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EX 9.41 (a) $X \sim f(x)$, f strictly $\downarrow$ pdf on $[0, \infty)$. For a fixed value of $1-\alpha$. $\forall\ [a,b]$ s.t. $\int_a^b f(x)\,dx = 1-\alpha$, the shortest is obtained by choosing $a=0$ & b s.t. $\int_0^b f(x)\,dx = 1-\alpha$

WANT TO SHOW : $a_0 = 0$, $\int_0^{b_0} f(x)\,dx = 1-\alpha$, $\quad b_0 - a_0 = \min\limits_{\int_a^b f(x)dx = 1-\alpha} b-a$

Namely to show : $\forall\ \epsilon > 0$, $\quad a_0+\epsilon,\ b_0+\epsilon$ with $\int_{a_0+\epsilon}^{b_0+\epsilon} f(x)\,dx < 1-\alpha$.

Then if $a = a_0+\epsilon$, b should ~~larger~~ larger than $b_0+\epsilon$, to keep $\int_a^b f(x)\,dx = 1-\alpha$, and $b-a > b_0-a_0$.

proof:

$$\int_{a_0+\epsilon}^{b_0+\epsilon} f(x)\,dx - (1-\alpha) = \int_{a_0+\epsilon}^{b_0+\epsilon} f(x)\,dx - \int_{a_0}^{b_0} f(x)\,dx$$

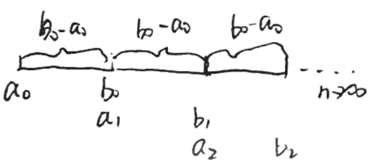
$$\overset{f(x)\downarrow}{=} \int_{b_0}^{b_0+\epsilon} f(x)\,dx - \int_{a_0}^{a_0+\epsilon} f(x)\,dx$$

$$\le f(b_0)[b_0+\epsilon-b_0] - f(a_0+\epsilon)[a_0+\epsilon-a_0]$$

$$= \epsilon\,[f(b_0) - f(a_0+\epsilon)]$$

Suppose $\epsilon < b_0 - a_0$, then $\int_{a_0+\epsilon}^{b_0+\epsilon} f(x)\,dx \le 1-\alpha$.

If $\epsilon \ge b_0-a_0$, let $n = 0, 1, 2, \dots, \infty$.



$a_{n+1} = b_n$, $\quad b_{n+1} = a_{n+1} + b_0 - a_0$, $\quad \forall\ 0 < \epsilon_{n+1} \le b_{n+1} - a_{n+1}$

$$\Rightarrow \int_{a_{n+1}+\epsilon_{n+1}}^{b_{n+1}+\epsilon_{n+1}} f(x)\,dx - \int_{a_{n+1}}^{b_{n+1}} f(x)\,dx \le \epsilon\,[f(b_{n+1}) - f(a_{n+1}+\epsilon_{n+1})] \le 0$$

$$\Rightarrow \forall n \quad \int_{a_{n+1}+\epsilon_{n+1}}^{b_{n+1}+\epsilon_{n+1}} f(x)\,dx \le \int_{a_{n+1}}^{b_{n+1}} f(x)\,dx \le \cdots \le \int_{a_0}^{b_0} f(x)\,dx$$

$n(b_0-a_0) \le \epsilon = (n+1)\epsilon_{n+1} \le (n+1)(b_0-a_0)$

$\Rightarrow \forall\ \epsilon > 0$, $\int_{a_0+\epsilon}^{b_0+\epsilon} f(x)\,dx < 1-\alpha$. $\quad \Rightarrow a_0 = 0$, $\int_0^{b_0} f(x)\,dx = 1-\alpha$

with $b_0 - a_0 = \min\limits_{\text{s.t. } \int_a^b f(x)dx = 1-\alpha} b-a$.

EX 9.36: $\{X_i\}$ iid $f_{X_i}(x|\theta) = e^{i\theta - x} I_{[i\theta,\infty)}(x)$. show: $T = \min_i (X_i/i)$ is suff stat for θ. Based on T, find the $1-\alpha$ confidence interval for θ of the form $[T+a, T+b]$ which is of min length

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① $f(x_1,\dots,x_n|\theta) = \prod_{i=1}^{n} e^{i\theta - x_i} I_{[i\theta,\infty)}(x_i) = e^{\Sigma(i\theta - x_i)} I_{[\theta,\infty)}[\min[x_i/i]]$

By factorization theorem, $T = \min(X_i/i)$ is sufficient statistics

$$\cancel{② \; P(T>t) = \prod_{i=1}^{n} P(X_i > it) = \prod_{i=1}^{n}\int_{it}^{\infty} e^{i\theta - x}\,dx = \prod_{i=1}^{n} e^{i(\theta-t)} = e^{-\frac{n(n+1)}{2}(t-\theta)}} \quad \cancel{t \geq \theta}$$

$$\cancel{f_T(t) = \frac{n(n+1)}{2} e^{-\frac{n(n+1)}{2}(t-\theta)}}$$

$$\cancel{P(T+a \leq \theta \leq T+b) = P(\theta-b \leq T \leq \theta-a) = \frac{n(n+1)}{2}\int_{\theta-b}^{\theta-a} e^{-\frac{n(n+1)}{2}(t-\theta)}\,dt}$$

$$\cancel{= e^{\frac{n(n+1)}{2}b} - e^{\frac{n(n+1)}{2}a} = 1-\alpha.}$$

$$\cancel{b = \frac{2}{n(n+1)}\ln\left[(1-\alpha) + e^{\frac{n(n+1)}{2}a}\right]} \qquad \cancel{\theta - b \geq \theta \Rightarrow b \leq 0.}$$

$$\cancel{\min_{a,b} b-a = \min_a \underbrace{\frac{2}{n(n+1)}\ln\left[(1-\alpha) + e^{\frac{n(n+1)}{2}a}\right] - a}_{g(a)}}$$

$$\cancel{g'(a) = \frac{n(n+1)}{2}\cdot\frac{1}{1-\alpha + e^{\frac{n(n+1)}{2}a}}\, e^{\frac{n(n+1)}{2}a} - 1 = 0}$$

$$\cancel{\min_{a,b} b-a.}$$

② let $Y := T-\theta$. Then $f_Y(y) = \frac{n(n+1)}{2} e^{-\frac{n(n+1)}{2}y}$ decreasing on $[0,\infty)$ $(T\geq\theta,\ Y-\theta\geq 0)$

$P(T+a \leq \theta \leq T+b) = P(-b \leq Y \leq -a)$

By Ex 9.41(a) $\Rightarrow$ $-b=0$, $-a$ such that: $\int_0^{-a} f_Y(y)\,dy = 1-\alpha$

$\Rightarrow b=0,\quad a = \frac{2\ln(1-\alpha)}{n(n+1)}$

3. (a) $P(X_{(n)} \le x) = \left(\frac{x}{\theta}\right)^n$, $f_{X_{(n)}}(x) = \frac{n x^{n-1}}{\theta^n}$

$E[(X_{(n)} - \theta)^2] = \int_0^\theta (x-\theta)^2 \frac{n}{\theta^n} x^{n-1} dx = \frac{2\theta^2}{(n+1)(n+2)}$

$\forall \varepsilon > 0$, $0 \le \lim_{n\to\infty} P(|X_{(n)} - \theta| > \varepsilon) \underset{\text{inequality}}{\overset{\text{Chebychev's}}{\le}} \lim_{n\to\infty} \frac{1}{\varepsilon^2} E[(X_n - \theta)^2] = \lim_{n\to\infty} \frac{1}{\varepsilon^2} \frac{2\theta^2}{(n+1)(n+2)} = 0$

By definition of weakly consistant $\Rightarrow$ $X_{(n)}$ is weakly consistent for θ

(b) ① $P(n(X_{(n)} - \theta) \le y) = P(X_{(n)} \le \frac{y}{n} + \theta) = \left(\frac{y}{n\theta} + 1\right)^n \xrightarrow{n\to+\infty} e^{\frac{y}{\theta}}$

$f_{n(X_{(n)}-\theta)}(y) = \frac{n}{n\theta}\left(\frac{y}{n\theta} + 1\right)^{n-1} = \frac{1}{\theta}\left(\frac{y}{n\theta} + 1\right)^{n-1} \xrightarrow{n\to+\infty} \frac{1}{\theta} e^{\frac{y}{\theta}}$

② $X_{(n)}$ is not asymptotically normal

$P(\sqrt{n}(X_{(n)} - \theta) \le t) = P(X_{(n)} \le \frac{t}{\sqrt{n}} + \theta) = \left(\frac{t}{\sqrt{n}\theta} + 1\right)^n \xrightarrow{n\to\infty} +\infty$

$f_{\sqrt{n}(X_{(n)}-\theta)}(t) = \frac{n}{\sqrt{n}\theta}\left(\frac{t}{\sqrt{n}\theta} + 1\right)^{n-1} = \frac{\sqrt{n}}{\theta}\left(\frac{t}{\sqrt{n}\theta} + 1\right)^{n-1} \xrightarrow{n\to\infty} +\infty$

$\Rightarrow \sqrt{n}\{X_{(n)} - \theta\} \overset{d}{\not\to} N(0, V(\theta))$

$\Rightarrow X_{(n)}$ is not asymptotically normal