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1. E x 7.5 $\{X_i\}_{i=1}^{n+1} \overset{iid}{\sim}$ Bernoulli(p), $h(p) := P(\sum_{i=1}^{n} X_i > X_{n+1} \mid p)$

(a) show that: $T(X_1, \cdots, X_{n+1}) = \begin{cases} 1 & \text{if } \sum_{i=1}^{n} X_i > X_{n+1} \\ 0 & \text{otherwise} \end{cases}$ unbiased estimator of $h(p)$

(b) Find the best unbiased estimator of $h(p)$

(a). $E_p T = 1 \cdot P_p(\sum_{i=1}^{n} X_i > X_{n+1}) + 0 \cdot (1 - P_p(\sum_{i=1}^{n} X_i > X_{n+1})) = P(\sum_{i=1}^{n} X_i > X_{n+1} \mid p) = h(p)$

(b) In order to find UMVUE for $h(p)$, only need to find a complete statistic W for p

$f(X; p) = p^{\sum_{i=1}^{n+1} X_i} (1-p)^{n - \sum_{i=1}^{n+1} X_i} = (1-p)^n \cdot \exp\left[\left(\sum_{i=1}^{n+1} x_i\right) \cdot \log \frac{p}{1-p}\right]$

By factorization theorem: $W(X) := \sum_{i=1}^{n+1} X_i$ is a sufficient statistic for p

By full rank, exponential family theorem, $W(X)$ is also a complete statistic

Then by Rao-Blackwell theorem:

$\phi(W) := E(T \mid W)$ is UMVUE for $h(p)$

$\phi(y) = E[T \mid \sum_{i=1}^{n+1} X_i = y) = P(\sum_{i=1}^{n} X_i > X_{n+1} \mid \sum_{i=1}^{n+1} X_i = y) = \dfrac{P(\sum_{i=1}^{n} X_i > X_{n+1}, \sum_{i=1}^{n+1} X_i = y)}{P(\sum_{i=1}^{n+1} X_i = y)}$

① if $y = 0$

$P(\sum_{i=1}^{n} X_i > X_{i+1}, \sum_{i=1}^{n+1} X_i = 0) = 0$

② if $y = 1$

$P(\sum_{i=1}^{n} X_i > X_{i+1}, \sum_{i=1}^{n+1} X_i = 1) = P(\sum_{i=1}^{n} X_i = 1, X_{n+1} = 0) \overset{X_i \text{ iid}}{=} (1-p) \cdot \binom{n}{1} p (1-p)^{n-1} = \binom{n}{1} p (1-p)^n$

③ if $y = 2$

$P(\sum_{i=1}^{n} X_i > X_{i+1}, \sum_{i=1}^{n+1} X_i = 2) = P(\sum_{i=1}^{n} X_i = 2, X_{n+1} = 0) \overset{X_i \text{ iid}}{=} (1-p) \cdot \binom{n}{2} p^2 (1-p)^{n-2} = \binom{n}{2} p^2 (1-p)^{n-1}$

③ if $y = 3, \cdots, n+1$

$P(\sum_{i=1}^{n} X_i > X_{i+1}, \sum_{i=1}^{n+1} X_i = y) = P(\sum_{i=1}^{n} X_i = y-1, X_{n+1} = 1) + P(\sum_{i=1}^{n} X_i = y, X_{n+1} = 0)$

$= p \binom{n}{y-1} p^{y-1} (1-p)^{n-(y-1)} + (1-p) \cdot \binom{n}{y} \cdot p^y (1-p)^{n-y}$

$= \left(\binom{n}{y-1} + \binom{n}{y}\right) \cdot p^y (1-p)^{n+1-y}$

$P(\sum_{i=1}^{n+1} X_i = y) = \binom{n+1}{y} \cdot p^y (1-p)^{n+1-y}$

$\phi(y) = \begin{cases} 0 & y = 0 \\ \binom{n}{y} / \binom{n+1}{y} & y = 1, 2. \\ \left(\binom{n}{y-1} + \binom{n}{y}\right) / \binom{n+1}{y} & y = 3, \cdots, n+1 \end{cases} \Rightarrow \phi(y) = \begin{cases} 0 & y = 0 \\ \frac{n-y+1}{n+1} & y = 1, 2. \\ 1 & y = 3, \cdots, n+1 \end{cases}$

Then $\phi(y)$ is the UMVUE for $h(p)$

2. EX7.58. $X \overset{pdf}{\sim} f(x|\theta) = \left(\frac{\theta}{2}\right)^{|x|}(1-\theta)^{1-|x|}$, $x=-1,0,1$. $0 \le \theta \le 1$

(a) $\hat{\theta}_{MLE}$ (b) $T(x) := \begin{cases} 2 & x=1 \\ 0 & \text{otherwise} \end{cases}$, show $T(x)$ unbaised for θ (c) find better than $T(x)$ & proof

(d) $X_i \overset{iid}{\underset{i=1}{\overset{n}{\sim}}} f(x_i|\theta)$, UMVUE for θ.

(a) $L(\theta;X) = f(X|\theta) = \left(\frac{\theta}{2}\right)^{|x|}(1-\theta)^{1-|x|}$

if $X=0$, $L(\theta;X) = 1-\theta$ $\quad \hat{\theta}_{MLE} = 0$ $\quad \Rightarrow \hat{\theta}_{MLE} = \begin{cases} 0 & X=0 \\ 1 & X=\pm 1 \end{cases}$

if $X=\pm 1$, $L(\theta;X) = \frac{\theta}{2}$ $\quad \hat{\theta}_{MLE} = 1$

(b) $E_\theta[T(X)] = 2 \cdot P_\theta(X=1) = 2 \cdot \frac{\theta}{2} = \theta \quad \Rightarrow T(X)$ unbaised for θ

(c) $\phi(x) := \begin{cases} 0 & X=0 \\ 1 & X=-1,+1 \end{cases}$

$E_\theta[\phi(X)] = P(X=1) + P(X=-1) = \frac{\theta}{2} + \frac{\theta}{2} = \theta \quad \Rightarrow \phi(X)$ unbaised for θ

$Var_\theta[\phi(X)] = E_\theta[(\phi(X) - E\phi(X))^2] = E_\theta[\phi(X)^2] - \theta^2$

$E_\theta[\phi^2(X)] = 1^2 \cdot P(X=1) + 1^2 \cdot P(X=-1) = \theta \quad \Rightarrow MSE(\phi(X)) = \theta - \theta^2$

$E_\theta[T^2(X)] = 4 \cdot P(X=1) = 4 \cdot \frac{\theta}{2} = 2\theta \quad \Rightarrow MSE(T(X)) = 2\theta - \theta^2$

$MSE(\phi(X)) < MSE(T(X)) \quad \Rightarrow \phi(X)$ is better than $T(X)$

(d) suppose $Y_i := |X_i|$ then $Y_i \overset{iid}{\sim}$ Bernoulli(θ) one parameter exponential family, with pdf $f(Y_i;\theta) = (1-\theta)\exp\left[Y_i \log\frac{\theta}{1-\theta}\right]$, $T(Y_i) = Y_i$, with

$E[T[Y_i]] = E[|X_i|] = \frac{\theta}{2} + \frac{\theta}{2} = \theta$, ...

then $\frac{1}{n}\sum_{i=1}^{n} Y_i$, by one-parameter full-rank exponential family theorem, as an unbiased estimator of θ, attains the C-R Lower Bond, i.e

$Var\left(\frac{1}{n}\sum_{i=1}^{n}|X_i|\right) = \frac{1^2}{I_n(\theta)}$

Namely, $\frac{\sum_{i=1}^{n}|X_i|}{n}$ is UMVUE of θ.

3. (a) $T(X,Y) := \frac{X+Y}{2}$ is the UMVUE for mean $1/\theta$

proof: $E_\theta[T(X,Y)] = \frac{1}{2}(E_\theta X + E_\theta Y) \overset{\text{expo}}{\underset{\text{distribution}}{=}} \frac{1}{2}(\frac{1}{\theta} + \frac{1}{\theta}) = \frac{1}{\theta}$ ⇒ unbaised for $1/\theta$

$Var_\theta[T(X,Y)] = Var_\theta(\frac{X+Y}{2}) = \frac{1}{4}(Var(X) + Var(Y)) \overset{\text{expo}}{=} \frac{1}{4}\cdot\frac{2}{\theta^2} = \frac{1}{2\theta^2} < \infty$

exponential distribution withis exponential family ⇒ interchangeability

$L(\theta;X,Y) = f(X;\theta)f(Y;\theta) = \theta^2 e^{-\theta(x+y)}$, $\frac{\partial}{\partial\theta}\log L(\theta;X,Y) = \frac{\partial}{\partial\theta}(2\log\theta - \theta(x+y))$

$= \frac{2}{\theta} - (x+y)$

$= -2(\frac{x+y}{2} - \frac{1}{\theta})$

$a(\theta) := -2$, by attainment of C-R Bond ⇒ $T(X,Y) = \frac{X+Y}{2}$ attains the C-R Bond

⇒ $T(X,Y) = \frac{X+Y}{2}$ is the UMVUE for mea $1/\theta$

(b) $Var(T(X,Y)) = Var(\frac{X+Y}{2}) = \frac{1}{4}(Var(X) + Var(Y)) \overset{\text{expo}}{=} \frac{1}{4}\cdot\frac{2}{\theta^2} = \frac{1}{2\theta^2}$

$MSE(\frac{X+Y}{2}) = \frac{1}{2\theta^2}$

$E[(XY)^{\frac{1}{2}}] = \int_0^\infty\int_0^\infty (xy)^{\frac{1}{2}}\theta^2 e^{-\theta(x+y)}\,dx\,dy = \theta^2\int_0^\infty x^{\frac{1}{2}}e^{-\theta x}dx\int_0^\infty y^{-\frac{1}{2}}e^{-\theta y}dy$

$MSE((XY)^{\frac{1}{2}}] = Var((XY)^{\frac{1}{2}}) + (E((XY)^{\frac{1}{2}}) - \frac{1}{\theta})^2$

$= E[XY] - (E[(XY)^{\frac{1}{2}}])^2 + [E((XY)^{\frac{1}{2}})]^2 + \frac{1}{\theta^2} - \frac{2}{\theta}E[(XY)^{\frac{1}{2}}]$

$\overset{X,Y\text{ iid}}{=} \frac{2}{\theta^2} - \frac{2}{\theta}E[(XY)^{\frac{1}{2}}]$

$\int_0^\infty t^{\frac{1}{2}}e^{-\theta t}dt \overset{z=t^{\frac{1}{2}}}{=} 2\int_0^\infty z^2 e^{-\theta z^2}dz = \int_0^\infty z\,d\frac{e^{-\theta z^2}}{-\theta} = \frac{1}{\theta}\int_0^\infty e^{-\theta z^2}dz.$

$= \frac{1}{\theta}\sqrt{2\pi}\sqrt{\frac{1}{2\theta}}\int_0^\infty \frac{e^{-\frac{(z)^2}{2\cdot\frac{1}{2\theta}}}}{\sqrt{2\pi}\sqrt{\frac{1}{2\theta}}}dz.$

$= \frac{1}{\theta}\sqrt{2\pi}\sqrt{\frac{1}{2\theta}}\cdot\frac{1}{2}$

$E[(XY)^{\frac{1}{2}}] = \theta^2\cdot(\frac{1}{\theta}\sqrt{2\pi}\sqrt{\frac{1}{2\theta}}\,\frac{1}{2})^2 = \frac{\pi}{4\theta}$

$MSE[(XY)^{\frac{1}{2}}] = \frac{2}{\theta^2} - \frac{2}{\theta}\cdot\frac{\pi}{4\theta} = \frac{1}{2\theta^2}(4-\pi) < \frac{1}{2\theta^2} = MSE(\frac{X+Y}{2})$ $\forall\theta$.

⇒ $\forall\theta$, $(XY)^{\frac{1}{2}}$ has smaller MSE than UMVUE $\frac{X+Y}{2}$