Let $\{X_n\}$ is a Markov chain

1) For j transient $\lim_{n\to\infty} P\{X_n=j \mid X_0=j\}=0$

2) For i and j recurrent, both within the same irreducible set, A, the
$\lim_{n\to\infty} P\{X_n=j \mid X_0=i\} = \pi(j)$ where
$\pi P = \pi$ and $\sum_{k\in A} \pi(k) = 1$
where P is the Markov matrix restricted to A.

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3) If j is recurrent then
$\lim_{n\to\infty} P\{X_n=j \mid X_0=i\} = F(i,j)\,\pi(j)$

4) If j is non-null recurrent with X_0 with the irreducible set containing j, then
$\underbrace{\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} I(k,j) = \pi(j)}_{\text{the Ergodic property}}$ a.s.

time avg = spatial avg.

5) If j is recurrent, $E[T^j \mid X_0=j] = \frac{1}{\pi(j)}$

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Two common criteria for dealing with cost or profit for irreducible M.C.

1) Long-run average

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} f(X_k) = \pi \cdot f$$

2) Total discounted cost.

Let α be a discount factor, in other words, α is present value of \$1 one period from now. $\Rightarrow 0 < \alpha < 1$

$$E_i\left[\sum_{k=0}^{\infty} \alpha^k f(X_k)\right] = \sum_{k=0}^{\infty} \alpha^k P^k f(i)$$

$$= \left[\sum_{k=0}^{\infty} \alpha^k P^k\right] f(i) = (I - \alpha P)^{-1} f(i)$$

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$$\pi P = \pi \longrightarrow \pi P = \pi I$$

$$\pi \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = 1 \qquad 0 = \pi (I - P)$$

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The random variable T is called a <u>stopping time</u> w.r.t. $\{N_t\}$ if one can determine whether the event $\{T \le t\}$ has occurred or not by knowing the history $\{N(u) \text{ for } u \le t\}$.

T is a stopping iff $\{T \le t\}$ is independent of $\{N(u) \text{ for } u > t\}$.

$$P\{N_{t+s} - N_t = k\} = \frac{e^{-\lambda s}(\lambda s)^k}{k!} \quad \text{for } k = 0, 1, \cdots$$

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The random variable N is a stopping time w.r.t. $\{X_n\}$ is one can determine whether or not $\{N=n\}$ has occurred based on $\{X_0, X_1, \cdots, X_n\}$.

N is a stopping time if $\{N=n\}$ is independent of $\{X_{n+1}, X_{n+2}, \cdots\}$

$$E\left[\sum_{k=1}^{N} X_k\right] = E[N]\,E[X_1]$$

true for N independent of $\{X_n\}$

Wald's Thm

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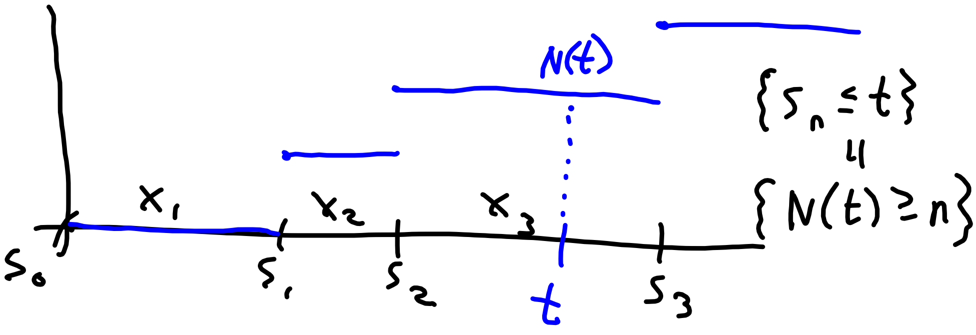
Renewal Processes



Assume $\{X_n\}$ are non-negative i.i.d. with d.f. $F(\cdot)$ such that $F(0) < 1$ and let $\mu = E[X_1]$

or $\mu = \int_{[0,\infty)} F(dx)\, x = \int_0^\infty [1 - F(x)]\, dx$

The $\{N(t)\}$ is called a renewal process
$\{S_n\}$

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$$N(t) = \sum_{n=1}^{\infty} I_{[0,t]}(S_n) \quad \text{also } N(t) = \sup\{n : S_n \le t\}$$

From basic prob.

Strong law of large numbers

$$\lim_{n\to\infty} \frac{S_n}{n} = \mu \quad \text{a.s. iff } \mu < \infty$$

$$\lim_{t\to\infty} \frac{N(t)}{t} = \frac{1}{\mu} \quad \text{a.s.}$$

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$$F * G(t) = \int_{[0,t]} F(ds)\,G(t-s) \qquad \begin{array}{l} F(t)=0 \text{ if } t<0 \\ G(t)=0 \text{ if } t<0 \end{array}$$

$$F * F(t) = F_2(t)$$
$$F * F_2(t) = F_3(t)$$
$$\vdots$$
$$\Rightarrow F_n(t) = P\{S_n \le t\}$$

$$P\{N(t)=n\} = P\{N(t) \ge n\} - P\{N(t) \ge n+1\}$$
$$= P\{S_n \le t\} - P\{S_{n+1} \le t\}$$
$$= F_n(t) - F_{n+1}(t)$$

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$$N(t) = \sum_{n=1}^{\infty} I_{[0,t]}(S_n)$$

$$m(t) = E[N(t)] = \sum_{n=1}^{\infty} E[I_{[0,t]}(S_n)]$$
$$= \sum_{n=1}^{\infty} P\{S_n \le t\} = \sum_{n=1}^{\infty} F_n(t)$$

the renewal function

$$\boxed{m(t) = \sum_{n=1}^{\infty} F_n(t)}$$

$$m * F(t) = \left(F_2(t) + F_3(t) + \cdots\right) + F_1(t) - F_1(t)$$

$$\boxed{m * F(t) = m(t) - F(t)}$$

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