Chapter 9: Interval Estimation

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Why Interval Estimators

Interval estimator: $[L(\mathbf{X}), U(\mathbf{X})]$.
Three types of intervals:

- Two-sided interval $[L(\mathbf{X}), U(\mathbf{X})]$
- $[L(\mathbf{X}), \infty]$ (call $L(\mathbf{X})$ the lower confidence bound)
- $[-\infty, U(\mathbf{X})]$ (call $U(\mathbf{X})$ the upper confidence bound)

Remark: By using the interval estimator, we give up some precision in our estimate, but gain confidence or assurance about our assertion.

Why Interval Estimators: Example

Example: $\mathbf{X}_1, \cdots, \mathbf{X}_4$ iid $N(\mu, 1)$. Compare two types of estimators.

- The point estimator of μ: $\bar{\mathbf{X}}$
- The interval estimator of μ: $[\bar{\mathbf{X}} - 1, \bar{\mathbf{X}} + 1]$

Interval Estimators: Definitions

Definition

Given an interval estimator $[L(\mathbf{X}), U(\mathbf{X})]$ of parameter θ, its coverage probability is defined as

$$P(\theta \in [L(\mathbf{X}), U(\mathbf{X})])$$

Definition

Given an interval estimator $[L(\mathbf{X}), U(\mathbf{X})]$ of parameter θ, its confidence coefficient is defined as

$$\min_{\theta} P(\theta \in [L(\mathbf{X}), U(\mathbf{X})])$$

Definition

Given an interval estimator $[L(\mathbf{X}), U(\mathbf{X})]$ of parameter θ, if its confidence coefficient is $1-\alpha$, we call it a $(1-\alpha)$ confidence interval or confidence set.

Interval Estimators: Example

Assume $X_1, \cdots, X_n$ Unif$(0, \theta)$. Let $Y = X_{(n)}$. Using Y , we construct two $(1 - \alpha)$ confidence intervals.

- (1) Interval $[aY, bY]$
- (2) Interval $[Y + c, Y + d]$

How to construct confidence intervals

Method 1: By inverting the acceptance region of tests;
Method 2: Using pivotal quantities.

How to construct confidence intervals: Inverting Test

Remark: Both hypotheses testing and CI look for consistency between samples and parameters, but from slightly different perspective.

- Hypothesis: Fix the parameter-asks what sample values (in the appropriate region) are consistent with that fixed value.
- Confidence set: Fix the sample value-asks what parameter value make this sample most plausible.

There is one-to-one correspondence between tests and confidence intervals.
Example: $X_1, \cdots, X_n$ iid $N(\theta, \sigma^2)$, σ known. Test or CI for θ

Inverting Test

Theorem

- *(1) For each* $\theta_0 \in \Theta$*, let* $A(\theta_0)$ *be the acceptance region of a level* α *test* $H_0 : \theta = \theta_0$*. Define a set* $C(\boldsymbol{x}) = \{\theta_0 : \boldsymbol{x} \in A(\theta_0)\}$*. Then the random set* $C(\boldsymbol{X})$ *is a* $(1-\alpha)$*-confidence set*
- *(2) Conversely, if* $C(\boldsymbol{x})$ *is a* $(1-\alpha)$ *confidence set for* θ*, for any* θ_0*, define the acceptance region of a test for the hypothesis* $H_0 : \theta = \theta_0$ *by* $A(\theta_0) = \{\boldsymbol{x} : \theta_0 \in C(\boldsymbol{x})\}$*. Then the test has level* α

Inverting Test

$A(\theta_0)$ is a set in the sample space, and $C(\mathbf{x})$ is a set in the parameter space. In the above normal example

$$A(\theta_0) = \{(x_1, \cdots, x_n) : \mu_0 - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \le \bar{x} \le \mu_0 + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\}$$

$$C(\mathbf{x}) = \{\mu_0 : \bar{x} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \le \mu_0 \le \bar{x} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\}$$

The $(1-\alpha)$ confidence interval is
$C(\mathbf{x}) = [\bar{\mathbf{X}} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}}, \bar{\mathbf{X}} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}]$

Inverting Test Statistic

We can invert an acceptance region of a test to get a confidence interval.

- To obtain a $(1-\alpha)$ two-sided confidence interval $[L(\mathbf{X}), U(\mathbf{X})]$, invert the acceptance region of a level α test for $H_0 : \theta = \theta_0$ vs $H_1 : \theta \neq \theta_0$
- To obtain a $(1-\alpha)$ one-sided confidence interval $[L(\mathbf{X}), \infty]$, we invert the acceptance region of a level α test for $H_0 : \theta = \theta_0$ vs $H_1 : \theta > \theta_0$
- To obtain a $(1-\alpha)$ one-sided confidence interval $[-\infty, U(\mathbf{X})]$, we invert the acceptance region of a level α test for $H_0 : \theta = \theta_0$ vs $H_1 : \theta < \theta_0$

Inverting Test Statistic: Example

$X_1, \cdots, X_n \sim N(\mu, \sigma^2)$, both unknown.

- Find $1 - \alpha$ confidence interval for μ
- Find $1 - \alpha$ upper confidence bound for μ
- Find $1 - \alpha$ lower confidence bound for μ

Inverting Test Statistic: Example

$X_1, \cdots, X_n \sim Bin(1, p)$. Find lower confidence bound.
Fact: Assume T has an MLR. Let $c_1(\theta)$ and $c_2(\theta)$ be the cut-off values satisfying

$$P_\theta(T(\mathbf{X}) > c_2(\theta)) = \alpha, \quad \text{and} \quad P_\theta(T(\mathbf{X}) < c_1(\theta)) = \alpha$$

Then both $c_1(\theta)$ and $c_2(\theta)$ are increasing in θ.

Note: Inverting LRT can be hard. See Example 9.2.3 in C&B.

Pivotal Quantity

A random quantity $Q(\mathbf{X}, \theta)$ is called pivotal (or a pivot) if the distribution of $Q(\mathbf{X}, \theta)$ is independent of θ.
Note: this is different from an ancillary statistic since $Q(\mathbf{X}, \theta)$ also depends on θ and hence is not a statistic.

- location family: iid $f(x - \theta)$. $\bar{\mathbf{X}} - \theta$
- scale family: iid $\frac{1}{\sigma} f(x/\sigma)$. $\bar{\mathbf{X}}/\sigma$
- location-scale family: iid $\frac{1}{\sigma} f(\frac{x-\mu}{\sigma})$. $\frac{\bar{\mathbf{X}} - \mu}{\sigma}$

Pivotal Quantity: Example

$X_1, \cdots, X_n \sim \exp(\lambda)$. Show the following are pivotal. What distributions do they have?

- (1) $\dfrac{X_1}{\lambda}$
- (2) $\sum_{i=1}^n X_i/\lambda$
- (3) $2\sum_{i=1}^n X_i/\lambda$

Pivotal Quantity

Theorem

If the pdf of T is expressed as

$$f(t|\theta) = g(Q(t,\theta))|(\partial/\partial t)Q(t,\theta)|$$

, and Q is monotone in t, then $Q(T,\theta)$ is a pivot.

Example: Let $X_1, \cdots, X_n \sim \exp(\lambda)$. Show $Q = \sum_{i=1}^{n} X_i/\lambda$ is pivotal.

Construct confidence set with pivotal quantity.

(i) Find a,b such that $P_\theta(a \leq Q(X,\theta) \leq b) = 1-\alpha$.
(ii) Define $C(x) = \{\theta : a \leq Q(x, \theta \leq b\}$.

$$P_\theta(\theta \in C(\mathbf{X})) = P_\theta(a \leq Q(\mathbf{X},\theta) \leq b) = 1-\alpha$$

When will $C(\mathbf{X})$ be an interval? Answer: If $Q(x,\theta)$ is monotone in θ, then $C(\mathbf{X})$ is an interval.

- If $Q(x,\theta)$ is increasing in θ, then

$$C(x) = \{\theta : L(x,a) \leq \theta \leq U(x,b)\}$$

- If $Q(x,\theta)$ is decreasing in θ, then

$$C(x) = \{\theta : L(x,b) \leq \theta \leq U(x,a)\}$$

Example: iid exponential

Normal Pivot for μ

- Example: iid $N(\mu, \sigma^2)$, σ known. Interval for μ.
- Example: iid $N(\mu, \sigma^2)$, σ unknown. Interval for μ.
- Example: iid $N(\mu, \sigma^2)$, μ known. Interval for σ.
- Example: iid $N(\mu, \sigma^2)$, μ unknown. Interval for σ.

Pivoting the CDFs

If T is statistic, its cdf $F(t|\theta) = P_\theta(T \leq t)$ follows

$$P_\theta(F(T|\theta) \leq u) = P_\theta(T \leq F^{-1}(u)) = F[F^{-1}(u)] = u$$

where $F(t|\theta) \in [0, 1]$ and is increasing in t.

- Define the function $Y = F(T|\theta)$. Then $Y \sim U(0, 1)$ and it is a pivot.
  Given α, we can choose α_1, α_2 such that
  $P(\alpha_1 \leq F(T|\theta) \leq 1 - \alpha_2) = 1 - \alpha$, with $\alpha_1 + \alpha_2 = \alpha$
- If $F(t|\theta)$ is decreasing in θ for all t, define θ_L, θ_U by

  $$F(t|\theta_L) = 1 - \alpha_2, F(t|\theta_U) = \alpha_1$$

  Then $[\theta_L(T), \theta_U(T)]$ is $(1 - \alpha)$ CI for θ.
- If $F(t|\theta)$ is increasing in θ for all t, define θ_L, θ_U by

  $$F(t|\theta_L) = \alpha_1, F(t|\theta_U) = 1 - \alpha_2$$

  Then $[\theta_L(T), \theta_U(T)]$ is $(1 - \alpha)$ CI for θ.

Pivoting the CDFs: Example

iid from $f(x|\mu) = \exp^{-(x-\mu)} I(x > \mu)$. $T = X_{(1)}$ sufficient. Construct the $100(1-\alpha)\%$ CI for μ by pivoting the cdf of T.

Evaluate and Compare Interval Estimators

- Coverage probabilities
- Expected length

Shortest Expected Length: Given α, the $(1-\alpha)$ CIs are not unique. Among many choices, we want to minimize expected length

$$\min E[U(\mathbf{x}) - L(\mathbf{x})]$$

Evaluate and Compare Interval Estimators: Example

Example: (Optimizing the length) Let $X_1, \cdots, X_n$ be iid $N(\mu, \sigma^2)$, where σ^2 is known. Using the pivotal

$$Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}}$$

we choose a and b such that

$$P(a \leq Z \leq b) = 1 - \alpha$$

The $(1-\alpha)$ confidence interval is given by $[\bar{X} - b\frac{\sigma}{\sqrt{n}}, \bar{X} - a\frac{\sigma}{\sqrt{n}}]$

- (1) What is the length of the interval?
- (2) How to choose a and b?

How to optimize the length

- Case 1: Unimodal case
- Case 2: Lagrange Method

Theorem

Let $f(x)$ be a unimodal pdf. If the interval $[a, b]$ satisfies

- *(i)* $\int_a^b f(x)dx = 1 - \alpha$
- *(ii)* $f(a) = f(b) > 0$ *and* $a \leq x^* \leq b$, *where* x^* *is a mode of* $f(x)$.

Then $[a, b]$ is the shortest among all intervals that satisfies (i).
Example: iid $N(\mu, \sigma^2)$ σ known.
Example: iid $N(\mu, \sigma^2)$ σ unknown.

How to optimize the length: Lagrange Method

Minimize $b - a$, subject to $\int_a^b f(x)dx = 1 - \alpha$

- Example: iid $N(\mu, \sigma^2)$ σ known.
- Example: iid exponential.