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EX 8.5 (a) & (b)

$\{X_i\}$ drawn from Pareto population with pdf $f(x|\theta,\nu) = \frac{\theta \nu^\theta}{x^{\theta+1}} \mathbb{1}_{[\nu,\infty)}(x)$, $\theta>0, \nu>0$.

(a) Find MLEs of $\theta$ & $\nu$.

(b). Show that the LRT of : $H_0: \theta=1$, $\nu$ unknown, vs, $H_1: \theta \neq 1$, $\nu$ unknown, has critical region of the form $\{x: T(x) \le c_1 \text{ or } T(x) \ge c_2\}$ where $0<c_1<c_2$ & $T:=\log\left[\frac{\prod_{i=1}^n X_i}{(\min_i X_i)^n}\right]$

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(a). $L(\theta,\nu|x) = \prod_{i=1}^n \frac{\theta \nu^\theta}{x_i^{\theta+1}} \mathbb{1}_{[\nu,\infty)}(x_i) = \frac{\theta^n \nu^{n\theta}}{(\prod_{i=1}^n x_i)^{\theta+1}} \mathbb{1}_{[\nu,\infty)}(\min_{i=1}^n X_i)$

Assume : $X_{(1)} := \min_{1\le i\le n} X_i$, then $\log L(\theta,\nu|X) = n\log\theta + n\theta\log\nu - (\theta+1)\log(\prod_{i=1}^n x_i)$, $X_{(1)} \ge \nu$

$\log L(\theta,\nu|X)$ increases with $\nu$ increas , $\Rightarrow \hat{\nu}_{MLE} = X_{(1)}$

$\frac{\partial \log L(\theta,\nu|X)}{\partial\theta} = \frac{n}{\theta} + n\log\nu - \log(\prod_{i=1}^n x_i) = 0 \Rightarrow \hat{\theta}_{MLE} = \frac{n}{\sum_{i=1}^n \log X_i - n\log X_{(1)}} = \frac{n}{T(X)}$

$\frac{\partial^2}{\partial\theta^2}\log L(\theta,\nu|X) = -\frac{n}{\theta^2} < 0$

$\Rightarrow \hat{\theta}_{MLE} = \frac{n}{T(X)} = \frac{n}{\sum_{i=1}^n \log X_i - n\log X_{(1)}}$ , $\hat{\nu}_{MLE} = X_{(1)}$

(b) By (a) $\sup_{\theta\in\Theta} L(\theta|X) = \max\left\{ \frac{\left(\frac{n}{T(X)}\right)^n (X_{(1)})^{n^2/T(X)}}{(\prod_{i=1}^n X_i)^{n/T(X)+1}}, 0 \right\}$

$\sup_{\theta\in\Theta_0} L(\theta|X) = L(\theta=1|X) = \frac{\nu^n}{(\prod_{i=1}^n X_i)^2} \mathbb{1}_{[\nu,\infty)}(X_{(1)}) \Longleftarrow \frac{X_{(1)}^n}{(\prod_{i=1}^n X_i)^2}$

LRT: $\lambda(X) = \frac{\sup_{\theta\in\Theta_0} L(\theta|X)}{\sup_{\theta\in\Theta} L(\theta|X)} = \left(\frac{T(X)}{n}\right)^n X_{(1)}^{n(1-\frac{n}{T(X)})} \left(\prod_{i=1}^n X_i\right)^{\frac{n}{T(X)}-1} = \left(\frac{T(X)}{n}\right)^n \left(\frac{X_{(1)}^n}{\prod_{i=1}^n X_i}\right)^{1-\frac{n}{T(X)}}$

$= \left(\frac{T(X)}{n}\right)^n \left(e^{-T(X)}\right)^{1-\frac{n}{T(X)}} = \left(\frac{T(X)}{n}\right)^n \cdot e^{-T(X)+n}$

Then. $\lambda(T) := \left(\frac{T}{n}\right)^n e^{-T+n}$

$\frac{\partial}{\partial T}\log\lambda(T) = \frac{\partial}{\partial T}\left(n\log T - n\log n + (-T+n)\right) = \frac{n}{T} - 1$

$\lambda(T)$ increases when $\frac{n}{T} - 1 \ge 0$, namely, $T \le n$, and decreases when $T > n$

Then the critical region for $R: \{X: \lambda(X) \le c\} \Longleftrightarrow R: \{x: \lambda(T(x)) \le c\}$

$\exists \cdot c_1, c_2$, s.t. $c_1 \le n$, $c_2 \ge n$, $\lambda(c_1) = \lambda(c_2) = c$.

and $\forall$ $T(X) \le c_1$, or $T(X) \ge c_2$, $\lambda(T(X)) \le c$

namely, $\exists$ critical region of the form $\{X: T(X) \le c_1 \text{ or } T(X) \ge c_2\}$

$\Longleftrightarrow \{X: \lambda(T(X)) = c\}$ with $0<c_1<c_2$.

& $T(X) := \log\left[\frac{\prod_{i=1}^n X_i}{(X_{(1)})^n}\right]$

EX 8.6. $\{X_i\}_{i=1}^{n} \overset{iid}{\sim} \exp(\theta)$, $\{Y_j\}_{j=1}^{m} \overset{iid}{\sim} \exp(\mu)$

(a) LRT of $H_0: \theta=\mu$, v.s. $H_1: \theta=\mu$ (b) show that (a) be based on stat $T := \sum X_i / (\sum X_i + \sum Y_j)$

(c) Find dist of $T$ when $H_0$ is true.

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(a) $$\lambda(X,Y) := \frac{\sup_{(\theta,\mu)\in\Theta_0} L(\theta,\mu|X,Y)}{\sup_{(\theta,\mu)\in\Theta} L(\theta,\mu|X,Y)} = \frac{\sup_{\theta} \prod_{i=1}^{n} \frac{1}{\theta} e^{-x_i/\theta} \prod_{j=1}^{m} \frac{1}{\theta} e^{-y_j/\theta}}{\sup_{\theta,\mu} \prod_{i=1}^{n} \frac{1}{\theta} e^{-x_i/\theta} \prod_{j=1}^{m} \frac{1}{\mu} e^{-y_j/\mu}} = \frac{\sup_{\theta} \theta^{-(n+m)} \exp[-(\sum_{i=1}^n x_i + \sum_{j=1}^m y_j)/\theta]}{\sup_{\theta,\mu} \theta^{-n}\mu^{-m} \exp[-(\sum_{i=1}^n x_i)/\theta - (\sum_{j=1}^m y_j)/\mu]}$$

suppose $h(\theta) := \theta^{-(n+m)} \exp[-(\sum_{i=1}^n x_i + \sum_{j=1}^m y_j)/\theta]$, $g(\theta,\mu) := \theta^{-n}\mu^{-m}\exp[-\frac{1}{\theta}\sum_{i=1}^n x_i - \frac{1}{\mu}\sum_{j=1}^m y_j]$

① $$\frac{\partial \log h(\theta)}{\partial\theta} = \frac{\partial}{\partial\theta}\left(-(n+m)\cdot\log\theta - \frac{1}{\theta}\left(\sum_{i=1}^n x_i + \sum_{j=1}^m y_j\right)\right) = -\frac{n+m}{\theta} + \frac{1}{\theta^2}\left(\sum_{i=1}^n x_i + \sum_{j=1}^m y_j\right) = 0 \Rightarrow \hat\theta = \frac{\sum_{i=1}^n x_i + \sum_{j=1}^m y_j}{n+m}$$

$$\left.\frac{\partial^2 \log h(\theta)}{\partial\theta^2}\right|_{\theta=\frac{\sum_{i=1}^n x_i+\sum_{j=1}^m y_j}{n+m}} = \frac{n+m}{\theta^2} - \frac{2}{\theta^3}\left(\sum_{i=1}^n x_i + \sum_{j=1}^n y_j\right) \overset{\theta=\frac{\sum_{i=1}^n x_i+\sum_{j=1}^m y_j}{n+m}}{=\!=\!=} -\frac{(n+m)^3}{(\sum_{i=1}^n x_i + \sum_{j=1}^n y_j)^2} < 0$$

$$\Rightarrow \hat\theta = \arg\sup_\theta \theta^{-(n+m)}\exp[-(\sum_{i=1}^n x_i + \sum_{j=1}^m y_j)/\theta] = \frac{\sum_{i=1}^n x_i + \sum_{j=1}^m y_j}{n+m}$$

② $$\frac{\partial \log g(\theta,\mu)}{\partial\theta} = \frac{\partial}{\partial\theta}\left(-n\log\theta - m\log\mu - \frac{1}{\theta}\sum_{i=1}^n x_i - \frac{1}{\mu}\sum_{j=1}^m y_j\right) = -\frac{n}{\theta} + \frac{1}{\theta^2}\sum_{i=1}^n x_i \Rightarrow \hat\theta = \frac{\sum_{i=1}^n x_i}{n}$$

$$\left.\frac{\partial^2 \log g(\theta,\mu)}{\partial\theta^2}\right|_{\theta=\frac{\sum_{i=1}^n x_i}{n}} = \frac{n}{\theta^2} - \frac{1}{\theta^3}\sum_{i=1}^n x_i \overset{\theta=\bar{X}}{=\!=\!=} \frac{n-n}{(\bar{X})^2} = 0$$

$$\frac{\partial \log g(\theta,\mu)}{\partial\mu} = -\frac{n}{\mu} + \frac{1}{\mu^2}\sum_{j=1}^n Y_j \Rightarrow \hat\mu = \frac{\sum_{j=1}^m x_j}{m} \qquad \left.\frac{\partial^2 \log g(\theta,\mu)}{\partial\mu^2}\right|_{\mu=\frac{\sum_{j=1}^m x_j}{m}} = 0$$

$$\Rightarrow \hat\theta, \hat\mu = \frac{\sum_{i=1}^n x_i}{n}, \frac{\sum_{j=1}^m y_j}{m}$$

①② $$\Rightarrow \lambda(X,Y) = \left(\frac{\sum_{i=1}^n x_i + \sum_{j=1}^m y_j}{n+m}\right)^{-(n+m)} \exp[-(n+m)] \Big/ \left[(\bar{x})^{-n}(\bar{Y})^{-m}\exp[-(n+m)]\right]$$

$$= \left(\frac{n\bar{x}+m\bar{Y}}{n+m}\right)^{-(n+m)} (\bar{x})^n(\bar{Y})^m = (n+m)^{n+m}\frac{(\bar{x})^n(\bar{Y})^m}{(n\bar{x}+m\bar{Y})^{n+m}}$$

LRT $\Rightarrow$ reject $H_0$ $R: \{(X,Y): \lambda(X,Y) \le c\}$

(b) $$\lambda(X,Y) = \frac{(n+m)^{n+m}}{n^n m^m}\frac{(\sum_{i=1}^n X_i)^n(\sum_{j=1}^m Y_j)^m}{(\sum_{i=1}^n X_i + \sum_{j=1}^m Y_j)^{n+m}} = \frac{(n+m)^{n+m}}{n^n m^m}(T(X,Y))^n(1-T(X,Y))^m$$

suppose $\lambda(T) := \frac{(n+m)^{n+m}}{n^n m^m}(T)^n(1-T)^m$, then $\frac{\partial \log\lambda(T)}{\partial T} = \frac{n}{T} - \frac{m}{1-T} = 0 \Rightarrow T = \frac{n}{n+m}$

$\frac{\partial^2 \log\lambda(T)}{\partial T^2} = -\frac{n}{T^2} - \frac{m}{(1-T)^2} < 0 \Rightarrow$ when $T \le \frac{n}{m+n}$, $\lambda$ increases with $T$ increases
when $T \ge \frac{n}{m+n}$, $\lambda$ decreases with $T$ increases

$R: \{(X,Y): \lambda(X,Y) \le c\} \Rightarrow \exists\ C_1, C_2$ s.t. $C_1 \le \frac{n}{m+n} \le C_2$, $\lambda(C_1) = \lambda(C_2) = c$

$\Leftrightarrow R: \{(X,Y): T(X,Y) \le C_1$ or $T(X,Y) \ge C_2\}$ with $C_1 \le \frac{n}{m+n} \le C_2$, $C_1^n(1-C_1)^m = C_2^n(1-C_2)^m$

(c) when $H_0$ is true, $\theta=\mu$, $\sum_{i=1}^n X_i \sim gamma(n,\theta)$, $\sum_{j=1}^m Y_j \sim gamma(m,\theta)$

$$f_T(t) = P\left(\sum_{i=1}^n X_i \Big/ \left(\sum_{i=1}^n X_i + \sum_{j=1}^m Y_j\right) = t\right) = \int P\left(\sum_{i=1}^n X_i = tz, \sum_{j=1}^m Y_j = (1-t)z\right)dz \overset{X_i, Y_j\ iid}{=\!=\!=} \int f_{\sum Y}((1-t)z)\cdot f_{\sum X}(tz)\cdot dz$$

$$= \int \frac{1}{\Gamma(m)\theta^m}[(1-t)z]^{m-1} e^{-\frac{(1-t)z}{\theta}} \cdot \frac{1}{\Gamma(n)\theta^n}(tz)^{n-1} e^{-\frac{tz}{\theta}}dz = \frac{\theta^{-(n+m)}}{\Gamma(m)\Gamma(n)}(1-t)^{m-1}t^{n-1}\int z^{m+n-2}e^{-\frac{z}{\theta}}dz$$

$$= \frac{\Gamma(m+n)}{\Gamma(m)\cdot\Gamma(n)} t^{n-1}(1-t)^{m-1} \sim Beta(n,m)$$