Homework #8

1. Consider a renewal process with the inter-renewal times being random numbers (i.e., continuous uniform between 0 and 1). Obtain an expression for $\lim_{t\to\infty}P\{V(t) <= x\}$ (i.e., give the limiting distribution for the time until the next renewal).

   $1-F(x) = 1-x$ for $0<x<1$, and $\mu = 0.5$ or $(1/\mu) = 2$.

   $\lim_{t\to\infty}P\{V(t) <= x\} = 1 - 2\int_x^1 (1-u)du = \ldots = x(2-x)$ for $0<x<1$.

2. Consider an item installed at time 0. When if fails, it is replaced by an identical item; when that item fails, it in turn is replaced by a new item; and so on. Suppose the lifetime of successive items are i.i.d. random variables $U_1, U_2, \ldots$ with distribution function given by $\varphi(.)$. Let the time to replace the items be denoted by i.i.d. random variables $V_1, V_2, \ldots$ with distribution function given by $\psi(.)$. Further assume that the lifetimes and replacement times are independent.

   Define the process $\{Z(t)\}$ such that $Z(t) = 1$ if the item is working at time t and $Z(t)=0$ if the item is being replaced at time t. First convince yourself that $\{Z(t)\}$ is a regenerative process and give an expression for the inter-renewal times.

   a. Obtain an expression for $P\{Z(t) = 1\}$.

Let $S_n = U_n + V_n$. $P\{Z(t) =1 \mid S_1 > t\} = P\{U_1 > t\}$ (Note also $\{U_1>t\}$ implies $\{S_1>t\}$.)

$P\{Z(t)=1\} = P\{U_1 > t\} + \int_{[0,t]} F(ds)P\{Z(t-s) = 1\}$ and also $P\{U_1>t\} = 1-\varphi(t)$

where $F(t) = \varphi * \psi(t) = P\{S_1<=t\}$ and let $m(t) = \Sigma F_n(t)$.

Therefore, $P\{Z(t)=1\} = 1-\varphi(t) + \int_{[0,t]} m(ds)[1-\varphi(t-s)]$

   b. Obtain an expression for $\lim_{t\to\infty}P\{Z(t) = 1\}$. Let $a=E[U_1]$ and $b=E[V_1]$.

Note that $\mu = E[S_1] = E[U_1+V_1] = a + b$.

$\lim_{t\to\infty}P\{Z(t) = 1\} = (1/(a+b)) \int_0^\infty (1-\varphi(t))dt = a/(a+b)$