Definition: The stochastic process $\{(X_n,T_n)\}$ is called a *Markov renewal process* with state space E if

$$P\{X_{n+1} = j, T_{n+1} - T_n \leq t \mid X_0, \ldots, X_n; T_0, \ldots, T_n\} = P\{ X_{n+1} = j, T_{n+1} - T_n \leq t \mid X_n\}$$

for all n=0, 1, …, $j \in E$, and $t \geq 0$.

We will always assume (1) the process is time homogeneous and (2) E is discrete.

Definition: The family of probabilities $Q = \{Q(i,j,t): i,j \in E, \text{ and } t \geq 0\}$ is called a *semi-Markov kernel* and is defined by

$$P\{ X_{n+1} = j, T_{n+1} - T_n \leq t \mid X_n = i\} = Q(i, j, t).$$

Definition: The process $\{Y(t)\}$ defined by

$$Y(t) = X_n \text{ for } T_n \leq t < T_{n+1} \text{ and } Y(t) = \Delta \text{ if } t \geq \sup \{T_n : n=0, 1, \ldots \},$$

is called a *semi-Markov process*, where Δ is a state not in E.

Two properties of a Markov renewal process:

1. {Xn} is a Markov chain.
2. $P\{ T_{n+1} - T_n \leq t \mid X_0, X_1, \dots ; T_0, \dots , T_n\} = P\{ T_{n+1} - T_n \leq t \mid X_n, X_{n+1}\}$

Counters of Type I: Arrivals to a particle counter form a Poisson process with rate λ. An arriving particle which finds the counter free gets registered and locks it for a random duration with distribution function ψ. Arrivals during a locked period have no effect. Define State 0 to be the state when the counter is unlocked and let State 1 be when the counter is locked. Let T_0=0, T_1, T_2, etc. be the successive instants of changes in the state of the counter and let X_n be the state immediately after T_n. Then $\{(X_n, T_n)\}$ is a Markov renewal process with state space E= {0, 1}. The semi-Markov process {Y(t)} associated with $\{(X_n, T_n)\}$ represents the state of the counter at time t. The semi-Markov kernel for this process is relatively simple.

M/G/1 Queueing System. An M/G/1 system represents a single-server queueing system with a Poisson arrival process with rate λ and independent service times with the common distribution φ. Let $T_0=0$, T_1, T_2, etc. be the successive instants of departures, and let Xn be the number of customers left behind by the nth departure. Then $\{(X_n, T_n)\}$ is a Markov renewal process with state space E= $\{0, 1, \ldots\}$. The semi-Markov kernel for this process is more complex than the previous example, but we do have $Q(i,j,t) = 0$ for $i - j \geq 2$ (in other words, the lower left portion of the kernel is zero exc).

G/M/1 Queueing System. A G/M/1 system represents a single-server queueing system with the arrival process being a renewal process with φ being the distribution of inter-arrival times and independent service times governed by an exponential distribution with mean rate μ. Let $T_0=0$, T_1, T_2, etc. be the successive instants of arrivals, and let X_n be the number of customers just **before** the n^{th} arrival. Then $\{(X_n, T_n)\}$ is a Markov renewal process with state space E= $\{0, 1, \ldots\}$. The semi-Markov kernel for this process is more complex than the previous example, but we do have $Q(i,j,t) = 0$ for $j - i \geq 2$.

Homework #15

Due by 7AM, Monday, April 6

***Instructions*:** Do your work on your own paper and give only the numerical answers in eCampus. Give your answers rounded to **three digits to the right of the decimal**.

1. Let {Xn,Tn} be a Markov renewal process with state space {a, b} and semi-Markov kernel Q given as

$$Q(t) = \begin{matrix} 0.6(1 - e^{-5t}) & 0.4 - 0.4e^{-2t} \\ 0.5 - 0.2e^{-3t} - 0.3e^{-5t} & 0.5 - 0.5e^{-2t} - te^{-2t} \end{matrix}$$

where t represents *days*.

a. If the process is currently in state a, what is the probability that the next jump will be back to itself (i.e., state a)?
b. If the process is currently in state b, what is the probability that the next jump will be to state a?
c. Given that the process has just made a jump to state a, what is the probability that the next jump will occur within six hours given that the next jump will be a return to state a?
d. Given that the process has just made a jump to state a, what is the probability that the next jump will occur within six hours given that the next jump will be to state b?

Given that the process starts in state a, then next moves to state b, and then back to state a, what is the probability that both initial sojourn times in state a and state b were less than six hours?