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# Worst-case robust design optimization under distributional assumptions

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## SUMMARY

Presented in this paper is a novel robust design optimization (RDO) methodology. The problem is reformulated in order to relax, when required, the assumption of normality of objectives and constraints, which often underlies RDO. In the second place, taking into account engineering considerations concerning the risk associated with constraint violation, suitable estimates of tail conditional expectations are introduced in the set of robustness metrics. A computationally affordable yet accurate implementation of the proposed formulation is guaranteed by the adoption of a reduced quadrature technique to perform the uncertainty propagation. The methodology is successfully demonstrated with the aid of an industrial test case performing the sizing of a mid-range passenger aircraft. Copyright © 2011 John Wiley & Sons, Ltd.

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## 1. INTRODUCTION

The strive for increased performance and reduced costs within the aerospace industry observed during the last decades has emphasized the importance of accounting for the interactions between disciplines such as aerodynamics, propulsion, structures and control as early as possible in the design lifecycle. The methods under the subject of multidisciplinary design analysis and optimization (MDAO) follow such paradigm by integrating the computational tools that model the various aspects of the product under development and by coupling them with an appropriate optimization algorithm to systematically explore the design space. However, the product development process is usually affected by severe uncertainty from the outset. For example, the complete problem frame might be only approximately known, the adopted computer models have low fidelity and assumptions based on previous experience have to be largely used. In such a context, the usefulness of deterministic optimization strategies is questionable. In fact, the nominal performance of the optimum design might be significantly degraded by unforeseen variations in the problem variable and parameters. Furthermore, if optimality is found on the design constraint boundaries, such variations may render the optimal design solution unfeasible. More articulated approaches, such as robust design optimization (RDO), are hence required, which are able to incorporate and exploit the available uncertain information regarding the design. Supposing that such information is expressed

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by means of stochastic models—either on the basis of sufficiently significant statistical data or by relying on expert opinion, or a combination of both—the problem is then tackled in two steps:

1. propagating the uncertainty through the analysis system, to obtain adequate metrics for assessing the non-deterministic behavior of the objective functions and constraints;
2. optimizing such metrics by means of an appropriate algorithm.

RDO objective and constraint functions are the mathematical expressions of the rationale behind the design approach to coping with uncertainty. The significance and interpretability of the optimal design solutions depends on how well such rationale is embodied in the problem. In addition, the formulation of objectives and constraints determines the choice of the propagation method and, indirectly, the choice of the optimization algorithm. This, in turn, highlights its importance for identifying a favorable trade-off between the accuracy and the efficiency of the entire RDO strategy. The main objective of this paper is therefore to propose a novel methodology for constructing RDO objective and constraint functions, which enables meaningful design choices and an affordable numerical implementation. The remainder of the paper is organized as follows. State-of-the-art objectives and constraints formulations are presented in detail in the following section. The novel methodology is presented in Section 3 and demonstrated in Section 4 with the aid of an aircraft sizing test case of industrial relevance. Section 5 presents the research conclusions and identifies the scope for further work.

## 2. BACKGROUND

Assume that the design analyses are performed by continuous functions $f(\mathbf{x})$ and $g_i(\mathbf{x})$, $i=1,2,\ldots,I$, where $\mathbf{x}\in[\mathbf{x}_L,\mathbf{x}_U]\subseteq\mathrm{R}^n$ is the vector of design variables. The *deterministic design optimization problem* is hence formulated as follows:

$$\begin{aligned}&\text{Find}\,\mathbf{x}\in\mathbb{R}^n \text{ to minimize } f(\mathbf{x}),\\&\text{subject to: } g_i(\mathbf{x})\leqslant 0, \quad i=1,2,\ldots,I,\\&\text{and: } \mathbf{x}_L\leqslant\mathbf{x}\leqslant\mathbf{x}_U.\end{aligned} \tag{1}$$

The *design optimization problem under uncertainty* resulting from the stochastic modeling of the design variables (an analogous reasoning could be made with respect to uncertain parameters) can be written as follows:

$$\begin{aligned}&\text{Find } \boldsymbol{\mu}_\mathbf{x}\in\mathbb{R}^n \text{ to minimize } F[f(\mathbf{x})],\\&\text{subject to: } P(g_i(\mathbf{x})\leqslant 0)\geqslant P_{0_i}, \quad i=1,2,\ldots,I,\\&\text{and: } P(\mathbf{x}_L\leqslant\mathbf{x}\leqslant\mathbf{x}_U)\geqslant P_\mathbf{b},\end{aligned} \tag{2}$$

where $\mathbf{x}$ is a vector of independent continuous random variables with mean $\boldsymbol{\mu}_\mathbf{x}$ and $P_{0i}$ and $P_\mathbf{b}$ are the desired probabilities of satisfying the $i$th constraint and the input bounds, respectively. The objective function $F$ depends, in general, on the probability density function (PDF) of the objective $p_f$ induced by the multivariate input PDF $p_\mathbf{x}$. In RDO, this dependence is usually expressed in terms of the expectation and variance of the objective $f$, and thus $F=F(\mu_f,\sigma_f^2)$. Such moments can be obtained through several available numerical methods, including Taylor-based method of moments, Monte Carlo simulation, Stochastic Expansion and Gaussian quadrature [1]. In such setting, the optimization of the expectation over the input distributions maximizes design performance, while variance minimization reduces the sensitivity of the solution to undesired change in the input variables. In the literature, the interplay of performance and robustness has been aggregated in a single response function by using loss or utility functions [2, 3]. Weighted sums of mean and standard deviation are also commonly adopted (see for example [4, 5]). They express the robust objective as follows:

$$F=\mu_f+k_f\sigma_f. \tag{3}$$



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$k_f$ is a positive real-valued coefficient which has to be chosen by the designer to quantify the desired robustness level of the sought solution. In fact, the use of weights implies that $\mu_f$ and $\sigma_f$ are thought of as representing two conflicting objectives, namely, design performance and sensitivity. This has been the main reason for the deployment of multi-objective optimization strategies to handle the robust counterpart of a single-objective deterministic optimization problem [5–9]. Please note that Equation (3) also represents a particular case of weighted-sum objective formulation for the multi-criteria optimization problem in which $\mu_f$ and $\sigma_f$ both are to be minimized. Alternative approaches [10, 11] include either the optimization of the expectation with the variance constrained or vice versa.

The constraints $g_i(\mathbf{x})$ identify crisp boundaries for the feasible region of the deterministic optimization problem. By contrast, a probabilistic definition of feasibility is required when dealing with engineering design under aleatory uncertainty, as given by the following integral:

$$P(g_i(\mathbf{x})\leqslant 0)=\Psi(\mathbf{x})=\int_{g_i(\mathbf{x})\leqslant 0} p_{\mathbf{x}}(\mathbf{x})\,\mathrm{d}\mathbf{x}, \tag{4}$$

where $\Psi(\mathbf{x})$ is the cumulative distribution function (CDF) for the constraint satisfaction. It is important to stress that in general $\Psi(\mathbf{x})\neq\Phi(\mathbf{x})$, where $\Phi(\mathbf{x})$ is the CDF for the normal distribution. The calculation of the integral in Equation (4) is obtained in an approximate fashion for most cases of interest. For example, Monte Carlo simulation can be employed for the purpose. However, a large number of samples would be required to obtain a suitable confidence level for demanding constraint satisfaction targets, even by using variance reduction techniques [12, 13] or quasi-Monte Carlo methods [14]. In the least expensive way, the probability of feasibility might be described as function of the constraint moments, as commonly done in RDO, under the hypothesis that the constraint distribution is normal [15]:

$$G_i=\mu_{g_i}+k_{g_i}\sigma_{g_i}\leqslant 0, \tag{5}$$

where $k_{g_i}=\Phi^{-1}(P_{0i})$. $\mu_{g_i}$ and $\sigma_{g_i}$ are usually estimated by Taylor-based method of moments. The assumption of normality can be supported in the following ways [16]: if $\mathbf{x}$ is normally distributed and $g_i$ is approximately linear in $\mathbf{x}$, then the PDF of the constraint, $p_{g_i}$, is approximately normal; if the dimension of $\mathbf{x}$ is large, affinity of $g_i$ is a sufficient condition to achieve asymptotic normality of $p_{g_i}$ under the central limit theorem, regardless of the distribution of $\mathbf{x}$. Other, more complex conditions can be established in some cases of non-linear $g_i$ [17, 18]. When $g_i$ is not normally distributed, moment-based constraint-handling strategies can only give approximate results. Such an approach is often adopted in the robust design literature [5, 19–22] since the estimation of the first moments of the constraints is less computationally demanding than the calculation of the CDF. However, it is deemed unacceptable for the degree of accuracy and the probability levels usually required by Reliability-Based Design Optimization (RBDO), where it is known under the name of mean value (MV) approach [23]. This difference is due to the peculiar scenarios of interest [24]: while RDO tries to minimize the sensitivity of the optimum with respect to routinary fluctuations degrading the performance of the designed system, RDBO is instead concerned with rare events, which have to be limited by a suitable probability of occurrence since they lead to system failure. More advanced Reliability-based approaches couple non-linear optimization methods and integration techniques. Such approaches consist of three stages. First, it is usually considered convenient to transform the random variables $\mathbf{x}$ into standard, uncorrelated normal variables, via the Rosenblatt or the Nataf transformation [25, 26]. Second, the most probable point of constraint activation (MPP) is sought by minimizing the distance between a linear or quadratic approximation of the constraint and the mean of the multivariate input distribution [23, 27]. Third, the probability of constraint satisfaction is estimated, through first- or second-order reliability methods formulas [23, 27], or by coupling the MPP search with variance reduction techniques [28, 29]. MPP-search-based methods rely on the assumption of monotonicity of the constraint function over the interval spanned by the input distributions [30]. Such approaches might overestimate the actual probability of feasibility in cases where the function is not monotonic, and the constraint function crosses the limit state multiple times [31]. In such a case, the efforts have to be focused on finding the global MPP, and the



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resulting accuracy has to be checked, for example by using MCS approaches. Reliability-based constraint formulations have been already considered in a unified approach together with robust objectives [28, 32]. The computational cost of such a strategy is clearly increased with respect to the moment-matching formulation. However, the achievable improvement is not usually considered worth the effort required, in applications for which reliability is not an issue.

Some significant drawbacks concerning the state-of-the-art formulations of robust objectives and constraints emerge from the above analysis. On the one hand, while the choice of the weights for objectives weighted-sum formulations is unclear, multi-objective strategies involve a steep increase in the required computational cost. On the other hand, there is a lack of intermediate options between the accurate but expensive formulations of the probabilistic constraints and the cheap but potentially unreliable moment-based formulations. Proposed in the following section is a methodological solution to tackle such limitations.

## 3. WORST-CASE RDO UNDER DISTRIBUTIONAL ASSUMPTIONS

Consider the case in which the robust objectives and constraints are expressed by weighted sums, as follows:

$$\begin{aligned}&\text{Find } \boldsymbol{\mu}_{\mathbf{x}} \in \mathbb{R}^n \text{ to minimize } F = \mu_f + k_f \sigma_f\\ &\text{subject to: } G_i = \mu_{g_i} + k_{g_i}\sigma_{g_i} \leqslant 0, \quad i = 1, 2, \ldots, I,\\ &\text{and: } \mathbf{x}_L + \mathbf{k}_{\mathbf{x}}\boldsymbol{\sigma}_{\mathbf{x}} \leqslant \boldsymbol{\mu}_{\mathbf{x}} \leqslant \mathbf{x}_U - \mathbf{k}_{\mathbf{x}}\boldsymbol{\sigma}_{\mathbf{x}}\end{aligned} \tag{6}$$

in which $\boldsymbol{\sigma}_{\mathbf{x}}$ is the vector of the standard deviations of the design variables, $\mathbf{k}_{\mathbf{x}}$ is a vector of $n$ positive real-valued components, while $k_f$ and $k_{g_i}$ are positive real-valued scalar coefficients. The key to our approach lies in the choice of $k_f$, $k_{g_i}$ and $\mathbf{k}_{\mathbf{x}}$, which departs from the assumption that the distributions of objectives, constraints and input variables are normal. While this might be true in the case of the input variables, since the distributions may be chosen to be normal, or transformed into standard normal variables (e.g. via the Rosenblatt transformation [25]), the same cannot be said about the objectives and constraints, since the underlying functions are not known *a priori*. In fact, the normality assumption might be unjustified, and negatively impact on the result of the optimization. If the distributional assumptions are completely removed, however, the knowledge of the first two moments of the output PDFs $p_f$ and $p_{g_i}$ would provide only a loose probability bound, which is given, for the generic function $y$, by the Chebyshev inequality [33]:

$$P(|y(\mathbf{x}) - \mu_y|) \geqslant k_y \sigma_y) \leqslant \frac{1}{k_y^2}, \tag{7}$$

where $k_y$ is a positive real number, $\mu_y < \infty$ and $\sigma_y \neq 0$ are mean and standard deviation of $y$. A one-sided version of this inequality, also known as Cantelli inequality [34], is given by the following formula:

$$P(y(\mathbf{x}) - \mu_y \geqslant k_y \sigma_y) \leqslant \frac{1}{1 + k_y^2}. \tag{8}$$

Such probability bound might be too conservative to be of practical use for design optimization. For example, the probability of satisfying a single constraint, imposed by shrinking the design space through a factor $k_y = 2$ would be just $1 - 1/(1 + k_y^2) = 0.8$, versus the value of 0.977 given by the normal assumption. Our aim is to find an intermediate approach which relaxes the normality assumption on the output distributions and can still supply meaningful bounds for the moment problem. For such purposes, some known mathematical results are recalled in the following subsection.

### 3.1. Probability bounds under distributional assumptions

The bound given by Equation (8) could be improved as follows, by assuming that the output PDF $p_y$ is symmetric [35]:

$$P(y(\mathbf{x})-\mu_y \geqslant k_y \sigma_y) \leqslant \begin{cases} \dfrac{1}{2k_y^2} & \text{if } k_y \geqslant 1; \\ \dfrac{1}{2} & \text{if } k_y \leqslant 1. \end{cases} \tag{9}$$

A second possible assumption is related to unimodality, which denotes distributions characterized by a CDF which is convex until the mode, and concave thereafter, such as the triangular, Student, chi-squared and uniform distributions. The Gauss inequality is useful in this direction. It holds for symmetric unimodal distributions and is given by the following formulas [36]:

$$P(|y(\mathbf{x})-\mu_y| \geqslant k_y \sigma_y) \leqslant \begin{cases} \dfrac{4}{9k_y^2} & \text{if } k_y \geqslant \sqrt{\dfrac{4}{3}}; \\ 1-\dfrac{k_y}{\sqrt{3}} & \text{if } k_y \leqslant \sqrt{\dfrac{4}{3}}. \end{cases} \tag{10}$$

A one-sided version of this inequality can be expressed as follows [35]:

$$P(y(\mathbf{x})-\mu_y \geqslant k_y \sigma_y) \leqslant \begin{cases} \dfrac{2}{9k_y^2} & \text{if } k_y \geqslant \dfrac{2}{3}; \\ \dfrac{1}{2} & \text{if } k_y \leqslant \dfrac{2}{3}. \end{cases} \tag{11}$$

The Gauss inequality has been generalized by Vysochanskij and Petunin to the case of asymmetric unimodal variables [33]:

$$P(|y(\mathbf{x})-\mu_y| \geqslant k_y \sigma_y) \leqslant \begin{cases} \dfrac{4}{9k_y^2} & \text{if } k_y \geqslant \sqrt{\dfrac{8}{3}}; \\ \dfrac{4}{3k_y^2}-\dfrac{1}{3} & \text{if } k_y \leqslant \sqrt{\dfrac{8}{3}}, \end{cases} \tag{12}$$

while the one-sided Vysochanskij–Petunin inequality is given by the following formulas [37]:

$$P(y(\mathbf{x})-\mu_y \geqslant k_y \sigma_y) \leqslant \begin{cases} \dfrac{4}{9(1+k_y^2)} & \text{if } k_y \geqslant \sqrt{\dfrac{5}{3}}; \\ 1-\dfrac{4}{3}\dfrac{k_y^2}{1+k_y^2} & \text{if } k_y \leqslant \sqrt{\dfrac{5}{3}}. \end{cases} \tag{13}$$

The impact of the different distributional assumptions can be reviewed in Figure 1, which illustrates the probability bounds expressed by the above inequalities, along with the exact value given by the normal distribution. Note that the two-tailed Vysochanskij–Petunin bound coincides with the Gauss bound for $k_y \geqslant \sqrt{\frac{8}{3}}$, despite relaxing the symmetric assumption, and with the Chebyshev bound for $k_y = 1$. For $k_y \leqslant 1$ neither the Chebyshev nor the Vysochanskij–Petunin bound is usable. In the case of one-tailed inequalities, it can be observed that for $k_y < \sqrt{\frac{3}{5}}$ the symmetrical Chebyshev bound lies below the Vysochanskij–Petunin bound. This means that the assumption of symmetry results, for those values of $k_y$, in a more informative bound than the one given by the unimodality assumption. The situation is reversed for $k_y > \sqrt{\frac{3}{5}}$. In the following subsection, the above presented set of probabilistic bounds are exploited to formulate the robust objectives and constraints.



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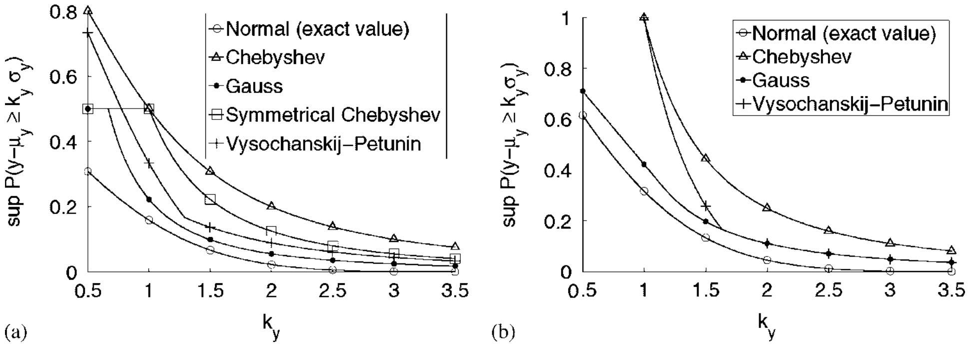


Figure 1. The effect of $k_y$ on the probability bounds: (a) one-tailed inequalities and (b) two-tailed inequalities.

Table I. Coefficient $k_{g_i}$ as a function of the probability of feasibility for Problem (6).

| | Distributional assumption | $k_{g_i}(P_{0i})$ | Validity |
|---|---|---|---|
| I | None | $k_{g_i}=\sqrt{\frac{P_{0i}}{1-P_{0i}}}$ | $0\leqslant P_{0i}\leqslant 1$ |
| II | Symmetry | $k_{g_i}=\frac{1}{\sqrt{2(1-P_{0i})}}$ | $P_{0i}\geqslant\frac{1}{2}$ |
| II | Unimodality | $k_{g_i}=\sqrt{\frac{9P_{0i}-5}{9(1-P_{0i})}}$ | $P_{0i}\geqslant\frac{5}{6}$ |
| | | $k_{g_i}=\sqrt{\frac{3P_{0i}}{(4-3P_{0i})}}$ | $P_{0i}\leqslant\frac{5}{6}$ |
| IV | Symmetry+unimodality | $k_{g_i}=\sqrt{\frac{2}{9(1-P_{0i})}}$ | $P_{0i}\geqslant\frac{1}{2}$ |

### 3.2. *Constraints and objectives formulation*

The probability bounds presented in the previous section translate into the constraint formulation as follows. Given the choice of the relevant assumption for the constraint $g_i$, for which one of the inequalities above holds, and also given the corresponding required probability of satisfaction $P_{0i}$, it holds that:

$$P(g_i(\mathbf{x})\geqslant\mu_{g_i}+k_{g_i}\sigma_{g_i})\leqslant 1-P_{0i}, \tag{14}$$

where the coefficient $k_{g_i}$ is chosen so that $1-P_{0i}$ equals the right hand side of the relevant inequality among those in Equations (8), (9), (11) and (13) (relationships $k_{g_i}=k_{g_i}(P_{0i})$ for the various distributional assumptions are summarized in Table I). Hence, by formulating the robust constraint as:

$$G_i=\mu_{g_i}+k_{g_i}\sigma_{g_i}\leqslant 0 \tag{15}$$

as required by Problem (2), it is imposed that:

$$P(g_i(\mathbf{x})\leqslant 0)\geqslant P_{0i}\Longleftrightarrow g_{i,P_{0i}}\leqslant 0, \tag{16}$$

where $g_{i,P_{0i}}$ is the $P_{0i}$-quantile of $g_i(\mathbf{x})$, defined for continuous random variables as the value such that $P(g_i(\mathbf{x})\leqslant g_{i,P_{0i}})=P_{0i}$. Since $k_{g_i}$ is obtained through a probabilistic bound and not through an exact relationship such as in the Gaussian case, Equation (15) constrains the worst case over the assumed class of distributions (e.g. the unimodal or the symmetric distributions), which have the given mean and standard deviation.

The objective formulation is also affected by such assumptions. In fact, the minimization of a weighted sum $\mu_f+k_f\sigma_f$ corresponds to searching the smallest threshold which is exceeded by $f$ with a probability not greater than a given probability $1-P_{0f}$. The relationship between such

 

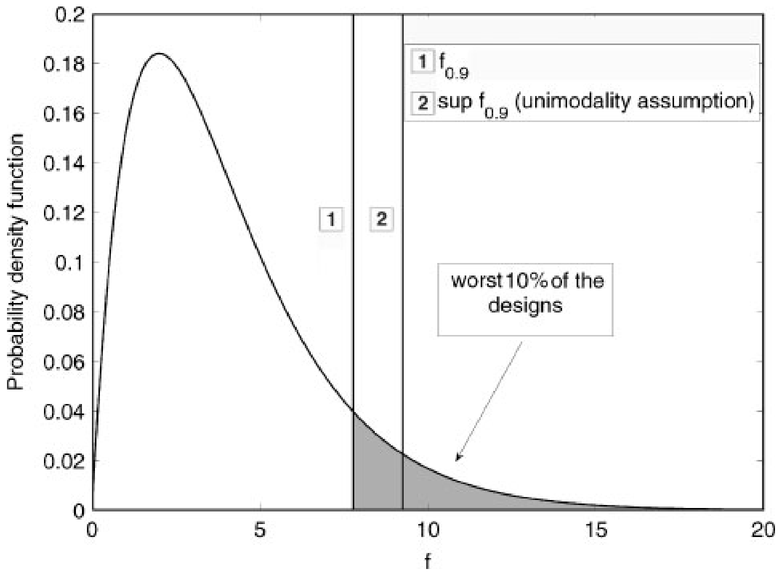


Figure 2. The robust objective $F$ as a worst-case threshold for the case of $f \sim \Gamma(2,2)$.

probability and the coefficient $k_f$ can be obtained from Table I, by substituting $k_{g_i}$ with $k_f$ and $P_{0i}$ with $P_{0f}$.

For example, suppose that the underlying (unknown) distribution of $f$ is a $\Gamma(2,2)$ distribution as shown in Figure 2. By assuming that such distribution is unimodal, to ensure that at least the 90% of the designs lie below the robust objective $F=\mu_f+k_f\sigma_f$, $k_f=1.856$ should be adopted from Table I, case III. Hence, by minimizing $F$, the worst-case 0.9-quantile threshold for $f$ is minimized, under a specified distributional assumption. This view allows to deal with the mean and standard deviation weighted sum formulation as a justifiable single objective. Therefore, despite adopting the same mathematical formalization of weighted sum multi-objective problems [5, 38], the worst-case RDO approach avoids the numerical difficulties inherent in the latter approach [39] by resorting to a different probabilistic design rationale.

The choice of $k_f$ and $k_{g_i}$ depends on the information available about the functions at hand. In many cases of interest, the designer has at least partial knowledge of the function behavior in the uncertain parameters domain. This knowledge might come from lower order theories, reduced order models, surrogate models and so on. Suppose for example that $\tilde{f}(\mathbf{x})$ is an analytical, coarse approximation for $f(\mathbf{x})$, which could be inexpensively interrogated by an MCS algorithm. While the probabilistic information this experiment could provide might not be sufficiently accurate for design purposes, it could still supply a qualitative information, which could help to frame $p_f$ within one of the considered classes of distributions. This would provide a guideline for the choice of $k_f$, given the desired $P_{0f}$.

The proposed formulation allows to increase the generality of robust design optimization by relaxing the assumptions on the output distributions. The fact that such assumptions are often implicit and overlooked is crucial to understand the importance of the reinterpretation of the RDO problem in terms of moments.

### *3.3. Extension to the tail conditional expectation*

The criteria driving the robust optimization should not overlook the behavior of the system response once the identified constraints' limits are exceeded. This might be relevant, in particular, for designs at the conceptual stage, for which probabilities of violating the constraints of about 10% or more might be deemed acceptable. Assessing such designs in terms of a probability $1-P_{0i}$ for which a prescribed limit $g_i=0$ does not have to be violated, as in Equation (16), may not enable the exercise of engineering judgment on a significant portion of the distribution tails. In fact, nothing



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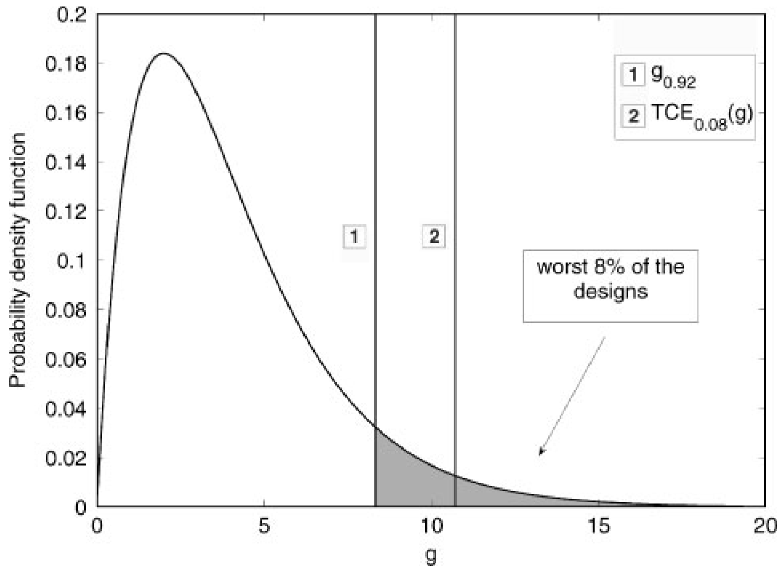


Figure 3. Comparison between TCE and quantile constraints for the case of $g \sim \Gamma(2,2)$.

is said about the extent of such constraint violation, and hence it is not possible to discern designs which are on average closer to the feasible region from designs which lie further away.

A possible solution is the adoption of a metric such as the tail conditional expectation (TCE), which for a probability of constraint satisfaction $P_{0i}$, is defined as follows:

$$\mathrm{TCE}_{1-P_{0i}}(g_i) = E[g_i | g_i \geqslant g_{i,P_{0i}}] = \frac{1}{1-P_{0i}} \int_{\Omega} g_i(\mathbf{x}) p_{\mathbf{x}}(\mathbf{x}) \, \mathrm{d}\mathbf{x}, \tag{17}$$

where $\Omega = \{\mathbf{x} : g_i(\mathbf{x}) \geqslant g_{i,P_{0i}}\}$ is the unfeasible region. Therefore TCE, which is equivalent to Conditional Value at Risk and Expected Shortfall for continuous output distributions [40], quantifies the expected functional value once the probabilistically defined feasible bounds are exceeded. For example, suppose that the true (but unknown in general) model for the random behavior of $g(\mathbf{x})$ is, in correspondence of a certain point $\mathbf{x}$ of the design space, a $\Gamma(2,2)$ distribution, and that a probability of feasibility $P_0 = 0.92$ is of interest. As shown in Figure 3, while the quantile $g_{0.92}$ identifies the specific design which can occur with probability 0.08, $\mathrm{TCE}_{0.08}(g)$ takes into account all the designs in the worst 8% range, by calculating their expected value. This entails a number of consequences for engineering design. Perhaps the most important is the consideration that the conditional expectation couples an information about the possible losses deriving from exceeding the constraints with the probability of such an event happening. It may then be interpreted as quantifying the risk associated with getting into the unfeasible region or, in short, the risk of unfeasibility. Such quantification can be useful for establishing suitable margins on design feasibility, or it can be interpreted within a probabilistic fail-safe perspective, which seeks to minimize the (expected) losses one would incur if the constraints are violated.

Fortunately, TCE metrics can be included within our framework without compromising the computational cost of the optimization. Under assumptions similar to the presented above, in fact, tractable mathematical expressions can be obtained for $\mathrm{TCE}_{1-P_{0i}}(g_i)$ in terms of the first two moments of $g_i$. For the normal case, the following exact result can be obtained [41]:

$$\mathrm{TCE}_{1-P_{0i}}(g_i) = \mu_{g_i} + \frac{\phi(\Phi^{-1}(P_{0i}))}{1-P_{0i}} \sigma_{g_i}, \tag{18}$$

where $\phi$ and $\Phi$ are, respectively, the density and cumulative density of the standard Gaussian variable. However, in the absence of sufficient information to completely characterize the constraint distribution, suitable inequalities could be adopted, which bound $\mathrm{TCE}_{1-P_{0i}}(g_i)$ under specific



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distributional assumptions. The most general expression is an inequality of the Chebyshev type applied to the conditional expectation [42, 43]:

$$\mathrm{TCE}_{1-P_{0i}}(g_i) \leqslant \mu_{g_i} + \sqrt{\frac{P_{0i}}{1-P_{0i}}}\sigma_{g_i}. \tag{19}$$

In the case where symmetry can be assumed for the output distribution, the following inequality can be used to bound $\mathrm{TCE}_{1-P_{0i}}(g_i)$ [43]:

$$\mathrm{TCE}_{1-P_{0i}}(g_i) \leqslant \begin{cases} \mu_{g_i} + \dfrac{\sigma_{g_i}}{\sqrt{2(1-P_{0i})}}, & \text{if } P_{0i} \geqslant \dfrac{1}{2}; \\ \mu_{g_i} + \dfrac{\sigma_{g_i}}{(1-P_{0i})}\sqrt{\dfrac{P_{0i}}{2}}, & \text{if } P_{0i} \leqslant \dfrac{1}{2}. \end{cases} \tag{20}$$

As for the quantile formulation, the adoption of bounds for the TCE allows to constrain the worst-case specimen over the assumed class of distribution. An interesting property of the TCE bounds appears from considering the quantile constraint formulation presented in the previous section:

$$g_{i,P_{0i}} \leqslant \mu_{g_i} + k_{g_i}\sigma_{g_i} \leqslant 0. \tag{21}$$

When $k_{g_i}$ is taken from the distributional assumptions I or II (for $P_{0i} > 1/2$) in Table I, such inequality gives the same bounds of Equations (19) and (20), respectively. Therefore, since $\mathrm{TCE}_{1-P_{0i}}(g_i) \geqslant g_{i,P_{0i}}$ by definition, the following inequalities hold:

$$g_{i,P_{0i}} \leqslant \mathrm{TCE}_{1-P_{0i}}(g_i) \leqslant \mu_{g_i} + k_{g_i}\sigma_{g_i} \leqslant 0. \tag{22}$$

This means that the TCE bound is tighter than the quantile bound, and in case no assumption can be made on the output distribution, or in case such distribution is assumed symmetric, it would be less conservative to adopt a TCE constraint rather than a quantile constraint. An example of this property is shown in Figure 4. Hence, suppose again that the underlying model for the random behavior of $g(\mathbf{x})$ is a $\Gamma(2,2)$ distribution, and that a probability of feasibility $P_0 = 0.92$ is of interest. The Chebyshev bounds for the 0.92-quantile $g_{0.92}$ and for the TCE, calculated by using Equations (8) and (19), respectively, coincide and are approximately equal to 13.59. However, the true values for the quantile and the TCE, which could be obtained if the distribution were known, are 8.33 and 10.72, respectively. This means that adopting the TCE bound to constrain $g(\mathbf{x})$ would result in a drastic reduction (45% in this case) of the overconservativeness yielded by the corresponding quantile bound. Such improvement is achieved by substituting quantile constraints with TCE constraints and requires no additional assumption nor further calculation. Note that the TCE formulation is not limited to the constraints $g_i$, but can also be extended to the objective $f$, in analogy with the reasoning in Section 3.2. In this case, the weighted sum objective can be interpreted as a threshold for the expectation of $f$ over a selected percentage of low performance designs. The proposed worst-case RDO methodology assumes that a stochastic description of the input variables is possible for the problem at hand. The worst-case metrics for objective and constraints are defined on the basis of the available partial knowledge on their distribution functions, i.e. as the worst quantile or TCE given a target probability and, possibly, a qualitative distributional assumption. This marks the difference with other worst-case approaches to design under uncertainty, such as those based on convex set representations, in which the worst case is identified by the least favorable combination of the input variables within their envelopes [44, 45].

### *3.4. Adopted propagation method*

The practical implementation of Problem (6) requires a trade-off between the accuracy of the objectives' and constraints' mean and variance estimates and the computational cost they involve. Suppose that the objective function $f$ (the same considerations hold for the constraint $g_i$) is differentiable a sufficient number of times with respect to the uncertain variables. Its statistical



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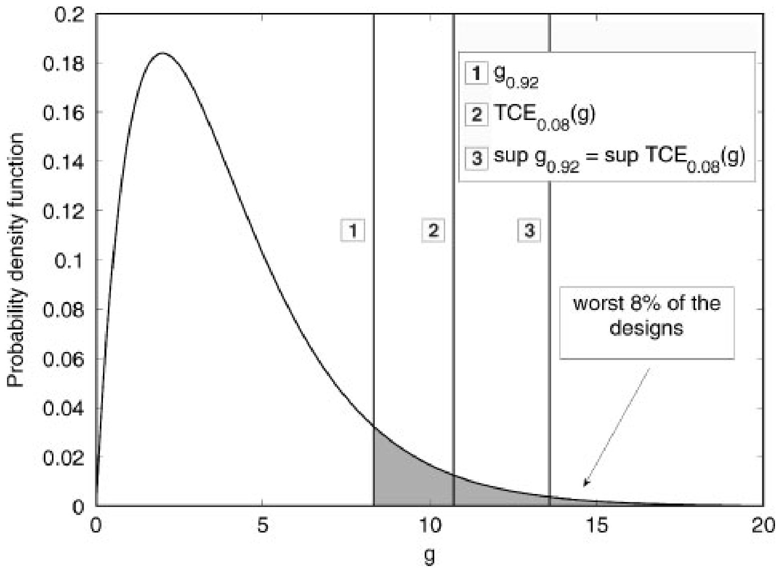


Figure 4. TCE bounds are tighter than quantile bounds in the Chebyshev (shown) and symmetric cases, as for this example, in which $g \sim \Gamma(2,2)$.

moments $\mu_f$ and $\sigma_f^2$ can then be calculated through a Taylor series expansion around the mean of the input variables. The resulting method is also known as the method of moments (MM). Consider the Taylor series expansion of $f$ truncated to the fourth order:

$$f_{MM} = f(\boldsymbol{\mu}_{\mathbf{x}}) + \sum_{p=1}^{n}\left(\frac{\partial f}{\partial x_p}\right)\Delta x_p + \frac{1}{2}\sum_{p=1}^{n}\sum_{q=1}^{n}\left(\frac{\partial^2 f}{\partial x_p \partial x_q}\right)\Delta x_p \Delta x_q$$
$$+\frac{1}{6}\sum_{p=1}^{n}\sum_{q=1}^{n}\sum_{r=1}^{n}\left(\frac{\partial^3 f}{\partial x_p \partial x_q \partial x_r}\right)\Delta x_p \Delta x_q \Delta x_r$$
$$+\frac{1}{24}\sum_{p=1}^{n}\sum_{q=1}^{n}\sum_{r=1}^{n}\sum_{s=1}^{n}\left(\frac{\partial^4 f}{\partial x_p \partial x_q \partial x_r \partial x_s}\right)\Delta x_p \Delta x_q \Delta x_r \Delta x_s + O(\Delta \mathbf{x}^5) \quad (23)$$

in which $\Delta x_p = x_p - \mu_{x_p}$ and the partial derivatives of $f$ with respect to the input variables are computed at $\mathbf{x} = \boldsymbol{\mu}_{\mathbf{x}}$. The term $O(\Delta \mathbf{x}^5)$ denotes the remainder, which includes all the neglected terms, of fifth order or higher. The following expressions can be obtained for mean and variance, for the case of independent input variables:

$$\mu_{f_{MM}} = \int f_{MM}\, p_{\mathbf{x}}\, \mathrm{d}\mathbf{x} = \overbrace{f(\boldsymbol{\mu}_{\mathbf{x}})}^{M_1} + \overbrace{\frac{1}{2}\sum_{p=1}^{n}\left(\frac{\partial^2 f}{\partial x_p^2}\right)\sigma_{x_p}^2}^{M_2} + \overbrace{\frac{1}{6}\sum_{p=1}^{n}\left(\frac{\partial^3 f}{\partial x_p^3}\right)\gamma_p \sigma_{x_p}^3}^{M_3}$$
$$+\overbrace{\frac{1}{24}\sum_{p=1}^{n}\left(\frac{\partial^4 f}{\partial x_p^4}\right)\Gamma_{x_p}\sigma_{x_p}^4}^{M_4} + \overbrace{\frac{1}{8}\sum_{p=1}^{n}\sum_{\substack{q=1\\ q\neq p}}^{n}\left(\frac{\partial^4 f}{\partial x_p^2 \partial x_q^2}\right)\sigma_{x_p}^2\sigma_{x_q}^2}^{M_5} + O(\boldsymbol{\sigma}_{\mathbf{x}}^5), \quad (24)$$

$$\sigma_{f_{MM}}^2 = \int (f_{MM} - \mu_{f_{MM}})^2 p_{\mathbf{x}}\, \mathrm{d}\mathbf{x} = \overbrace{\sum_{p=1}^{n}\left(\frac{\partial f}{\partial x_p}\right)^2 \sigma_{x_p}^2}^{V_1} + \overbrace{\sum_{p=1}^{n}\left(\frac{\partial^2 f}{\partial x_p^2}\right)\left(\frac{\partial f}{\partial x_p}\right)\gamma_{x_p}\sigma_{x_p}^3}^{V_2}$$

$$+\overbrace{\sum_{p=1}^{n}\sum_{\substack{q=1\\q\neq p}}^{n}\left(\frac{\partial^3 f}{\partial x_p^2\partial x_q}\right)\left(\frac{\partial f}{\partial x_q}\right)\sigma_{x_p}^2\sigma_{x_q}^2}^{V_3}+\overbrace{\frac{1}{2}\sum_{p=1}^{n}\sum_{\substack{q=1\\q\neq p}}^{n}\left(\frac{\partial^2 f}{\partial x_p\partial x_q}\right)^2\sigma_{x_p}^2\sigma_{x_q}^2}^{V_4}$$

$$+\overbrace{\frac{1}{3}\sum_{p=1}^{n}\left(\frac{\partial^3 f}{\partial x_p^3}\right)\left(\frac{\partial f}{\partial x_p}\right)\Gamma_{x_p}\sigma_{x_p}^4}^{V_5}+\overbrace{\frac{1}{4}\sum_{p=1}^{n}\left(\frac{\partial^2 f}{\partial x_p^2}\right)^2(\Gamma_{x_p}-1)\sigma_{x_p}^4}^{V_6}+O(\boldsymbol{\sigma}_{\mathbf{x}}^5), \tag{25}$$

where the term $O(\boldsymbol{\sigma}_{\mathbf{x}}^5)$ denotes all the neglected monomials of fifth order and higher, such as $\sigma_{x_p}^5$, $\sigma_{x_p}^3\sigma_{x_q}^2$, $\sigma_{x_p}^3\sigma_{x_q}^3$, etc. As in (23), the partial derivatives of $f$ with respect to the input variables are computed at $\mathbf{x}=\boldsymbol{\mu}_{\mathbf{x}}$. The skewness $\gamma_{x_p}$ and the kurtosis $\Gamma_{x_p}$ for the $p$th variable $x_p$ are defined as follows:

$$\gamma_{x_p}=\frac{E(x_p-\mu_{x_p})^3}{\sigma_{x_p}^3}, \tag{26}$$

$$\Gamma_{x_p}=\frac{E(x_p-\mu_{x_p})^4}{\sigma_{x_p}^4}. \tag{27}$$

First-order MM (I MM), which takes into account the terms $M_1$ and $V_1$ for mean and variance, respectively, is often quoted in the literature (see, for example [15, 21, 38, 46]). However, even for reduced spread of the input variables, the accuracy of the method may be severely spoiled by non-linearities in the system response. To overcome such limitations, the univariate reduced quadrature (URQ) propagation technique [1, 47] could be adopted. It estimates mean and variance of $f$ by using the following formulas:

$$\mu_{f_{URQ}}=W_0 f(\boldsymbol{\mu}_{\mathbf{x}})+\sum_{p=1}^{n}W_p\left[\frac{f(\mathbf{x}_{p_+})}{h_p^+}-\frac{f(\mathbf{x}_{p_-})}{h_p^-}\right], \tag{28}$$

$$\sigma_{f_{URQ}}^2=\sum_{p=1}^{n}\left\{W_p^+\left[\frac{f(\mathbf{x}_{p_+})-f(\boldsymbol{\mu}_{\mathbf{x}})}{h_p^+}\right]^2+W_p^-\left[\frac{f(\mathbf{x}_{p_-})-f(\boldsymbol{\mu}_{\mathbf{x}})}{h_p^-}\right]^2\right.$$
$$\left.+W_p^{\pm}\frac{[f(\mathbf{x}_{p_+})-f(\boldsymbol{\mu}_{\mathbf{x}})][f(\mathbf{x}_{p_-})-f(\boldsymbol{\mu}_{\mathbf{x}})]}{h_p^+h_p^-}\right\}. \tag{29}$$

The sampling points are found as follows:

$$\mathbf{x}_{p_\pm}=\boldsymbol{\mu}_{\mathbf{x}}+h_p^{\pm}\sigma_{x_p}\mathbf{e}_p, \tag{30}$$

where $\mathbf{e}_p$ is the $p$th vector of the identity matrix of size $n$ and $h_p^{\pm}$ are given as follows:

$$h_p^{\pm}=\frac{\gamma_{x_p}}{2}\pm\sqrt{\Gamma_p-\frac{3\gamma_{x_p}^2}{4}}. \tag{31}$$

The weights have to be chosen as:

- $W_0=1+\sum_{p=1}^{n}\frac{1}{h_p^+h_p^-}$;
- $W_p=\frac{1}{h_p^+-h_p^-}$;
- $W_p^+=\frac{(h_p^+)^2-h_p^+h_p^--1}{(h_p^+-h_p^-)^2}$;



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- $W_p^- = \frac{(h_p^-)^2 - h_p^+ h_p^- - 1}{(h_p^+ - h_p^-)^2}$;
- $W_p^{\pm} = \frac{2}{(h_p^+ - h_p^-)^2}$.

The URQ can be thought as a univariate version of the bivariate quadrature method proposed by Evans [48]. Evans's method exhibits an error of $O(\boldsymbol{\sigma}_{\mathbf{x}}^5)$ in the general case, which reduces to or $O(\boldsymbol{\sigma}_{\mathbf{x}}^6)$ for symmetric input distributions [48], while the URQ error is $O(\boldsymbol{\sigma}_{\mathbf{x}}^4)$ for the general case, and $O(\boldsymbol{\sigma}_{\mathbf{x}}^5)$ if the cross derivatives of order 2 and higher are negligible [1, 47]. However, Evans's method requires $2n^2+1$ function evaluations per iteration of optimization algorithm, while the URQ requires only $2n+1$ of such evaluations. The URQ has hence comparable cost to I MM, when the gradient for the latter is obtained through finite differences. However, as can be seen from Equations (24) and (25), I MM error is $O(\boldsymbol{\sigma}_{\mathbf{x}}^2)$ for the mean estimate, while the error of the variance estimate is $O(\boldsymbol{\sigma}_{\mathbf{x}}^3)$ in the general case, and $O(\boldsymbol{\sigma}_{\mathbf{x}}^4)$ in the case of symmetric input distributions. The derivative-free nature of the URQ is also advantageous with respect to I MM in the context of gradient-based optimization, since the resulting robust optimization procedure requires a single level of differentiation [1, 47].

## 4. AN INDUSTRIAL EXAMPLE: PASSENGER AIRCRAFT SIZING

Presented in this section is the application of the proposed methodology to an industrial aircraft sizing test case [49]. The original deterministic optimization problem is the following:
*Objective*: minimize maximum take-off weight $f = \text{MTOW}(\mathbf{x})$ with respect to the design variables $\mathbf{x}$.
*Constraints*:

1. Approach speed: $v_{\text{app}} < 120\,\text{kts} \Rightarrow g_1 = v_{\text{app}} - 120$.
2. Take-off field length: $\text{TOFL} < 2000\,\text{m} \Rightarrow g_2 = \text{TOFL} - 2000$.
3. Percentage of total fuel stored in wing tanks: $K_F > 0.75 \Rightarrow g_3 = 0.75 - K_F$.
4. Percentage of sea level thrust available in cruise: $K_T < 1 \Rightarrow g_4 = K_T - 1$.
5. Climb speed: $v_{\text{zclimb}} > 500\,\text{ft/min} \Rightarrow g_5 = 500 - v_{\text{zclimb}}$.
6. Range: $\text{RA} > 5800\,\text{km} \Rightarrow g_6 = 5800 - \text{RA}$.

Table II provides descriptions of the design variables with their permitted ranges. The problem's fixed parameters are given in Table III.

### 4.1. RDO with symmetric input distributions

The RDO problem in which the design variables are affected by uncertainty is termed sometimes 'type II robust design' [50]. Randomizing the design variables is the first step of a strategy which aims at rendering the optimal conceptual design insensitive to variations that are likely to occur downstream in the development process, with the purpose of avoiding nugatory iterations

Table II. Considered design variables, aircraft sizing test case.

| Design variable | Definition (units) | Bounds ($\mathbf{x}_L$, $\mathbf{x}_U$) |
|---|---|---|
| $S$ | Wing area ($\text{m}^2$) | [140, 180] |
| BPR | Engine bypass ratio ( ) | [5, 9] |
| $b$ | Wing span (m) | [30, 40] |
| $\Lambda$ | Wing sweep (deg) | [20, 30] |
| $t/c$ | Wing thickness to chord ratio ( ) | [0.07, 0.12] |
| $T_{\text{eSL}}$ | Engine sea level thrust (kN) | [100, 150] |
| FW | Fuel weight (kg) | [12 000, 20 000] |



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Table III. Fixed parameters.

| Parameter | Value |
|---|---|
| Number of passengers | 150 |
| Number of engines | 2 |
| Cruise Mach number | 0.75 |
| Altitude (ft) | 31 000 |

Table IV. Optimal deterministic design which serves as starting point for RDO.

| Design variables | | Obj./Constr. | |
|---|---|---|---|
| $S$ (m$^2$) | 140.00 | MTOW (kg) | 77 560.62 |
| BPR ( ) | 9.00 | $g_1$ (kts) | 0.00 |
| $b$ (m) | 40.00 | $g_2$ (m) | −80.00 |
| $\Lambda$ (deg) | 20.00 | $g_3$ ( ) | 0.00 |
| $t/c$ ( ) | 0.081 | $g_4$ ( ) | −0.12 |
| $T_{e_{SL}}$ (kN) | 101.60 | $g_5$ (ft/min) | 0.00 |
| FW (kg) | 14817.13 | $g_6$ (km) | 0.00 |

between design phases. The design variables are modeled by means of normal distributions truncated to the range $[\boldsymbol{\mu}_\mathbf{x} - 3\boldsymbol{\sigma}_\mathbf{x}, \boldsymbol{\mu}_\mathbf{x} + 3\boldsymbol{\sigma}_\mathbf{x}]$. The standard deviation of the design variables is chosen as: $\boldsymbol{\sigma}_\mathbf{x} = \{11.2\,(\mathrm{m}^2), 0.5(), 2.4\,(\mathrm{m}), 1.7\,(\mathrm{deg}), 0.007\,(), 8.8\,(\mathrm{kN}), 1120\,(\mathrm{kg})\}$. The mathematical problem to be solved is hence formulated as in Equation (6), where the adopted coefficients are: $\mathbf{k}_\mathbf{x} = \mathbf{1}$, to guarantee that each of the design variables will remain within its bounds with a probability of about 84.3%, and $k_f = k_{g_i} = 1.3$, for $i = 1, \ldots, 6$, which is justified later on in this section.

The first set of numerical experiments compares two gradient-based robust optimization strategies. In the first one, the robust objective and constraints are built by using the I MM propagation technique, while in the second these are built by adopting the URQ. The starting point considered for the optimizations is the result of the optimization of the deterministic problem, which is summarized in Table IV. The two robust optimizations differ only in their calculations of mean and variance, the first making use of I MM and the second the URQ method. Both optimization problems are solved using Matlab's gradient-based constrained optimizer `fmincon`. The derivatives are obtained through the automatic differentiation software MAD [22, 51].

Table V presents the results of the two optimizations. The nominal MTOW (not shown in Table V) for the two optimal values is 84.3 and 84.5 tons, respectively. Therefore, both robust optima are almost 10% heavier than the original deterministic optimum as shown in Table IV. The required fuel is about 3.5 tons more than for the deterministic optimum and the required thrust and wing area are significantly increased, to keep the wing loading and the thrust-weight ratio almost unvaried. This is an example of the margins that robust optimization procedures build into the design. Such margins may be substantial, and hence each of the steps leading to their estimation has to be justified. It can also be noted that the solution found by adopting the URQ is slightly more conservative with respect to the one given by the I MM.

A probabilistic post-optimal analysis is carried out by using MCS (random sampling with $2 \times 10^5$ samples) to validate the results in Table V with respect to the accuracy of the estimates and the probability of feasibility. Two simulations are performed, by choosing a truncated multivariate normal distribution with the mean corresponding to the optimal $\boldsymbol{\mu}_\mathbf{x}$ found by the two optimization processes, and standard deviation equal to $\boldsymbol{\sigma}_\mathbf{x}$ as given above. The results shown in Table VI confirm that the URQ can attain a higher mean estimate accuracy—at least one order of magnitude better—than the one achieved by the I MM. Variance estimates exhibit instead the same accuracy for the two methods. On the other hand, the analysis of the active constraints (which for both optimizations are the constraints 1, 3, 5 and 6) results in the probabilities of constraint satisfaction shown in Table VII. To assess the goodness of such results, further elaboration is required on



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Table V. Results of the robust optimizations.

| Mean design variables | I MM | URQ | Obj./Constr. | I MM | URQ |
|---|---|---|---|---|---|
| $S$ (m$^2$) | 165.18 | 166.02 | $F$ (kg) | 86453.81 | 86662.39 |
| BPR ( ) | 8.51 | 8.51 | $G_1$ (kts) | 0.00 | 0.00 |
| $b$ (m) | 37.55 | 37.55 | $G_2$ (m) | −200.10 | −196.14 |
| $\Lambda$ (deg) | 21.75 | 21.75 | $G_3$ ( ) | 0.00 | 0.00 |
| $t/c$ ( ) | 0.085 | 0.085 | $G_4$ ( ) | −11.30 | −0.10 |
| $T_{\mathrm{eSL}}$ (kN) | 125.30 | 126.09 | $G_5$ (ft/min) | 0.00 | 0.000 |
| FW (kg) | 18267.04 | 18363.38 | $G_6$ (km) | 0.00 | 0.000 |

Table VI. Post-optimality analysis: relative error on mean and variance estimation of objective and constraints with respect to MCS.

| | Mean estimation | | Variance estimation | |
|---|---|---|---|---|
| Obj./Constr. | $x_{\mathrm{opt,IMM}}$ | $x_{\mathrm{opt,URQ}}$ | $x_{\mathrm{opt,IMM}}$ | $x_{\mathrm{opt,URQ}}$ |
| MTOW | $-0.35\times10^{-4}$ | $0.26\times10^{-5}$ | $0.11\times10^{-1}$ | $0.11\times10^{-1}$ |
| $v_{\mathrm{app}}$ | $-0.15\times10^{-2}$ | $0.82\times10^{-6}$ | $0.18\times10^{-1}$ | $0.13\times10^{-2}$ |
| TOFL | $-0.87\times10^{-2}$ | $0.34\times10^{-4}$ | $0.60\times10^{-1}$ | $0.18\times10^{-1}$ |
| $K_F$ | $-0.82\times10^{-2}$ | $0.10\times10^{-3}$ | $0.11\times10^{-1}$ | $0.26\times10^{-1}$ |
| $K_T$ | $-0.87\times10^{-2}$ | $0.65\times10^{-4}$ | $0.51\times10^{-1}$ | $0.23\times10^{-1}$ |
| $v_{\mathrm{zclimb}}$ | $-0.13\times10^{-1}$ | $0.69\times10^{-4}$ | $0.90\times10^{-2}$ | $0.14\times10^{-1}$ |
| RA | $-0.24\times10^{-2}$ | $-0.55\times10^{-5}$ | $0.15\times10^{-2}$ | $0.14\times10^{-2}$ |

Table VII. Probabilities of feasibility verified by MCS.

| Active constraint | I MM | URQ |
|---|---|---|
| $P(v_{\mathrm{app}}<120\,\mathrm{kts})$ | 0.8893 | 0.9011 |
| $P(K_F>0.75)$ | 0.9235 | 0.9136 |
| $P(v_{\mathrm{zclimb}}>500\,\mathrm{ft/min})$ | 0.8949 | 0.8990 |
| $P(\mathrm{RA}>5800\,\mathrm{km})$ | 0.8988 | 0.9064 |

the choice of $k_{g_i}=1.3$, for $i=1,\ldots,6$. For example, had normal behavior been assumed for the constraints, such coefficients would have implied a target for the probability of feasibility for all the constraints equal to $\Phi(1.3)=0.9032$. However, from Table VII, it can be noticed that the constraints on $v_{\mathrm{app}}$, $v_{\mathrm{zclimb}}$ and RA in the case of I MM propagation, and on $v_{\mathrm{app}}$, $v_{\mathrm{zclimb}}$ in the case of URQ propagation, are slightly below such target. The fact that both optimal solutions turn out to be marginally unfeasible after MCS validation implies that increasing the accuracy of the uncertainty propagation phase might not be sufficient to guarantee better results. Hence, the two main RDO approximation levels, the estimation of the moments and the assumptions on the constraints' (and objectives') distributions, should be considered together in practice. The relative magnitude of the respective approximation errors is the key to understand the suitability of the proposed RDO methodology to conceptual design problems.

Within such perspectives, it is preferable to choose values for the coefficients $k_f$ and $k_{g_i}$ which are derived from the probability bounds described in Section 3.1. As explained above, considering bounds that correspond to the worst-case quantiles within a set of distributions (e.g. the set of symmetric distributions) is more conservative than assuming normal behavior for objective and constraints. For example, the probability of feasibility for the constraints assumed to have unimodal distributions for $k_{g_i}=1.3$ would have been 0.8348 (Table I, case III), while the one given by the unimodal symmetric assumption would have been 0.8685 (Table I, case IV). By comparison with the results in Table VII, for the URQ case, such overconservativeness is quantified to be around 8%



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and 4% for the unimodal and unimodal symmetric assumption, respectively. Finally, the possible advantage of substituting a quantile bound constraint with a TCE bound type of constraint can be analyzed. For example, to avoid any assumption on the output distribution, a Chebyshev TCE bound could have been used. Hence, $k_{g_i} = 1.3$ would have identified the upper bound for the average over the worst 37% of the designs via Equation (19). Such bound would then have been imposed to be not larger than the original deterministic limit. For example, in the case of $v_{\text{app}}$, the upper bound on TCE, i.e. $\sup[\text{TCE}_{0.37}(v_{\text{app}})]$, would have been imposed to be not larger than 120 kts, and the constraint would have been formulated again as:

$$G_1 = \sup[\text{TCE}_{0.37}(v_{\text{app}})] - 120 = \mu_{g_1} + 1.3\sigma_{g_1} \leqslant 0. \tag{32}$$

Since the constraint is active, $\sup[\text{TCE}_{0.37}(v_{\text{app}})] = 120$. A validation of the TCE approach on the MCS results shows that the average $v_{\text{app}}$ on the worst 37% of the samples is $\text{TCE}_{0.37}(v_{\text{app}}) = 119.28$ kts. Such result is still conservative, but physically very close to the estimated 120 kts. This is a sharp improvement with respect to imposing a quantile constraint. Without any assumption on $v_{\text{app}}$ PDF, in fact, the corresponding Chebyshev quantile bound would have guaranteed only that $P(v_{\text{app}} < 120) > 62.82\%$, while it is verified via MCS that $P(v_{\text{app}} < 120) = 90.11\%$, as reported in Table VII.

### *4.2. RDO with asymmetric input distributions*

Consider an application for which the input variables are modeled by triangular distributions having the same means and standard deviations of the distributions adopted in the previous section, but with a skewness $\gamma_{x_p} = -0.5$ for all the variables. The same coefficients $k_f = k_{g_i} = 1.3$, for $i = 1, \ldots, 6$, and $\mathbf{k_x} = \mathbf{1}$, used for the symmetric case, are adopted. Since the I MM technique can only exploit the knowledge of the mean and variance of the input variables, the optimal result of a I MM robust optimization for the present case would not differ from the result obtained in the previous section. By contrast, the URQ technique also incorporates the knowledge of skewness and kurtosis of the input variable distributions. Therefore, it can differentiate between the truncated normal distribution adopted in the previous example, and the asymmetric triangular distribution of the present one. To assess the performance of the proposed methodology for asymmetric input, the URQ is hence compared in this section to a third-order MM (III MM) robust optimization. III MM can model the terms $M_1$ to $M_3$ and $V_1$ to $V_6$ for mean and variance in Equations (24) and (25), respectively. Therefore, it requires the availability of third derivatives to estimate the robust objectives and constraints, and a fourth level of derivation if the gradients of such objectives and constraints are needed by the optimizer. Such computational effort is unaffordable for most practical applications. The method is used here only as a benchmark, within an auxiliary optimization study in which objectives and constraints are modeled through III MM, and the required derivatives are obtained through MAD [51]. The results, shown in Table VIII, have again been validated by means of MCS (random sampling with $2 \times 10^5$ samples). The probabilistic analysis, shown in Table IX, shows that the URQ can achieve a very good approximation of the first two moments when compared with III MM. The probabilities of satisfaction of the active constraints (which for both optimizations are the constraints 1, 3, 5 and 6) have also been quantified on the MCS samples, and the results are shown in Table X. From such results, and from Figure 5, where the MCS data regarding the URQ solution are shown together with their normal fits, it is clear that this time the normal assumption is unsatisfactory. Furthermore, the mismatch between the normal assumption and the probabilistic distributions of objectives and constraints does not occur only in correspondence of the optimum, but can also be observed throughout the optimization process. Hence, the result is not optimal in the sense sought by a formulation which makes use of the normal assumption.

As an alternative, the Vysochanskij–Petunin inequality in Equation (13) could have been used. In this case, relying on the unimodality of the probability distributions of the constraints, the optimal solution would have guaranteed a probability of feasibility of at least 0.8348, for $k_{g_i} = 1.3$. Likewise, $k_f = 1.3$ would have guaranteed that at least the 83.48% of the designs have MTOW below the threshold given by $F$ in Table VIII. Hence, the solution can be interpreted as optimal in a worst-case sense, for a given probability, under the unimodality assumption of objective



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Table VIII. Results of the robust optimizations, asymmetric case.

| Mean design variables | URQ | III MM | Obj./Constr. | URQ | III MM |
|---|---|---|---|---|---|
| $S$ (m$^2$) | 166.41 | 166.39 | $F$ (kg) | 86745.21 | 86742.78 |
| BPR ( ) | 8.51 | 8.51 | $G_1$ (kts) | 0.00 | −1.09 |
| $b$ (m) | 37.55 | 37.55 | $G_2$ (m) | −195.73 | −200.03 |
| $\Lambda$ (deg) | 21.75 | 21.75 | $G_3$ ( ) | 0.00 | 0.00 |
| $t/c$ ( ) | 0.095 | 0.095 | $G_4$ ( ) | −1.06 | −1.08 |
| $T_{\mathrm{eSL}}$ (kN) | 126.35 | 126.35 | $G_5$ (ft/min) | 0.00 | −2.78 |
| FW (kg) | 18400.59 | 18398.94 | $G_6$ (km) | 0.00 | −5.76 |

Table IX. Post-optimality analysis: relative error on mean and variance estimation of objective and constraints with respect to MCS.

| | Mean estimation | | Variance estimation | |
|---|---|---|---|---|
| Obj./Constr. | $x_{\mathrm{opt,URQ}}$ | $x_{\mathrm{opt,III\,MM}}$ | $x_{\mathrm{opt,URQ}}$ | $x_{\mathrm{opt,III\,MM}}$ |
| MTOW | $-0.35\times10^{-5}$ | $0.26\times10^{-5}$ | $0.37\times10^{-2}$ | $-0.23\times10^{-2}$ |
| $v_{\mathrm{app}}$ | $-0.37\times10^{-4}$ | $0.85\times10^{-4}$ | $0.46\times10^{-2}$ | $0.14\times10^{-2}$ |
| TOFL | $-0.13\times10^{-3}$ | $0.23\times10^{-4}$ | $0.15\times10^{-1}$ | $0.18\times10^{-1}$ |
| $K_F$ | $-0.29\times10^{-2}$ | $-0.28\times10^{-3}$ | $0.29\times10^{-1}$ | $0.24\times10^{-1}$ |
| $K_T$ | $-0.68\times10^{-4}$ | $0.21\times10^{-4}$ | $0.13\times10^{-1}$ | $0.13\times10^{-1}$ |
| $v_{\mathrm{zclimb}}$ | $-0.42\times10^{-4}$ | $-0.11\times10^{-3}$ | $0.39\times10^{-2}$ | $0.39\times10^{-2}$ |
| RA | $-0.25\times10^{-3}$ | $0.14\times10^{-3}$ | $0.16\times10^{-2}$ | $0.17\times10^{-2}$ |

Table X. Probability of feasibility verified by MCS, asymmetric case.

| Active constraint | URQ | III MM |
|---|---|---|
| $P(v_{\mathrm{app}} < 120\,\mathrm{kts})$ | 0.8746 | 0.8739 |
| $P(K_F > 0.75)$ | 0.9024 | 0.9026 |
| $P(v_{\mathrm{zclimb}} > 500\,\mathrm{ft/min})$ | 0.8819 | 0.8825 |
| $P(\mathrm{RA} > 5800\,\mathrm{km})$ | 0.8902 | 0.8901 |

and constraints. Removing all the distributional assumptions, the TCE Chebyshev inequality in Equation (19) could have been used to formulate the objective and constraints. To understand the convenience of such a choice, consider Table XI. The values of the objective and the constraints shown in the first column are obtained through weighted sums of mean and standard deviation, with $k_f = k_{g_i} = 1.3$. Such values could be interpreted as bounds either on the 0.6282-quantile (Equation (8)), or on the TCE over the 37.18% of the designs with the lowest performance (Equation (19)). The second and third columns of Table XI present the values of $\mathrm{TCE}_{0.3718}$ and the 0.6282-quantile calculated on the MCS samples, respectively. The comparison of the bounds with the actual TCE and quantile estimates shows that, while the TCE bounds are still conservative, as expected, they lie much closer to the corresponding MCS estimates than the equivalent quantile bounds. Such results confirm the usefulness of TCE bounds for RDO.

## 5. CONCLUSIONS AND FUTURE WORK

Presented in this paper is a novel methodology for robust design optimization (RDO), which has been termed *worst-case RDO under distributional assumptions*. It stems from the recognition that adopting weighted sums of mean and standard deviation estimates of the original objective $f$ and

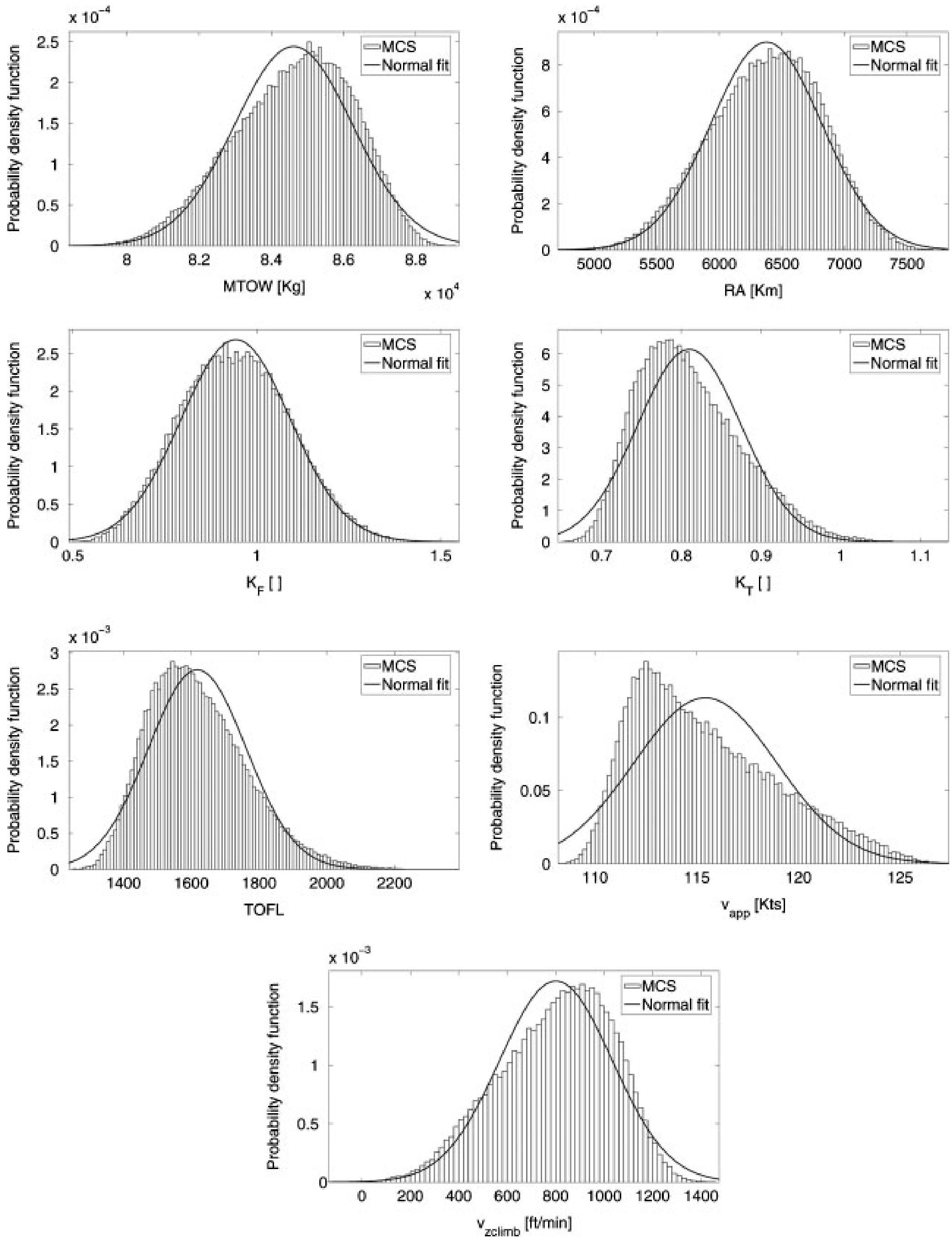


Figure 5. MCS validation of optimal results.

constraints $g_i$ to build their robust counterparts $F$ and $G_i$ is among the most computationally convenient robust optimization options, and hence of great relevance for practical applications. The proposed solution reformulates the RDO problem in order to relax, when required, the assumption of normality of $f$ and $g_i$, which often underlies RDO. This is done by exploiting a set of probabilistic inequalities, which can supply bounds for specified quantiles of $f$ and $g_i$ under a range of possible assumptions for their distributions. The TCE is also introduced as a useful metric for RDO, in order to minimize or constrain the average of a selected percentage of low performance designs. Furthermore, in analogy with the quantile formulation, TCE objectives and constraints



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Table XI. MCS validation of the TCE and quantile bounds corresponding to the optimal URQ solution.

| Obj./Constr. | $\mu_{y,\mathrm{URQ}}+1.3\sigma_{y,\mathrm{URQ}}$ | $\mathrm{TCE}_{0.3718,\mathrm{MCS}}(y)$ | $y_{0.6282,\,\mathrm{MCS}}$ |
|---|---|---|---|
| MTOW (kg) | 86745.21 | 86259.90 | 85298.61 |
| $v_{\mathrm{app}}$ (kts) | 120.00 | 119.26 | 116.12 |
| TOFL (m) | 1804.30 | 1769.36 | 1651.55 |
| $K_F$ ( ) | 0.75 | 0.79 | 0.88 |
| $K_T$ ( ) | 0.89 | 0.88 | 0.82 |
| $v_{\mathrm{zclimb}}$ (ft/min) | 500.00 | 553.37 | 738.94 |
| RA (km) | 5800.00 | 5910.53 | 6240.01 |

The bounds are based on the Chebyshev inequality for $k=1.3$.

are formulated through weighted sums of mean and standard deviation estimates of $f$ and $g_i$, by exploiting probabilistic bounds under suitable distributional assumptions. Resorting to bounds for quantile or TCE metrics identifies the most conservative cases among those possible within the chosen distributional assumptions, and hence the denomination for the proposed methodology. A computationally affordable yet sufficiently accurate implementation of the proposed formulation is guaranteed by the adoption of the Univariate Reduced Quadrature technique to enable the uncertainty propagation. Such technique is as efficient as the commonly used first-order method of moments, but exhibits a higher accuracy and can also account for asymmetric input distributions. An industrial RDO problem regarding the sizing of a passenger aircraft at conceptual design stage has been used to demonstrate the practical value of the proposed methodology.

The skewness and kurtosis of objectives and constraints distributions are not taken into account explicitly in the current formulation. The future work will investigate the adoption of higher order quadrature schemes, in order to estimate such moments with sufficient accuracy. This would enable the formulation of tighter inequality bounds as function of the first four moments, and would therefore be particularly beneficial for the case of more complex distribution types such as the bimodal distribution, for which only the most conservative results (e.g. the Chebyshev inequality) would be applicable in the proposed framework. Furthermore, the proposed approach could be extended to the case of vector-valued objectives and constraints functions by adopting multivariate probabilistic inequalities.

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## REFERENCES

1. Padulo M. Computational engineering design under uncertainty. An aircraft conceptual design perspective. *Ph.D. Thesis*, Cranfield University, Cranfield, Bedfordshire, U.K., 2009.
2. Box G, Jones S. Designing products that are robust to the environment. *Total Quality Management and Business Excellence* 1992; **3**(3):265–282.
3. Murphy TE, Tsui KL, Allen KJ. A review of robust design methods for multiple responses. *Research in Engineering Design* 2005; **16**:118–132.
4. Su J, Renaud JE. Automatic differentiation in robust optimization. *AIAA Journal* 1997; **35**(6):1072–1079.
5. Park GJ, Lee TH, Lee KH, Hwang KH. Robust design: an overview. *AIAA Journal* 2006; **44**(1):181–191.
6. Chen W, Wiecek MM, Zhang J. Quality utility—a compromise programming approach to robust design. *Journal of Mechanical Design* 1999; **121**(2):179–187.
7. Das I. Robustness optimization for constrained nonlinear programming problems. *Engineering Optimization* 2000; **32**(5):585–618.
8. Messac A, Ismail-Yahaya A. Multiobjective robust design using physical programming. *Structural and Multidisciplinary Optimization* 2002; **23**:357–371.

9. Jin Y, Sendhoff B. Trade-off between performance and robustness: an evolutionary multiobjective approach. *Proceedings of Second International Conference on Evolutionary Multi-criteria Optimization*. Lecture notes in Computer Science, vol. 2632. Springer: Berlin, 2003; 237–251.
10. Deb K, Gupta H. Introducing robustness in multiple-objective optimization. *KanGAL Report Number 2004016*, Kanpur Genetic Algorithms Laboratory, Indian Institute of Technology, Kanpur, India, 2004.
11. Molina-Cristobal A, Parks G, Clarkson P. Finding robust solutions to multi-objective optimisation problems using polynomial chaos. *The Sixth ASMO UK/ISSMO Conference on Engineering Design Optimization*, Oxford, U.K., 2006.
12. Hammersley JM, Handscomb DC. *Monte Carlo Methods*. Chapman and Hall: New York, 1964.
13. Helton JC, Davis FJ. Latin hypercube sampling and the propagation of uncertainty in analyses of complex systems. *Reliability Engineering and System Safety* 2003; **81**(1):23–69.
14. Caflisch RE. Monte Carlo and quasi-Monte Carlo methods. *Acta Numerica* 1998; **7**:1–49.
15. Parkinson A, Sorensen C, Pourhassan. A general approach for robust optimal design. *Journal of Mechanical Design* 1993; **115**(1):74–80.
16. Cramer H. *Mathematical Methods of Statistics* (11th edn). Princeton University Press: Princeton, 1966.
17. Von Mises R. Les lois de probabilite pour les fonctions statistiques. *Annales de l'Institut Henri Poincaré* 1936; **6**:185–212.
18. Von Mises R. Sur les fonctions statistiques. *Bulletin de la Société Mathématique de France* 1939; **67**:177–184. Supplement.
19. Tappeta R, Rao G, Milanowski P. Practical implementation of robust design optimization. *The 11th AIAA/ISSMO Multidisciplinary Analysis and Optimization Conference*, Austin, TX, 2005.
20. Rangavajhala S, Mullur A, Messac A. The challenge of equality constraints in robust design optimization: examination and new approach. *Structural and Multidisciplinary Optimization* 2007; **34**:381–401.
21. Putko M, Newman P, Taylor A, Green L. Approach for uncertainty propagation and robust design in CFD using sensitivity derivatives. *The 15th AIAA Computational Fluid Dynamics Conference*, Anaheim CA, 2001. AIAA 2001-2528.
22. Padulo M, Forth SA, Guenov MD. Robust aircraft conceptual design using automatic differentiation in Matlab. In *Advances in Automatic Differentiation*, Bischof CH, Bücker HM, Hovland PD, Naumann U, Utke J (eds). Springer: Berlin, 2008; 271–280.
23. Eldred MS, Agarwal H, Perez VM, Wojtkiewicz SFJ, Renaud JE. Investigation of reliability method formulations in DAKOTA/UQ. *The 9th ASCE Joint Specialty Conference on Probabilistic Mechanics and Structural Reliability*, Albuquerque, NM, 2004.
24. Huyse L. Free-form airfoil shape optimization under uncertainty using maximum expected value and second-order second-moment strategies. *Rept. 2001-18/NASA CR 2001-211020*, Inst. for Computer Applications in Science and Engineering, Hampton, VA, 2001.
25. Rosenblatt M. Remarks on a multivariate transformation. *Annals of Mathematical Statistics* 1952; **23**(3):470–472.
26. Ditlevsen O, Madsen H. *Structural Reliability Methods*. Internet edition 2.3.7., http://www.web.mek.dtu.dk/staff/od/books.htm., 2007. First edition published by Wiley, New York, 1996.
27. Youn BD, Choi KK. Selecting probabilistic approaches for reliability-based design optimization. *AIAA Journal* 2004; **42**(1):124–131.
28. Du X, Chen W. Towards a better understanding of modeling feasibility robustness in engineering design. *1999 ASME Design Technical Conference*, Las Vegas, NV, 1999. Paper No. DAC-8565.
29. Ayyub BM, Chia C. Generalized conditional expectation for structural reliability assessment. *Structural Safety* 1992; **11**:131–146.
30. Wu YT, Millwater H, Cruse T. An advanced probabilistic structural analysis method for implicit performance functions. *AIAA Journal* 1990; **28**(9):1663–1669.
31. Kiureghian AD, Dakessian T. Multiple design points in first and second-order reliability. *Structural Safety* 1998; **20**(1):37–49.
32. Mourelatos ZP, Liang J. An efficient unified approach for reliability and robustness in engineering design. *NSF Workshop on Reliable Engineering Computing*, August 2004.
33. Pukelsheim F. The three sigma rule. *The American Statistician* 1994; **48**:88–91.
34. Cantelli FP. Intorno ad un teorema fondamentale della teoria del rischio. *Bollettino dell'Associazione degli Attuari Italiani* 1910; **24**:1–23.
35. Popescu I. A semidefinite programming approach to optimal-moment bounds for convex classes of distributions. *Mathematics of Operations Research* 2005; **30**(3):632–657.
36. Sellke T. Generalized Gauss-Chebyshev inequalities for unimodal distributions. *Metrika* 1996; **43**(1):107–121.
37. Vysochanskij DF, Petunin YI. Improvement of the unilateral 3 $\sigma$-rule for unimodal distributions. *Doklady Akademii Nauk Ukrainy SSR*, *Series A* 1985; (1):6–8.
38. Du X, Chen W. Efficient uncertainty analysis methods for multidisciplinary robust design. *AIAA Journal* 2002; **40**(3):545–552.
39. Das I, Dennis J. A closer look at drawbacks of minimizing weighted sums of objectives for Pareto set generation in multicriteria optimization problems. *Structural Optimization* 1997; **14**:63–69.
40. Xusong X, Pin W. A study on risk measurements exceeding VaR: TCE, CVaR and ES. *Proceedings of the International Conference on Wireless Communications, Networking and Mobile Computing*, Shangai, 2007; 4039–4042.

41. Landsman Z, Valdez EA. Tail conditional expectations for elliptical distributions. *North American Actuarial Journal* 2003; **7**(4):55–71.
42. Mallows CL, Richter D. Inequalities of Chebyshev type involving conditional expectations. *Annals of Mathematical Statistics* 1969; **40**:1922–1932.
43. Cerbakova J. Worst-case VaR and CVaR. In *Operations Research Proceedings*, Haasis HD, Kopfer H, Schonberger J (eds). Springer: Berlin, 2005; 817–822.
44. Ben-Haim Y, Elishakoff I. *Convex Models of Uncertainty in Applied Mechanics*. Elsevier Science: Amsterdam, 1990.
45. Elishakoff I, Ohsaki M. *Optimization and Anti-optimization of Structures Under Uncertainty*. Imperial College Press: London, 2010.
46. Lewis L, Parkinson A. Robust optimal design using a second order tolerance model. *Research in Engineering Design* 1994; **6**(1):25–37.
47. Padulo M, Campobasso MS, Guenov MD. Novel uncertainty propagation method for robust aerodynamic design. *AIAA Journal* 2011; **49**(3):530–543.
48. Evans DH. Statistical tolerancing: the state of the art, part II. *Journal of Quality Technology* 1975; **7**(1):1–12.
49. Guenov MD, Fantini P, Balachandran L, Maginot J, Padulo M. MDO at predesign stage. In *Advances in Collaborative Civil Aeronautical Multidisciplinary Design Optimization*, Kesseler E, Guenov MD (eds). AIAA: Washington, DC, 2010; 83–108.
50. Chen W, Allen J. A procedure for robust design: minimizing variations caused by noise factors and control factors. *Journal of Mechanical Design* 1996; **118**(4):478–493.
51. Forth SA. An efficient overloaded implementation of forward mode automatic differentiation in MATLAB. *ACM Transactions on Mathematical Software* 2006; **32**(2):195–222.



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