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Distributionally Robust Optimization of Two-Stage Lot-Sizing Problems

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This paper studies two-stage lot-sizing problems with uncertain demand, where lost sales, backlogging and no backlogging are all considered. To handle the ambiguity in the probability distribution of demand, distributionally robust models are established only based on mean-covariance information about the distribution. Based on shortest path reformulations of lot-sizing problems, we prove that robust solutions can be obtained by solving mixed 0-1 conic quadratic programs (CQPs) with mean-risk objective functions. An exact parametric optimization method is proposed by further reformulating the mixed 0-1 CQPs as single-parameter quadratic shortest path problems. Rather than enumerating all potential values of the parameter, which may be the super-polynomial in the number of decision variables, we propose a branch-and-bound-based interval search method to find the optimal parameter value. Polynomial time algorithms for parametric subproblems with both uncorrelated and partially correlated demand distributions are proposed. Computational results show that the proposed models greatly reduce the system cost variation at the cost of a relative smaller increase in expected system cost, and the proposed parametric optimization method is much more efficient than the CPLEX solver.

Key words: distributionally robust optimization; two-stage lot-sizing; parametric search; demand correlation; mean-covariance
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1. Introduction

Dynamic lot-sizing (DLS) is one of the most important problems in production planning, and has been extensively studied since the pioneering work of Wagner and Whitin (1958) and Manne (1958). Classical DLS determines production periods, when production will take place, and production quantities in these periods to satisfy known deterministic demand over T periods.

In many applications, these two types of decisions are made at different time points. For example, the production on/off decisions are made in the planning stage before the demand is fully revealed, while the final production quantities are determined by customer orders. For example, consider a flexible production line facing uncertain demand. If the setup time for each product is long, then the setup decision and rough production quantities must be determined only based on estimated demand, and the final production quantities can be adjusted based on more accurate demand information. Compared with the classical lot-sizing model, the proposed two-stage model also provides more flexility to adjust production quantities and robustness against the demand fluctuation. Two-stage LS problems have been studied by Zhang (2011), Lia and Moralesa (2013) and Guan and Miller (2008) using minimax regret robust optimization (RO) and stochastic programming (SP) models. RO models assume that demand in each period can take any value in a given interval, which neglect other distribution information about demand, while SP models require an exact demand distribution, which is usually hard to obtain, and thus may lead to fragile solutions when the predefined distribution is inadequate.

To enhance the robustness of solutions against the ambiguity in the probability distribution of demand, this paper studies distributionally robust models (DRMs) of two-stage lot-sizing (LS) problems. DRM describes the demand uncertainty by a distributional set, which is a set of probability distributions with specific properties. The selection of the distributional set is a key factor for the effectiveness and tractability of DRM. Various information has been used to construct the distributional set, such as exact first and second moments (Bertsimas et al. 2010, Mak et al. 2014, Popescu 2007, Scarf et al. 1958), inexact first and second moments (Delage and Ye 2010), directional deviations (Chen et al. 2007), probability measures (Ben-Tal et al. 2013, Klabjan et al. 2013) and conic representable confidence sets (Wiesemann et al. 2014). For our problems, we will show that the expectation of the second-stage cost function only depends on the first moment of the random demand. Thus, to describe the demand uncertainty, it is sufficient to consider a distributional set based on first moment information. Specifically, we restrict the first moment of demand to an ellipsoid of size ε centered at an estimated mean, and its shape is controlled by an estimated covariance matrix. A tradeoff between robustness and the expected performance of the optimal solution can be made by adjusting the value of ε.

The proposed DRMs are min-max-min problems where the uncertainty lies in the right-hand side of inner constraints. Such problems are very hard to solve. For example, Bertsimas et al. (2010) show that min-max-min problems with exact first and second moments and the uncertainty in the right-hand side of inner constraints are NP-hard in general. Note that binary constraints make our problems even harder. However, by reformulating the inner minimization problems as shortest path problems, we show that robust solutions can be obtained by solving mixed 0-1 conic quadratic programs (CQPs). The objective function of the mixed 0-1 CQPs have interesting mean-risk structures, where the risk is measured by the standard deviation of the random system cost.

Shortest path formulations are critical for deriving the equivalent mixed 0-1 CQPs and designing efficient solution methods. Shortest path formulations of LS problems with either backlogging (LS-B) or no backlogging (LS-NB) have been proposed by Evans (1985) and Pochet and Wolsey (1988), respectively. However, the shortest path formulation of the LS problem with lost sales (LS-LS) has not been reported although an $\mathcal{O}(T^2)$ time algorithm is proposed by (Aksen et al. 2003). LS-LS is first studied by Sandbothe and Thompson (1990). Since then, studies concentrate on designing efficient algorithms for LS-LS with more operational constraints, such as bounded production and inventory capacity (Sandbothe and Thompson 1993), lower bounded inventory (Loparic et al. 2001), bounded inventory and special cost structures (Chu and Chu 2008, Liu and Tu 2008, Liu et al. 2007), both loss sale and backlogging (Absi et al. 2011), and bounded inventory and general cost structures (Hwang et al. 2013). This study proposes a shortest path formulation over a directed acyclic graph with $T + 1$ vertices for LS-LS with T periods.

In general, mixed 0-1 CQPs can be solved by standard optimization software such as CPLEX using the branch-and-bound method, and valid inequalities have been developed to improve computational results (Atamtürk et al. 2012). However, our experiments show that as the problem size increases, CPLEX becomes quite inefficient. To address this issue, we propose an exact parametric optimization method (POM) by reformulating the mixed 0-1 CQPs into single-parameter parametric optimization problems (POPs). To solve POPs, effective parameter search methods and algorithms for parametric subproblems are developed.

A direct method to find the optimal parameter value is the enumeration method, which is effective when the number of potential parameter values is small in problem size. There are many successful applications of this method, such as the production-transportation problem (Tuy et al. 1993a,b, 1996), the stochastic inventory-transportation network design problem (Shu et al. 2005), the stochastic supply-chain design problem (Shen 2007) and the strategic location problem (Shen 2007). A closely related research is conducted by Nikolova et al. (2006), where an uncorrelated stochastic shortest path problem with a quasi-convex objective function is solved by the enumeration method. However, as we see later, there may be super-polynomial number of potential parameter values needed to be considered in our problems. Thus, the enumeration method may be inefficient. Besides the enumeration method, there are other parameter search methods based on problem properties. For example, Popescu (2007) shows that certain robust mean-covariance problems can be reformulated as single parameter maximization problems with continuous and unimodal objective functions, which can be solved efficiently.

To avoid the enumeration, we exploit properties of our POPs and propose an effective branch-and-bound-based interval search method. A lower bound of the objective function when the parameter belongs to a given interval can be obtained using the concavity of objective functions. A branch strategy is proposed to avoid probing of unnecessary parameter values. To speed up the parameter search, the initial interval is determined based on an estimation of optimal original solutions. The proposed interval search method has a clear geometric interpretation and can also be used to solve POPs with similar concave objective functions. Computational experiments show that the proposed method converges fast and only needs to solve very few number of parameter subproblems (no more than 10 in all the test instances).

We further propose polynomial time algorithms for certain parametric subproblems. Note that the parametric subproblem with a general covariance matrix Σ is a quadratic shortest path problem, which is strongly NP-hard (Rostami et al. 2015). Thus we focus on parametric problems with uncorrelated and partially correlated demand distributions (UDD and PCDD). UDD models assume demand over different periods is uncorrelated, that is, Σ is diagonal. For such models, we propose $\mathcal{O}(T^2)$ time algorithms for parametric subproblems of LS-LS, LS-B, and LS-NB. PCDD models allow demand correlations over adjacent periods, where Σ is tri-diagonal or five-diagonal. Although an algorithm proposed by Rostami et al. (2015) can solve our subproblems with a tri-diagonal Σ in $\mathcal{O}(T^3)$ time, by exploiting the special cost structure of our problems, we design $\mathcal{O}(T^2)$ time algorithms for subproblems of both LS-LS with a tri-diagonal Σ and LS-NB with a five-diagonal Σ. Both algorithms use the idea of constructing extended graphs. Rostami et al. (2015)'s algorithm introduces an auxiliary vertex for each pair of adjacent arcs in the original graph, while our algorithms only introduce an auxiliary vertex for each vertex.

The main contributions of this study can be summarized as follows:

1. We propose DRMs for two-stage LS problems with lost sales, backlogging and no backlogging. A shortest path formulation of the LS problem with lost sales is proposed. We show how to reformulate these problems as the mixed 0-1 CQPs with mean-risk objective functions and path constraints.
2. An effective parametric optimization method is proposed. A branch-and-bound-based interval search method is designed to solve our POPs. This method has a clear geometric interpretation and can be used to solve similar concave network minimization problems. Computational experiments show that it converges after testing very few parameter values.
3. We propose $\mathcal{O}(T^2)$ time algorithms for parametric subproblems of both LS-LS with a tri-diagonal Σ and LS-NB with a five-diagonal Σ by constructing extended graphs.

The rest of the study is organized as follows. In section 2, we present DRMs for two-stage LS problems with lost sales, backlogging and no backlogging, and derive equivalent mixed 0-1 CQPs. In section 3, we reformulate the mixed 0-1 CQPs into single parameter POPs. An effective branch-and-bound-based interval search method and polynomial time algorithms for certain parametric subproblems are proposed. Section 4 reports computational results. In section 5, we conclude with directions for future research.

2. Models and Reformulations

2.1. DRM for Two-Stage LS Problems

Consider uncapacitated two-stage LS problems with random demand and zero lead-time. In the first stage, one needs to decide production periods when the production will take place before the realization of random demand. In the second stage, production quantities in each production period are determined based on known demand. For each period $t \in \{1, 2, \ldots, T\}$, we define the following notations:

D_t	nonnegative random demand in period t and $D = (D_1, \ldots, D_T)^{\mathrm{T}}$,
d_t	a realization of D_t and $d = (d_1, \ldots, d_T)^{\mathrm{T}}$,
c_t	setup cost,
p_t	unit production cost,
h_t	unit holding cost. Let $h_{k,t} = \sum_{i=k}^{t} h_i$, if $t \geq k$; otherwise, $h_{k,t} = 0$,
π_t	penalty cost for unit lost sale or backlogging.

Let F be the joint probability distribution function of D. We assume that F is unknown but belongs to a predefined distributional set $\mathcal{D}$. A distributional set is a collection of probability distributions with specific properties. The distributional set $\mathcal{D}$ will be specified in section 2.3.

We consider the following DRM of two-stage LS problems:

$$\text{(DRM)} \quad \min_{y \in \{0,1\}^T} \left(\sum_{t=1}^{T} c_t y_t + \sup_{F \in \mathcal{D}} \mathbb{E}_F[f(y, D)] \right), \tag{1}$$

where y_t is the production on/off decision in period t and $f(y, d)$ is the second-stage cost for given y and $D = d$. According to different treatments of the unsatisfied demand, three types of second-stage cost functions will be considered.

When the unsatisfied demand leads to immediate loss sales, the second-stage cost function is given as follows:

$$\begin{aligned} f_{LS}(y,d) = \min_{x,s,L} \sum_{t=1}^{T} (p_t x_t + h_t s_t + \pi_t L_t) \\ \text{s.t. } s_t = s_{t-1} + x_t - (d_t - L_t), &\quad 1 \leq t \leq T, \\ x_t \leq M y_t, &\quad 1 \leq t \leq T, \\ L_t \leq d_t, &\quad 1 \leq t \leq T, \\ x_t, s_t, L_t \geq 0, s_0 = 0, &\quad 1 \leq t \leq T, \end{aligned} \tag{2}$$

where x_t, s_t and L_t denote the production quantity, end-of-period inventory level and the amount of unmet demand in period t, respectively, and M is a sufficiently large positive number. If the demand is known before production on/off decisions are made, our DRM for two stage LS-LS reduces to the model studied by Aksen et al. (2003).

When backlogging is allowed, based on the formulation of LS problems with backlogging (LS-B) given by Pochet and Wolsey (1988), the second-stage cost function is given as follows:

$$
\begin{aligned}
f_B(y,d) = \min_{x,s,r} & \sum_{t=1}^{T}(p_t x_t + h_t s_t + \pi_t r_t) \\
\text{s.t. } & s_t - r_t = s_{t-1} - r_{t-1} + x_t - d_t, \ 1 \le t \le T, \\
& x_t \le M y_t, \qquad 1 \le t \le T, \\
& x_t, s_t, r_t \ge 0, s_0 = r_0 = s_T = r_T = 0, \ 1 \le t \le T,
\end{aligned} \tag{3}
$$

where s_t and r_t denote the stock and shortage in period t, respectively.

Note that the LS with no backlogging (LS-NB) can be treated as a special case for both LS-LS and LS-B. The second-stage cost function f_{NB} of LS-NB is equal to f_{LS} with $L_t = 0$ $(1 \le t \le T)$, or f_B with $r_t = 0$ $(1 \le t \le T)$.

2.2. Reformulations of Second-Stage Cost Functions

In this subsection, we give shortest path reformulations for second-stage cost functions of LS-LS, LS-B and LS-NB.

We first give a facility location formulation for f_{LS}. From Lemmas 1, 2 and 3 of Aksen et al. (2003), we know that when y is fixed, the demand in each period is either lost or fully satisfied by a single production made in a specific period. Let the binary variable $w_{k,t} \in \{0, 1\}$ indicate whether d_t is satisfied by a production made in period k, and let $w_{0,t} \in \{0, 1\}$ indicate whether d_t is lost where $1 \le k \le t \le T$. Since the unsatisfied demand leads to immediate loss sales and backlogging is not allowed, we have $\sum_{k=0}^{t} w_{k,t} = 1$ for $1 \le t \le T$. Therefore, we have the following result.

LEMMA 1. *For any $y \in \{0, 1\}^T$ and $d \in \mathbb{R}_+^T$, we have*

$$
\begin{aligned}
f_{LS}(y,d) = \min_{w} & \sum_{t=1}^{T} d_t \left\{ \pi_t w_{0,t} + \sum_{k=1}^{t}(p_k + h_{k,t-1}) w_{k,t} \right\} \\
\text{s.t. } & \sum_{k=0}^{t} w_{k,t} = 1, \ w_{k,t} \le y_k, \ 1 \le k \le t \le T, \\
& w_{0,t}, w_{k,t} \in \{0,1\}, \qquad 1 \le k \le t \le T.
\end{aligned} \tag{4}
$$

Furthermore, there exists an optimal w^*, such that if $w^*_{k,t} = 1$ for some $1 \le k \le t \le T$, then for any $\tau \in \{k, k+1, \cdots, t-1\}$, we have either $w^*_{k,\tau} = 1$ or $w^*_{0,\tau} = 1$.

PROOF. Since both the objective function and constraint sets of (4) are separate in t, we have $f(y, d) = \sum_{t=1}^{T} \alpha_t d_t$, where $\alpha_t = \min\{\pi_t, \min_{1 \le k \le t}\{p_k + h_{k,t-1} + (1 - y_k)M\}\}$. If $w^*_{k,t} = 1$, then for any $l \in \{1, \ldots, t\}$, we have $p_k + h_{k,t-1} \le p_l + h_{l,t-1}$. Thus for any $\tau = \max\{l, k\}, \ldots, t$, we have $p_k + h_{k,\tau-1}, \le p_l + h_{l,\tau-1}$. Therefore, we have either $w^*_{k,\tau} = 1$ if $p_k + h_{k,\tau-1} \le \pi_\tau$, or $w^*_{0,\tau} = 1$ if $p_k + h_{k,\tau-1} > \pi_\tau$. □

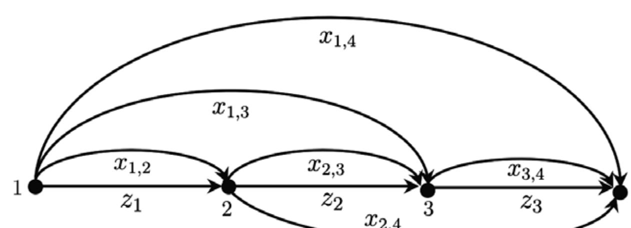
By exploiting the special cost structure of LS-LS, the following theorem shows that problem (4) can be further reformulated as a shortest path problem in a directed acyclic graph $G = (V, A)$, where $V = \{1, \ldots, T + 1\}$, $A = A_1 \cup A_Z$, $A_1 = \{(k, t) : 1 \le k < t \le T + 1\}$ and $A_Z = \{(t, t+1) : t = 1, \ldots, T\}$. Figure 1 gives an example of the graph $G = (V, A)$ when $T = 3$. Note that there are two paralleled arcs between nodes k and $k + 1$ with different arc costs for any $k = 1, \ldots, T$. We associate binary variables $x_{k,t}$ and z_t with arcs $(k, t) \in A_1$ and $(t, t+1) \in A_Z$, respectively. The binary variable $x_{k,t} = 1$ indicates that the last period, in which the demand is satisfied by the production made in period k, is t, and $z_t = 1$ indicates that d_t is lost. Note that when $x_{k,t} = 1$, it is possible that demand in a certain period s is also lost where $k \le s \le t - 1$.

THEOREM 1. *For any $y \in \{0, 1\}^T$ and $d \in \mathbb{R}_+^T$, we have*

$$
\begin{aligned}
f_{LS}(y,d) = \min_{x,z} & \sum_{t=1}^{T} d_t \left(\sum_{k=1}^{t} \sum_{\tau=t+1}^{T+1} \theta_{k,t} x_{k,\tau} + \pi_t z_t \right) \\
\text{s.t. } & \sum_{t=2}^{T+1} x_{1,t} + z_1 = 1, \\
& \sum_{k=1}^{T} x_{k,T+1} + z_T = 1, \\
& \sum_{k=1}^{t-1} x_{k,t} + z_{t-1} = \sum_{\tau=t+1}^{T+1} x_{t,\tau} + z_t, \ 2 \le t \le T, \\
& x_{k,t}, z_k \in \{0,1\}, \qquad 1 \le k < t \le T+1, \\
& x_{k,t} \le y_k, \qquad 1 \le k < t \le T+1,
\end{aligned} \tag{5}
$$

where $\theta_{k,t} = min\{p_k + h_{k,t-1}, \pi_t\}$ for $1 \le k \le t \le T$. Furthermore, for a given $y \in \{0, 1\}^T$, there exists an

Figure 1 An Example of the Graph $G = (V, A)$ when $T = 3$. There are two paralleled arcs between each pair of consecutive nodes



optimal solution (x^*, z^*), such that $f_{LS}(y, d) = \sum_{t=1}^{T} d_t \left(\sum_{k=1}^{t} \sum_{\tau=t+1}^{T+1} \theta_{k,t} x^*_{k,\tau} + \pi_t z^*_t\right)$, for any $d \in \mathbb{R}^T_+$.

PROOF. From the separability of Equation (4) in t, we have

$$\begin{aligned} f_{LS}(y,d) = \min_{w} \quad & \sum_{t=1}^{T} d_t \left\{ \pi_t w_{0,t} + \sum_{k=1}^{t} \theta_{k,t} w_{k,t} \right\} \\ \text{s.t.} \quad & \sum_{k=0}^{t} w_{k,t} = 1, \ w_{k,t} \le y_k, \quad 1 \le k \le t \le T, \\ & w_{0,t}, w_{k,t} \in \{0,1\}, \qquad 1 \le k \le t \le T, \end{aligned} \tag{6}$$

where $\theta_{k,t} = \min\{p_k + h_{k,t-1}, \pi_t\}$ for $1 \le k \le t \le T$, and there exists an optimal solution w^* of Equation (6), such that if $w^*_{k,t} = 1$ for some $1 \le k \le t \le T$, then for $\tau \in \{k, k+1, \ldots, t-1\}$, we have $w^*_{k,\tau} = 1$. From the definitions of z_t and $x_{k,t}$, we have $w_{0,t} = z_t$ and $w_{k,t} = \sum_{s=t+1}^{T+1} x_{k,s}$ where $1 \le k \le t \le T$. Therefore, we have the shortest path formulation (5). The desired optimal solution (x^*, z^*) is given as follows: for any $1 \le k < t \le T+1$, $z^*_k = w^*_{0,k}$ and $x^*_{k,t} = w^*_{k,t-1} - w^*_{k,t}$, where $w^*_{k,T+1} = 0$. □

Pochet and Wolsey (1988) have provided facility location and shortest path formulations for LS-B. Using a similar analysis and Theorem 5 in Pochet and Wolsey (1988), we have the following theorem. The first four constraints are flow conservation constraints, and an illustration of the corresponding graph can be found in Pochet and Wolsey (1988).

THEOREM 2. *For any $y \in \{0,1\}^T$ and $d \in \mathbb{R}^T_+$, we have*

$$\begin{aligned} & f_B(y,d) \\ & = \min_{w,z,v} \sum_{t=1}^{T} d_t \left(\sum_{k=1}^{t-1} \alpha_{k,t} \sum_{s=t}^{T} w_{k,s} + p_t z_{t,t} + \sum_{k=t+1}^{T} \beta_{t,k} \sum_{s=1}^{t} v_{s,k} \right) \\ & \text{s.t.} \sum_{s=1}^{T} v_{1,s} = 1, \\ & \sum_{s=k}^{T} v_{k,s} = \sum_{s=1}^{k-1} w_{s,k-1}, \qquad 2 \le k \le T, \\ & \sum_{s=1}^{k} v_{s,k} = z_{k,k}, \qquad 1 \le k \le T, \\ & z_{k,k} = \sum_{s=k}^{T} w_{k,s}, \qquad 1 \le k \le T, \\ & v_{k,t}, w_{k,t}, z_{k,k} \in \{0,1\}, \qquad 1 \le k \le t \le T, \\ & z_{k,k} \le y_k, \ 1 \le k \le T, \end{aligned} \tag{7}$$

where $alpha_{k,t} = p_k + \sum_{l=k}^{t-1} h_l$ for $1 \le k < t \le T$ and $\beta_{t,k} = p_k + \sum_{l=t}^{k-1} \pi_l$ for $1 \le t < k \le T$. Furthermore, for a given $y \in \{0,1\}^T$, there exists an optimal solution (w^*, z^*, v^*), such that $f_B(y,d) = \sum_{t=1}^{T} d_t (\sum_{k=1}^{t-1} \alpha_{k,t} \sum_{s=t}^{T} w^*_{k,s} + p_t z^*_{t,t} + \sum_{k=t+1}^{T} \beta_{t,k} \sum_{s=1}^{t} v^*_{s,k})$, for any $d \in R^T_+$.

Since LS-NB is a special case of LS-LS, its shortest path formulation can be obtained from Equation (5) with $z_t = 0$ $(1 \le t \le T)$. Although replacing the binary constraints by nonnegative constraints in both (5) and (7) does not change optimal objective values due to the totally unimodularity of constraint matrices, our experiments show that formulations with binary constraints are more favourable for CPLEX.

2.3. Mixed 0-1 CQPs

Theorems 1 and 2 show that when the setup decision y is fixed, optimal solutions of the second stage problems LS-LS, LS-B and LS-NB are only determined by the cost parameters and independent of the realization of the random demand. Since the second-stage functions, $f_{LS}(y,d)$, $f_B(y,d)$ and $f_{NB}(y,d)$, are linear in d, the expectations of these second-stage cost functions only depend on the expected demand. Therefore, to deal with the demand uncertainty, it is sufficient to consider the distributional set with uncertain first moment. In this study, we suppose the first moment $\mathbb{E}_F[D]$ of D belongs to an ellipsoid centered at an estimated mean $\mu \in \mathbb{R}^T$. Therefore, the distribution function F of D belongs to the following distributional set:

$$\mathcal{D} = \left\{ F : (\mathbb{E}_F[D] - \mu)^{\mathrm{T}} \Sigma^{-1} (\mathbb{E}_F[D] - \mu) \le \epsilon^2 \right\}, \tag{8}$$

where Σ is an estimation of the covariance matrix of D and $\varepsilon > 0$ controls the size of $\mathcal{D}$. In the following, we assume that $\mu > 0$, Σ is positive definite and $\Lambda = \{x \in \mathbb{R}^T : (x - \mu)^{\mathrm{T}} \Sigma^{-1} (x - \mu) \le \epsilon^2\} \subseteq \mathbb{R}^T_+$.

In the distributional set $\mathcal{D}$, μ and Σ can be chosen as the sample mean and covariance. The selection of ε relies on the estimation accuracy of the first moment. We will test the effect of the estimation accuracy on the performance of our models by numerical experiments in section 4.

A data-driven method has been proposed by Delage and Ye (2010) to determine the value of ε based on a set of M independent samples $\{d^i\}_{i=1}^M$ of D. In particular, suppose $\mathbb{E}[D] = \mu_0$ and $\mathrm{Cov}(D) = \Sigma_0$. The Corollary 2 in Delage and Ye (2010) shows that for any $\delta > 0$, if $\mathrm{Prob}((D - \mu_0)^{\mathrm{T}} \Sigma_0^{-1} (D - \mu_0) \le R^2) = 1$, then with probability greater than $1 - \delta$ over the choice of $\{d^i\}_{i=1}^M$, we have $(\mu_0 - \hat{\mu})^{\mathrm{T}} \Sigma_0^{-1} (\mu_0 - \hat{\mu}) \le \epsilon(\delta)$ where $\hat{\mu} = \frac{1}{M} \sum_{i=1}^{M} \xi_i$ and $\epsilon(\delta) =$

$\frac{R^2}{M}(2 + \sqrt{2\ln(1/\delta)})^2$. Corollary 4 in Delage and Ye (2010) further shows how to choose ε when R and Σ_0 are also unknown.

With this distributional set, our original min-max-min problems can be simplified into mixed 0-1 CQPs. Unlike results obtained by Delage and Ye (2010) for general min-max problems with moment uncertainty, our shortest path formulations enable us to further eliminate the first stage variable y.

THEOREM 3. *If the distributional set $\mathcal{D}$ is given by Equation (8), then DRMs of two-stage LS-LS, LS-B and LS-NB can be equivalently reformulated as the following mixed 0-1 CQPs:*

$$(P_{LS})\ \min_{r,q,x,z} \epsilon r + \mu^{\mathrm{T}} q + \sum_{k=1}^{T}\sum_{t=k+1}^{T+1} c_k x_{k,t}$$
$$s.t.\ q_t = \sum_{k=1}^{t}\sum_{\tau=t+1}^{T+1} \theta_{k,t} x_{k,\tau} + \pi_t z_t, \qquad 1 \le t \le T,$$
$$\|\Sigma^{1/2} q\| \le r,\ q \in \mathbb{R}^T, (x,z) \in \Delta_{LS},$$

where the path set Δ_{LS} of LS-LS is defined by the first four constraints of problem (5),

$$(P_B)\ \min_{r,q,w,z,v} \epsilon r + \mu^{\mathrm{T}} q + \sum_{k=1}^{T} c_k z_{k,k}$$
$$s.t.\ q_t = \sum_{k=1}^{t-1} \alpha_{k,t} \sum_{s=t}^{T} w_{k,s} + p_t z_{t,t} + \sum_{k=t+1}^{T} \beta_{t,k} \sum_{s=1}^{t} v_{s,k},\ 1 \le t \le T,$$
$$\|\Sigma^{1/2} q\| \le r,\ q \in \mathbb{R}^T,\ (w,z,v) \in \Delta_B,$$

where the path set Δ_B of LS-B is defined by the first five constraints of problem (7), and

$$(P_{NB})\ \min_{r,q,x} \epsilon r + \mu^{\mathrm{T}} q + \sum_{k=1}^{T}\sum_{t=k+1}^{T+1} c_k x_{k,t}$$
$$\text{s.t. } q_t = \sum_{k=1}^{t}\sum_{\tau=t+1}^{T+1} \theta_{k,t} x_{k,\tau}, \qquad 1 \le t \le T,$$
$$\|\Sigma^{1/2} q\| \le r,\ q \in \mathbb{R}^T, \quad x \in \Delta_{NB},$$

where the path set Δ_{NB} of LS-NB is defined by

$$\Delta_{NB} = \left\{ (x_{k,t} \in \{0,1\} : 1 \le k < t \le T+1) : \sum_{t=2}^{T+1} x_{1,t} = \sum_{k=1}^{T} x_{k,T+1} = 1, \sum_{k=1}^{t-1} x_{k,t} = \sum_{\tau=t+1}^{T+1} x_{t,\tau}, \quad \forall 2 \le t \le T \right\}.$$

PROOF. We first consider the DRM of two-stage the LS-LS. For a given $y \in \{0,1\}^T$, let (x^*, z^*) be the optimal solution of problem (5), which is independent of d, and let $\Omega_{LS}(y) = \Delta_{LS} \cap \{(x,z) : x_{k,t} \le y_k, \forall 1 \le k < t \le T+1\}$. For any $1 \le t \le T$, let $q_t = \sum_{k=1}^{t}\sum_{\tau=t+1}^{T+1} \theta_{k,t} x_{k,\tau} + \pi_t z_t$ and $q_t^* = \sum_{k=1}^{t}\sum_{\tau=t+1}^{T+1} \theta_{k,t} x_{k,\tau}^* + \pi_t z_t^*$.

From Theorem 1, it is easy to see $\mathbb{E}_F[f_{LS}(y,D)] = (q^*)^{\mathrm{T}}\mathbb{E}_F[D] = \min_{(x,z)\in\Omega_{LS}(y)} q^{\mathrm{T}}\mathbb{E}_F[D]$ since (x^*, z^*) is also an optimal solution of problem (5) with $d = \mathbb{E}_F[D] \in \mathbb{R}_+^T$. Thus, we have

$$\sup_{F\in\mathcal{D}} \mathbb{E}_F[f_{LS}(y,D)] = \sup_{F\in\mathcal{D}} \min_{(x,z)\in\Omega_{LS}(y)} q^{\mathrm{T}}\mathbb{E}_F[D] = \max_{\mathbb{E}[D]\in\Lambda} \min_{(x,z)\in\Omega_{LS}(y)} q^{\mathrm{T}}\mathbb{E}_F[D]. \qquad (9)$$

Next, we interchange the order of the min-max operation in the last term of Equation (9) where $\Omega_{LS}(y)$ is a discrete (non-convex) set. Theorem 1 in Fan (1953) provides a sufficient and necessary condition for the equivalence between min-max problems and max-min problems. However, rather than checking the condition (2) in Fan (1953), we show the equivalence by using the existence of the optimal solution (x^*, z^*), which is independent of d. In fact, since $\min_{(x,z)\in\Omega_{LS}(y)} q^{\mathrm{T}}\mathbb{E}_F[D] = (q^*)^{\mathrm{T}}\mathbb{E}_F[D]$, taking the maximum over $\mathbb{E}[D] \in \Lambda$ on both sides, we have

$$\max_{\mathbb{E}[D]\in\Lambda} \min_{(x,z)\in\Omega_{LS}(y)} q^{\mathrm{T}}\mathbb{E}_F[D] = \max_{\mathbb{E}[D]\in\Lambda} (q^*)^{\mathrm{T}}\mathbb{E}_F[D]. \qquad (10)$$

Since $(x^*, z^*) \in \Omega_{LS}(y)$, we have

$$\max_{\mathbb{E}[D]\in\Lambda} (q^*)^{\mathrm{T}}\mathbb{E}_F[D] \ge \min_{(x,z)\in\Omega_{LS}(y)} \max_{\mathbb{E}[D]\in\Lambda} q^{\mathrm{T}}\mathbb{E}_F[D]. \qquad (11)$$

From Equations (10) and (11), we have

$$\max_{\mathbb{E}[D]\in\Lambda} \min_{(x,z)\in\Omega_{LS}(y)} q^{\mathrm{T}}\mathbb{E}_F[D] \ge \min_{(x,z)\in\Omega_{LS}(y)} \max_{\mathbb{E}[D]\in\Lambda} q^{\mathrm{T}}\mathbb{E}_F[D] \quad (12)$$

The other direction is straightforward from the minimax weak duality.

Because of $\max_{x\in\Lambda} q^{\mathrm{T}} x = \epsilon\sqrt{q^{\mathrm{T}}\Sigma q} + \mu^{\mathrm{T}} q$, the DRM of the two-stage LS-LS can be reformulated as

$$\min_{r,q,x,z,y} \epsilon\sqrt{q^{\mathrm{T}}\Sigma q} + \mu^{\mathrm{T}} q + \sum_{k=1}^{T} c_k y_k$$
$$\text{s.t. } q_t = \sum_{k=1}^{t}\sum_{\tau=t+1}^{T+1} \theta_{k,t} x_{k,\tau} + \pi_t z_t,\ 1 \le t \le T, \qquad (13)$$
$$x_{k,t} \le y_k, \qquad 1 \le k < t \le T+1$$
$$q \in \mathbb{R}^T, (x,z) \in \Delta_{LS}, y \in \{0,1\}^T.$$

Finally, we show there exists an optimal solution of problem (13), such that $y_k = \sum_{t=k+1}^{T+1} x_{k,t}$ for all $1 \le k \le T$. Indeed, it is easy to show that for any feasible solution (r, q, x, z, y) of problem (13), due to the path constraints, we have $\sum_{t=k+1}^{T+1} x_{k,t} \in \{0, 1\}$. If $\sum_{t=k+1}^{T+1} x_{k,t} = 0$, it is optimal to set $y_k = 0$; otherwise, we must have $y_k = 1$. Replacing y_k with $\sum_{t=k+1}^{T+1} x_{k,t}$ in problem (13) gives (P_{LS}).

Using a similar analysis, we have equivalent (P_B) and (P_{NB}) for DRMs of the two-stage LS-B and LS-NB, respectively. □

The equivalent mixed 0-1 CQPs show how the mean-covariance information affects optimal decisions. Objective functions of the mixed 0-1 CQPs have mean-risk structures, where the risk is measured by the standard deviation of the random system cost. The parameter ε plays a role of the weight on the risk.

The mixed 0-1 CQPs can be solved by the branch-and-bound method based on conic quadratic relaxation and valid inequalities as suggested in Atamtürk et al. (2012). However, by exploiting the properties of two-stage LS models with special demand distributions, we can design more efficient algorithms. Specifically, section 3 provides an exact optimization method for DRMs with the uncorrelated demand distribution (UDD) or partially correlated demand distribution (PCDD). In the first model, the demand distributions over different periods are assumed to be uncorrelated, that is, $\Sigma = \mathrm{Diag}(\sigma_{1,1}, \ldots, \sigma_{T,T})$. In the second model, we allow demand correlations between adjacent periods, that is, Σ is a tri-diagonal matrix ($\Sigma_{i,j} = 0$ for all $|i - j|\ \ 2$) or a five-diagonal matrix ($\Sigma_{i,j} = 0$ for all $|i - j|\ \ 3$).

3. Parametric Optimization Method

This section proposes a parametric optimization method (POM) to solve the mixed 0-1 CQPs. In order to find a parametric subproblem, such that its optimal solutions are also optimal for the original problem, POM searches in the parameter space and solves a series of relatively easy parametric subproblems.

POM has been used to solve many optimization problems. For example, Shu et al. (2005) and Shen and Qi (2007) use this method to solve the stochastic inventory-transportation network design problem and the strategic location problem, respectively. For these problems, the number of potential parameter values is polynomial in the problem size, and the corresponding parametric subproblems are easy to solve. Thus, they propose polynomial time algorithms by systematically enumerating all the potential parameter values. However, as we see later, in our problems, there may be super-polynomial number of potential parameter vales, and the difficulty in solving the parametric subproblems depends on the structure of Σ.

In section 3.1, we formulate equivalent POPs for the mixed 0-1 CQPs. In section 3.2, an effective interval search method is proposed to avoid the enumeration of all potential parameter values. Although the proposed method is applicable to the mixed 0-1 CQPs with any positive definite matrix Σ, in general the resulted parametric subproblems may be difficult to solve. Thus, in sections 3.3 and 3.4, we provide polynomial time algorithms for parameter subproblems of UDD and PCDD models.

3.1. Parametric Optimization Problem

We show how to reformulate (P_{LS}) as a POP. A similar analysis applies to (P_B) and (P_{NB}).

For any $(x, z) \in \Delta_{LS}$, let $a_{x,z} = \epsilon^2 q^{\mathrm{T}} \Sigma q$ and $b_{x,z} = \mu^{\mathrm{T}} q + \sum_{k=1}^{T} \sum_{t=k+1}^{T+1} c_k x_{k,t}$, where $q_t = \sum_{k=1}^{t} \sum_{\tau=t+1}^{T+1} \theta_{k,t} x_{k,\tau} + \pi_t z_t$ for $t = 1, \ldots, T$. Define a discrete set $H = \{(a_{x,z}, b_{x,z}) : (x, z) \in \Delta_{LS}\}$ and a bivariate concave function $h(a, b) = \sqrt{a} + b$. Thus, (P_{LS}) can be reformulated as:

$$\min_{(x,z)\in\Delta_{LS}} h(a_{x,z}, b_{x,z}) = \min_{(a,b)\in H} h(a,b) = \min_{(a,b)\in \mathbf{conv}(H)} h(a,b),$$

where the last equation uses the concavity of h. To solve this concave minimization problem, we need to enumerate all the extreme points of $\mathrm{conv}(H)$. These extreme points can be obtained by solving the following parametric subproblems:

$$(P_{\lambda,\gamma}) \min_{(a,b)\in\mathrm{conv}(H)} \lambda a + \gamma b = \min_{(x,z)\in\Delta_{LS}} \lambda a_{x,z} + \gamma b_{x,z}, \quad \forall(\lambda,\gamma) \in \mathbb{R}^2.$$

Thus, to solve (P_{LS}), it is equivalent to select a proper parameter (λ^*, γ^*). In fact, if we know an optimal solution (x^*, z^*) of (P_{LS}), the desired parameter (λ^*, γ^*) can be selected as $(\nabla h(a_{x^*,z^*}, b_{x^*,z^*}))^{\mathrm{T}} = (\frac{1}{2\sqrt{a_{x^*,z^*}}}, 1)$. This property has been used in the parametric optimization method, such as Shu et al. (2005) and Shen and Qi (2007). The following lemma gives this property formally.

Lemma 2. *Suppose* $x^* \in \arg\min\{f(x) : x \in X \subseteq \mathbb{R}^n\}$. *If* $f : R^n \to R$ *is a concave function, then* $x^* \in \arg\min_{x\in X}\{\nabla f(x^*)^{\mathrm{T}} x\}$; *if* $f(x)$ *is strictly concave, then* $\{x^*\} = \arg\min_{x\in X}\{\nabla f(x^*)^{\mathrm{T}} x\}$.

Proof. Define an affine function $g(x) = f(x^*) + \nabla f(x^*)^{\mathrm{T}}(x - x^*)$. Due to the concavity of f, for any $x \in X$, we have $g(x) \ge f(x) \ge f(x^*) = g(x^*)$. Therefore, $x^* \in \arg\ min_{x\in X} g(x) = \arg\min_{x\in X}\{\nabla f(x^*)^{\mathrm{T}} x\}$. If

f is strictly concave, for any $x \in X$ and $x \neq x^*$, we have $g(x) > f(x) \geq f(x^*) = g(x^*)$. □

Based on Lemma 2, we only need to consider the extreme points of conv(H) corresponding to the following parametric subproblems with $\lambda > 0$:

$$(P_\lambda) \quad \min_{(a,b)\in \mathrm{conv}(H)} \lambda a + b = \min_{(x,z)\in\Delta_{LS}} \lambda a_{x,z} + b_{x,z}.$$

We may expect that the number of such extreme points of conv(H) is polynomial in T. Unfortunately, even for the UDD Model, we will see that (P_λ) is a parametric shortest path problem with $T + 1$ vertices, and Carstensen (1983) and Mulmuley and Shah (2000) prove that for such problems, the number may be $T^{\Theta(\log T)}$. Thus, methods based on the enumeration of extreme points may not work efficiently for our problems.

To address this issue, we consider the following problem:

$$\min_{\lambda > 0} \min_{(a_\lambda,b_\lambda)\in\Phi_\lambda} h(a_\lambda, b_\lambda),$$

where Φ_λ is the set of optimal solutions of (P_λ). Since $\sqrt{x}$ is strictly concave over $(0, +\infty)$, it is easy to see that there exists a desired λ^*, such that $(a_{\lambda^*}, b_{\lambda^*})$ is the unique optimal solution of (P_{λ^*}), i.e., $\Phi_{\lambda^*} = \{(a_{\lambda^*}, b_{\lambda^*})\}$. Thus, with a little abuse of notation, it is sufficient to consider the following POP:

$$(\mathrm{POP}) \quad \min_{\lambda > 0} h(a_\lambda, b_\lambda),$$

where (a_λ, b_λ) is any optimal solution of (P_λ). In next subsection, we will exploit the concavity of our problems and propose an effective parameter search method to avoid the enumeration of all extreme points of conv(H).

3.2. Interval Search Method

The proposed interval search method is based on the branch and bound. Specifically, for a given interval $[\lambda_1, \lambda_2]$, where $0 < \lambda_1 < \lambda_2$, it calculates a lower bound of $h(a_\lambda, b_\lambda)$ over this interval, and partitions this interval into two subintervals $[\lambda_1, \lambda_3]$ and $[\lambda_3, \lambda_2]$. The following Lemma 3 shows how to obtain the lower bound and λ_3.

Lemma 3. *For any $0 < \lambda_1 < \lambda_2$, we have that*

(1) $a_{\lambda_1} \geq a_{\lambda_2}$ and $b_{\lambda_1} \leq b_{\lambda_2}$. If $a_{\lambda_1} = a_{\lambda_2}$, then $b_{\lambda_1} = b_{\lambda_2}$, and vice versa.

(2) If $a_{\lambda_1} \neq a_{\lambda_2}$, let $\lambda_3 = \frac{b_{\lambda_2} - b_{\lambda_1}}{a_{\lambda_1} - a_{\lambda_2}}$, then $\lambda_1 \leq \lambda_3 \leq \lambda_2$. If there is an extreme point $(a_{\lambda'}, b_{\lambda'})$ of conv(H) between $(a_{\lambda_1}, b_{\lambda_1})$ and $(a_{\lambda_2}, b_{\lambda_2})$, such that $\lambda_1 < \lambda' < \lambda_2$, then neither $(a_{\lambda_1}, b_{\lambda_1})$ nor $(a_{\lambda_2}, b_{\lambda_2})$ is an optimal solution of (P_{λ_3}).

(3) $\min_{\lambda_1 \leq \lambda \leq \lambda_2} h(a_\lambda, b_\lambda) \geq \min_{i=0,1,2} h(a_{\lambda_i}, b_{\lambda_i})$, where

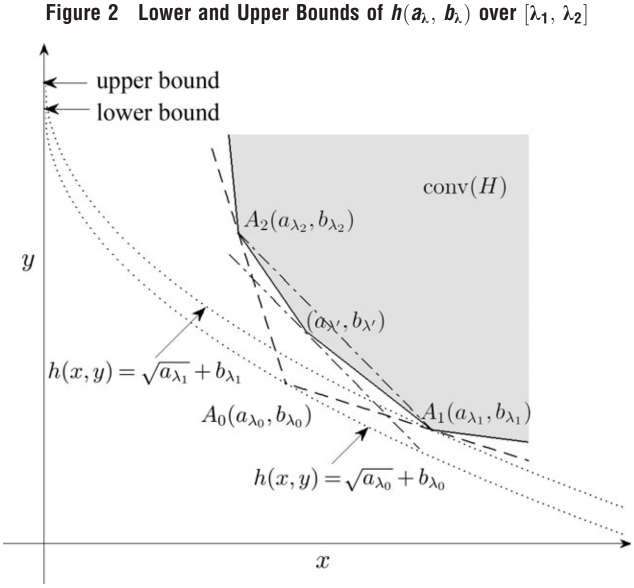
$$a_{\lambda_0} = \frac{\lambda_2 a_{\lambda_2} + b_{\lambda_2} - \lambda_1 a_{\lambda_1} - b_{\lambda_1}}{\lambda_2 - \lambda_1} \quad \textit{and}$$
$$b_{\lambda_0} = \frac{\lambda_1\lambda_2(a_{\lambda_1} - a_{\lambda_2}) + \lambda_2 b_{\lambda_1} - \lambda_1 b_{\lambda_2}}{\lambda_2 - \lambda_1}.$$

Proof. From the definition of (a_λ, b_λ), we have $\lambda_1 a_{\lambda_2} + b_{\lambda_2} \geq \lambda_1 a_{\lambda_1} + b_{\lambda_1}$ and $\lambda_2 a_{\lambda_1} + b_{\lambda_1} \geq \lambda_2 a_{\lambda_2} + b_{\lambda_2}$. Therefore, we have $\lambda_1(a_{\lambda_2} - a_{\lambda_1}) \geq b_{\lambda_1} - b_{\lambda_2} \geq \lambda_2(a_{\lambda_2} - a_{\lambda_1})$, which gives property (1).

If $a_{\lambda_1} \neq a_{\lambda_2}$, from $a_{\lambda_1} - a_{\lambda_2} > 0$, we have $\lambda_1 \leq \frac{b_{\lambda_2} - b_{\lambda_1}}{a_{\lambda_1} - a_{\lambda_2}} \leq \lambda_2$. Since $(a_{\lambda'}, b_{\lambda'})$ is an extreme point of conv(H) between $(a_{\lambda_1}, b_{\lambda_1})$ and $(a_{\lambda_2}, b_{\lambda_2})$, it locates strictly below the line $A_1A_2 = \{(x, y) : \lambda_3 x + y = \lambda_3 a_{\lambda_1} + b_{\lambda_1}\}$. Therefore, $\lambda_3 a_{\lambda'} + b_{\lambda'} < \lambda_3 a_{\lambda_1} + b_{\lambda_1} = \lambda_3 a_{\lambda_2} + b_{\lambda_2}$, which indicates that neither $(a_{\lambda_1}, b_{\lambda_1})$ nor $(a_{\lambda_2}, b_{\lambda_2})$ is an optimal solution of (P_{λ_3}). For any $\lambda_1 \leq \lambda \leq \lambda_2$, from its definition, (a_λ, b_λ) is a point on the boundary of conv(H). Thus the minimum value of $h(a_\lambda, b_\lambda)$ over $[\lambda_1, \lambda_2]$ is obtained at some points on the boundary of conv(H) between $(a_{\lambda_1}, b_{\lambda_1})$ and $(a_{\lambda_2}, b_{\lambda_2})$ (see Figure 2). Note that for any $(x, y) \in$ conv(H), we have $\lambda_1 x + y \geq \lambda_1 a_{\lambda_1} + b_{\lambda_1}$ and $\lambda_2 x + y \geq \lambda_2 a_{\lambda_2} + b_{\lambda_2}$. Therefore, a lower bound of $h(a_\lambda, b_\lambda)$ over $[\lambda_1, \lambda_2]$ can be obtained by minimizing $h(x, y)$ over the triangle $A_0A_1A_2$, where $A_i = (a_{\lambda_i}, b_{\lambda_i})$ $(i = 0, 1, 2)$ and $(a_{\lambda_0}, b_{\lambda_0}) = \left(\frac{\lambda_2 a_{\lambda_2} + b_{\lambda_2} - \lambda_1 a_{\lambda_1} - b_{\lambda_1}}{\lambda_2 - \lambda_1}, \frac{\lambda_1\lambda_2(a_{\lambda_1} - a_{\lambda_2}) + \lambda_2 b_{\lambda_1} - \lambda_1 b_{\lambda_2}}{\lambda_2 - \lambda_1}\right)$ is the intersection point of $\lambda_1 x + y = \lambda_1 a_{\lambda_1} + b_{\lambda_1}$ and $\lambda_2 x + y = \lambda_2 a_{\lambda_2} + b_{\lambda_2}$. Due to the concavity of h, we have property (3). □

To improve the efficiency of the proposed method, the initial interval $(0, +\infty)$ can be further reduced to a smaller interval. Specifically, since $0 < \lambda^* \leq \infty$, we have $a_0 \leq a_{\lambda^*} \leq a_\infty$, where (a_0, b_0) and (a_∞, b_∞) are optimal solutions of (P_0) and (P_∞), respectively. From

Figure 2 Lower and Upper Bounds of $h(a_\lambda, b_\lambda)$ over $[\lambda_1, \lambda_2]$



Lemma 1, it is sufficient to search over $[\lambda_{min}, \lambda_{max}]$, where $\lambda_{min} = \frac{1}{2\sqrt{a_\infty}}$ and $\lambda_{max} = \frac{1}{2\sqrt{a_0}}$. Algorithm 1 outlines the proposed interval search method.

Algorithm 1 Interval search method for (POP)

1. Initialization: Initialize the set of active intervals as $I = \{[\lambda_{min}, \lambda_{max}]\}$. Solve $(P_{\lambda_{min}})$ and $(P_{\lambda_{max}})$ to obtain $(a_{\lambda_{min}}, b_{\lambda_{min}})$ and $(a_{\lambda_{max}}, b_{\lambda_{max}})$. Set the incumbent solution $\bar{\lambda} = \arg\min\{h(a_\lambda, b_\lambda) : \lambda = \lambda_{min} \text{ or } \lambda_{max}\}$ and the upper bound $\bar{h} = h(a_{\bar{\lambda}}, b_{\bar{\lambda}})$.
2. Termination: If $I = \emptyset$, then return the incumbent solution $\bar{\lambda}$.
3. Interval selection: Select and delete an active interval $[\lambda_1, \lambda_2]$ from I.
4. Pruning: If $a_{\lambda_1} = a_{\lambda_2}$, goto Step 2. Let $\lambda_3 = \frac{b_{\lambda_2} - b_{\lambda_1}}{a_{\lambda_1} - a_{\lambda_2}}$. If $\lambda_3 = \lambda_1$ or λ_2, or $h(a_{\lambda_0}, b_{\lambda_0}) \geq \bar{h}$, goto Step 2, where $a_{\lambda_0} = \frac{\lambda_2 a_{\lambda_2}^2 + b_{\lambda_2} - \lambda_1 a_{\lambda_1} - b_{\lambda_1}}{\lambda_2 - \lambda_1}$ and $b_{\lambda_0} = \frac{\lambda_1\lambda_2(a_{\lambda_1} - a_{\lambda_2}) + \lambda_2 b_{\lambda_1} - \lambda_1 b_{\lambda_2}}{\lambda_2 - \lambda_1}$.
5. Branching: Solve (P_{λ_3}) to obtain $(a_{\lambda_3}, b_{\lambda_3})$. If points $(a_{\lambda_i}, b_{\lambda_i})$, $i = 1, 2, 3$, are co-linear, goto Step 2. If $h(a_{\lambda_3}, b_{\lambda_3}) < \bar{h}$, update $\bar{h} = h(a_{\lambda_3}, b_{\lambda_3})$, $\bar{\lambda} = \lambda_3$ and $I = I \cup \{[\lambda_1, \lambda_3], [\lambda_3, \lambda_2]\}$, goto Step 2.

Suppose there are N extreme points between $(a_{\lambda_{min}}, b_{\lambda_{min}})$ and $(a_{\lambda_{max}}, b_{\lambda_{max}})$. The property (2) in Lemma 3 shows that in the worst case the proposed interval search method at most tests $2N - 1$ points before finding an optimal λ^*. Therefore, we have the following theorem.

THEOREM 4. *If for any $\lambda \geq 0$, the parametric subproblem (P_λ) can be solved, then the interval search method finds an optimal λ^* after solving finite parametric subproblems.*

3.3. Parametric Subproblems of the UDD Models

In this subsection, we show that for a given $\lambda > 0$, the parametric subproblems of the UDD models are equivalent to single source single destination shortest path problems in directed acyclic graphs with $\mathcal{O}(T)$ vertices. In particular, the graphs corresponding to the parametric subproblems of (P_{LS}), (P_B) and (P_{NB}) have $T + 1$, $3T$ and $T + 1$ vertices, respectively. Thus, all these parametric problems can be solved in $\mathcal{O}(T^2)$ time by dynamic programming.

PROPOSITION 1. *When $\Sigma = \mathrm{Diag}(\sigma_{1,1}, \ldots, \sigma_{T,T})$, parametric subproblems of (P_{LS}), (P_B) and (P_{NB}) are equivalent to the following shortest path problems:*

$(P_{LS,\lambda})$

$$\min\left\{\sum_{k=1}^{T}\sum_{t=k+1}^{T+1}\kappa^1_{k,t}(\lambda)x_{k,t} + \sum_{t=1}^{T}\kappa^2_t(\lambda)z_t : (x,z) \in \Delta_{LS}\right\},$$

where $\kappa^1_{k,t}(\lambda) = c_k + \sum_{\tau=k}^{t-1}\mu_\tau\theta_{k,\tau} + \lambda\epsilon^2\sum_{\tau=k}^{t-1}\sigma_{\tau,\tau}\theta^2_{k,\tau}$ *and* $\kappa^2_t(\lambda) = \mu_t\pi_t + \lambda\epsilon^2\sigma_{t,t}\pi_t^2$,

$$(P_{B,\lambda}) \quad \min\left\{\sum_{k=1}^{T-1}\sum_{t=k+1}^{T}\kappa^3_{k,t}(\lambda)w_{k,t} + \sum_{t=1}^{T}\kappa^4_t(\lambda)z_{t,t} + \sum_{k=1}^{T}\sum_{t=1}^{k-1}\kappa^5_{k,t}(\lambda)v_{t,k} : (w,z,v) \in \Delta_B\right\},$$

where $\kappa^3_{k,t}(\lambda) = \sum_{s=k+1}^{t}\left(\mu_s\alpha_{k,s} + \lambda\epsilon^2\sigma_{s,s}\alpha^2_{k,s}\right)$, $\kappa^4_t(\lambda) = c_t + \mu_t p_t + \lambda\epsilon^2\sigma_{t,t}p_t^2$ *and* $\kappa^5_{k,t}(\lambda) = \sum_{s=t}^{k-1}(\mu_s\beta_{s,k} + \lambda\epsilon^2\sigma_{s,s}\beta^2_{s,k})$ *and*

$$(P_{NB,\lambda}) \quad min\left\{\sum_{k=1}^{T}\sum_{t=k+1}^{T+1}\kappa^1_{k,t}(\lambda)x_{k,t} : x \in \Delta_{NB}\right\}.$$

PROOF. We first transform the parametric subproblem of (P_{LS}) into a shortest path problem. For any $1 \leq t \leq T$, we have

$$q_t^2 = \left(\sum_{k=1}^{t}\sum_{\tau=t+1}^{T+1}\theta_{k,t}x_{k,\tau} + \pi_t z_t\right)\left(\sum_{l=1}^{t}\sum_{s=t+1}^{T+1}\theta_{l,t}x_{l,s} + \pi_t z_t\right) = \sum_{k=1}^{t}\sum_{\tau=t+1}^{T+1}\theta^2_{k,t}x_{k,\tau} + \pi_t^2 z_t,$$

where we use the fact that for any $1 \leq k,l \leq t$ and $t + 1 \leq \tau,s \leq T + 1$, we have $z_t x_{k,\tau} = 0$, and $x_{k,\tau}x_{l,s} = 0$ if either $k \neq l$ or $\tau \neq s$. Thus, we have

$$q^{\mathrm{T}}\Sigma q = \sum_{t=1}^{T}\sigma_{t,t}q_t^2 = \sum_{k=1}^{T}\sum_{t=k+1}^{T+1}\sum_{\tau=k}^{t-1}\sigma_{\tau,\tau}\theta^2_{k,\tau}x_{k,t} + \sum_{t=1}^{T}\sigma_{t,t}\pi_t^2 z_t.$$

Therefore, the objective function of the parametric subproblem of (P_{LS}) can be simplified as

$$\lambda\epsilon^2 q^{\mathrm{T}}\Sigma q + \mu^{\mathrm{T}}q + \sum_{k=1}^{T}\sum_{t=k+1}^{T+1}c_k x_{k,t} = \sum_{k=1}^{T}\sum_{t=k+1}^{T+1}\kappa^1_{k,t}(\lambda)x_{k,t} + \sum_{t=1}^{T}\kappa^2_t(\lambda)z_t,$$

where $\kappa^1_{k,t}(\lambda) = c_k + \sum_{\tau=k}^{t-1}\mu_\tau\theta_{k,\tau} + \lambda\epsilon^2\sum_{\tau=k}^{t-1}\sigma_{\tau,\tau}\theta^2_{k,\tau}$ and $\kappa^2_t(\lambda) = \mu_t\pi_t + \lambda\epsilon^2\sigma_{t,t}\pi_t^2$.

Using a similar analysis, for the parametric subproblem of (P_B), we have

$$q^{\mathrm{T}}\Sigma q = \sum_{t=1}^{T}\sigma_{t,t}q_t^2 = \sum_{k=1}^{T-1}\sum_{t=k+1}^{T}\sum_{s=k+1}^{t}\sigma_{s,s}\alpha^2_{k,s}w_{k,t} + \sum_{t=1}^{T}\sigma_{t,t}p_t^2 z_{t,t} + \sum_{k=1}^{T}\sum_{t=1}^{k-1}\sum_{s=t}^{k-1}\sigma_{s,s}\beta^2_{s,k}v_{t,k}.$$

Therefore, the parametric subproblem of (P_B) can also be reformulated as a shortest path problem. The shortest path reformulation of the parametric subproblem of (P_{NB}) can be obtained by removing z from $(P_{LS,\lambda})$. □

Proposition 1 shows that for a given $\lambda > 0$, $(P_{LS,\lambda})$, $(P_{B,\lambda})$ and $(P_{NB,\lambda})$ can be solved as shortest path problems in directed acyclic graphs, where the cost on each arc is a piecewise affine function of λ.

It is known that there exists a parametric shortest path problem with n vertices and its arc costs being linear functions in a parameter, such that it has $n^{\Theta(\log n)}$ different shortest paths when the parameter varies over R (see Carstensen 1983, Mulmuley and Shah 2000). Therefore, even for the UDD models, the enumeration method may need to test super-polynomial number of parameter values before finding an optimal λ^*.

3.4. Parametric Subproblems of the PCDD Models

Note that the parametric subproblem with a general covariance matrix Σ is a quadratic shortest path problems, which is strongly NP-hard (Rostami et al. 2015). Thus, in this subsection, we focus on developing polynomial time algorithms for parametric subproblems of the PCDD models where Σ is tri-diagonal or five-diagonal. The PCDD models are useful when the random demand between adjacent periods shows a strong correlation.

An $\mathcal{O}(T^3)$ time algorithm has been developed by Rostami et al. (2015) to solve quadratic shortest path problem only considering correlation between adjacent arcs. Rostami et al. (2015)'s algorithm uses the idea of constructing an equivalent shortest path problem on an extended graph. Rostami et al. (2015) introduce an auxiliary vertex for each pair of adjacent arcs in the original graph, and thus construct an extended graph with $O(T^3)$ arcs. However, by exploiting the special cost structure of the parametric subproblems of (P_{LS}) and (P_{NB}), the number of adjacent arcs are substantially reduced in the following Lemmas 4 and 5. Furthermore, we construct extended graphs with only $\mathcal{O}(T^2)$ arcs by only introducing an auxiliary vertex for each vertex in the original graph. Therefore, we can design $\mathcal{O}(T^2)$ time algorithms for our parametric subproblems of (P_{LS}) with a tri-diagonal Σ and (P_{NB}) with a five-diagonal Σ.

We first consider the parametric subproblem of (P_{LS}) with a tri-diagonal Σ.

Lemma 4. *When Σ is a tri-diagonal matrix, the parametric subproblem of (P_{LS}) is equivalent to the following quadratic shortest path problem:*

$$\min\left\{\sum_{k=1}^{T}\kappa_k^6(\lambda)z_k+\sum_{k=1}^{T}\sum_{t=k+1}^{T+1}\kappa_{k,t}^7(\lambda)x_{k,t}\right.$$
$$\left.+\sum_{k=1}^{T-1}\lambda e_k z_k z_{k+1}+\sum_{k=1}^{T-1}\sum_{t=k+1}^{T}\lambda v_{k,t}x_{k,t}z_t:\ (x,z)\in\Delta_{LS}\right\},$$

where $\kappa_k^6(\lambda)=\mu_k\pi_k+\lambda u_k$ for $1\le k\le T$, $\kappa_{k,t}^7(\lambda)=c_k+\sum_{\tau=k}^{t-1}\mu_\tau\theta_{k,\tau}+\lambda b_{k,t}$ for $1\le k<t\le T+1$ and parameters b, u, e, v are defined in the proof.

Proof. We first simplify the following cross term:

$$q_t q_{t+1}=\left(\sum_{k=1}^{t}\sum_{\tau=t+1}^{T+1}\theta_{k,t}x_{k,\tau}+\pi_t z_t\right)$$
$$\times\left(\sum_{l=1}^{t+1}\sum_{s=t+2}^{T+1}\theta_{l,t+1}x_{l,s}+\pi_{t+1}z_{t+1}\right).$$

For any $1\le t\le T-1$, we have

$$\left(\sum_{k=1}^{t}\sum_{\tau=t+1}^{T+1}\theta_{k,t}x_{k,\tau}\right)\left(\sum_{l=1}^{t+1}\sum_{s=t+2}^{T+1}\theta_{l,t+1}x_{l,s}\right)$$
$$=\sum_{k=1}^{t}\theta_{k,t}\theta_{t+1,t+1}x_{k,t+1}\sum_{\tau=t+2}^{T+1}x_{t+1,\tau}+\sum_{k=1}^{t}\sum_{\tau=t+2}^{T+1}\theta_{k,t}\theta_{k,t+1}x_{k,\tau}\tag{14}$$

$$=\theta_{t+1,t+1}\sum_{k=1}^{t}\theta_{k,t}x_{k,t+1}-\theta_{t+1,t+1}z_{t+1}\sum_{k=1}^{t}\theta_{k,t}x_{k,t+1}$$
$$+\sum_{k=1}^{t}\sum_{\tau=t+2}^{T+1}\theta_{k,t}\theta_{k,t+1}x_{k,\tau},\tag{15}$$

where Equation (14) uses the fact that for any $1\le l\le t$ and $t+2\le s\le T+1$, $x_{k,t+1}x_{l,s}=0$ and for any $1\le k\le t$, $1\le l\le t+1$ and $t+2\le\tau,s\le T+1$, $x_{k,\tau}x_{l,s}=0$ if either $k\ne l$ or $\tau\ne s$, and Equation (15) uses the fact that $x_{k,t+1}\sum_{\tau=t+2}^{T+1}x_{t+1,\tau}=x_{k,t+1}(1-z_{t+1})$. Similarly, we have

$$\pi_t z_t\left(\sum_{l=1}^{t+1}\sum_{s=t+2}^{T+1}\theta_{l,t+1}x_{l,s}\right)=\theta_{t+1,t+1}\pi_t z_t\sum_{\tau=t+2}^{T+1}x_{t+1,\tau}$$
$$=\theta_{t+1,t+1}\pi_t z_t-\theta_{t+1,t+1}\pi_t z_t z_{t+1},$$

and

$$\left(\sum_{k=1}^{t}\sum_{\tau=t+1}^{T+1}\theta_{k,t}x_{k,\tau}\right)\pi_{t+1}z_{t+1}=\pi_{t+1}z_{t+1}\sum_{k=1}^{t}\theta_{k,t}x_{k,t+1}.$$

Therefore, we have

$$\sum_{t=1}^{T-1}\sigma_{t,t+1}q_t q_{t+1}=\sum_{k=1}^{T-1}\sum_{t=k+1}^{T}\sigma_{t-1,t}\theta_{t,t}\theta_{k,t-1}x_{k,t}$$
$$+\sum_{k=1}^{T-1}\sum_{t=k+2}^{T+1}\sum_{s=k}^{t-2}\sigma_{s,s+1}\theta_{k,s}\theta_{k,s+1}x_{k,t}+\sum_{t=1}^{T-1}\sigma_{t,t+1}\theta_{t+1,t+1}\pi_t z_t$$
$$+\sum_{t=1}^{T-1}\sigma_{t,t+1}\pi_t(\pi_{t+1}-\theta_{t+1,t+1})z_t z_{t+1}$$
$$+\sum_{k=1}^{T-1}\sum_{t=k}^{T-1}\sigma_{t,t+1}\theta_{k,t}(\pi_{t+1}-\theta_{t+1,t+1})x_{k,t+1}z_{t+1}.$$

After introducing auxiliary parameters b, u, e and v, we have

$$\begin{aligned}\epsilon^2 q^{\mathrm{T}}\Sigma q =& \epsilon^2\sum_{t=1}^{T}\sigma_{tt}q_tq_t + 2\epsilon^2\sum_{t=1}^{T-1}\sigma_{t,t+1}q_tq_{t+1}\\ =& \sum_{k=1}^{T}\sum_{t=k+1}^{T+1} b_{k,t}x_{k,t} + \sum_{k=1}^{T}u_kz_k + \sum_{k=1}^{T-1}e_kz_kz_{k+1}\\ &+\sum_{k=1}^{T-1}\sum_{t=k+1}^{T} v_{k,t}x_{k,t}z_t,\end{aligned}$$

where $b_{k,k+1} = \epsilon^2\sigma_{k,k}\theta_{k,k}^2 + 2\epsilon^2\sigma_{k,k+1}\theta_{k,k}\theta_{k+1,k+1}$ for $1 \le k \le T-1$, $b_{T,T+1} = \epsilon^2\sigma_{T,T}\theta_{T,T}^2$, $b_{k,t} = \epsilon^2\sum_{s=k}^{t-1}\sigma_{s,s}\theta_{k,s}^2 + 2\epsilon^2\sum_{s=k}^{t-2}\sigma_{s,s+1}\theta_{k,s}\theta_{k,s+1}, + 2\epsilon^2\sigma_{t-1,t}\theta_{k,t-1}\theta_{t,t}$ for $1 \le k \le T-1$ and $k+2 \le t \le T$, $b_{k,T+1} = \epsilon^2\sum_{s=k}^{T}\sigma_{s,s}\theta_{k,s}^2 + 2\epsilon^2\sum_{s=k}^{T-1}\sigma_{s,s+1}\theta_{k,s}\theta_{k,s+1}$ for $1 \le k \le T-1$; $u_T = \epsilon^2\sigma_{T,T}\pi_T^2$, $u_k = \epsilon^2\sigma_{kk}\pi_k^2 + 2\epsilon^2\sigma_{k,k+1}\pi_k\theta_{k+1,k+1}$ for $1 \le k \le T-1$; $e_k = 2\epsilon^2\sigma_{k,k+1}\pi_k(\pi_{k+1} - \theta_{k+1,k+1})$ for $1 \le k \le T-1$, $e_T = 0$; $v_{k,t} = 2\epsilon^2\sigma_{t-1,t}\theta_{k,t-1}(\pi_t - \theta_{t,t})$ and $1 \le k < t \le T$, and $v_{k,T+1} = 0$ for $1 \le k \le T$.

Therefore, the objective function of the parametric subproblem of (P_{LS}) can be simplified as follows

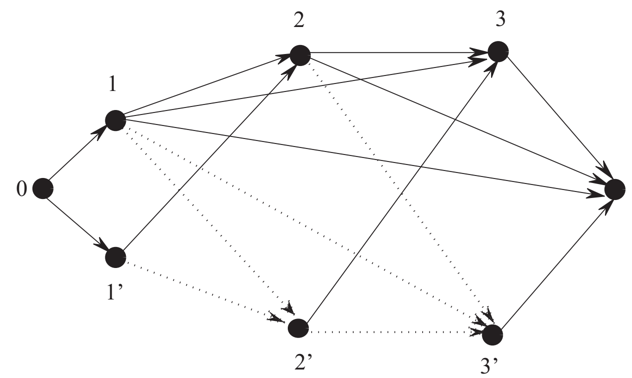
$$\sum_{k=1}^{T}(\mu_k\pi_k + \lambda u_k)z_k + \sum_{k=1}^{T}\sum_{t=k+1}^{T+1}\left(c_k + \sum_{\tau=k}^{t-1}\mu_\tau\theta_{k,\tau} + \lambda b_{k,t}\right)x_{k,t} + \sum_{k=1}^{T-1}\lambda e_kz_kz_{k+1} + \sum_{k=1}^{T-1}\sum_{t=k+1}^{T}\lambda v_{k,t}x_{k,t}z_t. \qquad \square$$

Note that the parametric subproblem of (P_{LS}) is a quadratic shortest path problem from vertex 1 to vertex $T+1$ in graph $G = (V, A)$; see Figure 1. Based on Lemma 4, we will construct an extended directed acyclic graph $\tilde{G} = (\tilde{V}, \tilde{A})$ with $2(T+1)$ vertices and $T^2 + 2T + 1$ arcs, and show that the parametric subproblem of (P_{LS}) is equivalent to a shortest path problem in $\tilde{G}$. The vertex set of $\tilde{G}$ is $\tilde{V} = \{0\} \cup V \cup V'$, where $V' = \{1', \ldots, T'\}$. The arc set of $\tilde{G}$ is $\tilde{A} = A_0 \cup A_1 \cup A_2 \cup A_3 \cup A_4$, where $A_0 = \{(0, 1), (0, 1')\}$, $A_2 = \{(1, 2'), (1, 3'), \ldots, (1, T'), (2, 3'), (2, 4'), \ldots, (2, T'), \ldots, (T-1, T')\}$ and $A_3 = (1', 2), (2', 3), \ldots, (T', T+1)\}$ and $A_4 = \{(1', 2'), (2', 3'), \ldots, ((T-1)', T')\}$. The cost on the arc $(k, t) \in \tilde{A}$ is defined as

$$d_{k,t}^{\lambda} = \begin{cases} 0, & \text{if } (k,t) \in A_0,\\ c_k + \sum_{\tau=k}^{t-1}\mu_\tau\theta_{k,\tau} + b_{k,t}\lambda, & \text{if } (k,t) \in A_1,\\ c_k + \sum_{\tau=k}^{t-1}\mu_\tau\theta_{k,\tau} + (b_{k,t} + v_{k,t})\lambda, & \text{if } (k,t') \in A_2,\\ \mu_k\pi_k + u_k\lambda, & \text{if } (k', k+1) \in A_3,\\ \mu_k\pi_k + (u_k + e_k)\lambda, & \text{if } (k', (k+1)') \in A_4. \end{cases}$$

Figure 3 gives an example of the extended graph $\tilde{G}$ when $T = 3$, where the arcs in A_2 and A_4 are denoted by dashed lines to emphasize that their costs are modified to incorporate quadratic cost terms.

Figure 3 An Example of the Extended Graph $\tilde{G}$ when $T = 3$. The costs on dashed arcs are modified to incorporate quadratic cost terms



Proposition 2. *When Σ is a tri-diagonal matrix, an optimal solution of the parametric subproblem of (P_{LS}) can be obtained in $\mathcal{O}(T^2)$ time by solving the shortest path problem from vertex 0 to vertex $T+1$ in $\tilde{G}$.*

Proof. We first show that for any path P from 1 to $T+1$ in G, there exists a path Q from 0 to $T+1$ in $\tilde{G}$ with the same path cost. Let $P = \{(k_1, k_2), (k_2, k_3), \cdots, (k_m, T+1)\}$ be a path in G where $k_1 = 1$. The path Q in $\tilde{G}$ is defined as $Q = \{(0, l_1), (l_1, l_2), \cdots, (l_m, T+1)\}$ where for each $i = 1, \cdots, m$, if $(k_i, k_{i+1}) \in A_1$, we have $l_i = k_i$; otherwise, $l_i = (k_i)'$. It is easy to check that both P and Q have the same path cost.

Using a similar construction, we can also show that for any path Q from 0 to $T+1$ in $\tilde{G}$, there exists a path P from 1 to $T+1$ in G with the same path cost. Thus, an optimal solution of the parametric subproblem of (P_{LS}) corresponds to a shortest path problem from vertex 0 to vertex $T+1$ in $\tilde{G}$.

Note that the order $0, 1, 1', \ldots, T, T', T+1$ is a topological order of this extended graph. Therefore, the shortest path problem from vertex 0 to vertex $T+1$ in $\tilde{G}$ can be solved in $\mathcal{O}(T^2)$ time by dynamic programming in $\tilde{G}$. $\square$

Next we consider the parametric subproblem of (P_{NB}) with Σ being a tri-diagonal or five-diagonal matrix. Since LS-NB is a special case of LS-LS, the parametric subproblem of (P_{NB}) with a tri-diagonal Σ can also been solved in $\mathcal{O}(T^2)$. In fact, the special structure of LS-NB enables us to solve the parametric subproblem of (P_{NB}) with a five-diagonal matrix Σ

(that is, for any $|i - j| \;\; 3$, $\Sigma_{i,j} = 0$) in $\mathcal{O}(T^2)$. The following lemma shows that the parametric subproblem of (P_{NB}) with a five-diagonal Σ corresponds to a special quadratic shortest path problem.

Lemma 5. *When Σ is a five-diagonal matrix, the parametric subproblem of (P_{NB}) is equivalent to the following quadratic shortest path problem:*

$$\min\left\{\sum_{k=1}^{T}\sum_{t=k+1}^{T+1}\left(c_k + \sum_{\tau=k}^{t-1}\mu_\tau\theta_{k,\tau} + \lambda\rho_{k,t}\right)x_{k,t} + \sum_{k=1}^{T-2}\sum_{t=k+1}^{T-1}\lambda\omega_{k,t}x_{k,t}x_{t,t+1} : x \in \Delta_{NB}\right\},$$

where parameters ρ and ω are defined in the proof.

Proof. Note that $q_t q_{t+1} = \left(\sum_{k=1}^{t}\sum_{\tau=t+1}^{T+1}\theta_{k,t}x_{k,\tau}\right)\left(\sum_{l=1}^{t+1}\sum_{s=t+2}^{T+1}\theta_{l,t+1}x_{l,s}\right)$. From Equation (14) and using the fact that $x_{k,t+1}\sum_{\tau=t+2}^{T+1}x_{t+1,\tau} = x_{k,t+1}$, we have that

$$q_t q_{t+1} = \sum_{k=1}^{t}\theta_{k,t}\theta_{t+1,t+1}x_{k,t+1} + \sum_{k=1}^{t}\sum_{\tau=t+2}^{T+1}\theta_{k,t}\theta_{k,t+1}x_{k,\tau}. \tag{16}$$

Furthermore, for any $1 \le t \le T-2$, we have

$$\begin{aligned} q_t q_{t+2} &= \left(\sum_{k=1}^{t}\sum_{\tau=t+1}^{T+1}\theta_{k,t}x_{k,\tau}\right)\left(\sum_{l=1}^{t+2}\sum_{s=t+3}^{T+1}\theta_{l,t+2}x_{l,s}\right)\\ &= \sum_{k=1}^{t}\theta_{k,t}x_{k,t+1}\left(\theta_{t+1,t+2}\sum_{s=t+3}^{T+1}x_{t+1,s} + \theta_{t+2,t+2}\sum_{s=t+3}^{T+1}x_{t+2,s}\right) + \sum_{k=1}^{t}\theta_{k,t}x_{k,t+2}\left(\theta_{t+2,t+2}\sum_{s=t+3}^{T+1}x_{t+2,s}\right)\\ &\quad + \sum_{k=1}^{t}\sum_{\tau=t+3}^{T+1}\theta_{k,t}\theta_{k,t+2}x_{k,\tau} \end{aligned}\tag{17}$$

$$= \sum_{k=1}^{t}\theta_{k,t}x_{k,t+1}\left(\theta_{t+1,t+2}(1-x_{t+1,t+2}) + \theta_{t+2,t+2}x_{t+1,t+2}\right) + \sum_{k=1}^{t}\theta_{k,t}\theta_{t+2,t+2}x_{k,t+2} + \sum_{k=1}^{t}\sum_{\tau=t+3}^{T+1}\theta_{k,t}\theta_{k,t+2}x_{k,\tau}, \tag{18}$$

where Equation (17) is obtained using a similar analysis to Equation (14), and Equation (18) uses the fact that $x_{k,t+1}\sum_{s=t+3}^{T+1}x_{t+1,s} = x_{k,t+1}(1 - x_{t+1,t+2})$,

$$\begin{aligned} x_{k,t+1}\sum_{s=t+3}^{T+1}x_{t+2,s} &= x_{k,t+1}x_{t+1,t+2}\sum_{s=t+3}^{T+1}x_{t+2,s}\\ &= x_{k,t+1}x_{t+1,t+2}, \end{aligned}$$

and $x_{k,t+2}\sum_{s=t+3}^{T+1}x_{t+2,s} = x_{k,t+2}$. Based on Equations (16) and (18) and $q_t q_t = \sum_{k=1}^{t}\sum_{\tau=t+1}^{T+1}\theta_{k,t}^2 x_{k,\tau}$, we have

$$\begin{aligned} \epsilon^2 q^{\mathrm{T}}\Sigma q &= \epsilon^2\sum_{t=1}^{T}\sigma_{tt}q_t q_t + 2\epsilon^2\sum_{t=1}^{T-1}\sigma_{t,t+1}q_t q_{t+1}\\ &\quad + 2\epsilon^2\sum_{t=1}^{T-2}\sigma_{t,t+2}q_t q_{t+2}\\ &= \sum_{k=1}^{T}\sum_{t=k+1}^{T+1}\rho_{k,t}x_{k,t} + \sum_{k=1}^{T-2}\sum_{t=k+1}^{T-1}\omega_{k,t}x_{k,t}x_{t,t+1}, \end{aligned}$$

where $\rho_{k,t} = \epsilon^2\Big(\sum_{s=k}^{t-1}\sigma_{s,s}\theta_{k,s}^2 + 2\sigma_{t-1,t}\;\;\theta_{k,t-1}\theta_{t,t} + 2\sum_{s=k}^{t-2}\;\;\sigma_{s,s+1}\theta_{k,s}\;\;\theta_{k,s+1}\;\; + 2\sigma_{t-1,t+1}\theta_{k,t-1}\theta_{t,t+1} + 2\sigma_{t-2,t}\;\;\theta_{k,t-2}\;\;\theta_{t,t} + 2\sum_{s=k}^{t-3}\sigma_{s,s+2}\theta_{k,s}\theta_{k,s+2}\Big)$ for $1 \le k < t \le T+1$ and $\omega_{k,t} = 2\epsilon^2\sigma_{t-1,t+1}\;\;\theta_{k,t-1}(\theta_{t+1,t+1} - \theta_{t,t+1})$ for $1 \le k < t \le T-1$. Note that $\theta_{k,t} = 0$ for any $k > t$. Reformulating the objective function finishes the proof. □

Based on Lemma 5, we can construct an extended directed acyclic graph $\bar{G} = (\bar{V}, \bar{A})$ with $2T - 1$ vertices and $T^2 - T + 1$ arcs, such that the parametric subproblem of (P_{NB}) is equivalent to a shortest path problem in $\bar{G}$. The vertex set of $\bar{G}$ is $\bar{V} = V \cup \{2', \ldots, (T-1)'\}$. The arc set of $\bar{G}$ is $\bar{A} = A_5 \cup A_6 \cup A_7 \cup A_8$, where $A_5 = \{(1,2), (1,3), \ldots, (1, T+1), \ldots, (k, k+2), \ldots, (k, T+1), \ldots, (T, T+1)\}$, $A_6 = \{(1, 2'), \ldots, (1, (T-1)'), \ldots, (k, (k+2)'), \ldots, (T-3, (T-1)')\}$, $A_7 = \{(2', 3), \ldots, (k', k+1), \ldots, ((T-1)', T)\}$ and $A_8 = \{(2', 3'), \ldots, (k', (k+1)'), \ldots, ((T-2)', (T-1)')\}$. The cost on the arc $(k, t) \in \bar{A}$ is defined as

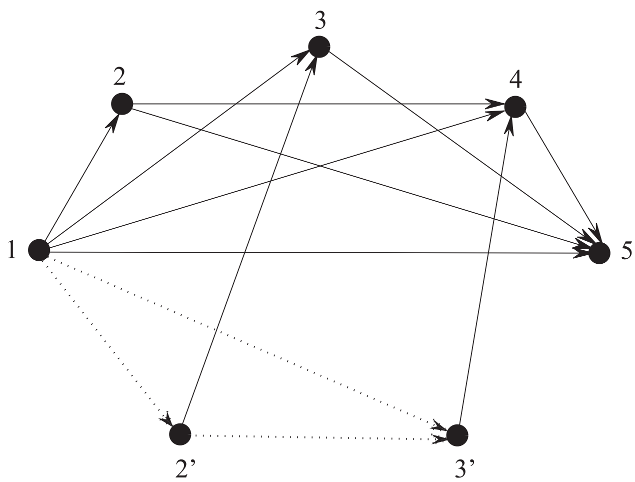
$$d_{k,t}^{\lambda} = \begin{cases} c_k + \sum_{\tau=k}^{t-1}\mu_\tau\theta_{k,\tau} + \rho_{k,t}\lambda, & \text{if } (k,t) \in A_5 \text{ or } (k',t) \in A_7,\\ c_k + \sum_{\tau=k}^{t-1}\mu_\tau\theta_{k,\tau} + (\rho_{k,t} + \omega_{k,t})\lambda, & \text{if } (k,t') \in A_6 \text{ or } (k',t') \in A_8. \end{cases}$$

Figure 4 gives an example of the extended graph $\bar{G}$ when $T = 4$, where the arcs in A_6 and A_8 are also denoted by dashed lines. Using a similar analysis to the extended graph $\tilde{G}$, we have the following result.

Proposition 3. *When Σ is a five-diagonal matrix, an optimal solution of the parametric subproblem of (P_{NB}) can be obtained in $\mathcal{O}(T^2)$ time by solving the shortest path problem from vertex 1 to vertex $T+1$ in $\bar{G}$.*

However, the parametric subproblem of (P_B) is harder than that of (P_{LS}) and (P_{NB}) in the sense that an $\mathcal{O}(T^2)$ time algorithm may not be possible even when Σ is tri-diagonal. In particular, although an analysis of its cross terms enables us to construct an extended graph with less arcs than that given by Rostami et al. (2015), the constructed extended graph still

Figure 4 An Example of the Extended Graph $\bar{G}$ when $T = 4$. The costs on dashed arcs are modified to incorporate quadratic cost terms



has $\mathcal{O}(T^3)$ arcs, which can only be solved in $\mathcal{O}(T^3)$ time.

4. Experiments

In sections 2 and 3, we proposed DRMs of two-stage lot-sizing problems and a parametric optimization method to solve the problems with both uncorrelated and partially correlated demand distributions. In this section, we report experimental results that validate the effectiveness of the proposed solution method and models. In section 4.1, we test the performance of the proposed POM by comparison with the CPLEX solver (CPX) 12.6, and compare the proposed $\mathcal{O}(T^2)$ time algorithm for the subproblem of the PCCD models with the algorithm proposed by Rostami et al. (2015). In section 4.2, we compare the proposed DRMs with a stochastic model (SM). Since all two-stage LS-LS, LS-B and LS-NB share similar structures, we only test the model and method for the two-stage LS-LS. All the instances are solved on an Intel Core i5-4570 CPU at 3.2 GHz.

In the experiments, we suppose that the true demand distribution in each period follows an uniform distribution over $[0, a]$, where a is uniformly drawn from [5, 15]. For the PCCD models with a tridiagonal Σ, the correlation coefficient $\rho_{t,t+1}$ of demand in period t and $t + 1$ is generated as follows: $\rho_{1,2}$ is generated from an uniform distribution over $[-1, 1]$ and for $2 \le t \le T - 1$, $\rho_{t,t+1}$ is generated from an uniform distribution over $[|\rho_{t-1,t}| - 1, 1 - |\rho_{t-1,t}|]$, such that Σ is positive definite. Parameters p_t, h_t, c_t and π_t are also generated from the uniform distribution. We use the data-driven method proposed by Delage and Ye (2010) to construct the distributional set $\mathcal{D}$; see section 2.3. Specifically, we first generate M independent samples $\{d^i\}_{i=1}^M$ of D and calculate the sampled $\bar{\mu}$ and $\bar{\Sigma}$. Then we choose $\delta = 5\%$ and $R = \max_{i=1,\ldots,M} (d^i - \bar{\mu})^{\mathrm{T}}\bar{\Sigma}^{-1}(d^i - \bar{\mu})$ and set $\epsilon = \frac{R}{\sqrt{M}}(2 + \sqrt{2\ln(1/\delta)})$. Experimental parameters are summarized in Table 1.

Table 1 Model Parameters for Test Instances

Parameter	Value	Parameter	Value
a	Uniformly drawn from [5, 15];	M	From 200 to 10,000;
p_t	Uniformly drawn from [0, 1];	h_t	Uniformly drawn from [0, 1];
π_t	Uniformly drawn from [0, 5];	c_t	Uniformly drawn from [0, 10];

4.1. Computational Results of the Proposed Solution Method

Tables 2 and 3 report the averaged computational results of POM and CPX over 20 randomly generated instances of the UDD and PCDD models, respectively. For each instance, we randomly generate model parameters as described in Table 1. The second column gives the number of binary variables of test instances. The third and fourth columns give the average and maximum number of parametric subproblems solved by POM. The fifth and sixth columns give

Table 2 Comparison of POM and CPX for UDD Models

		POM			CPX	
T	Binary	Average #	Max #	CPU time (ms)	CPU time (ms)	Gap (%)
50	1,326	2.1	3	0.4	224.5	0
100	5,151	2.4	3	1.6	847.4	0
200	20,301	2.8	4	2.0	7,039.0	0
400	80,601	3.5	5	33.2	102,434.0	0
600	180,901	4.5	6	694.4	574,903.0	0
800	321,201	4.8	7	5,219.4	643,167.8	0.14
1,000	501,501	4.9	6	6,306.0	699,050.1	0.11

Table 3 Comparison of POM and CPX for PCCD Models

		POM			CPX	
T	Binary	Average #	Max #	CPU time (ms)	CPU time (ms)	Gap (%)
50	1,326	2.1	3	1.9	166.7	0
100	5,151	2.2	3	5.7	848.3	0
200	20,301	2.7	4	93.5	9,107.1	0
400	80,601	3.6	5	115.1	171,592.2	0
600	180,901	3.6	5	1,684.4	486,364.4	0
800	321,201	4.8	6	5,547.0	605,757.6	0.14
1,000	501,501	5.4	6	7,297.2	638,406.4	0.12

the CPU time of POM and CPX, where the time limit of CPX is 600 seconds. The final column gives the optimality gap of the solutions given by CPX.

From Tables 2 and 3, we see that the proposed POM finds optimal solutions of both the UDD and PCDD models in very short time, and outperforms CPX for all the test instances. Note that when T is greater than 800, CPX fails to find an optimal solution for both the UDD and PCDD models in 600 seconds. To solve the parametric optimization problems, POM only needs to test no more than 10 parameter values in all the test instances. Therefore, the proposed interval search method is very efficient.

We report the performance of the proposed algorithm (labelled "Alg-1") and the algorithm (labelled "Alg-2") proposed by Rostami et al. (2015) for subproblems of the PCDD models in Table 4. The third and fourth columns of Table 4 give the average run time of both algorithms over 10 randomly generated subproblems. The computational results show that the proposed algorithm, which exploits the special cost structure of subproblems, is more efficient, and validate the computation complexity analysis of both algorithms.

4.2. Robustness of DRM

In this subsection, we compare the proposed DRM with SM for two-stage LS-LS. SM uses the sampled mean as the true first moment of the random demand. Thus, SM is a special case of DRM in the sense that the distributional set contains probability distributions with exact first moment.

The procedure of the computational experiments includes the following steps: (i) calculate the sampled mean and covariance matrix based on sampling, and construct the distributional set using the data-driven method; (ii) obtain the robust and stochastic solutions by solving corresponding DRM and SM; (iii) randomly generate a set of 10,000 samples and evaluate the performance of DRM and SM solutions. Specifically, in the first step, we vary the sample size M from 200 to 10,000. In the last step, we calculate the mean and variance of the system cost, and the robust objective cost (the summation of the mean cost and weighted square root term) of both solutions. Further we repeat this procedure 20 times and calculate the average values of these indices. To highlight the comparison, the final results are translated into the ratio form, that is, DRM/SM.

Table 4 Comparison of Algorithms for Subproblems of PCCD Models

T	Binary	Alg-1 (ms)	Alg-2 (ms)
50	1,326	0.7	1.2
100	5,151	2.2	5.1
200	20,301	10.0	34.1
400	80,601	28.6	220.4
800	321,201	435.9	2,038.9
1,600	1,282,401	1,829.2	17,355.7
3,200	5,124,801	8,749.5	178,126.2

Table 5 Comparison of Distributionally Robust Model and Stochastic Model

	M	200	500	1000	2000	5000	10,000
$T = 50$	$\kappa = \frac{\epsilon}{\sqrt{T}}$	2.668	1.778	1.251	0.899	0.591	0.426
	Mean	1.081	1.053	1.033	1.021	1.011	1.007
	Variance	0.655	0.665	0.724	0.758	0.828	0.864
	Robust	0.925	0.943	0.964	0.978	0.988	0.994
$T = 200$	$\kappa = \frac{\epsilon}{\sqrt{T}}$	4.654	3.118	2.093	1.534	0.995	0.702
	Mean	1.108	1.085	1.057	1.045	1.020	1.014
	Variance	0.623	0.651	0.677	0.712	0.790	0.811
	Robust	0.887	0.919	0.937	0.959	0.977	0.985
$T = 400$	$\kappa = \frac{\epsilon}{\sqrt{T}}$	6.296	4.041	2.971	2.057	1.329	0.930
	Mean	1.132	1.101	1.079	1.054	1.037	1.021
	Variance	0.600	0.628	0.655	0.680	0.717	0.788
	Robust	0.866	0.896	0.921	0.940	0.962	0.979

Table 5 reports the performance of DRM and SM on problems with uncorrelated demand distributions when $T = 50$, 200 and 400. We introduce an index $\kappa = \frac{\epsilon}{\sqrt{T}}$ to measure the estimation accuracy of the expectation of demand. For a fixed problem size T, as the sample size M increases, both ε and κ become smaller. From Table 5, we have the following observations. First, for all test instances, compared with stochastic solutions, robust solutions greatly decrease the variance of the system cost at the cost of a relatively smaller increase in expected system cost, and also have smaller robust objective cost. For example, when $\kappa \approx 0.95$, robust solutions decrease the variance of the system cost by about 20% while only increase the expected system cost by about 2%, and robust solutions also decrease the robust objective cost by about 2%. Second, when the distribution information can be well estimated, that is, M is large and κ is small, robust solutions have comparable performance to stochastic solutions in terms of expected system cost and also provides substantial robustness. For example, when $T = 50$ and $M = 10{,}000$, the robust solution only increases the expected system cost by 0.7% while decreases the variance by 13.6%.

From the experimental results, we conclude that DRMs can greatly reduce the system cost variation at the cost of a relative smaller increase in expected system cost, especially when the distribution information of the random demand can be well estimated. Experimental results also show that it is critical to select a proper vale of ε. Note that ε also plays a role of the weight on the risk term $\sqrt{q^T \Sigma q}$, which is the standard deviation of the random system cost. Therefore, the value of ε mainly depends on how much historical information is available to estimate the first moment of the random demand, and the attitude of decision makers towards the risk. If the size of the historical data is small, it is better to choose a large ε, and *vice versa*. To balances the expected cost and risk, a trial-and-error procedure can be used to determine the best value of ε for decision makers.

5. Conclusion

This study employs the distributionally robust optimization method to handle the ambiguity in the probability distribution of the uncertain demand for two-stage LS problems, where lost sales, backlogging and no backlogging are all considered. Although the resulted min-max-min problems are hard to solve directly, we derive equivalent mixed 0-1 CQPs by exploiting shortest path reformulations of LS problems. Experimental results show that the system cost variance of the robust solution is much smaller than that of the stochastic solution, and the proposed models also provide decision makers with more flexibility to balance the expected system cost and risk.

A parametric optimization method is proposed to solve the mixed 0-1 CQPs. The effectiveness of the parametric optimization method depends on its parameter search method and efficient algorithms for parametric subproblems. We propose an effective branch-and-bound-based interval search method to avoid the enumeration of all potential parameter values, and polynomial time algorithms for the UDD and PCDD models. The proposed parametric optimization method can be further used to solve special concave network minimization problems with the form: $\min_{x\in X}\{g(x^{\mathrm{T}}Ax)+x^{\mathrm{T}}Bx\}$, where $g:\ \mathbb{R}\rightarrow\mathbb{R}$ is an increasing concave function and $X\subseteq\{0,1\}^n$ represents network flow constraints, such as the constraints in the shortest path problem, assignment problem and matching problem. By exploiting properties of certain parametric subproblems, efficient algorithms may be designed. Even if the parametric subproblems cannot be solved exactly, the parametric optimization method still provides an approximate solution for the original problem. Properties of such approximate parametric optimization method will be studied in the future.

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References

Absi, N., S. Kedad-Sidhoum, S. Dauzére-Pèrés. 2011. Uncapacitated lot-sizing problem with production time windows, early productions, backlogs and lost sales. *Int. J. Prod. Res.* **49**(9): 2551–2566.

Aksen, D., K. Altinkemer, S. Chand. 2003. The single-item lot-sizing problem with immediate lost sales. *Eur. J. Oper. Res.* **147**(3): 558–566.

Atamtürk, A., G. Berenguer, Z. J. M. Shen. 2012. A conic integer programming approach to stochastic joint location-inventory problems. *Oper. Res.* **60**(2): 366–381.

Ben-Tal, A., D. Den Hertog, A. De Waegenaere, B. Melenberg, G. Rennen. 2013. Robust solutions of optimization problems affected by uncertain probabilities. *Management Sci.* **59**(2): 341–357.

Bertsimas, D., X. V. Doan, K. Natarajan, C. P. Teo. 2010. Models for minimax stochastic linear optimization problems with risk aversion. *Math. Oper. Res.* **35**(3): 580–602.

Carstensen, P. J. 1983. The complexity of some problems in parametric linear and combinatorial programming. Unpublished doctoral dissertation, University of Michigan, Ann Arbor, MI.

Chen, X., M. Sim, P. Sun. 2007. A robust optimization perspective on stochastic programming. *Oper. Res.* **55**(6): 1058–1071.

Chu, F., C. B. Chu. 2008. Single-item dynamic lot-sizing models with bounded inventory and outsourcing. *IEEE Trans. Syst Man Cybern A Syst. Hum.* **38**(1): 70–77.

Delage, E., Y. Y. Ye. 2010. Distributionally robust optimization under moment uncertainty with application to data-driven problems. *Oper. Res.* **58**(3): 595–612.

Evans, J. R. 1985. An efficient implementation of the wagner-whitin algorithm for dynamic lot-sizing. *J. Oper. Manag.* **5**(2): 229–235.

Fan, K. 1953. Minimax theorems. *Proc. Natl Acad. Sci. USA* **39**(1): 42.

Guan, Y. P., A. J. Miller. 2008. Polynomial-time algorithms for stochastic uncapacitated lot-sizing problems. *Oper. Res.* **56**(5): 1172–1183.

Hwang, H. C., VAN DEN Heuvel, A. P. Wagelmans. 2013. The economic lot-sizing problem with lost sales and bounded inventory. *IIE Trans.* **45**(8): 912–924.

Klabjan, D., D. Simchi-Levi, M. Song. 2013. Robust stochastic lot-sizing by means of histograms. *Prod. Oper. Manag.* **22**(3): 691–710.

Lia, D., D. R. Moralesa. 2013. Polynomial time algorithms for the minimax regret uncapacitated lot sizing model. *Optim. Online* **12**(4149): 1–6.

Liu, X., Y. Tu. 2008. Production planning with limited inventory capacity and allowed stockout. *Int. J. Prod. Econ.* **111**(1): 180–191.

Liu, X., F. Chu, C. Chu, C. Wang. 2007. Lot sizing with bounded inventory and lost sales. *Int. J. Prod. Res.* **45**(24): 5881–5894.

Loparic, M., Y. Pochet, L. A. Wolsey. 2001. The uncapacitated lot-sizing problem with sales and safety stocks. *Math. Program.* **89**(3): 487–504.

Mak, H. Y., Y. Rong, J. W. Zhang. 2014. Appointment scheduling with limited distributional information. *Management Sci.* **61**(2): 316–334.

Manne, A. S. 1958. Programming of economic lot sizes. *Management Sci.* **4**(2): 115–135.

Mulmuley, K., P. Shah. 2000. A lower bound for the shortest path problem. Proceedings 15th Annual Conference on Computational Complexity, IEEE, Florence, Italy, 14–21.

Nikolova, E., J. A. Kelner, M. Brand, M. Mitzenmacher. 2006. Stochastic shortest paths via maximization. Y. Azar, T. Erlebach, eds. *European Symposium on Algorithms*. Springer, Cham, Switzerland, 552–563.

Pochet, Y., L. A. Wolsey. 1988. Lot-size models with backlogging: Strong reformulations and cutting planes. *Math. Program.* **40**(1–3): 317–335.

Popescu, I. 2007. Robust mean-covariance solutions for stochastic optimization. *Oper. Res.* **55**(1): 98–112.

Rostami, B., F. Malucelli, D. Frey, C. Buchheim. 2015. On the quadratic shortest path problem. E. Bampis, ed. *International Symposium on Experimental Algorithms*. Springer, Berlin, Germany, 379–390.

Sandbothe, R. A., G. L. Thompson. 1990. A forward algorithm for the capacitated lot size model with stockouts. *Oper. Res.* **38**(3): 474–486.

Sandbothe, R. A., G. L. Thompson. 1993. Decision horizons for the capacitated lot size model with inventory bounds and stockouts. *Comput. Oper. Res.* **20**(5): 455–465.

Scarf, H., K. J. Arrow, S. Karlin. 1958. A min-max solution of an inventory problem. K. J. Arrow, S. Karlin, H. Scarf, eds. *Studies in the Mathematical Theory of Inventory and Production*, vol. 10. Stanford, CA, USA, 201–209.

Shen, Z. J. M. 2007. Integrated stochastic supply-chain design models. *Comput. Sci. Eng.* **9**(2): 50–59.

Shen, Z. J. M., L. Qi. 2007. Incorporating inventory and routing costs in strategic location models. *Eur. J. Oper. Res.* **179**(2): 372–389.

Shu, J., C. P. Teo, Z. J. M. Shen. 2005. Stochastic transportation-inventory network design problem. *Oper. Res.* **53**(1): 48–60.

Tuy, H., N. D. Dan, S. Ghannadan. 1993a. Strongly polynomial time algorithms for certain concave minimization problems on networks. *Oper. Res. Lett.* **14**(2): 99–109.

Tuy, H., S. Ghannadan, A. Migdalas, P. V. VÄarbrand. 1993b. Strongly polynomial algorithm for a production-transportation problem with concave production cost. *Optimization* **27**(3): 205–227.

Tuy, H., S. Ghannadan, A. Migdalas, P. Värbrand. 1996. A strongly polynomial algorithm for a concave production-transportation problem with a fixed number of nonlinear variables. *Math. Program.* **72**(3): 229–258.

Wagner, H. M., T. M. Whitin. 1958. Dynamic version of the economic lot size model. *Management Sci.* **5**(1): 89–96.

Wiesemann, W., D. Kuhn, M. Sim. 2014. Distributionally robust convex optimization. *Oper. Res.* **62**(6): 1358–1376.

Zhang, M. H. 2011. Two-stage minimax regret robust uncapacitated lot-sizing problems with demand uncertainty. *Oper. Res. Lett.* **39**(5): 342–345.