Distributional Robust Optimization theorem and application

Abstract— Distributional Robust Optimization(DRO) is a method combining stochastic programming(SP) and robust optimization(RO). Since RO can't get feasible solution of worst case in some situation and SP need to know exactly distribution function, DRO is introduced to solve the above problem. In this work, the Theorem related to DRO is shown and simple application by using DRO is presented.

Keywords— Distributional Robust Optimization, Robust optimization

1 Introduction

In decision making problems, " quite small(just 0.1%) pertubations of 'obviously uncertain' data coefficients can make the 'nominal' optimal solution x^* heavily infeasible and thus practically meaningless". For example, in a real simple production problem, the solution is quite sensitive to the choice of parameters that only 2% error in the estimation of the conversion can results in 22% drop for profit which make the solution meaning less. Thus, the method RO comes out to overcome the dependence on parameters, which mean to give an uncertainty set for parameters. The following picture shows out the popularity of "robust optimization" in the decision making problems.

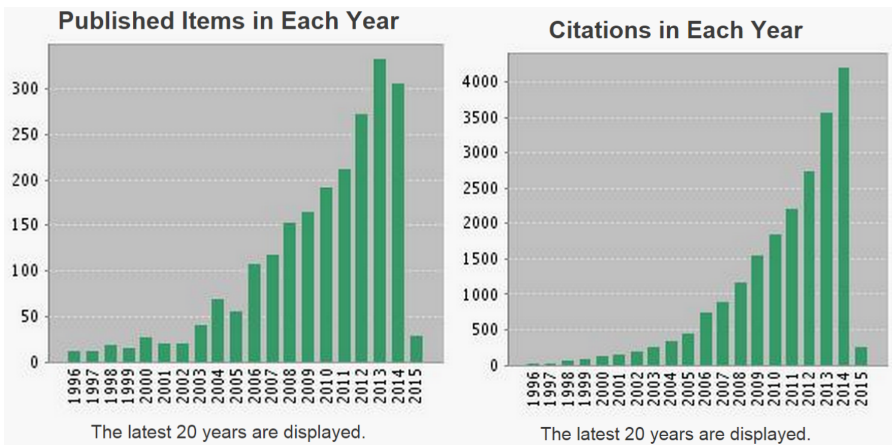

Figure 1: Rise in popularity of "robust optimization" in the scientific literature

From the picture, we can see that, in order to solve decision making problems with large data sets, RO method is a more useful and popular choice in the recent years, especially when the number of values or parameters is large enough (e.g. semi-infinite dimension problem whose number of decision variables is infinite and of constraints is finite).

The traditional RO method can be known as Static Robust Optimization(SRO). It can be applied to inventory and logistics problem, finance area, revenue management area, queneing network, machine learning, energy systems and public good area. According to purpose of problems, SRO can be divided into 3 situation. (1) solution is feasibility for uncertain parameters; (2) guarantee objective value; (3) guarantee distance to optimality.

In the first situation, 'uncertainty' can be understood into two ways: (1) uncertainty on the feasibility of the solution; (2) uncertainty on its objective value. Since 'uncertainty' set is smaller than 'realistic' set, SRO is realizated in most cases. It can ensuring computational tractability and limiting deterioration of the objective at optimality. However, when it comes to worst-case, way (1) may get infeasible solution and way(2) may get sub-optimality solution. Namely

SRO is still not enough realization and is over-conservatism. In recent years, people focuse large on worst case.

In 2008, Averbakh and Zhao construct the uncertainty set by a system of convex inequatlities. In 2009, Fischetti and Monaci shows a method, Light Robustness namely add simplified 'SP' to SRO, to improve the flexibility for worst case. In 2011 Ben-Tal and Den Hertog proofs that robust convex quadratic constraint with ellipsoidal implementtion error can equal to conic quadratic constraints. In 2012, Sniedovich proofs that local robustness doesn't equal to global robustness and Nemirovski shows sate tractable approximations. It's all the way to DRO.

DRO comes from RO, used directly for SP model. Namely, based on traditional SP model, DRO introduce RO method with an uncertaity set of distributional functions.

The rest of the paper is organized as follows. Difference among SP, SRO and DRO is showned in Section 2. Basic theorem related to DRO is presented in Section 3. Some simple cases are given in section 4.

2 Comparison among SP, SRO and DRO

In this section, some basic definitions related are shown .

2.1 SRO

SRO is a traditional method for signal processing problem. In this section, the basic definition of SRO is shown.

$$\begin{array}{lll} \min_{x\in X} \max_{\epsilon\in D} h(x,\epsilon) \quad or & \min_{x\in X} & h_1(x) \\ & s.t. & h_2(x,\epsilon) \le 0, \forall \epsilon \in D \end{array} \tag{1}$$

Note:

- D includes any feasible parameter.
- D,h are not related to probability.

In this form, we can know without lots of generalizations that we can always choose one situation for parameter to be the worst case. In the case, the solution may act infeasible in real life. Here comes the cons that SRO is too conservation.

2.2 SP

SP model is also known as "stochastic programming" model. The definition is as follow:

$$\min_{x\in X} \mathbb{E}_{F_\epsilon}[h(x,\epsilon)] \tag{2}$$

Here, F_ϵ is a fixed distribution. In most cases we can't know exactly the distribution F_ϵ and it may result in sub-optimal or meaningless solution.

Since SP problem need to decide an infinite time's activity which satisfies the decision variable is infinite, RO method can be used on SP problem.

2.3 DRO

DRO is a special case of RO which is just used for SP problem. It overcomes the conservation in RO problems and the dependence of fixed distribution in SP problems. The definition for DRO is shown as follow:

$$\min_{x \in X} \max_{F_\epsilon \in D} \mathbb{E}_{F_\epsilon}[h(x, \epsilon)] \tag{3}$$

Note:

- Here, $D, \mathbb{E}_{F_\epsilon}[h(x, \epsilon)]$ are related to probability.
- D is the uncertainty set of distributions, which only includes part of distributionally functions. For example, $D := \{p : dist|p - p_{ref}| \leq \sigma\}$. Here, p_{ref} is based on historical data choosing from basic known distributions(e.g. Guassian). Wasserstein metric is used for probability distance. σ is choosen to suit well for some percentage interval.

3 Theorem to solve DRO problem

In this section, we focus on the following problem:

$$\min_{x \in X} \sup_{F \in D} \mathbb{E}_F[h(x, \epsilon)] \tag{4}$$

Here ϵ is the random vector drawn from a distribution F.

There are mainly three kinds of different ways to construct uncertainty set D: moment based models, Scenario-based models and Wasserstein distance based models. We will talk more about the first way in the following section.

3.1 Moment based models

Moment based models are mostly used method to construct the uncertainty set. Here, we assume that random variable ϵ has a continuous support and only a number of moments are known for the distribution F. We divided moment based models into 3 kinds of different situations: mean and support models, mean and variance models, moment uncertainty models.

3.1.1 mean and support models

In this method, we only used the support and mean value to construct the set D. D is shown as follow:

$$D(Z, \mu) := \left\{ F \in M \;\middle|\; \begin{array}{l} \mathbb{P}(\epsilon \in Z) = 1 \\ \mathbb{E}[\epsilon] = \mu \end{array} \right\}$$

Here, M is the set of all probability measures on the measurable space $(\mathbb{R}^m, B)$, B is the Borel $\sigma-$algebra on $\mathbb{R}^m$, and Z is a Borel set in $\mathbb{R}^m$.

Then we can rewrite the part $\sup_{F \in D} \mathbb{E}_F(h(x,\epsilon))$ of equation (4) in the following form:

$$\begin{array}{cl} \max_{F\in M} & \int_Z h(x,\epsilon)dF(\epsilon) \\ s.t. & \int_Z dF(\epsilon) = 1 \\ & \int_Z \epsilon dF(\epsilon) = \mu \end{array} \tag{5}$$

Here, we further assume that $h(x,.)$ is real-valued measurable in $(\mathbb{R}^m, B)$.

Then we have the following theorem to solve the equation (4).

Theorem 3.1. *Let $D(Z,\mu)$ be a distribution set for which there exists a feasible solution $F_0 \in D(Z,\mu)$, then the moment problem (5) is equivalent to the following robust optimization problem:*

$$\min_q \sup_{z\in Z} h(x,z) + (\mu - z)^T q \tag{6}$$

In this theorem, Z is arbitary Borel set in $\mathbb{R}^m$. If we simplify and limit Z to be points in $\mathbb{R}^m$, we will have the following therorem 3.2.

proof. *Here, we only show the basic idea to proof the theorem 3.1. We mainly have 4 steps.*

Step 1: *Construct Lagrangean equation*

Here, we can instance formulate the Lagrangean equation for the moment problem(5) and the equation is as follow:

$$\begin{array}{rcl} L(F,r,q) & = & \int_Z h(x,\epsilon)dF(\epsilon) + r(1 - \int_Z dF(\epsilon)) + q^T(\mu - \int_Z \epsilon dF(\epsilon)) \\ & = & r + \mu^T q + \int_Z (h(x,\epsilon) - r - q^T\epsilon)dF(\epsilon) \end{array}$$

Step 2: *Analysis properties of L*

By dual theorem, we can equal (5) with $\sup_F \inf_{r,q} L(F,r,q)$. Then we have the following inequality:

$$\sup_R \inf_{r,q} L(F,r,q) \leq \inf_{r,q} \sup_F L(F,r,q) = \inf_{r,q} \begin{cases} r + \mu^T q & if\ h(x,z) - r - q^T z \leq 0, \forall z \in Z \\ \infty & otherwise \end{cases}$$

Here, we have the assumption that $D(Z,\mu) \neq \emptyset$

Step 3: *Translate the right part into program*

Rewrite the right part in Step 2, we can get the following program, i.e. if $\exists F \in D(Z,\mu)$, then we have:

$$\begin{array}{cl} \min_{r,q} & \mu^T q + r \\ s.t. & z^T + r \geq h(x,z), \forall z \in Z, \end{array}$$

where $r \in \mathbb{R}, q \in \mathbb{R}^m$ are the dual variables associated with constraints (5).

Step 4: *Translate into (6)*

By dual theorem, we can assume that the optimal satisfy equation:

$$z^T + r = h(x,z), \forall z \in Z.$$

Replace r in the object function, we have the following problem:

$$\min_q \sup_{z\in Z} h(x,z) + (\mu - z)^T q$$

Theorem 3.2. *Let $Z \in \mathbb{R}^m$ be a Borel set, and F_0 be some feasible distribution according to $D(Z,\mu)$, then problem (5) is equivalent to the following nite dimensional optimization problem*

$$\begin{array}{cl} \max_{p,\{z_i\}_{i=1}^{m+1}} & \sum_{i=1}^{m+1} p_i h(x,z_i) \\ s.t. & \sum_{i=1}^{m+1} p_i = 1 \& p \geq 0 \\ & \sum_{i=1}^{m+1} p_i z_i = \mu \\ & z_i \in Z, \forall i = 1,...,m+1, \end{array} \tag{7}$$

where $p \in \mathbb{R}^{m+1}$ and each $z_i \in \mathbb{R}^m$.

Moreover,if we simplify Z to be a convex set, the following theorem 3.3 is shown:

Theorem 3.3. *When Z is a convex set and $h(x,z) := \max_{k=1,...,K} h_k(x,z)$ for some K with each $h_k(x,z)$ a concave function of z, then problem (5) is equivalent to*

$$\begin{array}{cl} \max_{p,\{z_k\}_{k=1}^{K}} & \sum_{k=1}^{K} p_k h_k(x,z_k) \\ s.t. & \sum_{k=1}^{K} p_k = 1, p \geq 0 \\ & \sum_{k=1}^{K} p_k z_k = \mu \\ & z_k \in Z, \forall k = 1,...,K. \end{array} \tag{8}$$

From theorem 3.3, we can get the following corollary:

Corollary. *When Z is a convex set and h(x,z) is a concave function of z, then the DRO problem presented in (4) is equivalent to*

$$\min_{x\in X} h(x,\mu) \tag{9}$$

Then we get the following result for problem(4):

Theorem 3.4. *Let $D(Z,\mu)$ be a distribution set for which there exists a feasible solution $F_0 \in D(Z,\mu)$, the DRO problem presented in (4) is equivalent to the following robust optimization problem:*

$$\min_{x\in X,q} \sup_{z\in Z} h(x,z) + (\mu - z)^T q \tag{10}$$

Moreover, the problem can be reformulated as follows when Z is a convex set and $h(x,z) := \max_k h_k(x,z)$ where each $h_k(x,z)$ is a concave function of z:

$$\begin{array}{cl} \min_{x,q,\{v_k\}_k,t} & t \\ s.t. & t \geq \delta^*(v_k|Z) + \mu^T q - h_*^k(x, v_k + q), \forall k \end{array} \tag{11}$$

where for each k, $v_k \in \mathbb{R}^m$, while $\delta^*(v|Z)$ is the support function of Z and $h_*^k(x,v)$ is the partial concave conjugate function of $h_k(x,z)$.

Finally, we can solve the problem (3) by using dual theorem in the equivalent form equation (11).

3.1.2 mean and variance models

In this method, assume we have know the mean and variance value of the distribution function, then we can have the following uncertain set:

$$D(\mu, \sigma^2) := \left\{ F \in M \middle| \begin{array}{l} \mathbb{P}(\epsilon \in \mathbb{R}) = 1 \\ \mathbb{E}[\epsilon] = \mu \\ \mathbb{E}[(\epsilon - \mu)^2] = \sigma^2 \end{array} \right\}$$

By assuming that $Z' := \{z' \in \mathbb{R}^2 | z'_2 = (z'_1 - \mu)^2\}$, we can translate the mean and variance models into the mean and support models. Namely the following set are equivalent:

$$D(\mu, \sigma) = D(Z', [\mu, \sigma^2]^T)$$

Then by the above theorem 3.4, we can change DRO problem (4) into the following non-linear robust optimization model:

$$\begin{array}{ll} \min_{x \in X, q_1, q_2 \geq 0, t} & t \\ s.t. & t \geq \sup_{z_1 \in \mathbb{R}} h_k(x, z) + \mu q_1 + (\sigma^2 - \mu^2) q_2 - (q_1 - 2q_2\mu) z_1 - q_2 z_1^2, \forall k, \end{array} \tag{12}$$

when $h(x, z) := \max_k h_k(x, z)$

In higher dimension space, uncertain set is shown as follow:

$$D(Z, \mu, \Sigma) := \left\{ F \middle| \begin{array}{l} \mathbb{P}(\epsilon \in Z) = 1 \\ \mathbb{E}[\epsilon] = \mu \\ \mathbb{E}[\epsilon\epsilon^T] \leq \Sigma \end{array} \right\}$$

Here, $\mathbb{E}[(\epsilon\epsilon^T] \leq \Sigma$ mean $\Sigma - \mathbb{E}[(\epsilon\epsilon^T]$ is positive semi-definite. Similarly, suppose that:

$$Z' := \{(z_1, Z_2) \in \mathbb{R}^m \times S^{m \times m} | z_1 \in Z, Z_2 \geq z_1 z_1^T\}. \tag{13}$$

Here $S^{m \times m}$ is the space of all $m \times m$ symmetric matrices. Similarly, we can reduce equation (4) to the following problem with $h(x, z) := \max_k h_k(x, z), Z_2 = z_1 z_1^T$

$$\begin{array}{ll} \min_{x \in X, q, Q, r} & r \\ s.t. & r \geq \sup_{z1 \in Z} h_k(x, z_1) + (\mu - z_1)^T q + \Sigma Q - z_1^T Q z_1, \forall k \\ & Q \geq 0 \end{array}$$

3.1.3 moment uncertainty models

When our moments come from historical data, the chosen for moments shouldn't be precisely known but lie in some confidence region U instead. Here, we change our model(3) to be the following one:

$$\min_{x \in X} \sup_{\mu \in U, F \in D(Z, \mu)} \mathbb{E}_F[h(x, z)] \tag{14}$$

To solve the above problem, we need the following corollary:

Corollary. *Let $D(Z, \mu)$ be a distribution set and $U \in \mathbb{R}^m$ be a bounded and convex uncertainty set for the moment vector μ. Given that for all $\mu \in U$, there exists an $F \in D(Z, \mu)$, the DRO problem presented in (14) is equivalent to the following robust optimization problem:*

$$\min_{x \in X, q} \sup_{z \in Z} h(x, z) - z^T q + \delta^*(q|U) \tag{15}$$

Moreover, the problem can be reformulated as follows when Z is a convex set and $h(x, z) := \max_k h_k(x, z)$ where each $h_k(x, z)$ is a concave function:

$$\begin{array}{ll} \min_{x \in X, q, \{v_k\}_k, t} & t + \delta^*(q|U) \\ s.t. & t \geq \delta^*(v_k|Z) - h^k_*(x, v_k + q), \forall k \end{array} \tag{16}$$

where for each k, $v_k \in \mathbb{R}^m$, while $\delta^*(v|Z)$ is the support function of Z and $h^k_*(x, v)$ is the partial concave conjugate function of $h_k(x, z)$.

3.2 Scenario-based models

This approach starts with a set of scenarios $Z := \{z^1, z^2, ..., z^K\}$ and has the following DRO form:

$$\min_{x \in X} \sup_{p \in U} \sum_{k=1}^{K} p_k h(x, z^k),$$

where $p \in \mathbb{R}^K$ is a vector describing the probability of obtaining each of the K scenarios for ϵ while $U \subseteq \{p \in \mathbb{R}^K | p \geq 0, \sum_{k=1}^K p_k = 1\}$ is the uncertainty set for the distribution, which can also be calibrated using historical data.

3.3 Wassertein distance based models

In [35], the authors propose a general data-driven DRO formulation that can achieve the three most valued properties of such models:

- **Finite sample guarantee:** The property that the optimal value of the DRO model is guaranteed with high probability to bound from above the expected cost when a finte number of i.i.d. realizations have been observed.
- **Consistency:** The property that the optimal solution will eventually converge to the optimal solution of the stocastic program(2) as more i.i.d. realizations are used to construct the distribution set D.
- **Tractability:** The DRO model can be solved using convex optimization algorithms for a large class of problems.

4 Application

Here we show two simple examples of Distributionally Robust Optimization

4.1 Example 1

Suppose that there is one random variable $\bar{z}$, and the adjustable decision made after the observation of uncertainty realization z is denoted by $y(z)$. This distributionally robust optimization model is expressed by the following problem.

$$\begin{array}{lll} \min & \max_{\mathbb{P}\in\mathbb{F}} \mathbb{E}_{\mathbb{P}}[y(\bar{z})] & \\ s.t. & y(z) \geq z, & \forall z \in Z \\ & y(z) \geq -z, & \forall z \in Z \end{array}$$

where $\mathbb{P}$ is the distribution of random variable $\bar{z}$, Z is the support set of $\bar{z}$, which is set to be $[-2, 2]$, and $\mathbb{F}$ is the ambiguity set that characterizes a collection of distributions, which is expressed in the equation below.

$$\mathbb{F} = \left\{ \mathbb{P} \in P_0(\mathbb{R}) : \begin{array}{l} \bar{z} \in \mathbb{R} \\ \mathbb{E}_{\mathbb{P}}(\bar{z}) = 0 \\ \mathbb{E}_{\mathbb{P}}(\bar{z}^2) \leq 1 \\ \mathbb{P}\{\bar{z} \in\} = \mathbb{P}\{-2 \leq \bar{Z} \leq 2\} = 1 \end{array} \right\}$$

For this simple case, it is apparent that the optimal adustable decision follows $y(z) = |z|$, and the objective value is 1. However, for a general problem, such adjustable decisions are intractable to identify, so the linear decision rule is applied to approximate the actual adjustable decisions by linear affine functions of random variable $\bar{z}$ and some auxiliary variables. The ambiguity set is extended to the following lifted form by introducing auxiliary variable $\bar{u}$ into the set.

$$\mathbb{G} = \left\{ \mathbb{Q} \in P_0(\mathbb{R}^2) : \begin{array}{l} \bar{z} \in \mathbb{R}, \bar{u} \in \mathbb{R} \\ \mathbb{E}_{\mathbb{Q}}(\bar{z}) = 0 \\ \mathbb{E}_{\mathbb{Q}}(\bar{u}) \leq 1 \\ \mathbb{Q} \left\{ \begin{array}{c} -2 \leq \bar{z} \leq 2 \\ \bar{z}^2 \leq \bar{u} \leq 4 \end{array} \right\} = 1 \end{array} \right\}$$

The decision rule is a sunction of uncertainty z and the auxiliary variable u, expressed as $\bar{y}(z, u)$ in the equation below.

$$\bar{y}(z, u) = y^0 + y^z z + y^u u$$

After replacing the recourse decision by the linear decision rule, we can transform the distributionally robust model into a tractable robust counterpart, ad solves it by the external solver CPLEX. Details f defining the extended ambiguity sets and generalized decision rule are explained in the following code. The solution suggests that the optimal decision rule is $\bar{y} = \frac{1+z^2}{2}$. The actual optimal recourse decision and the optimal decision rule, together with the worst-case distribution, are displayed by the figure below.

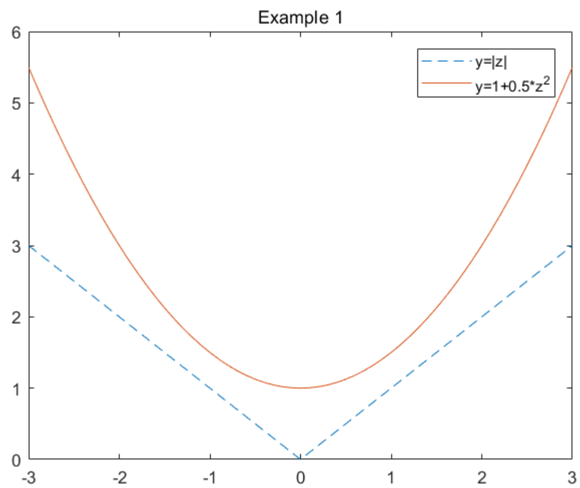

Figure 2: Example 1

It can be seen that although $\bar{y}(z,u)$ is a conservative approximation, it attaches to the optimal recourse decision as the probability is nonzero, so the expected value of the objective $\bar{y}(z,u)$ is still the same as 1.

4.2 Example 2

Now consider the case with two confidence sets incorporated into the ambiguity set in order to better capture the ambiuous information of distribution. The updated ambiguity set is expressed as follows.

$$\mathbb{F} = \left\{ \mathbb{P} \in P_0(\mathbb{R}) : \begin{array}{l} \bar{z} \in \mathbb{R} \\ \mathbb{E}_{\mathbb{P}}(\bar{z}) = 0 \\ \mathbb{E}_{\mathbb{P}}(\bar{z}^2) \leq 1 \\ \mathbb{P}\{\bar{z} \in Z\} = \mathbb{P}\{-2 \leq \bar{z} \leq 2\} = 1 \\ \mathbb{P}\{\bar{z} \in Z_1\} = \mathbb{P}\{-1 \leq \bar{z} \leq 1\} = 0.9 \\ \mathbb{P}\{\bar{z} \in Z_2\} = \mathbb{P}\{-0.5 \leq \bar{z} \leq 0.5\} \in [0.6, 0.7] \end{array} \right\}$$

Similarly the extended ambiguity set an be formulated as follows:

$$\mathbb{G} = \left\{ \mathbb{Q} \in P_0(\mathbb{R}^2) : \begin{array}{l} \bar{z} \in \mathbb{R}, \bar{u} \in \mathbb{R} \\ \mathbb{E}_{\mathbb{Q}}(\bar{z}) = 0 \\ \mathbb{E}_{\mathbb{Q}}(\bar{u}) \leq 1 \\ \mathbb{Q}\left\{ \begin{array}{c} -2 \leq \bar{z} \leq 2 \\ \bar{z}^2 \leq \bar{u} \leq 4 \end{array} \right\} = 1 \\ \mathbb{Q}\left\{ \begin{array}{c} -1 \leq \bar{z} \leq 1 \\ \bar{z}^2 \leq \bar{u} \leq 1 \end{array} \right\} = 0.9 \\ \mathbb{Q}\left\{ \begin{array}{c} -0.5 \leq \bar{z} \leq 0.5 \\ \bar{z}^2 \leq \bar{u} \leq 0.25 \end{array} \right\} \in [0.6, 0.7] \end{array} \right\}$$

The updated objective value becomes 0.9920. The optimal recourse decision, and the decision rule, as well as the worst-case distribution, are depicted by the figure below.

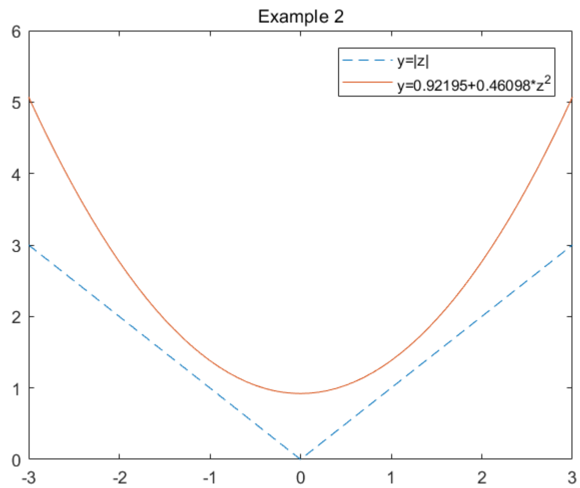

Figure 3: Example 2

Finally, we can show the results of example 1 and example 2 into the same plot.

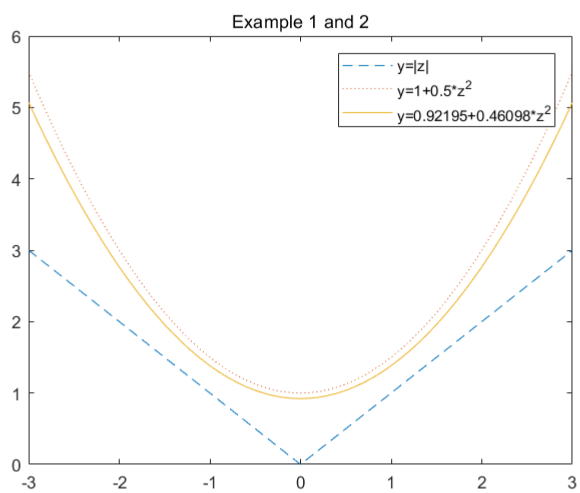

Figure 4: Caption

5 Appendix

5.1 proof for theorem

5.2 code in application

http://xprog.weebly.com/examp6.html

```
model=xprog('simple_dro');                    % create a model ...
    named 'simple_dro'

y=model.recourse(1,1);                        % define decision ...
    rule y for the adjustable decision

z=model.random(1);                            % define random ...
    variable z
u=model.random(1);                            % define auxiliary ...
    variable u

y.depend(z);                                  % define ...
    dependency of decision rule y on z
y.depend(u);                                  % define ...
    dependency of decision rule y on u

model.uncertain(expect(z)==0);                % 2nd line of set ...
    G: mean value of z
model.uncertain(expect(u)≤1);                % 3rd line of set ...
    G: variance of z
model.uncertain(z≤ 2);                       % 4th line of set ...
    G: support set of z
model.uncertain(z≥-2);                       % 4th line of set ...
    G: support set of z

model.uncertain(z.^2≤u);                     % 4th line of set ...
    G: u is the upper bound of z^2
model.uncertain(u≤2^2);                      % 4th line of set ...
    G: the upper limit of u is 2^2

model.min(expect(y));                         % objective ...
    function is the expected value of y

model.add(y≥z);                              % 1st constraint
model.add(y≥-z);                             % 2nd constraint

model.solve;                                  % solve the problem
```

```
model=xprog('simple_dro');                    % create a model ...
    named 'simple_dro'

y=model.recourse(1,1);                        % define decision ...
    rule y for the adjustable decision y

z=model.random(1);                            % define random ...
    variable z
```

```
u=model.random(1);                              % define auxiliary ...
    variable u

y.depend(z);                                    % define ...
    dependency of y on z
y.depend(u);                                    % define ...
    dependency of y on u

model.uncertain(expect(z)==0);                  % 2nd line of set ...
    G: mean value of z
model.uncertain(expect(u)≤1);                  % 3rd line of set ...
    G: variance of z

model.uncertain(z≤ 2);                         % 4th line of set ...
    G: support set Z
model.uncertain(z≥-2);                         % 4th line of set ...
    G: support set Z
model.uncertain(z.^2≤u);                       % 4th line of set ...
    G: u is the upper bound of z^2
model.uncertain(u≤2^2);                        % 4th line of set ...
    G: upper limit of u

Z1=model.subset(0.9);                           % define Z1 to be ...
    a proper subset of support set, P{Z1} is 0.9
model.uncertain(z≤ 1,Z1);                      % 5th line of set ...
    G: confidence set Z1
model.uncertain(z≥-1,Z1);                      % 5th line of set ...
    G: confidence set Z1
model.uncertain(z.^2≤u,Z1);                    % 5th line of set ...
    G: u is the upper bound of z^2
model.uncertain(u≤1^2,Z1);                     % 5th line of set ...
    G: upper limit of u

Z2=model.subset([0.6 0.7],Z1);                  % define Z2 to be ...
    a proper subset of Z1, P{Z2} is between 0.6 and 0.7
model.uncertain(z≤ 0.5,Z2);                    % 6th line of set ...
    G: confidence set Z2
model.uncertain(z≥-0.5,Z2);                    % 6th iine of set ...
    G: confidence set Z2
model.uncertain(z.^2≤u,Z2);                    % 6th line of set ...
    G: u is the upper bound of z^2
model.uncertain(u≤0.5^2,Z2);                   % 6th line of set ...
    G: upper limit of u

model.min(expect(y));                           % the objective ...
    function is the expected value of y

model.add(y≥z);                                % 1st constraint
model.add(y≥-z);                               % 2nd constraint

model.solve;                                    % solve the problem
```

```
example_4_2_1_simple_dro_1
p1=model.Solution.Decision(1:2);
xyz=-3:0.1:3;
figure;
```

```
plot(xyz,abs(xyz),'--',xyz,p1(1)+p1(2)*xyz.^2);
legend('y=|z|',['y=',num2str(p1(1)),'+',num2str(p1(2)),'*z^2']);
title('Example 1');


example_4_2_1_simple_dro_2
p2=model.Solution.Decision(1:2);
figure;
plot(xyz,abs(xyz),'--',xyz,p2(1)+p2(2)*xyz.^2);
legend('y=|z|',['y=',num2str(p2(1)),'+',num2str(p2(2)),'*z^2']);
title('Example 2');

figure;
plot(xyz,abs(xyz),'--',xyz,p1(1)+p1(2)*xyz.^2,':',xyz,p2(1)+p2(2)*xyz.^2);
legend('y=|z|',['y=',num2str(p1(1)),'+',num2str(p1(2)),'*z^2'],['y=',num2str(p2(1)),'+',num2str(p
title('Example 1 and 2');
```