) - x_i$, then x is stable.

It is noted that in the Shapley value, blocking and counter-blocking involve the subgames of the entire coalitional game, and the previously discussed blocking and counter-blocking are limited only to the entire game. We first have the following definition.

Definition 8.4.6 (Balanced Contribution) We say that a solution ψ satisfies the property of *balanced contribution*, if for $i, j \in N, i \neq j$,

$$\psi_j(N, v) - \psi_j(N/\{i\}, v^{N/\{i\}}) = \psi_i(N, v) - \psi_i(N/\{j\}, v^{N/\{j\}}).$$

Shapley (1953) proposed the concept of the Shapley value, which is based on individuals' marginal contributions. The marginal contribution of player i to the coalition S is defined as $\Delta_i(S) = v(S \cup i) - v(S)$.

The Shapley value is defined as $\psi_j(N, v) = \frac{1}{|N|!} \sum_{R \in \mathfrak{R}} \Delta_i(S(R))$, in which R is a permutation of the set of players N , the number of all possible permutations is $|N|!$, the set of all permutations is denoted by \mathfrak{R} , and $S(R)$ is all players before player i in permutation R .

Therefore, the payoff that the Shapley value assigns to each player equals the average of his or her marginal contribution to all possible coalitions.

It can be shown that the Shapley value is the only solution that satisfies the property of balanced contribution and, in addition, satisfies the following three properties.

- (1) Symmetry (SYM): for any $S \subseteq N, i \notin S, j \notin S, \Delta_i(S(R)) = \Delta_j(S(R))$ implies $\varphi_i(N, n) = \varphi_j(N, n)$;
- (2) Dummy players (DUM): if $\Delta_i(S) = v(j)$ for any $S \subseteq N, i \notin S$, then $\varphi_i(N, n) = v(i)$;
- (3) Additivity (ADD): for any two games $\langle N, v \rangle$ and $\langle N, w \rangle$, and for any $i \in N$, we have

$$\varphi_i(N, v + w) = \varphi_i(N, v) + \varphi_i(N, w),$$

where $\langle N, v + w \rangle$ is defined as: for any $S \subseteq N, (v + w)(S) = v(S) + w(S)$.

Now, we show how to solve for the Shapley value through some examples.

Example 8.4.4 (Continued from Example 8.2.2 (3)) There are three players, and a total of 300 units of resources that are available for allocation. For coalitional game $\langle N, v \rangle$, $N = \{1, 2, 3\}$, if $S \subseteq N$ and $|S| \geq 2$, then $v(S) = 300$; if $S \subseteq N$ and $|S| = 1$, then $v(S) = 0$.

By the definition of the Shapley value, there are six possible permutations of the set of players N , i.e., there are only two possible permutations in which each player i can have a positive marginal contribution of 300. Consequently, the Shapley value of this game is $\varphi(N, n) = (100, 100, 100)$. The Shapley value can also be obtained with symmetry. Because the contribution of each player in the coalition is symmetric, the payoffs that they receive are also the same.

Example 8.4.5 (Weighted Majority Game) A weighted majority game is a kind of simple game, in which the weight of player i is defined as w_i , and the fixed amount q represents the lower bound of weight required to become the winning coalition:

$$v(S) = \begin{cases} 1, & \sum_{i \in S} w(i) \geq q, \\ 0, & \text{otherwise.} \end{cases}$$

Assume that the set of players is $N = \{1, 2, 3, 4\}$, and their weight is:

$$(w_i) = \begin{cases} 0.2, & i = 1, \\ 0.4, & \text{otherwise.} \end{cases}$$

In the 24 possible permutations, calculate the average of the players' marginal contribution, and we obtain the Shapley value of the entire game:

$$\varphi\langle N, n \rangle = \left\{ \frac{1}{2}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6} \right\}.$$

Example 8.4.6 Consider the previous simple game, $N = \{1, 2, 3, 4\}$. It is a simple game if and only if $S = \{2, 3, 4\}$, or $\{1, i\} \subseteq S$, for any $i \in \{2, 3, 4\}$. We know that the game's kernel and nucleolus allocation outcome is $x_1 = \frac{2}{5}$, $x_2 = x_3 = x_4 = \frac{1}{5}$. Next, we calculate the Shapley value of this game. In all 24 possible permutations, player 1's marginal contribution is 1 in 12 of them and 0 in the rest; the other players' marginal contribution is 1 in 4 of them and 0 in the rest. In this way, the Shapley value for the entire game is: $\varphi\langle N, n \rangle = \{\frac{1}{2}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}\}$.

It can be seen that the Shapley value is different from the allocation of the nucleolus discussed earlier. They establish blocking and counter-blocking mechanisms based on different environments, and thus have different applicability under various environments. Moreover, the coalitional games discussed above are based on transfers. For coalitional games without transferable payoff, there are also corresponding solution concepts, which usually involve a more mathematical and technical background. Readers can refer to certain books, such as Myerson (1991) and Peleg and Sudholter (2007).

8.5 Biographies

8.5.1 Robert J. Aumann

Robert J. Aumann (1930—), an Israeli-American economist, made fundamental contributions to decision theory and played an important, or even

indispensable, role in the formation of game theory and many other economic theories. In 2005, he shared the Nobel Memorial Prize in Economic Sciences with Thomas C. Schelling (see his biography in section 20.6.2), a Professor at the School of Public Policy of the University of Maryland, College Park, for “having enhanced our understanding of conflict and cooperation through game-theory analysis” .

In June 1930, Robert Aumann was born in a traditional Jewish family in Frankfurt am Main, Germany. He moved to the United States in 1938, received a bachelor’s degree from the City College of New York, and received a Ph.D. degree in mathematics from the Massachusetts Institute of Technology in 1955. He then served as a strategic advisor and did two years of post-doctoral research at Princeton University. After that, he moved to Israel. Aumann was elected as a Fellow of the American Academy of Arts and Sciences in 1974; received the Harvey Prize in Science and Technology from the Technion–Israel Institute of Technology in 1983; has been a Member of the U.S. National Academy of Sciences since 1985; has been a Member of the Israel Academy of Sciences and Humanities since 1989; and received the Israel Prize for Economics in 1994. He is currently a Professor at the Center for the Study of Rationality at the Hebrew University of Jerusalem in Israel.

Aumann is the first to define the concept of correlated equilibrium in game theory, which is a type of equilibrium in non-cooperative games and more flexible than the classical Nash equilibrium. He has also proposed a market model with a continuum of traders, the most natural mathematical model for a market with perfect competition, and demonstrated that the core of such a market coincides with the set of its equilibrium allocations. The introduction of “continuum” has a great influence on the development of economics. Aumann points out that continuum can be viewed as an approximation to the true situation where there is a great, but finite, number of particles (or economic agents, strategies, or possible prices). Using continuum as a rough approximation makes it possible to apply a powerful, precise method called “mathematical analysis” , while a discrete method would be more difficult or even useless.

Aumann has also made numerous important contributions in the field

of set-valued functions (i.e., functions map to multiple points rather than a single point), such as the “Aumann Measurable Choice Theorem” and the theory of integration of set-valued functions. Most of the problems that he investigated arise from the study of different game theories and economic models, while the continuum of economic agents and mathematical theory constitute important tools for the evolution and analysis of these models. The results obtained by Aumann regarding general equilibrium, optimal allocation, nonlinear programming, control theory, measure theory, fixed-point theory, etc., are fundamental, and they are applied to numerous fields, such as economics, mathematics, operations research, etc. In addition, Aumann has extended the equilibrium outcome of the Kuhn’s theorem on optimal behavior strategies in finite games of perfect recall to an infinite situation, overcoming complex technical difficulties.

8.5.2 Reinhard Selten

Reinhard Selten (1930-2016) was the founder of the subgame perfect Nash equilibrium, the founder of experimental economics, and winner of the 1994 Nobel Memorial Prize in Economic Sciences.

Selten was born in Breslau, a German city that became part of Poland and was renamed Wroclaw after World War II. In 1951, Selten graduated from high school, and although he had considered studying economics or psychology in university, he finally decided to study mathematics. In 1951, Selten was admitted to the Department of Mathematics at Goethe University Frankfurt. He graduated in 1957 with a master’s degree in mathematics, and was later engaged in academic research in game theory and its applications, experimental economics, etc. In 1961, Selten received a Ph.D. degree in mathematics from Goethe University Frankfurt. In the early 1960s, Selten conducted experiments on an oligopoly game; from 1967 to 1968, he was a visiting professor at the University of California, Berkeley. He transferred to work at the University of Bielefeld in Germany in 1972 and began working at the University of Bonn in Germany in 1984. In 1994, Selten received the Nobel Memorial Prize in Economic Sciences for his “pioneering analysis of equilibria in the theory of non-cooperative games” .

Having obtained the master's degree in 1957, Selten was hired as an assistant by Professor Heinz Sauermann, an economist at Goethe University Frankfurt. Sauermann was among the first economists to advocate Keynesianism in Germany. At first, Selten was arranged to study the application of game theory to industrial organization, but he soon became fascinated with economic laboratory experimentation. His work received support from Sauermann. Then, Selten and several colleagues started working on experimental research of economics. In 1959, Selten published his first academic paper together with Sauermann - "An Oligopoly Experiment". At that time, the discipline of experimental economics did not yet exist.

In conducting experiments on an oligopoly game, Selten found there were many equilibria in this game. In order to solve the problem of multiple equilibria, Selten introduced the concept of subgame perfectness, and published his most famous paper of game theory in 1965 - "An Oligopoly Model with Demand Inertia". Selten did not expect that his article would often be quoted, almost exclusively for the definition of subgame perfectness which laid the foundation for his winning the Nobel Prize in Economics. In 1964, Selten published the paper "Valuation of n-Person Games". This is an important paper on game theory, and it is another major contribution to game theory. In 1975, Selten published another well-known paper, "Reexamination of the Perfectness Concept for Equilibrium Points in Extensive Games". In this paper, Selten proposed the concept of "trembling hand perfect equilibrium". As the University of Bielefeld encouraged cross-disciplinary research, in exchange with biologists, Selten realized that game theory could be applied to the study of biology. Selten familiarized himself with the notion of evolutionary stability and developed a strong interest in biological game theory. He investigated evolutionary stability in extensive games and wrote a series of papers.

8.6 Exercises

Exercise 8.1 In a cooperative game with transferable payoff, use linear programming and the duality theorem to prove the Bondereva-Shapley Theorem: a sufficient and necessary condition for the existence of a nonempty core in a coalitional game with transferable payoff is that the game is balanced.

Exercise 8.2 In a two-person bargaining game, $x = (x_1, x_2)$ represents a payoff allocation, F represents the feasible set of allocations, and v represents the utility function. The Nash bargaining solution (x_1, x_2) is defined as the solution that maximizes the Nash product $(x_1 - v_1)(x_2 - v_2)$.

1. Verify whether the Nash bargaining solution satisfies individual rationality and Pareto efficiency.
2. Let (F, v) be an allocation of 100 dollars between two players. If the two players cannot reach an agreement, then they will receive nothing; if an agreement can be reached, then player 1 receives x , and player 2 receives $100 - x$. Assume $v_1(x) = x$ and $v_2(100 - x) = \sqrt{100 - x}$. Solve for the Nash bargaining solution of the problem.

Exercise 8.3 Suppose that two players perform a bargaining game on 1 unit of divisible goods. The utility functions for player 1 and player 2 are $u_1(\alpha) = \alpha/2$ and $u_2(\beta) = 1 - (1 - \beta)^2$, respectively, where $\alpha, \beta \in [0, 1]$.

1. Derive the set of feasible utilities and represent them with figures.
2. Derive Nash bargaining outcome, and provide the allocation plan of the goods and the utility of each player.
3. Suppose that the player's utility has a discount factor of $\delta \in [0, 1]$. Derive Rubinstein bargaining outcome, i.e., the subgame perfect Nash equilibrium solution of the infinite alternating-proposal bargaining game.

4. Use the results of questions 2-3 to derive the solution to the Rubinstein bargaining problem when δ approaches 1.

Exercise 8.4 Suppose that two players perform a bargaining game on 1 unit of divisible goods. The utility functions for player 1 and player 2 are $u_1(\alpha) = \alpha$ and $u_2(\beta) = \beta^{\frac{1}{2}}$, respectively, where $\alpha, \beta \in [0, 2]$. Assume that they have a discount factor of $\delta \in [0, 1]$.

1. Derive Rubinstein bargaining outcome.
2. According to the result of question 1, derive Nash bargaining outcome.
3. Suppose that player 2's utility function is unchanged, but player 1's utility function become:

$$u_1(\alpha) = \begin{cases} \alpha, & \text{if } \alpha \in [0, 1], \\ 1, & \text{if } \alpha \in [1, 2]. \end{cases}$$

Derive Nash bargaining outcome, and provide the allocation plan of the goods and the utility of each player.

Exercise 8.5 (Weighted Majority Game) A weighted majority game is a simple game $\langle N, v \rangle$, such that for some $q \in \mathcal{R}$ and weights $w \in \mathcal{R}_+^N$,

$$v(S) = \begin{cases} 1, & \text{if } w(S) \geq q, \\ 0, & \text{if } w(S) < q, \end{cases}$$

where $w(S) = \sum_{i \in S} w_i$. w_i can be interpreted as the number of votes that player i owns, and q is the number of votes needed to win. A weighted majority game is homogeneous if $w(S) = q$ for any minimum winning coalition; it is zero-sum if for each coalition, either $v(S) = 1$ or $v(N) = 1$, but the two cannot hold at the same time.

Consider a zero-sum homogeneous weighted majority game $\langle N, v \rangle$, where for each player i who does not belong to any minimum winning coalition, $w_i = 0$. Prove that the core of $\langle N, v \rangle$ includes all allocations that satisfy $x_i = w_i/w(N)$ for any $i \in N$.

Exercise 8.6 Given the following men and women (3 women and 5 men) marriage market preference structure:

Women's preferences are as follows:

	<i>A</i>	<i>B</i>	<i>C</i>	<i>D</i>	<i>E</i>
<i>a</i>	1	1	2	3	3
<i>b</i>	2	3	1	1	2
<i>c</i>	3	2	3	2	1

Men's preferences are as follows:

	<i>A</i>	<i>B</i>	<i>C</i>	<i>D</i>	<i>E</i>
<i>a</i>	2	1	3	4	5
<i>b</i>	3	1	2	5	4
<i>c</i>	3	1	4	2	5

1. Use women-proposing Gale-Shapley algorithm to find out a stable matching.
2. Which women do not have a spouse in this stable matching? Do these women get spouses in other stable matchings? Please explain.

Exercise 8.7 Consider the following leader-follower game:

$$\langle I, J, X_i, Y_j, f_i, g_j \rangle,$$

where I is the finite set of leaders i , J is the finite set of followers j , X_i is the action set of leader i , Y_j is the action set of follower j , $f_i : X \times Y \rightarrow \mathcal{R}$ is the payoff function of leader i , and $g_j : X \times Y \rightarrow \mathcal{R}$ is the payoff function of follower j .

In the multi-leader-follower game, the leaders make decisions first, and then the followers play the game after obtaining information about the leaders' action. For the leaders' action $x \in X$, let $\mathcal{C}(x)$ represent the cooperative equilibrium core of the followers' game. It becomes a correspondence relationship $\mathcal{C} : X \rightrightarrows Y$, i.e., $y \in \mathcal{C}(x)$ means that for any $B \subseteq J$, there exists no $u^B \in Y^B$ satisfying:

$$g_j(x, u^B, v^{-B}) > g_j(x, y), \forall v^{-B} \in Y^{-B}, \forall j \in B.$$

In the multi-leader-follower game, a coalition $B \subseteq I$ is considered to **block** $x \in X$, if there exists $u^B \in X^B$, such that:

$$f_i(u^B, z^{-B}, y) > f_i(x, y), \forall y \in \mathcal{C}(x), \forall z^{-B} \in X^{-B}, \forall i \in B.$$

We say that action $x \in X$ is the **cooperative equilibrium of the multi-leader-follower game**, if there exists no coalition $B \subseteq I$ that can block x .

Prove the following theorem:

For the multi-leader-follower game

$$\langle I, J, X_i, Y_j, f_i, g_j \rangle,$$

it satisfies the following conditions:

- (1) For each $i \in I$ and each $j \in J$, both X_i and Y_j are nonempty compact convex subsets in normed linear spaces.
- (2) For each $i \in I$, f_i is continuous on $X \times Y$.
- (3) For each $i \in I$ and each $y \in Y$, $f_i(\cdot, y)$ is quasi-concave on X .
- (4) For each $j \in J$, g_j is continuous on $X \times Y$.
- (5) For each $j \in J$ and each $x \in X$, $g_j(x, \cdot)$ is quasi-concave on Y .

Then, the set of cooperative game equilibrium solutions of this multi-leader-follower game is not empty.

Exercise 8.8 Determine if true or false for the following three propositions and provide corresponding explanations.

1. The core is a subset of each stable set.
2. A stable set may be a proper subset of another stable set.
3. If the core is a stable set, then there is no other stable set.

Exercise 8.9 An allocation set of a three-person game can be geometrically represented as an equilateral triangle with a height of $v(N)$. Each side represents a player, and each point in the triangle represents an allocation. The distance from a point to each side represents the payoff of everyone in

the allocation (for example, the vertex corresponds to the allocation that uniquely assigns $v(N)$ to the player who is represented by the opposite side of this vertex).

1. Use this figure to find the general form of stable set of the following three-person game: $v(1, 2) = \beta < 1$, $v(1, 3) = v(1, 2, 3) = 1$, and $v(S) = 0$ for any other subset S .
2. We can interpret the game in question 1 as a market in which player 1 is the seller, while player 2 and 3 are buyers with reserve prices of β and 1, respectively. Explain the stable set of this game in terms of this market.

Exercise 8.10 Three cities can establish connections with a new power source, P , to increase power supply. The utilities of increased power supply for the three cities A , B , and C are $u_A = 100$, $u_B = 140$, and $u_C = 130$, respectively. Suppose that any established connection has the transmission capacity to meet the requirement of simultaneous power supply for the three cities, and the costs of establishing a direct connection between two points are as follows:

Connection	AB	BC	CA	AP	BP	CP
Cost	50	20	30	100	140	130

The net value of power transmission for each city equals $v_i = u_i - c_i$ for $i = A, B, C$.

1. Use a cooperative game to set up this problem.
2. Solve for the core, give the set, and draw figures to represent it.
3. Solve for the Shapley value. Is it in the core?
4. Solve for the nucleolus. Is it in the core?

Exercise 8.11 There are three players, each denoted by Z , M , and H . Each of them has a different skill: Z is strong, M moves fast, and H has good stamina. They plan to work together to hunt antelopes. They need to decide how to share the work and allocate the reward with antelope as the unit of measurement. Antelopes can be used for transfers, and their coalitions and

payoffs are shown in the table below.

Coalition	Payoff
$\{ZMH\}$	(6)
$\{ZM\}\{H\}$	(4)(1)
$\{ZH\}\{M\}$	(3)(1)
$\{MH\}\{Z\}$	(3)(2)
$\{Z\}\{M\}\{H\}$	(2)(1)(1)

1. Write the characteristic function of the coalition, and verify that it is superadditive.
2. Write the core allocation of the game.
3. Derive the Shapley value, and check whether it is in the core.

Exercise 8.12 Three players, A , B , and C , consider establishing a company. A is good at technology, B is good at design, and C is good at sales. The characteristic function of the coalitional game is shown in the following table.

A, B, C	50
A, B	25
B, C	20
A, C	30
A	15
B	10
C	5

Use the core concept of cooperative game to answer the following questions:

1. Which coalitions may appear? Why?
2. Someone suggests that members of the coalition should divide their income equally. Is this a stable allocation?
3. Find the Shapley value, and show whether it is in the core.

Exercise 8.13 Three individuals, A , B , and C , make appointments for treatment at a clinic on Monday, Tuesday, and Wednesday, respectively. The utility of their treatment at each time is given in the following table:

	Monday	Tuesday	Wednesday
A	2	4	8
B	10	5	2
C	10	6	4

Everyone can benefit by exchanging their time for doctor visits. In this case, consider the following questions:

1. Use the cooperative game to model this problem.
2. Solve for the core and give the set.
3. Solve for the Shapley value. Is it in the core?
4. Solve for the nucleolus. Is it in the core?

Exercise 8.14 Consider a three-person coalitional game with transferable payoff. For each real number a and v_a , given the following:

$$\begin{aligned}
 v_a(i) &= 0, \quad i = 1, 2, 3; \\
 v_a(\{1, 2\}) &= 3, \quad v_a(\{1, 3\}) = 2, \quad v_a(\{2, 3\}) = 1; \\
 v_a(\{1, 2, 3\}) &= a.
 \end{aligned}$$

Answer the following questions:

1. What is the minimum value of a , such that the core of this cooperative game is nonempty?
2. Calculate the Shapley value for $a = 6$.
3. What is the minimum value of a , such that the Shapley value is in the core?

Exercise 8.15 Consider a company with multiple shareholders, two of whom have $\frac{1}{3}$ shares each, and the other $n - 2$ shareholders own the remaining shares evenly. First, model this situation as a weighted majority game. Then, answer the following questions:

1. When n approaches infinity, what are the limits of the Shapley values for these two major shareholders?
2. According to the Shapley value, is it desirable for the $n - 2$ small shareholders to form the only coalition?

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Chapter 9

Market Theory

9.1 Introduction

In Part II, we studied the rational behavior of individual consumers and firms when market prices are taken as given and beyond individuals' control. In this chapter, we explore the consequences of that behavior when consumers and firms come together in markets. We will also consider the determination of equilibrium price and quantity in a single market or a group of related markets under different market structures.

The key insight of Adam Smith's *The Wealth of Nations* is simple: if an exchange between two parties is voluntary, it will not take place unless both parties believe that they will benefit from it. How is this also true for any number of parties and for production cases? The price system is the mechanism that performs this task well without government intervention in many situations. In Chapter 1, we have already highlighted that price performs three functions in organizing economic activities in a free market economy: (1) it transmits information about production and consumption in an efficient manner; (2) it provides appropriate incentives. The subtlety of a free price system is that the price that transmits the information also provides incentives for individuals who receive the information to adjust demand and supply accordingly. (3) It determines the distribution of income. When price is utilized to transmit information and provide incentives, it will inevitably affect the distribution of income. If one's income

does not depend on prices, what incentive does he or she have to seek out price information or to respond to that information?

This chapter focuses on partial equilibrium analysis that involves a single market or a group of related markets, implicitly assuming that changes in the markets under consideration do not change the prices of commodities in other markets. We will treat all markets simultaneously in general equilibrium theory in the next part. Here, we mainly discuss the market behavior of firms, focusing on how they determine the price at which they will sell the output or at which they are willing to purchase the input. In fact, this is the main focus of microeconomics, which can be essentially characterized in one word — pricing. Price-taking behavior may be a reasonable approximation to some optimal behaviors, but there are numerous other cases in which we have to explore the price-setting process.

A firm has the power of free pricing only when the market competition is imperfect. In an imperfectly competitive market, firms can influence demand, and thus have the ability to determine prices and quantities of the products that they produce. Such ability is known as **market power** or the ability to influence the market, or as a competitive advantage. Such market power may come from unique resources, intellectual property, governments, product differentiation, or cost advantages from economies of scale.

The main feature of an imperfectly competitive market is that the demand curve faced by a single firm is downward, and its elasticity is related to the market structure (degree of competition). Imperfect competition includes monopoly (seller's and buyer's), oligopoly (seller's and buyer's), and monopolistic competition. We will first consider the situation of perfect competition as a benchmark and then, based on this benchmark market structure, turn to investigate situations in which individuals have the market power. These situations include several market structures, i.e., pure monopoly, monopsony, monopolistic competition, and oligopoly.

9.2 Perfect Competition

We begin by considering a perfectly competitive market, which constitutes an ideal and extreme market structure. Just like free fall in physics, although it basically does not exist in reality, it is crucial to fully comprehend and study a competitive market, as it serves as a benchmark for investigating more realistic market structures.

The basic feature of a **perfectly competitive market** is that many firms sell homogeneous products in the market, and each consumer or producer, irrespective of how much he or she consumes or produces, does not have an impact on the market price. From the perspective of maximizing social welfare, it is the most efficient market.

For perfectly competitive markets, we make the following assumptions:

- (1) **A large number of buyers and sellers** that take prices as given.
- (2) **Unrestricted mobility of resources** among industries: no barrier to entry into, or exit from, the market.
- (3) **Homogeneous product**: the products of all of the firms in an industry are identical to consumers.
- (4) **All relevant information is common knowledge**: firms and consumers have all of the information necessary to make optimal economic decisions.

9.2.1 The Competitive Firm

The above four assumptions portray the basic nature of firms in a perfectly competitive market, i.e., the firm's impact on the market is zero. Next, we discuss the characteristics of firms in a perfectly competitive industry.

A **competitive firm** is free to set whatever price that it wants and produce whatever quantity that it is able to produce. However, since all products are homogeneous, each firm that sells the product must sell it at the same price, and thus the inverse demand function is a horizontal line parallel to the consumption axis. If any firm attempted to set its price at a level

higher than the market price, it would immediately lose all of its customers; if it sets its price at a level below the market price, all of the consumers would immediately come to it, but it can also sell its output at a higher price. Therefore, as a profit-maximizing firm, it must use the market price for its products. When making supply decisions, each firm must take the market price as given, and thus price is an exogenous variable.

Next, we analyze the behavior of perfectly competitive firms. The analysis of firms' market behavior is usually based on the short-run or long-run time frame, according to the scope and degree of firms' decision-making. Short-run analysis means that certain production factors of the firm, such as manufacturing plants and production lines, are constrained, and thus the scope of its decision-making is also constrained accordingly. In the short-run, the structures of firms in the market are immutable. In the long-run, all production factors can be changed. For example, a firm can expand its factory buildings, set up new production lines, and establish new business outlets. It can also decide whether to enter or exit from a market. In the long-run, the structures of firms in the market are variable.

9.2.2 The Competitive Firm's Short-Run Supply Function

Assuming that a firm only produces one product, we want to know how its supply function and the market equilibrium price are determined. Since a perfectly competitive firm must take the market price as given, its profit maximization problem is simple. The firm only needs to choose an output level y to solve:

$$\max_y py - c(y),$$

where p is the price of the product, and $c(y)$ is the cost function of production.

The first-order condition (FOC for short) for the interior solution of the above optimization problem gives:

$$p = MC(y),$$

i.e., the market price equals the marginal cost.

In the first-order condition of profit maximization, the marginal revenue equals the marginal cost. In a perfectly competitive market, the revenue is $R = py$, and the marginal revenue is $MR = \frac{dR}{dy} = p$. Therefore, $MR = MC$ means $p = MC(y)$.

The second-order condition for the above optimization problem is $c''(y) > 0$, which means that the associated production function is concave.

The above two conditions determine the supply decision of the competitive firm. At any price p , the firm will supply an amount of output $y(p)$. According to the first-order condition, we have $p = c'(y(p))$. Further derivation yields $1 = c''(y)y'(p)$. From the second-order condition, we know that $y'(p) > 0$, which means that the law of supply holds.

When the market price p is sufficiently low, the firm may not produce. The firm's short-run cost function is $c(y) = c_v(y) + F$, where F denotes the fixed cost. If $py(p) - c_v(y) - F \geq -F$, then the firm should choose to produce a positive amount of output. This means that $p \geq \frac{c_v(y(p))}{y(p)} \equiv AVC$. In other words, in the short-run, the necessary condition for the firm to produce is that the market price of the product is not less than the minimum average variable cost (see Figure 9.1).

In general, the **supply curve** for the competitive firm is given by:

$$y = \begin{cases} (c')^{-1}(p), & \text{if } p \geq \frac{c_v(y(p))}{y(p)}, \\ 0, & \text{otherwise,} \end{cases}$$

where $(c')^{-1}$ is the inverse function of c' .

As long as the price is greater than the average variable cost, the firm's supply curve coincides with the upward sloping portion of the marginal cost curve. If the price is lower than the (smallest) average variable cost, the firm's supply is zero.

Suppose that there are J firms in the market. The **industry supply function** is simply the sum of all firms' supply functions, i.e., $\hat{y}(p) = \sum_{j=1}^J y_j(p)$, where $y_j(p)$ is the supply function of firm j . Since each firm chooses a level of output at which price equals marginal cost, each firm that produces a positive amount of output must have the same marginal cost. The industry supply function measures the relationship between industry output and

the common marginal cost of producing this output.

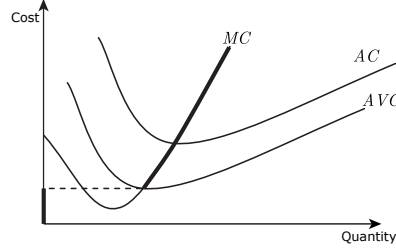


Figure 9.1: Firm's supply curve, AC, AVC, and MC curves.
The bold curve is the supply curve.

The **industry aggregate demand function** measures the total output demanded at any price, which is given by $\hat{x}(p) = \sum_{i=1}^n x_i(p)$, where $x_i(p)$ is the demand function of consumer i , and n denotes the number of consumers.

9.2.3 Single-Commodity Market Equilibrium

The market price of a commodity is determined by the requirement that the total quantity of output that the firms wish to supply is equal to the total quantity of output that the consumers wish to consume. Formally, we have the following definition.

Definition 9.2.1 (Equilibrium Price) The *market equilibrium price* p^* of a single product in a perfectly competitive market is a price at which the aggregate quantity demanded equals the aggregate quantity supplied, i.e., it is the solution of the following equation:

$$\sum_{i=1}^n x_i(p) = \sum_{j=1}^J y_j(p).$$

Once this equilibrium price is determined, we can go back to look at the individual supply schedules of each firm and determine its output level, revenue, and profit. In Figure 9.2, we have depicted cost curves for three firms. The first has positive profit, the second has zero profit, and the third has negative profit. Even though the third firm has negative profit, it may

make sense for it to continue production as long as its revenues cover its variable costs, i.e., $p \geq AVC$; otherwise, it will suffer even greater losses (which equals the fixed cost).

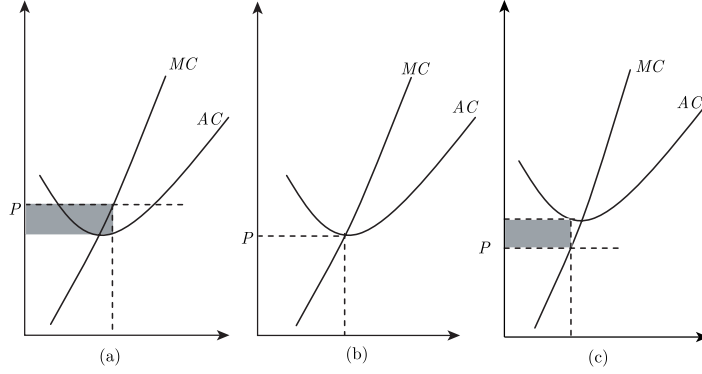


Figure 9.2: Positive, zero, and negative profits.

9.2.4 Competitive Market and Returns to Scale of Production Technology

From the producer theory, we know that the feature of returns to scale of technology can be inferred from the cost function. If the average cost decreases (increases or remains unchanged) as the output increases, then the technology exhibits increasing (decreasing or constant) returns to scale.

The following cost function exhibits the typical feature of increasing returns to scale:

$$C(q) = \begin{cases} F + cq, & \text{if } q > 0, \\ 0, & \text{if } q = 0. \end{cases}$$

The corresponding average cost function is $AC(q) = \frac{F}{q} + c$. Figure 9.3 illustrates the average cost and the marginal cost of the technology: the average cost decreases with the increase in output; and when the output goes to infinity, the average cost converges to the marginal cost.

However, a technology with increasing returns to scale is not compatible with a perfectly competitive market. Let us use an example to demonstrate this statement.

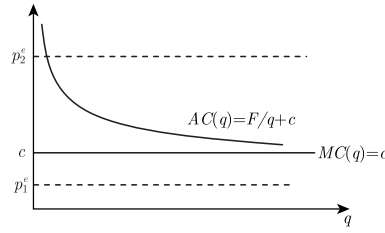


Figure 9.3: Increasing returns to scale of production technology

Let the market demand function be $P(Q) = a - bq$, where $b > 0, a > c$.

Suppose that the equilibrium of the competitive market exists, and let the equilibrium price be p^e . Then, there are only two possibilities for the equilibrium price: either $p^e \leq c$ or $p^e > c$.

When $p^e = p_1^e \leq c$, for any positive amount of output q , $p^e = p_1^e \leq c < \frac{F}{q} + c = AC(q)$, the firm's profit and the producer surplus are both less than zero. For a firm that pursues profit maximization, its production can only be zero. However, if the firm's output is zero, the price p_1^e cannot be the market equilibrium, because at p_1^e the market demand is greater than zero, while the market supply is zero.

When $p^e = p_2^e > c$, when output q exceeds a certain limit, we will have $p_2^e > \frac{F}{q} + c = AC(q)$, and $\frac{d(p_2^e - AC(q))}{dq} > 0$. Thus, at market price p_2^e , the competitive firm that pursues profit maximization will choose infinitely large output (because the producer surplus is greater than zero at any output). At this point, the market supply is infinitely large. However, at p_2^e , the market demand is limited. Therefore, p_2^e can also not be the market equilibrium price.

In conclusion, if the firm's production technology displays increasing returns to scale, the market structure cannot be perfectly competitive. In the following chapter on general equilibrium theory, we will discuss the incompatibility between increasing returns to scale and perfectly competitive market equilibrium from a different perspective.

9.2.5 Long-Run Equilibrium

The long-run behavior of a competitive industry is determined by two effects. Consider that all firms can choose other firms' production technologies, or that production technologies are common knowledge. If production technology cannot be replicated, in the case of patents for example, then the market structure is not a perfectly competitive market in the typical sense. The first effect is free entry and exit that make the long-run profits of all firms zero. If a firm is making negative profits, it would eventually exit from the market. Conversely, if a firm is making positive profits, other firms would enter the industry. If we have an industry characterized by free entry and exit, all firms surely will make the same level of profits in the long-run. As a result, every firm makes zero profit at the long-run competitive equilibrium as illustrated in Figure 9.4.

The second effect on the long-run behavior of a competitive industry is that of technological adjustment. In the long-run, firms will attempt to adjust technologies and so their fixed factors in order to produce the equilibrium level of output in the least expensive way. If every firm attempts to do this, the equilibrium price will change.

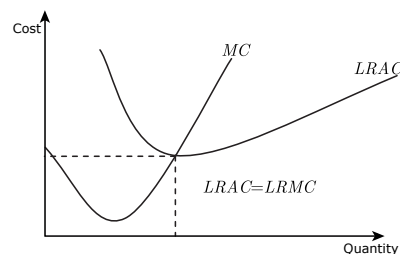


Figure 9.4: Long-run competitive equilibrium where every firm makes zero profit.

In the case of free entry or exit, the equilibrium number of firms is determined by the following principle: at the equilibrium, the entry of new firms will make the profits of all firms less than zero. In other words, when free entry and exit makes the profits of firms approach zero at the equilibrium price, the number of firms is determined.

Example 9.2.1 $c(y) = y^2 + 1$. The equilibrium level of output is the solution of the following equation:

$$AC(y) = MC(y).$$

Therefore, at $y = 1$, the average cost reaches its minimum, which is 2. If the price is higher than 2, the firm is making a positive profit. If the price is lower than 2, its profit is less than zero. The supply function satisfies $p = MC(y) = 2$, and thus we have $y = \frac{p}{2}$.

Suppose that the demand is linear: $D(p) = a - bp$. Then, the equilibrium price will be the smallest p^* that satisfies the condition: $p^* = \frac{a}{b + J/2} \geq 2$, and thus $J^* = [a - 2b]$, where $[\cdot]$ is a rounding function that takes a number to the nearest integer. If $j > J^*$, the entry of firms would make the market price less than 2, and firms' profits less than zero; if $j < J^*$, firms entering the market would make positive profits.

9.2.6 Social Welfare under Perfect Competition

In a single-commodity market, participants are consumers and producers, whose social welfare is made up of the net benefits of these participants. In consumer theory, as previously discussed, the concept for the net benefit of the consumer in market transactions is consumer surplus; in producer theory, the concept for the net benefit of the producer in market transactions is net profit (equal to producer surplus minus fixed cost). Given J firms in a certain market, at market price p , we define social welfare as $W(p) = CS(p) + \sum_{j=1}^J PS(p)$. In the long-run, since the fixed cost is zero, the producer surplus equals the profit of the firm, so that $W(p) = CS(p) + \sum_{j=1}^J \pi_j(p)$.

In the perfectly competitive market, the market equilibrium is the result of market transactions with maximum social welfare. Therefore, it is frequently adopted as a benchmark to analyze the loss or change of social welfare in other market structures.

We employ a simple example to illustrate this. Let the unit cost (or marginal cost) of production be c and the market demand be $P(Q) = a - bq$,

where $b > 0$, $a > c$. Figure 9.5 depicts the market demand curve and the marginal cost curve. At market price $p = p_0$ and market transaction quantity Q_0 , the consumer surplus is α , the firm's producer surplus or profit is β , and social welfare is $\alpha + \beta$. We find that in the process when the price falls from p_0 to the marginal cost c , the consumer surplus increases, the firm's profit declines, and social welfare also increases. When the price equals c , the consumer surplus is $\alpha + \beta + \gamma$, the firm's profit is zero, and social welfare is $\alpha + \beta + \gamma$. When the price falls further, we find that the decrease of the firm's profit exceeds the increase of the consumer surplus, and thus social welfare decreases. As a consequence, when price equals marginal cost, social welfare reaches the maximum at the perfectly competitive equilibrium where the market price equals the marginal cost.

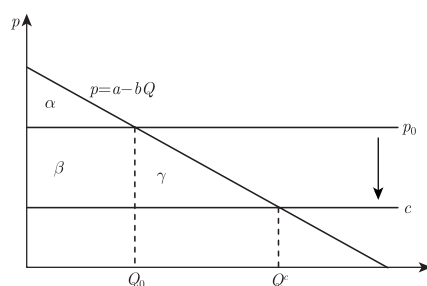


Figure 9.5: Social welfare in the perfectly competitive market

The market price describes consumers' willingness to pay for extra product, while the marginal cost is the cost that a firm spends to produce extra product. As long as the social welfare generated by the production of an extra product (depicted by the consumer's willingness to pay) exceeds the social cost (depicted by the marginal cost if there is no externality) of it, trading this extra product will increase social welfare. When consumers' willingness to pay is lower than the marginal cost, trading this extra product in the market will reduce social welfare. When consumers' willingness to pay equals the marginal cost, the market transaction is optimal. The perfectly competitive market happens to be the place where the two are equal. In other market structures, firms have the ability to make the market price higher than the marginal cost, which means that firms possess the market

power to make consumers' willingness to pay higher than firms' marginal cost. As a result, social welfare is lower than that of the perfectly competitive market, and thus social welfare loss results. In the remainder of this chapter, we further discuss social welfare in imperfectly competitive markets.

9.3 Pure Monopoly

The other extreme of market structure that is the opposite of perfect competition is pure monopoly, referred to as **monopoly**. In the monopoly market, there is only one seller. If there is only one buyer in the factor market, it is called **monopsony market**, which will be discussed later. The sources of monopoly are from: (1) economies of scale; (2) barriers to market entry; and (3) exclusive possession of rare factors of production. In this section, we classify monopoly into two categories: monopoly in the product market; and monopsony in the factor market. Monopolies in the two types of markets have different characteristics. We first discuss monopoly in the product market, and then discuss monopoly (i.e., monopsony) in the factor market.

9.3.1 Monopoly in the Product Market

A monopolistic firm of a single commodity faces two types of decisions, i.e., how much to produce and at what price it should sell this output. Of course, the two decisions of the monopolist are interrelated. Different from a competitive firm, who takes the market price as given and decides its output at the price, the monopolistic firm needs to choose the output and price of its product. The price is determined by the demand function $q(p)$ together with the marginal cost function of production. Sometimes, it is more convenient to consider the inverse demand function $p(q)$, which indicates the price that consumers are willing to pay for the output of the monopolist. We have already given conditions under which the inverse demand function exists in Chapter 3. The revenue that the firm receives can be expressed as a function of the output, i.e., $R(q) = p(q)q$.

The production cost of the firm also depends on the quantity of output. We have studied in depth the characteristics of the cost function in producer theory. Here, we assume that the factor market is perfectly competitive, and thus the factor price can be set as constant (we will discuss below that in a monopoly market of factors, the factor prices are determined by the monopoly buyer), and so the conditional cost function can be written as a function of the level of output of the firm.

The profit maximization problem of the firm can then be written as:

$$\max_q R(q) - C(q) = \max_q p(q)q - C(q).$$

The first-order condition for profit maximization is that marginal revenue equals marginal cost, or $p(q^*) + p'(q^*)q^* = C'(q^*)$, for positive production, the left side of which is the marginal revenue.

The economic implication of this condition is the following: if the monopolist considers producing one extra unit of output, then, on the one hand, the sale of more goods (when the price is greater than the marginal cost) will increase his or her revenue; on the other hand, according to the law of demand, the increase of demand will force the price down, thereby reducing the monopolist's revenue. The sum of these two effects in opposite directions gives the marginal revenue. When the output is small, the former effect dominates, and revenue increases with output. When the output is large, the latter effect dominates, and revenue decreases with output. If the marginal revenue exceeds the marginal cost of production, the monopolist will expand production; otherwise, the output will be reduced until the marginal revenue and the marginal cost balance out.

The first-order condition for profit maximization can be reformulated through the use of the price elasticity of demand.

The **price elasticity of demand** is defined as $\varepsilon(q) = \frac{p}{q(p)} \frac{dq(p)}{dp}$, which is always a negative number since $\frac{dq(p)}{dp} < 0$.

Simple algebra shows that the condition for marginal cost to equal marginal

revenue can be expressed as:

$$p(q^*) \left[1 + \frac{q^*}{p(q^*)} \frac{dp(q^*)}{dq} \right] = p(q^*) \left[1 + \frac{1}{\varepsilon(q^*)} \right] = C'(q^*),$$

or

$$p(q^*) = \frac{C'(q^*)}{1 + \frac{1}{\varepsilon(q^*)}}.$$

Since $\varepsilon(q^*) < 0$, in order to ensure that the price is non-negative, the firm needs to produce within the arrangement of the elastic demand, i.e., $\varepsilon(q^*) < -1$. Therefore, we have $[1 + (1/\varepsilon(q^*))] \leq 1$, which implies that the price is not less than the marginal cost at profit maximization.

This formula is very useful, and provides a basic pricing formula for any market. It shows that the equilibrium price of a product depends on its marginal cost and price elasticity of demand. It should be noted that the price is dependent on the price elasticity of demand. When the marginal cost remains unchanged, the optimal pricing of the product is inversely proportional to the price elasticity of demand, which means that the smaller is the elasticity, the bigger is the market power and thus the higher is the price. When the market is perfectly competitive, the price elasticity of demand is infinitely large, and thus the price equals the marginal cost.

For a monopoly market, the monopoly price equals the marginal cost multiplied by a markup. This markup is a decreasing function of the price elasticity of demand. Meanwhile, the smaller is the absolute value of the price elasticity, the stronger is the monopoly power of the firm and the higher is the markup. Price elasticity measures the substitutability of a product. The smaller is the absolute value of price elasticity, the lower is the consumer's sensitivity to the price of the product, like in the case of salt. If such commodity as salt is subject to monopoly pricing, its price could be very high (e.g., salt monopolized by the government in China's feudal society). This is because there are no other substitutes, and firms are not concerned that price increases may bring about decline in sales.

A graphical illustration of the profit maximization condition is shown in Figure 9.6. Suppose for simplicity that the inverse demand function is linear, i.e., $p(q) = a - bq$. Therefore, the revenue function is $R(q) = qp(p)$,

and the marginal revenue function is $R'(q) = a - 2bq$. The marginal revenue curve has the same vertical intercept as the demand function, but the former is twice as steep as the latter. Figure 9.6 illustrates the relationship between price elasticity and monopoly price. Figure (a) shows the case of low price elasticity of demand, in which the difference between monopoly price and marginal cost is large; Figure (b) shows the case of high price elasticity of demand, in which the difference between monopoly price and marginal cost is small.

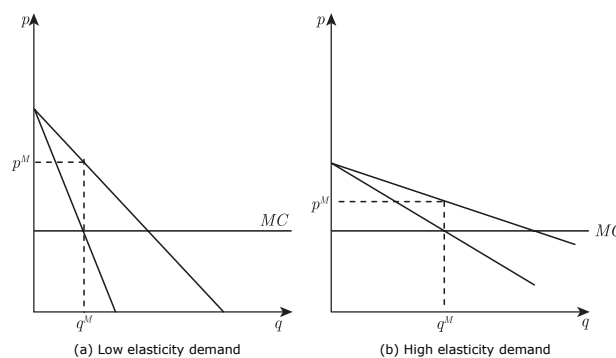


Figure 9.6: Low price elasticity of demand and high price elasticity of demand

9.3.2 Monopoly in the Long Run

We have seen how the long-run and the short-run behavior of a competitive industry may differ due to changes in technology and entry. There are similar effects in a monopolistic industry. The technological effect is the simplest: the monopolist will choose the level of fixed factors in order to maximize her long-run profits. Therefore, she will operate where the marginal revenue equals the long-run marginal cost. The entry effect is slightly more subtle. Presumably, if the monopolist is earning positive profits, other firms would like to enter the industry. If the monopolist remains a monopolist, there must be some sort of barrier to entry for the industry, so that it still can make positive profits even in the long run.

These barriers to entry may be of a legal sort, but often they are due to the fact that the monopolist owns some unique factor of production. For

example, a firm might own a patent on a certain product, or might own a certain proprietary process or factor of production, such as in the case of South Africa having the largest diamond supply in the world. If the monopoly power of the firm is due to a unique factor, we must be especially careful when measuring the monopoly profit. Indeed, because this factor has an opportunity cost, we need to deduct all explicit or implicit costs when calculating the profit.

9.3.3 Disadvantage of Monopoly: Social Welfare Losses

We say that a situation is **Pareto efficient** if there is no way to make one agent better off without making the others worse off. Pareto efficiency is a major theme in the discussion of welfare economics. Here, we only provide a simple illustration of this concept. In partial markets, according to the social welfare function as previously defined, although social welfare maximization and Pareto efficiency are distinct, a link exists between the two. If an allocation maximizes social welfare, then it must be Pareto optimal when the social welfare function is strictly increasing in individuals' utilities. This means that the allocation in the competitive market is a Pareto optimal allocation, but the reverse may not necessarily be true.

Because the monopoly price of firms exceeds the marginal cost, from the previous discussion of social welfare in a competitive market, it can be inferred that monopoly markets suffer social welfare losses. According to the Pareto criterion, the monopoly allocation is a Pareto inefficient allocation, which means that there is a way to improve the situation of the monopolist without negatively affecting the situation of the consumers.

To illustrate this, we consider the production decision of a monopolist. At a monopoly price of p^m , its sales volume is q^m . Assume that the monopolist intends to produce a quantity of extra output Δq and sell it to consumers. How much will consumers pay for this extra output? Clearly, they will pay a price $p(q^m + \Delta q)$. What is the additional cost of producing this extra output? The answer is the marginal cost $MC(q)$. Under such changes in production, the consumers are at least not worse off, while the monopolist is better off since she sells the extra output at a price that is greater

than the cost of production. Here, we allow the monopolist to discriminate in his or her pricing (we will further discuss differential pricing later). She firstly sells an output of quantity q^m at a certain price, and then sells more at some other (lower) price.

How long will this process continue? Once the output reaches the competitive level, the firm will not be able to improve its situation further. At this point, the competitive levels of price and output are Pareto efficient for this industry. The welfare losses of the monopolist relative to the competitive market are revealed by Figure 9.7. At the monopoly price p^m , the consumer surplus is the area of the triangle DEC , the monopoly profit is the area of the trapezoid $CEFA$, and the total social welfare is the area of the trapezoid $DEFA$. In a competitive market, the total social welfare is the area of the triangle DGA . Therefore, the loss of social welfare resultant from monopoly is the area of triangle EGF .

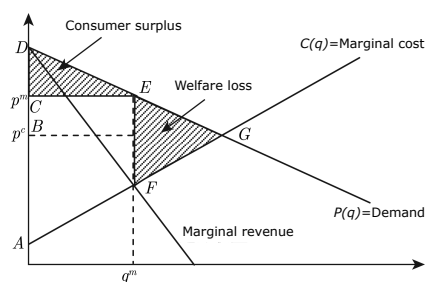


Figure 9.7: Social welfare losses of monopoly

9.3.4 Advantage of Monopoly: Corporate Innovation

From the above discussion, we know that monopoly means that firms possess market power in setting prices. They can use their monopoly power to raise the price of their products above the competitive equilibrium price, and make their output lower than that of the perfectly competitive market. As such, this results in Pareto inefficient allocations, incurring certain social welfare loss. Government regulations may then be needed, such as anti-trust laws, to increase market competition. However, because some industries are naturally monopolistic due to economies of scale, technology

innovation, or other reasons, even if private firms are allowed to produce, the problem of monopoly pricing remains. In order to achieve economies of scale, some market entities frequently monopolize the market by means of alliances, mergers, and acquisitions, thereby distorting the market competition mechanism and preventing it from playing a spontaneous and efficient role in resource allocation. Nevertheless, the problem can be solved through government regulations to realize compatibility of firms' interest and social welfare.

However, monopoly is not without certain merit. In terms of resource allocation efficiency of the entire society, since monopoly generates efficiency loss, competition is clearly a superior choice. Nonetheless, in the view of firms, they hope to achieve a monopoly. Since their profits decrease as competition increases, due to the profit-seeking incentive of private firms, they often have a strong incentive to innovate continuously to conduct research and develop new products, and thus they can set the price of new products above the competitive equilibrium price to obtain great profits. Once doing so, other firms in the same industry will soon develop similar products to share the market and profits. Such market competition leads to the decline of firms' profits, and thus forces firms to engage in further innovation.

Corporate innovation then leads to monopoly profits, and considerable profits will attract other firms to enter and compete again. In this way, market competition leads to the decrease of profits and firms obtain monopoly profits through innovation, which forms a **repeated cycle of competition-innovation-monopoly-competition**. In this cycle, market competition produces market equilibrium, while innovation disrupts this equilibrium. This repeated game in the market motivates firms to constantly pursue innovation. Through this repeated game process, the market economy maintains long-term vitality, increases social welfare and promotes economic development, revealing the unique beauty and power of the market system. Of course, in order to encourage innovation, the government should enact intellectual property protection laws. At the same time, to encourage competition and form externalities of technology innovation, the protection given by anti-monopoly laws and intellectual property rights legislation should

not be permanent, but rather for a limited time, in case fixed or permanent oligopolies and monopolies may appear.

Overall, competition and monopoly are two sides of the same entity, like supply and demand, the two of which can become an awe-inspiring dialectical unity of opposites under market forces, showing the beauty and power of the market system. Without competition, just like state-owned enterprises with a government monopoly, they would not possibly have the incentive to innovate. Entrepreneurs' pursuit of reasonable profit constitutes the power source for market progress. Where does the pricing power come from? The monopoly of a product generates pricing power. On what does the ability of a monopoly rely? In addition to government protection, which can lead to monopolies and thus produce low efficiency and incentive distortions, innovation and unique products are needed to gain the preemptive advantage.

Schumpeter's (see his biography in section 2.12.2) "Innovation Theory" informs us that valuable competition is not a price war, but instead competition of new products, technologies, markets, sources of supply, and combination forms. The root of the long-term vitality of the market economy lies in innovation and creativity, which originates from entrepreneurship and entrepreneurs' constant "creative destruction" of market equilibria. Apple Inc., a thriving company today, was once on the verge of bankruptcy in the late 20th century. What did Steve Jobs rely on to turn the tide when he returned to Apple as its CEO? It was innovation that met and stimulated demand in the market. Its series of products that combined human aspects with science and technology profoundly affected people's consumption preferences and lifestyle. Innovation is the result of flashes of inspiration, yet it cannot be simply achieved from a single brainstorming session. Essentially, innovation comprises many existing ideas. Therefore, entrepreneurs are crucial, but "it is not part of his function to 'find' or 'create' new possibilities. They are always present, abundantly accumulated by all sorts of people". The function of an entrepreneur is to put those possibilities into practice before they disappear by conceiving and developing new combinations of factors.

It should also be pointed out that innovation mainly relies on private

firms. Because innovation means breaking with routines, it inevitably implies high risk. High-tech innovation, in particular, is a high-risk venture with a very low success rate. However, once it succeeds, there may be parabolic profits, which will then attract additional funds for investment. For state-owned enterprises (SOEs), however, due to the congenital lack of risk-taking incentive mechanisms, the SOE leaders are neither courageous nor able to assume such responsibility. Moreover, irrespective of how great the profit is, SOE leaders themselves do not profit from it, and thus they have no incentive to take such high risks, and it is unrealistic to look to SOEs for innovation.

The private economy, on the other hand, is the most willing to take risks because of the strong incentive to pursue self-interest, and thus the most innovation-driven. Therefore, as shown clearly from the situations in different countries, the main agents of innovation (non-basic scientific research) are all private firms. For instance, the most widely-recognized innovative companies even in China, such as Alibaba, Tencent and Huawei, are also private firms. The emergence of Internet finance, such as Yu'eobao, is the result of barring the private economy from entering the financial industry under the control of state-owned enterprises, and thus private firms have no choice but to resort to financial technology innovation for survival.

9.3.5 Price Discrimination of Monopolies

The pricing of firms discussed above is uniform price, but in reality, firms usually adopt differential pricing, known as **price discrimination**. Under differential pricing, the firm may charge varied prices to different customers or different purchase quantities, even though the production cost is the same. The premise of differential pricing is that the purchase of the demander will generate consumer surplus. In the analysis of monopoly pricing, the firm obtains monopoly profit by setting a uniform price, while different consumers also obtain consumer surplus to some extent. To further increase profits, a monopolist may obtain some, or even all, of the consumer surplus through price discrimination.

There are many ways that price discrimination appears in the real world.

For instance, magazines offer discounts to students; airlines charge different prices at different times of purchase; some associations, such as the American Economic Association, charge membership fees according to the category of their members; many stores offer discounts or coupons on holidays; some product packages offer coupons, which allow consumers to purchase the product at a lower price next time and enable sellers to charge first-time customers a higher price than repeat visitors; for telecommunications services, there are many kinds of packages and different charging standards in different time periods. These are all various forms of price discrimination.

However, as Dennis W. Carlton and Jeffrey M. Perloff (1998) pointed out,¹ *movies and television shows often portray, as great heroes, physicians who charge poorer patients lower rates. In very old movies, a country doctor may accept a chicken as payment, instead of insisting on cash. Are doctors selfless creatures or profit maximizers who engage in price discrimination? Certainly, some physicians see indigent patients at no cost or nominal fees as an act of altruism. Others, however, may be engaging in price discrimination.*

There are many broadly adopted forms of price discrimination.

The first kind is **two-part tariff**. The firm first charges the consumer a fixed fee (the first charge); after paying the fee, the consumer then has the right to purchase the goods at a specified price, such as monthly charges by cell phone service providers in many countries. Some amusement parks, as another example, charge an admission fee first and then a surcharge for each event.

The second is **quantity discount**. In this case, different unit prices are set according to the quantity of purchase. For instance, the sales of numerous products discriminate between group purchasing and individual purchases.

The third is **tie-in sale**. In this case, a firm only sells the product to the customers they want to purchase when they also buys some other product(s) at the same time. For example, when a consumer wants to purchase a durable machine, the consumer must purchase the maintenance service

¹An excerpt from Dennis W. Carlton and Jeffrey M. Perloff (1998), Chapter 11, Page 640.

or maintenance parts together. When selling copiers, the firm may require that the customer purchase ink, copy paper, etc., at the same time. When the consumer buys a cell phone, he or she also needs to purchase a charger from the seller. Indeed, it is sometimes the case that consumers have no choice due to technological compatibility requirements, as in the case of chargers for Apple products.

The fourth is **quality discrimination**. Quality discrimination may occur when a firm offers commodities with different profiles of price and quality. For example, a firm can offer high-quality products at a high price to some customers who greatly value quality and low-quality products at a low price to the others, thus dividing the customers into two markets. As another example, airline tickets are divided into first class, business class, and economy class. A similar situation is found in the case of train and performance tickets. In the following, a real-world example in Dupuit's discussion of railroad tariffs is given:

It is not because of the few thousand francs which would have to be spent to put a roof over the third-class carriages or to upholster the third-class seats that some company or other has open carriages with wooden benches.... What the company is trying to do is prevent the passengers who can pay the second-class fare from traveling third-class; it hits the poor, not because it wants to hurt them, but to frighten the rich.... And it is again for the same reason that the companies, having proven almost cruel to third-class passengers and mean to second-class ones, become lavish in dealing with first-class passengers. Having refused the poor what is necessary, they give the rich what is superfluous.²

In addition, it should be noted that not every non-uniform pricing constitutes price discrimination. There are numerous other reasons that firms would charge different consumers varied prices. For example, when firms offer quantity discounts to consumers, it may be due to cost savings of large orders that firms would return to bulk buyers. Sometimes, even for the same price, price discrimination can also exist. For example, a firm may charge consumers at different locations the same price for a home delivery service, in which the price cannot fully reflect the cost difference, which

²An excerpt from Tirole (1988), Chapter 3, Page 150.

comprises the cost of production and transportation.

Principle of Price Discrimination

As mentioned above, the increment of revenue of a monopolist generated by one extra unit of production is the summation of two effects. One is the incremental revenue p generated by selling one extra unit of product at price p . The other is the reduction in the total revenue resulting from the lowered price, the amount of which is $Q\Delta p$. Consequently, when the total revenue is $p(Q)Q$, its marginal revenue is $p(Q) + Q\frac{dp(Q)}{dQ}$.

If a monopolist only needs to lower the price of the extra product, then the firm will constantly expand production as long as the price of the last unit of product is higher than the marginal cost. Until the price of the last unit is equal to the marginal cost, the monopolist will stop producing and turn to differential pricing for additional profits. In fact, all kinds of price discrimination can be viewed as a marketing design of the firm to reduce the negative impact of the second effect of expanding production on marginal revenue.

Conditions of Price Discrimination

It is not unconditional for a firm to practice price discrimination because consumers can arbitrage between different prices to undermine the intention of the firm. The important conditions for price discrimination are as follows.

Firstly, the firm must possess certain power to manipulate the market (or the ability to obtain profit by setting the price above the marginal cost); otherwise, the firm cannot charge consumers a price that is higher than that in a competitive market.

Secondly, the firm needs to know or infer consumers' willingness to pay for each unit of product, which varies with consumers and sales volume. In other words, the firm should be able to determine which type of consumers can be charged a high price.

Thirdly, the firm should be able to prevent or restrict resale. Resale means that low-price buyers sell the commodities to customers who are

willing to pay a higher price. If it is easy to resale, then the firm's attempt to charge higher prices to one consumer group than the other will not succeed. Moreover, in the case of quantity discount, the firm has to prevent bulk buyers from reselling the commodities to small buyers.

Resale can be prevented or reduced in the following ways.

(1) **Service.** The vast majority of services cannot be resold.

(2) **Guarantee.** Taking after-sale service as an example, a firm may announce that only first-time buyers will be provided a guarantee or free after-sale service. For example, black market electronic products sold via unauthorized channels are apparently not guaranteed.

(3) **Mixture.** A firm may include certain other substances in a product, so that the product cannot be used for other purposes. For instance, medicinal alcohol cannot be converted into alcoholic beverages.

(4) **Transaction costs.** A large transaction cost can be charged for resale. Taking the resale of coupons (which allows consumers to purchase a product at a lower price) as an example, it costs too much to search for buyers without coupons.

(5) **Contract remedy.** The firm may ban resale as a condition of sale in the purchase contract. For example, if one uses a student sports game ticket, one needs to present one's student ID. In addition, in some universities, students and teachers can purchase computers below the market price, but resales are banned.

(6) **Vertical integration.** Upstream firms can integrate some downstream firms to charge higher prices than do other downstream firms.

Three Typical Price Discriminations

Price discrimination can be divided into three types.

- (1) **First-degree price discrimination**, also known as **perfect price discrimination**. By charging a different price for each unit of product, the producer can obtain the entire consumer surplus. Such price discrimination generally requires the producer to know the reserve price of each consumer, and to prevent resale or arbitrage between consumers.

- (2) **Second-degree price discrimination.** The firm prices the same product differently based on the quantity of purchase. In reality, the two conditions above for perfect price discrimination are rarely fully satisfied. At this time, the producer can design different profiles of quantity and price from which the consumer can freely choose. The producer can still obtain the consumer surplus (more than that of uniform pricing).
- (3) **Third-degree price discrimination.** The firm divides the market into two or more groups, and prices the product for each group. The producer can observe certain signals related to consumer preferences (e.g., age, occupation, location, etc.), and utilize these signals to divide customers into different markets for differential pricing.

The difference between second-degree and third-degree price discrimination is that in third-degree price discrimination, firms can take advantage of the signals of consumer types to divide the market into many independent markets for differential pricing; in second-degree price discrimination, because firms cannot observe the signals of consumer types, there is only one market, and thus firms implement price discrimination by setting different purchase contracts from which consumers can choose.

Besides, we also have the **two-part tariff** that is related to price discrimination and provides another means of extracting consumer surplus. A two-part tariff consists in charging consumers with a lump sum fee for the right to purchase the product and then a uniform price per unit consumed.

9.3.6 First-Degree (Perfect) Price Discrimination

First-Degree Price Discrimination means to price each unit of commodity differently. Let us consider a simple case. There is only one type of consumer in the market, and they all have unit demand (purchasing either nothing or 1 unit of product). Let their valuation of unit product be V . Then, the monopolist can obtain all of the consumer surplus by setting the price $p = V$.

Now, consider an extended case. There are n types of consumers in the market, consumers of each type all have unit demand, and their valuation of the unit product is $V_i, i = 1, \dots, n$. Then, the monopolist can price, respectively, for the n types of consumers. For consumers of type i , the price is set as V_i . Similarly, the monopolist can extract all the consumer surplus.

For the case of non-unit demand, suppose that there are n consumers in the market. The market demand function is $D(p)$, and each consumer's demand function is $\frac{D(p)}{n}$. If only uniform price is allowed, the maximum profit obtained by the monopolist is $p^m D(p^m) - C(D(p^m))$, where p^m is the monopoly price. If a more flexible pricing strategy can be adopted, the monopolist will be able to make greater profits.

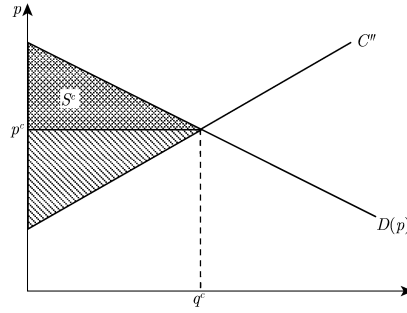


Figure 9.8: Perfect price discrimination

Suppose that the monopolist adopts a two-part tariff which is quite the same as first-degree price discrimination, which charges a fixed fee and a same unit price, i.e., $T(q) = A + pq$. If the monopolist adopts competitive pricing, then $T(q) = p^c q$. At this point, the net surplus of all consumers is $S^c = \int_0^{q^c} [p(q) - p^c] dq$, where $q^c = p^{-1}(p^c)$, and the net surplus of each consumer is $\frac{S^c}{n}$. If $A = \frac{S^c}{n}$, $p = p^c$, the two-part tariff $T(q) = A + pq$ will be accepted by the consumer, at which the net surplus left for the consumer is 0, and the monopoly profit equals the optimal social welfare. Setting aside the issue of allocation and equity, and only taking efficiency into consideration, perfect price discrimination increases social welfare, and the decrease of consumer surplus is offset by the increase of producers' profits. Figure

9.8 shows the change of consumer surplus and firm profit under perfect price discrimination.

We can extend the above analysis to the case of n types of consumers. Each type of consumers' demand function is $Q^i(p)$. In this way, each type of consumer makes one separate market. Similar to the previous analysis, if different two-part tariffs are adopted for each type of consumer, then in each market, the unit price corresponds to the competitive market price, while the fixed fee is the surplus of consumers of each type corresponding to the competitive price.

A serious information problem exists in perfect price discrimination, as the firm does not know the consumers' type, and the consumers are also reluctant to disclose their type in most cases. In this situation, perfect price discrimination is impossible, and the firm cannot obtain all of the consumer surplus.

9.3.7 Second-Degree (Self-Selection) Price Discrimination

Second-Degree Price Discrimination means to price the same product differently based on the quantity of purchase. If the information of the consumer's type cannot be observed, the monopolist can also design various consumption bundles from which consumers can choose.

Suppose that there are two types of consumers in the market, whose valuations of the commodity are $\theta_i V(q)$, $i = 1, 2$. Figure 9.9 describes the demands of the two types of consumers and their consumer surplus.

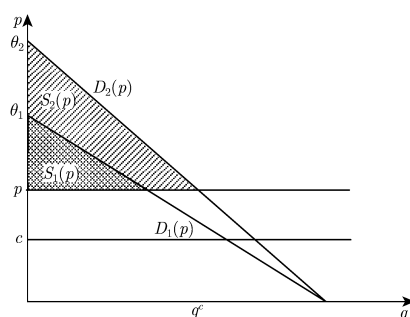


Figure 9.9: Demands of two types of consumers and their consumer surpluses

θ_i is the preference parameter which characterizes type i consumers' preferences for commodities. Assume that in the market, the proportion of consumers with preference θ_1 is λ , and the rest are consumers with preference θ_2 . Let $\theta_2 > \theta_1$, $V(q) = \frac{1 - (1 - q)^2}{2}$. The firm cannot identify the type of consumers, and the marginal cost is c .

We first look at the demands of different types of consumers.

First, for type i of consumers, in the face of the market price p , their decision is to solve the following maximization problem

$$\max \theta_i V(q) - pq.$$

Its first-order condition is

$$\theta_i V'(q) = p,$$

i.e.,

$$\theta_i(1 - q) = p.$$

Therefore, the demand function of type i consumers is

$$D_i(p) = 1 - \frac{p}{\theta_i}.$$

The consumer surplus of type i consumers is

$$S_i(p) = \theta_i \left[\frac{1 - (1 - D_i(p))^2}{2} \right] - pD_i(p) = \frac{(\theta_i - p)^2}{2\theta_i}.$$

The market demand function is

$$D(p) = \lambda D_1(p) + (1 - \lambda) D_2(p) = 1 - \frac{p}{\theta},$$

where $\frac{1}{\theta} \equiv \frac{\lambda}{\theta_1} + \frac{1 - \lambda}{\theta_2}$.

Now, we discuss how firms can implement second-degree price discrimination for consumers. Assume that a two-part tariff is adopted for differential pricing. In order to understand the difference between second-degree price discrimination and other types of price discrimination, we use perfect price discrimination and uniform monopoly pricing as a bench-

mark.

In the previous discussion of perfect price discrimination, we know that the firm gains the revenue of perfect price discrimination by setting various types of two-part tariffs for different consumers. In perfect price discrimination, for type i of consumers, the corresponding two-part tariff set by the monopolist is $T_i(q) = A_i + cq$, satisfying $p^{fd} = c$ and $A_i = S_i(c)$. The monopolist gains all consumer surplus, and the monopolist profit is

$$\pi^{fd} = \lambda \frac{(\theta_1 - p)^2}{2\theta_1} + (1 - \lambda) \frac{(\theta_2 - p)^2}{2\theta_2}.$$

If the monopolist adopts uniform monopoly pricing, then the monopoly price can only be a linear price, i.e., $T(q) = pq$. The monopolist's best choice of price is to solve the following maximization problem

$$\max(p - c) \left(1 - \frac{p}{\theta}\right).$$

The monopoly price and profit are $p^m = \frac{c + \theta}{2}$ and $\pi^m = \frac{(\theta - c)^2}{4\theta}$, respectively.

When the monopolist does not know the type of the consumer, how should it implement second-degree price discrimination? We start with a simple way of using a single two-part tariff for second-degree price discrimination, i.e., $T(q) = A + pq$. Since the firm cannot distinguish the type of consumers, this pricing is for consumers of all types.

When the unit price is p , consumers with preference θ_1 may still purchase, and the maximum fixed fee that they are willing to pay is $A = S_1(p)$. Then, the consumer surplus of consumers with preference θ_1 is completely extracted, but consumers with preference θ_2 still have positive consumer surplus. Since $S_1(p) > 0$, given the price p , the optimal fixed fee is $A = S_1(p)$. The subsequent question is how much the unit price p should be.

For the monopolist, the most profitable two-part tariff is found by solving the following optimization problem,

$$\max_p S_1(p) + (p - c)D(p) = \frac{(\theta_1 - p)^2}{2\theta_1} + (p - c) \left(1 - \frac{p}{\theta}\right).$$

The first-order condition is,

$$-\frac{\theta_1 - p}{\theta_1} + \left(1 - \frac{p}{\theta}\right) - \frac{p - c}{\theta} = 0,$$

so the optimal unit price in second-degree price discrimination is,

$$p^{sd} = \frac{c}{2 - \theta/\theta_1},$$

and the profit is denoted by π^{sd} .

Obviously, $c < p^{sd} < p^m$, and $\pi^{fd} \geq \pi^{sd} \geq \pi^m$, i.e., the profit of perfect price discrimination is higher than that of second-degree price discrimination, the latter of which is higher than the monopoly profit under uniform pricing. Social welfare under perfect price discrimination is the highest. For second-degree price discrimination and uniform monopoly pricing, given any price p , social welfare is

$$TW(p) = \lambda S_1(p) + (1 - \lambda) S_2(p) + (p - c) [\lambda D_1(p) + (1 - \lambda) D_2(p)],$$

where

$$TW'(p) = (p - c) [\lambda D'_1(p) + (1 - \lambda) D'_2(p)].$$

When $p \geq c$, social welfare decreases as p increases. Since $c < p^{sd} < p^m$, compared with uniform monopoly pricing, second-degree price discrimination improves social welfare.

In second-degree price discrimination, the monopolist can actually have more choices, meaning that she can adopt more complicated nonlinear pricing methods to obtain higher profits. In the following, we discuss the nonlinear pricing that maximizes the monopolist's profit. This constitutes a standard principal-agent problem. In Part VI, we discuss the principal-agent theory in detail.

Suppose that the commodity bundle designed by the firm for consumers with preference θ_i is (q_i, T_i) , where the purchase quantity is q_i and the total payment is T_i , and meanwhile, $(q_i, T_i) \neq (q_j, T_j), i \neq j$. This design needs to meet two conditions: first, choosing (q_i, T_i) over (q_j, T_j) can bring more utility to consumers with preference θ_i , which is called the **incentive**

compatibility (IC) constraint; second, the net consumer surplus cannot be less than zero when consumers with preference θ_i choose (q_i, T_i) , which is called the **participation constraint**.

The objective of the monopolist is,

$$\max_{(q_i, T_i)} \lambda(T_1 - cq_1) + (1 - \lambda)(T_2 - cq_2), \quad (9.3.1)$$

subject to

$$\theta_1 V(q_1) - T_1 \geq 0 \quad (\text{Type } \theta_1 \text{'s participation constraint}), \quad (9.3.2)$$

$$\theta_2 V(q_2) - T_2 \geq 0 \quad (\text{Type } \theta_2 \text{'s participation constraint}), \quad (9.3.3)$$

$$\theta_1 V(q_1) - T_1 \geq \theta_1 V(q_2) - T_2 \quad (\text{Type } \theta_1 \text{'s IC constraint}), \quad (9.3.4)$$

$$\theta_2 V(q_2) - T_2 \geq \theta_2 V(q_1) - T_1 \quad (\text{Type } \theta_2 \text{'s IC constraint}). \quad (9.3.5)$$

At the optimal point, at least one equality holds in inequalities (9.3.2) and (9.3.4); otherwise, we can increase the value of T_1 to increase the objective value. Similarly, at least one equality holds in the inequalities (9.3.3) and (9.3.5).

For inequality (9.3.5), since $\theta_2 V(q_2) - T_2 \geq \theta_2 V(q_1) - T_1 > \theta_1 V(q_1) - T_1 \geq 0$, inequality (9.3.3) is strict, and thus we have $\theta_2 V(q_2) - T_2 > 0$, which means that $\theta_2 V(q_2) - T_2 = \theta_2 V(q_1) - T_1$, i.e., (9.3.5) is binding. If the equality holds in inequality (9.3.4), add it to inequality (9.3.5) and we obtain $V(q_1) = V(q_2)$, i.e., $q_1 = q_2$, which implies that $T_1 = T_2$, and thus contradicts the previous assumption. Therefore, inequality (9.3.4) is strict, which implies that (9.3.2) is binding.

To sum up the above discussion, only the inequalities (9.3.2) and (9.3.5) hold with equality, which implies that

$$T_1 = \theta_1 V(q_1)$$

and

$$\theta_2 V(q_2) - T_2 = \theta_2 V(q_1) - T_1 = (\theta_2 - \theta_1)V(q_1).$$

Substituting them into the objective function,

$$\max\{\lambda(\theta_1 V(q_1) - cq_1) + (1 - \lambda)(\theta_2 V(q_2) - cq_2 - (\theta_2 - \theta_1)V(q_1))\}.$$

The first-order condition is then given by

$$\theta_1 V'(q_1) = \frac{c}{1 - \frac{1 - \lambda}{\lambda} \frac{\theta_2 - \theta_1}{\theta_1}},$$

$$\theta_2 V'(q_2) = c.$$

Because of $V(q) = \frac{1-(1-q)^2}{2}$, we have

$$q_1 = 1 - \frac{c}{\theta_1 - \frac{1 - \lambda}{\lambda}(\theta_2 - \theta_1)},$$

$$q_2 = 1 - \frac{c}{\theta_2}.$$

Therefore, the consumption for high-demand type consumers is socially optimal (the marginal utility of commodities equals the marginal cost), but the consumption for low-demand type consumers is lower than the socially optimal level. Meanwhile, the high-demand type consumers receive positive consumer surplus.

Figure 9.10 depicts the above optimal nonlinear pricing. Point B_1 in the figure corresponds to the purchase profile (q_1, T_1) of consumers with preference θ_1 , where the consumer surplus is zero; point C_2 corresponds to the purchase profile (q_2, T_2) of consumers with preference θ_2 , whose consumer surplus is greater than zero. In the previous second-degree price discrimination with two-part tariffs, the purchase profile for consumers with preference θ_1 is B_1 and for consumers with preference θ_2 is B_2 . Compared with the above single two-part tariff, consumers with preference θ_2 purchase more commodities in nonlinear second-degree price discrimination, and the monopolist's profits are also higher. Since the second-degree price discrimination with two-part tariffs is a form of nonlinear pricing, the monopolist under the above optimal nonlinear pricing chooses a form of discrimination that is different from the two-part tariff, resulting in a higher

profit.

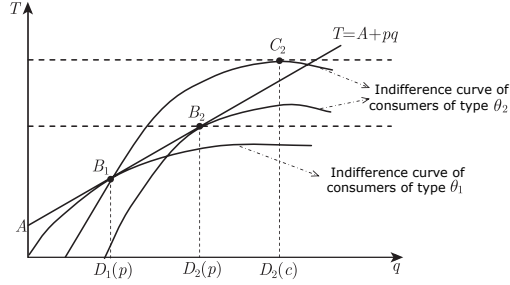


Figure 9.10: Second-degree price discrimination

9.3.8 Third-Degree (Multi-Market) Price Discrimination

Third-degree Price Discrimination means to divide the market into two or more groups, for which the same product is priced differently. The monopolist can divide the aggregate demand into m markets based on the information of consumers, and knows the demand curves of these markets. If all markets are independent and there is no arbitrage between any two markets, the monopolist can set different prices for different markets, but a unified price within each market. What the monopolist practices is termed **multi-market monopoly pricing**.

Let $\{p_1, \dots, p_i, \dots, p_m\}$ be the prices of m markets, and the corresponding demands of the m markets be

$$\{D_1(p_1), \dots, D_i(p_i), \dots, D_m(p_m)\}.$$

The monopolist chooses m monopoly prices for the m markets.

The objective function of the monopolist is

$$\max \sum_i p_i D_i(p_i) - C \left(\sum_i D_i(p_i) \right).$$

The first-order condition is:

$$\frac{p_i - C'(q)}{p_i} = \frac{1}{\varepsilon_i},$$

where $\varepsilon_i = -\frac{p_i D'_i(p_i)}{D_i(p_i)}$ is the absolute value of the elasticity of demand of market i .

From the above condition, we know that, compared with uniform pricing, consumers with higher elasticity of demand face lower prices under price discrimination. As a consequence, this kind of consumers prefer differential pricing. Less-elastic consumers will be charged higher prices due to price discrimination, resulting in reduced welfare. However, for the monopolist, differential pricing will certainly bring more profits; otherwise, there is no need for it. Because of the pair of opposite effects, the change of social welfare in third-degree price discrimination is uncertain.

Now, let us consider a more general case. Suppose that there are m markets, the marginal cost is constant c , the demand of the i th market is $q_i = D_i(p_i)$, the corresponding consumer surplus is $S_i(p_i)$, and the firm's profit is $(p_i - c)q_i$. If differential pricing is not allowed, the monopolist sets a uniform price \bar{p} for all markets. Then, the corresponding demand of the i th market is $\bar{q}_i = D_i(\bar{p})$, the consumer surplus is $S_i(\bar{p})$, and the profit is $(\bar{p} - c)\bar{q}_i$. After adopting differential pricing, the demand of the i th market will become $\Delta q_i \equiv q_i - \bar{q}_i$.

The change of social welfare after adopting price discrimination is:

$$\Delta W = \left\{ \sum_i (S_i(p_i) - \bar{S}_i(\bar{p})) \right\} + \left\{ \sum_i (p_i - c)q_i - \sum_i (\bar{p} - c)\bar{q}_i \right\}. \quad (9.3.6)$$

The (net) consumer surplus is $S(p) = \int_p^{\hat{p}} D(\xi) d\xi$, where \hat{p} is the maximum price, i.e., $D(\hat{p}) = 0$. Since $S'(p) = -D(p) < 0$ and $S''(p) = -D'(p) > 0$, $S(p)$ is a convex function. Thus, we have

$$S_i(p_i) - S_i(\bar{p}) \geq S'_i(\bar{p})(p_i - \bar{p}), \quad (9.3.7)$$

$$S_i(p_i) - S_i(\bar{p}) \leq S'_i(p_i)(p_i - \bar{p}). \quad (9.3.8)$$

Substituting formulas (9.3.7) and (9.3.8) into (9.3.6), we obtain

$$\Delta W \geq \sum_i (p_i - c)\Delta q_i$$

and

$$\Delta W \leq (\bar{p} - c) \sum_i \Delta q_i,$$

where $\Delta q_i = q_i(p_i) - q_i(\bar{p})$. If $\Delta q_i \geq 0, \forall i$, then $\Delta W \geq 0$. In other words, after adopting differential pricing, the demand of each type of consumers increases, and thus differential pricing increases social welfare.

If $\sum_i \Delta q_i \leq 0$, then $\Delta W \leq 0$, which means that the aggregate demand of consumers decreases, and thus price discrimination reduces social welfare.

Therefore, whether the effect of differential pricing on social welfare is positive or negative depends on its influence on sales volume.

Example 9.3.1 (Linear demand) Suppose that there are two markets, the demand functions are $q_1 = a_1 - b_1 p$ and $q_2 = a_2 - b_2 p$, respectively, and the marginal cost is 0. In addition, suppose that $a_1 \geq a_2$ and $b_1 \leq b_2$, which means that market 1 is larger than market 2.

Suppose that the monopolist can adopt price discrimination. In market i , the objective of the monopolist is

$$\max p_i(a_i - b_i p_i).$$

The first-order condition of profit maximization is

$$p_i = \frac{a_i}{2b_i},$$

$$q_i = \frac{a_i}{2}.$$

Suppose that the monopolist can only set a uniform price instead of differential pricing. If the monopolist sets a uniform price \bar{p} in all markets, and at the same time all markets have positive demands (of course, the demand of some markets can be zero at a certain price level, which will be discussed later), then the aggregate market demand is

$$Q = \bar{q}_1 + \bar{q}_2 = (a_1 + a_2) - (b_1 + b_2)\bar{p}.$$

As a result, the objective function of the monopolist is

$$\max \bar{p}[(a_1 + a_2) - (b_1 + b_2)\bar{p}].$$

The first-order condition of profit maximization is

$$\bar{p} = \frac{(a_1 + a_2)}{2(b_1 + b_2)}.$$

Thus, the aggregate demand is

$$Q = \bar{q}_1 + \bar{q}_2 = \frac{(a_1 + a_2)}{2} = q_1 + q_2,$$

and the profit is

$$\bar{\pi} = \frac{(a_1 + a_2)^2}{4(b_1 + b_2)^2}.$$

Since $\bar{q}_1 + \bar{q}_2 = q_1 + q_2$, i.e., $\Delta q_1 + \Delta q_2 = 0$, price discrimination lowers social welfare.

At uniform pricing, if some markets no longer have positive demands, then it can be assumed that only market 1 is still open, while market 2 is closed. The monopoly price, output, and profit are $\tilde{p} = \frac{a_1}{2b_1}$, $\tilde{q} = \frac{a_1}{2}$, and $\tilde{\pi} = \frac{a_1^2}{4b_1}$, respectively. Then, we have $\tilde{\pi} = \frac{a_1^2}{4b_1} > \frac{(a_1 + a_2)^2}{4(b_1 + b_2)^2} = \bar{\pi}$. For example, if $a_1 = 2$, $a_2 = 1$, $b_1 = 1$ and $b_2 = 2$, then $\tilde{\pi} = 1 > \frac{3}{4} = \bar{\pi}$. We have $\Delta q_1 = 0$, $\Delta q_2 = q_2 > 0$. Consequently, price discrimination increases social welfare in this case.

Therefore, the effect of third-degree price discrimination on social welfare depends on the specific demand and characteristics of technology, and thus is indeterminate.

Different properties of commodities, such as those of durable goods and non-durable goods, will also affect the choice of the monopolist. In the next subsection, we discuss the monopolist's choice in the durable goods market.

9.3.9 Monopolist of Durable Goods

When the monopolist provides a durable good, intertemporal demand will change the interaction between the monopolist and consumers, because intertemporal demand will generate a dynamic game structure. In the pre-

vious discussion of game theory, we know that the commitment of players is usually an important factor in a dynamic game. When consumers have purchased a durable good, they may not purchase it again in the future, and thus the monopolist will face consumers that are different from the previous ones, who we call the remaining consumers.

Since the willingness to pay of the remaining consumers is lower than that of the previous consumers, as long as the marginal cost of the monopoly is sufficiently low, the monopolist will have the incentive to lower the price in order to obtain additional profits. When expecting intertemporal price discrimination from the monopolist, rational consumers will change their purchase decisions. In fact, concerning the issue of durable goods, the pricing flexibility of the monopolist will harm her own profits. When the monopolist adjusts the price frequently, the monopoly profit approaches zero. This conclusion is called the Coase conjecture.

We will identify some means to restrict the flexibility of price adjustment in business, such as the most-favored-customer clause for price difference compensation. Sometimes, the firm will choose to rent instead of selling the goods, in order to enhance the credibility of price adjustment. In other situations, the firm will choose uneconomical durability, also called planned abandonment, as in the case of the regular publication of new versions of textbooks with approximately identical contents.

To illustrate this issue, let us examine a simple example. Assume that there are seven consumers, their willingness to pay for a durable good is 1, 2, ..., 7, respectively, and the production cost is 0. Let time be discrete, and the time discount rate be δ . Let us assume that there is only one period, and the monopolist charges a monopoly price 4 and sells the good to consumers with willingness to pay no less than 4. At this point, the monopoly profit is 16. Now, consider multiple periods. If the monopolist charges 4 in the first period and consumers with willingness to pay at least 4 accept this price, then in the second period, in the face of consumers whose willingness to pay is 1, 2, and 3, the monopolist will choose the monopoly price 2, and thus consumers with willingness to pay no less than 2 will accept this price. However, the above outcome cannot be a stable equilibrium, because if expecting price reduction in the second period, consumers who

have willingness to pay at least 4 will not purchase in the first period, and the demand in the first period will decrease.

Such examples are ubiquitous in reality. For example, many electronic products are durable goods, such as the iPad. When the product is newly released, only consumers with a high willingness to pay will purchase it. As people expect that the firm will inevitably lower the price to attract medium- and low-end consumers in the future, consumers with a lower willingness to pay will wait for future price reductions.

Sale and Rental of Durable Goods

Here, we use an example to explore how the monopolist's pricing commitment affects its profits.

Assume that the lifespan of a certain product is two periods, consumers also live for only two periods, and each consumer purchases one unit of product at most. The value of the product to consumers is v per period, which is uniformly distributed over $[0, 100]$. The time discount rate for consumers is 1, and the marginal cost of production is zero. Here, we define the game structure of the sale of durable goods as follows:

(1) Players: the monopolist and consumers (who have different values v of the product).

(2) A set of strategies: the producer chooses prices p_1 and p_2 for the first and second period, respectively, which indirectly determines the demands q_1 and q_2 in the two periods, meaning that q_1 consumers purchase in the first period and q_2 consumers purchase in the second period. Consumers choose whether and when to purchase.

(3) Payoffs: first, for consumers (whose valuation of product is v in every period), $u(v, 1) = 2v - p_1$ if purchasing in the first period, $u(v, 2) = v - p_2$ if purchasing in the second period, and $u(v, \phi) = 0$ if not purchasing at all. Then, for the monopolist, her payoff is made up of the profits in the two periods, i.e., $\pi = p_1 q_1 + p_2 q_2$.

In the following, we use backward induction to solve the above game.

Assume that \hat{v} is the value of the product for marginal buyers who are indifferent between purchasing in the first period and purchasing in the

second period, and thus $\hat{v} = p_1 - p_2$. Consumers whose valuation of the durable good is $v > \hat{v}$ will choose to purchase in the first period, and consumers whose valuation of the durable good is $v > \hat{v}$ and $v \geq p_2$ will choose to purchase in the second period.

In the second period, the inverse demand function is $p_2 = 100 - q_1 - q_2$, where $q_1 = 100 - \hat{v}$ and $q_2 = \hat{v} - p_2$. The optimization condition of the second period means that the marginal revenue equals the marginal cost, i.e., $MR = MC$, and thus we obtain $100 - q_1 - 2q_2 = 0$, i.e.,

$$p_2 = q_2 = \frac{100 - q_1}{2}.$$

Now, consider demand q_1 or price p_1 in the first period of the game. Marginal consumers' valuation of the durable good is \hat{v} , satisfying $\hat{v} = p_1 - p_2$, and meanwhile, $q_1 = 100 - \hat{v}$, and thus we obtain

$$p_1 = \frac{300 - 3q_1}{2}.$$

Therefore, the monopolist's total profit of two periods is

$$\pi = \frac{300 - 3q_1}{2} q_1 + \left(\frac{100 - q_1}{2} \right)^2.$$

The choice of maximizing profits is $q_1 = 40, p_1 = 90$ and $q_2 = 30, p_2 = 30$ when the profits are $\pi = 4,500$.

If the monopolist chooses to rent instead of selling, because renting does not involve the issue of product durability, the demands for each period are identical. From the above model setting, we know that the inverse demand function for each period is $p_t = 100 - q_t, t = 1, 2$. The total profits of the two periods of the monopolist are $\pi = p_1 q_1 + p_2 q_2$. From the first-order condition, we obtain that $q_1 = 50, p_1 = 50; q_2 = 50, p_2 = 50; \pi = 5,000$.

We find that, in this example, the profit from the rental of durable goods is higher than the profit from the sale. The reason for this phenomenon is that during the selling process, the dynamic commitment issue of the monopolist may affect her profit from the first-period consumers, while renting avoids the issue of pricing commitment for durable goods.

9.3.10 Monopoly in an Input Market

A similar classification of market structures exists for input markets. If firms take the factor prices as given, then we have competitive factor markets. If instead there is only one firm which demands some factor of production, then we have a **monopolistic factor market**, and this kind of market structure is called a **monopsony**. In the monopolistic factor market, the behavior of a monopsonist determines the price of the purchased factor. Now, we discuss the rational decisions of the monopsonist.

Let us consider a simple example of a firm that is a competitor in its output market, but the sole purchaser of some input good. Let $w(x)$ be the (inverse) supply curve of this factor of production. Then, the profit maximization problem for the firm is:

$$\max_x pf(x) - w(x)x,$$

where $f(x)$ is a production function, and p is the price of a competitive product. The first-order condition is:

$$pf'(x^*) - w(x^*) - w'(x^*)x^* = 0.$$

$pf'(x^*)$ refers to the marginal value of product, or the marginal revenue of product, and $w(x^*) + w'(x^*)x^*$ is the marginal purchase cost of the factor.

We can rewrite the above condition as

$$pf'(x^*) = w(x^*) \left(1 + \frac{1}{\varepsilon(x)} \right),$$

where $\varepsilon(x) = \frac{dx}{dw(x)} \frac{w(x)}{x}$ is the price elasticity of supply. As the elasticity goes to infinity, the monopsonist becomes perfectly competitive in the input market.

Recall that, in Chapter 4, we considered only the behavior of a firm in competitive factor markets. Similarly, it is possible to define the cost function for a monopsonistic firm. For example, suppose that $x_i(w)$ is the supply function for factor i . Then, we can define $C(y) = \min \sum w_i x_i(w)$,

such that $f(x(w)) = y$. At this time, $C(y)$ is the minimum cost of producing y in the monopolistic factor market.

9.4 Monopolistic Competition

The previous section assumes that the demand curve for a product depends only on the price set by the monopolist. However, this constitutes an extreme case. Most commodities have some substitutes, and the prices of those substitutes will affect the demand of a commodity. In this section, we consider what occurs when the prices and quantities of similar products produced by many firms affect one another, in what we term a **monopolistically competitive market**.

We assume that a group of n “monopolists” sell similar, but not identical, products. The price that consumers are willing to pay for the output of firm i depends on not only the level of output of firm i , but also the levels of output of the other firms: we write this inverse demand function as $p_i(q_i, \mathbf{q}_{-i})$, where $\mathbf{q}_{-i} = (q_1, \dots, q_{i-1}, q_{i+1}, \dots, q_n)$.

Each firm i chooses output level q_i in order to maximize its profits,

$$\max p_i(q_i, \mathbf{q}_{-i})q_i - C_i(q_i).$$

Since the demand facing firm i also depends on what the other firms do, how is firm i supposed to forecast the other firms’ behavior? We will adopt a very simple behavioral hypothesis that firm i assumes that the other firms’ behavior will be constant and thus we can use Nash equilibrium as solution concept. Then, each firm i will take the output level of other firms as given and choose its level of output q_i^* . We then have the following first-order condition:

$$p_i(q_i^*, \mathbf{q}_{-i}) + \frac{\partial p_i(q_i^*, \mathbf{q}_{-i})}{\partial q_i} q_i^* - C_i'(q_i^*) \leq 0, \quad \text{with equality if } q_i^* > 0.$$

The optimal output of all firms is denoted by $\mathbf{q} = (q_1, \dots, q_n)$. For firm i , there will be some optimal output level, denoted by $Q_i(\mathbf{q}_{-i})$.

In order for the market to be in equilibrium, each firm’s forecast about

the behavior of the other firms must be compatible with what the other firms actually do. As a consequence, if $\mathbf{q}^* = (q_1^*, \dots, q_n^*)$ is the vector of equilibrium output, it must satisfy:

$$q_i^* = Q_i(\mathbf{q}_{-i}^*), i \in \{1, 2, \dots, n\},$$

i.e., q_1^* must be the best response of firm 1 if it assumes that other firms are going to produce q_2^*, \dots, q_n^* .

For each firm, its marginal revenue equals the marginal cost given the actions of all of the other firms. This is illustrated in Figure 9.11. At the point of equilibrium under monopolistic competition depicted in Figure 9.11, firm i is making positive profits. In a monopolistically competitive industry, if there is no barrier and firms can freely enter and exit, it is necessary to consider long-run equilibrium.

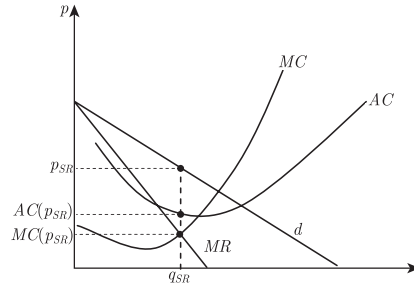


Figure 9.11: Short-run equilibrium under monopolistic competition

9.4.1 Long-run Equilibrium under Monopolistic Competition

Since firms can freely enter and exit, the profits of the monopolistically competitive industry will be zero in the long run. This means that firm i must set a price p_i^* and choose an output level q_i^* , such that

$$p_i^* q_i^* - C_i(q_i^*) = 0,$$

or

$$p_i = \frac{C_i(q_i^*)}{q_i^*}.$$

Therefore, in long-run equilibrium, the price must be equal to the average cost, but higher than the marginal cost. This means that there is excessive accumulation of production capacity, and the choice of the firm's output level is not efficient, i.e., it does not produce at the lowest point of its average cost. Figure 9.12 depicts the long-run equilibrium of such industry.

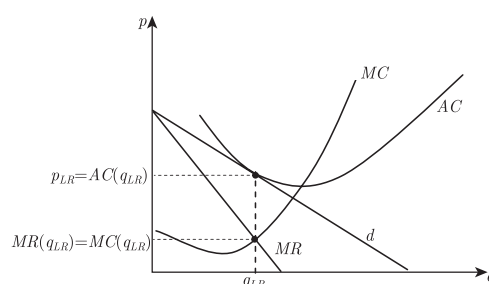


Figure 9.12: Long-run equilibrium under monopolistic competition

9.4.2 Social Welfare in Monopolistic Competition

Since the price is higher than the marginal cost, there is a loss of social welfare in monopolistic competition compared with perfect competition. In addition, as their products are different, the product varieties may also lead to an efficiency loss. Since each firm cannot obtain all of the consumer surplus, the positive externality shows the possibility of insufficient entry. Moreover, the entry of a firm will reduce the profits of others, and thus the negative externality shows the possibility of excessive entry. As a result, a monopolistically competitive industry may have too many or too few product varieties. In the following, we discuss a classic monopolistic competition model.

9.4.3 Dixit-Stiglitz Model of Monopolistic Competition

Assume that there is a representative consumer who prefers diverse products. There are L kinds of differentiated products, and L is endogenously determined. Assume that each firm can only produce one product. Then, how many firms or types of products will the market have in the long run?

Here, we analyze this problem according to the Dixit-Stiglitz classic model. We assume that consumer preference is a CES (constant elasticity of substitution) utility function,

$$U(q_1, \dots, q_L) = \left(\sum_{l=1}^L q_l^\rho \right)^{1/\rho}, \quad \rho \leq 1,$$

where q_l denotes the quantity of differentiated products. The consumer prefers diverse products, i.e., $\frac{\partial U(q_1, \dots, q_L)}{\partial q_l} \rightarrow \infty$ when $q_l \rightarrow 0$.

The budget constraint of the consumer is $\sum_{l=1}^L p_l q_l \leq I$, where p_l is the price of differentiated product l , and I is the exogenously-given income of the representative consumer. There are two parts of costs for firms to produce differentiated products: one is the fixed cost F ; and the other is the marginal cost c . Assume that these costs are both cost of labor. Therefore, the cost function for producing q_l differentiated products is

$$TC_l(q_l) = F + cq_l.$$

An equilibrium under monopolistic competition must satisfy the following conditions:

- (1) given income and market price, the consumer chooses a consumption bundle that maximizes his or her utility;
- (2) given the consumer's choice, firms of differentiated products (each of whom is a monopolist of its own product) chooses a monopoly price or output to maximize its profit;
- (3) firms can enter and exit freely: the point at which the market profits are zero determines how many firms enter the market.

First, for consumer maximization,

$$\begin{aligned} & \max_{q_1, \dots, q_L} \left(\sum_{l=1}^L q_l^\rho \right)^{1/\rho} \\ \text{s.t.} \quad & \sum_{l=1}^L p_l q_l = I. \end{aligned} \tag{9.4.9}$$

The Lagrangian function of the above optimization problem is

$$L(q_1, \dots, q_L; \lambda) = \left(\sum_{l=1}^L q_l^\rho \right)^{1/\rho} - \lambda \left(\sum_{l=1}^L p_l q_l - I \right).$$

The first-order condition is:

$$\left(\sum_{l=1}^L q_l^\rho \right)^{\frac{1-\rho}{\rho}} q_l^{\rho-1} = \lambda p_l, \quad l = 1, \dots, L;$$

$$\sum_{l=1}^L p_l q_l = I,$$

thus, we obtain that

$$\lambda = \frac{\left(\sum_{l=1}^L q_l^\rho \right)^{\frac{1}{\rho}}}{I}$$

and

$$q_l = \left(\frac{p_l}{I} \right)^{\frac{1}{\rho-1}} \left(\sum_{l=1}^L q_l^\rho \right)^{\frac{1}{\rho-1}}.$$

From the demand function above, we can solve for the price elasticity of demand for products in monopolistic competition as

$$\eta \equiv - \frac{\partial \ln q_l}{\partial \ln p_l} = \frac{1}{1 - \rho}.$$

Second, for each firm, its decision is to solve the following optimization problem,

$$\max_{p_l} D_l(p_l) p_l - c D_l(p_l) - F.$$

We then obtain that

$$p_l = \frac{c}{1 - \frac{1}{\eta}} = \frac{c}{\rho}.$$

By symmetry, we have

$$q_l = q = \frac{I}{L p_l} = \frac{I \rho}{L c}.$$

At the equilibrium point, the profit of each monopolistic competitive

firm is 0, and thus we obtain the equilibrium number of firms by

$$\frac{I}{L}(1 - \rho) = F.$$

Consequently,

$$L^* = \frac{I(1 - \rho)}{F}$$

and

$$q_l^* = \frac{F\rho}{(1 - \rho)c}.$$

From the above analysis, we can conclude that the larger is the elasticity of substitution ρ , the lower is the price, the smaller is the number of firms, and the greater is the output of differentiated products; the higher is the fixed cost, the smaller is the number of firms, or the fewer is the product varieties, the greater is the output; and the rise of revenue will increase the number of firms, but it has no effect on the price and quantity of the products.

9.5 Oligopoly

Since the action of each firm can influence the market price and actions of other firms, an oligopolistic industry generally does not exhibit the characteristics of perfect competition. In perfect competition, the interaction among firms is indirectly affected by the relationship between market price and profit. Now we discuss the direct interaction of a few firms in a market. Such a market is called an **oligopoly**. The study of this issue is grounded almost entirely on the game theory. Many of the specifications of market interactions are clarified with the concepts of game theory. The purpose of this section is to elucidate how the market power of a firm is determined under different circumstances. The so-called market power refers to the extent to which the firm's pricing can deviate from the marginal cost. It will also affect the welfare level of the market. In the following, we will first discuss static oligopolistic competition, then dynamic oligopolistic competition, and finally oligopolistic competition under asymmetric information.

9.5.1 Price Competition: Bertrand Model

The simplest and most basic static price competition model of oligopoly was proposed by the French economist Joseph Bertrand in 1883, so is called the **Bertrand model**. When the market structure of a monopoly is broken, competition arises among firms. If their strategic means is price competition, what will the interactions among them be like? What is the market equilibrium?

We first assume that there are only two firms in the market which are symmetric to each other. They produce homogeneous products with the same marginal cost c , and the market demand function is $q = D(p)$. We will find that in such a symmetric price competition and interactive equilibrium, the equilibrium prices of the two firms are both the marginal cost, which is the same as the outcome of perfect competition. At this point, adding just one competitor will make the market power of the original monopolist completely reduce to zero. The enlightenment of this model is that when two rival firms compete, they should not immediately engage in a price war. The price war is the most direct form of competition, which often results in a lose-lose outcome.

When firm 1 and firm 2 are in price competition, the profit of firm i is

$$\pi^i(p_i, p_j) = (p_i - c)D_i(p_i, p_j),$$

where $D_i(p_i, p_j)$ is the demand faced by firm i given the prices of itself and its opponent, i.e., p_i and p_j , respectively. Since the products of the two firms are homogeneous, the demand faced by firm i is characterized as follows:

$$D_i(p_i, p_j) = \begin{cases} D(p_i), & \text{if } p_i < p_j, \\ \frac{1}{2}D(p_i), & \text{if } p_i = p_j, \\ 0, & \text{if } p_i > p_j. \end{cases}$$

The objective of firm i is to choose p_i to maximize $\pi^i(p_i, p_j)$.

The Bertrand equilibrium (p_i^*, p_j^*) is a Nash equilibrium, i.e., when the

opponent chooses p_j^* , the best response function of firm i is

$$p_i^* = \arg \max \pi^i(p_i, p_j^*) = (p_i - c)D_i(p_i, p_j^*).$$

Formally, the Bertrand price competition equilibrium is defined as:

Definition 9.5.1 (Bertrand Equilibrium) (p_1^b, p_2^b) is a *Bertrand equilibrium*, if it satisfies the following conditions:

- (i) given $p_2 = p_2^b, p_1^b = \arg \max_{p_1} \pi_1(p_1, p_2^b)$;
- (ii) given $p_1 = p_1^b, p_2^b = \arg \max_{p_2} \pi_2(p_1^b, p_2)$.

We find that the Bertrand equilibrium outcome is unique, i.e., $(p_1^*, p_2^*) = (c, c)$. To see this, we consider three cases.

- (1) When $p_i^* > p_j^* > c$, the strategy profile is not a Nash equilibrium. This is because, in this case, the profit of firm i is zero, and at least firm i has the incentive to deviate from the choice. If the strategy of firm j is unchanged while $p_i = p_j^* - \varepsilon > c$, then the profit of firm i is $(p_j^* - \varepsilon - c)D(p_j^* - \varepsilon) > 0$.
- (2) When $p_i^* = p_j^* > c$, the strategy profile is not a Nash equilibrium. Suppose that the strategy of firm j is unchanged. Then, if the strategy of firm i also remains unchanged, its profit is $\frac{1}{2}(p_j^* - c)D(p_j^*)$; if firm i chooses a price $p_i^* = p_j^* - \varepsilon > c$, its profit becomes $(p_j^* - \varepsilon - c)D(p_j^* - \varepsilon) > 0$. As long as the positive number ε is sufficiently small, we will have $(p_j^* - \varepsilon - c)D(p_j^* - \varepsilon) > \frac{1}{2}(p_j^* - c)D(p_j^*)$.
- (3) When $p_i^* > p_j^* = c$, the strategy profile is not a Nash equilibrium. At this time, the profit of firm j is zero. If the strategy of firm i is unchanged, when firm j chooses $p_j = c + \varepsilon < p_i^*$, where $\varepsilon > 0$, its profit becomes $\varepsilon D(c + \varepsilon) > 0$.

Therefore, the only possible equilibrium is $p_1^* = p_2^* = c$, which is indeed a Nash equilibrium, because no firm can obtain higher profit by unilaterally changing its pricing strategy.

The equilibrium outcome of Bertrand competition is that all symmetric firms set the price of the product at the marginal cost so that the profit of each firm is zero, which is the same as in the case of perfect competition. However, it is difficult to conceive that, in reality, firms do not have any market power and cannot obtain positive profits in an industry with only a few firms. As such, this is termed the “Bertrand Paradox” . The problem is that the assumption of the Bertrand price competition model is far from reality in several ways.

First, the two firms may not be identical or have asymmetric production technology, which will affect the pattern of market competition. Suppose that we allow an asymmetric Bertrand competition, $c_1 < c_2$. When the marginal cost of firm 1 is much lower than that of firm 2 so that $c_2 > p^m(c_1)$, firm 1 will obviously choose the monopoly price and firm 2 will exit from the market. Indeed, the market will evolve into a monopoly. When $c_2 \leq p^m(c_1)$, whereas no pure-strategy Nash equilibrium exists, there exists a mixed strategy Nash equilibrium where $p_1^* = c_2$, and p_2^* randomizes on an interval above c_2 . In this case, the profit of firm 1 is $(c_2 - c_1)D(c_2) > 0$, the profit of firm 2 is 0, and the market price is the higher marginal cost of the two firms. This can be regarded as a variant of the symmetrical outcome.

Second, in addition to the above symmetry assumption of production technology, it also relies on other implicit and explicit assumptions that will affect the Bertrand equilibrium outcome:

It is implicitly assumed that an oligopolistic firm can supply all of the demands that it faces at the same marginal cost, i.e., when the price is the marginal cost c , the oligopolistic firm can supply $D(c)$. However, in reality, firms are usually limited by their productivity. When the production capacity of firm 1 is less than $D(c)$, $p_1^* = p_2^* = c$ is not an equilibrium. The reason for this is as follows: if the pricing decisions of both firms are unchanged, their profits are both zero; however, since firm 1's production capacity is less than $D(c)$, the product at price c is not available to some consumers who will turn to firm 2 instead even if firm 2 raises the price slightly. In this way, firm 2 can choose a price that is higher than the marginal cost, and it has an incentive to deviate from marginal-cost pricing. In practice, the production capacity needs to be accumulated through investment in advance.

At the same time, in many industries, the marginal cost will rise when the output reaches a certain level. We often see the weakening of competition due to the limitation of production capacity.

It is also assumed that a firm only makes one price decision. In reality, however, firms usually can adjust prices. We learned in the previous repeated game that if the interaction between firms is repeated, two firms may choose to cooperate after all. For example, through price collusion, they can both obtain greater benefits.

In addition, it is assumed that the products of firms are homogeneous. In reality, there will be differences between products produced by different firms. Some products have their own adherents, and thus even if their prices are higher than other similar products, consumers will not change the target of purchase. For example, suppose that there are two stores in different locations. A slight decrease in price will not assist one store to occupy the entire market, nor will a slight increase of price make the other store lose all of its customers. As a result, $p_1^* = p_2^* = c$ cannot be the price competition equilibrium of differentiated products. Under the extreme circumstance, the product difference is so great that every firm is a monopolist and each will choose a monopoly price, which is similar to the monopoly market structure, as previously discussed.

As a consequence, the outcome of oligopolistic competition depends on the intensity of competition, the latter of which further depends on various environmental factors. Next, we discuss strategic choices and equilibria when the above three assumptions are relaxed. Meanwhile, we retain the symmetry assumption for the sake of discussion.

9.5.2 Price Competition with Production Capacity Constraints

Suppose that the cost function $C(q)$ satisfies $C'(q) > 0$ and $C''(q) > 0$. In this way, the production cost of the firm exhibits the property of decreasing returns to scale, and the extreme case of decreasing returns to scale is constrained production capacity. The production capacity \bar{q} implies that the marginal cost will be infinitely large when the output exceeds \bar{q} . In the following, we examine price competition among firms under production

capacity constraints through an example.

We investigate the residual demand functions of firms under production capacity constraints. If firm i charges a low price p_i and the supply of firm i under this price is less than the market demand, i.e., $S_i(p) < D(p)$, then there are consumers who cannot purchase from firm i , and other firms will face positive residual demand.

Let $p_1 < p_2$. $\bar{q}_1 \equiv S_1(p_1)$ denote the supply of firm 1 or the production capacity constraint of firm 1. When $\bar{q}_1 \equiv S_1(p_1) < D(p_1)$, the residual demand function of firm 2 is

$$\tilde{D}_2(p_2) = \begin{cases} D(p_2) - \bar{q}_1, & \text{if } D(p_2) > \bar{q}_1, \\ 0, & \text{if } D(p_2) \leq \bar{q}_1. \end{cases}$$

This rationing rule for residual demand is called the **efficient rationing rule**, which means that consumers will first purchase products from low-priced producers. Tirole (1988) also discussed some other rationing rules in his classic textbook about industrial organization.

Let the market demand function be $D(p) = 1 - p$, and the inverse demand function be $p = P(q_1 + q_2)$. Assume that both firms are subject to production capacity constraints, i.e., the output of firm i satisfies $q_i \leq \bar{q}_i$. The production capacity \bar{q}_i was obtained by firm i prior to price competition at a unit cost $c_0 \in [3/4, 1]$. Once the production capacity is completed, the marginal cost is 0 at an output below \bar{q}_i and becomes infinitely large at an output greater than \bar{q}_i . Because of the monopoly profit $\max_p p(1-p) = \frac{1}{4}$, the ex-ante profit of firm i does not exceed $\frac{1}{4}$, and meanwhile $c_0 \in [3/4, 1]$, and thus $\bar{q}_i \leq \frac{1}{3}$.

It can be shown that the Nash equilibrium caused by the price competition between the two firms is that both firms charge $p^* = 1 - (\bar{q}_1 + \bar{q}_2)$. In other words, in the price competition, the two firms are doing their best to produce in the market, and all of the demand can be satisfied. This is because even if one firm lowers the price, it cannot sell more products, which means that lowering the price will only bring about less revenue, and thus no firm will choose to lower the price. Now, we discuss whether the firms

have the incentive to raise the price.

There are two effects of raising the price. One effect is to obtain more revenue from consumers falling inside of the margin who have positive net consumer surplus. The other effect is to decrease the sales volume of the product, or to drive away cutoff consumers whose net consumer surplus is zero, and some other consumers falling inside of the margin. If the price set by firm i satisfies $p \geq p^*$, the residual demand of firm i (here, we employ the efficient rationing rule) is $q = 1 - \bar{q}_j - p$, and thus its profit is $p(1 - \bar{q}_j - p) = (1 - q - \bar{q}_j)q$ given $p \geq p^*$ and $q \leq \bar{q}_i$. In this way, the derivative of profit $(1 - q - \bar{q}_j)q$ with respect to q is $1 - 2q - \bar{q}_j \geq 0$, since $q \leq \bar{q}_i \leq \frac{1}{3}$ and $\bar{q}_j \leq \frac{1}{3}$.

This means that increasing the sales volume will increase the profits of firms, or raising the price will have a negative impact on profits, and thus we conclude that $p^* = 1 - (\bar{q}_1 + \bar{q}_2)$ is the Nash equilibrium of price competition for the two firms. Thus, under production capacity constraints, the price competition of firms will lead to a price that just clears the market. Meanwhile, the price is higher than the marginal cost and the firms obtain a certain degree of market power. This outcome is the same as the Cournot competition model, which will be discussed below. Under a more general assumption, Kreps and Scheinkman (1983) proved that the two-stage (production capacity accumulation and price competition) oligopolistic competition equilibrium outcome is consistent with that of Cournot competition. In the following, we discuss Cournot competition, which is an oligopolistic interaction model of quantity competition.

9.5.3 Quantity Competition: Cournot Model

Now, we discuss quantity competition. This model is a simple duopoly model introduced by the French economist Cournot in 1838, so known as the **Cournot model**. For simplicity, suppose that two firms, i, j , are in quantity competition, and their strategies are both to choose an output level so that the price just clears the market. We can regard the quantity competition of firms as a two-stage competition. In the first stage, the two firms choose their profit-maximizing production levels q_i, q_j . In the second stage, the two firms engage in price competition. According to the price compe-

tition with production constraints discussed above, we know that the equilibrium price can just clear the market at this moment. This two-stage explanation was proposed by Kreps and Scheinkman (1983).

The profit of firm i can be written as

$$\pi^i(q_i, q_j) = q_i P(q_i + q_j) - C_i(q_i).$$

Assume that the profit function π^i is strictly concave and twice differentiable to q_i . Through the first-order condition for the output from the profit function, the reaction function of firm i can be obtained, which is $q_i = q_i^*(q_j)$, such that $\pi^i(q_i^*(q_j), q_j) = 0$.

Deriving the first-order condition for q_i from the above profit function, we have

$$P(q_i + q_j) - C'_i(q_i) + q_i P'(q_i + q_j) = 0.$$

We then obtain firm i 's Lerner index, which measures the market power of the firm,

$$L_i \equiv \frac{P - C'_i}{P} = -\frac{q_i P'}{P} = -\frac{P' Q}{P} \frac{q_i}{Q} = \frac{d_i}{\varepsilon},$$

where d_i is the market share of firm i , and ε is the absolute value of demand elasticity. Given the linear demand and cost, the equilibrium outcome can be easily deduced.

For discussion purposes, we set a specific market demand function, $P(Q) = 1 - Q$. The cost function of each firm is $C_i(q_i) = c_i q_i$, and the first-order condition at this time is

$$1 - (q_i + q_j) - c_i - q_i = 0,$$

and thus we obtain that

$$q_i = q_i^*(q_j) = \frac{1 - q_j - c_i}{2}$$

Solving the simultaneous reaction functions, we can derive

$$q_i = \frac{1 - 2c_i + c_j}{3}.$$

In the equilibrium of quantity competition, the equilibrium profit of each firm is

$$\pi_i = \frac{(1 - 2c_i + c_j)^2}{9}.$$

From the above equilibrium we find that $\frac{\partial q_i}{\partial c_j} > 0$ and $\frac{\partial \pi_i}{\partial c_j} > 0$, i.e., the output and profit of firm i increase with the increase of the cost of its opponent. When its opponent firm j becomes weaker (i.e., its marginal cost increases), the output of firm j will decrease, and thus firm i faces more residual demand and higher production.

We can extend the competition between the two firms to n firms' quantity competition. For convenience of discussion, assume that firms are symmetric, and thus $P(Q) = 1 - Q$, $C_i(q_i) = cq_i$ and $Q = \sum_{i=1}^n q_i$. Then, the first-order condition becomes $1 - Q - c - q_i = 0$. By symmetry, we obtain the equilibrium output as

$$q_i^c = q = \frac{1 - c}{n + 1},$$

the equilibrium profit as

$$\pi_i^c = \frac{(1 - c)^2}{(n + 1)^2},$$

the consumer surplus as

$$CS = \frac{n^2(1 - c^2)}{2(n + 1)^2},$$

and the market price as

$$p = 1 - nq = c + \frac{1 - c}{n + 1}.$$

Note that $p = c$ when $n \rightarrow \infty$, this is exactly the case of perfect competition.

We can also investigate the relationship between the number of firms and social welfare, defined as

$$W(N) \equiv CS(N) + N\pi(N) = \frac{(1 - c)^2}{2} \left(1 - \frac{1}{(N + 1)^2} \right).$$

Then, it is easy to show that $\frac{d}{dn} W(n) > 0$, i.e., the more are the oligopolistic

firms, the higher is the social welfare.

Dynamic Explanation of the Cournot Equilibrium

The Cournot equilibrium is a steady-state solution. Once the equilibrium is reached, no firm will unilaterally choose to deviate. However, can the Cournot equilibrium explain the actual output choices of firms in competition? Alternatively, if the output choices of firms are not at a Cournot equilibrium, will the two firms adjust their strategies? Below, we explain the equilibrium of static quantity competition with dynamic adjustment. In period 1, firm 1 chooses a certain output. In period 2, firm 2 adjusts its output to an optimal level according to the choice of firm 1. In period 3, firm 1 adjusts its output to a new level according to the choice of firm 2. Repeat this process infinitely, and we find that, regardless of what output level the initial firm chooses, the final outcome of dynamic adjustment is a Cournot equilibrium output. Figure 9.13 shows this dynamic adjustment process. We also find that the convergence outcome does not depend on the initial output choice.

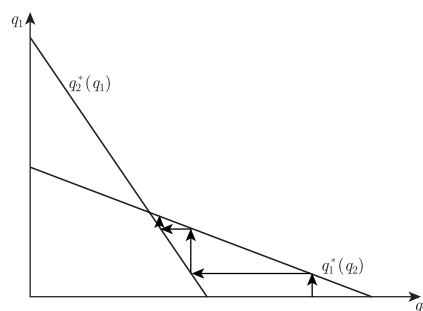


Figure 9.13: The convergence of the Cournot equilibrium

If the two firms choose the output level not simultaneously, but sequentially, is the competition equilibrium still a Cournot equilibrium? Next, we discuss quantity competition equilibrium under sequential decision-making.

9.5.4 Sequential Quantity Competition: Stackelberg Model

Stackelberg (1934) discussed sequential quantity competition, so known as the **Stackelberg Model**. In an industry, the dominant firm 1 takes the lead in choosing its output. After observing the output of firm 1, firm 2 chooses its own output, and then the market clears.

For convenience, we assume that the market demand is $Q(p) = 1 - p$, and the average cost of each firm is c . This is a standard dynamic game. So we use subgame perfect Nash equilibrium as solution concept and adopt the backward induction technique to find the equilibrium.

First, we analyze the action of firm 2. After observing the output of firm 1, firm 2 makes the optimal output decision as follows:

$$\max_{q_2} (1 - q_1 - q_2 - c)q_2,$$

from which we can obtain the best response of firm 2,

$$q_2 = q_2^*(q_1) = \frac{1 - c - q_1}{2}.$$

Given the reaction function of firm 2, the optimal output choice of firm 1 is to solve the problem

$$\max_{q_1} (1 - q_1 - q_2^*(q_1) - c)q_1,$$

from which we obtain that

$$q_1^s = \frac{1 - c}{2} > q_1^c,$$

$$q_2^s = \frac{1 - c}{4} < q_2^c,$$

$$p^s < p^c,$$

and the profits satisfy

$$\pi_1^s > \pi_1^c,$$

$$\pi_2^s < \pi_2^c.$$

From comparison of the above market equilibria, we find that there is a first-mover advantage in the quantity competition.

Having discussed the one-shot interaction of firms, we next explore the impact of multi-period interactions on market competition.

9.5.5 Dynamic Price Competition and Firm Collusion

When firms can conduct multiple pricing, an early price may affect subsequent ones. For instance, if a firm lowers its price, it may trigger a chain reaction of price cuts, which as a result poses an incentive that constrains the price cuts of firms. Chamberlin's oligopoly model pointed out that in the oligopoly of homogeneous products, when recognizing the strategic dependence between them, firms can maintain the monopoly price through tacit agreement without taking any specific measures so that each firm can obtain higher profits. The tacit collusion between them is based on revenge for unilateral deviations, such as in a ruthless price war.

The validity of collusion between firms depends on the effectiveness of the penalty mechanism for deviations. The so-called effectiveness refers to the intensity of punishment against deviant participants, such that no firm is willing to deviate from the collusive price. This depends on numerous factors, such as observability of information and coordination during the punishment process. Here, we adopt a simple infinitely repeated game to describe the collusion mechanism between firms.

When interactions are repeated infinitely, a significant change occurs in comparison with the static interaction because there is no explicit final period at this time. Every firm, when making decisions at any period, needs to consider the impact of previous decisions on subsequent price competition. Of course, there is more than one equilibrium strategy at this time.

For example, the firm charges the marginal-cost pricing in each period, i.e., $p_{1t} = p_{2t} = \dots = c$, $t = 1, 2, \dots$, which is a Nash equilibrium of the infinite-period price competition. This is because if the opponent sets a price at the marginal cost in each period, then the firm will not obtain a higher profit by unilaterally changing the price. However, the above equilibrium is not the only equilibrium. When $\delta \geq \frac{1}{2}$, the grim strategy

discussed in Chapter 7 is also a Nash equilibrium (in fact it is a subgame perfect Nash equilibrium) in infinite-period price competition.

The grim strategy: in the first period, each firm chooses $p_{11} = p_{21} = p^m$, where p^m is the monopoly price; in period $t \geq 2$, for firm i , if the prices previously chosen by its opponent are the monopoly price (also known as the **collusive price**), i.e., $p_{js} = p^m, \forall s < t$, then firm i chooses $p_{it} = p^m$; if the price previously chosen by its opponent deviates from the monopoly price, i.e., there exists $s < t$, such that $p_{js} \neq p^m$, then firm i chooses $p_{it} = c$ in period t .

This price strategy is called the “**grim strategy**” because once the opponents have ever deviated from the collusion price, the firm will never forgive them, i.e., in each subsequent period, it will choose the marginal-cost pricing to punish the opponents.

Let us now determine whether such a strategy can become a Nash equilibrium. If the firm decides to follow the collusive arrangement and chooses the monopoly price in each period, then the total discounted profit of firm i is $\frac{\pi^m}{2}(1 + \delta + \delta^2 + \dots) = \frac{\pi^m}{2(1 - \delta)}$. If firm i deviates from the collusive arrangement in period t , then the maximal profit that it can obtain in this period is π^m . In subsequent periods, because its opponent chooses to price the product at the marginal cost, its profit is zero. Then, the total discounted profit of firm i is

$$\begin{aligned} \frac{\pi^m}{2}(1 + \delta + \delta^2 + \dots + \delta^{t-1}) + \delta^t \pi^m &\leq \frac{\pi^m}{2}(1 + \delta + \delta^2 + \dots + \delta^{t-1}) \\ &\quad + \delta^t \left[\frac{\pi^m}{2}(1 + \delta + \delta^2 + \dots) \right] \\ &= \frac{\pi^m}{2}(1 + \delta + \delta^2 + \dots). \end{aligned}$$

$$\text{If } \delta \geq \frac{1}{2},$$

$$\frac{\pi^m}{2}(1 + \delta + \delta^2 + \dots) = \frac{\pi^m}{2(1 - \delta)} \geq \pi^m,$$

then the grim strategy is a subgame perfect Nash equilibrium.

From the above analysis, it can be seen that there is a possibility of price collusion between oligopolists to make the market price the same as

the monopoly price. In the collusion mechanism, some strict punishment mechanisms exist to restrict the deviation of firms.

Although the infinite-period assumption seems to be unrealistic, as long as there is no clear cut-off time in the market competition, assuming that there is a probability \hat{p} that firm i will coexist with its opponents in the market in the next period, we can write the discount rate as $\hat{\delta} = \delta\hat{p}$. Then, the total discounted profit of firm i is $\sum_t \delta\hat{p}^t \pi_t^i(p_{it}, p_{ut}) = \sum_t \hat{\delta}^t \pi_t^i(p_{it}, p_{ut})$. Therefore, as long as $\hat{\delta} = \delta\hat{p} \geq \frac{1}{2}$, there will exist the same equilibrium of price collusion as in the above case.

By the Folk Theorem in Chapter 7, we know that repeated games usually have infinitely many equilibria. In the above context, we find that firms choosing the price $p_{1t} = p_{2t} = p \in [c, p^m]$ in each period also constitute a (subgame perfect) Nash equilibrium. A similar grim strategy exists, i.e., each firm chooses the price p in the beginning, and if any opponent deviates in a period, then in each period afterwards, other firms choose marginal-cost pricing as a punishment for the opponent.

From the above multi-period price competition model, we find that in the case of repeated games, firms have the incentive and ability to choose collusion, which further explains why in reality we do not see the same intense price competition as in the Bertrand equilibrium. Next, we will relax the assumption of the homogeneity of products. We will find that in the market of heterogeneous products, price competition still brings market power to firms.

9.5.6 Price Competition under Horizontal Product Differentiation: Hotelling Model

The characterization of product differentiation is usually based on differences in consumer valuations. There are primarily two kinds of value differentiations. One is **horizontal differentiation**, i.e., the value of products is different for different consumers. The other is **vertical differentiation**, i.e., the value of a product is the same for all consumers, but is different from that of other products. For price competition of differentiated products, we mainly adopt the method of the spatial location model. First, we

discuss the horizontal differentiation model.

For many commodities in real life, such as clothes with different colors and styles, different people have heterogeneous preferences, just as consumers in different locations have different preferences for stores in different locations. Harold Hotelling (1895—1973, see his biography in Section 9.6.1) developed a location model to characterize the price competition of differentiated products in a paper published in the *Journal of Political Economy* in 1929.

Assume that the city is a line of length 1, and consumers are uniformly distributed in the city with a density of 1. Two stores are located at both ends of the city. Store 1 is located at $x = 0$, and store 2 is located at $x = 1$. The marginal cost of each store is c . Assume that the traffic cost is a quadratic function of distance, i.e., for consumers located at x , the traffic cost of going to store 1 is tx^2 , and the traffic cost of going to store 2 is $t(1 - x)^2$. The product prices of the two stores are p_1 and p_2 , respectively. Assume that the price difference is not large to the extent that one store faces no demand, i.e., $|p_1 - p_2| < t$. This assumption can be verified at the Nash equilibrium point. If there is no demand for one store, as long as the price is not lower than the marginal cost, the store has an incentive to lower the price. The consumer has a unit demand for the product, and his valuation of the product \bar{s} is sufficiently large. This assumption ensures that all consumers will purchase at the equilibrium point. In addition, in the Hotelling model, the products of the two stores are homogeneous, except the location.

According to these assumptions, there always exists one consumer in the city who is indifferent between the two stores. This consumer is called the cutoff consumer, whose position depends on the prices of the two stores. Let the cutoff consumer's position be $x(p_1, p_2)$, satisfying

$$p_1 + tx^2(p_1, p_2) = p_2 + t(1 - x(p_1, p_2))^2.$$

The demand for store 1 is

$$D_1(p_1, p_2) = x(p_1, p_2) = \frac{p_2 - p_1 + t}{2t}.$$

The demand for store 2 is

$$D_2(p_1, p_2) = 1 - x(p_1, p_2) = \frac{p_1 - p_2 + t}{2t}.$$

The profit of store i is

$$\pi^i(p_i, p_j) = (p_i - c) \frac{p_j - p_i + t}{2t}.$$

Solving the first-order condition with respect to p_i , $p_j + c + t - 2p_i = 0$. Then, the reaction function of store i is

$$R_i(p_j) = \frac{p_j + c + t}{2}.$$

By symmetry, we can obtain the Nash equilibrium price as $p_1^c = p_2^c = c + t$, and the profits as $\pi^1 = \pi^2 = \frac{t}{2}$.

Here, the value t characterizes the degree of product differentiation. When $t = 0$, the products supplied by the two stores are homogeneous, and the equilibrium price is equal to that of the Bertrand equilibrium. When $t > 0$, both stores obtain positive profits under price competition, and the greater is the product differentiation, the greater is the market power of the stores.

9.5.7 Vertical Product Differentiation Model

Consumers usually have a preference ordering for the characteristics of products. A typical example of this is quality. Although consumers generally like high-quality products in practice, the value of product quality possessed by each consumer is different. Use x to describe different types of consumers. For convenience of discussion, assume that the consumer type x is uniformly distributed in $[0, 1]$, and the marginal cost of firms is zero. Here, we discuss not only price competition under product differentiation, but also under product positioning.

In price competition under vertical product differentiation, suppose that the product qualities of firm A and B are a and b , respectively, where $a < b$, and the prices of the two firms are p_A and p_B , respectively. Let the utilities

of the two products for consumers of type x be

$$U_x(i) \equiv \begin{cases} ax - p_A, & i = A, \\ bx - p_B, & i = B. \end{cases}$$

Consider the competitive equilibrium when both firms have positive market shares. Let \hat{x} be cutoff consumers, satisfying

$$\hat{x} = \frac{p_B - p_A}{b - a}.$$

Then, \hat{x} is the market share of firm A , and $1 - \hat{x}$ is that of firm B .

The competitive equilibrium prices of the two firms are

$$p_A^* = \arg \max_{p_A} p_A \hat{x}; \quad p_B^* = \arg \max_{p_B} p_B (1 - \hat{x}).$$

Their reaction functions are

$$p_A = R_A(p_B) = \frac{p_B}{2}; \quad p_B = R_B(p_A) = \frac{b - a + p_A}{2}.$$

As a consequence, we obtain that

$$p_A^* = \frac{b - a}{3}; \quad p_B^* = \frac{2(b - a)}{3}.$$

Therefore, $\hat{x} = \frac{p_B^* - p_A^*}{b - a} = \frac{1}{3}$. At Nash equilibrium, the profits of the two firms are

$$\pi_A(a, b) = \frac{b - a}{9}; \quad \pi_B(a, b) = \frac{4(b - a)}{9}.$$

We further discuss the choice of product positioning. Consider the following two-stage game and find the subgame perfect Nash equilibrium. In the first stage, the two firms choose product quality, and in the second stage, they engage in price competition.

If the two firms choose product quality a and b , respectively, in the first stage where $a \leq b$, then the equilibrium of the above price competition is the Nash equilibrium in the second stage of the game.

In the first stage, the profits of firm A and B are

$$\pi_A(a, b) = \frac{b - a}{9}; \quad \pi_B(a, b) = \frac{4(b - a)}{9}.$$

Since

$$\frac{\partial \pi_A(a, b)}{\partial a} = -\frac{1}{9}, \quad \frac{\partial \pi_B(a, b)}{\partial b} = \frac{4}{9},$$

we have $a^* = 0$ and $b^* = 1$, which means that in price competition under vertical product differentiation, firms will maximize their product differences from others.

In the second stage, the prices of the two firms are $p_A^* = \frac{1}{3}$ and $p_B^* = \frac{2}{3}$, both larger than the marginal cost, which means that in the price competition under vertical product differentiation, firms will weaken the intensity of price competition.

The above oligopolistic competition models assume that the market structure is exogenous. However, in practice, entering or exiting the market is an important dimension of firms' strategy. In the following, we will consider competition in the dynamic market structure.

9.5.8 Market Entry Deterrence

Bain (1956) summarized four factors that affect the market structure, which also affect the ability of incumbent firms to maintain market power.

The first factor is economies of scale. The minimum efficient scale of a firm (or the output at the lowest average cost) is a crucial factor to determine the consumer demand of the industry. For example, in a firm that exhibits increasing returns to scale, its minimum efficient scale may be infinitely large. If we have $C(q) = f + cq$, $AC(q) = \frac{f}{q} + c$ as the decreasing function of the output, then only a few firms can survive in the market. In an industry with increasing returns to scale, the cost is the least when all of the output is produced by one firm.

The second is the absolute advantage of cost. Incumbent firms may have more advanced technologies obtained through experience (the process of learning by doing) or R&D (patenting or innovation). Incumbent firms may accumulate capital to lower their production costs, or prevent

entrants from acquiring important inputs by contracts with suppliers, or raise the cost of their competitors.

The third is the advantage of product differentiation. Incumbent firms obtaining patents for their products can prevent other firms from using or imitating their technologies. They can also use the first-mover advantage to gain brand loyalty of consumers.

The fourth is capital input requirement. Before a firm enters the market, it needs to finance the investment. When financing is difficult, for example, banks are unwilling to lend due to risk considerations, or if incumbent firms increase the intensity of potential competition in the product market and thus lower the potential entrant's expectation for profitability, the willingness of the potential entrant to enter the market will decrease.

Some barriers to entry are exogenous, such as those granted by law, and not controlled by incumbent firms; whereas, some others are endogenous, and caused by strategic choices of incumbent firms. When an incumbent faces the threat of entry, she may take the following three actions.

- (1) **Entry blockade:** it is difficult for the entrant to obtain the resources needed to establish a business, and the incumbent faces little threat of entry.
- (2) **Entry deterrence:** the incumbent has successfully thwarted the entry by adjusting her strategy to lower the entrant's expectation of profitability.
- (3) **Entry accommodation:** the incumbent finds it more profitable to allow (a few) entrants to enter the market than setting barriers to entry.

Below, we investigate an example of endogenous barrier.

Consider an industry consisting of two firms. Firm 1 (the incumbent) chooses a capital level k_1 (or the previous "production capacity") and fixes on it. Firm 2 (the potential entrant) observes k_1 and chooses a capital level k_2 . If $k_2 = 0$, this means that firm 2 has not entered the market. Under the dynamic market structure, the competition of the two firms is to choose their own capital levels successively. Let the output be the previous pro-

duction capacity. The game between the two firms is a two-stage game and then we can use subgame perfect Nash equilibrium as solution concept.

The first stage: firm 1 chooses a capital level of k_1 .

The second stage: having observed k_1 , firm 2 chooses a capital level k_2 . If firm 2 chooses to enter, i.e., $k_2 > 0$, it must bear a fixed entry cost f .

If not considering whether firm 2 enters, this model is the previously discussed Stackelberg model.

Let market demand be linear, and the market price be $p = 1 - k_1 - k_2$. At this point, the profit of firm 1 is $\pi^1(k_1, k_2) = k_1(1 - k_1 - k_2)$; the profit of firm 2 is $\pi^2(k_1, k_2) = k_2(1 - k_1 - k_2) - f$ if it enters; otherwise, it is zero.

In the second stage, if firm 2 enters, i.e., $k_2 > 0$, its choice is the best response to firm 1's action, i.e.,

$$k_2 = R_2(k_1) = \frac{1 - k_1}{2}.$$

In the first stage, expecting the response of firm 2, firm 1 chooses the optimal capital level from the problem:

$$\max_{k_1} k_1(1 - k_1 - R_2(k_1))$$

or

$$\max_{k_1} k_1 \left(1 - k_1 - \frac{1 - k_1}{2} \right).$$

We use the Stackelberg model to obtain the subgame perfect Nash equilibrium, $k_1^* = \frac{1}{2}$, $k_2^* = \frac{1}{4}$, $\pi^1 = \frac{1}{8}$, $\pi^2 = \frac{1}{16} - f$. When $f > \frac{1}{16}$, we know that firm 2 will not enter, or there is a market entry blockade. When $f < \frac{1}{16}$, if firm 1 allows firm 2 to enter, then it will choose the outcome of the Stackelberg equilibrium, i.e., $k_1^* = \frac{1}{2}$, and firm 2 will also choose $k_2^* = \frac{1}{4}$. The profit of firm 1 is then $\pi^1 = \frac{1}{8}$, and the profit of firm 2 is $\frac{1}{16} - f > 0$.

If firm 1 deters the entry of firm 2, it will choose a capital level k_1^b , such that the best choice of firm 2 is $k_2^* = 0$. k_1^b satisfies $\max_{k_2} k_2(1 - k_1^b - k_2) - f = 0$, and thus we obtain $k_1^b = 1 - 2\sqrt{f} > 1/2$. The profit of firm 1 is

$\pi^1 = (1 - 2\sqrt{f})[1 - (1 - 2\sqrt{f})]$. When $f \rightarrow \frac{1}{16}$, $\pi^1 \rightarrow \frac{1}{4} = \pi^m$, a deterrence strategy is a better choice for firm 1. When $f \rightarrow 0$, an accommodation strategy is a better choice for firm 1.

We can verify that, when $f \leq \left(\frac{1}{4} - \frac{\sqrt{2}}{8}\right)^2$, firm 1 will choose an accommodation strategy; otherwise, it will choose a deterrence strategy.

9.5.9 Price Competition with Asymmetric Information

The above discussion of oligopolistic competition is based on the assumption that the competition game is one of complete information. However, in practice, firms usually face some state variables that cannot be accurately observed when making decisions, such as the cost of an opponent, the market demand or market potential, because some information is privately owned, such as the case in which a firm knows its own production technology better than its opponents do. George J. Stigler (1911-1991, see his biography in Section 9.6.2) is one of the founders of information economics. He believed that it costs too much for consumers to obtain information about quality, price and purchase timing of goods, so that consumers are neither able nor willing to obtain sufficient information, resulting in different prices for the same commodity. According to Stigler, such result was inevitable and did not require human intervention. In this subsection, we discuss oligopolistic competition and dynamic market entry under asymmetric information. For a detailed discussion of such principal-agent issues under incomplete information, see the chapters on incentive mechanism design theory in Part VI.

In real market competition, firms have information that their opponents do not possess. In game theory, we can characterize the types of firms according to the information that is not known to their competitors. Below, we discuss a simple price competition model with asymmetric information.

Suppose that there are two firms in the market, firm 1 and firm 2, which produce differentiated products and engage in price competition with each other. Firms' demand functions are public information in the form of linear function $D_i(p_i, p_j) = a - bp_i + dp_j$, $0 < d < b$. The information on the

cost of firm 2 is public, but the cost information of firm 1, also called its type, is private. There are two possibilities for the marginal cost of firm 1, $c_1^l < c_1^h$, with a prior distribution $\text{prob}(c_1 = c_1^l) = \beta$ and $\text{prob}(c_1 = c_1^h) = 1 - \beta$. Therefore, firm 2's expectation for firm 1's marginal cost is $c_1^e \equiv \beta c_1^l + (1 - \beta)c_1^h$. The marginal cost of firm 2 is c_2 .

The ex-post profit of firm i is

$$\pi^i(p_i, p_j) = (p_i - c_i)(a - bp_i + dp_j).$$

The two firms choose the prices at the same time. To solve for the Bertrand price equilibrium under asymmetric information, we use Bayesian-Nash equilibrium as a solution concept. Let p_2^* be the equilibrium price of firm 2, and p_1^l and p_1^h be the equilibrium price strategies of firm 1 at marginal costs c_1^l and c_1^h , respectively.

Now, we solve for the best response functions of the two firms.

For firm 1, given marginal cost c_1 and the pricing of firm 2, p_2^* , from profit maximization, we obtain

$$a - 2bp_1 + dp_2^* + bc_1 = 0,$$

and then

$$p_1 = R_1(p_2; c_1) = \frac{a + dp_2^* + bc_1}{2b}.$$

Therefore,

$$p_1^l = \frac{a + dp_2^* + bc_1^l}{2b}.$$

Similarly, given marginal cost c_2 and the pricing of firm 2, p_2^* , we have

$$p_1^h = \frac{a + dp_2^* + bc_1^h}{2b}.$$

The expectation of firm 2 for firm 1's pricing is then

$$\begin{aligned} p_1^e &= \beta p_1^l + (1 - \beta)p_1^h \\ &= \frac{a + dp_2^* + bc_1^e}{2b}. \end{aligned}$$

For firm 2, given firm 1's choice, its objective is to choose a price p_2 to

maximize the expected profit

$$\begin{aligned} E\pi^2 &= \beta(p_2 - c_2)(a - bp_2 + dp_1^l) + (1 - \beta)(p_2 - c_2)(a - bp_2 + dp_1^h) \\ &= (p_2 - c_2)(a - bp_2 + dp_1^e). \end{aligned}$$

From the first-order condition, we can obtain the reaction function of firm 2:

$$p_2^* = R_2(p_1) = \frac{a + bp_1^e + bc_2}{2b},$$

and thus we obtain the Bayesian-Nash equilibrium:

$$\begin{aligned} p_2^* &= \frac{2ab + ad + 2b^2c_2 + bdc_1^e}{4b^2 - d^2}; \\ p_1^l &= \frac{a + dp_2^* + bc_1^l}{2b}; \\ p_1^h &= \frac{a + dp_2^* + bc_1^h}{2b}. \end{aligned}$$

We compare the above case with the case of symmetric information. When information is symmetric, we can obtain that

$$\begin{aligned} p_2^*(c_1 = c_1^l) &= \frac{2ab + ad + 2b^2c_2 + bdc_1^l}{4b^2 - d^2} < p_2^*; \\ p_2^*(c_1 = c_1^h) &= \frac{2ab + ad + 2b^2c_2 + bdc_1^h}{4b^2 - d^2} > p_2^*. \end{aligned}$$

We can verify that, in the case of symmetric information, the profit of firm 2 is higher because the equilibrium profit of firm 2 is a convex function of firm 1's cost.

For firm 1, we have that $\frac{d\pi_1}{dp_2} = \frac{\partial\pi^1}{\partial p_2} = d(p_1 - c_i) > 0$. Consequently, when the cost of firm 1 is c_1^h , it will have an incentive to disclose its type information if there is an opportunity to disclose; when the cost of firm 1 is c_1^l , it will deliberately conceal its cost information. This is because a firm with high cost can avoid competitors from choosing more aggressive low-price strategies by disclosing information, since the reaction functions of firms are increasing functions in the costs of its opponents. If a firm experiences a stage of information disclosure prior to setting its price, the

act of releasing (verifiable) information conveys that it is a high-cost type.

9.5.10 Limit Pricing with Asymmetric Information: Dynamic Market Structure

The traditional view is that an incumbent firm can block entry through low prices. Bain proposed the concept of **limit pricing**. If there is a positive relationship between the price prior to entry and the speed or degree of entry, the incumbent firm will have an incentive to lower the price. A pertinent question becomes: why can low prices prevent firms from entering? Bain asserted that the information delivered to potential entrants by low prices was poor market profitability, or high competition intensity, or that incumbent firms have low-cost advantages.

Consider the situation in which price acts as a signal mechanism. Assume that, in order to convey her private information of competitiveness or cost level, the incumbent firm chooses a price to deliberately reveal its type information to form an entry barrier. This game constitutes a typical signaling game.

Assume that there are two periods. In the first period, there is a monopolistic incumbent firm 1 that chooses a price. In the second period, firm 2 decides whether or not to enter. If firm 2 enters, there is a duopoly market structure in the second period; if not, firm 1 remains a monopolist. The cost of the incumbent firm is private information, and it has two possible values. Assume that the probability of low cost is β , and the probability of high cost is $1 - \beta$. If firm 2 does not enter the market, it does not know the private information of firm 1. If it enters, firm 2 will get the cost information of firm 1. Meanwhile, entry is irreversible. The significance of introducing private information here is that we can study how private information influences the market entry decision and no longer has an impact post-entry.

Let $M_1^t(p_1)$ be the profit of the incumbent firm when her type is $t \in \{l, h\}$ and price is p_1 . Let p_m^l and p_m^h be the monopolistic prices when the incumbent has low cost and high cost, respectively. From the monopoly pricing, we know that $p_m^l < p_m^h$. Let M_1^l and M_1^h be the monopoly profits at low cost and high cost, respectively, i.e., $M_1^t \equiv M_1^t(p_m^t)$, $t \in \{l, h\}$.

If firm 2 enters, the cost information of firm 1 can be observed by firm 2, and the two firms engage in price competition in the second period. Let D_1^t and D_2^t be the profits of the two firms after entry, where $t \in \{l, h\}$. To make the discussion meaningful, suppose $M_1^t > D_1^t, \forall t; D_2^h > 0 > D_2^l$. In other words, for the incumbent, monopoly is superior to oligopolistic competition; if the incumbent is a high-cost type, the entrant possesses an advantage and can earn positive profits, while if the incumbent is a low-cost type, firm 2 will not wish to enter. The discount rate is δ for both firms.

Since the incumbent aims to obtain a monopoly profit, she hopes that her price choice in the first period can make the potential entrant believe that she is a low-cost type. The problem here is whether or not the incumbent has the ability to achieve the goal. Consider the following signaling mechanism. Firm 1 chooses a low price p_1^l to send the low-cost signal. As a result, the profit of the incumbent may decrease in first period, but such a reduction can bring returns in the second period, i.e., the incumbent can obtain a monopoly profit, as the potential entrant believes that the incumbent is a low-cost type and thus abandons entering the market. Now, the issue becomes whether the potential entrant will believe that the incumbent is low-cost without observing p_1^l . The answer may not be readily apparent. This is because even a high-cost incumbent may be able to imitate the low-price type, so that price signals may not reflect the type information. To analyze this problem, we usually use equilibrium concepts of the incomplete information dynamic game, such as “perfect Bayesian equilibrium”.

In Chapter 7, we divided signaling game equilibria into three categories: separating equilibrium; pooling equilibrium; and semi-separating equilibrium (in which some types of signal senders will adopt mixed strategies). Here, we focus on the first two categories.

First, we analyze the separating strategy. The separating strategy means that different types of the incumbent will send different signals and satisfy incentive compatibility constraints. Since a separating equilibrium is a perfect Bayesian equilibrium, it is necessary to define the belief of the signal receiver off the equilibrium path. In separating equilibrium, a firm of high-cost type does not need to imitate a low-cost type through low prices. She will choose a high-cost monopoly price p_m^h in the first period where her

profit is M_1^h , while in the second period, her profit is the oligopoly profit D_1^h , and thus the total discounted profit is $M_1^h + \delta D_1^h$. Let p_1^l be the price of the low-cost incumbent in the first period. Since it is a separating equilibrium, it should be ensured that the high-cost type does not choose p_1^l to imitate the pricing strategy of the low-cost type.

Therefore, the incentive compatibility constraint for the high-cost type is

$$M_1^h + \delta D_1^h \geq M_1^h(p_1^l) + \delta M_1^h,$$

or

$$M_1^h - M_1^h(p_1^l) \geq \delta(M_1^h - D_1^h), \quad (9.5.10)$$

where $M_1^h(p_1^l)$ is the profit of the high-cost type in the first period when it chooses price p_1^l .

In the following, we consider the strategic choice of a low-cost type. Since the incumbent can at least set a monopoly price p_m^l in the first period and the worst case in the second period is that there are entrants, the minimum profit that he or she can obtain is $M_1^l + \delta D_1^l$. If choosing p_1^l in the first period, the total discounted profit of the low-cost type of the incumbent is $M_1^l(p_1^l) + \delta M_1^l$. As a consequence, the incentive compatibility constraint for the low-cost type to choose p_1^l is

$$M_1^l(p_1^l) + \delta M_1^l \geq M_1^l + \delta D_1^l,$$

or

$$M_1^l - M_1^l(p_1^l) \leq \delta(M_1^l - D_1^l). \quad (9.5.11)$$

Assume that the separating equilibrium under asymmetric information is different from the pricing of various types of the incumbent under symmetric information (when the high-cost and low-cost types choose p_m^h and p_m^l , respectively). In other words, under asymmetric information, if the low-cost type chooses p_m^l , the high-cost type has an incentive to pool, i.e.,

$$M_1^h - M_1^h(p_m^l) < \delta(M_1^h - D_1^h). \quad (9.5.12)$$

In (9.5.10) and (9.5.11), p_1^l is in a range $[\tilde{p}_1, \tilde{p}_1]$, where $\tilde{p}_1 < p_m^l$. Con-

sequently, to achieve the separating equilibrium, the pricing of a low-cost type must be sufficiently lower than her monopoly price, which is the cost of preventing the high-cost firm from pooling, in order to prevent market entry in the second period.

The reason why a high-cost type does not mimic the pricing of a low-cost type is that imitation leads to more decrease of profit in the first period than the gain.

When $M_1^l - D_1^l > M_1^h - D_1^h$, i.e., the revenue obtained by the low-cost type through low prices to prevent entry (the increase in profit from oligopoly to monopoly) is more than that of a high-cost type, or

$$\frac{d}{dc_1}[M_1(c_1) - D_1(c_1)] < 0,$$

where

$$M_1(c_1) = \max_{p_1}[(p_1 - c_1)D_1^m(p_1)],$$

$$D_1(c_1) = \max_{p_1}(p_1 - c_1)D_1(p_1, p_2^d).$$

According to the envelope theorem, we have

$$\frac{dM(c_1)}{dc_1} = -D_1^m(p_1^m),$$

$$\frac{dD_1(c_1)}{dc_1} = -D_1(p_1^d, p_2^d) + (p_1^d - c_1) \frac{\partial D_1(p_1^d, p_2^d)}{\partial p_2} \frac{\partial p_2^d}{\partial c_1}.$$

Thus,

$$\frac{d}{dc_1}[M_1(c_1) - D_1(c_1)] = -D_1^m(p_1^m) + D_1(p_1^d, p_2^d) - (p_1^d - c_1) \frac{\partial D_1(p_1^d, p_2^d)}{\partial p_2} \frac{\partial p_2^d}{\partial c_1}.$$

If the monopoly demand of the incumbent is greater than the duopoly demand, and $\frac{\partial p_2^d}{\partial c_1} > 0$, which means that the increasing cost of the incumbent will make the opponent raise the price, then we have $\frac{d}{dc_1}[M_1(c_1) - D_1(c_1)] < 0$.

Figure 9.14 illustrates the separating equilibrium. In the separating interval, a low-cost type is most likely to choose \tilde{p}_1 , because this is the separating price closest to the low-cost monopoly price in the interval, which

is of the lowest (credible) signaling cost and also will not be mimicked by the high-cost type. Since the high-cost type's price choice will always reveal her own type, she will choose the monopoly price in the first period. Therefore, the most reasonable separating equilibrium in this equilibrium series (the continuum of separating equilibria) is that in the first period, the low-cost type chooses \tilde{p}_1 , and the high-cost type chooses p_m^h .

We next examine whether the pooling equilibrium exists. One condition for the existence of pooling equilibrium is $\beta D_2^l + (1 - \beta)D_2^h < 0$, or $\beta > \frac{D_2^h}{D_2^h - D_2^l}$, i.e., the probability of the low-cost type is sufficiently large, which means that firm 2 will not choose to enter the market if there is no information. If both types choose the same price, the potential entrant cannot distinguish the types of the incumbent based on the price. In this case, the potential entrant in the second period has the same belief for both types as the initial belief. Another condition for pooling equilibrium to exist is that both types have the incentive to choose the same price. For the high-cost incumbent, the condition of choosing the same price p_1 is $M_1^h - M_1^h(p_1) \geq \delta(M_1^h - D_1^h)$. For the low-cost type, it has no incentive to show the difference from the high-cost type, because firm 2 will not enter when firm 1's type cannot be distinguished.

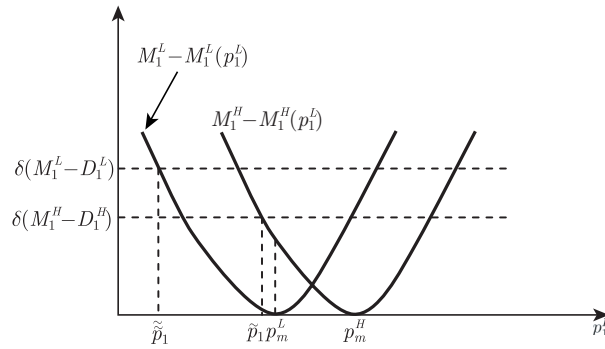


Figure 9.14: Separating equilibrium

One conclusion in Chapter 6 on game theory is that only separating equilibrium satisfies the **intuitive criterion** in the signaling game.

Here, we use a simple example to solve the above signaling game in which the price transmits the cost information. Assume that the inverse

demand function is $p = 10 - Q$, and the incumbent firm 1's marginal cost c_1 is private information. Assume that the prior distribution of the type of firm 1 is $\text{prob}(c_1 = 0) = \text{prob}(c_1 = 4) = 0.5$. The cost of the potential entrant firm 2 is $c_2 = 1$, which is public knowledge. The market entry cost is 9, and the time discount rate is $\delta = 1$. Suppose that the oligopolistic market competition is quantity competition. The time structure of the game is as follows: in the first period, firm 1 is the market monopolist and chooses the price p_1 ; in the second period, firm 2 chooses whether to enter the market. If firm 2 enters, it will pay a fixed cost 9, and the two oligopolists will engage in quantity competition; if not, firm 1 is still a monopolist and chooses the price p_2 in the second period.

We use simple formulas for equilibrium profits of quantity competition under linear demand, which are

$$\pi_1(c_1, c_2) = \frac{(a - 2c_1 + c_2)^2}{9},$$

$$\pi_2(c_1, c_2) = \frac{(a - 2c_2 + c_1)^2}{9} - f,$$

where $a = 10$. Table 9.1 depicts the profits of the two firms.

Type of the incumbent:	The potential entrant	
	Enter	Not enter
Low cost $c_1 = 0$	$\pi_1^c(0) \approx 13, \pi_2^c(0) \approx -1.9$	$\pi_1^m(0) = 25, \pi_2(0) = 0$
High cost $c_1 = 4$	$\pi_1^c(4) = 1, \pi_2^c(4) = 7$	$\pi_1^m(4) = 9, \pi_2(4) = 0$

Table 9.1: Firms' profits under limit pricing

If under price p_1 in the first period, firm 2 cannot distinguish the type of the firm 1 (which is the pooling equilibrium), the expected revenue of firm 2 to enter the market is $0.5 \times (-1.9) + 0.5 \times 7 > 0$. As a result, firm 2 will choose to enter.

We discuss $p_1(c_1)$ in the first period that satisfies the intuitive criterion. In other words, when $p_1 = p_1(c_1 = 0)$, firm 2 believes that firm 1 is the low-cost type, or the high-cost type will not mimic the low-cost type, and meanwhile, $p_1(c_1 = 0)$ is the price that brings the highest profits to the low-cost type among all separating equilibria. We can verify that $p_1(c_1 = 0) =$

4.17, because $9.99 = \pi_1^m(p_1 = 4.17, c_1 = 4) + \pi_1^m(4) = (10 - 4.17)(4.17 - 4) + 9 < \pi_1^m(4) + \pi_1^c(4) = 10$.

9.5.11 Concluding Remarks on the Oligopolistic Market

From the above all kinds of models in the oligopolistic market, we find that different market environments, including information distribution (symmetric or asymmetric), one-shot or repeated games, product properties (homogeneous or heterogeneous), sequence of actions, different strategy spaces (price decisions, quantity decisions, product decisions), etc., will all affect the final strategic choices of the firms, and impact the final market price and the profits of the firms. Because the firms possess market power, i.e., the pricing ability of deviating from the marginal cost, generally the more are the market transactions, the greater is the social welfare. Therefore, an in-depth analysis of the oligopolistic market may be beneficial to provide certain useful suggestions for government policies. For example, it can provide a logical basis for the design of competition policy to control and restrain the excessive market power of incumbents and control price collusion among firms.

We have a simple classification of market structure, but in practice, an industry may experience different market structures in various periods. For instance, when a new product is just released, the market is likely to be a monopoly. With the imitation of other firms, the market structure slowly becomes an oligopoly. In the mature period of the industry, firms will increasingly emerge and the market may be transformed into a monopolistically competitive market, or even close to a perfectly competitive market. In the evolution process, the welfare of consumers and producers will change, as well.

The models discussed above only describe a small part of firm interactions in the market. Indeed, many other interesting problems exist, such as corporate R&D and innovation incentives, related patent system design, compatibility of technical standards in the network economy, network externality, relationships between upstream and downstream firms, platform economy, etc., which constitute an extensive branch of microeconomics, i.e.,

industrial organization. Readers who are interested in these issues can refer to certain classic textbooks, including Tirole's seminal work *The Theory of Industrial Organization* published in 1988. Some of the examples in our discussion are based on this book.

9.6 Biographies

9.6.1 Harold Hotelling

Harold Hotelling (1895—1973) is a widely recognized figure in the fields of statistics, economics, and mathematics. Although his papers on economics were not many, he made a profound contribution to economic sciences. He is considered one of the leaders of the Pareto school.

Hotelling originally majored in journalism when studying at the University of Washington, but later turned to mathematics to carry out related research in the field of topology, and received his Ph.D. degree in 1924. He then received an appointment at Stanford University. His most important contribution to statistical theory was multivariate analysis and probability. His most influential paper is “The Generalization of Student's Ratio”, which is now known as Hotelling's T^2 . He also played a key role in the development of principal component analysis and canonical correlations. He was elected as a Fellow of the U.S. National Academy of Sciences in 1972 and a member of the Accademia Nazionale dei Lincei in Rome in 1973.

The Edgeworth model described the instability of the market with only two sellers, but in 1929, Hotelling challenged this view and proposed the Hotelling model. He believed that price or output instability was not the basic feature of oligopoly. The Hotelling model is obviously a criticism of Edgeworth and Bertrand. Hotelling disagreed with the idea that consumers' sudden change of choice from one seller to another constituted a feature of the market. He expected that the decrease in price would, in fact, not attract a great number of consumers. Therefore, he asserted that as long as consumers turned to other sellers gradually, the market would remain stable. At the same time, he proposed the theory of spatial competition, which divides product difference into different points in the line segment of space,

and thus product difference has a testable empirical meaning. One well-known example of this is the previously discussed Hotelling model that was published in the *Journal of Political Economy* in 1929. In 1931 he published “The Economics of Exhaustible Resources” , which is considered to be a hallmark of the birth of resource economics.

Hotelling taught Milton Friedman statistics and Kenneth J. Arrow mathematical economics. He also helped Arrow to transfer from the Department of Mathematics to the Department of Economics, and was an important influence in his change of research interest from mathematical statistics to economic theory.

9.6.2 George J. Stigler

George J. Stigler (1911-1991) was a prominent American economist, economic historian, and professor at the University of Chicago. He and Milton Friedman were known as the leaders of the Chicago School of Economics. He was the 1982 Laureate in the Nobel Memorial Prize in Economic Sciences. Stigler grew up in Seattle, in the U.S., where he received his education until he graduated from the University of Washington with a bachelor's degree in business administration, and later earned a master's degree in business administration from Northwestern University. After that, he stayed at the University of Washington for more than one year before he went to the University of Chicago to pursue his doctoral degree. In 1936—1938, he served as an Assistant Professor at Iowa State University. In 1938—1946, he taught at the University of Minnesota, and was promoted to Full Professor in 1941. In 1946, Stigler learned that his alma mater, the University of Chicago, wanted him to attend an interview for the recruitment of professors, where he met another professor candidate, Friedman, who finally received the only professor vacancy. Stigler then came to Brown University and taught there until 1947. From 1947 to 1958, he taught at Columbia University, during which time his economic thought gradually matured. In 1958, when a professor vacancy came up at the University of Chicago, Stigler finally received the position of Full Professor there. He spent more than 20 years at the University of Chicago, during which time

the Chicago School of Economics took the lead in academia.

Stigler believed that it was a pleasant and uniquely stimulating life to be a devoted intellectual and dedicate himself to “boring” economics research. Indeed, he intended to avoid all non-academic occupations and activities. He had endless enjoyment from teaching, research and academic exchanges, and left numerous invaluable works, including *Production and Distribution Theories* (1941), *The Theory of Competitive Price* (1942), *The Theory of Price* (1946, 1952, 1964), *Domestic Servants in the United States, 1900-1940* (1947), *Trends in Output and Employment* (1947), *Employment and Compensation in Education* (1950), *The Price Statistics of the Federal Government* (1961), *The Intellectual and the Marketplace* (1962), *A Dialogue on the Proper Economic Role of the State* (coauthored work, 1963), *Capital and Rates of Return in Manufacturing Industries* (1963), *Essays in the History of Economics* (1965), *The Organization of Industry* (coauthored work, 1968), *The Behavior of Industrial Prices* (coauthored work, 1970), *Modern Man and His Corporations* (1971), *The Citizen and the State: Essays on Regulation* (1975), etc.

Stigler was a representative of the Chicago School of Economics in the area of microeconomics. He was one of the founders of information economics. He contended that it was too costly for consumers to obtain information on quality, price and timing of purchasing, so that consumers were neither able to nor willing to obtain sufficient information, thus resulting in different prices for the same commodity. According to him, this was inevitable and a normal market phenomenon that did not require human intervention. Stigler’s view renewed the assumption that there was only one price for a commodity in the market theory of microeconomics. In the research process, Stigler also extended this analysis to the labor market. These studies established a new research area called information economics. Since his paper *The Economics of Information* was published in 1961, the study has become a prominent subject in today’s economic discipline, producing many Laureates of the Nobel Memorial Prize in Economic Sciences. Another contribution of Stigler was his criticism of social regulation policies. His frequent comments on public policies were often cited by political figures. His most well-known contribution was to demonstrate that the free market mechanism remains the most efficient system in existence

today. He utilized the latest research results of econometrics, and provided many examples in which government regulations that aimed to improve efficiency were actually ineffective or deleterious. He also advocated the free market system, and opposed monopoly and state intervention. He was the primary founder of a new and important research area called regulatory economics. Friedman praised Stigler as a pioneer who used economic and analytical methods to study legal and political issues.

9.7 Exercises

Exercise 9.1 In an industry with unchanged technology, the long-run cost function of perfectly competitive firms is $LTC = q^3 - 10q^2 + 175q$, and the market demand function is $Q = 1000 - 2P$.

1. Find the long-run supply function of the industry.
2. Find the number of firms at the long-term equilibrium.
3. If a consumption tax of \$50 per unit is levied, find the number of firms at the new long-run equilibrium.
4. If the above consumption tax is replaced by a sales tax of 50% of the product price, find the number of firms at the new long-run equilibrium.

Exercise 9.2 The demand function of a perfectly competitive market is $q = a - bp$, and the supply function is $q = c + dq$, where $a, b, c, d > 0$.

1. If consumers are levied a specific duty t , how will social welfare change? Why?
2. If the above specific duty t is now levied on producers instead, how will social welfare change? Why?
3. If consumers are given a specific subsidy s instead, how will social welfare change? Why?

Exercise 9.3 The demand function of a perfectly competitive market is $q^d(p)$, and the supply function is $q^s(p)$, where $q^d(p)$ is a decreasing function of p , and $q^s(p)$ is an increasing function of p .

1. Consider the case of specific duty. Are the equilibrium quantity and price finally obtained by producers when taxing consumers the same as those when taxing producers? Justify your answer.

2. Consider the case of specific subsidy. Are the equilibrium quantity and price finally obtained by producers when subsidizing consumers the same as those when subsidizing producers? Justify your answer.
3. Consider the case of ad valorem duty. Are the equilibrium quantity and price finally obtained by producers when taxing consumers the same as those when taxing producers? Justify your answer.

Exercise 9.4 There is a unique monopolistic firm in the market, who can adopt first-degree price discrimination.

1. Will the firm choose the output level according to the decision principle $MR = MC$? Why?
2. Will the firm choose to produce where the market demand is inelastic? If it is possible, give an example; if not, justify your answer.

Exercise 9.5 A monopolist has two geographically separated markets with demand functions $q_1 = 30 - 2p_1$ and $q_2 = 25 - p_2$, respectively, its marginal cost is 3, and the fixed cost is zero.

1. If the monopolist can implement third-degree price discrimination, what will its price, output, and profit be?
2. If price discrimination is prohibited by law, then what will its price, output, and profit be?
3. If the demand of market 2 increases and its demand function becomes $q_2 = a - p_2$ for $a > 25$, answer questions 1 and 2 again.

Exercise 9.6 There is only one firm in the market with a demand function $q = a - bp$ and a marginal cost c satisfying $c < a/b$. The firm sells through a unique retailer. It first sets a wholesale price w , and then the retailer sets a retail price p after observing the wholesale price. The retailer's cost is zero. Prove the following: The retail price in the market is higher than the price set by the vertically integrated monopolist.

Exercise 9.7 There is a unique monopolistic firm in the market, and its inverse demand function (in each period) is $p = a - bq$. The marginal cost

in period 1 is c_1 ; as the monopolist is “learning by doing”, the marginal cost in period 2 is $c_2 = c_1 - mq_1$. Suppose that $a > c$ and $b > m$.

1. What is the output of the monopolist in each period?
2. If the output in each period is chosen by a social planner, will she choose the output on the principle that “the price equals the marginal cost” in period 1? Why?
3. If the output in period 2 is determined by the monopolist, will the social planner choose an output level in period 1 higher than the result obtained in question 1? Why?

Exercise 9.8 Suppose that the monopolist and consumers both live for infinite periods. The value v by consumers is uniformly distributed over $[0, 1/(1 - \delta)]$ (i.e., the valuation in each period is subject to a uniform distribution on $[0, 1]$). If a consumer with value v purchases at price p_t at time t , his utility is $\delta^t(v - p_t)$. The monopolist’s intertemporal profit is $\sum_{t=1}^{\infty} \delta^t p_t q_t$. Find the linear stable equilibrium: in a certain period, when the price is p , any consumer with a value higher than $w(p) = \lambda p$ will purchase, where $\lambda > 1$, while consumers with lower valuations will not. Conversely, in a certain period, if consumers with value higher than v purchase while others do not purchase, then the monopolist charges a price $p(v) = \mu v$, where $\mu < 1$.

1. When only consumers with values lower than v purchase, the monopolist charges p_t, p_{t+1}, \dots , and consumers follow the above linear rule. Find the intertemporal profit of the monopolist starting from period t .
2. Prove the following: The optimization of p_t by the monopolist leads to a linear rule, where λ is determined by $1 - 2\lambda\mu + \delta\lambda^2\mu^2 = 0$.
3. Write down the indifference equation of consumers with valuation $w(p)$.
4. Prove the following: When δ approaches 1, the monopolist’s profit approaches 0.

Exercise 9.9 Consider an economy consisting of two consumers and two commodities. The utility function of type A consumers is $u(x, y) = 6x - x^2 + y$, and the utility function of type B consumers is $v(x, y) = 8x - x^2 + 2y$. The price of commodity y is 1, the income of each consumer is 5,000, and the numbers of the two types of consumers are both n .

1. Suppose that the monopolist's marginal cost of producing commodity x is c , and it cannot implement any price discrimination. Find the optimal price and output. What range is c in when the monopolist sells the commodity to both types of consumers?
2. Suppose that the monopolist adopts a "two-part tariff" under which consumers must first pay a lump-sum fee k , so that they can purchase at a unit price p . If $p < 4$, what is the highest lump-sum fee k for type A consumers? If a type A consumer pays k and then purchases at a unit price p , how many units will he purchase?
3. If the economy only has n type A consumers and no type B consumers, what will p and K be when the profit is maximized?
4. If $c < 1$, and both types of consumers purchase, what will p and k be when the profit is maximized?

Exercise 9.10 A retailer purchases products from a wholesaler and sells the products to consumers. The retailer holds all sales channels, and thus it is a monopolist in the retail market. The market demand is $p = 20 - q$. Suppose that the retailer cannot implement price discrimination. The wholesaler is a monopolist in the wholesale market, and its production cost is $c(Q) = 3Q^2 + 10$. The wholesaler charges the retailer a two-part tariff: in addition to the wholesale price w per unit of product, there is a fixed entry fee F (note that when the retailer does not purchase products from the wholesaler, i.e., when the retailer chooses to withdraw from the market, there is no need to pay the entry fee F). Because the wholesaler does not have retail channels, it cannot directly sell its products to consumers. The retailer's goal is to maximize her profit $p(q) = wq - F$ by choosing q under the premise of given (w, F) . To facilitate the solution, we assume that the retailer will exit

from the market if and only if its profit is negative, and the wholesaler chooses a wholesale price $w < 20$.

1. Express the retailer's profit as a function of w , q , and F .
2. If the retailer chooses to enter the market, find the output q^* when the retailer's profit is maximized, where q^* is a function of w .
3. After obtaining q^* , express the retailer's price p and its profit as functions of w .
4. What conditions shall F and w satisfy so that the retailer will not exit from the market?

Exercise 9.11 Suppose that the product market is perfectly competitive, and that the production of this product requires two factors of production, i.e., labor and capital. A firm is a price-taker in the labor market, but it is the only buyer in the capital market, and thus can determine the price of capital.

1. Write the firm's profit maximization problem and give the first-order condition.
2. If the firm becomes a price-setter in both labor and capital factor markets, how will the first-order condition of its profit maximization problem change?
3. If the firm becomes a monopolist in the product market and a price-setter in the capital factor market, how will the first-order condition of its profit maximization problem change?

Exercise 9.12 There are two oligopolists in the market who engage in a Bertrand competition. The market demand function $x(p)$ is continuous and strictly decreasing in p , and there exists a $\bar{p} < \infty$, such that $x(p) = 0$ for all $p \geq \bar{p}$. The marginal costs of the two firms are both $c > 0$.

1. Prove that there is a unique pure strategy Nash equilibrium $p_1^* = p_2^* = c$.

2. Prove that the above pure strategy Nash equilibrium is also the unique Nash equilibrium.
3. If the market demand function becomes $x(p) = p^{-1}$, prove that there is a mixed strategy Nash equilibrium, such that both firms have positive profits.

Exercise 9.13 Consider the following product differentiation model: $p_1 = a - bq_1 - dq_2$ and $p_2 = a - dq_1 - bq_2$, where $a, d > 0, b \geq d$. Suppose that the marginal costs of the two firms are c_1 and c_2 , respectively.

1. If the two firms engage in quantity competition, find the equilibrium outputs.
2. Derive the demand functions of the two firms.
3. If the two firms engage in price competition, find the equilibrium prices.

Exercise 9.14 Consider the following product differentiation model: $p_1 = a - b(q_1 + \lambda q_2)$ and $p_2 = a - b(\lambda q_1 + q_2)$, where $a, b > 0$ and $0 \leq \lambda \leq 1$. Suppose that the marginal costs of the two firms are both c , and the two firms make decisions simultaneously, where firm 1 sets the price and firm 2 sets the output. Find the equilibrium outputs and prices of the two firms.

Exercise 9.15 There exist two oligopolists in the market, and their cost functions are $c_1 = 2q_1^2$ and $c_2 = q_2^2$, respectively. The demand function is $q = 10 - 2p$.

1. If the two firms do not form a cartel, solve for the optimal outputs and profits of the two in a Cournot competition.
2. If the two firms form a cartel, solve for the profit-maximizing outputs. How will the two firms distribute the profits?
3. If the two firms' cost functions change to $c_1 = 2q_1$ and $c_2 = q_2$, answer questions 1 and 2 again.

Exercise 9.16 There are two oligopolists in the market who engage in a Cournot competition. The marginal costs of the two firms are both 2. The market demand function is $q = 14 - 3q$. Before they decide on their outputs, firm 1 can choose whether to adopt a new technology with a fixed cost of 10 to reduce its marginal cost to 0. Firm 2 can observe whether firm 1 has adopted the technology.

1. Write the strategy sets of the two firms in this game.
2. Will firm 1 use this technology? What are the equilibrium outputs of the two firms, respectively?

Exercise 9.17 There are J oligopolists in the market who engage in a Cournot competition. The market demand function is $q = a - 2p$, and the marginal cost of firm i ($i = 1, 2, \dots, J$) is c_i .

1. Find the equilibrium outputs of the J firms.
2. Suppose that an upcoming policy will increase the marginal costs of all firms by a constant c_0 . Will they support the policy? Why?
3. Suppose that an upcoming policy will increase the marginal costs of all firms by a fixed percentage of t . Will they support the policy? Why?

Exercise 9.18 There are two oligopolists in the market. Their marginal costs are both c , and the inverse demand function is $p = a - bq$. The two firms compete in a Stackelberg game, where firm 1 is the leader, and firm 2 is the follower.

1. Solve for the output levels in the subgame perfect equilibrium, i.e., (q_1^*, q_2^*) .
2. Is (q_1^*, q_2^*) a Nash equilibrium of this game? Why?
3. If these two firms now compete in a Cournot game, prove that the Nash equilibrium of the Cournot game is also the Nash equilibrium of the Stackelberg game.

Exercise 9.19 There are three oligopolists in the market. The demand function is $q = a - bp$, their marginal costs are all c , and the fixed costs are all zero. If the three firms make output decisions, respectively, in the following orders, solve for the optimal output level of each firm.

1. The three firms choose their outputs simultaneously.
2. Firm 1 first chooses its output, firm 2 chooses its output after observing firm 1's output, and firm 3 chooses its output after observing the outputs of firm 1 and firm 2.
3. Firm 1 first chooses its output, and after observing firm 1's output, firm 2 and firm 3 choose their outputs simultaneously.
4. Firm 1 and firm 2 first choose their outputs simultaneously, and firm 3 chooses its output after observing the outputs of firm 1 and firm 2.

Exercise 9.20 There are two oligopolists in the market. The market demand function is $q = 500 - 4p$. Firm 1 and firm 2 have constant marginal costs of 6 and 10, respectively.

1. If firm 1 sets the market price of the product and firm 2 is a price-taker, find the equilibrium outputs and profits of the two firms.
2. If firm 2 sets the market price of the product and firm 1 is a price-taker, find the equilibrium outputs and profits of the two firms.
3. Is firm 1 or firm 2 willing to become a price leader? Why?
4. If the two firms engage in a Cournot competition, find the equilibrium outputs and profits, and compare them with the results of questions 1 and 2. What do you find?

Exercise 9.21 There is a leader firm and a follower firm in the market. The market demand function is $q = 10 - 2p$. The leader firm 1 sets the market price of the product, and then the follower firm 2 takes the price as given. This is called the “price leadership model”. The cost functions of the two firms are $TC_1 = 0$ and $TC_2 = 2q_2^2$, respectively.

1. Explain the difference between the price leadership model and the Stackelberg model.
2. Solve for the outputs and profits of the leader firm and the follower firm.
3. If the follower firm's cost function becomes $TC_2 = aq_2$, where a is a constant, solve for the output and profit of the leader firm.

Exercise 9.22 Suppose that the cost for each firm to enter the market is $c > 0$, and the following conditions are satisfied: (1) the equilibrium output of a single firm decreases in the number of firms; (2) the total output increases in the number of firms; and (3) the equilibrium price is always above the marginal cost. Prove the following: From the perspective of social welfare, the symmetric Cournot model of free entry leads to excessive entry.

Exercise 9.23 Consumers are uniformly distributed on an axis of length 1, and each consumer only purchases one unit of goods. In the first stage, two firms choose their locations. Firm 1 is $a > 0$ away from the left end of the axis, firm 2 is $b > 0$ away from the right end, and $a + b \leq 1$. In the second stage, the two firms choose their prices p_1 and p_2 , respectively. The transportation cost is quadratic, i.e., the transportation cost for a consumer at x to go purchase from firm 1 is cx^2 , while that for a consumer at y to go purchase from firm 2 is cy^2 . The fixed costs and marginal costs of the two firms are all zero. Prove the following: $a = b = 0$.

Exercise 9.24 (Rotemberg and Saloner, 1986) There are two oligopolists in the market who produce homogeneous products, and both have a constant marginal cost c . The two firms engage in infinitely repeated Bertrand competition with a discount factor δ . The market demand function is $q = a - p$, and the intercept a randomly fluctuates: in each period, the probability of $a = a_h$ is λ , and the probability of $a = a_l$ is $1 - \lambda$. $a_l < a_h$ and the demands in different periods are independent. The monopoly prices at two demand levels are denoted by p_h and p_l .

1. Solve for the δ^* , such that for $\delta \geq \delta^*$, the two firms can employ the trigger strategy to maintain the above-mentioned monopoly prices in the subgame perfect Nash equilibrium.

2. For $1/2 < \delta < \delta^*$, find the highest price $p(\delta)$, such that in the subgame perfect Nash equilibrium, the two firms can use the trigger strategy to maintain the price $p(\delta)$ at the high demand level, and p_l at the low demand level.

Exercise 9.25 There are two oligopolists in the market who produce homogeneous products and engage in a Cournot competition. The market demand function is $q = a - bp$. The marginal cost of firm 1 is c_h with a probability λ and c_l with a probability $1 - \lambda$. The marginal cost of firm 2 is c_h with a probability η and c_l with a probability $1 - \eta$.

1. If both firms know exactly their own marginal costs and know only the probability distribution of the opponent's marginal cost, find the outputs of the two firms.
2. If the two firms know only the probability distribution of the marginal costs of their own and their opponents, find the outputs of the two firms.

Exercise 9.26 Suppose that consumers and firms are uniformly distributed on a unit circle. A consumer chooses one firm to purchase one unit of its product, and his transportation cost is tx , where x represents the distance between the consumer and the chosen firm. The market is free to enter, and the fixed entry cost for each firm is f .

1. Find the equilibrium number J of firms.
2. Find the socially optimal number m of firms. What is the numerical relation between m and J ? Provide an explanation for your answer.
3. If the consumer's transportation cost is changed to tx^2 , answer questions 1 and 2 again.

Exercise 9.27 (Selten, 1973) There are J firms in the market, and their marginal costs are all zero. The market demand function is $q = 1 - p$. The firms decide whether to join a cartel. The cartel members determine the output distribution standard and strictly enforce it, and they engage in a Cournot competition with other firms. Prove the following: If $J \leq 4$, then all firms

will join the cartel; but if $J \geq 6$, the cartel will only include some of the firms.

Exercise 9.28 There are J firms in the market, their marginal costs are all c , and the market demand function is $q = a - p$.

1. Consider a merger of two firms. Prove the following: If $J = 2$, they can profit from it; but if $J \geq 3$, the merger is unprofitable.
2. Now, consider a merger of k firms. Find the necessary and sufficient conditions for them to profit from the merger.

Exercise 9.29 In the limit pricing model in this chapter, it is assumed that the marginal cost of the potential entrant is public knowledge. Now, consider the following model, in which the marginal cost of the potential entrant is private information. Specifically, let the demand function be $p = 10 - 2q$, the prior distribution of the incumbent firm 1's type be $p(c_1 = 0) = \alpha$ and $p(c_1 = 4) = 1 - \alpha$, and the prior distribution of the potential entrant firm 2's type be $p(c_2 = 1) = \beta$ and $p(c_2 = 2) = 1 - \beta$. The market entry cost is F , and the time discount rate is $\delta = 1$. Find possible pooling equilibria and separating equilibria, and verify whether they satisfy the intuitive criterion.

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Part V

Externalities and Public Goods

The rest of the book will examine the allocation of resources in more realistic economic environments. The main theme is how to solve the issue of “market failure” to remedy market defects. These will be the topics discussed in the remaining chapters of this book.

The market we have discussed so far is basically a frictionless ideal market economy. In addition, we directly or implicitly assume that the markets are perfectly competitive except for monopolistic competition, oligopoly and monopoly discussed in Chapter 9.

Chapters 3-10 and Chapter 13 are mainly descriptive positive analysis of the market economy, discussing how rational consumers and firms make optimal decisions and how the market operates in various structures (perfect competition, monopolistic competition, oligopoly). Chapters 11-12 mainly make normative analysis on perfectly competitive market, and discuss the market’s optimality, rationality, uniqueness and universality from different aspects.

In Chapters 11 and 12, we have discussed the internal logic connections between competitive equilibrium (Walrasian equilibrium) and efficiency (Pareto optimality). The concept of competitive equilibrium provides us with an appropriate notion of market equilibrium for competitive market economies. The concept of Pareto optimality offers a minimal and uncontroversial test that any socially optimal economic outcome should pass since it is a formulation of the idea that there is no further improvement within a given social resources, and it conveniently separates the issue of economic efficiency from more controversial (and political) questions regarding the ideal distribution of well-being across individuals.

The important results and insights obtained in Chapters 11-12 are the First and Second Fundamental Theorems of Welfare Economics, Economic Core Theorem, Core Equivalence Theorem, and the Fairness Theorem. These results demonstrate the optimality, rationality and uniqueness of a perfectly competitive free market economy from different perspectives. The First Fundamental Theorems of Welfare Economics reveals how a market economy results in Pareto optimal allocations under certain conditions such as perfect competition, personal interests-pursuing (local non-satiation preferences), no externalities, no public goods, no increasing returns to s-

cale, complete information, etc. When these conditions are not satisfied, market often fails, resulting in inefficient allocations of resources. It is, in some sense, the formal expression of Adam Smith's claim about the "invisible hand" of the market.

The Second Fundamental Theorems of Welfare Economics goes even further. It states that under the same set of conditions plus convexity and continuity conditions, all Pareto optimal outcomes can in principle be implemented through market by appropriate redistribution of initial endowments and then "letting the market work". The First and Second Fundamental Theorems of Welfare Economics shows that Pareto optimal allocations and competitive market equilibrium allocations are equivalent in a certain sense, while the Economic Core Theorem reveals that a free competitive market institution is also socially stable.

Equity of resource allocations is also an important criterion to an economy. The Fairness Theorem discussed in Chapter 12 gives specific suggestions on how to achieve efficiency and equity of resource allocations simultaneously. If all economic agents have equal starting point for competition, even if this starting point is not Pareto efficient, but through market competition, it can achieve both Pareto efficient and equitable allocations. The Core Equivalence Theorem is even deep and profound which reveals the market economic institution is a unique institution that results in efficient allocations in the presence of sufficient economic freedom and competition. That is, it demonstrates that under the objective and realistic constraints of individual pursuing their self-interests, as long as individuals choose to voluntarily cooperate and exchange freely under sufficient competition, even if no economic system arrangement is arranged beforehand, the result approaches to a competitive market equilibrium allocation.

All these conclusions show the effective role of market economy in resolving problems associated to fairness and justice. Through the joint role of government, market and society, we can achieve resource allocations with both efficiency and equity. These results provide theoretical support for inheritance tax, compulsory education, environmental protection, anti-monopoly, and financial regulation, thus providing inspirations for solving the problem of excessive gap between the rich and the poor as well as mar-

ket failures.

As mentioned above, all these conclusions are based on their preconditions, and the perfectly competitive market is just an ideal state without friction, which basically does not exist in reality and belong to benchmark theories. But this research method, like natural science, has its necessity. Like the ideal state without friction in physics, it does not exist at all in practice, but it provides a benchmark or a reference system for the study of practical problems with various frictions. Similarly, the perfect competition situation analyzed by the general equilibrium theory has important theoretical and practical significance for us to think, study and test a real market economy. It provides direction and strategy for the choice and reform of economic systems, establishes a benchmark or a reference system for us to study more realistic markets, and it is the starting point people to think of and test the results of market economy. In particular, if a market economy fails to achieve the efficiency of resource allocation, the market that is intervened to achieve Pareto improvement will inevitably violate at least some of the conditions of the First Fundamental Theorem of Welfare Economics.

Of course, from the micro- and incomplete information perspectives, market still faces many problems that often lead to “market failure”. It is important to analyze the circumstances under which a market may fail and how a government should perform. Provided the governance boundary of the market mechanism is clarified, we do not unconditionally decline the role of the market, or go from one extreme to the other, but know the circumstances under which the market can play a decisive role in allocating resources, and how a government could play a good, appropriated but not unlimited role in cases of market failures. Of course, the basic premise is still that individuals are rational, and government should not directly intervene in economic activities, but establish rules or systems to solve the problem of market failure. This is due to that with incomplete information, direct intervention in economic activities (such as a large number of state-owned enterprises and arbitrary restrictions on market access and interference with commodity prices) often does not work well. In this regard, the last two parts of the book, which are devoted to the study of how to design

rules and institutional incentive compatible mechanisms, can play a significant role in making the market more efficient and solving market failure problems.

Therefore, the contents of the rest of the book can be seen as a further expansion of these topics. From Chapter 14 to the end of the book, our discussion will turn from how the superiority of the frictionless and freely competitive market system to the issues that may arise, and discuss how to remedy the failures. To this end, we will examine the various situations in which the actual market deviates from the ideal perfect competition situation and the so-called “market failure” issues resulting in Pareto inefficient allocations, and we will provide solutions for market failure.

In the current part, we will study externalities and public goods in Chapter 13 and Chapter 14, respectively. In both cases, the actions of one agent directly affect the utility or production of other individuals in the economy. We will see these nonmarketed “goods” or “bads” lead to Pareto inefficiency in general. It turns out that private markets often do not work well in the presence of externalities and public goods. We will consider situations of incomplete information which also result in Pareto inefficient outcomes in general in Parts ?? and ??.

Chapter 14

Externalities

14.1 Introduction

In this chapter we deal with economic environments with externalities. The so-called externality refers to situations that economic activities (production or consumption activities) of some individuals affect the utility or production levels of other individuals, which further affect their own economic activities. The basic conclusion is that the existence of externality generally leads to Pareto inefficient outcomes, leading to a market failure. The fundamental reason is that due to externality in economy, some factors affecting economic activities are not properly considered, and thus even if perfect competition and free choice are present, allocation is often inefficient, and thus other institutional arrangements or mechanism designs are necessary to improve the allocation of resources.

Externality is an objective and ubiquitous phenomenon. Specifically, externality consists of two categories: consumer externality and production externality, and they are the same in nature.

14.1.1 Consumption Externality

In the previous analysis, utility, satisfaction or welfare of individuals are only related to their own consumption levels, but not to the consumption of others. But in fact, in many situations, the utility level of an agent will also be affected by the consumptions of others, while the agent cannot con-

trol the consumption of others, thereby the agent's utility level is passively affected. Externality can be either negative, which may hurt an agent, or positive, which may benefit the agent. Such examples abound.

Example 14.1.1 The following examples about consumption externalities are commonly seen in practice.

- (i) One person's quiet environment is disturbed by another person's noise.
- (ii) Mr. A hates Mr. D smoking next to him.
- (iii) Mr. B's satisfaction decreases as Mr. C's consumption level increases, because Mr. B envies Mr. C's affluent life, which leads to the mentality of hating the rich.
- (iv) It does not matter if you are feeling well or not. You do want to others feel well. If someone else is feeling well, you will not feel well. Seeing others feel well makes you unhappy.
- (v) You watch the TV bought by your roommates.
- (vi) Take your colleague's free ride to work.
- (vii) Your dress is influenced by the style of other people.
- (viii) The utility from using phone, WhatsApp and email depends on whether other people have the corresponding device.

Environmental impacts or pollution in (i) and (ii) are typical externalities, and individual behaviors that pollute the environment can have a negative impact on the health of others. Also, there are always some people who are jealous of others around us. Benefiting oneself at the cost of others in (iv) may be understandable, and harming others at the cost of oneself (all perish together) may also be understandable, but (iii) harming others without benefiting oneself is commonplace. Although it is somewhat hard to understand, it may be interpreted by negative consumption externalities. (v) and (vi) are examples of positive externalities, while (vii) and (viii) are examples of consumption network externalities.

These examples illustrate that the existence of externalities is a very common phenomenon. Some may deny the existence of externality with the example of looking at beauty without cost. In fact, there are two misunderstandings in this assertion. Firstly, whether the price is zero is not used in defining externality. Secondly, to see a beauty more often, such as model shows or stars in the movie, you need to pay.

Formally, we express the existence or absence of consumption externalities as:

$$\begin{aligned} u_i(\mathbf{x}_i) &: \text{without preference externality;} \\ u_i(\mathbf{x}_1, \dots, \mathbf{x}_n) &: \text{with preference externality,} \end{aligned}$$

in which other individuals' consumptions affect an individual's utility.

14.1.2 Production Externality

Production externality means that the level of one firm's output is affected by the production activities of other economic agents. Production externality effects can be either negative or positive.

The following lists some examples for production externality:

Example 14.1.2 Here are some typical examples about production externalities in practice.

- (i) Sewage discharge from chemical plants can affect the production of surrounding fishermen. In particular, downstream fishing can be adversely affected by pollutants emitted from an upstream chemical plant.
- (ii) The machine noise of the factory near to you disturbs your quietness.
- (iii) Smog from factories was somehow equivalent to smoking a pack of cigarettes a day.
- (iv) Problem with a bank has caused panic and a significant decline in currency liquidity. In the face of bank failures caused

by the financial crisis, everyone is afraid to deposit, thus affecting the real economy.

- (v) Beekeeping and farms have mutually positive production externalities. The flowers in the farm benefit the beekeepers, which in turn facilitates pollination of crops.
- (vi) University brands can benefit all students, including those with low GPA, making it relatively easier to find jobs.
- (vii) R&D of an enterprise may increase the productivity and in turn output levels of other firms. In the IT industry, the fixed cost of production is large, and the marginal cost is small, but the positive externality of the product is very large. As such, knowledge has a typical externality. In order to facilitate innovation, intellectual property protection is required; otherwise, a enterprise may not have incentives to conduct R&D. A monopoly is not altogether a bad thing under competition environment. One of its benefits is that it can stimulate firms to carry out R&D and innovations (and thus obtain monopoly profits).
- (viii) The output of a firm is influenced by the stock of knowledge in the entire economy (production network externality).

In addition, the decision of government and its officials might have a huge positive or negative externality to individuals, both in terms of production and consumption. This is the fundamental reason why there is a need for supervision or checks and balances on government and its officials.

This leads to an examination of various suggestions for alternative ways to allocate resources that may lead to more efficient outcomes. Achieving an efficient allocation in the presence of externalities essentially involves making sure that agents face the correct pricing for their activities. Ways of solving externality problems include taxation, regulation, property rights, merges, mechanism design, market design, etc.

14.2 Consumption Externality

When there are no consumption externalities, agent i 's utility function is a function of only his own consumption:

$$u_i(\mathbf{x}_i). \quad (14.2.1)$$

In this case, for any $l = 1, 2, \dots, L$ and $h = 1, 2, \dots, L$, the first-order conditions for the competitive equilibrium are given by

$$MRS_{x_1^l, x_1^h} = MRS_{x_2^l, x_2^h} = \dots = MRS_{x_n^l, x_n^h} = \frac{p^l}{p^h}, \quad (14.2.2)$$

and from Chapter 11, we know that the first-order conditions for Pareto efficiency are also given by:

$$MRS_{x_1^l, x_1^h} = \dots = MRS_{x_n^l, x_n^h}, \quad (14.2.3)$$

As a result, because of price-taking behavior, every competitive equilibrium results in Pareto efficiency when utility functions are locally non-satiated.

The main purpose of this section is to show that a competitive equilibrium allocation is not in general Pareto efficient when there exists an externality in consumption. We show this by examining that the first-order conditions for a competitive equilibrium are not in general the same as the first-order conditions for Pareto efficient allocations in the presence of consumption externalities. The following discussion is mainly drawn from Tian and Yang (2009).

Consider the following simple two-person and two-good exchange economy.

$$u_A(x_A, x_B, y_A), \quad (14.2.4)$$

$$u_B(x_A, x_B, y_B), \quad (14.2.5)$$

which are assumed to be strictly increasing in his own goods consumption, quasi-concave. To have interior solutions, we assume that u_i satisfies the Inada condition $\frac{\partial u}{\partial x_i}(0) = +\infty$, and also $\lim_{x_i \rightarrow 0} \frac{\partial u}{\partial x_i} x_i = 0$. Note that good

x results in consumption externalities.

The first order conditions for the competitive equilibrium are the same as before:

$$MRS_{xy}^A = \frac{p_x}{p_y} = MRS_{xy}^B. \quad (14.2.6)$$

We now find the first order conditions for Pareto efficient allocations in exchange economies with consumption externalities. Pareto efficient allocations x^* can be completely characterized by the FOCs of the following problem.

$$\begin{aligned} & \max_{x \in \mathcal{R}_{++}^4} u_B(x_A, x_B, y_B) & (14.2.7) \\ \text{s.t.} \quad & x_A + x_B \leq w_x, \\ & y_A + y_B \leq w_y, \\ & u_A(x_A, x_B, y_A) \geq u_A(x_A^*, x_B^*, y_A^*). \end{aligned}$$

The first order conditions are

$$x_A : \frac{\partial u_B}{\partial x_A} - \lambda_x + \mu \frac{\partial u_A}{\partial x_A} = 0, \quad (14.2.8)$$

$$y_A : -\lambda_y + \mu \frac{\partial u_A}{\partial y_A} = 0, \quad (14.2.9)$$

$$x_B : \frac{\partial u_B}{\partial x_B} - \lambda_x + \mu \frac{\partial u_A}{\partial x_B} = 0, \quad (14.2.10)$$

$$y_B : \frac{\partial u_B}{\partial y_B} - \lambda_y = 0, \quad (14.2.11)$$

$$\lambda_x : w_x - x_A - x_B \geq 0, \lambda_x \geq 0, \lambda_x (w_x - x_A - x_B) = 0, \quad (14.2.12)$$

$$\lambda_y : w_y - y_A - y_B \geq 0, \lambda_y \geq 0, \lambda_y (w_y - y_A - y_B) = 0, \quad (14.2.13)$$

$$\mu : u_A - u_A^* \geq 0, \mu \geq 0, \mu (u_A - u_A^*) = 0. \quad (14.2.14)$$

By (14.2.11), $\lambda_y = \frac{\partial u_B}{\partial y_B} > 0$, and thus by (14.2.13),

$$y_A + y_B = w_y, \quad (14.2.15)$$

which means there is never disposal (or destruction) of the good that has

no externality. Also, by (14.2.9) and (14.2.11), we have

$$\mu = \frac{\frac{\partial u_B}{\partial y_B}}{\frac{\partial u_A}{\partial y_A}}. \quad (14.2.16)$$

Then, by (14.2.8) and (14.2.9), we obtain

$$\frac{\lambda_x}{\lambda_y} = \left[\frac{\frac{\partial u_A}{\partial x_A}}{\frac{\partial u_A}{\partial y_A}} + \frac{\frac{\partial u_B}{\partial x_A}}{\frac{\partial u_B}{\partial y_B}} \right], \quad (14.2.17)$$

and by (14.2.10) and (14.2.11), we have

$$\frac{\lambda_x}{\lambda_y} = \left[\frac{\frac{\partial u_B}{\partial x_B}}{\frac{\partial u_B}{\partial y_B}} + \frac{\frac{\partial u_A}{\partial x_B}}{\frac{\partial u_A}{\partial y_A}} \right]. \quad (14.2.18)$$

Thus, by (14.2.17) and (14.2.18), we get

$$\frac{\frac{\partial u_A}{\partial x_A}}{\frac{\partial u_A}{\partial y_A}} + \frac{\frac{\partial u_B}{\partial x_A}}{\frac{\partial u_B}{\partial y_B}} = \frac{\frac{\partial u_B}{\partial x_B}}{\frac{\partial u_B}{\partial y_B}} + \frac{\frac{\partial u_A}{\partial x_B}}{\frac{\partial u_A}{\partial y_A}}, \quad (14.2.19)$$

which means the equality of the social marginal rates of substitution of good x for good y for the two consumers at Pareto efficient allocations. We call it social marginal rate of substitution because the social welfare function can be written as the sum of individual utilities. From the above marginal equality condition, we know that, in order to evaluate the relevant marginal rates of substitution for the optimality conditions, we must take into account both the direct and indirect effects of consumption activities in the presence of externalities. That is, to achieve Pareto optimality, when one consumer increases the consumption of good x , not only does the consumer's consumption of good y need to change, the other consumer's consumption of good y also need to change. Therefore the social marginal rate of substitution of good x for good y by consumer i equals $\frac{\frac{\partial u_i}{\partial x_i}}{\frac{\partial u_i}{\partial y_i}} + \frac{\frac{\partial u_j}{\partial x_i}}{\frac{\partial u_j}{\partial y_j}}$. Thus, since the first order conditions for competitive equilibrium and Pareto optimality are not the same, we immediately have the following conclusion:

Proposition 14.2.1 *If there is consumption externality, competitive equilibrium allocations are in general not Pareto efficient.*

From the above marginal equality conditions, we can see that in order to correctly calculate the relevant marginal rate of substitution for the marginal equality conditions, we must consider the direct and indirect effects of consumption activities in the presence of consumption externalities.

Solving (14.2.8) and (14.2.10) for μ and λ_x , we have

$$\mu = \frac{\frac{\partial u_B}{\partial x_B} - \frac{\partial u_B}{\partial x_A}}{\frac{\partial u_A}{\partial x_A} - \frac{\partial u_A}{\partial x_B}} > 0 \quad (14.2.20)$$

and

$$\lambda_x = \frac{\frac{\partial u_A}{\partial x_A} \frac{\partial u_B}{\partial x_B} - \frac{\partial u_A}{\partial x_B} \frac{\partial u_B}{\partial x_A}}{\frac{\partial u_A}{\partial x_A} - \frac{\partial u_A}{\partial x_B}}. \quad (14.2.21)$$

When the consumption externality is positive, from (14.2.17) or (14.2.18), we can easily see that λ_x is always positive since $\lambda_y = \frac{\partial u_B}{\partial y_B} > 0$. Also, when no externality or a one-sided externality¹ exists, by either (14.2.17) or (14.2.18), λ_x is positive. Thus, the marginal equality condition (14.2.19) and the balanced conditions completely determine all Pareto efficient allocations for these cases. However, when there are negative externalities for both consumers, the Kuhn-Tucker multiplier λ_x directly given by (14.2.21) or indirectly given by (14.2.17) or (14.2.18) is the sum of negative and positive terms, and thus the sign of λ_x may be indeterminate. Then (14.2.19) and the balanced conditions along may not guarantee finding Pareto efficient allocations correctly.

To guarantee that an allocation be Pareto efficient in the presence of negative externalities, we must require $\lambda_x \geq 0$ at efficient points, which in turn requires that social marginal rates of substitution of good x for good y be nonnegative, that is,

$$\frac{\frac{\partial u_A}{\partial x_A}}{\frac{\partial u_A}{\partial y_A}} + \frac{\frac{\partial u_B}{\partial x_A}}{\frac{\partial u_B}{\partial y_B}} = \frac{\frac{\partial u_B}{\partial x_B}}{\frac{\partial u_B}{\partial y_B}} + \frac{\frac{\partial u_A}{\partial x_B}}{\frac{\partial u_A}{\partial y_A}} \geq 0, \quad (14.2.22)$$

¹Only one consumer imposes an externality on another consumer.

or equivalently requires both (14.2.19) and

$$\underbrace{\frac{\partial u_A}{\partial x_A} \frac{\partial u_B}{\partial x_B}}_{\text{(joint marginal benefit)}} \geq \underbrace{\frac{\partial u_A}{\partial x_B} \frac{\partial u_B}{\partial x_A}}_{\text{(joint marginal cost)}} \quad (14.2.23)$$

for all Pareto efficient points.

We can interpret the term in the left-hand side of (14.2.23), $\frac{\partial u_A}{\partial x_A} \frac{\partial u_B}{\partial x_B}$, as the joint marginal benefit of consuming good x , and the term in the right-hand side, $\frac{\partial u_A}{\partial x_B} \frac{\partial u_B}{\partial x_A}$, as the joint marginal cost of consuming good x because the negative externality hurts the consumers. To consume the goods efficiently, a necessary condition is that the joint marginal benefit of consuming good x should not be less than the joint cost of consuming good x .

Thus, the following conditions

$$(PO) \left\{ \begin{array}{l} \frac{\frac{\partial u_A}{\partial x_A}}{\frac{\partial u_A}{\partial y_A}} + \frac{\frac{\partial u_B}{\partial x_A}}{\frac{\partial u_B}{\partial y_B}} = \frac{\frac{\partial u_B}{\partial x_B}}{\frac{\partial u_B}{\partial y_B}} + \frac{\frac{\partial u_A}{\partial x_B}}{\frac{\partial u_A}{\partial y_A}} \geq 0, \\ y_A + y_B = w_y, \\ x_A + x_B \leq w_x, \\ \left(\frac{\partial u_A}{\partial x_A} \frac{\partial u_B}{\partial x_B} - \frac{\partial u_A}{\partial x_B} \frac{\partial u_B}{\partial x_A} \right) (w_x - x_A - x_B) = 0, \end{array} \right.$$

constitute a system (PO) from which all Pareto efficient allocations should be satisfied. We can further analyse the solution for Pareto efficiency by considering three cases.

Case 1. When $\frac{\partial u_A}{\partial x_A} \frac{\partial u_B}{\partial x_B} > \frac{\partial u_A}{\partial x_B} \frac{\partial u_B}{\partial x_A}$, or equivalently $\frac{\frac{\partial u_A}{\partial x_A}}{\frac{\partial u_A}{\partial y_A}} + \frac{\frac{\partial u_B}{\partial x_A}}{\frac{\partial u_B}{\partial y_B}} = \frac{\frac{\partial u_B}{\partial x_B}}{\frac{\partial u_B}{\partial y_B}} + \frac{\frac{\partial u_A}{\partial x_B}}{\frac{\partial u_A}{\partial y_A}} > 0$, we have $\lambda_x > 0$ and thus the last two conditions in the above system (PO) reduce to $x_A + x_B = w_x$. In this case, there is no destruction for good x . Substituting $x_A + x_B = w_x$ and $y_A + y_B = w_y$ into the marginal equality condition (14.2.19), it provides a relationship between x_A and y_A , from which we can find Pareto efficient allocations.

Case 2. When the joint marginal benefit equals the joint marginal cost:

$$\frac{\partial u_A}{\partial x_A} \frac{\partial u_B}{\partial x_B} = \frac{\partial u_A}{\partial x_B} \frac{\partial u_B}{\partial x_A}, \quad (14.2.24)$$

then

$$\frac{\frac{\partial u_A}{\partial x_A} + \frac{\partial u_B}{\partial x_A}}{\frac{\partial u_A}{\partial y_A} + \frac{\partial u_B}{\partial y_B}} = \frac{\frac{\partial u_B}{\partial x_B} + \frac{\partial u_A}{\partial x_B}}{\frac{\partial u_B}{\partial y_B} + \frac{\partial u_A}{\partial y_A}} = 0 \quad (14.2.25)$$

and thus $\lambda_x = 0$. In this case, when $x_A + x_B \leq w_x$, the necessity of destruction is indeterminant (Note that no destruction means that $x_A + x_B = w_x$). However, even when destruction is necessary, we can still determine the set of Pareto efficient allocations by using $y_A + y_B = w_y$ and the zero social marginal equality condition (14.2.25). Indeed, after substituting $y_A + y_B = w_y$ into (14.2.25), we can solve for x_A in terms y_A .

Case 3. When $\frac{\partial u_A}{\partial x_A} \frac{\partial u_B}{\partial x_B} < \frac{\partial u_A}{\partial x_B} \frac{\partial u_B}{\partial x_A}$, for any allocation that satisfies $x_A + x_B = w_x$, $y_A + y_B = w_y$, and the marginal equality condition (14.2.19), the social marginal rates of substitution is negative. Hence, the allocation will not be Pareto efficient. Thus, there must be a destruction (free disposal) for good x for Pareto efficiency, and there exists Pareto efficient allocations that satisfy (14.2.25).

Summarizing the above three cases. We conclude that one can use the following set of conditions

$$\left\{ \begin{array}{l} \frac{\frac{\partial u_A}{\partial x_A} + \frac{\partial u_B}{\partial x_A}}{\frac{\partial u_A}{\partial y_A} + \frac{\partial u_B}{\partial y_B}} = \frac{\frac{\partial u_B}{\partial x_B} + \frac{\partial u_A}{\partial x_B}}{\frac{\partial u_B}{\partial y_B} + \frac{\partial u_A}{\partial y_A}} \\ x_A + x_B = w_x \\ y_A + y_B = w_y \end{array} \right. ,$$

together with whether $\frac{\partial u_A}{\partial x_A} \frac{\partial u_B}{\partial x_B} \geq \frac{\partial u_A}{\partial x_B} \frac{\partial u_B}{\partial x_A}$, to determine whether there is destruction (free disposal) or not:

Indeed, if $\frac{\partial u_A}{\partial x_A} \frac{\partial u_B}{\partial x_B} \geq \frac{\partial u_A}{\partial x_B} \frac{\partial u_B}{\partial x_A}$ is also satisfied, then no free disposal in achieving Pareto efficient allocations. If $\frac{\partial u_A}{\partial x_A} \frac{\partial u_B}{\partial x_B} < \frac{\partial u_A}{\partial x_B} \frac{\partial u_B}{\partial x_A}$, although utility functions are strictly increasing in x , there must be the case of destruction of some amount of x in achieving Pareto efficient allocations.

We then have the following proposition that characterizes whether or not there is destruction (free disposal) of endowments w_x in achieving Pareto efficient allocations although utility functions are strictly increasing in x .

Proposition 14.2.2 For 2×2 pure exchange economies, suppose that utility functions $u_i(x_A, x_B, y_i)$ are continuously differentiable, strictly quasi-concave, and $\frac{\partial u_i(x_A, x_B, y_i)}{\partial x_i} > 0$ for $i = A, B$.

- (1) If the social marginal rates of substitution of good x for good y are positive at a Pareto efficient allocation x^* ,² then there is no free disposal for w_x in achieving Pareto efficient allocation x^* .
- (2) If the social marginal rates of substitution of good x for good y are negative for any allocation (x_A, x_B) satisfying $x_A + x_B = w_x$, $y_A + y_B = w_y$, and the marginal equality condition (14.2.19), then there must be free disposal for w_x in achieving any Pareto efficient allocation x^* . That is, $x_A^* + x_B^* < w_x$ and x^* is determined by $y_A + y_B = w_y$ and (14.2.25).

Example 14.2.1 Consider the following utility function:

$$u_i(x_A, x_B, y_i) = \sqrt{x_i y_i} - x_j, \quad i \in \{A, B\}, j \in \{A, B\}, j \neq i.$$

By the marginal equality condition (14.2.19), we obtain

$$\left(\sqrt{\frac{y_A}{x_A}} + 1\right)^2 = \left(\sqrt{\frac{y_B}{x_B}} + 1\right)^2 \quad (14.2.26)$$

and thus

$$\frac{y_A}{x_A} = \frac{y_B}{x_B}. \quad (14.2.27)$$

Let $x_A + x_B \equiv \bar{x}$. Substituting $x_A + x_B = \bar{x}$ and $y_A + y_B = w_y$ into (14.2.27), we have

$$\frac{y_A}{x_A} = \frac{w_y}{\bar{x}}. \quad (14.2.28)$$

Then, by (14.2.27) and (14.2.28), we get

$$\frac{\partial u_A}{\partial x_A} \frac{\partial u_B}{\partial x_B} = \frac{1}{4} \sqrt{\frac{y_A}{x_A}} \sqrt{\frac{y_B}{x_B}} = \frac{y_A}{4x_A} = \frac{w_y}{4\bar{x}} \quad (14.2.29)$$

and

$$\frac{\partial u_A}{\partial x_B} \frac{\partial u_B}{\partial x_A} = 1. \quad (14.2.30)$$

Thus, $\bar{x} = w_y/4$ is the critical point that makes $\frac{\partial u_A}{\partial x_A} \frac{\partial u_B}{\partial x_B} - \frac{\partial u_A}{\partial x_B} \frac{\partial u_B}{\partial x_A} = 0$, or equivalently $\frac{\frac{\partial u_A}{\partial x_A}}{\frac{\partial u_A}{\partial y_A}} + \frac{\frac{\partial u_B}{\partial x_B}}{\frac{\partial u_B}{\partial y_B}} = \frac{\frac{\partial u_B}{\partial x_B}}{\frac{\partial u_B}{\partial y_B}} + \frac{\frac{\partial u_A}{\partial x_B}}{\frac{\partial u_A}{\partial y_B}} = 0$. Hence, if $w_x > \frac{w_y}{4}$, then $\frac{\partial u_A}{\partial x_A} \frac{\partial u_B}{\partial x_B} -$

²As we discussed above, this is true if the consumption externality is positive, or there is no externality or only one side externality.

$\frac{\partial u_A}{\partial x_B} \frac{\partial u_B}{\partial x_A} < 0$, and thus there must be destruction in any Pareto efficient allocation. If $w_x < \frac{w_y}{4}$, then $\frac{\partial u_A}{\partial x_A} \frac{\partial u_B}{\partial x_B} - \frac{\partial u_A}{\partial x_B} \frac{\partial u_B}{\partial x_A} > 0$, and thus no Pareto optimal allocation requires no free disposal. Finally, when $w_x = \frac{w_y}{4}$, any allocation that satisfies the marginal equality condition (14.2.19) and the balanced conditions $x_A + x_B = w_x$ and $y_A + y_B = w_y$ also satisfies (14.2.23) since $\frac{\partial u_A}{\partial x_A} \frac{\partial u_B}{\partial x_B} - \frac{\partial u_A}{\partial x_B} \frac{\partial u_B}{\partial x_A} = 0$, and thus it is a Pareto efficient allocation with no free disposal.

Note that, since $\frac{\partial u_A}{\partial x_A}$ and $\frac{\partial u_B}{\partial x_B}$ represent marginal utility, they are usually diminishing with increase in consumption in good x . Since $\frac{\partial u_A}{\partial x_B}$ and $\frac{\partial u_B}{\partial x_A}$ are in the form of marginal cost, their absolute values are typically increasing in good x . Hence, when total endowment w_x is small, the social marginal benefit would exceed the social marginal cost so that there is no destruction of good x . As the total endowment of w_x increases with the total endowment of w_y fixed (i.e., y becomes relatively scarce when x becomes abundant), the social marginal cost will ultimately outweigh the marginal social benefit, which results in the destruction of the endowment of w_x .

Alternatively, we can get the same result by using social marginal rates of substitution. When utility functions are strictly quasi-concave, marginal rates of substitution are diminishing. Therefore, in the presence of negative consumption externalities, social marginal rates of substitution of good x for good y may become negative when the consumption of good x becomes sufficiently large. When this occurs, it is better to destroy some resources for good x . As the destruction of good x increases, which will, in turn, decrease the consumption of good x , social marginal rates of substitution will increase. Eventually they will become nonnegative.

When there is a negative externality, it is seemingly strange that some commodities need to be free disposed or destroyed in order to achieve Pareto efficient allocations of resources, but this phenomenon is not only important in theory, but also related to practice. Indeed, Tian and Yang (2012) used the above theoretical results to explain a well-known puzzle of the happiness-income relationship in the economics and psychology literatures: people's happiness rises with income up to a point, but not beyond it. For example,

mean life satisfaction in the United States has been declining in roughly past sixty years, while that in the United Kingdom has run approximately flat across the same period in Britain. If we interpret income as a good, when the good becomes an inferior good or people envy each other's income level (e.g. low-income people envy high-income people), then according to the above results, when income exceeds a certain threshold level, if all income is spent, people's happiness decreases with the increase of consumption, which leads to Pareto inefficient allocations. Therefore, when economic growth reaches a certain level, if other aspects (such as spiritual civilization and political civilization) can not follow up, increase in income does not increase people's satisfaction, which is the so-called puzzle of the happiness-income.

To illustrate this point, go back to Example 14.2.1, as analyzed by Tian and Yang (2012), and interpret x as the composite of material goods or GDP index, and y as the composite non-material goods or non-GDP index. If we do not increase the level of non-material goods w_y in a comprehensive and balanced way, and only focus on the economic growth of GDP, eventually we will have $w_x > \frac{w_y}{4}$. As a result, we can see in practice that as people's income level constantly rises, but happiness continues to decline.

The above results have a strong policy implication, that is, the government's pursuit of GDP growth does not always improve people's happiness, but may reduce people's satisfaction, resulting in Pareto inefficient allocations. This is the fundamental reason that in the past few decades, people's happiness in many countries has risen and then began to decline as income continue to rise. A similar phenomenon is also recently seen in China. But according to the above discussions, an individual's happiness comes from two levels: material and non-material (spiritual civilization and political civilization). In fact, the level of an individual's happiness is determined by multiple factors: (1) material factors, such as income levels and differences; (2) mental factors, such as career achievement, work stress, unemployment, leisure time, friendships and family harmony; (3) social and political factors, such as social equity, political stability, democratic rights, etc.; (4) ecological factors, such as control of environmental pollution and ecological damage, which are related to individual health

and even survival. It can be seen that the factors listed in (1) are GDP products and the factors listed in (2)-(4) are non-material goods or non-GDP products.

Therefore, happiness comes from material civilization, spiritual civilization, political civilization, and ecological civilization. When people's living standards are limited, people care more about pursuing material civilization. When their living standard reaches a certain level, people will tend to pursue spiritual civilization, political civilization and ecological civilization. That is, people first need to solve the problem of food, clothing, shelter, and transportation, then the superstructure of art, poetry, philosophy, life comfort and quality, physical health, democratic politics, and protection of one's own rights. Due to the negative externalities resulting from envying other people's living stand, the construction of spiritual civilization, political civilization and ecological civilization is also very important. Therefore, both material and non-material goods need to be balanced and fully developed, otherwise, it will not lead to social harmony and efficiency.

Happiness level is the subject of psychology, ethics, and economics. Many economists believe that mainstream economics cannot solve the problem of human happiness. However, the above results show that happiness economics can also be incorporated into the framework of mainstream economics. It can still be assumed that individuals are self-interested in pursuing their personal interests, and Pareto optimality or social welfare maximization is still a necessary and basic criterion of whether the resource allocation is efficient. It just adds to the assumption that people's income generally has negative externalities (this is a reasonable assumption, which is common in practice, for example, some people think that their utility will be decreased if others' income is higher than their own income). For a detailed discussion of this issue, see Tian and Yang in (2009, 2012).

14.3 Production Externality

We now discuss that when production externalities exist, competitive markets may also lead to inefficient allocations of resources. To illustrate this, consider a simple economy with two firms. Firm 1 produces output x that

will be sold in a competitive market. However, production of x imposes an externality cost, denoted by $e(x)$, to firm 2, which is assumed to be convex and strictly increasing.

Let y be the output produced by firm 2, which is sold in competitive market. Let $c_x(x)$ and $c_y(y)$ be the cost functions of firms 1 and 2 which are both convex and strictly increasing.

The profits of the two firms are:

$$\pi_1 = p_x x - c_x(x), \quad (14.3.31)$$

$$\pi_2 = p_y y - c_y(y) - e(x), \quad (14.3.32)$$

where p_x and p_y are the prices of x and y , respectively. Then, by the first order conditions, we have for positive amounts of outputs:

$$p_x = c'_x(x), \quad (14.3.33)$$

$$p_y = c'_y(y). \quad (14.3.34)$$

However, the profit maximizing output x_c from the first order condition is over produced from a social point of view. The first firm only takes account of its own production cost – the cost that is imposed on itself – but it ignores the social cost – the private cost plus the cost it imposes on the other firm.

What's the socially efficient output?

The social profit, $\pi_1 + \pi_2$, is not maximized at x_c and y_c which satisfy (14.3.33) and (14.3.34). If the two firms merged so as to internalize the externality

$$\max_{x,y} p_x x + p_y y - c_x(x) - e(x) - c_y(y) \quad (14.3.35)$$

which gives the first order conditions:

$$p_x = c'_x(x^*) + e'(x^*),$$

$$p_y = c'_y(y^*),$$

where x^* is an efficient amount of output; it is characterized by price being equal to the marginal social cost. Thus, production of x^* is less than the competitive output x_c in the externality case by the convexity of $e(x)$ and

$c_x(x)$.

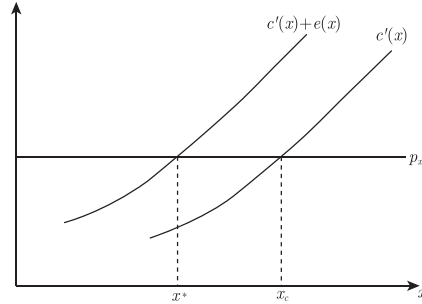


Figure 14.1: The efficient output x^* is less than the competitive output x_c .

14.4 Solutions to Externalities

From the above discussion, we know that a competitive market in general may not result in Pareto efficient outcomes in the presence of externalities, and one needs to seek some other alternative mechanisms to solve the market failure problem. There are many ways proposed to remedy the market failure of externality such as:

1. Pigovian taxes;
2. Voluntary negotiation (Coase Approach);
3. Compensatory tax/subsidy;
4. Creating a missing market with property rights;
5. Direct intervention;
6. Merges of firms;
7. Creating a market for the exchange of emission rights;
8. Incentive mechanism design.

Any of the above solution may result in Pareto efficient outcomes, but may lead to different income distributions. Also, it is important to know what kinds of information are required to implement a solution listed above.

Most of the above proposed solutions need to make the following assumptions:

1. The source and degree of the externality are identifiable.
2. The recipients of the externality are identifiable.
3. The causal relationship of the externality can be established objectively.
4. The cost of preventing (by different methods) an externality is perfectly known to everyone.
5. The cost of implementing taxes and subsidies is negligible.
6. The cost of voluntary negotiation is negligible.

We will discuss the advantages and disadvantages of the above schemes below, and find out which are feasible and which are not. In addition, it is important to know what kind of information is needed to perform each of these solutions. Most of the above schemes require information symmetry, such as Pigovian tax, Coase theorem, and so on. There will be major problems in the implementation of these rules in case of incomplete information. Therefore, incentive mechanism design is necessary to implement these solutions or to provide new solutions.

14.4.1 Pigovian Tax

The Pigovian tax was proposed by Arthur Cecil Pigou (1877–1959, see his biography in Section 14.6.1). For those externality-producing firms, the government imposes a tax on the marginal cost of externality as the tax rate $t = e'(x^*)$. In the case of complete information, the externality and the tax rate t can be determined, and the first-order optimal conditions of the enterprise are the same as the socially optimal first-order conditions, so as to achieve efficient allocation of resources.

To see this, set a tax rate, t , such that $t = e'(x^*)$. This tax rate to firm 1 would internalize the externality. Indeed, the net profit of firm 1 is:

$$\pi_1 = p_x \cdot x - c_x(x) - t \cdot x, \quad (14.4.36)$$

which leads to the first order condition:

$$p_x = c'_x(x) + t = c'_x(x) + e'(x^*), \quad (14.4.37)$$

which is the same as the one for social optimality. That is, when firm 1 faces the wrong pricing of its action, a tax $t = e'(x^*)$ should be imposed for each unit of firm 1's production. This will lead to a socially optimal outcome that is less than that of competitive equilibrium outcome. Such correction taxes are called Pigovian taxes.

The problem with this solution is that it requires that the taxing authority knows the externality cost $e(x)$. But, how does the authority know the externality and how do they estimate the value of externality in real world? If the authority knows this information, it would work such as imposing a Pigovian tax on gasoline since emission of automobile is relatively easier to determine. But, in most cases, it does not work well. Therefore, it is only applicable to scenarios where $e(x)$ is relatively easier to identify. For example, the emissions of a car can be easily quantified, so that the emission tax per gallon of gasoline can be easily determined.

In addition, as pointed out by Ng (2004), if $e(x)$ is an assessment function of environmental disruption, this often involves many people (even globally) and the future, thus it is difficult to estimate. But if $e(x)$ is a cost function on abatement spending, it is often easier to estimate. Ng argues that in the case of serious pollution (and therefore there is abatement investment), it is not necessary to estimate the damage of pollution, but to tax according to the marginal cost of abatement spending. However, when $e(x)$ is private information and is difficult to identify, it is hard for the tax collector to accurately obtain information about the cost $e(x)$, so this solution cannot be directly adopted. In order to obtain information, some effective means are necessary, but they all involve a cost. If the cost is too high, it is difficult to adopt in practice.

14.4.2 Coase's Approach

A different approach to the externality problem relies on the parties to negotiate a solution to the problem themselves.

Nobel laureate Ronald Harry Coase (1910-2013, see his biography in Section 14.6.2) raised two problems of Pigou's tax: first, government intervention interferes with economic freedom; second, taxpayers are unlikely

to know about the cost of $e(x)$ in most situations.

The greatest novelty of Nobel laureate Ronald Coase's contribution was the systematic treatment of trade in property rights. To solve the externality problem, Coase in his famous article, "The Problem of Social Cost", in 1960 emphasized that whether externality can be effectively solved depends on whether property rights are clearly defined. For this reason, Coase put forward clear definition of property rights and methods of voluntary exchange and negotiation. The so-called **Coase Theorem** asserts that as long as property rights are clearly defined, the outcome of negotiations between the two parties will result in the efficient level of production in the presence of production externality.

The term "Coase Theorem" originated with George Stigler, who explained Coase's ideas in his textbook, "The Theory of Price", pp. 110-114. Stigler asserted that the Coase theorem actually contains two claims in the absence of transaction costs:

Claim 1 (**Coase Efficiency Theorem**): Voluntary negotiations over externalities will lead to a Pareto-optimal outcome.

Claim 2 (**Coase Neutrality Theorem or Independence Theorem**): The level of externality is the same regardless of who the property rights are given and how they are allocated.

Stigler's Claim 2 would follow from Claim 1 if it were true that every Pareto optimal allocation has the same level of externality, regardless of the way that private goods are distributed. Thus, the so-called **Coase's Theorem** assesses that as long as property rights are clearly assigned and transition cost is zero, the two parties will negotiate in such a way that the optimal level of the externality-producing activity is implemented. As a policy implication, a government should simply rearrange property rights with appropriately designed property rights. Market then could take care of externalities without direct government intervention.

Coase illustrates his assertion through various examples of the two-person economy with externalities. The following simple examples depict Coase's core ideas and ideas.

Example 14.4.1 Two firms: One is chemical factory that discharges chemicals into a river and the other is the fisherman. Suppose that the river can produce a value of \$100,000. If the chemicals pollute the river, the fish cannot be eaten. How does one solve the externality? Coase's solution states that as long as the property rights of the river are clearly assigned, it results in efficient outcomes. That is, the government should give the ownership of the lake either to the chemical firm or to the fisherman, then it will yield an efficient output. To see this, assume that:

The cost of the filter is denoted by c_f .

Case 1: The lake is given to the factory.

- i) $c_f < \$100,000$. The fisherman is willing to buy a filter for the factory. The fisherman will pay for the filter so that the chemical cannot pollute the lake.
- ii) $c_f > \$100,000$. The chemical is discharged into the lake. The fisherman does not want to install any filter.

Case 2: The lake is given to the fisherman, and the firm's net product revenue is greater than \$100,000.

- i) $c_f < \$100,000$. The factory buys the filter so that the chemical cannot pollute the lake.
- ii) $c_f > \$100,000$. The firm pays \$100,000 to the fisherman before the chemical is discharged into the lake.

In this way, no matter who owns the property, two cases lead to the same efficient result: as long as $C_f < \$500,000$, pollution will not happen, otherwise it will happen. Only differences is the income distribution effect is different.

Like the above example, Cases's own examples supporting his claims are about negotiations between firms rather than those between individuals. Because firm are profit maximization rather than utility maximization, their economic behavior seem to be **fiduciaries** behavior. This difference is important because profit maximization has no income effect, while utility

maximization generally has income effect. So, we have to make some restricted assumptions on consumers' utility functions to make Coase's Theorem be true.

Now consider an economy with two consumers with L goods. Furthermore, consumer i has initial wealth w_i , and his utility function is given by

$$u_i(x_i^1, \dots, x_i^L, h).$$

That is, the utility of each consumer is related to the quantity of goods consumed as well as the activity h carried out by consumer 1.

Activity h is something that has no direct monetary cost for person 1. For example, h is the quantity of loud music played by person 1. In order to play it, the consumer must purchase electricity, but electricity can be captured as one of the components of x_i . From the point of view of consumer 2, h represents an external effect of consumer 1's action. In the model, we assume that

$$\frac{\partial u_2}{\partial h} \neq 0.$$

Thus the externality in this model lies in the fact that h affects consumer 2's utility, but it is not priced by the market. Let $v_i(p, w_i, h)$ be consumer i 's indirect utility function:

$$\begin{aligned} v_i(w_i, h) &= \max_{x_i} u_i(x_i, h) \\ \text{s.t.} \quad & \mathbf{p}x_i \leq w_i. \end{aligned}$$

To rule out income effect resulted from the assignment of property rights, we assume that utility functions are quasi-linear with respect to some numeraire commodity. Thus, the consumer's indirect utility function takes the form:

$$v_i(w_i, h) = \phi_i(h) + w_i.$$

We further assume that utility is strictly concave in h : $\phi_i''(h) < 0$. Again, the competitive equilibrium outcome in general is not Pareto optimal. In order to maximize utility, consumer 1 chooses h in order to maximize v_1 so that the interior solution satisfies $\phi_1'(h^*) = 0$. Even though consumer 2's

utility depends on h , it cannot affect the choice of h .

On the other hand, the socially optimal level of h will maximize the sum of the consumers' utilities:

$$\max_h \phi_1(h) + \phi_2(h).$$

The first-order condition for an interior maximum is:

$$\phi'_1(h^{**}) + \phi'_2(h^{**}) = 0,$$

where h^{**} is the Pareto optimal amount of h . Thus, the social optimum is where the sum of the marginal benefit of the two consumers equals zero. In the case where the externality is bad for consumer 2 (loud music), the level of $h^* > h^{**}$. That is, too much h is produced. In the case where the externality is good for consumer 2, too little will be provided, $h^* < h^{**}$.

Now we show that, as long as property rights are clearly determined, the two parties will negotiate in such a way that the optimal level of the externality-producing activity is implemented. We first consider the case where consumer 2 has the right to prohibit consumer 1 from undertaking activity h . But, this right is contractible. Consumer 2 can sell consumer 1 the right to undertake h_2 units of activity h in exchange for some transfer, T_2 . The two consumers will bargain both over the size of the transfer T_2 and over the number of units of the externality-producing good, h_2 .

In order to determine the outcome of the bargaining, we first specify the bargaining mechanism as follows:

1. Consumer 2 offers consumer 1 a take-it-or-leave-it contract specifying a payment T_2 and an activity level h_2 .
2. If consumer 1 accepts the offer, that outcome is implemented.
If consumer 1 does not accept the offer, consumer 1 cannot produce any of the externality-producing good, i.e., $h_2 = 0$.

To analyze this, begin by considering which offers, (h_2, T_2) , will be accepted by consumer 1. Since in the absence of agreement, consumer 1 must have $h_2 = 0$ by noting that the rights are given to consumer 2, consumer 1 will accept (h_2, T_2) if and only if it satisfies the participation constraint, that

is, consumer 1 accepts the offer if and only if:

$$\phi_1(h_2) - T_2 \geq \phi_1(0).$$

Given this constraint on the set of acceptable offers, consumer 2 will choose (h_2, T_2) in order to solve the following problem:

$$\begin{aligned} \max_{h_2, T_2} & \phi_2(h_2) + T_2 \\ \text{s.t.} & \phi_1(h_2) - T_2 \geq \phi_1(0). \end{aligned}$$

Since consumer 2 prefers higher T_2 , the constraint is binding at the optimum. Thus the problem becomes:

$$\max_{h_2} \phi_1(h_2) + \phi_2(h_2) - \phi_1(0).$$

The first-order condition for this problem is given by:

$$\phi_1'(h_2) + \phi_2'(h_2) = 0.$$

This is the same condition that results in the socially optimal level of h_2 . Thus consumer 2 chooses $h_2 = h^*$, and, using the constraint, $T_2 = \phi_1(h^*) - \phi_1(0)$. And, the offer (h_2, T_2) is accepted by consumer 1 and the bargaining process implements the social optimum.

Now we consider the case where consumer 1 has the right to produce as much of the externality as she wants. We maintain the same bargaining mechanism. Consumer 2 can make consumer 1 a take-it-or-leave-it offer (h_1, T_1) , where the subscript indicates that consumer 1 has the property right in this situation. However, now, in the event that consumer 1 rejects the offer, she can choose to produce as much of the externality as she wants, which means that she will choose to produce h^* . Thus the only change between this situation and the first case is what happens in the event that no agreement is reached. In this case, consumer 2's problem is:

$$\begin{aligned} \max_{h_1, T_1} & \phi_2(h_1) - T_1 \\ \text{s.t.} & \phi_1(h_1) + T_1 \geq \phi_1(h^*). \end{aligned}$$

Again, we know that the constraint is binding, and so consumer 2 chooses h_1 and T_1 in order to maximize

$$\max \phi_1(h_1) + \phi_2(h_1) - \phi_1(h^*),$$

which is also maximized at $h_1 = h^{**}$, since the first-order condition is the same. The only difference is in the transfer. Here $T_1 = \phi_1(h^*) - \phi_1(h^{**})$.

While the outcomes of both property-rights implement h^{**} , they have different distributional consequences. The transfer payment is positive in the case where consumer 2 has the property rights, while it is negative when consumer 1 has the property rights. The reason for this is that consumer 2 is in a better bargaining position when the non-bargaining outcome is that consumer 1 is forced to produce 0 unit of the externality-producing good.

However, note that in the quasi-linear framework, redistribution of the numeraire commodity has no effect on social welfare. The fact that regardless of how the property rights are assigned, that bargaining leads to Pareto optimal allocations is an example of the Coase Theorem: If trade of the externality can occur, then bargaining will lead to an efficient outcome no matter how property rights are assigned (as long as they are clearly assigned). Note that well-defined, enforceable property rights are essential for bargaining to work. If there is a dispute over who has the right to pollute (or not pollute), then bargaining may not lead to efficiency. An additional requirement for efficiency is that the bargaining process itself is costless. Note that the government does not need to know about individual consumers here - it only needs to define property rights. However, it is critical that they are clearly defined. Thus the Coase Theorem provides an argument in favor of having clear laws and well-developed courts.

However, we know that quasi-linear function is a very restrictive assumption, which means there is no income effect. If Coase's theorem is only valid for quasilinear utility function, then it has great limitations on the conclusion that consumption externality is valid. So, a natural question is, does Coase Theorem hold for other types of utility functions? Hurwicz's conclusions are surprising and regrettable. Hurwicz (*Japan and the World E-*

conomy, 7, 1995, pp. 49-74) argued that, even when the transaction cost is zero, the absence of income effects in the demand for the good with externality is not only sufficient (which is well known) but also necessary for Coase Neutrality Theorem to be true, i.e., when the transaction cost is negligible, the level of pollution will be independent of the assignments of property rights if and only if preferences of the consumers are quasi-linear with respect to the private good, leading to absence of income effects in the demand for the good with externality.

Unfortunately, as shown by Chipman and Tian (2012), the proof of Hurwicz's claim on the necessity of parallel preferences for "Coase's conjecture" is not valid. To see this, consider the following class of utility functions $U_i(x_i, h)$ that have the functional form:

$$U_i(x_i, h) = x_i e^{-h} + \phi_i(h), \quad i = 1, 2 \quad (14.4.38)$$

where

$$\phi_i(h) = \int e^{-h} b_i(h) dh. \quad (14.4.39)$$

$U_i(x_i, h)$ is clearly not quasi-linear in x_i . It is further assumed that for all $h \in (0, \eta]$, $b_1(h) > \xi$, $b_2(h) < 0$, $b'_i(h) < 0$ ($i = 1, 2$), $b_1(0) + b_2(0) \geq \xi$, and $b_1(\eta) + b_2(\eta) \leq \xi$.

We then have

$$\begin{aligned} \partial U_i / \partial x_i &= e^{-h} > 0, \quad i = 1, 2, \\ \partial U_1 / \partial h &= -x_1 e^{-h} + b_1(h) e^{-h} > e^{-h} [\xi - x_1] \geq 0, \\ \partial U_2 / \partial h &= -x_2 e^{-h} + b_2(h) e^{-h} < 0 \end{aligned}$$

for $(x_i, h) \in (0, \xi) \times (0, \eta)$, $i = 1, 2$. Thus, by the mutual tangency equality condition for Pareto efficiency, we have

$$0 = \frac{\partial U_1}{\partial h} \Big/ \frac{\partial U_1}{\partial x_1} + \frac{\partial U_2}{\partial h} \Big/ \frac{\partial U_2}{\partial x_2} = -x_1 - x_2 + b_1(h) + b_2(h) = b_1(h) + b_2(h) - \xi, \quad (14.4.40)$$

which is independent of x_i . Hence, if (x_1, x_2, h) is Pareto optimal, so is (x'_1, x'_2, h) provided $x_1 + x_2 = x'_1 + x'_2 = \xi$. Also, note that $b'_i(h) < 0$

($i = 1, 2$), $b_1(0) + b_2(0) \geq \xi$, and $b_1(\eta) + b_2(\eta) \leq \xi$. Then $b_1(h) + b_2(h)$ is strictly monotone and thus there is a unique $h \in [0, \eta]$, satisfying (14.4.40). Thus, the contract curve is horizontal even though individuals' preferences need not be parallel.

Example 14.4.2 Suppose that $b_1(h) = (1+h)^\alpha \eta^\eta + \xi$ with $\alpha < 0$, and $b_2(h) = -h^\eta$. Then, for all $h \in (0, \eta]$, $b_1(h) > \xi$, $b_2(h) < 0$, $b'_i(h) < 0$ ($i = 1, 2$), $b_1(0) + b_2(0) > \xi$, and $b_1(\eta) + b_2(\eta) < \xi$. Thus, $\phi_i(h) = \int e^{-h} b_i(h) dh$ is concave, and $U_i(x_i, h) = x_i e^{-h} + \int e^{-h} b_i(h) dh$ is quasi-concave, $\partial U_i / \partial x_i > 0$ and $\partial U_1 / \partial h > 0$, and $\partial U_2 / \partial h < 0$ for $(x_i, h) \in (0, \xi) \times (0, \eta)$, $i = 1, 2$, but it is not quasi-linear in x_i .

Chipman and Tian (2012) then investigate the necessity for the “Coase conjecture” that the level of pollution is independent of the assignments of property rights. This reduces to developing the necessary and sufficient conditions that guarantee that the contract curve is horizontal so that the set of Pareto optima for the utility functions is h -constant. This in turn reduces to finding the class of utility functions such that the mutual tangency (first-order) condition does not contain x_i and consequently it is a function, denoted by $g(h)$, of h only:

$$\frac{\partial U_1}{\partial h} \bigg/ \frac{\partial U_1}{\partial x_1} + \frac{\partial U_2}{\partial h} \bigg/ \frac{\partial U_2}{\partial x_2} = g(h) = 0. \quad (14.4.41)$$

Let $F_i(x_i, h) = \frac{\partial U_i}{\partial h} / \frac{\partial U_i}{\partial x_i}$ ($i = 1, 2$), which can be generally expressed as

$$F_i(x_i, h) = x_i \psi_i(h) + f_i(x_i, h) + b_i(h),$$

where $f_i(x_i, h)$ are nonseparable and nonlinear in x_i . $\psi_i(h)$, $b_i(h)$, and $f_i(x_i, h)$ will be further specified below.

Let $F(x, h) = F_1(x, h) + F_2(\xi - x, h)$. Then the mutual tangency equality condition can be rewritten as

$$F(x, h) = 0. \quad (14.4.42)$$

Thus, the contract curve, i.e., the locus of Pareto-optimal allocations, can be expressed by a function $h = f(x)$ that is implicitly defined by (14.4.42).

Then, the Coase Neutrality Theorem, which is characterized by the condition that the set of Pareto optimal allocations (the contract curve) in the (x, h) space for $x_i > 0$ is a horizontal line $h = \text{constant}$, implies that

$$h = f(x) = \bar{h}$$

with \bar{h} constant, and thus we have

$$\frac{dh}{dx} = -\frac{F_x}{F_h} = 0$$

for all $x \in [0, \xi]$ and $F_h \neq 0$, which means that the function $F(x, h)$ is independent of x . Then, for all $x \in [0, \xi]$,

$$F(x, h) = x\psi_1(h) + (\xi - x)\psi_2(h) + f_1(x, h) + f_2(\xi - x, h) + b_1(h) + b_2(h) \equiv g(h). \quad (14.4.43)$$

Since the utility functions U_1 and U_2 are functionally independent, and x disappears in (14.4.43), we must have $\psi_1(h) = \psi_2(h) \equiv \psi(h)$ and $f_1(x, h) = -f_2(\xi - x, h) = 0$ for all $x \in [0, \xi]$. Therefore,

$$F(x, h) = \xi\psi(h) + b_1(h) + b_2(h) \equiv g(h), \quad (14.4.44)$$

and

$$\frac{\partial U_i}{\partial h} \bigg/ \frac{\partial U_i}{\partial x_i} = F_i(x_i, h) = x_i\psi(h) + b_i(h), \quad (14.4.45)$$

which is a first-order linear partial differential equation. It can be verified that the principal integral $U_i(x_i, h)$ of (14.4.45) is given by

$$U_i(x_i, h) = x_i e^{\int \psi(h) dh} + \phi_i(h), \quad i = 1, 2 \quad (14.4.46)$$

with

$$\phi_i(h) = \int e^{\int \psi(h) dh} b_i(h) dh. \quad (14.4.47)$$

The general solution of (14.4.45) is then given by $\bar{U}_i(x, y) = \psi(U_i)$, where ψ is an arbitrary function. Since a monotonic transformation preserves orderings of preferences, we can regard the principal solution $U_i(x_i, h)$ as a general functional form of utility functions that is fully characterized

by (14.4.45).

Note that (14.4.46) is a general utility function that contains quasi-linear utility in x_i and the utility function given in (14.4.38) as special cases. Indeed, it reduces to quasi-linear utility function when $\psi(h) \equiv 0$ and to the utility function given by (14.4.38) when $\psi(h) = -1$.

To make the the mutual tangency (first-order) condition (14.4.41) be also sufficient for the contract curve to be horizontal in a pollution economy, we assume that for all $h \in (0, \eta]$, $x_1\psi(h) + b_1(h) > 0$, $x_2\psi(h) + b_2(h) < 0$, $\psi'(h) \leq 0$, $b'_i(h) < 0$ ($i = 1, 2$), $\xi\psi(0) + b_1(0) + b_2(0) \geq 0$, and $\xi\psi(\eta) + b_1(\eta) + b_2(\eta) \leq 0$.

We then have for $(x_i, h) \in (0, \xi) \times (0, \eta)$, $i = 1, 2$,

$$\begin{aligned}\partial U_i / \partial x_i &= e^{\int \psi(h)} > 0, \quad i = 1, 2, \\ \partial U_1 / \partial h &= e^{\int \psi(h)} [x_1\psi(h) + b_1(h)] > 0, \\ \partial U_2 / \partial h &= e^{\int \psi(h)} [x_2\psi(h) + b_2(h)] < 0,\end{aligned}$$

and thus

$$\begin{aligned}0 &= \frac{\partial U_1}{\partial h} \bigg/ \frac{\partial U_1}{\partial x_1} + \frac{\partial U_2}{\partial h} \bigg/ \frac{\partial U_2}{\partial x_2} \\ &= (x_1 + x_2)\psi(h) + b_1(h) + b_2(h) \\ &= \xi\psi(h) + b_1(h) + b_2(h),\end{aligned}\tag{14.4.48}$$

which does not contain x_i . Hence, if (x_1, x_2, h) is Pareto optimal, so is (x'_1, x'_2, h) provided $x_1 + x_2 = x'_1 + x'_2 = \xi$. Also, note that $\psi'(h) \leq 0$, $b'_i(h) < 0$ ($i = 1, 2$), $\xi\psi(0) + b_1(0) + b_2(0) \geq 0$, and $\xi\psi(\eta) + b_1(\eta) + b_2(\eta) \leq 0$. Then $\xi\psi(h) + b_1(h) + b_2(h)$ is strictly monotone and thus there is a unique $h \in [0, \eta]$ that satisfies (14.4.48). Thus, the contract curve is horizontal even though individuals' preferences need not be parallel.

The formal statement of Coase Neutrality Theorem obtained by Chipman and Tian (2012) thus can be set forth as follows:

Proposition 14.4.1 (Coase Neutrality Theorem) *In a pollution economy considered in the chapter, suppose that the transaction cost equals zero, and that the utility functions $U_i(x_i, h)$ are differentiable and such that $\partial U_i / \partial x_i > 0$, and $\partial U_1 / \partial h > 0$ but $\partial U_2 / \partial h < 0$ for $(x_i, h) \in (0, \xi) \times (0, \eta)$, $i = 1, 2$. Then,*

the level of pollution is independent of the assignments of property rights if and only if the utility functions $U_i(x, y)$, up to a monotonic transformation, have a functional form given by

$$U_i(x_i, h) = x_i e^{\int \psi(h)} + \int e^{\int \psi(h) dh} b_i(h) dh, \quad (14.4.49)$$

where h and b_i are arbitrary functions such that the $U_i(x_i, h)$ are differentiable, $\partial U_i / \partial x_i > 0$, and $\partial U_1 / \partial h > 0$ but $\partial U_2 / \partial h < 0$ for $(x_i, h) \in (0, \xi) \times (0, \eta)$, $i = 1, 2$.

Although the above Coase neutrality theorem includes quasilinear utility function as a special case, due to the special form of such function, Coase Neutrality Theorem has no generality, which is not true for general utility function. Therefore, Hurwicz's insight on the limitations of Coase's theorem is still valid. Coase's theorem is more applicable to production externalities rather than consumption externalities.

It is important to understand the limitations of the Coase Theorem. One might think that with clear property rights, free exchange and voluntary cooperation, the market can operate efficiently without considering the preconditions of Coase's Theorem, especially the two basic prerequisites: (1) zero transaction cost; (2) no income effect. In practice, costs of negotiation and organization, in general, are not negligible, and the income effect may not be zero. Thus, a privatization is optimal only in case of zero transaction cost, no income effect, and perfectly competitive economic environments. But in practice, these conditions are often not satisfied. For example, the privatization of state-owned enterprises tends to be costly, and there is a lot of debate about how to privatize and who should get a share or benefit. If there is no corresponding system as the basis and the transaction cost is too large, then radical privatization may not be desirable.

Does the clearly defined private property rights necessarily lead to the optimal allocation of resources, and is it impossible for other ownership? Tian (2000, 2001) showed that private ownership (resp. state ownership and collective ownership) may be (resp. relatively or constrainedly) efficient, depending on the development level of underlying institutional environment. If an institutional environment is much underdeveloped, state

and collective ownership could be sub-optimal (relatively more efficient) compared to private ownership. Only when the market system is mature, and has a good governance, the private property right system can be efficient, or the private property rights system is (globally) optimal. So, all three property rights systems may be (sub-)optimal, depending on the regularity of institutional environments. Therefore, instead of rapid privatization of state-owned enterprises, it is better to continuously improve the institutional environment and allow private enterprises to flourish. China's economic reform and opening-up over the past 40 years has fully demonstrated this point.

The problem of the Coase Efficiency Theorem is more serious. First, as Arrow (1979, p. 24) pointed out, the basic postulate underlying Coase's theory appears to be that the process of negotiation over property rights can be modelled as a cooperative game, and this requires the assumption that each player know the preferences or production functions of each of the other individuals. When information is not complete or asymmetric, in general, it results in Pareto inefficient outcomes. For instance, when there is one polluter and there are many pollutees, a "free-rider" problem arises and there is an incentive for pollutees to misrepresent their preferences. Whether the polluter is liable or not, the pollutees may be expected to overstate the amount they require to compensate for the externality. Thus, we may need to design an incentive compatible mechanism to solve the free-rider problem.

Secondly, even if the information is complete, there are several circumstances that have led a number of economists to question the conclusion in the Coase Efficiency Theorem:

- (1) The economic core may be empty, and hence no Pareto optimum exists. An example of this for a three-agent economy was presented by Aivazian and Callen (1981).
- (2) There may be a fundamental non-convexity that prevents a Pareto optimum from being supported by a competitive equilibrium. Starrett (1972) showed that externalities are characterized by "fundamental non-convexities" that may

preclude existence of competitive equilibrium.

- (3) When an agent possesses the right to pollute, there is a built-in incentive for extortion. As Andel (1966) pointed out, anyone with the right to pollute has an incentive to extract payment from potential pollutees, e.g., threat to blow a bugle in the middle of the night.
- (4) Arrow (1979) argued that Coase's theorem relies on a bargaining process and finally forms a cooperative game, which depends on the assumption of complete information. Obviously in practice the information is not complete and may lead to free-rider problems.

Thus, the hypothesis that negotiations over externalities will mimic trades in a competitive equilibrium is, as Coase himself has conceded, not one that can be logically derived from his assumptions, but must be regarded as an empirical conjecture that may or may not be confirmed by the data. A lot of theoretical work therefore still remains in order to provide Coasian economics with the rigorous underpinning.

14.4.3 Missing Market

We can regard externality as a lack of a market for an "externality". For the above example in Pigovian taxes, a missing market is a market for pollution. Adding a market for firm 2 to express its demand for pollution - or for a reduction of pollution - will provide a mechanism for efficient allocations. By adding this market, firm 1 can decide how much pollution it wants to sell, and firm 2 can decide how much pollution it wants to buy.

Let r be the price of abatement of pollution.

x_1 = the units of pollution that firm 1 wants to produce;

x_2 = the units of pollution that firm 2 wants to get compensation.

Normalize the output of firm 1 to x_1 .

The profit maximization problems become:

$$\begin{aligned}\pi_1 &= p_x x_1 - r x_1 - c_1(x_1), \\ \pi_2 &= p_y y + r x_2 - e_2(x_2) - c_y(y).\end{aligned}$$

The first order conditions are:

$$\begin{aligned}p_x - r &= c'_1(x_1) \quad \text{for Firm 1,} \\ p_y &= c'_y(y) \quad \text{for Firm 2,} \\ r &= e'(x_2) \quad \text{for Firm 2.}\end{aligned}$$

At the market equilibrium, $x_1^* = x_2^* = x^*$, we have

$$p_x = c'_1(x^*) + e'(x^*) \tag{14.4.50}$$

which results in a socially optimal outcome.

In this model, we assume that the price of abatement of pollution is taken as given to all firms. When the number of firms is small, this assumption is not necessarily true, and thus may still result in inefficiency of pollution level. Then, in the real world, the method of auctioning pollution rights is used to greatly increase competition.

The basic idea of this model has been used to deal with a large number of externalities in practice, establishing a warrants market for the consumption and production of many externality-producing goods. In addition to establishing a market for pollution rights transactions, it is also widely applied in establishing licenses trading markets for many externality-producing goods such as radio frequency spectrum. As an application, we will discuss this issue by considering emissions trading in the end of the chapter.

14.4.4 The Compensation Mechanism

The Pigovian taxes were not adequate in general to solve externalities due to incomplete information: the tax authority cannot know the cost imposed by the externality. How can one solve this incomplete information prob-

lem?

Varian (1994) proposed an incentive mechanism which encourages the firms to correctly reveal the costs they impose on the others. Here, we discuss this mechanism. In brief, a mechanism consists of a message space and an outcome function (rules of game). We will introduce in detail the mechanism design theory in Part VI. Varian's incentive mechanism allows firms to form a Pareto efficient tax rate through the game. The regulatory department does not know the individual's information, so it is necessary to induce individuals' information about their economic characteristics to implement the efficient tax rates t_1 and t_2 through an incentive compatible mechanism.

Varian's mechanism is designed in a way that firms propose a tax rate for each other. If the tax rates set by the two parties are different, they then will be punished. The mechanism proposed by Varian is divided into two stages. In the first stage, firms independently propose tax rates for each other, which captures the idea of competitive markets, and no firm can control the tax rate imposed on itself. If one can determine its tax rate, then rent-seeking appears. In the second stage, the mechanism designer distributes interests according to the information of both parties. Finally, the individuals make the decision of production and output according to the rules determined by the mechanism, and the equilibrium outcome is Pareto efficient.

Strategy Space (Message Space): $M = M_1 \times M_2$ with $M_i = \{(t_i, x_i)\}$, $i = 1, 2$, where t_1 is interpreted as a Pigovian tax proposed by firm 1 and x_1 is the proposed level of output by firm 1, and t_2 is interpreted as a Pigovian tax proposed by firm 2 and y_2 is the proposed level of output by firm 2.

The mechanism has two stages:

Stage 1 (Announcement stage): Firms 1 and 2 name Pigovian tax rates each other, t_i , $i = 1, 2$, which may or may not be the efficient level of such a tax rate.

Stage 2 (Choice stage): If firm 1 produces x units of pollution, firm 1 must pay t_2x to firm 2. Thus, each firm takes the tax rate as given. Firm 2 receives t_1x units as compensation. Each firm pays a penalty, $(t_1 - t_2)^2$, if they announce different tax rates.

Thus, the payoffs of two firms are:

$$\begin{aligned}\pi_1^* &= \max_x p_x x - c_x(x) - t_2 x - (t_1 - t_2)^2, \\ \pi_2^* &= \max_y p_y y - c_y(y) + t_1 x - e(x) - (t_1 - t_2)^2.\end{aligned}$$

Since this is a two-stage game, we may use the subgame perfect equilibrium, i.e., an equilibrium in which each firm takes into account the repercussions of its first-stage choice on the outcomes in the second stage. As usual, we solve this game by looking at stage 2 first.

At stage 2, firm 1 will choose $x(t_2)$ to satisfy the first order condition:

$$p_x - c'_x(x) - t_2 = 0. \quad (14.4.51)$$

Note that, by the convexity of c_x , i.e., $c''_x(x) > 0$, we have

$$x'(t_2) = -\frac{1}{c''_x(x)} < 0. \quad (14.4.52)$$

Firm 2 will choose y to satisfy $p_y = c'_y(y)$.

Stage 1: Each firm will choose the tax rates t_1 and t_2 to maximize their payoffs.

For Firm 1,

$$\max_{t_1} p_x x - c_x(x) - t_2 x(t_2) - (t_1 - t_2)^2, \quad (14.4.53)$$

which leads to the following first order condition:

$$2(t_1 - t_2) = 0,$$

and thus

$$t_1^* = t_2. \quad (14.4.54)$$

For Firm 2,

$$\max_{t_2} p_y y - c_y(y) + t_1 x(t_2) - e(x(t_2)) - (t_1 - t_2)^2 \quad (14.4.55)$$

so that the first order condition is

$$t_1 x'(t_2) - e'(x(t_2))x'(t_2) + 2(t_1 - t_2) = 0,$$

and then we have

$$[t_1 - e'(x(t_2))]x'(t_2) + 2(t_1 - t_2) = 0. \quad (14.4.56)$$

By (14.4.52), (14.4.54) and (14.4.56), we get

$$t^* = e'(x(t^*)) \text{ with } t^* = t_1^* = t_2^*. \quad (14.4.57)$$

Substituting the equilibrium tax rate, $t^* = e'(x(t^*))$, into (14.4.51), we obtain

$$p_x = c'_x(x^*) + e'(x^*), \quad (14.4.58)$$

which is the condition for social efficiency of production.

Remark 14.4.1 This mechanism works by setting opposing incentives for two agents. Firm 1 always has an incentive to match the announcement of firm 2. But consider firm 2's incentive. If firm 2 thinks that firm 1 will propose a large compensation rate t_1 for him, he wants firm 1 to be taxed as little as possible so that firm 1 will produce as much as possible. On the other hand, if firm 2 thinks firm 1 will propose a small t_1 , it wants firm 1 to be taxed as much as possible. Thus, the only point where firm 2 is indifferent about the level of production of firm 1 is where firm 2 is exactly compensated for the cost of the externality.

In general, individuals' personal goals are different from a social goal, which are in general incentive incompatible. However, we may be able to construct an appropriate incentive-compatible mechanism so that individuals' personal interests-maximizing goals are consistent with a social goal such as efficient allocations. Tian (2003) also gave the solution to the consumption externalities by giving the incentive mechanism that results in Pareto efficient allocations. Tian (2004) studied the informational efficiency problem of the mechanisms that results in Pareto efficient allocations for consumption externalities.

14.5 Emissions Trading and Efficient Allocation of Pollution Rights

This section deals with emissions trading. **Emissions trading** is also called **cap and trade (CAT)** that is a market-based approach to controlling pollution by providing economic incentives for achieving reductions in the emissions of pollutants. In contrast to command-and-control environmental regulations such as **best available technology (BAT)** standards and government subsidies, emissions trading programs are a type of flexible environmental regulation that allows organizations to decide how best to meet policy targets.

We will discuss the governance of pollution and focuses on how to achieve efficient allocation of pollution rights through markets, so that pollution can be efficiently controlled. In the 1990s, some European and American countries established emission permit markets for pollutants and gained somewhat successful experience. Their application has also been gradually extended to other countries in the world. This section focuses on the intrinsic mechanism, efficiency, and possible limitations of the emissions trading market. The discussion in this section refers to the analysis of emissions trading markets by Leach (2004) and Newell and Stavins (2003).

An important issue in the way of pollution charges and emission cap is information. Since the government does not have the information on technology of firms, discharge of pollutants usually do not reach the level of Pigovian tax, nor can it achieve the most efficient level of pollution abatement. In addition, in the environmental pollution, the negotiations between firms and residents are faced with excessive transaction costs, such as free-rider, and thus cannot achieve efficient pollution control. Cap-and-trade can usually reduce information requirements and transaction costs through market mechanism.

Here is a simple example to discuss the efficiency of the emission rights market.

Consider an economy with two firms. There is no externality between two firms and they may discharge pollutants. Without control, firm i will produce pollution of \bar{e} . Let a_i denote the volume of emission abatement.

Assume that the cost of emission abatement is $C_i = c_i \frac{a_i^2}{2}$. Two firms have different costs of emission reduction. Suppose that $c_1 < c_2$, the total social emission regulated by the government is $2\bar{e}$, and the emission cap for every firm is $\hat{e} < \bar{e}$.

14.5.1 Emission Reduction without Trading Market

If the emission rights of two firms cannot be exchanged, then emission abatement cost for firm i is $C_i = c_i \frac{(\bar{e} - \hat{e})^2}{2}$, and the total cost of emission abatement is $(c_1 + c_2) \frac{(\bar{e} - \hat{e})^2}{2}$. The marginal costs of emission abatements for firm 1 and firm 2 are $c_1(\bar{e} - \hat{e})$ and $c_2(\bar{e} - \hat{e})$, respectively. Since two firms' marginal costs of emission abatement are different, the total cost of social emission abatement is not minimized. If firm 2 transfers 1 unit of emission rights to firm 1, social emission abatement cost is reduced by $(c_2 - c_1)(\bar{e} - \hat{e})$. The establishment of an emission rights market will reduce the total cost of emission abatement without affecting the total emissions.

14.5.2 Emissions Trading

Below we discuss market equilibrium under emissions trading. Assume that the emissions trading market is competitive, and the number of firm i is continuum. The total number of firms in each category is standardized to 1.

For firm i , the optimization problem is as below:

$$\min_{a_1} c_1 \frac{a_1^2}{2} + p(\bar{e} - \hat{e} - a_1) \quad (14.5.59)$$

$$\text{s.t.} \quad a_1 \leq \bar{e}, \quad (14.5.60)$$

where a_1 is the actual emission abatement by firm 1 and p is the price of emission rights. Emission abatement needed for firm 1 is $\bar{e} - \hat{e}$. If the actual emission abatement is $a_1 < \bar{e} - \hat{e}$, firm 1 needs to purchase $\bar{e} - \hat{e} - a_1$ of emission rights. If its actual emission abatement is $a_1 > \bar{e} - \hat{e}$, firm 1 can supply $-(\bar{e} - \hat{e} - a_1)$ of emission rights. Therefore, the objective function for firm 1 is to minimize the pollution cost under emission cap \hat{e} .

Condition (14.5.60) represents that the highest possible abatement of emissions by firm 1 will not exceed the pollution it produces.

So, the optimal decision of firm 1 is

$$a_1 = \begin{cases} \frac{p}{c_1}, & \text{if } p < c_1 \bar{e}, \\ \bar{e}, & \text{otherwise.} \end{cases} \quad (14.5.61)$$

The demand of emission rights for firm 1 is denoted as $d_1(p) = \bar{e} - \hat{e} - a_1$. Here we see the supply as a negative demand. By (14.5.61), we obtain

$$d_1(p) = \begin{cases} \bar{e} - \hat{e} - \frac{p}{c_1}, & \text{if } p < c_1 \bar{e}, \\ -\hat{e}, & \text{otherwise.} \end{cases} \quad (14.5.62)$$

Similarly, we can get the demand of emission rights for firm 2, $d_2(p)$.

When $d_1(p) + d_2(p) = 0$, emissions rights market reaches equilibrium and there are two cases of equilibrium.

(1) Equilibrium 1: firm 1 reserves part of emission rights. That is, $a_1 < \bar{e}$.

Market clears and satisfies

$$2(\bar{e} - \hat{e}) - \left[\frac{p}{c_1} + \frac{p}{c_2} \right] = 0,$$

the equilibrium price is

$$p = 2(\bar{e} - \hat{e}) \left[\frac{1}{c_1} + \frac{1}{c_2} \right]^{-1}, \quad (14.5.63)$$

and satisfies

$$p < c_1 \bar{e}.$$

Substituting the equation (14.5.63) into the above equation gives

$$\hat{e} > \bar{e} \frac{c_2 - c_1}{2c_2}.$$

In other words, if firm 1's cap is large enough, then the firm will retain part of the emission rights. In this equilibrium, emissions trading is

$$-d_1 = d_2 = (\bar{e} - \hat{e}) \frac{c_2 - c_1}{c_1 + c_2}. \quad (14.5.64)$$

From the formula (14.5.64), we can see that the greater the difference in the emissions technologies of the two firms, the greater the scale of the emissions trading. Through emissions trading, the social emission abatement costs can be minimized. In equilibrium, the emission abatement cost of the society is

$$C_1 + C_2 = 2(\bar{e} - \hat{e})^2 \left[\frac{1}{c_1} + \frac{1}{c_2} \right]^{-1} < (\bar{e} - \hat{e})^2 \frac{c_1 + c_2}{2}.$$

(2) Equilibrium 2: firm 1 sells all emission rights.

Market clears and satisfies

$$\bar{e} - 2\hat{e} - \frac{p}{c_2} = 0,$$

the equilibrium price is

$$p = c_2(\bar{e} - 2\hat{e}),$$

and satisfies

$$p > c_1\bar{e},$$

getting

$$\hat{e} < \bar{e} \frac{c_2 - c_1}{2c_2}.$$

The volume of emissions trading is

$$-d_1 = d_2 = \hat{e}.$$

The total social emission abatement cost is

$$\frac{c_1}{2}(\bar{e})^2 + \frac{c_2}{2}(\bar{e} - 2\hat{e})^2 < (\bar{e} - \hat{e})^2 \frac{c_1 + c_2}{2}.$$

In the emissions trading market, when firms have different technologies in emission abatement, the transaction will allow them to remain the same in the marginal cost of emission abatement, resulting in the efficient allocation of pollution rights. In addition, in the case of emission rights market, firms will also have greater incentives in the innovation of emission abatement technologies. This is because, market transactions can make the benefits of emission abatement innovations greater, rather than merely

lower their own emission abatement costs. Many studies have found that through market transactions, emission abatement costs of the society can be greatly saved. For example, the study on sulfur dioxide emissions trading among electric utilities by Carlson et al. (2000) shows that emissions trading can lower abatement cost curves for the American power industry by over 50% since 1985, as opposed to the command-and-control approach (requiring a uniform emission rate standard), and the trading can reduce annual abatement costs by 700 million ~ 800 million dollars.

However, the operation of the emission rights market will also have its costs. For example, how to allocate emission rights among firms and how to allocate for new participants. Of course, a way to deal with the distribution on emission rights is by auction. However, setting up an auction market among different pollutants is also a big challenge, because in some industries, there may be intrinsic links among different pollutants. Joskow et al. (1998) studied the transaction costs in the operation of emission rights market, and discussed the impact of auctions on the price of emission rights in the sulfur dioxide emission rights market in the context of the United States.

In addition, the allocation of emission rights may lead to rent-seeking and social risks. Different firms may have different borrowing capabilities and may suffer from efficiency losses in auctions and market trading. Besides, for many developing countries, the supervision of emission has always been a ticklish problem. If the pollution discharge cannot be efficiently monitored, then the market for emission rights will inevitably lack clear property rights. Stavins (1995) discussed the impact of transaction costs of emission rights on pollution control efficiency. In addition, Tietenberg (1995) discussed the spatial allocation of emission rights. For more information on the emissions trading market, readers can refer to Gayer and Horowitz (2005). They discussed in detail some of the important theoretical and practical issues on the emissions trading market.

14.6 Biographies

14.6.1 Arthur Pigou

Arthur Cecil Pigou (1877 - 1959) was an English economist, the “father of welfare economics”, and one of the leading representatives of the Cambridge School. Pigou was born in a military family in England and was admitted to the University of Cambridge where he first studied history. Later, he came to economics under the influence of Marshall. In 1908 Pigou became a Professor of Political Economy at the University of Cambridge in succession to Alfred Marshall, and held the post until 1943. Pigou inherited Marshall’s academic tradition and analytical framework to a large extent. In addition, he also served as a Fellow of the Royal Society, Honorary President of the International Economic Association, member of the Cunliffe Committee on the Currency and Foreign Exchange and the Royal Commission on Income Tax. He proposed the concept of “economic welfare” in his representative works *The Economics of Welfare*, *Industrial Fluctuations*, and *A Study in Public Finance*. He advocated the equalization of national income and established the cardinal utility theory. Pigou was the first to systematically study externality from the perspective of welfare economics. Based on the concept of “external economy” put forward by Marshall, he expanded the concept and content of “external diseconomy”. The study turned from the effect of external factors on the business to the impact of the business or residents on other businesses or residents.

The *The Economics of Welfare* published in 1920 was Pigou’s most famous representative work. This book systemized welfare economics and marked the establishment of Pigou’s complete theoretical system. Its interpretation of welfare economics has always been regarded as “classic”, and Pigou was therefore also known as the “father of welfare economics”. Pigou believed that the purpose of this book was to study the important factors that affect economic welfare in real life. The whole book was centered on how to increase social welfare. Pigou proposed the Pigovian Tax, which advocates subsidies for activities that have positive externalities. Pigou was known for his “Pigovian tax”.

Pigou was an important model of neoclassical thought. In fact, Keynes criticized Pigou as a representative of a full employment analysis perspective in the neoclassical school. Pigou also replied by saying that Keynes's *The General Theory of Employment, Interest, and Money* was a mixture of wrong ideas. To respond to Keynes, he tried to restore the position of neoclassical employment theory through a logically complete demonstration under the classical assumption about wages and price elasticity.

14.6.2 Ronald Coase

Ronald Harry Coase (1910 - 2013) was the originator of the new institutional economics, the founder of the property rights theory, and one of the representatives of the Chicago School of Economics. He was awarded the 1991 Nobel Memorial Prize in Economic Sciences for his discovery and analysis of the role of transaction costs and property rights in the institutional structure and operation.

Coase was born on 29 December 1910 in a small town named Willesden outside London. In his childhood, Coase had to wear leg-irons to help support his legs. Due to physical limitations, young Coase had to attend the school for physical defective. Through his own unremitting efforts, Coase successfully entered the London School of Economics and obtained a Bachelor of Commerce degree at the age of 22. After 6 years of teaching at this school, Coase received his doctoral degree from the University of London in 1951. He then came to the United States and taught at the University at Buffalo and the University of Virginia. After that, he became a professor at the University of Chicago.

Coase had written only a few papers in his life. The most famous of these were "The Nature of the Firm" published in 1937 and "The Problem of Social Cost" published in 1960. They were probably the most widely cited among all economics literature. Although he seldom used mathematics, his articles were logically clear. He introduced and adopted the concept of transaction costs and clarified property rights to study the boundaries and externalities of the firm. He introduced institution and firm into the mainstream economics that previously focused on interpreting how the market

price system works, and demonstrated the relationship of firm, property right, contract, and market, as well as the important role of these factors in economic development. His economic ideas were extremely profound, and had far-reaching influence on the development of modern economics. Many economic disciplines, such as the economics of property rights, information economics, mechanism design theory, contract theory, and transition economics, have all been influenced by Coase's ideas which promoted the development of these fields.

In his paper titled "The Nature of the Firm", he explained how the firm was formed from a distinctive perspective. This paper was later widely considered as having epoch-making significance on economics. From the perspective of "transaction costs", Coase gave his reasons for how firms emerged. Coase believed that there were costs in the market transaction behavior. These costs include bargaining, costs of contracts formation and implementation, and time costs. Coase also believed that when market transaction costs are higher than coordination costs within the firm, then firm emerges. The existence of the firm is to save market transaction costs by replacing higher-cost market transactions with lower-cost intra-firm transactions. This distinctive research perspective has still been marveled by the economics community.

Coase's research seldom involves mathematics. In the 1960 famous paper "The Problem of Social Cost", he used a written discourse to deal with the economic problem of externalities, and to demonstrate the definition of property rights and the importance of property rights arrangement in economic transactions. George Joseph Stigler (1911 - 1991), winner of the 1982 Nobel Prize in Economics, further classified Coase's theory as "under perfect competition, private and social costs will be equal", and eventually formed the "Coase Theorem". The importance of Coase Theorem lies in that apart from the price, the theorem reveals the influence of property rights arrangement and transaction costs on institutional arrangements. Coase Theorem is divided into two parts. As long as the transaction cost is zero and the property rights are clearly defined, (1) the level of the externality will be the same regardless of the assignment of property rights, known as the Coase Neutrality Theorem; (2) with voluntary exchanges

and voluntary negotiations, clearly defined property rights will lead to efficient allocation of resources. That is, with market mechanisms, through voluntary trading and negotiations, the contractual arrangements that implement the best interests of all individuals can be found. This conclusion is called the Coase Efficiency Theorem. Coase further argued that even if there are transaction costs, the parties involved in the interaction will find a less costly institutional arrangement through the contract when the property rights are clearly defined.

Coase's economic theory and his insights have spread widely in China, which has always been in the process of economic reforms, making him one of the most cited contemporary economists among Chinese economists. Coase also enjoyed a long life and died at the age of 102.

14.7 Exercises

Exercise 14.1 There is an orchard next to an apiary. The orchard produces fruit, and the apiary supplies honey. The flowers of the fruit tree provide honeybees with nectar, and the bees also promote pollen transmission. Suppose that the price of fruit is \$2 per unit, and the price of honey is \$8 per unit. Let H be the output of honey and A be the yield of fruit. The orchard's cost function is $C_A(A, H) = A^2/2 - 6H$ and the apiary's cost function is $C_H(A, H) = H^2/2 - 3A$.

1. If the orchard and the apiary make independently decisions, what is the output of fruit and honey, respectively?
2. If the orchard and the apiary merge, what is the output of fruit and honey, respectively?
3. Do the markets for fruit and honey result in Pareto efficient allocations? Why?

Exercise 14.2 Consider an economy with two goods and two consumers. The utility functions of the two consumers are

$$u_1(\mathbf{x}) = 0.5 \ln(x_1^1 + x_2^1) + 0.5 \ln x_1^2,$$

$$u_2(\mathbf{x}) = 0.5 \ln x_2^1 + 0.5 \ln x_2^2.$$

Their consumption spaces $Z_i = \mathcal{R}_+^2$, $i = 1, 2$, and the initial endowments are $\mathbf{w}_1 = (1, 2)$ and $\mathbf{w}_2 = (2, 1)$.

1. Solve for Pareto efficient allocations and competitive equilibria.
2. Is a competitive equilibrium allocation Pareto efficient? Why?

Exercise 14.3 Consider a pure exchange economy with two goods and two consumers. The first good is “music” and the second good is “bread”. Consumption space is $X_i = \mathcal{R}_+^2$, $i = 1, 2$. The aggregate initial endowment is (w_m, w_b) . The utility functions of the two consumers are

$$u_1(m_1, b_1) = m_1^{3/5} b_1^{2/5} - k_1,$$

$$u_2(m_2, b_2) = m_2^{3/5} b_2^{2/5} - k_2,$$

in which m_1 and m_2 are music consumptions, and b_1 and b_2 are bread consumptions of consumer 1 and 2, respectively, and k_1 and k_2 are constant parameters.

1. What is the set of Pareto optimal allocations?
2. Suppose that the initial endowments of consumers 1 and 2 are $\mathbf{w}_1 = (3/2, 1/2)$ and $\mathbf{w}_2 = (1/2, 3/2)$. Solve for the competitive equilibrium.
3. Verify whether the equilibrium allocation in the question 2 is Pareto optimal.
4. Now suppose that the consumers’ utility functions change to

$$\hat{u}_1(m_1, m_2, b_1) = m_1^{3/5} b_1^{2/5} - m_2,$$

$$\hat{u}_2(m_1, m_2, b_2) = m_2^{3/5} b_2^{2/5} - m_1.$$

One explanation to the above utility functions is that while one person’s consumption of music increases his own utility, it also interferes with the quiet environment of the other, thereby reducing the utility of the other.

- (a) What is the critical value \bar{w}^m of the aggregate initial endowment such that exhaustion of resources beyond the critical value will result in Pareto inefficient allocation?
- (b) What is the set of interior-point Pareto efficient allocations? (Discuss two situations: $w^m \leq \bar{w}^m$ and $w^m > \bar{w}^m$). Compare the result with that in question 1 above.
- (c) Suppose that $w_1 = (3/2, 1/2)$ and $w_2 = (1/2, 3/2)$. Solve for the competitive equilibrium. Is the competitive allocation Pareto optimal?

Exercise 14.4 Consider the pure exchange economy of two commodities and two consumers. Consumption space is $X_i = \mathcal{R}_+^2$, $i = 1, 2$, and the aggregate endowment is given by (w_x, w_y) , where x and y represent two commodities. The utility functions of the two consumers are

$$u_1(x_1, y_1) = x_1^{0.3} y_1^{0.7} - x_2,$$

$$u_2(x_2, y_2) = x_2^{0.3} y_2^{0.7} - x_1.$$

1. Solve for \bar{w}_x such that if $w_x > \bar{w}_x$, any attainable (balanced) allocation is not Pareto efficient.
2. Solve for interior-point Pareto efficient allocations.
3. Suppose that $w_1 = (2, 1)$, $w_2 = (1, 2)$, solve for competitive equilibrium. Is the competitive allocation Pareto efficient? Why?

Exercise 14.5 Consider the economy of two commodities and two consumers. Consumption space is $X_i = \mathcal{R}_+^2$, $i = 1, 2$. Commodity m represents all commodities that can be purchased with money. Commodity n characterizes all commodities that cannot be purchased with money, such as freedom, and family life. In other words, commodity m represents the composite of all goods that can be included in GDP, while commodity n can not. The utility function of consumer i is:

$$u_i(m_i, n_i, m_j) = m_i^\alpha n_i^{1-\alpha} - \beta m_j, \quad 0 < \alpha < 1, \beta > 0.$$

1. Prove that if the total endowment of m exceeds a certain amount, there must be some free disposal in m (i.e., m can not be exhausted completely) in reaching Pareto optimal allocations.
2. Solve for Pareto efficient allocations.
3. Do you think the conclusion of this question can explain the following paradox: The individual's happiness index may not increase with the country's wealth?

Exercise 14.6 Consider the economy of n consumers. Each consumer i chooses an action $h_i \in R_+$ and his utility function is $\phi_i(h_i, \sum_i h_i) + w_i$. Suppose that $\phi_i(\cdot)$ is strictly concave. w_i is his initial endowment.

1. Characterize Pareto optimal actions h_1, \dots, h_n .
2. Characterize Nash equilibrium on actions.
3. Compare the Pareto efficient outcome with the Nash equilibrium outcome. What kind of tax rate can lead to Pareto optimal outcomes?

Exercise 14.7 There is a common grazing land in a mountain village where the villager can herd sheep. The cost of raising each sheep is 4. The total revenue of raising sheep on this grazing land is $f(x) = 20x - 3x^2/2$, where x denotes the number of sheep.

1. Prove that free grazing does not maximize the total welfare of the grazing land.
2. The government now decides that a license is necessary for raising sheeps. How should the government decide on the price of the license so as to maximize its revenue?

Exercise 14.8 Two persons have to decide separately how fast they should drive the car. The individual i chooses driving speed x_i to obtain the utility of $u_i(x_i)$ and $u'_i(x_i) > 0$. However, the faster the car, the more likely there will be a car accident. Let $P(x_1, x_2)$ be the probability of an accident and it is an increasing function of x_1 and x_2 . Let $c_i > 0$ be the cost to individual i in the event of the accident. Each person's utility is linear with regard to currency.

1. Prove that the individual's choice of driving speed is faster than the requirement for social welfare maximization.
2. If the penalty for the individual i was t_i in the event of accident, solve for the t_i that can internalize the externality.
3. Now suppose that the utility of individual i changes to 0 in the event of an accident. Find the penalty for the internalization of the externality.

Exercise 14.9 A manufacturer's cost function $c(q, h)$ is differentiable and strictly convex, $q \geq 0$ is its output level, and h is the negative externality level of production. This externality affects the consumer, whose indirect utility function is $\phi(h) + w$, where $\phi(h)$ is differentiable and satisfies $\phi'(h) < 0$.

1. Derive the first-order conditions that the manufacturer chooses q and h .
2. Derive the Pareto-optimal first-order conditions of q and h .
3. Suppose that the government imposes a tax on the producer's output. Prove that it cannot achieve Pareto optimality.
4. Suppose that the government directly levies taxes on the externality and prove that this approach can achieve Pareto optimality.
5. Suppose that $h = \gamma q$ is constant for $\gamma > 0$. Prove that the taxation on production can achieve Pareto optimality.

Exercise 14.10 (Tragedy of the Commons) Fishermen can fish freely in the lake. The cost of a fishing boat is $c > 0$. When there are b fishing boats in the lake, a total of $f(b)$ fish are captured. The fishing amount for each fishing boat is $f(b)/b$. For $b \geq 0$, we have $f'(b) > 0$ and $f''(b) < 0$. The price of fish is $p > 0$ per unit.

1. Solve for the equilibrium quantity of boats.
2. Solve for the Pareto optimal quantity of fishing boats and prove that it is less than the equilibrium quantity.

3. What kind of fishing tax should be imposed on fishing boats to achieve the Pareto optimal quantity?
4. Suppose that the lake belongs to someone. How does he choose the number of fishing boats?

Exercise 14.11 Consider an economy with two consumers, A and B , and two commodities, 1 and 2. y_i^j represents individual i 's consumption of the commodity j , and I_i represents the individual's income level. The prices of the two commodities are p^1 and p^2 , respectively. The utility function for consumer A is $u_A(y_A^1, y_A^2, y_B^2)$, and the utility function for consumer B is $u_B(y_B^1, y_B^2)$, which means the consumption of good 2 by consumer B has an externality to consumer A .

1. Write down the utility maximization problems of consumers and get the conditions that should be satisfied under an equilibrium allocation.
2. If $\partial u_A / \partial y_B < 0$, is the equilibrium allocation Pareto efficient?
3. Now suppose that specific duty, denoted t_B^2 , is levied on the consumption of commodity 2 by consumer B , what will happen to the utility maximization problem of consumer B ?
4. Can the method of taxing restore the Pareto efficient allocation? Why?

Exercise 14.12 A local government proposes to implement a sewage tax system with a minimum discharge standard. Each firm is allowed to emit a certain amount \bar{h} of pollutants without being taxed, while those beyond \bar{h} will be taxed.

1. Write down the objective function of the firm.
2. Explain why the system is not efficient in general and cannot encourage minimal-cost abatements, and under what circumstances the system is efficient.

Exercise 14.13 Consider the constant marginal abatement cost function of two firms:

$$\begin{aligned} -C'_1(h_1) &= a, \quad h_1 < \hat{h}_1; \\ -C'_2(h_2) &= b, \quad h_2 < \hat{h}_2. \end{aligned}$$

1. When the damage function of pollution is convex and linear, find socially optimal emissions of the two firms.
2. Under these circumstances, is it possible to use economic incentive policy tools to achieve the socially optimal allocation?

Exercise 14.14 (Macho-Stadler and Pérez-Castrillo, 2006) Under the linear sewage tax rate and incomplete supervision, a firm decides to report its emissions of z while its actual emissions are h . The linear tax rate is τ , and the firm pays a sewage fee of $z\tau$. The firm's return is a function of the actual amount of pollutants discharged, which is denoted by $g(h)$. When firms are not subject to any regulation, their emissions are \bar{h} , and $g'(\bar{h}) = 0$; when $h \in [0, \bar{h})$, $g(h)$ is an increasing and concave function: $g'(h) > 0$, $g''(h) < 0$. The probability that a firm's real emissions are detected by environmental protection agencies is $\rho \in [0, 1]$. A penalty to a firm that is found to make a false report is $\theta(h - z)$, and the penalty is a monotonically increasing and convex function of the difference between actual emissions and reported emissions: for $x > 0$, $\theta'(x) > 0$ and $\theta''(x) > 0$. Since the penalty of unit false report should be higher than the tax rate, we assume $\theta'(0) > \tau$.

1. How does the firm's optimal reported emissions and actual emissions change with the regulatory intensity ρ ?
2. Does the increase of regulatory intensity for the firm that reports greater than zero emissions reduce its actual emissions?

Exercise 14.15 There is a chemical plant in the upper reaches of a river, and its production will cause pollution to two downstream fishermen. The chemical plant can spend \$5,000 to purchase equipment to avoid pollution. The pollution will result in losses of \$2,500 and \$4,000 for the two fisher-

men, respectively. The fishermen can purchase the decontamination unit alone or jointly for \$6,000 to eliminate the pollution.

1. Suppose that the property rights are not clearly defined and pollution has already occurred, and the two fishermen can negotiate. What is the result?
2. Suppose that the property right is owned by the chemical plant. Chemical plant and the two fishermen can negotiate. What is the result?
3. Now suppose that property right belong to the two fishermen. Chemical plant and the two fishermen can also negotiate. What is the result?
4. Which result of the above three questions is Pareto efficient?
5. If the “tax-subsidy” mechanism is introduced, how should the regulator achieve Pareto efficient outcome?

Exercise 14.16 (Coase Neutrality Theorem) Coase Neutrality Theorem concludes that as long as the property rights are clearly defined, the equilibrium level of the externality will be the same regardless of the assignment of property rights. Consider the pure exchange economy of two types of commodities and two consumers. One commodity is “money” and both of them are fond of it. The other commodity is “music”. Music consumption will increase one’s own utility while reduce the utility of the other.

1. What special assumptions about consumer preferences will lead to the Coase neutrality theorem? Demonstrate your claim on two situations concerning the definition of property rights: (a) the musician has the right to play music without his neighbors’ approval; (b) the musician must be authorized by his neighbor to have music. (In order to reach an agreement, one person can compensate another person.) Use a diagram to illustrate your answer.
2. Suppose that both have the Cobb-Douglas utility function. Does the argument of Coase Neutrality Theorem still hold? Illustrate your answer with a diagram.

Exercise 14.17 The Coase efficiency theorem states as follows: If the property rights are clearly defined and the transaction costs are zero, the negotiation of externalities will lead to Pareto optimal outcomes.

1. Prove that the quasilinear utility function is a sufficient condition for the Coase Theorem to hold.
2. Is the quasilinear utility function a necessary condition for the Coase Theorem to hold? If yes, give a proof; if not, give a counterexample.

Exercise 14.18 (Kolstad, 2000) Suppose that there are two polluting firms with hidden characteristics θ . For both firms, θ does not have to be equal. Suppose that θ can take a value of 1 or 2. The revenue of firm i is $S_i(h_i, \theta_i) = 1 - \frac{(1-\theta_i h_i)^2}{2\theta_i}$. The damage caused by pollution is $D(h_1 + h_2) = (h_1 + h_2)^2/2$.

1. Suppose that the regulator knows each firm's θ : θ_1 and θ_2 . For all possible combinations of θ_1 and θ_2 , what is the socially optimal pollution for each firm: $h_1^*(\theta_1, \theta_2)$, $h_2^*(\theta_1, \theta_2)$?
2. Now suppose the regulator does not know θ , but asks each firm report θ . After receiving reports from each firm, each firm i will be charged a fee of $T_i(h_i, \theta_i)$. This fee is based on the reported θ_i by firm i , the reported θ_j by firm j , and real emissions h_i :

$$T_i(h_i, \theta_i) = D[h_i + h_j^*(\theta_1, \theta_2)] - S_j[h_j^*(\theta_1, \theta_2), \theta_j].$$

Before firms report their θ values, all of above specifications are common knowledge. Prove that it is in the best interest of each firm to report the true θ and take the socially optimal pollution level h^* .

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Chapter 15

Public Goods

15.1 Introduction

The previous chapter discusses the economic environments with externalities. It is very important to consider this issue when analysing the efficiency of a market. The possibility of externality determines whether or not intervening and supervisory measures are needed. In the presence of externalities, the market may fail to achieve efficient allocation even under perfect competition and freedom of choice, and then appropriated remedies need to be adopted. These measures include: Pigouvian tax, Coasian's approach, building a market for emissions trading, and designing a proper incentive mechanism, etc.

The presence of public goods is another significant situation that results in market failure. Once public goods are present in an economy, externalities and thus market failure may occur. It is well-known that financing a public project via voluntary donation is difficult. This is because public good is essentially different from private goods. Two main differences are the **non-exclusivity** and **non-rivalry**. A good is **excludable** if other individuals can be excluded from consuming it when an individual consumes it. A good is **non-rival** if one person's consumption does not reduce the amount available to other consumers.

A **pure public good** is a good in which **consuming one unit of the good by an individual in no way prevents others from consuming the**

same unit of the good. Thus, the good is nonexcludable and non-rival. Examples of public goods include street lights, policemen, fire protection, highway system, national defence, flood-control project, public television and radio broadcast, public parks, and a public project, etc. The purest (most representable) public good is national defense. It protects all citizens from aggression.

Local Public Goods: when there is a location restriction for the service of a public good.

The non-exclusivity of public good may result in free-rider problem. For example, individuals want to get benefits from but do not want to contribute to a public project. The inefficiency of state-owned enterprises also originated from the free-rider problem. These enterprises usually lack proper incentive to make efforts — everyone wants to enjoy the efforts provided by others. Even if the competitive market is an efficient social institution for allocating private goods, but is not an efficient mechanism for allocating public goods.

There are three possible ways that might be used to solve this problem: (1) forming social norms and cultures of donation habits, but it is difficult to achieve in the short term, and the effect is limited; (2) remoulding one's ideology by taking altruism and work as happiness and joy, but the practice is cruel and it is ineffective unless the genes that pursue personal interests are altered; (3) in situations where the social norms and cultures such as donation habits are difficult to form in a short run and individuals' ideological consciousness cannot be greatly improved, incentive mechanism design shall be adopted by taking the fact that individuals' ideological realm is limited as a constraint condition on a case-by-case basis. A comprehensive governance approach that combines incentive mechanism with social norms and regulations can better solve the free-rider problem in the presence of goods.

15.2 Notations and Basic Settings

A general setting of public goods economy includes consumers, producers, private goods, public goods, and economic characteristics of consumers

and produces.

Let

- n : the number of consumers.
- L : the number of private goods.
- K : the number of public goods.
- $Z_i \subseteq \mathbb{R}_+^L \times \mathbb{R}_+^K$: the consumption space of consumer i .
- $Z \subseteq \mathbb{R}_+^{nL} \times \mathbb{R}_+^K$: consumption space.
- $\mathbf{x}_i \in \mathbb{R}_+^L$: a consumption of private goods by consumer i .
- $\mathbf{y} \in \mathbb{R}_+^K$: a consumption/production of public goods.
- $\mathbf{w}_i \in \mathbb{R}_+^L$: the initial endowment of private goods for consumer i . For simplicity, it is assumed that there is no public goods endowment, but they can be produced from private goods.
- $Y \subseteq \mathcal{R}^{L+K}$: the set of production possibilities of the firm. For simplicity, we assume there is only one firm to produce the public goods.
- $(\mathbf{y}, -\mathbf{v}) \in Y$: a production plan, where $\mathbf{v} \in \mathbb{R}_+^L$ is the vector of private goods input.
- $f: \mathbb{R}_+^L \rightarrow \mathbb{R}_+^K$: production function with $\mathbf{y} = f(\mathbf{v})$.
- θ_i : the profit share of consumer i from the production.
- $(\mathbf{x}_i, \mathbf{y}) \in Z_i$: a consumption of private goods and public goods by consumer i .
- $(\mathbf{x}, \mathbf{y}) = (\mathbf{x}_i, \dots, \mathbf{x}_n, \mathbf{y}) \in Z$: an allocation.
- \succsim_i (or u_i if exists): a preference ordering (it is complete and transitive).
- $e_i = (Z_i, \succsim_i, \mathbf{w}_i, \theta_i)$: the characteristic of consumer i .
- $e = (e_1, \dots, e_n, f)$: a public goods economy.

The above is a relatively simple depicted class of public goods economic environments. Analogous to the discussion in general equilibrium problem, the economic environments can be more general to allow for production possibility sets of general form, an arbitrary number of firms, either public or private goods as input or output, see Foley (1970) and Milleron (1972) as detailed discussion.

Definition 15.2.1 Allocation $z \equiv (x, y) = (x_1, \dots, x_n, y) \in Z$ is *feasible*, if $(\sum_{i=1}^n x_i - \hat{w}, y) \in Y$, where $\hat{w} = \sum_{i=1}^n w_i$.

If technology can be represented by function $y = f(v)$, feasibility condition can be written as:

$$\sum_{i=1}^n x_i + v \leq \sum_{i=1}^n w_i, \quad (15.2.1)$$

and

$$y = f(v). \quad (15.2.2)$$

Definition 15.2.2 An allocation (x, y) is *Pareto efficient* for a public goods economy e if it is feasible and there is no other feasible allocation (x', y') such that $(x'_i, y') \succ_i (x_i, y)$ for all consumers i and $(x'_k, y') \succ_k (x_k, y)$ for some k .

Definition 15.2.3 An allocation (x, y) is *weakly Pareto efficient* for the public goods economy e if it is feasible and there is no other feasible allocation (x', y') such that $(x'_i, y') \succ_i (x_i, y)$ for all consumers i .

Remark 15.2.1 Unlike private goods economies, even though under the assumptions of continuity and strong monotonicity, a weakly Pareto efficient allocation may not be Pareto efficient for the public goods economies. The following proposition is due to Tian (1988).

Proposition 15.2.1 *For the public goods economies, a weakly Pareto efficient allocation may not be Pareto efficient even if preferences satisfy strong monotonicity and continuity.*

PROOF. The proof is by way of a counter-example. Consider an economy with $(n, L, K) = (3, 1, 1)$, constant returns in producing y from

x (the input-output coefficient normalized to one), and the following endowments and utility functions: $w_1 = w_2 = w_3 = 1$, $u_1(x_1, y) = x_1 + y$, and $u_i(x_i, y) = x_i + 2y$ for $i = 2, 3$. Then $z = (x, y)$ with $x = (0.5, 0, 0)$ and $y = 2.5$ is weakly Pareto efficient but not Pareto efficient because $z' = (x', y') = (0, 0, 0, 3)$ Pareto-dominates z by consumers 2 and 3. \square

However, under an additional condition of strict convexity, they are equivalent. The proof is left to readers.

15.3 Discrete Public Goods

15.3.1 Efficient Provision of Public Goods

For simplicity, consider a public good economy with n consumers and two goods: one private good and one public good.

Discrete public good, also called public project, is indivisible. It is assumed that the units of public goods provided is normalized to 1. This can also be interpreted as a logical variable of 0 or 1, 1 for providing public projects, and 0 for not providing public project.

Let g_i be the contribution made by consumer i , so that

$$\begin{aligned} x_i + g_i &= w_i; \\ \sum_{i=1}^n g_i &= v. \end{aligned}$$

Let c be the cost of providing the public project so that the production technology is given by

$$y = \begin{cases} 1 & \text{if } \sum_{i=1}^n g_i \geq c \\ 0 & \text{otherwise} \end{cases}.$$

Assume that $u_i(x_i, y)$ is strictly monotonically increasing and continuous. We first want to know under what conditions providing the public good Pareto dominates not providing it, i.e., there exist (g_1, \dots, g_n) such that $\sum_{i=1}^n g_i \geq c$ and

$$u_i(w_i - g_i, 1) > u_i(w_i, 0), \quad \forall i. \quad (15.3.3)$$

Let r_i be the **maximum willingness-to-pay** (reservation price) of consumer i , i.e., r_i must satisfy

$$u_i(w_i - r_i, 1) = u_i(w_i, 0). \quad (15.3.4)$$

Inequality (15.3.3) implies that providing the public project will bring higher utilities for all consumers than not providing the public project, and then from the perspective of social optimality, public goods should be provided. Therefore, as long as we know the utility function of each individual, we know their willingness-to-pay, thus knowing whether providing the public good is Pareto dominated.

If providing the public project Pareto dominates not providing the public project, we have

$$u_i(w_i - g_i, 1) > u_i(w_i, 0) = u_i(w_i - r_i, 1), \quad \forall i. \quad (15.3.5)$$

By strong monotonicity of u_i , we have

$$w_i - g_i > w_i - r_i \quad \forall i. \quad (15.3.6)$$

Then, we have

$$r_i > g_i, \quad (15.3.7)$$

and thus

$$\sum_{i=1}^n r_i > \sum_{i=1}^n g_i \geq c. \quad (15.3.8)$$

That is, the sum of the willingness-to-pay for the public good must exceed the cost of providing it. This condition is necessary. In fact, this condition is also sufficient. In summary, we have the following proposition.

Proposition 15.3.1 *Providing a public good Pareto dominates not providing the public good if and only if $\sum_{i=1}^n r_i > \sum_{i=1}^n g_i \geq c$.*

The problem is that individual preferences/utility functions are unknown to a social planner and thus it may result in inefficient outcomes as we discussed below. It then becomes an important issue how to design an incentive mechanism to induce individuals to truthfully report their pri-

vate information.

15.3.2 Free-Rider Problem

First of all, we want to know whether a free competitive market leads to the efficient provision of public goods. The following example shows that, due to the problem of free-rider, we generally cannot expect individual decision-making to lead to the efficient provision of public projects.

To see this, consider a simple economy with only two participants, each of whom pays $r_i = 100$, $i = 1, 2$. Suppose that the cost for providing the public project is $c = 150$. The participants decide on their own how much they will contribute for the public project. If both are willing to contribute 75, the public project will be provided, and each participant will receive 25 units of benefit. If only one person contributes 150, the public project is provided, but the benefit is 50, while the benefit of another person is 100. Formally, we have

$$\begin{aligned} r_i &= 100 \quad i = 1, 2; \\ c &= 150 \text{ (total cost);} \\ g_i &= \begin{cases} 150/2 = 75 & \text{if both agents make contributions;} \\ 150 & \text{if only agent } i \text{ makes contribution.} \end{cases} \end{aligned}$$

Each person decides independently whether or not to contribution for providing the public good. As a result, each one has an incentive to be a free-rider on the other as shown by the payoff matrix in Table 15.1.

		Person 2	
		Contribute	Not Contribute
Person 1	Contribute	(25,25)	(-50,100)
	Not Contribute	(100,-50)	(0,0)

Table 15.1: Private Provision of a Discrete Public Good

Note that net payoffs are defined by $r_i - g_i$. Thus, it is given by $100 - 150/2 = 25$ when both consumers are willing to produce the public project, and $100 - 150 = -50$ when only one person wants to contribute, but the other person does not.

The dominant strategy equilibrium in this game is (no contribution, no contribution). Thus, although the public project benefits both agents, no body wants to share the cost of producing the public project, but wants to free-ride on the other consumer. As a result, the public good is not provided at all even though it would be more efficient to do so. Thus, voluntary contribution in general does not result in the efficient level of the public good.

The above-mentioned problem is typically a prisoner's dilemma. This phenomenon is common in practice, causing both worse off. For example, if two firms conspire to monopolize prices, they can gain higher profits, but if one side lowers prices slightly, they can attract more customers. Therefore, as Bertrand Model predicted, each party has an incentive to reduce the price, and ultimately the price reaches the marginal cost level. It illustrates the basic conclusion that inefficient allocation often results from conscious and self-dedication only.

15.3.3 Voting for a Discrete Public Good

The amount of a public good is also often determined by a voting. Will this generally result in an efficient provision? The answer in general is negative.

Voting does not result in efficient provision. Consider the following example.

Example 15.3.1

$$c = 120$$

$$r_1 = 80, r_2 = 35, r_3 = 35.$$

Clearly, $r_1 + r_2 + r_3 > c$. $g_i = 120/3 = 40$. So the efficient provision of the public project should be yes. However, under the majority rule, only consumer 1 votes "yes" since she receives a positive net benefit if the good is provided. The 2nd and 3rd persons vote "no" to provide public project, and therefore, the public project will not be provided so that we have inefficient provision of the public project. The problem with the majority rule is that it only measures the net benefit for the public good, whereas the ef-

efficient condition requires a comparison of willingness-to-pay, resulting in inconsistent of individual rationality and collective rationality.

This example also shows that democracy and efficiency are often incompatible in decentralized decision-making, because the voters are usually driven by their own interests, that is so-called buttock often determines the head. Through democratic voting, it often leads to inefficient provision of public goods. To overcome the possible inconsistency between democracy and efficiency, a criterion whether democratic decision-making should be adopted is that the higher the level, the more respect for public opinion in the election of leaders, and the more democratic decision-making should be adopted in the selection of leaders. Because leaders' deciding direction and major decisions has tremendous externalities, we need to elect/select a person who respects public opinion, cares about the total welfare of a society, and be responsible to the voters. Otherwise, they will not be elected/selected in the next term for public office.

However, once a person is elected/selected, since she is accountable to the voters, the implementation of his goals and specific decisions should be efficient. Otherwise, her decisions are always rejected by her staff or team members. How can she be responsible for the voters? So constantly applying the simple majority rule in every specific issue may often lead to inefficient outcomes. Therefore, even in a democratic system, the major leader of an organization (say, presidents of a nation or of a university) usually has the power to nominate his deputy and the whole team of leadership. Of course, if a unit fails to improve and does not do well, the people will not be satisfied after the end of a term of office, and then the existing leaders may not be reelected. Thus the top leaders have incentives to fulfill their commitments to the people. The structure of government departments and enterprises is basically like this.

An example is the professors' committee in universities. Its duty is to evaluate the academic performance and promotion of faculties rather than getting involved in the detailed or all daily executive works. If every professor has voting right to support his own field of speciality, then the inefficient outcome as described in the above example may arise.

The above analysis shows that neither market nor democratic voting procedure could lead to efficient provision of public goods. The solution to this problem is much hard, which depends on the design of proper incentive mechanisms. We will discuss the VCG (Vickrey-Clarke-Groves) mechanism in Chapter 18, which may elicit efficient provision of public goods and truth-telling of voters.

15.4 Continuous Public Goods

15.4.1 Efficient Provision of Public Goods

Similar results to continuous public goods can also be obtained. Again, for simplicity, we assume there is only one public good and one private good that may be regarded as money, and $y = f(v)$, where y is the production of public good, and v is the input of private good used in producing public good.

The welfare maximization approach shows that Pareto efficient allocations can be characterized by

$$\begin{aligned} & \max_{(x,y)} \sum_{i=1}^n a_i u_i(x_i, y) \\ \text{s.t.} \quad & \sum_{i=1}^n x_i + v \leq \sum_{i=1}^n w_i, \\ & y \leq f(v). \end{aligned}$$

Define the Lagrangian function:

$$L = \sum_{i=1}^n a_i u_i(x_i, y) + \lambda \left(\sum_{i=1}^n w_i - \sum_{i=1}^n x_i - v \right) + \mu (f(v) - y). \quad (15.4.9)$$

When u_i is strictly quasi-concave and differentiable and $f(v)$ is concave and differentiable, the set of Pareto optimal allocations is characterized by

the first-order conditions:

$$a_i \frac{\partial u_i}{\partial x_i} - \lambda \leq 0, \quad \text{with equality if } x_i > 0; \quad (15.4.10)$$

$$\mu f'(v) - \lambda \leq 0, \quad \text{with equality if } v > 0; \quad (15.4.11)$$

$$\sum_{i=1}^n a_i \frac{\partial u_i}{\partial y} - \mu \leq 0, \quad \text{with equality if } y > 0. \quad (15.4.12)$$

So at an interior solution, by (15.4.10) and (15.4.11)

$$\frac{a_i}{\mu} = \frac{f'(v)}{\frac{\partial u_i}{\partial x_i}}. \quad (15.4.13)$$

Substituting (15.4.13) into (15.4.12), we have

$$\sum_{i=1}^n \frac{\frac{\partial u_i}{\partial y}}{\frac{\partial u_i}{\partial x_i}} = \frac{1}{f'(v)}. \quad (15.4.14)$$

Thus, we obtain the well-known Lindahl-Samuelson condition. This condition is different from the Pareto optimality for economies with private goods only. This condition indicates that the sum of the marginal rate of substitution of a public good for a private good of all economic agents is equal to the marginal rate of technical substitution, while for the private goods economy, the marginal rate of substitution of any two goods for every agent i is equal to the marginal rate of technical substitution at the Pareto optimality.

Thus, the conditions for Pareto efficiency are given by

$$\begin{cases} \sum_{i=1}^n MRS_{yx_i}^i = MRTS_{yv}, \\ \sum x_i + v \leq \sum_{i=1}^n w_i, \\ y = f(v). \end{cases} \quad (15.4.15)$$

The result shows that the provision level of public goods and the consumption of private goods are jointly determined.

Example 15.4.1 Consider an economy with one public good, one private

good, and n consumers. The utility function of consumer i is:

$$\begin{aligned} u_i &= a_i \ln y + \ln x_i, \\ y &= v. \end{aligned}$$

The Lindahl-Samuelson condition is

$$\sum_{i=1}^n \frac{\frac{\partial u_i}{\partial y}}{\frac{\partial u_i}{\partial x_i}} = 1, \quad (15.4.16)$$

and thus

$$\sum_{i=1}^n \frac{\frac{a_i}{y}}{\frac{1}{x_i}} = \sum_{i=1}^n \frac{a_i x_i}{y} = 1 \Rightarrow \sum a_i x_i = y, \quad (15.4.17)$$

which implies the level of the public good depends on the private good consumptions of all agents and is not uniquely determined.

Thus, in general, the marginal willingness-to-pay for a public good depends on the amount of private goods consumption, and therefore, the efficient level of y depends on x_i . However, in the case of quasi-linear utility functions,

$$u_i(x_i, y) = x_i + u_i(y), \quad (15.4.18)$$

the Lindahl-Samuelson condition becomes

$$\sum_{i=1}^n u'_i(y) = \frac{1}{f'(v)} \equiv c'(y), \quad (15.4.19)$$

and thus y is uniquely determined.

Example 15.4.2 Suppose that

$$\begin{aligned} u_i &= a_i \ln y + x_i, \\ y &= v. \end{aligned}$$

The Lindahl-Samuelson condition is

$$\sum_{i=1}^n \frac{\frac{\partial u_i}{\partial y}}{\frac{\partial u_i}{\partial x_i}} = 1, \quad (15.4.20)$$

and thus

$$\sum_{i=1}^n \frac{a_i}{y} = 1 \Rightarrow \sum a_i = y, \quad (15.4.21)$$

which implies the level of public good is uniquely determined.

15.4.2 Lindahl Mechanism and Equilibrium

We gave the conditions for Pareto efficiency in the presence of public goods. The next issue is how to achieve Pareto efficient allocations under decentralized decision-making of individuals. In an economy with only private goods, as long as the local non-satiation assumption is satisfied, every competitive equilibrium allocation is Pareto efficient.

However, with public goods, a competitive mechanism in general cannot result in Pareto efficient allocations. Indeed, if public goods are allocated through competitive market, the equilibrium outcome is the same as the one for private goods economies, that is, the marginal rate of substitution of two goods for all individuals is equal to the price ratio of the corresponding goods and then equal to the marginal rate of technical substitution, which does not satisfy the Lindahl-Samuelson condition. As such, competitive mechanism leads to inefficient allocations in the presence of public goods.

For instance, if we solve for competitive equilibrium in an economy with one private good, one public good and two consumers, then utility maximizing behavior would equalize the MRS of y for x and its relative price, e.g.,

$$MRS_{yx}^A = MRS_{yx}^B = \frac{p_y}{p_x},$$

which violates the Lindahl-Samuelson condition. Thus, in public goods economy, market failure occurs.

Then, what an economic mechanism should be adopted to achieve Pareto efficient allocations in public goods economies? We know that in the private goods economy, the Walras mechanism can lead to the efficient resource allocation. In the presence of public goods, one possible institutional arrangement is the Lindahl mechanism.

At the earlier of the 20th century, Lindahl proposed an institutional arrangement based on the Lindahl-Samuelson condition. Lindahl suggested

to use a taxation approach to provide public goods by which the tax rates may be different for different individuals. Each individuals should pay a specific “**personalized price**” for public goods, which means, for the same amount of the consumption of public goods, different prices are assigned to different individuals for consuming public goods. Thus, the Lindahl solution is a way to mimic the competitive solution in the presence of public goods with a difference that the consumption level of a public good is the same to all consumers, but the prices for consuming public goods are personalized due to different preferences of consumers.

To see this, consider an economy with $x_i \in \mathcal{R}_+^L$ (private goods) and $y \in \mathcal{R}_+^K$ (public goods). For simplicity, we assume the production possibility set of public goods Y is a closed convex cone. Thus, production technologies characterized by $y = f(v)$ exhibit constant returns to scale (CRS). A feasible allocation satisfies

$$\sum_{i=1}^n x_i + v \leq \sum_{i=1}^n w_i. \quad (15.4.22)$$

Let $q_i \in \mathcal{R}_+^K$ be the personalized price vector of consumer i for consuming the public goods y .

Let $\hat{q} = \sum_{i=1}^n q_i$ be the market price vector of y .

Let $p \in \mathcal{R}_+^L$ be the price vector of private goods.

The profit is defined as $\pi = qy - pv$.

Definition 15.4.1 (Lindahl Equilibrium) We say that an allocation $(x^*, y^*) \in Z$, a price vector of private goods $p^* \in \mathcal{R}_+^L$, and a personalized price vector of public goods $q_i^* \in \mathcal{R}_+^K$, one for each individual $i = 1, \dots, n$, constitute a *Lindahl equilibrium* if the following conditions are satisfied:

- (i) $p^* x_i^* + q_i^* y^* \leq p^* w_i$ for all i ;
- (ii) If $(x_i, y) \succ_i (x_i^*, y^*)$, then $p^* x_i + q_i^* y > p^* w_i$ for all i ;
- (iii) for all $(y, -v) \in Y$, there is $\hat{q}^* y^* - p^* v^* \geq \hat{q}^* y - p^* v$;
- (iv) $(y^*, -v^*) \in Y$,

where $v^* = \sum_{i=1}^n w_i - \sum_{i=1}^n x_i^*$, $\sum_{i=1}^n q_i^* = \hat{q}^*$.

The first condition above is budget constraint, the second is utility maximization condition, the third is profit maximization condition, and the fourth is feasibility condition.

Remark 15.4.1 Because the production function exhibits constant returns to scale, the maximum profit is zero at the Lindahl equilibrium. That is, $\hat{q}^* y^* - p^* v^* = 0$, therefore

$$\sum_{i=1}^n p^* x_i^* = \sum_{i=1}^n p^* w_i + \hat{q}^* y^*.$$

Thus the budget constraint (i) holds with equality at Lindahl equilibrium for every consumer.

We may regard a Walrasian equilibrium as a special case of a Lindahl equilibrium when there are no public goods. In fact, the concept of Lindahl equilibrium in economies with public goods is, in many ways, a natural generalization of Walrasian equilibrium in private goods economies, with attention to the well-known duality that reverses the role of prices and quantities between private and public goods, and between Walrasian and Lindahl allocations. In Walrasian mechanism, prices for all commodities are the same for all consumers, but the quantities of their private goods consumption are personalized; in Lindahl mechanism, the quantities of public goods consumed are the same for all consumers, while prices charged for public goods are personalized. In addition, the concepts of Walrasian and Lindahl equilibria are both relevant to private-ownership economies. Moreover, they are both characterized by purely price-taking behavior on the part of consumers. The Lindahl solution for efficient provision of public goods is essentially an informationally decentralized decision-making process.

Similar to Walrasian equilibrium, Lindahl equilibrium has similar properties. In fact, by redefining the consumption space, an economy with public goods can be regarded as an economy with private goods only. So, a Lindahl equilibrium can then be regarded as a Walrasian equilibrium under this redefinition of the consumption space. This method is adopted in the following to prove the existence of Lindahl equilibrium, where the

first and second fundamental theorems of welfare economics still hold. The proof is similar to that of Walrasian equilibrium.

Theorem 15.4.1 (Existence Theorem on Lindahl Equilibrium) *For a public goods economy $\mathbf{e} = (\{X_i, \mathbf{w}_i, \succsim_i\}, \{Y_j\}, \{\theta_i\})$, there exists a Lindahl equilibrium if the following conditions are satisfied:*

- (i) $Z_i = \mathcal{R}_+^{L+K}$;
- (ii) $\mathbf{w}_i > 0$;
- (iii) \succsim_i is continuous, strictly convex, and monotonic;
- (iv) Y is a closed and convex cone, $0 \in Y$, $(-\mathbb{R}_+^L, 0) \subseteq Y$ (free disposal property).

PROOF. We prove this theorem by constructing an economy with only private goods to which the existence theorem of Walrasian equilibrium (CE) can be applied. Specifically, treating the consumptions of different consumers of public goods as different commodities and changing the original commodity space of consumer i to $\bar{Z}_i = (Z_i, \{0\}) \subseteq \mathcal{R}^{L+K} \times \mathcal{R}^{(n-1)K}$, where 0 is null element of $(n-1)K$ dimensional space. The consumption bundle of i is $\bar{z}_i = (\mathbf{x}_i, \mathbf{y}_i, 0, \dots, 0)$. The consumption space constructed above and the conditions of this theorem satisfy all requirements of Theorem 10.4.7 (Existence Theorem III for competitive equilibrium), and thus the existence of CE is guaranteed. Therefore, a Lindahl equilibrium exists for the original public goods economy. \square

Similarly, we can enhance the monotonicity assumption to strong monotonicity and relax the assumption of interior-point initial endowments of private goods.

For a public goods economy with one private good and one public good $y = \frac{1}{q}v$, the definition of Lindahl equilibrium becomes much simpler.

Definition 15.4.2 An allocation (\mathbf{x}^*, y^*) is a *Lindahl equilibrium allocation* if there exist q_i^* , $i = 1, \dots, n$ such that

$$(i) \ x_i^* + q_i^* y^* \leq w_i;$$

- (ii) if $(x_i, y) \succ_i (x_i^*, y^*)$, then $x_i + q_i^* y > w_i$;
 (iii) $\sum_{i=1}^n q_i^* = \hat{q}$.

In fact, the feasibility condition is automatically satisfied when the budget constraints (i) is satisfied.

If (\mathbf{x}^*, y^*) is an interior point of the Lindahl equilibrium allocation, then we can have the first order condition of utility maximization:

$$\frac{\frac{\partial u_i}{\partial y}}{\frac{\partial u_i}{\partial x_i}} = \frac{q_i}{1}, \quad (15.4.23)$$

which means the Lindahl-Samuelson condition holds:

$$\sum_{i=1}^n MRS_{yx_i} = \hat{q},$$

which is the necessary condition for Pareto efficiency.

Example 15.4.3 Solve for the Lindahl equilibrium of the following public goods economy:

$$u_i(x_i, y) = x_i^{\alpha_i} y^{(1-\alpha_i)}, \quad 0 < \alpha_i < 1, \\ y = \frac{1}{\hat{q}} v.$$

The budget constraint is:

$$x_i + q_i y = w_i.$$

The demand functions for private goods x_i and public goods y_i of consumer i are given by

$$x_i = \alpha_i w_i, \quad (15.4.24)$$

$$y_i = \frac{(1 - \alpha_i) w_i}{q_i}. \quad (15.4.25)$$

Since $y_1 = y_2 = \cdots = y_n = y^*$ at Lindal equilibrium, we have by (15.4.25)

$$q_i y^* = (1 - \alpha_i) w_i. \quad (15.4.26)$$

Making summation leads to

$$\hat{q}y^* = \sum_{i=1}^n (1 - \alpha_i)w_i.$$

Then, we have

$$y^* = \frac{\sum_{i=1}^n (1 - \alpha_i)w_i}{\hat{q}}.$$

Thus, by (15.4.26), we have

$$q_i = \frac{(1 - \alpha_i)w_i}{y^*} = \frac{\hat{q}(1 - \alpha_i)w_i}{\sum_{i=1}^n (1 - \alpha_i)w_i}. \quad (15.4.27)$$

If we want to find a Lindahl equilibrium, we must know the preferences or MRS of each consumer. But, the information about individual-s' preferences are in general unknown, because of the free-rider problem, each consumer will have the incentive to not truthfully reveal their preferences in order to contribute less. What is more, as each consumer has a personalized price system, when the preferences of consumers are not public information, it is difficult to regard the personalized price system of each consumer as given because his report will affect his price.

15.5 Welfare Properties of Lindahl Equilibrium

15.5.1 The First Fundamental Theorem of Welfare Economics

Similarly, we have the following First Fundamental Theorem of Welfare Economics for public goods economies.

Theorem 15.5.1 (The First Welfare Theorem for Public Goods Economy)

For a public goods economy $\mathbf{e} = (e_1, \dots, e_n, Y)$, every Lindahl allocation $(\mathbf{x}^, \mathbf{y}^*)$ with the price system $(\mathbf{q}_1^*, \dots, \mathbf{q}_n^*, \mathbf{p}^*)$ is weakly Pareto efficient. Furthermore, if local non-satiation is satisfied, it is Pareto efficient.*

PROOF. The proof of the first conclusion of the theorem is included in the proof of the second one. We only need to prove that under local non-satiation, every Lindahl allocation $(\mathbf{x}^*, \mathbf{y}^*)$ is Pareto efficient. Suppose not. Then there exists another feasible allocation $(\mathbf{x}_i, \mathbf{y})$ such that $(\mathbf{x}_i, \mathbf{y}) \succsim_i (\mathbf{x}_i^*, \mathbf{y}^*)$ for all i and $(\mathbf{x}_k, \mathbf{y}) \succ_k (\mathbf{x}_k^*, \mathbf{y}^*)$ for some k .

We first show

$$p^* x_i + q_i^* y \geq p^* w_i, \text{ for all } i = 1, 2, \dots, n,$$

If not, then there is some i such that

$$p^* x_i + q_i^* y < p^* w_i.$$

Then, by local non-satiation, there is (x'_i, y') such that $(x'_i, y') \succ (x_i, y) \succsim_i (x_i^*, y^*)$ and $p^* x'_i + q_i^* y' < p^* w_i$, contradicting the fact that (x_i^*, y^*) is consumer i 's utility maximizing consumption bundle.

For consumer k , since (x_k^*, y^*) is consumer k 's optimal choice, by $(x_k, y) \succ_k (x_k^*, y^*)$, we must have

$$p^* x_k + q_k^* y > p^* w_k \text{ for some } k.$$

Thus, making summation leads to

$$\sum_{i=1}^n p^* x_i + \sum_{i=1}^n q_i^* y > \sum_{i=1}^n p^* w_i, \quad (15.5.28)$$

that is,

$$\hat{q}^* y + p^* \sum_{i=1}^n (x_i - w_i) = \hat{q}^* y + p^* v > 0.$$

However, according to profit maximization condition, for all $(y, v) \in Y$, we have

$$\hat{q}^* y + p^* v \leq 0.$$

This contradiction proves the theorem. \square

Similarly, we can define Lindahl equilibrium with transfers.

Definition 15.5.1 (Lindahl Equilibrium with Transfers) For a public goods economy $e = (e_1, \dots, e_n, Y)$, an allocation $(x^*, y^*) \in Z$, a price vector of private goods $p^* \in \mathcal{R}_+^L$ and a personalized price vector of public good $q_i^* \in \mathcal{R}_+^K, \forall i$, constitute a *Lindahl equilibrium with transfers* if there exists an assignment of wealth levels (I_1, \dots, I_n) with $\sum_i I_i = p \cdot \sum_i w_i$, such that

$$(i) \ p^* x_i^* + q_i^* y^* \leq I_i;$$

- (ii) if $(x_i, y) \succ_i (x_i^*, y^*)$, then $p^* x_i + q_i^* y > I_i$;
- (iii) for all $(y, -v) \in Y$, $\hat{q}^* y^* - p^* v^* \geq \hat{q}^* y - p^* v$;
- (iv) $(y^*, -v^*) \in Y$,

where $v^* = \sum_{t=1}^n w_t - \sum_{i=1}^n x_i^*$, $\sum_{t=1}^n q_t^* = \hat{q}^*$.

Theorem 15.5.2 (The First Welfare Theorem of LE with Transfers) *For a public goods economy $e = (e_1, \dots, e_n, Y)$, every Lindahl equilibrium allocation (x^*, y^*) with transfers and price system $(q_i^*, \dots, q_n^*, p^*)$ is weakly Pareto efficient. Moreover, if consumers' preferences are locally non-satiated, then it is Pareto efficient.*

PROOF. The proof is analogous to the proof of Theorem 15.5.1 and is thus omitted. \square

15.5.2 Economic Core in the Presence of Public Goods

Similar to private goods economy, we can define **economic core** in the presence of public goods as follows:

Definition 15.5.2 (Blocking Coalition) A group of economic agents $S \subseteq N$ is said to *block (improve upon)* a given allocation (x, y) , if the coalition can be Pareto improved within their own endowments, i.e., there exists another allocation (x', y') such that

- (1) (x', y') is feasible with respect to S , that is, $(y, \sum_{i \in S} (x'_i - w_i)) \in Y$,
- (2) $(x'_i, y') \succ_i (x_i, y)$ for all $i \in S$, and $(x'_k, y') \succ_k (x_k, y)$ for some $k \in S$.

Definition 15.5.3 (Economic Core) A feasible allocation (x, y) is said to have the *core property* if it cannot be blocked by any coalition. The set of all allocations in core is called *economic core* or simply *core*.

Remark 15.5.1 Every allocation in core is Pareto efficient and individually rational, that is, $(x_i, y) \succ_i (w_i, y), \forall i = 1, 2, \dots, n$.

Similar to the proof of economic core theorem for private goods, we can show that every Lindahl equilibrium allocation has the core property under local non-satiation of preferences.

Theorem 15.5.3 *Suppose that local non-satiation condition is satisfied. If $(\mathbf{x}, \mathbf{y}, \mathbf{p})$ is a Lindahl equilibrium, then (\mathbf{x}, \mathbf{y}) is in the core.*

Though every Lindahl equilibrium allocation is in the core under local non-satiation of preferences, the core convergence theorem does not hold necessarily. See Milleron (1972) for a counterexample.

15.5.3 The Second Fundamental Theorem of Welfare Economics

Similarly, we have the Second Fundamental Theorem of Welfare Economics for public goods economy.

Theorem 15.5.4 (The Second Welfare Theorem for Public Goods Economy)

For a public goods economy $\mathbf{e} = (e_1, \dots, e_n, \{Y_j\})$, suppose that \succsim_i is continuous, convex, and strongly monotonic, Y is a closed convex set, and $0 \in Y$. Then for every Pareto efficient allocation $(\mathbf{x}^, \mathbf{y}^*)$ with interior-point private consumption (that is $\mathbf{x}^* \in \mathcal{R}_{++}^{nL}$), there exists a nonzero price vector $(\mathbf{q}_1, \dots, \mathbf{q}_n, \mathbf{p}) \in \mathcal{R}_+^{L+nK}$ such that $((\mathbf{x}, \mathbf{y}), (\mathbf{q}_1, \dots, \mathbf{q}_n), \mathbf{p})$ is a Lindahl equilibrium with transfers, that is, there exists an assignment of wealth levels after transfers (I_1, \dots, I_n) satisfying $\sum_i I_i = \mathbf{p} \cdot \sum_i \mathbf{w}_i$ such that*

- (1) if $(\mathbf{x}_i, \mathbf{y}) \succsim_i (\mathbf{x}_i^*, \mathbf{y}^*)$, then $\mathbf{p}\mathbf{x}_i + \mathbf{q}_i\mathbf{y} > I_i \equiv \mathbf{p}\mathbf{x}_i^*, i = 1, \dots, n$,
- (2) for all $(\mathbf{y}, -\mathbf{v}) \in Y$, $\hat{\mathbf{q}}\mathbf{y}^* - \mathbf{p}\mathbf{v}^* \geq \hat{\mathbf{q}}\mathbf{y} - \mathbf{p}\mathbf{v}$,

where $\mathbf{v}^* = \sum_{i=1}^n \mathbf{w}_i - \sum_{i=1}^n \mathbf{x}_i^*$ and $\sum_{i=1}^n \mathbf{q}_i = \hat{\mathbf{q}}$.

PROOF. Define two subsets W and $P(\mathbf{x}^*, \mathbf{y}^*)$ of space \mathcal{R}^{nK+L} as follows:

$$W = \{(\mathbf{y}, \dots, \mathbf{y}; -\mathbf{v}) : (\mathbf{y}, -\mathbf{v}) \in Y\}.$$

Because Y is a closed convex set and $0 \in Y$, W is non-empty, closed and convex.

$$P(\mathbf{x}^*, \mathbf{y}^*) = \{(\mathbf{y}'_1, \dots, \mathbf{y}'_n; -\mathbf{v}') : \mathbf{v}' = \sum_{i=1}^n (\mathbf{x}'_i - \mathbf{w}_i) \text{ \& } (\mathbf{x}'_i, \mathbf{y}'_i) \succsim_i (\mathbf{x}_i^*, \mathbf{y}^*)\}.$$

Since \succsim_i is convex, $P(\mathbf{x}^*, \mathbf{y}^*)$ is convex.

Since $(\mathbf{x}^*, \mathbf{y}^*)$ is a Pareto optimal allocation, we must have $W \cap P(\mathbf{x}^*) = \emptyset$; otherwise, there may exist a Pareto improvement. Therefore, applying

the hyperplane separating theorem introduced in Chapter 2, we obtain that there exists $(q_1, \dots, q_n, p) \neq 0$ such that for all $(y, \dots, y; -v) \in W$ and for all $(y'_1, \dots, y'_n; -v') \in P(x^*, y^*)$, we have

$$\sum_{i=1}^n q_i y'_i - p v' \geq \sum_{i=1}^n q_i y_i - p v. \quad (15.5.29)$$

We proceed with the proof in five steps.

1. Profit-maximization, that is, for all $(y, -v) \in Y$, we have $\hat{q}y^* - pv^* \geq \hat{q}y - pv$.

Let $\hat{z} = (y^*, \dots, y^*; -v')$, where $v' = \sum_{i=1}^n (w_i - x'_i)$ and $x'_i \geq x_i^*$ (i.e., $x'_i \geq x_i^*$ and $x'_i \neq x_i^*$). Then, by strong monotonicity of preferences, we have $(x'_i, y^*) \succ_i (x_i^*, y^*)$, and thus $(y^*, \dots, y^*; -v') \in P(x^*, y^*)$. Hence, it follows from (15.5.29) that, for any $(y, -v) \in Y$, we have

$$\hat{q}y^* - pv' \geq \hat{q}y - pv.$$

Let $v' \rightarrow v^*$, then we have $\hat{q}y^* - pv^* \geq \hat{q} \cdot y - pv$, $\forall (y, -v) \in Y$.

2. $(q_1, \dots, q_n, p) \geq 0$, and $p \neq 0$.

Firstly, we prove $q_i \geq 0$, $i = 1, \dots, n$. Let

$$\hat{z} = (y, \dots, y; -v) + e_{yi}^k,$$

where $(y, \dots, y; -v)$ is an element in W , $e_{yi}^k = (0, \dots, 1, 0, \dots, 0)$ is a vector in \mathcal{R}^{nK+L} such that the element of public goods k associated with q_i^k is 1 and the other elements are all zero. Then, by the strong monotonicity of preferences and the fact that e_y^k is equally distributed among all economic agents, we have $\hat{z} \in P(x^*, y^*)$. Therefore, from (15.5.29), we have

$$\hat{q}y - pv + q_i^k \geq \hat{q}y - pv. \quad (15.5.30)$$

Consequently,

$$q_i^k \geq 0, \quad k = 1, 2, \dots, K; i = 1, 2, \dots, n. \quad (15.5.31)$$

Now we prove that $p \geq 0$. Let $e_x^l = (0, \dots, 1, 0, \dots, 0)$ be a vector in \mathcal{R}^{nK+L} such that the element associated with good l is 1 and other elements are zero. Repeating the above procedure, we have

$$p_i^l \geq 0, \quad l = 1, 2, \dots, L. \quad (15.5.32)$$

Lastly, we prove $p \neq 0$ by contradiction. If $p = 0$, since $(q_1, \dots, q_n, p) \neq 0$, then for some public good k , we must have its price $q^k = \sum_{i=1}^n q_i^k > 0$. Since the production of public goods exhibits constant returns to scale, when $p = 0$, the costs for all private goods as inputs are zero. Then the profit may be infinitely large, which contradicts with the fact that y is a profit-maximizing plan.

3. For all i , if $(x_i, y) \succ (x_i^*, y^*)$, then $\sum_i p x_i + \hat{q} y \geq \sum_i p x_i^* + \hat{q} y^*$.

For every i and $(x_i, y) \succ (x_i^*, y^*)$, by strong monotonicity of preferences, there exists (x'_i, y') that is arbitrarily close to (x_i, y) , such that $(x'_i, y') \succ_i (x_i, y) \succ_i (x_i^*, y^*)$, and thus $(y', \dots, y'; \sum_i (x'_i - w_i)) \in P(x^*, y^*)$. Also, note that $(y^*, \dots, y^*; \sum_i (x_i^* - w_i)) \in W$. Thus, by (15.5.29), we have

$$\sum_i p x'_i + \hat{q} y' \geq \sum_i p x_i^* + \hat{q} y^*.$$

Let $x'_i \rightarrow x_i$. We have $\sum_i p x_i + \hat{q} y \geq \sum_i p x_i^* + \hat{q} y^*$.

4. For all i , if $(x_i, y) \succ (x_i^*, y^*)$, then $p x_i + q_i y \geq p x_i^* + q_i y^*$. Let

$$\begin{aligned} (x'_i, y') &= (x_i, y), \\ (x'_m, y') &= (x_m^*, y^*), \quad m \neq i. \end{aligned}$$

Then, it follows from step 3 that

$$px_i + q_i y + \sum_{m \neq i} (px_m^* + q_m y^*) \geq \sum_j px_j^* + \sum_i q_i y^*,$$

therefore

$$px_i + q_i y \geq px_i^* + q_i y^*.$$

5. For all i , if $(x_i, y) \succ (x_i^*, y^*)$, then $px_i + q_i y > px_i^* + q_i y^* \equiv I_i$.

If the conclusion does not hold, then

$$px_i + q_i y = px_i^* + q_i y^*. \quad (15.5.33)$$

Since $(x_i, y) \succ (x_i^*, y^*)$, when $0 < \lambda < 1$ is sufficiently close to 1, it follows from the continuity of preferences that $(\lambda x_i, \lambda y) \succ (x_i^*, y^*)$. From the conclusion of step 4, we have $\lambda(px_i + q_i y) \geq px_i^* + q_i y^* = px_i + q_i y$.

Since $x^* \in \mathcal{R}_{++}^{nL}$, from step 2 where we have $(q_1, \dots, q_n, p) \geq 0$ and $p \neq 0$, we already know that $px_i + q_i y = px_i^* + q_i y^* > 0$; therefore, $\lambda \geq 1$, which contradicts the fact that $\lambda < 1$.

Thus all the conditions of Lindahl equilibrium with transfers are satisfied. \square

15.6 Free-Rider Problem

When the MRS is known, Pareto efficient allocation (x, y) can be determined from the Lindahl-Samuelson condition or the Lindahl solution. After that, the contribution of each consumer is given by $g_i = w_i - x_i$. However, since individuals' preferences are private information, normally a social planner does not know the information about MRS. Of course, it will be naive to think that each individual will truthfully reveal his preferences and determine the willingness-to-pay based on the revealed information. Since all economic agents are self-interested, generally they will not tell the true MRS so they may be able to make less contribution.

Indeed, if consumers realize that their shares of the contribution for producing public goods (or the personalized prices) depend on their report of MRS, they have “incentives to deceive.” That is, when the consumers are asked to report their preferences or MRSs, they will have incentives to report their economic characteristics so that they can pay less to consume the public goods, resulting in insufficient provision of public goods and leading to Pareto inefficient allocations. This is the so-called problem of free riders. That is why it is difficult to raise enough fund for the provision of public goods through voluntary contribution.

To see this, notice that the social goal is to reach Pareto efficient allocations for the public goods economy, but from the perspective of personal interest, the utility maximization problem of each person is the following:

$$\max u_i(x_i, y) \quad (15.6.34)$$

subject to

$$\begin{aligned} g_i &\in [0, w_i]; \\ x_i + g_i &= w_i; \\ y &= f\left(g_i + \sum_{j \neq i}^n g_j\right). \end{aligned}$$

That is, each consumer i maximizes his payoffs when others' strategies \mathbf{g}_{-i} are given. From this problem, we can form a non-cooperative game:

$$\Gamma = (G_i, \phi_i)_{i=1}^n,$$

where $G_i = [0, w_i]$ is the strategy space of consumer i and $\phi_i : G_1 \times G_2 \times \cdots \times G_n \rightarrow R$ is the payoff function of consumer i which is defined by

$$\phi_i(g_i, \mathbf{g}_{-i}) = u_i\left[(w_i - g_i), f\left(g_i + \sum_{j \neq i}^n g_j\right)\right]. \quad (15.6.35)$$

According to the definition of Nash equilibrium in Chapter 5, for the game $\Gamma = (G_i, \phi_i)_{i=1}^n$, the strategy $\mathbf{g}^* = (g_1^*, \dots, g_n^*)$ is a **Nash Equilibrium**

if

$$\phi_i(g_i^*, \mathbf{g}_{-i}^*) \geq \phi_i(g_i, \mathbf{g}_{-i}^*) \quad \forall g_i \in G_i, \forall i = 1, 2, \dots, n,$$

and \mathbf{g}^* is a **dominant strategy equilibrium** if

$$\phi_i(g_i^*, \mathbf{g}_{-i}) \geq \phi_i(g_i, \mathbf{g}_{-i}) \quad \forall g_i \in G_i, \forall i = 1, 2, \dots, n.$$

Remark 15.6.1 Note that the difference between Nash equilibrium (NE) and dominant strategy is that at NE, given optimal strategy of others, each consumer chooses his optimal strategy and so one's own optimal strategy depends on others' strategies; while dominant strategy means that the strategy chosen by each consumer is optimal regardless of others' strategies so that each one's optimal strategy is independent of any strategy of others. Thus, a dominant strategy equilibrium is clearly a Nash equilibrium, but the converse may not be true. Only for a very special payoff functions, there is a dominant strategy while a Nash equilibrium exists for a continuous and quasi-concave payoff functions that are defined on a compact set.

For Nash equilibrium, if u_i and f are differentiable, then the first order condition is:

$$\frac{\partial \phi_i(\mathbf{g}^*)}{\partial g_i} \leq 0, \text{ with equality if } g_i > 0, \quad \forall i = 1, \dots, n. \quad (15.6.36)$$

Thus, we have

$$\frac{\partial \phi_i}{\partial g_i} = \frac{\partial u_i}{\partial x_i}(-1) + \frac{\partial u_i}{\partial y} f' \left(g_i^* + \sum_{j \neq i}^n g_j \right) \leq 0, \text{ with equality if } g_i > 0.$$

So, at an interior-point solution \mathbf{g}^* , we have

$$\frac{\frac{\partial u_i}{\partial y}}{\frac{\partial u_i}{\partial x_i}} = \frac{1}{f'(g_i^* + \sum_{j \neq i}^n g_j)},$$

and thus

$$MRS_{yx_i}^i = MRTS_{yv},$$

which does not satisfy the Lindahl-Samuelson condition. Hence, the Nash equilibrium in general does not result in Pareto efficient allocations.

The above equation implies that a lower level of public good is provided rather than the Pareto efficient level of the public good when utility functions are quasi-concave because of diminishing MRS (see Figure 15.1). Therefore, Nash equilibrium allocations are in general not consistent with Pareto efficient allocations. How can one solve this free-rider problem? We will answer this question in Part VI on mechanism design theory. Vickrey-Clarke-Groves mechanism of demand revelation can solve the problem of efficient provision of public goods.

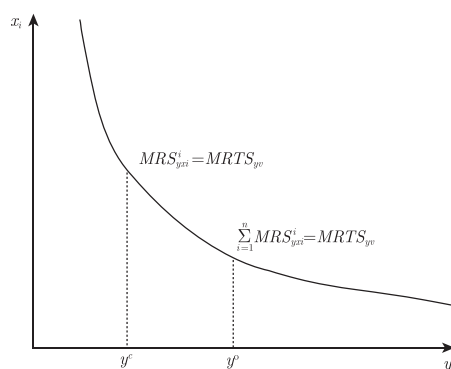


Figure 15.1: Free-rider results in a lower provision of public goods than the level of Pareto efficient provision of public goods.

15.7 Biographies

15.7.1 Ludwig Mises

Ludwig von Mises (1881-1973), the third generation head of the Austrian School and a member of the Mont Pelerin Society, enrolled at the University of Vienna in 1900, where he was greatly influenced by Carl Menger (1840 - 1921) and got his doctoral degree from the school of law in 1906. From 1909 to 1934, he was an economist for the Vienna Chamber of Commerce. After the World War I, he also served as a legal advisor to a government agency, responsible for drafting the terms of the final-war treaty - to resolve pre-war

private debt problem between belligerents. On New Year's Day in 1927, Austrian Institute for Business Cycle Research he founded was formally established and Hayek became the first director.

In 1934 - 1940, he moved to Geneva as a professor at the Graduate Institute of International Studies. In 1940 he moved to New York in USA. At pre, Keynesianism in the American academic world was prevalent, Mises' liberalism was clearly out of the mainstream, and he was not employed by any academic organization. In 1945, through the recommendation of the Lawrence Fertig & William Volker Foundation, Mises entered New York University, but he could only serve as a visiting professor. In 1949, Mises published "Human Behavior." Even so, he was only able to find a visiting professor position until retirement in 1969.

For a long time, even though Mises' ideas had not been accepted by mainstream economists, his ideological influence and knowledge contribution to the 20th century human society cannot be ignored. To a certain extent, it could be said that the history of economic thought of human society in the 20th century can not tell a complete story if Mises is missing. In 2000, America's "Freedom" magazine referred to Mises as "the century figure of libertarianism."

The reason why Mises occupied such an important position in the history of contemporary human society is mainly because Mises made numerous remarkable theoretical contributions in understanding the basic principles of human economic and social operations. In addition to his theoretical contributions in inflation, economic cycles, economic epistemology and methodology, and his own unique catallactics (i.e., a theory of the way the free market system reaches exchange ratios and prices) and praxeology, his main theoretical contributions lie in the early 1920s. He presented such a major theoretical insight with a discerning eye: In the absence of a market price mechanism, the impossibility of economic computations will result in the infeasibility of the centrally planned economy.

15.7.2 Douglass North

Douglass C. North (1920 - 2015), as the founder and pioneer of New Economic History (Cliometrics), New Institutional Economics, and New Political Economy, was one of the most important and influential economists in the late 20th century. In 1942 and 1952, respectively, he received a bachelor degree and a Ph.D. degree from the University of California, Berkeley. He began teaching at the University of Washington in 1951, taught at Rice University in 1979, at Cambridge University in 1981-1982, and returned to the University of Washington in 1982. North was awarded the Nobel Memorial Prize in Economic Sciences for renewed research in economic history by applying economic theory and quantitative methods in order to explain economic and institutional change in 1993.

The main contribution of North was innovation in research methodologies, that is, to study new objects with the methods of neoclassical economics. In other words, he used neoclassical economics and econometrics to study the issue of economic history. His early research on ocean shipping and the balance of payments of the U.S., in step with the school of new economic history represented by Robert W. Fogel (1926 - 2013), combined the neoclassical production theory with the data found in the research of economic history. This new method has revolutionized the study of economic history. North was not satisfied with this. He also used property rights theory to explain the impact of institutional change on economic performance in the American history. North's early work, such as **The Economic Growth of the United States: 1790-1860** and **Growth and Welfare in the American Past: A New Economic History**, have fully reflected this.

From the 1980s, North began to use the property rights theory of the new institutional economics to analyze the more general theory of industrialization in the western world in the last two centuries with the purpose to explore the causes of economic growth in the western world, the internal relation between economic growth and institutional changes, the trend of interaction between property rights system and economic development, and the inherent requirements of economic development for the institution. North's works in this area include **The Rise of the Western World: A New**

Economic History and Institutional Change and the American Economic Growth.

From the 1990s, North began to summarize his experience of 30 years' research on economic history and extracted some theories that became important contribution to economics, especially the new institutional economics. His works in this area mainly include **Institutions, Institutional Change, and Economic Performance**. To sum up, North's contribution to economics mainly includes three aspects: first, he used the method of institutional economics to explain economic growth in history; second, as one of the founders of the new institutional economics, North re-examined the role of institutions including the property rights system; third, as an economist, North applied institutions, a content that was not involved in neoclassical economics, as an endogenous variable in economic research. In particular, property rights system, ideology, state, and ethics are taken as variables in economic evolution and economic development. The theory of institutional change has thus been greatly developed.

Property rights theory, state theory and ideology theory are the three cornerstones of North's institutional change theory. Through the examination and analysis of the evolution of market economy, North produced the thought of institutional change theory and built his analytical framework with the three cornerstone theories. As he stated, "My research focuses on institutional theory. The cornerstones of this theory are: the theory of property rights that describes incentives for individuals and groups in institution; the theory of the state that defines and enforces property rights; the theory of ideology that influences people's different reaction on the change of objective existence, which explains why people have different understandings in practice." It is worth mentioning that, in the process of clarifying the above analytical framework, North always used the cost-benefit analysis to demonstrate the rationality of the choice of property rights structure, the necessity of the existence of state, and the importance of ideology. Such analysis makes North's institutional change theory enormously persuasive. In 1993, North was awarded the Nobel Memorial Prize in Economic Sciences for his establishment of the "Institutional Change Theory" including property rights theory, state theory, and ideolo-

gy theory.

15.8 Exercises

Exercise 15.1 Prove that for the public goods economy, weak Pareto efficient allocation is Pareto efficient when preferences satisfy strong monotonicity, continuity, and strict convexity.

Exercise 15.2 (Pareto efficiency of Lindahl equilibrium) Consider a public goods economy with n individuals who consume one private good x and one public good y in consumption space $Z_i = \mathcal{R}_+^2$. Each individual i is endowed with w_i units of private good. There are no initial endowments for the public good, but the public good can be produced from the private good, according to a production technology

$$y = \frac{1}{q}v.$$

The utility function of individual i is denoted by $u_i(x_i, y)$ which is continuously differentiable and satisfies

$$\frac{\partial u_i}{\partial x_i} > 0 \quad \text{for all } i = 1, 2, \dots, n.$$

1. Define the Lindahl equilibrium and Pareto efficiency for this economy, without assuming representation of preferences by utility functions.
2. For interior Pareto efficiency, is it necessary that all individuals have the same marginal rate of substitution, i.e., that

$$\frac{\frac{\partial u_1}{\partial y}}{\frac{\partial u_1}{\partial x_1}} = \frac{\frac{\partial u_2}{\partial y}}{\frac{\partial u_2}{\partial x_2}} = \dots = \frac{\frac{\partial u_n}{\partial y}}{\frac{\partial u_n}{\partial x_n}}$$

If yes, give proof. If not, give a counter example.

3. Now suppose the utility functions are of the quasi-linear form $u_i = x_i + v_i(y)$ with v_i strictly increasing, strictly concave and continuously

differentiable.

- (a) At interior Pareto efficient solutions $(x_1^*, x_2^*, \dots, x_n^*, y^*)$, does y^* vary with $(x_1^*, x_2^*, \dots, x_n^*)$? Why or why not?
- (b) Is the answer to part (a) different when quasi-linearity is not assumed? Why or why not?
- (c) Suppose that $n = 2$, $w_1 = 5$, $w_2 = 4$, production function is given by $y = 1/2v$, utility function is given by

$$u_1 = x_1 + \alpha \ln y$$

$$u_2 = x_2 + \beta \ln y$$

Find the Lindahl equilibrium for this economy.

Exercise 15.3 Consider an economy with two goods, a private (rivalrous) good x , say leisure, and a public (non-rivalrous) good y , say radio broadcast music. Both goods are measured in hours per day. There are two consumers, 1 and 2, and one firm. The firm produces y , using labor v as input. (Thus, if consumer i supplies v_i units of labor to the firm, then the amount of leisure left to i is $x_i = w_i - v_i$, where w_i is i 's initial endowment of leisure.) Let the production function of the firm be linear (constant returns to scale), with k units of v needed to produce one unit of y at any scale of output ($k > 0$). There is no free disposal. The initial endowments w_i of x are positive, but the initial endowment of y is zero.

Assume that each consumption set is $Z_i = \mathcal{R}_+^{L+K}$, and the utility function of consumer i is given by:

$$u_i = x_i + \phi_i(y), \tag{15.8.37}$$

where the valuation function $\phi_i(y)$ is twice differentiable, and has a positive derivative $\phi_i'(y) > 0$, and a negative second derivative $\phi_i''(y) < 0$ for all $y \geq 0$, for $i = 1, 2$. (Remember it is assumed $w_i > 0$ for $i = 1, 2$.)

Suppose that each consumer i voluntarily chooses to contribute an amount of labor $v_i \geq 0$ toward the production of the public good y , with $v_i < w_i$. By definition, at a Nash equilibrium allocation, written $(\bar{x}_1, \bar{x}_2, \bar{v}_1, \bar{v}_2, \bar{y})$,

each consumer i maximizes u_i , treating v_j as given (for $j \neq i$), and taking into account the equality

$$ky = v_1 + v_2. \quad (15.8.38)$$

Answer the following questions (a)-(e) and explain your answers to these questions as fully as possible.

1. Find the conditions that characterize Pareto efficient allocations. (These will be equations in x_1, x_2, y and the original endowments.)
2. Suppose that, at a Nash equilibrium, consumer 2 contributes a positive amount of labor, but is still left with positive amounts of leisure, i.e., $w_2 > \bar{v}_2 > 0$, while consumer 1 contributes nothing, i.e., $\bar{v}_1 = 0$. Could such an equilibrium be Pareto optimal?
3. Suppose that, at a Nash equilibrium, both consumers contribute positive amounts of labor, but are still left with positive amounts of leisure. Could such an equilibrium be Pareto optimal?
4. Suppose that, at a Nash equilibrium, neither consumer contributes any labor. Could such an equilibrium be Pareto optimal?
5. Define the Lindahl equilibrium and prove that every Lindahl equilibrium is Pareto efficient under local non-satiation.

Exercise 15.4 Consider a public economy with n consumers, one private goods x and one public goods y . Each consumer has a consumption space $Z_i = \mathcal{R}_+^2$, an endowment of private goods $w_i = 10$, and his preferences can be represented by $u_i(x_i, y) = x_i + \theta_i \ln y$. The production technology of public goods is $f(q) = q$, where q represents the input of private good in production.

1. Find the Pareto optimal allocation. How does this answer change with n ?
2. If each person contributes some of his endowment to produce public good, what is the Nash equilibrium of voluntary contributions? How does this answer change with n ?

3. If a government chooses to impose an individual tax of ϵ on each person to produce $n\epsilon$ units of public good, and each individual decides whether to make an extra contribution to public good, what is the total amount of public good provided in this way (Assume that ϵ is very small)?
4. If the government can only collect an income tax at the rate of τ , and all taxes are used to produce public goods, what is the tax rate that can guarantee the efficient provision of public goods? If consumers vote to determine the tax rate, what is the tax rate determined by the majority voting? What is the difference compared to the situation where all consumers have the same preference parameter θ ?

Exercise 15.5 (Public goods and group size) Consider a public good economy with n identical economic agents, one private good and one public good. Suppose that consumer i 's consumption space $Z_i = \mathcal{R}_+^2$ and the utility function is $u_i(x_i, y) = x_i + h(y)$, where x_i represents private good consumed by consumer i and y represents public good. Suppose that h is concave, differentiable, monotonically increasing, and satisfies $\lim_{y \rightarrow 0} h'(y) > 1$ and $\lim_{y \rightarrow \infty} h'(y) = 0$. Each agent has an endowment ω of private good, and the amount of endowment is sufficiently large such that the non-negativity constraint of private consumption is always not binding. The public goods production exhibits constant returns to scale. One unit of private goods can produce one unit of public goods, and only symmetric allocations are discussed.

1. Prove that the optimal provision level of public goods is an increasing function of n .
2. The level of public good provided under the voluntary contribution equilibrium is independent of the number of people n . And comment on this.

Exercise 15.6 Consider a public good economy with n identical economic agents, one private good and one public good. One way to resolve the free-rider problem proposed in the economics literature is to prize the contribution for public goods in the form of lottery: if the individual i contributes

z_i , then there is a probability of $\frac{z_i}{\sum_{j=1}^n z_j}$ to win lottery worthy R units of private good. Personal contributions are used both to raise funds for public good and to provide lottery bonuses, so only $\sum_{j=1}^n z_j - R$ of the contribution is put into public good production. Suppose that the lottery bonus R is independent of n . Prove that under the symmetric Nash equilibrium in which each individual contributes z (z is a function of n and R):

1. If $n > 1$, the provision level of public good $y = nz - R$ is always greater than the voluntary provision level in the previous question.
2. The provision level of public good $y = nz - R$ increases with n .
3. When $n \rightarrow \infty$, the provision level of public good approaches to a finite value.
4. Now suppose that when $n \rightarrow \infty$, the level of optimal public good also approaches to infinity. The conclusion in the above question 3 is disappointing. So assume that the total bonus increases with n : $R = nr$. Prove:
 - (a) The provision level of public good $y = n(z - r)$ increases with n .
 - (b) When $n \rightarrow \infty$, the public good level also approaches to infinity, and comment on this result.

Exercise 15.7 (Tragedy of the Commons) Suppose that there are n peasant households in a village and each household has the right to raise dairy cows in a public pasture. The number of dairy cows raised by the farmer i is x_i . The amount of milk a cow can produce depends on the number of cows \hat{x} grazing on the pasture. Assume that the income of the farmer i raising x_i cows is $x_i v(\hat{x})$. $v(\hat{x}) > 0$ when $\hat{x} < \hat{x}_0$, $v(\hat{x}) = 0$ when $\hat{x} > \hat{x}_0$, where $v(0) > 0$, $v' < 0$ and $v'' < 0$. The cost per cow is c , and assume that the cow can be perfectly segmented and that $v(0) > c$. Each household also decides how many cows to buy at the same time, and all the cows bought will graze on the public pasture.

1. Determine the game for peasant households.

2. Find the Nash equilibrium and compare it with the socially optimal result.

Exercise 15.8 Consider a public good economy with n individuals, one private good x and one public good y . The total endowment of private good is w , and public good can be produced from the private good with cost function $c(y)$. The utility function for consumer i is $u_i(x_i, y) = x_i + v_i(y)$, where v_i is an arbitrary function defined on \mathcal{R}_+ . Each consumer's consumption set is $\mathcal{R} \times \mathcal{R}_+$, and the consumption of public good is non-negative. The private good cannot be used for public consumption.

1. Suppose that destruction is cost-free, write down all inequalities that describe the feasible allocation (x_1, \dots, x_n, y) .
2. The public good consumption y is said to be "surplus maximizing" if the following condition is satisfied:

$$y \in \arg \max_{y' \geq 0} \sum_i v_i(y') - c(y').$$

Consider whether the proposition is correct:

"If y is the surplus maximizing public good consumption, then any feasible allocation that produces y units of public good must be Pareto optimal."

Prove your answer.

3. Consider whether the proposition is correct:

"If the allocation (x_1, \dots, x_n, y) is Pareto optimal, then y is surplus-maximizing."

Prove your answer.

Exercise 15.9 Suppose that there are n fishermen in a fishing village. Some fishermen are fishing in the sea. Since the sea is big enough, no matter how many fishermen go fishing, every fisher can catch k fish. There are some other fishermen who go fishing in a lake (The fish in the sea and the fish in

the lake are perfect substitutes). If x fishermen go fishing in the lake, then each fisherman can catch $x^{-1/2}$ fish (That is, these fishermen can catch $x^{1/2}$ fish in total, and every fisherman catches the same amount of fish).

1. For fishermen, it is cost-free to choose between fishing in the lake or in the sea, and no fishermen go where they believe they will catch less fish. How many fishermen will go fishing in the sea? How many fishermen will go fishing in the lake? How many fish will they catch on average?
2. If the government restricts fishing in the lake, how many fishermen should be allowed to go fishing in the lake so as to maximize the fishing capacity in this community?
3. If the demand function of the fish is assumed to be

$$Q = A - BP,$$

compare the price of fish in the market without restriction to that under efficient allocation.

4. Now suppose that the fish in the lake and the fish in the sea are not perfect substitutes. The price of marine fish is \$20 each, and the demand for fish in the lake is

$$Q_L = A' - B'P_L.$$

If there are no restrictions on fishermen, how many fishermen will go fishing in the lake at equilibrium? If the government collects a fixed license fee on fishermen who go fishing in the lake, will the price of fish in the lake rise or fall? Write the derivation process.

Exercise 15.10 Consider a public good economy with one private good x , one public good y , and n consumers whose consumption choice sets are non-negative in each dimension. Each consumer i owns w_i units of private good, and they do not own public good which can be produced with production function is $y = v$. The utility function for each consumer i is represented by $u_i(x_i, y)$, which may not be differentiable (note that it is then

not possible to answer the following questions 1 and 2 with differential methods).

1. Define Lindahl equilibrium and Pareto efficient allocation of the economy.
2. Prove that every Lindahl equilibrium allocation is Pareto efficient. (Hint: If you need additional assumptions, make it clear in the proof.)
3. Now suppose that u_i is differentiable. Then give the Lindahl-Samuelson first-order conditions for Pareto efficient allocation.
4. When the consumer's utility function is $u_i(x_i, y) = (x_i + 1)^{\alpha_i} y^{(1-\alpha_i)}$ and $0 < \alpha_i < 1$, find Lindahl equilibrium. Is it Pareto efficient?

Exercise 15.11 Suppose that n economic agents have the same Cobb-Douglas utility function $u_i(x_i, y) = x_i^\alpha y^{1-\alpha}$ and the consumption set $Z_i = \mathcal{R}_+^2$. The total amount of wealth is w , and they are divided among $k \leq n$ individuals. How many public goods are provided? How does the quantity of public goods change when k increases?

Exercise 15.12 An ancient village uses some goods (say sheep) for two purposes: either as food or as a public religious sacrifice. Suppose that villager i 's initial endowment of sheep is $w_i > 0$. Let $x_i \geq 0$ be the consumption of sheep, and $g_i \geq 0$ be the amount for public sacrifice. The total amount of sheep used for sacrifice is $y = \sum_{i=1}^n g_i$. The utility function for villager i is given by:

$$u_i(x_i, y) = x_i + a_i \ln y,$$

where $a_i > 1$.

1. When deciding on their sacrifice, each villager i regards that the sacrifice of other villagers remain fixed, and on this basis he decides the sacrifice he would offer. Let

$$y_{-i} = \sum_{j \neq i} g_j$$

be the sacrifice except villager i . Write out the utility-maximizing problem that determines the sacrifice of villager i .

2. Recall that for all individuals i , $y = g_i + y_{-i}$. What is the equilibrium amount of public good? (Hint: Not everyone will contribute positive public good.)
3. Who will be a free-rider in this problem?
4. In this economy, what is the Pareto efficient quantity of public good to be provided?

Exercise 15.13 A town has a population of 1,000, and each citizen's utility function is $u_i(x_i, y) = (x_i + k_i)y^\alpha$, where y is the size of the town's ice-skating rink measured in square meters, and x_i is the number of bread consumed each year. Suppose that the price of bread is 1 each, and the price of maintaining a square meter of skating rink is also 1. Different citizens have different income w_i . Find the Lindahl equilibrium for this town. Under the Lindahl equilibrium, how much should the government raise from resident i ?

Exercise 15.14 (The First Welfare Theorem of Lindahl Allocation with Transfers) Prove Theorem 15.4.3: given the public goods economy $e = (e_1, \dots, e_n, \{Y_j\})$, every Lindahl equilibrium allocation (x^*, y^*) with transfers and the price system $(q_i^*, \dots, q_n^*, p^*)$, is weakly Pareto efficient. If the consumers' preferences also satisfy local non-satiation, it is Pareto efficient.

Exercise 15.15 (The First Welfare Theorem of Welfare Economics with strictly convex preference) Suppose that \succsim_i are strictly convex. Let allocation $(x, y) \in X \times Y$ and non-zero price vector $(q_1, \dots, q_n, p) \in \mathcal{R}_+^{L+nK}$ constitute a Lindahl equilibrium. Prove the Lindahl equilibrium allocation is Pareto efficient.

Exercise 15.16 (Lindahl equilibrium, constrained Lindahl equilibrium, Lindahl quasi-equilibrium and Pareto optimality) For the public goods economy $e = (e_1, \dots, e_n, \{Y_j\})$, suppose that for all i , $0 \neq w_i \in X_i = \mathcal{R}_+^L$ and \succsim_i is preference ordering. Allocation $(x, y) \in X \times Y$ and non-zero price vector $(q_1, \dots, q_n, p) \in \mathcal{R}_+^{L+nK}$ constitute a **constrained Lindahl equilibrium**, if the other conditions in the definition remain the same, except that (ii) is replaced by

(ii') $(x_i, y) \succsim_i (x_i^*, y^*)$ and $x_i + y \leq \sum_{t=1}^n w_t$ implies $p^* x_i + q_i^* y > p^* w_i, \forall i = 1, \dots, n$.

Allocation $(x, y) \in X \times Y$ and non-zero price vector $(q_1, \dots, q_n, p) \in \mathcal{R}_+^{L+nK}$ constitute a Lindahl quasi-equilibrium, if the other conditions in the definition remain the same, except that (ii) is replaced by

(ii'') $(x_i, y) \succsim_i (x_i^*, y^*)$ implies $p^* x_i + q_i^* y \geq p^* w_i, \forall i = 1, \dots, n$.

1. We know that if preferences \succsim_i satisfy local non-satiation, every Lindahl equilibrium allocation is Pareto optimal. What if the local non-satiation is not satisfied?
2. Prove: If \succsim_i satisfies convexity, then the interior-point constrained Lindahl equilibrium is a Lindahl equilibrium. Can the convexity be relaxed to local non-satiation?
3. Prove: If \succsim_i satisfies strong monotonicity, then Lindahl equilibrium is Lindahl quasi-equilibrium. Can strong monotonicity be relaxed to monotonicity?
4. Prove: If a Lindahl equilibrium allocation is a Lindahl quasi-equilibrium allocation, it is Pareto efficient.
5. From the previous question, if \succsim_i are strictly convex, the Lindahl equilibrium allocation is Pareto optimal. Then, if \succsim_i is strictly convex, is Lindahl equilibrium necessarily a Lindahl quasi-equilibrium?
6. Suppose that \succsim_i satisfies continuity for every individual i . Prove: If $p \in \mathcal{R}_{++}^L$, the Lindahl quasi-equilibrium is a Lindahl equilibrium.
7. Suppose that for any individual i , \succsim_i satisfies continuity and strong monotonicity. Prove: If (p, x) is a Lindahl quasi-equilibrium and $x_i \in \text{int } \mathcal{R}_+^L$ for some i , then $p \in \mathcal{R}_{++}^L$.

Exercise 15.17 (Economic core theorem in public economy) Prove Theorem 15.4.4: Under the local non-satiation of preferences, if (x, y, p) is a Lindahl equilibrium, then (x, y) is in the core.

Exercise 15.18 (The Second Theorem of Welfare Economics in public goods economy with non-satiated preferences) Prove the theorem: for given public goods economy $e = (e_1, \dots, e_n, \{Y_j\})$, suppose that preferences \succsim_i are continuous, convex, and non-satiated. Y is a closed convex set and $0 \in Y$. Then, for any Pareto optimal allocation (x^*, y^*) , there exists non-zero price vector $(q_1, \dots, q_n, p) \in \mathcal{R}^{L+nK}$ such that $((x, y), (q_1, \dots, q_n), p)$ is a Lindahl quasi-equilibrium with transfers. That is, there is an assignment of wealth levels (I_1, \dots, I_n) with $\sum_i I_i = p \sum_i w_i$ such that

- (1) if $(x_i, y) \succsim_i (x_i^*, y^*)$, then $px_i + q_i y \geq I_i \equiv px_i^* + q_i y^*$,
 $i = 1, \dots, n$;
- (2) for all $(y, -v) \in Y$, we have $\hat{q}y^* - pv^* \geq \hat{q}y - pv$,

where $v^* = \sum_{i=1}^n w_i - \sum_{i=1}^n x_i^*$, $\sum_{i=1}^n q_i = \hat{q}$.

Furthermore, if for all i , $0 \in X_i$ and $px_i^* + q_i y^* > 0$, then (x^*, y^*, p) is a Lindahl equilibrium with transfers.

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