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ORIGINAL PAPER

A robust optimization approach to dispatching technicians under stochastic service times

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Abstract We consider the problem of dispatching technicians to service/repair geographically distributed equipment. This problem can be cast as a vehicle routing problem with time windows, where customers expect fast response and small delays. Estimates of the service time, however, can be subject to a significant amount of uncertainty due to misdiagnosis of the reason for failure or surprises during repair. It is therefore crucial to develop routes for the technicians that would be less sensitive to substantial deviations from estimated service times. In this paper we propose a robust optimization model for the vehicle routing problem with soft time windows and service time uncertainty and solve real-world instances with a branch and price method. We evaluate the efficiency of the approach through computational experiments on real industry routing data.

Keywords VRP with time windows · *k*-repairmen problem · Service time uncertainty · Robust optimization

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1 Introduction

In this paper we are concerned with the routing problem faced by maintenance and repair service providers for a certain type of equipment (copiers, washers, refrigerators). In this problem, requests are generated by the distributed owners of the equipment due to regular maintenance or failure of the equipment. Each request for service will include an estimated service time and a time window constraint, or more specifically a deadline for the start of service. The repair service center decides on a schedule for the technicians to service these requests in order to minimize travel time and delays at customers. Such a problem can be formulated as a classic vehicle routing problem (VRP) with soft time windows [49]. When the focus is just to minimize the delays at customers this problem is also referred to as the k-traveling repairmen problem [28].

In the case of repair services, the a-priori service time estimates can be radically different from the actual service times, due to misdiagnoses of the reason for service or unexpected situations during repair. This uncertainty in the service time of one client can result in a large delay for customers scheduled later for the same technician. Therefore it becomes important to construct technician dispatching strategies that will be efficient even in the presence of significant uncertainty in service times.

The motivation for this work is a real industrial application (repair technicians for business copiers) which services each day about 40 requests with 10 technicians in a large urban setting [51]. Given that travel times between customers are much smaller than service times and that variations on these travel times are usually small, we consider travel times deterministic. The estimated service times on the other hand can be significantly different from the actual service times. In this application about 30% of the requests have estimates for service time that significantly differ from actual repair times. In such cases the actual service times can be several times the estimated service times. Therefore, service time is the only relevant source of uncertainty. Similar conditions are encountered in other problems that route services, such as doctor house calls, as long as service times are uncertain and delays are costly.

In this work we are interested in the stochastic aspects of this sequencing problem. To be more precise, we focus on the VRP with soft time windows and uncertainty in service times, which is representative of the repair technician routing problem that motivates us. Because customer satisfaction is important in a repair service industry, our objective is to obtain an a-priori scheduling strategy that will obtain a small cost (travel time) and small delays regardless the outcome of the uncertainty. In practice, such an a-priori scheduling strategy could be implemented dinamically by resolving the problem as the day progresses and the uncertainty is revealed. However, our focus is to develop a method that obtains routing solutions that not only have good performance measures on average, but also have a small deviation of these measures under uncertainty.

The bulk of research on stochastic VRP (SVRP) has concentrated on stochastic customers and demands, see [14,30] for surveys of methods and algorithms. The few existing works on SVRP with stochastic service times and/or travel times formulate the problem as a stochastic program either with recourse or chance constraints, or both, see [34,41,43]. Typically the stochastic optimization solution approach assumes that the distribution of the uncertainty is known, which is difficult in practice. In addition, solution methods may require the distribution of the sum of the travel and service times along routes [30,43]. However, such a distribution may not be easy to compute, in particular for problems with hard time windows. Also, representing the distribution of the uncertainty through scenarios typically leads to a large increase in the dimension of the problem.

In this paper we present an alternative model for this SVRP, whose objective is to find an efficient solution that is insensitive to the uncertainty in service times. This *robust solution* is obtained by solving a robust counterpart problem that optimizes the worst case value over all data uncertainty. To formulate and solve the robust version of the VRP with stochastic service times (VRPSST) we borrow a formulation from the mathematical programming literature which defines the robust counterpart for convex optimization problems [4,7,8] later extended to integer programming problems [12]. Robust solutions have the potential to be viable solutions in practice, since they tend not to be far from the deterministic optimal solution but can significantly outperform it in the worst case [13,32].

The paper continues as follows, in the following section we provide a review of relevant literature in stochastic VRP, robust optimization, and recent work combining these two. In Sect. 3 we introduce the problem and its robust counterpart in general form. In Sect. 4 we discuss the uncertainty assumptions on service times and present the robust counterpart problem in detail. We then present a column generation solution approach with a constraint programming subproblem for this problem, adapted from [51] in Sect. 5, and present computational results and conclusions in Sects. 6 and 7 respectively.

2 Literature review

The VRP determines the set of routes to be performed by a fleet of vehicles to serve a dispersed set of customers at minimum travel cost, see [42,48] for comprehensive overviews of the VRP. The SVRP differs from its deterministic counterpart in several aspects, such as problem formulation approaches and solution techniques. A general review on SVRP appears in [30]. In broad terms SVRPs can be classified by the type of uncertainty into VRP with stochastic customers (VRPSC) [10,37–39,50], VRP with stochastic demands (VRPSD) [10,11,14,21–23,45,46], VRP with stochastic travel time (VRPSTT) [18,40,41,43] and VRP with stochastic service time (VRPSST) [34].

The VRP with stochastic travel time (VRPSTT) models the uncertainty in moving from client to client due to traffic conditions. Some first heuristics for the TSP with stochastic travel time, based on dynamic programming and implicit enumeration are proposed in [40]. A generalized dynamic programming methodology is used to solve the TSP with uncertainty in travel times in [18]. Systematic research on VRP with stochastic service time and travel time (VRPSSTT) is done in [43]. That work proposed three models for VRPSSTT: a chance constrained model, a 3-index recourse model and a 2-index recourse model. A general branch-and-cut algorithm is applied to solve all 3 models to optimality for up to 20 vertices. The VRPSSTT model was applied to a banking context in [41]. The problem is to design money collection routes through bank branches in the presence of stochastic travel times. Late arrival at the depot means that all money contained in the vehicles loses one day's interest. An adaptation of the Clarke and Wright Savings algorithm, [19] is used for this problem. The VRP with stochastic service time was used in [34] to model and solve a repair service. They used a two stage recourse model and a paired tree search algorithm to solve it.

When the vehicle routing problem focuses only on minimizing the delay (or latency) at customers, it is also referred to as the repairmen problem, [28]. Previous work has considered the stochastic version of the problem and developed offline, online, and distributed control strategies for the repairmen problem, [16,29].

In this paper we use the robust optimization methodology to obtain solutions for the VRPSST that perform well for all possible data uncertainty. The robust optimization approach was introduced for convex optimization problems in a series of papers [7,9,26], and extended and adapted to a number of applications, including least squares problems [24], structural truss design [6], portfolio optimization [25,32] and supply chain management [5,15]. It has also been extended to general combinatorial optimization problems [12] and applications in network problems [1], transportation [27], and routing [47]. The general approach of robust optimization is to optimize against the worst instance that might arise due to data uncertainty by using a min-max objective. The resulting solution from the robust counterpart problem is insensitive to the data uncertainty as it is the one that minimizes the worst case. The robust optimization methodology assumes the uncertain parameters belong to an uncertainty set, without additional distribution assumptions. Recent work has developed richer models of uncertainty sets that fit historic data without fixing the distribution of uncertain parameters, see for example [3,20,31]. The theoretical and applied work on robust optimization has shown that for many classes of problems and uncertainty sets the robust counterpart problem is only modestly larger than the original deterministic problem, and therefore of comparable complexity. Furthermore, the robust solution can provide significant protection for the worst case outcome at a modest loss in the expected objective value [13,32].

In particular the work on robust routing [47] considers a capacitated VRP with uncertainty in demand. The problem is shown to be equivalent to a deterministic instance with modified demand values, similar to a chance constrained model, if the Miller-Tucker-Zemlin (MTZ) formulation of the VRP is used. The approach is also applicable for uncertainty in travel times in VRPTW given the similarity of the MTZ constraints and the arrival time definition constraints. Furthermore, that paper notes that uncertainty in travel cost could be handled using the robust combinatorial optimization approach from [12], which would require solving m deterministic routing problems, where m is the number of arcs. An alternative form of robust routing solution is the concept of consistent routing [33], where routes in addition aim to visit random customers by the same driver and in the same order.

3 Notation and problem definition

The repair scheduling problem considered in this work is based on the problem introduced in [51] with the addition of uncertain service times. This problem considers K

technicians and I customers, which can be separated into $I_1 = \{1, \ldots, K\}$, the set of customers currently being serviced by the K technicians, and $I_2 = \{K+1, \ldots, I\}$, the set of customers that need to be scheduled. We also denote by $\mathcal{K} = \{1, \ldots, K\}$ the set of technicians, letting technician $k \in \mathcal{K}$ begin its route at client $k \in I_1$ without loss of generality. We assume that all technicians are sent to a dummy depot client $I+1$ after their schedule is finished. We define the arc set $A = \{(i, j) \mid i \in I_1 \cup I_2, j \in I_2 \cup \{I+1\}, \ i \neq j\}$ that represents all feasible trips among clients. Let W_i be the upper bound of the time window for service at client i, and s_i the estimated service time at that client.

The travel time from client i to client j is t_{ij} and all jobs have to be scheduled before the end of the work day L. We let variable x_{ij}^k indicate whether technician k services client i and then client j, or not. In addition we make use of some auxiliary variables to represent the time technician k starts service at client i, w_{ik}, and penalty for having technician k violate the soft time window constraint at client i, δ_{ik}. Finally, we define a binary variable v_i to represent when demand is not served during the day with the available fleet and is scheduled for the next day, at a high penalty P. Then, v_i is equal to one if request i is assigned to the next day, and equal to zero otherwise. The deterministic problem can then be expressed by

$$
\begin{array}{lll}
\min_{x,v,w,\delta} & \beta \sum\limits_{k\in\mathcal{K}} \sum\limits_{i\in I_2} \delta_{ik} + (1-\beta) \sum\limits_{k\in\mathcal{K}} \sum\limits_{(i,j)\in A} t_{ij} x_{ij}^k + \sum\limits_{i\in I_2} v_i P & \\
\text{s.t.} & \sum\limits_{k\in\mathcal{K}} \sum\limits_{j:(i,j)\in A} x_{ij}^k = 1 & i \in I_1 \\
& \sum\limits_{k\in\mathcal{K}} \sum\limits_{j:(i,j)\in A} x_{ij}^k = 1 - v_i & i \in I_2 \\
& \sum\limits_{j:(i,j)\in A} x_{ij}^k - \sum\limits_{j:(j,i)\in A} x_{ji}^k = b_i^k & i \in I_1 \cup I_2 \cup \{I+1\}, \ k \in \mathcal{K} \\
& x_{ij}^k \in \{0,1\} & (i,j) \in A, \ k \in \mathcal{K} \\
& v_i \in \{0,1\} & i \in I_2 \\
& w_{ik} + s_i + t_{ij} - w_{jk} \leq (1 - x_{ij}^k) M & (i,j) \in A, \ k \in \mathcal{K} \\
& w_{ik} \leq L \sum\limits_{j:(j,i)\in A} x_{ji}^k & i \in I_2, \ k \in \mathcal{K} \\
& w_{ik} - \delta_{ik} \leq W_i & i \in I_2, \ k \in \mathcal{K} \\
& w_{ik}, \delta_{ik} \geq 0 & i \in I_2, \ k \in \mathcal{K}\,.
\end{array}
\tag{1}
$$

The objective function is made up of a convex combination between the total delay and the total travel time, with $\beta \in [0, 1]$, and a penalty for not serving clients. The first five sets of constraints define feasible routes and identify which customers are left for the next day, if any. We first require that customers currently being serviced are visited by exactly one technician. The second constraint ensures the remaining customers are served by at most one technician or will be left for the next day. The third set of constraints are the technician flow constraints. Here the demand/supply vectors b^k are defined by

$$b_i^k = \begin{cases} 1 & i \in I_1, \quad i = k \\ -1 & i = I + 1 \\ 0 & \text{otherwise,} \end{cases}$$

where $k = i$ and $i \in I_1$ indicate that technician k starts its route at client i.

The sixth set of constraints define the arrival time of technician k to client j, forcing it to be greater than the arrival at i (w_{ik}) plus the service time at i (s_i) and the travel time to j (t_{ij}) if technician k goes from i to j ($x_{ij}^k = 1$). The parameter M is a large constant that makes this constraint redundant when $x_{ij}^k = 0$. We also need to set $w_{ik} = 0$ for all $i \in I_1$ and $k \in \mathcal{K}$. The seventh set of constraints enforces that all technicians arrive at customers before the end of the work day L. The next set of constraints define the delay of technician k at client i when $w_{ik} > W_i$.

The model and the results in this paper extend to more general problems with some simple adjustments. For instance, heterogeneous technicians that travel at different speeds or with restricted capabilities can be modeled by indexing travel times on k (t_{ij}^k) or eliminating variables x_{ij}^k for clients i and j where technician k is not able to provide service. We can consider a different travel cost, modifying the travel time coefficient in the objective to a given cost c_{ij} that could also be indexed on the technician. Finally we note that since the model considers the set I_1 of customers currently being serviced, this model can readily be used to re-optimize during the day when sufficient information of the uncertainty is revealed.

A model with uncertain service times aims to capture the discrepancy between the service times s_i that have been estimated for $i \in I_2$, known at the time of scheduling, and the service times that actually occur. Which uncertainty model is used to represent this discrepancy is greatly determined by the type and quality of information available. For instance, if the exact probability distribution of service times is known at every client, it is possible to develop chance constrained models that would minimize travel time and a delay that can be satisfied with a prescribed probability. This is achieved by replacing constraint six in problem (1) with the following constraint for a given confidence level $\alpha \in (0, 1)$:

$$\mathcal{P}(w_{ik} + s_i + t_{ij} - w_{jk} \leq (1 - x_{ij}^k)M) \geq 1 - \alpha \quad (i, j) \in A, \; k \in \mathcal{K},$$

for the known probability distribution of s_i. Such chance constraints can be linearized using the inverse cumulative density function. In particular, in our technician routing problem, as in many industrial applications, there is historic data on service times that can be used to build models of its variability.

In robust optimization, the uncertainty is represented by closed, convex and bounded uncertainty sets, which can also be estimated from the historic data. Accordingly we consider that $s \in \mathcal{U}$, a given uncertainty set. We now present the robust repair service scheduling problem and later mention two cases of uncertainty sets that lead to tractable robust counterpart problems. We consider robust solution as the solution that achieves the best worst case outcome with respect to the uncertainty. In other words the solution is robust if its worst case is minimized. Since we consider uncertainty only on the estimated service times s this leads to the following problem:

$$
\begin{array}{lll}
\min_{x,v,w,\delta} & \beta \sum\limits_{k\in\mathcal{K}} \sum\limits_{i\in I_2} \delta_{ik} + (1-\beta) \sum\limits_{k\in\mathcal{K}} \sum\limits_{(i,j)\in A} t_{ij} x_{ij}^k + \sum\limits_{i\in I_2} v_i P & \\
\text{s.t.} & (x^1,\dots,x^K,v) \in \mathcal{X} & \\
& w_{ik} + s_i + t_{ij} - w_{jk} \le (1-x_{ij}^k)M & (i,j)\in A,\ k\in\mathcal{K},\ s\in\mathcal{U} \\
& w_{ik} \le L \sum\limits_{j:(j,i)\in A} x_{ji}^k & i\in I_2,\ k\in\mathcal{K} \\
& w_{ik} - \delta_{ik} \le W_i & i\in I_2,\ k\in\mathcal{K} \\
& w_{ik}, \delta_{ik} \ge 0 & i\in I_2,\ k\in\mathcal{K}
\end{array}
\tag{2}
$$

For clarity, here we used the set $\mathcal{X}$ to represent feasible paths for the K technicians to service all customers. In other words $(x^1,\dots,x^K,v)\in\mathcal{X}$ if and only if the variables x_{ij}^k and v_i satisfy the first five sets of constraints in problem (1).

4 Robust technician dispatching variants

In this section we present the robust counterpart problem (2) of the technician dispatching problem under different types of uncertainty sets. The objective is to obtain robust counterparts that are not harder to solve than the original deterministic version of the problem. We present two robust counterpart models that increasingly take into account the correlation between uncertain parameters. Our first model effectively assumes that the uncertainty in service time is independent between customers while the second takes into account the correlation between the service times that each technician faces. We note that the formulation used is central to the ability to obtain a robust problem that takes into account correlation between uncertain parameters. We conclude by pointing out the difficulty in considering correlations between service times at all customers.

4.1 Independent uncertainty per client

First we assume only that the uncertainty set $\mathcal{U}$ is bounded. In this case, for all $i\in I_2$ there is a finite maximal value $\hat{s}_i = \sup_{s\in\mathcal{U}} s_i$. Note that only one of these bounded uncertain parameters (service times) appears in each constraint. Therefore the robust problem is equivalent to the deterministic problem with the maximum service time ($\hat{s}_i$) in each client, i.e.:

$$
\begin{array}{lll}
\min_{x,v,w,\delta} & \beta \sum\limits_{k\in K} \sum\limits_{i\in I_2} \delta_{ik} + (1-\beta) \sum\limits_{k\in\mathcal{K}} \sum\limits_{(i,j)\in A} t_{ij} x_{ij}^k + \sum\limits_{i\in I_2} v_i P & \\
\text{s.t.} & (x^1,\dots,x^K,v) \in \mathcal{X} & \\
& w_{ik} + \hat{s}_i + t_{ij} - w_{jk} \le (1-x_{ij}^k)M & (i,j)\in A,\ k\in\mathcal{K} \\
& w_{ik} \le L \sum\limits_{j:(j,i)\in A} x_{ji}^k & i\in I_2,\ k\in\mathcal{K} \\
& w_{ik} - \delta_{ik} \le W_i & i\in I_2,\ k\in\mathcal{K} \\
& w_{ik}, \delta_{ik} \ge 0 & i\in I_2,\ k\in\mathcal{K}.
\end{array}
$$

Note that because in this formulation of the routing problem there is only one uncertain parameter per constraint, this robust counterpart is exactly the same as the problem obtained when the uncertainty set is a box $\mathcal{U} = \{s \mid \bar{s}_i - \gamma_i \leq s_i \leq \bar{s}_i + \gamma_i,\ i \in I_2\}$, replacing $\hat{s}_i = \bar{s}_i + \gamma_i$. Therefore, if the set $\mathcal{U}$ has correlations between the random variables they are not reflected in this robust counterpart. To take into account the correlations between uncertain parameters at different customers a different formulation should be used, where multiple uncertain parameters appear on the same constraint. One possibility is switching to a path or route based formulation of the technician dispatching problem. These formulations use variables $x_r = (x^1, \ldots, x^K)$, such that $(x_r, v) \in \mathcal{X}$, that represent feasible routes for each technician. We consider such formulations next.

4.2 Independent uncertainty per technician

We now consider that service times can vary between $\bar{s}_i - \gamma_i \leq s_i \leq \bar{s}_i + \gamma_i$ and that there is an upper bound on the total sum of service time deviations, say U. So far this is a special case of the bounded uncertainty sets discussed above. In this subsection we assume in addition that the uncertainty does not concentrate on any one route, and that there is a bound U_k on the sum of the deviations on route k. The idea is that each technician will see at most U_k of the uncertainty and the worst case will not concentrate on a single technician. For example, if the uncertainty is distributed uniformly across technicians we can set $U_k = U/K$. This is different from the uncertainty sets from the previous subsection as the uncertainty set depends on the routes that each technician takes. Let $(x_r, v) \in \mathcal{X}$ represent a feasible routing solution and $P^k(x_r)$ the route taken by technician k. The route is represented by the set of arcs traversed by technician k, that is $P^k(x_r) = \{(i, j) \in A \mid x_{ij}^k = 1\}$. Furthermore if technician k visits client i, we denote the partial route of technician k until client i by the set of arcs $P_{ki}^k(x_r)$. Given the routes x_r the uncertainty set is given by:

$$\mathcal{U}(x_r) = \left\{ s \mid \bar{s}_i \leq s_i \leq \bar{s}_i + z_i,\ 0 \leq z_i \leq \gamma_i,\ i \in I_2, \quad \sum_{(q,l)\in P^k(x_r)} z_q \leq U_k \right\}.$$

Using this uncertainty set and denoting c the vector of travel costs, so $c^T x_r = \sum_{k\in\mathcal{K}} \sum_{(i,j)\in A} t_{ij} x_{ij}^k$, the robust counterpart problem is written as

$$\begin{aligned}
\min_{x_r, v}\ & (1-\beta)c^T x_r + \beta \sum_{k\in\mathcal{K}} \Gamma_k + \sum_{i\in I_2} v_i P \\
\text{s.t.}\ & (x_r, v) \in \mathcal{X} \\
& \Gamma_k \geq \sum_{i\in I_2} \left(\sum_{(q,l)\in P_{ki}^k(x_r)} \left(\bar{s}_q + z_q + t_{ql}\right) - W_i \right)^+ \quad k \in \mathcal{K} \\
& \qquad \forall\, 0 \leq z_i \leq \gamma_i,\ \textstyle\sum_{(q,l)\in P^k(x_r)} z_q \leq U_k.
\end{aligned}$$

Here Γ_k is an auxiliary variable that represents the worse case time windows violations for technician k given route x_r and uncertainty set $\mathcal{U}(x_r)$, that is Γ_k will be set to

$$
\begin{aligned}
\max_z \; & \sum_{i \in I_2} \left(\sum_{(q,l) \in P^k_{ki}(x_r)} \left(\bar{s}_q + z_q + t_{ql} \right) - W_i \right)^+ \\
\text{s.t.} \; & \sum_{(q,l) \in P^k(x_r)} z_q \leq U_k \\
& 0 \leq z_i \leq \gamma_i .
\end{aligned}
\tag{3}
$$

The next result shows that the solution to (3) has variations in service times as early as possible.

Proposition 1 *If the path $P^k(x_r)$ visits customers $1, 2, \ldots, q$, then the optimal solution to problem (3) is such that, for any $i = 2, \ldots, q$, if $z_i > 0$ then $z_{i-1} = \gamma_{i-1}$.*

Proof Assume that z is a feasible solution to problem (3) that satisfies $z_i > 0$ and $z_{i-1} < \gamma_{i-1}$ for some $i \in \{2, \ldots, q\}$. We define $\tilde{z}$ equal to z except for $\tilde{z}_i = z_i - \varepsilon$ and $\tilde{z}_{i-1} = z_{i-1} + \varepsilon$. The solution $\tilde{z}$ is feasible for any $0 < \varepsilon \leq \min\{\gamma_{i-1} - z_{i-1}, z_i\}$. The difference between the objective functions becomes:

$$
\sum_{j=2}^{q} \left(\sum_{l=1}^{j-1} (\bar{s}_l + \tilde{z}_l + t_{l\,l+1}) - W_j \right)^+ - \sum_{j=2}^{q} \left(\sum_{l=1}^{j-1} (\bar{s}_l + z_l + t_{l\,l+1}) - W_j \right)^+
$$

which due to the definition of $\tilde{z}$ and the fact that $\tilde{z}_{i-1} + \tilde{z}_i = z_{i-1} + z_i$, simplifies to

$$
\begin{aligned}
& \left(\sum_{l=1}^{i-1} (\bar{s}_l + \tilde{z}_l + t_{l\,l+1}) - W_i \right)^+ - \left(\sum_{l=1}^{i-1} (\bar{s}_l + z_l + t_{l\,l+1}) - W_i \right)^+ \\
& = \left(\sum_{l=1}^{i-1} (\bar{s}_l + z_l + t_{l\,l+1}) + \varepsilon - W_i \right)^+ - \left(\sum_{l=1}^{i-1} (\bar{s}_l + z_l + t_{l\,l+1}) - W_i \right)^+ \geq 0
\end{aligned}
$$

which shows that $\tilde{z}$ can only improve the objective. □

We obtain the robust counterpart by taking into account this result and the fact that in the worst case technician k will face the maximum uncertainty, which equals $\min\{U_k, \sum_{(q,l) \in P^k(x_r)} \gamma_q\}$. This leads to representing problem (3) as the following mixed integer program. Here $y_{ji} \in \{0, 1\}$ is an auxiliary variable that is equal to one when customer j has $z_j = \gamma_j$ and the path visits i right after j.

$$
\begin{aligned}
\min_{z,y} \; & \sum_{i \in I_2} \left(\sum_{(q,l) \in P^k_{ki}(x_r)} \left(\bar{s}_q + z_q + t_{ql} \right) - W_i \right)^+ \\
\text{s.t.} \; & \sum_{(q,l) \in P^k(x_r)} z_q \geq \min\{U_k, \sum_{(q,l) \in P^k(x_r)} \gamma_q\} \\
& 0 \leq z_i \leq \gamma_i \sum_{j:(j,i) \in A} y_{ji} && i \in I_2 \\
& y_{ji} \leq x^k_{ji} && (j,i) \in A \\
& y_{ji} \leq \frac{z_j}{\gamma_j} && (j,i) \in A \\
& y_{ji} \in \{0, 1\} && (j,i) \in A
\end{aligned}
$$

The second set of constraints make sure that variations are possible on a customer in I_2 that is immediately preceded by a customer with full variation in service time. The last three constraints enforce the definition of y_{ji}. The robust counterpart problem is obtained by using the above formulation of problem (3), which gives:

$$\begin{aligned}
\min_{x_r,v,z,y}\ & (1-\beta)c^T x_r+\beta \sum_{k\in K}\sum_{i\in I_2}\left(\sum_{(q,l)\in P^k_{ki}(x_r)}\left(\bar{s}_q+z_q+t_{ql}\right)-W_i\right)^+ +\sum_{i\in I_2} v_i P \\
\text{s.t. } & (x_r, v)\in \mathcal{X} \\
& \sum_{(q,l)\in P^k(x_r)} z_q \geq \min\{U_k, \sum_{(q,l)\in P^k(x_r)} \gamma_q\} && k\in\mathcal{K} \\
& 0\leq z_i\leq \gamma_i \sum_{j:(j,i)\in A} y_{ji} && i\in I_2 \\
& y_{ji}\leq \sum_{k\in\mathcal{K}} x^k_{ji} && (j,i)\in A \\
& y_{ji}\leq \frac{z_j}{\gamma_j} && (j,i)\in A \\
& y_{ji}\in\{0,1\} && (j,i)\in A\ .
\end{aligned} \tag{4}$$

This is slightly more complicated than the deterministic problem. The robust counterpart is a routing problem with an additional $|A|+|I_1\cup I_2|$ new variables ($|A|$ of them binary) and $2|A|+|I_1\cup I_2|+1$ new constraints. We discuss in Sect. 5 that we solve this problem with a similar column generation method to the one previously used to solve a deterministic version of this problem. In this solution method all the new constraints and variables are part of the column generation subproblem.

4.3 Correlated uncertainty sets

A robust optimization model that takes into account correlations of service times at all customers has been much harder to tackle. For this we consider the following uncertainty set

$$\mathcal{U}=\{s\mid \bar{s}_i\leq s_i\leq \bar{s}_i+z_i,\ i\in I_2,\ \sum_{i\in I_2} z_i\leq U,\ 0\leq z_i\leq \gamma_i\},$$

which includes a global bound on the variation of service times at all customers. Again, this is a special case of the bounded uncertainty sets discussed in Sect. 4.1. However, the objective now is to find a robust counterpart formulation that does not ignore the correlation in the uncertainty. Since time window violations are taken into account in the objective and the arrival time at a customer depends on the stops visited by that technician, uncertain parameters on the same route are naturally related.

Indeed, as in Proposition 1 we can show that, given a routing solution $x_r\in\mathcal{X}$, any variation assigned to a particular technician will occur at the start of that route in the worst case. Therefore, a robust counterpart would distribute the global uncertainty U between technicians (u_k for $k\in\mathcal{K}$) and use the previous approach for each technician. The challenge is to find the worst case division of this global uncertainty budget U among technicians. Possible alternatives either approximate this worst case value by arbitrarily separating the budget or lead to non-linear problems when using duality of the linear program in variables u_k to obtain robust bounds.

Recent work, [17], has analyzed a rolling horizon robust model with correlated uncertainty sets for multi period linear programming. The heuristic methods developed, which use duality and therefore do not readily apply for integer routing problems, highlight the difficulty in addressing correlated uncertainty. Finding a reformulation that does not significantly increase the difficulty of solving this routing problem with uncertainty is left for future work.

5 Branch and price solution method

To be able to solve large instances of this problem exactly we implemented a branch and price solution method. The algorithm here follows what was implemented for the deterministic version of this problem and described in [51]. In particular our Branch and Price scheme [2] uses the branching strategy proposed by Ryan and Foster [44]. The remaining key aspect of this algorithm is the column generation method implemented to solve the linear programming relaxations at each node. It is this column generation that has to be modified to adapt the solution method for the robust formulation of the problem.

The column generation method is applied to a path formulation of the routing problem. Analogous to the algorithm for the deterministic version of the problem, the master problem is a set partitioning model that chooses what routes will be used given a set R of robust routes:

$$\begin{array}{lll}
\min_{x_r, v} & \sum_{r\in R} \hat{c}_r x_r + \sum_{i\in I_1\cup I_2} v_i P & \\
\text{s.t.} & \sum_{r\in \mathrm{R}} a_{ir} x_r = 1 - v_i & i \in I_1 \cup I_2 \\
 & x_r \in \{0, 1\} & r \in R \\
 & v_i \in \{0, 1\} & i \in I_1 \cup I_2
\end{array}$$

In this model, the binary variable x_r is 1 if route r is used and 0 otherwise. Note that route r has a cost $\hat{c}_r = \sum_{k\in\mathcal{K}} \sum_{(q,l)\in P^k(x_r)} \beta\delta_l^r + (1-\beta)t_{ql}$. The soft time window penalty that route r incurs at client l, δ_l^r, is a master problem parameter, computed as part of the subproblem resolution. The parameter a_{ir} is equal to 1 if customer i is visited by route r and 0 otherwise. Therefore, the set partitioning constraint ensures that each customer is visited by only one route. The column generation procedure can be initialized with an empty set R, making the initial feasible solution $v_i = 1$ for all clients $i \in I_1 \cup I_2$, with a cost $I \times P$. If the column generation procedure ends with an optimal solution where one or more clients have a variable $v_i = 1$, then there is no feasible routing solution that visits all clients on the same day. Clients with $v_i = 1$ are scheduled for the next day.

At each iteration of the column generation, the subproblem generates a new robust route that could improve the actual routing solution. The model is a shortest path problem with negative costs. The cost of the path is equal to the total travel time, plus the soft time windows penalties, minus the shadow prices from the set partitioning constraint of the customers that belong to the path. As the following formulation will result in a single route for a specific technician, we omit the k index in all variables. In addition, since the route can start from any technician in the original problem, we add a dummy origin node $\{0\}$, from where the shortest path route will start. This dummy depot is connected to all nodes belonging to I_1, with travel time equal to 0. The new set of arcs A' then becomes $A' = A \cup \{(0, j) \mid j \in I_1\}$. The column generation subproblem is formulated as follows,

$$
\begin{array}{lll}
\min_{x,w,\delta,z,y} & \beta \sum_{i \in I_2} \delta_i + (1-\beta) \sum_{(i,j)\in A'} t_{ij} x_{ij} - \sum_{(i,j)\in A'} \alpha_i x_{ij} & \\
\text{s.t.} & \sum_{j:(i,j)\in A} x_{ij} - \sum_{j:(j,i)\in A} x_{ji} = b_i & i \in \{0\} \cup I_1 \cup I_2 \cup \{I+1\} \\
& w_i + s_i + z_i + t_{ij} - w_j \le (1 - x_{ij}) M & (i,j) \in A' \\
& w_i \le L \sum_{j:(j,i)\in A} x_{ji} & i \in I_2 \\
& w_i - \delta_i \le W_i & i \in I_2 \\
& \sum_i z_i \ge \min\{U, \sum_i \gamma_i x_{ij}\} & \\
& 0 \le z_i \le \gamma_i \sum_{j:(j,i)\in A} y_{ji} & i \in I_2 \\
& 0 \le z_i \le \gamma_i \sum_{j:(i,j)\in A} x_{ij} & i \in I_1 \\
& y_{ji} \le x_{ji} & (j,i) \in A' \\
& y_{ji} \le \frac{z_j}{\gamma_j} & (j,i) \in A' \\
& x_{ij} \in \{0,1\} & (i,j) \in A' \\
& y_{ji} \in \{0,1\} & (j,i) \in A' \\
& w_i, \delta_i, z_i \ge 0 & i \in I_2
\end{array}
\tag{5}
$$

The objective function is the reduced cost of the new route created: the sum of soft time windows penalties plus the total travel time minus the sum of the shadow prices α_i of the customers that belong to the new created route that come from the master problem's partition constraints. For the dummy origin 0 we set $\alpha_0 = 0$. The first set of constraints ensures the flow conservation for each client, the starting condition for dummy origin 0 and the final arrival to the fictitious node $I + 1$ (b_i is equal to 1 for $i = 0$, -1 for $i = I + 1$, and 0 otherwise). The second to fourth constraint sets impose the arrival times and the relation with the time windows penalties and extra service time. And the fifth to ninth constraint sets impose the relation between the variables defined in formulation (4).

The solution procedure used is similar to what was implemented in [51], simply modifying the column generation subproblem to problem (5) above, which considers additionally variables z, γ and y, and the fifth to ninth constraint sets. We solve the master problem using a linear programming solver (CPLEX 9.0 [35]) and the subproblem with a constraint programming (CP) solver (SOLVER 6.0 [36]).

6 Computational experiments

In this section we compare the solutions obtained by the robust model with uncertainty bounds per technician for different levels of robustness. The robustness level corresponds to the amount of uncertainty that the model takes into consideration when planning the routes, if this level is zero we are looking for the deterministic optimal solution. In the approach we are proposing here, the level of robustness is given by the parameters U_k and γ_i. The objective of these simulation experiments is to measure how the solutions behave in terms of configuration and service level for different values of the robust parameters. Notice that γ_i is a parameter associated with the customer while U_k is related to the maximum uncertainty faced by technician k, which originates from the group of customers that are serviced by that technician during the day.

To organize the experiments, we define a base case taken from a real-world instance comprising 41 customers and 15 technicians. The soft time window deadlines, W_i, are those reported by the company and are linked with the priority of each customer and the accumulated wait of that customer so far. The travel and service times were estimated from historical data corresponding to all service requests in one year of operation of this company. This data suggests that service times per customer approximately follow a Weibull distribution with a mean of $s_i = 77$ minutes and a standard deviation of $\sigma_i = 44$ minutes. In the experiments that follow we use a constant mean, s, and standard deviation, σ, equal to these estimated values to build the robust routing model. We evaluate the solutions found by simulating actual service times from a Weibull distribution with mean s and standard deviation σ.

For a given security factor $\rho \geq 0$, we define the parameters γ_i and U_k that describe the uncertainty model as follows: $\gamma_i = \sigma\rho$ and $U_k = \sqrt{|I_2|/K}\sigma\rho$, where $|I_2|/K$ is an estimate of the number of customers that will be visited by every technician. In these experiments we use $\sqrt{|I_2|/K} = 2.45$. The parameter γ_i therefore is also constant over all customers in these experiments. We note that, if service times were normally distributed, a security factor of 1.96 would ensure that in approximately 95 % of the cases the actual service time is less than or equal to $s + \rho\sigma$. Note that the original deterministic model is recovered by setting $\rho = 0$. Thus, ρ is the parameter used to calibrate the desired level of robustness. Our experiments consider $\rho \in \{0, 0.3, 0.6, 0.9, 1.2, 1.5\}$, obtaining six solutions for each scenario below.

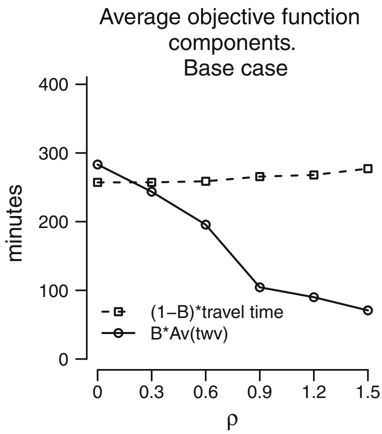
Each solution is evaluated on 1,000 simulated instances with service times for every customer drawn from a Weibull distribution with mean s and standard deviation σ. For each different routing solution we evaluate the objective function under the simulated service times. Note that although the travel costs remain constant, the time window violation and penalty for pushing clients to the next day can vary in each simulation. In fact if a customer cannot be served because of the simulated service times, then that customer is pushed to the next day with a penalization equal to a complete day of violation. In that case, the extra delay is added to the time window violation, weighted by β. The route is not recomputed in the simulation and still considers the travel time to that client.

We conduct a sensitivity analysis from the base case, varying each of the following twice: 1) the penalty on the time window violation β, 2) the number of technicians K, 3) the standard deviation of service times σ, and 4) tightness of the time windows. The complete set of experiments, where we vary one parameter at a time, is summarized in Table 1, with the base case highlighted in bold. For each combination of parameters we solve the robust routing problem, with a stopping criteria of an optimality gap ≤ 0.3 %. We evaluate each solution found by simulating 1,000 scenarios with service times randomly chosen from a Weibull distribution function as described above.

Table 1 Summary of sensitivity parameters

Description	Parameter	Values		
Time window violation penalty	β	0.3	**0.6**	0.9
Number of technicians	K	13	**15**	17
Standard deviation of the service times	σ	22	**44**	70
Time window deadlines relative to base case	ω	**1**	0.85	0.6

Fig. 1 Average objective function components



In Fig. 1 we plot the average value of the objective function (from the 1,000 repetitions) split in the two major components, travel time and time window violation, where the latter includes the penalty of sending customers to the next day. We can see that solutions that take less uncertainty into account (i.e. when ρ is close to zero) have a higher the time window violation. The reason for this is that in solutions that take little uncertainty into account many clients can be scheduled with little slack. This causes more clients to be sent to the next day when simulating service times. Note also that the solutions with smaller time window violations incur a slight increase in travel cost. In Figs. 2 to 5 below we present a summary of these results. We plot the mean value of the objective function ($av(of)$), the mean value of the time window violation ($av(twv)$) and the standard deviation computed for the objective function ($sd(of)$) as a function of ρ for each instance solved. The value of $sd(of)$ can be interpreted as a measure of the reliability of the solution. A low value of $sd(of)$ allows the dispatcher to have a good estimation of the arrival time of the technicians to each of the clients.

Figure 2 shows the sensitivity with respect to β of the performance measures discussed above. Here and in the subsequent results, the base case is represented with the solid line. Recall that the solutions for the deterministic case are obtained when $\rho = 0$.

Notice that as β increases, the quality of the solutions improves with the level of robustness. This effect can be visualized in $av(of)$, where clearly the line with the lowest β value ($\beta = 0.3$) does not improve with ρ in the same magnitude as the lines

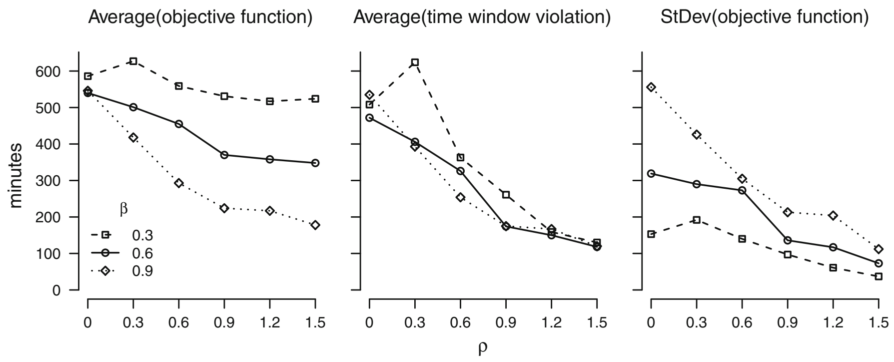


Fig. 2 Sensitivity to parameter (β)

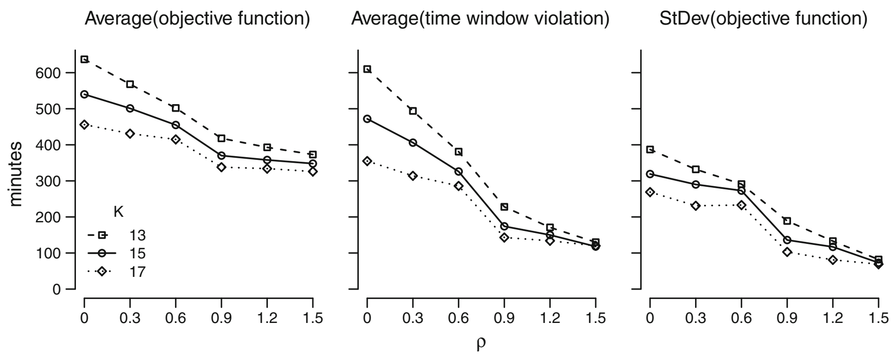


Fig. 3 Sensitivity to the number of technicians

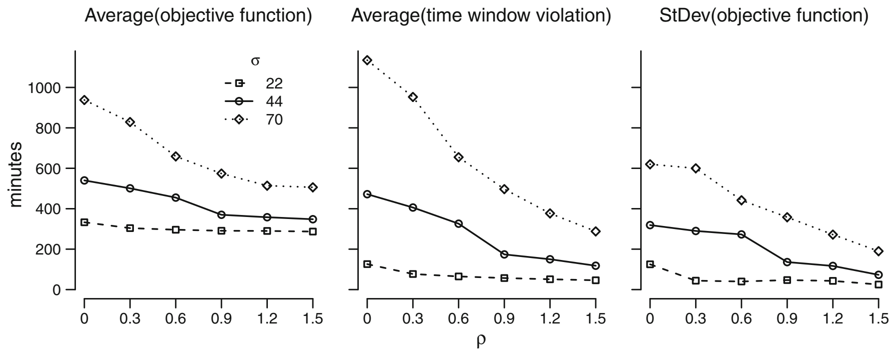


Fig. 4 Sensitivity to service time standard deviation (σ)

corresponding to $\beta = 0.6$ and $\beta = 0.9$. In addition, both $av(twv)$ and $sd(of)$ show the same tendency as $av(of)$.

This is expected for the robust formulation, as the robust counterpart tries to protect the system against the violation of the time windows, which becomes more important as β increases. We note that the line for $\beta = 0.3$ is not monotonically decreasing.

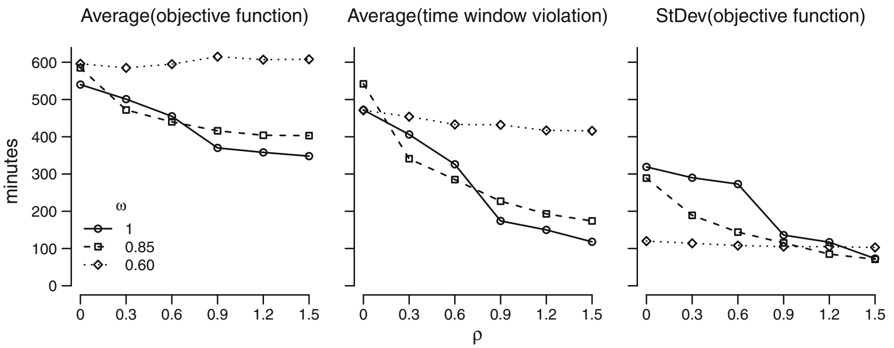


Fig. 5 Sensitivity to time window deadlines

The reason for this is that the dispatcher schedules more customers with little slack up to the end of the day. This causes these clients to be pushed beyond the end of day limit under simulated service times. The resulting values for $sd(of)$ show an inverse relation to what is observed in the function average value. In the case $\beta = 0.9$ we observe the largest standard deviation, while it has the smallest average objective function. Finally we note that as ρ increases, the mean objective function for the three β values becomes quite different, while the standard deviation decreases for all β and become quite similar for higher values of ρ.

Figure 3 shows the sensitivity analysis with respect to the number of technicians of the performance measures discussed above. More technicians implies better quality solutions albeit more expensive. This trade off must be carefully analyzed in terms of desired robustness as a guide to determine the size of the work force (equivalent to planning the fleet size if one route is assigned to one technician). For example, if the level of robustness is high (namely, ρ values above 0.6), there is not much difference in hiring either 15 or 17 technicians. However, if the solution is not robust enough (small values of ρ) there is substantial difference in performance for the two extra technicians (from 15 to 17). The results also show that the real jump in terms of system benefit from the robust formulation is observed when ρ moves from 0.6 to 0.9 for all curves, beyond that the benefits decrease considerably, stabilizing the results with ρ around 1.2 or 1.5.

Figure 4 illustrates the system response when we artificially add more uncertainty to the system, reflected in the value of the standard deviation in service times (σ). The larger the σ value, the more relevant the effect of the robustness parameter ρ in the solution quality. If the standard deviation of the uncertain parameter is high, the robust formulation can improve the solution quality notoriously. Moreover, if σ is low, the robust formulation does not improve much the solution quality. Then, in that case it seems better to use the deterministic model for finding the routes, which is easier to solve as it has fewer integer variables. Actually, for a standard deviation of 22, the implementation of the robust solutions does not produce significant effect at all in the performance indicators.

Finally, Fig. 5 shows the effect of having tighter time window deadlines. As expected the results show that a tighter time window implies a worse solution quality. This is

Table 2 Average number of customers serviced on the next day

w	ρ					
	0	0.3	0.6	0.9	1.2	1.5
1.00	0.275	0.209	0.170	0.047	0.030	0.007
0.85	0.149	0.062	0.029	0.015	0.005	0.002
0.60	0.000	0.000	0.000	0.000	0.000	0.000

what can be seen in the plots for $av(of)$ and $av(twv)$ in Fig. 5, with the exception of a few cases where the different configuration of the robust routes resulted in a difference in the number of customers scheduled on the next day. This effect is not simple to visualize in the figures. Actually, the crossing of the curves associated with $\omega = 1$ and $\omega = 0.85$, in both figures ($av(of)$ and $av(twv)$) occurs when $\rho = 0.6$ and 0.3. In these cases, the number of customers left for the next day is considerably larger for $\omega = 1$ compared with $\omega = 0.85$, see Table 2. This leads to a larger time window violation in the base case scenario, producing a larger average objective function in the graph. Note also that the average number of clients pushed to the next day decreases as ρ increases.

The four previous experiments show that as the robustness parameter increases, in almost all cases the quality of the solution improves: the average objective function decreases, as well as the average violation of the time windows and the standard deviation of the objective function. Recall that for a given routing solution, the total travel time is constant no matter what the values of the service times are. We also can say that from the results, the tendency observed in the different curves showed stable results after $\rho > 1.5$ (higher robustness levels are not relevant any further), making those results not very interesting, and that is why we chose that threshold in all previous figures.

7 Conclusions

In this paper we develop a model for the VRP with stochastic service times (VRPSST) to find an efficient solution that is insensitive to the uncertainty in service times. This problem is motivated by a real routing problem faced by maintenance and repair service providers where service times can show high variability depending on the features of each specific service request. We propose a robust solution that is obtained by solving a robust counterpart problem that optimizes the worst-case value over all data uncertainty. To formulate and solve the robust version of the VRPSST we follow the robust optimization literature for integer programming problems. The contribution here is to explore problem formulations that allow to capture the correlation between uncertain parameters. We show that we can find a robust solution when the uncertainty faced by each technician is bounded. Robust solutions have the potential to be preferable in practice because uncertain parameters differ from estimated values on day to day operations and a robust solution tends to be not far from the solution that optimizes the average scenario, and can significantly outperform that solution in the worst case.

The robust formulation we present in this paper considers that the uncertainty is represented by having the uncertain parameters belong to a closed, convex, and bounded set. This uncertainty set could be estimated from historical data on the uncertain parameters. Other modeling approaches require building detailed probability density functions of service times, which is not an easy task considering the quality and accuracy in databases of service time variability from real companies. In this paper, we present a model for the robust repair service scheduling problem and later mention two versions of uncertainty sets that lead to tractable robust counterpart problems. We leave for future work a comparison of this robust model to other existing optimization models under uncertainty for the VRP. Such comparison should make careful decisions on which models to consider, how to represent the uncertainty for each, and how to fairly evaluate their performance. Another topic for future research is to understand the merits of using a robust model to re-optimize during operations.

With regard to the solution method, the structure of our model allows to easily adapt a branch and price methodology (originally developed to solve the deterministic problem) to efficiently solve our problem in case of real instances. The results show that we can solve instances of size relevant for real-world problems. Our sensitivity analysis evaluates the practical quality of the robust solutions found seeing how they perform under simulated service times. We find that solutions that take uncertainty into account exhibit a smaller sensitivity to the uncertainty but also have smaller time window violations. The reason for this drop in time window violations is due to the fact that a solution that incorporates slack in its schedule will tend to send less customers to the next day.

The proposed approach nicely fits the conditions of service scheduling problem we are working with; however the model and the results of the paper can be extended to more general problems with minor adjustments (heterogeneous technicians, different travel costs, and so on). The robust approach can also be extended to include other sources of uncertainty, such as travel times which are uncertain due to traffic conditions, congestion, and time of the day.

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