Why is $\lim_{t\to\infty} \int_{[0,t]} m(ds)\, g(t-s) = \frac{1}{\mu}\int_0^\infty g(x)\,dx$

for g directly Riemann integrable and $\{N(t)\}$ recurrent?

detailed proof in Chap. 9 of Çinlar (1975) Intro to Stochastic Processes.

Intuitive proof:

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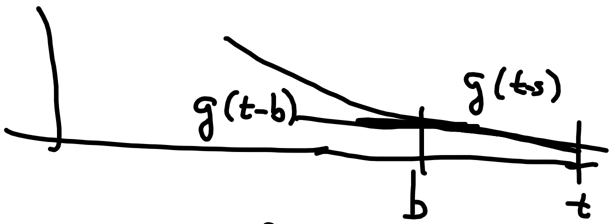
$$\int_{[0,t]} m(ds)\, g(t-s) = \int_0^b m(ds)\, g(t-s) + \int_b^t m(ds)\, g(t-s)$$



$m(t) \approx \frac{1}{\mu} t$

$$\int_{[0,b]} m(ds)\, g(t-s) \le \int_{[0,b]} m(ds)\, g(t-b) = g(t-b)\, m(b) \to 0$$

$$\int_{(b,t]} m(ds)\, g(t-s) = \int_{(b,t]} \frac{1}{\mu} g(t-s)\, ds \qquad \text{let } x = t-s$$

$$= \frac{1}{\mu}\int_0^{t-b} g(x)\,dx$$

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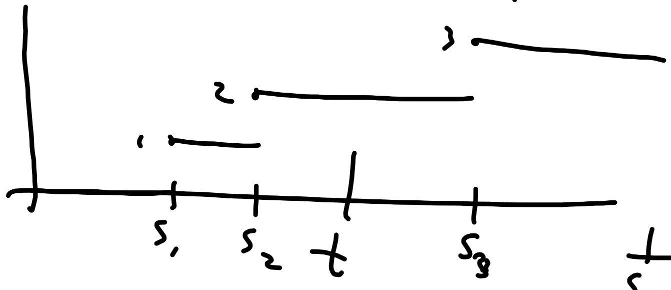
If N is independent of $\{X_n\} \Rightarrow$

$$E\left[\sum_{n=1}^{N} X_n\right] = E[N]\,E[X_1]$$

If N is a stopping time of $\{X_n\} \Rightarrow$

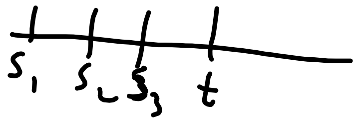
$$E\left[\sum_{n=1}^{N} X_n\right] = E[N]\,E[X_1] \longleftarrow \text{Wald's Lemma}$$

N is a stopping time iff the event $\{N=n\}$ is independent of $\{X_{n+1}, X_{n+2}, \dots\}$



$N(t)$

$N(t)=3$



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Conclusion: $N(t)+1$ is a stopping time

$N(t)$ is not a stopping time

$$V(t) = S_{N(t)+1} - t$$

$$E[V(t)] = E[S_{N(t)+1}] - t$$

$$= E[N(t)+1]\,E[X_1] - t = \mu(m(t)+1) - t$$

Poisson: $\mu(\frac{1}{\mu}t+1) - t = \mu$

$$U(t) = t - S_{N(t)}$$

$$E[U(t)] = t - E[S_{N(t)}]$$

Incorrect $\rightarrow$ $\cancel{t - E[N(t)]\,E[X_1]}$

$\cancel{= t - m(t)\cdot\mu}$

$$E[U(t)] = \int_0^\infty P\{U(t) > x\}\,dx$$

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$$P\{Z(t)=1\} = P\{Z(t)=1, S_1>t\} + P\{Z(t)=1, S_1\le t\}$$

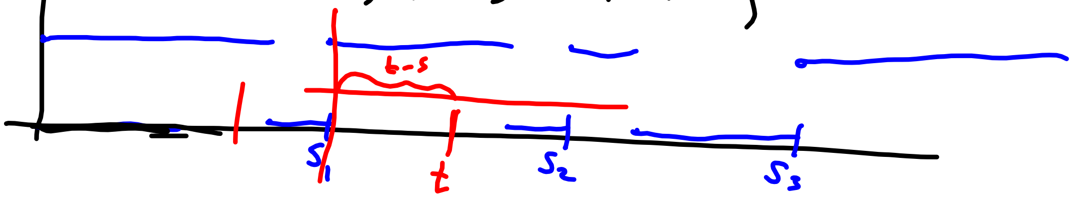
$$= P\{U_1>t\} + \int_{[0,t]} F(ds)\, P\{Z(t-s)=1\}$$

$$= 1-\varphi(t) + \int_{[0,t)} m(ds)\,(1-\varphi(t-s))$$

where $F=\varphi * \psi$ and $m=\sum_{n=1}^{\infty} F_n$

$$\lim_{t\to\infty} P\{Z(t)=1\} = \frac{1}{\mu}\int_0^\infty (1-\varphi(t))\,dt = \frac{a}{a+b}$$

$$P\{Z(t)=1, S_1>t\} = P\{U_1>t\}$$



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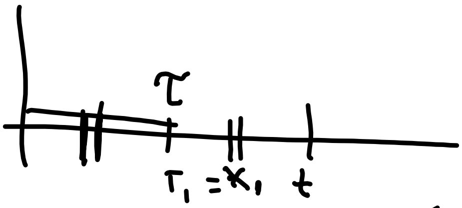
Assume $\varphi(\tau)<1$

$$P\{T_1>t\} = P\{T_1>t, S_1^*>t\} + P\{T_1>t, S_1^*\le t\}$$

where $S_n^* = X_1+\cdots+X_n$



$$P\{T_1>t, S_1^*>t\} = P\{S_1^*>t\} = 1-\varphi(t)$$

$$P\{T_1>t, S_1^*\le t\} = \begin{cases} \int_{[0,t]} \varphi(ds)\, P\{T_1>t-s\} & \text{if } t<\tau \\ \int_{[0,\tau]} \varphi(ds)\, P\{T_1>t-s\} & \text{if } t>\tau \end{cases}$$

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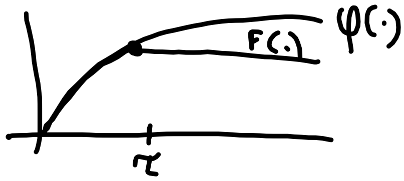
$$P\{T_1 > t, X_1 > t\} = 1 - \varphi(t)$$

$$P\{T_1 > t, X_1 \le t\} = \int_{[0,\tau]} \varphi(ds)\, P\{T_1 > t-s\}$$

$$= \int_{[0,t]} \varphi(ds)\, I_{[0,\tau]}(t)\, P\{T_1 > t-s\} \leftarrow$$

Let $F(t) = \begin{cases} \varphi(t) & \text{for } t \le \tau \\ \varphi(\tau) & \text{for } t > \tau \end{cases}$



$$P\{T_1 > t\} = 1 - \varphi(t) + \int_{[0,t]} F(ds)\, P\{T_1 > t-s\} -$$

$$P\{T_1 > t\} = 1 - \varphi(t) + \int_{[0,t]} m(ds)\, [1 - \varphi(t-s)]$$

where $m(t) = \sum_{n=1}^{\infty} F_n(t)$

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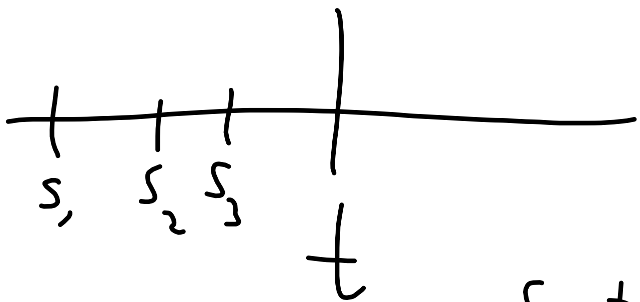

$N(t) = 3 \rightarrow$ if stopping time do not need $S_4, S_5, \ldots$

$N(t)+1=3 \leftarrow$ No

$\cancel{N(t)+1=3}$

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$$\int_{[0,t]} \varphi(ds)\, I_{[0,\tau]}(t) \left[\qquad \right]$$

$$F(ds) = \begin{cases} \varphi(ds) & \text{on } [0,\tau] \\ 0 & \text{on } (\tau, t] \end{cases}$$

$$\int_{[0,t]} F(ds) \left[\qquad \right]$$

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