Renewal Eq.: Find $h(\cdot)$ such that

$$h(t) = g(t) + \int_{[0,t]} F(ds)\, h(t-s)$$

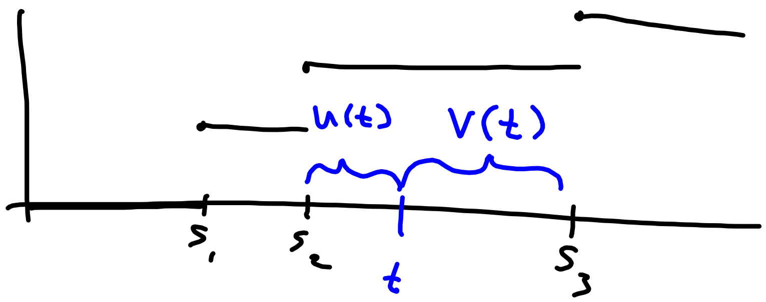
Solution

$$h(t) = g(t) + \int_{[0,t]} m(ds)\, g(t-s)$$

For g directly Riemann integrable and $\{N(t)\}$ recurrent, aperiodic

$$\lim_{t\to\infty} h(t) = \frac{1}{\mu}\int_0^\infty g(t)\,dt \qquad \mu = E[X_1]$$

Feb 12-7:55 AM



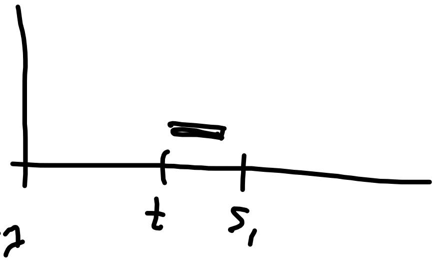
$$V(t) = S_{N(t)+1} - t$$

$$U(t) = t - S_{N(t)}$$

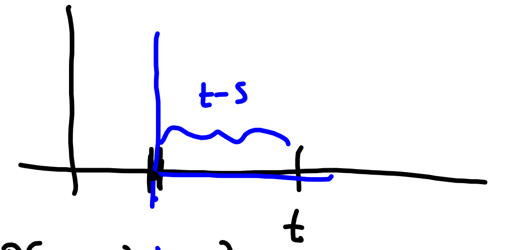
Feb 12-8:03 AM

$P\{V(t) > y\}$

on $\{S_1 > t\}$



$$P\{V(t) > y \mid S_1\} = \begin{cases} 1 & \text{if } S_1 > t+y \\ 0 & \text{if } S_1 \le t+y \end{cases}$$



$$P\{V(t) > y\} = 1 \cdot P\{S_1 > t+y\} + \int_{[0,t]} F(ds)\, P\{V(t-s) > y\}$$

Feb 12-8:05 AM

$$P\{V(t) > y\} = P\{S_1 > t+y\} + \int_{[0,t]} F(ds)\, P\{V(t-s) > y\}$$

$h(t) = P\{V(t) > y\}$ and $g(t) = 1 - F(t+y)$

$$P\{V(t) > y\} = 1 - F(t+y) + \int_{[0,t]} m(ds)\,[1 - F(t-s+y)]$$

$$\lim_{t \to \infty} P\{V(t) > y\} = \frac{1}{\mu} \int_0^\infty [1 - F(t+y)]\, dt$$

let $u = t+y \Rightarrow du = dt$

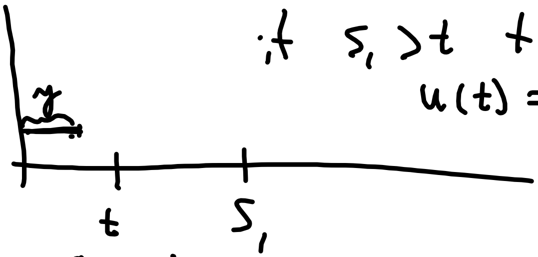
$$\lim_{t \to \infty} P\{V(t) > y\} = \frac{1}{\mu} \int_y^\infty [1 - F(u)]\, du$$

For Poisson process with rate λ,

$$P\{V(t) > y\} = e^{-\lambda y}$$

Feb 12-8:10 AM

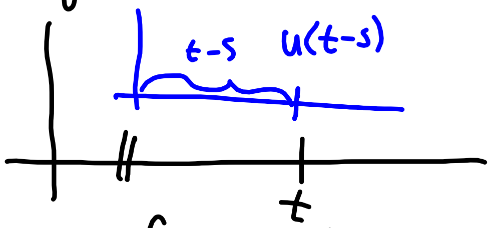
$U(t) = t - S_{N(t)}$



if $S_1 > t$ then $u(t) = t$

$\{S_1 > t\}$

$$P\{u(t) > y \mid S_1\} = \begin{cases} 1 & \text{if } y < t \\ 0 & \text{if } y \geq t \end{cases}$$



$$P\{u(t) > y\} = P\{S_1 > t\} I_{(y,\infty)}(t) + \int_{[0,t]} F(ds) P\{u(t-s) > y\}$$

Feb 12-8:35 AM

$$P\{u(t) > y\} = [1 - F(t)] I_{(y,\infty)}(t) + \int_{[0,t]} F(ds) P\{u(t-s) > y\}$$

$h(t) = P\{u(t) > y\}$ and $g(t) = [1 - F(t)] I_{(y,\infty)}(t)$

$$\therefore P\{u(t) > y\} = [1 - F(t)] I_{(y,\infty)}(t) + \int_{[0,t]} m(ds)[1 - F(t-s)] I_{(y,\infty)}(t-s)$$

need $t - s > y \Rightarrow s < t - y$

if $t - y < 0 \Rightarrow$ the integral is zero

$$P\{u(t) > y\} = \begin{cases} 0 & \text{if } t < y \\ 1 - F(t) + \int_{[0,t-y]} m(ds)[1 - F(t-s)] \end{cases}$$

Feb 12-8:43 AM

for $t < y$ $\quad P\{u(t) > y\} = 0$

for $t \ge y$ $\quad P\{u(t) > y\} = [1 - F(t)] + \int_{[0, t-y]} m(ds)[1 - F(t-s)]$

$$\lim_{t \to \infty} P\{u(t) > y\} = \lim_{t \to \infty} \left\{ [1 - F(t)] + \int_{[0,t)} m(ds)[1 - F(t-s)] I_{(y,\infty)}(t-s) \right\}$$

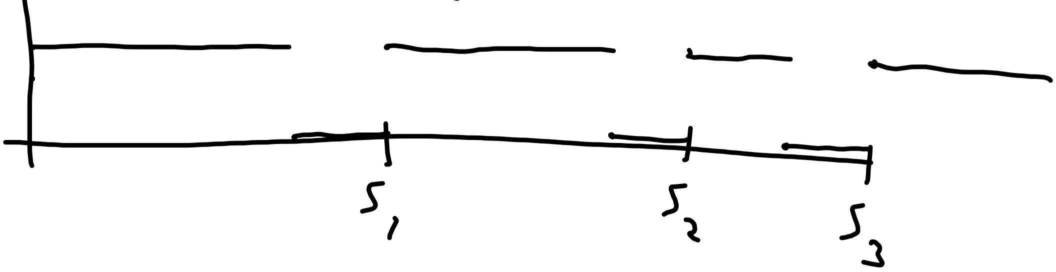
$$= \frac{1}{\mu} \int_0^\infty [1 - F(t)] I_{(y,\infty)}(t)\, dt$$

$$= \frac{1}{\mu} \int_y^\infty (1 - F(t))\, dt$$

Feb 12-8:49 AM

$$a = \int_0^\infty [1 - \varphi(t)]\, dt$$

$$b = \int_0^\infty [1 - \psi(t)]\, dt$$



$P\{Z(t) = 1\}$ $\quad$ on $\{s_1 > t\}$

$P\{Z(t) = 1 \mid s_1\}$

Feb 12-8:59 AM