Homework #14

Consider a small machine shop. Customers arrive to the shop according to Poisson process with mean rate of three per day. Because the shop is small, they only work on one job at a time and if they stay busy they can average two completed jobs per day. If two jobs are in the system and a customer arrives with a job, that customer is sent to another shop; thus, there are at most two jobs in the shop at any time. Let Z(t) be a process with state space {0,1,2} where Z(t) is the number of jobs in the shop at time t.

First draw the state diagram showing rates of one-step transitions. Usually it is easiest to write the G matrix by first giving the off-diagonal elements, then summing the rows to obtain the diagonal elements. Below is G and G^2 since the G^2 is needed for part c.

$$G = \begin{matrix} -3 & 3 & 0 \\ 2 & -5 & 3 \\ 0 & 2 & -2 \end{matrix} \qquad G\text{^}2 = \begin{matrix} 15 & -24 & 9 \\ -16 & 37 & -21 \\ 4 & -14 & 10 \end{matrix}$$

a. Notice that {Z(t)} is a Markov process. Form the generator matrix for this process. What is G(0,1)? (So that our answers have the same units, use days for your time units.) G(0,1) = 3

b. What is G(1,1)? G(1,1) = -5

c. One eigenvalue of G is a number between -2 and -3. What is its value rounded to four digits to the right of the decimal. The eigenvalues are 0 and -5 +/- SQRT(6); thus the eigenvalue between -3 and -2 is -2.5505.

d. What is $\lim_{t\to\infty} P\{Z(t) = 0 \mid Z(0) = 0\}$ rounded to four digits to the right of the decimal? This is the left eigenvector associated with 0. Since it is the left eigenvector, each column yields and equation. The first two equations give $p_1=(3/2)p_0$ and $p_2 = (5/2)p_1 – (3/2)p_0 = (9/4)p_0$. Finally we use the norming equation to give $p_0 = 4/19 =$ 0.2105.

e. Given that the shop was empty at the start of the day, what is the probability that it will be empty at the end of the day? Assume each day is from 8AM to 4PM so that noon is t=0.5. Round your answer to three digits to the right of the decimal.

$$N = \begin{matrix} 1 & 1 & 1 \\ 1 & -1.4480 & 0.1498 \\ 1 & 0.5443 & -0.5443 \end{matrix} \qquad D = \begin{matrix} 0 & 0 & 0 \\ 0 & -7.4495 & 0 \\ 0 & 0 & -2.5505 \end{matrix}$$

Since t=0.5, the diagonal elements of e^{Dt} are 1, 0.024119, and 0.27936. Then answer is the upper left element of the $Ne^{Dt}N^{-1}$ yielding 0.379679. Rounding this yields 0.380.