Markov chains

One-step probabilities: P

Initial probabilities: μ

Reward function: f

First passage probabilities: F

$$F(i,j) = P_i\{T^j < \infty\}$$

Expected number of visits

$$R(i,j) = E_i[N^j]$$

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Two states communicate ($i \longleftrightarrow j$) iff there is a path with positive probability of going from i to j and from j to i.

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$$\begin{array}{c} a \\ b \\ c \\ d \\ e \end{array}\begin{bmatrix} 0.3 & 0 & 0 & 0 & 0.7 \\ 0.1 & 0.4 & 0.1 & 0.3 & 0.1 \\ 0.1 & 0 & 0.9 & 0 & 0 \\ 0 & 0.4 & 0.3 & 0.1 & 0.2 \\ 0 & 0 & 0.2 & 0 & 0.8 \end{bmatrix}$$

$a \leftrightarrow c$, $a \leftrightarrow e$, $c \leftrightarrow e$, $b \leftrightarrow d$

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1) A state j is recurrent if $F(j,j)=1$ (or equivalently if $R(j,j)=\infty$)
2) A state j is transient if $F(j,j)<1$ (or equivalently if $R(j,j)<\infty$)
3) A recurrent state j is null recurrent if $E_j[T^j]=\infty$
4) A recurrent state j is periodic with period $d \geq 2$ if d is the largest integer for which $P_j\{T^j = nd \text{ for some } n \geq 1\} = 1$. In other words, the chain can only return to state j at steps numbered $d, 2d, 3d$, etc.
5) State j is absorbing if $P(j,j)=1$

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6) A set of states is closed if once the process is in the set, it cannot leave.

7) A set is irreducible if it is closed and contains no proper subset that is also closed.

8) A set of states is irreducible if it is closed and all states communicate with all other states in the set

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$$F = \begin{array}{c} \\ a \\ c \\ e \\ d \\ b \end{array} \begin{array}{c} \begin{array}{ccccc} a & c & e & d & b \end{array} \\ \left[\begin{array}{ccc|c|c} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 \\ \hline 0.5 & 0.5 & 0.5 & 0.5 & 0.4 \end{array}\right] \end{array} \quad \begin{array}{c} a \\ c \\ e \\ d \\ b \end{array} \left[\begin{array}{ccc|c|c} 0.3 & 0 & 0.7 & 0 & 0 \\ 0.1 & 0.9 & 0 & 0 & 0 \\ 0 & 0.2 & 0.8 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 \\ \hline 0.1 & 0.1 & 0.1 & 0.3 & 0.4 \end{array}\right]$$

$$R = \begin{array}{c} a \\ c \\ e \\ d \\ b \end{array} \left[\begin{array}{ccc|c|c} \infty & \infty & \infty & 0 & 0 \\ \infty & \infty & \infty & 0 & 0 \\ \infty & \infty & \infty & 0 & 0 \\ \hline 0 & 0 & 0 & \infty & \\ \hline \infty & \infty & \infty & \infty & \end{array}\right]$$

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If X is an irreducible non-null recurrent M.C. with Markov matrix P, then

- $\lim_{n\to\infty} P_i\{X_n=j\} = \pi(j)$ where

$$\pi P = \pi \text{ and } \sum \pi(j) = 1$$

Also if f is a reward vector

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(X_k) = \pi f \quad \text{a.s.}$$

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$N \to$ # tries until first head

X is return for game $X = 2^N$

$P\{N=n\} = \left(\frac{1}{2}\right)^n$

$E\{X\} = \sum_{n=1}^{\infty} 2^n \cdot \left(\frac{1}{2}\right)^n = \sum_{n=1}^{\infty} 1$

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