Useful Facts Regarding Eigenvalues for Finite Matrices[1]

The below discussion assumes that the matrix A is square of dimension n×n. First are given the definitions of eigenvalues and eigenvectors, then we give some useful facts regarding eigenvalues.

For a fixed scalar x, the characteristic polynomial of A is defined by

$$\varphi(x) = \det(xI - A).$$

The polynomial φ is of degree n; therefore, there are n roots to the equation $\varphi(x) = 0$. Call these (possibly complex and not necessarily unique) roots, $\lambda_1, \ldots, \lambda_n$ eigenvalues. If an eigenvalue is unique among the list of n eigenvalues, it is called a simple eigenvalue.

A vector, v, such that $Av = \lambda v$ is called an eigenvector and v is unique up to a multiplicative constant for each eigenvalue. Sometimes this eigenvector is called a right eigenvector.

A row vector, π, such that $\pi A = \lambda\pi$ is called a left eigenvector. The following information is relevant for Markov chains and/or Markov processes.

$(1, 1, \ldots, 1)A = (s, s, \ldots, s) = s(1, 1, \ldots, 1)$

e, v

1. If each row sums to the same value, call the sum s, then s is an eigenvalue.
2. If each column sums to the same value, call the sum s, then s is an eigenvalue
3. The trace of A, denoted tr(A), is the sum of its diagonal elements and
   a. $tr(A) = \lambda_1 + \ldots + \lambda_n$ $\quad tr(AB) = tr(BA) \quad A = N^{-1} D N$

   $tr(A) = tr(NN^{-1}D) = tr(D)$

   $\sum_{i=1}^{n} \lambda_i \xrightarrow{k\to+\infty} \sum \pi_i = 1 \Rightarrow |\lambda_i| < 1$

   b. $tr(A^k) = \lambda_1^k + \ldots + \lambda_n^k$ for each k = 1, 2, … by 1.
4. For an irreducible, aperiodic Markov matrix, P, the value 1 is a simple eigenvalue of P. If λ is an eigenvalue of P with $\lambda \neq 1$, then $|\lambda| < 1$. $(\pi A)_i \le \lambda \pi_i$ (∵ A ⇒ prob ≤ 1)
5. Let P be an irreducible, aperiodic Markov matrix and let π be the left eigenvector associated with the eigenvalue of 1 with $\pi\mathbf{1} = 1$. Let $\beta = \max\{ |\lambda_j| : \lambda_j \neq 1$ and λ_j is an eigenvalue of P$\}$, then there exists a constant α such that $\quad \pi P = \pi \quad \|\pi\| = 1$

   ? $i = ?$ proof

   $$| P^k(i,j) - \pi(j) | = \alpha\, \beta^k \quad \text{for } k = 1, 2, \ldots \quad \forall i \quad \forall j$$
6. If all the eigenvalues of A are unique, then A is diagonalizable. Let $v_1, \ldots, v_n$ be the eigenvectors associated with $\lambda_1, \ldots, \lambda_n$. Form the matrix N by letting its k^{th} column be v_k. Let the matrix D be a matrix such that D(i,j)=0 if i≠j and $D(i,i) = \lambda_i$ .for i=1, ..., n. Then

   $$A = N D N^{-1}.$$

   It might also be noted that the rows of N^{-1} are left eigenvectors of A. (The uniqueness of the eigenvalues is sufficient but not necessary for a matrix to be diagonalizable. Under certain conditions it is possible to diagonalize the matrix A even when the eigenvectors are not all unique.)
7. If A is diagonalizable, then

   $e^A = N e^D N^{-1}$ where $\quad \lambda_i \Rightarrow$ a number

   $e^D(i,j) = 0$ if i≠j and $e^D(i,i) = e^{\lambda i}$ for i = 1, …, n. Note that $e^A = \sum_{k=0}^{\infty} A^k/k!$

[1] Material taken from the appendix in E. Cinlar (1975). *Introduction to Stochastic Processes*, Prentice-Hall, Inc.