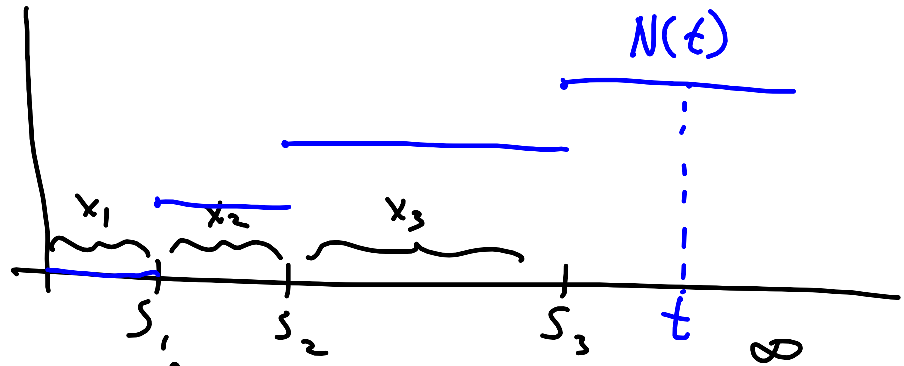

$$S_n = \sum_{k=1}^{n} X_k \quad \text{also} \quad N(t) = \sum_{k=1}^{\infty} I_{(0,t]}(S_k)$$

$$I_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

Jan 27-7:58 AM

$P\{S_n \leq t\}$

First note: $\{S_n \leq t\} = \{N(t) \geq n\}$

$$P\{S_n \leq t\} = 1 - P\{N(t) \leq n-1\} = 1 - \sum_{k=0}^{n-1} P\{N(t) = k\}$$

$$= 1 - \sum_{k=0}^{n-1} \frac{e^{-\lambda t}(\lambda t)^k}{k!} \quad \text{for } t \geq 0$$

Erlang type-k

$$E[S_n] = \frac{n}{\lambda} \qquad Var(S_n) = \frac{n}{\lambda^2}$$

Jan 27-8:08 AM

Conditional distribution of arrival times

Fix t, find $P\{X_1 \le s \mid N(t)=1\}$ for $s \le t$

$$P\{X_1 \le s \mid N(t)=1\} = \frac{P\{X_1 \le s,\ N(t)=1\}}{P\{N(t)=1\}}$$

$$= \frac{P\{N(s)=1\}\, P\{N(t)-N(s)=0\}}{P\{N(t)=1\}}$$

$$= \frac{\lambda s e^{-\lambda s} \cdot e^{-\lambda(t-s)}}{\lambda t e^{-\lambda t}} = \frac{s}{t}$$

Given n events in $(0,t)$, then the un-ordered arrival times are uniformly and independently distributed on interval $(0,t)$

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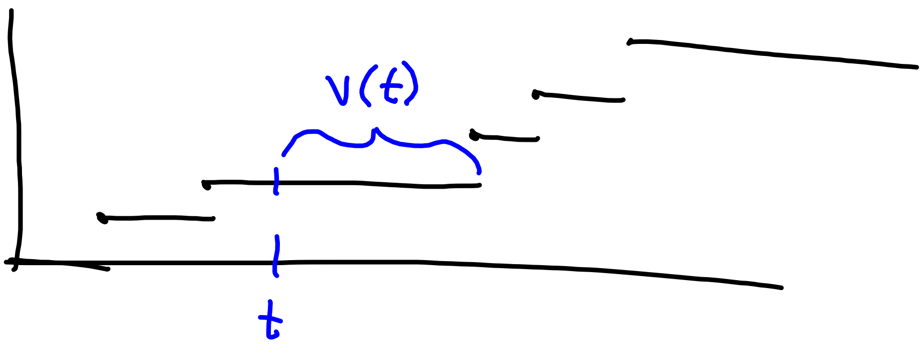

$$P\{V(t)>s \mid N(u);\ u\le t\} = P\{N(t+s)-N(t)=0 \mid N(u);\ u\le t\}$$

by independent increments

$$= P\{N(t+s)-N(t)=0$$

by stationarity

$$= P\{N(s)=0\} = e^{-\lambda s}$$

Jan 27-8:22 AM

Compound Poisson Process

$\{N(t)\}$ is Poisson process with rate λ

$\{Y_n\}$ i.i.d. nonnegative indep. of $\{N(t)\}$
with $E[Y_1] = \mu$ and $Var(Y_1) = \sigma^2$

Let $M(t) = \begin{cases} 0 & \text{if } N(t) = 0 \\ \sum_{k=1}^{N(t)} Y_k & \text{if } N(t) > 0 \end{cases}$

$\{M(t)\}$ is called a compound Poisson process

$$E[M(t)] = \lambda \mu t$$

$$Var(M(t)) = (\sigma^2 + \mu^2)\lambda t$$

Jan 27-8:25 AM

Markov chains $\{X_n\}$ on state space E

$$P\{X_{n+1} = j \mid X_1, X_2, \cdots, X_n\} = P\{X_{n+1} = j \mid X_n\}$$

$$P\{X_{n+1} = j \mid X_n = i\} = P\{X_1 = j \mid X_0 = i\} = P_i\{X_1 = j\} = P(i,j)$$

indicates initial state

Markov Matrix

$$P_i\{X_n = j\} = P^n(i,j)$$

if f is a reward vector $\Rightarrow f(i)$ is reward for for each visit to state i

$$E_i[f(X_n)] = P^n f(i)$$

if μ gives initial probabilities $\Rightarrow \mu(i) = P\{X_0 = i\}$

$$P_\mu\{X_1 = j\} = \mu P(j)$$

$$E_\mu[f(X_n)] = \mu P^n f$$

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Let $T^j = \min\{n \geq 1 : X_n = j\}$ ← first passage time

$$N^j = \sum_{n=0}^{\infty} I(X_n, j)$$ ← number of visits

$$P = \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & .3 & .7 & 0 \\ 0 & .5 & .5 & 0 \\ .2 & 0 & .1 & .7 \end{bmatrix}$$

$X_0 = 4, \; X_1 = 4, \; X_2 = 1, \; \ldots$

$T^1 = 2 \qquad N^1 = \infty$

$T^2 = \infty \qquad N^2 = 0$

$T^3 = \infty \qquad N^3 = 0$

$T^4 = 1 \qquad N^4 = 2$

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$$F(i,j) = P_i\{T^j < \infty\}$$

$$R(i,j) = E_i[N^j \mid X_0 = i]$$

$$P_j\{N^j = k\} = F(j,j)^{k-1}(1 - F(j,j))$$

$$P_i\{N^j = k\} = F(i,j)\, F(j,j)^{k-1}(1 - F(j,j))$$

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if $F(j,j) = 1 \Rightarrow P_j\{N^j = \infty\} = 1$

$R(j,j) = \dfrac{1}{1 - F(j,j)}$ and $R(i,j) = F(i,j) R(j,j)$ for $i \neq j$

Jan 27-9:00 AM