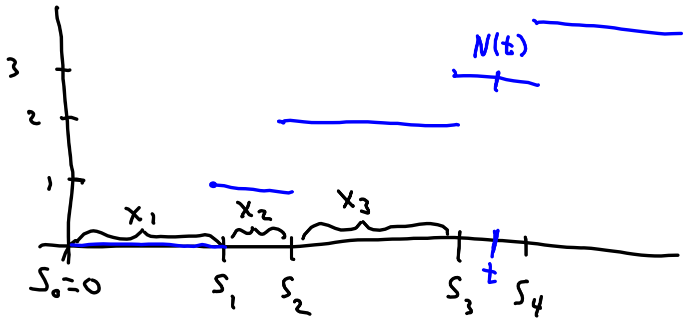
Review of Renewal Processes



Renewal process is a generalization of a Poisson process

$\{X_n\}$ nonnegative i.i.d. with $\mu = \int_{[0,\infty)} x F(dx) = \int_0^\infty [1-F(x)]\,dx$

$F(x) = P\{X_1 \le x\}$, assume $F(0) < 1$

Aside: Non-negative reals $[0,\infty)$

Non-negative extended real numbers $[0,\infty]$

Intro to prob. + stat $\rightarrow$ Range for R.V. $[0,\infty)$

Stochastic processes $\rightarrow$ Range is extended reals

$\lim_{t\to\infty} F(t) < 1$

$P\{X \le x\} = F(x)$

$\lim_{x\to\infty} F(x) = F(\infty) = P\{X < \infty\}$

$$S_n = \sum_{k=1}^{n} X_n$$

$$N(t) = \sum_{n=1}^{\infty} I_{[0,t]}(S_n) = \sup\{n : S_n \le t\}$$

$\{N(t)\} = \{N(t) ; t \geq 0\} \leftarrow$ renewal process

$\{S_n\} \leftarrow$ renewal process

Renewal function $\rightarrow m(t) = E[N(t)] \leftarrow$ definition

$m(t) = \sum_{n=1}^{\infty} F_n(t)$ where $F_n(t)$ is n-fold convolution of F

$m * F(t) = m(t) - F(t)$ OR $m * F = m - F$

Renewal Equation

Find function h such that

$$h = g + F * h$$

Sol: $h = g + m * h$ where $m = \sum_{n=1}^{\infty} F_n$

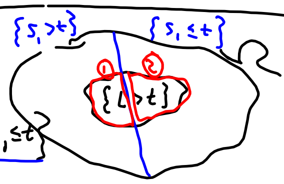
If $\lim_{t\to\infty} F(t) = 1 \iff \{N(t)\}$ is recurrent

$\lim_{t\to\infty} F(t) < 1 \iff \{N(t)\}$ is transient

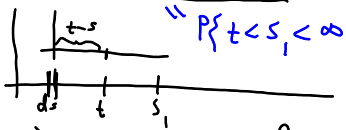
Let $\{N(t)\}$ be a transient renewal process

$$\lim_{t\to\infty} m(t) = \frac{F(\infty)}{1-F(\infty)}$$

$$L = \sup\{S_n : S_n < \infty\}$$



1. $P\{L>t\} = \underbrace{P\{L>t,\, S_1>t\}}_{= P\{t<S_1<\infty\}} + P\{L>t,\, S_1\le t\}$



$$P\{L>t\} = P\{t<S_1<\infty\} + \int F(ds)\, P\{L>t-s\}$$

2. $$\underbrace{P\{L>t\}}_{h(t)} = \underbrace{F(\infty)-F(t)}_{g(t)} + \underbrace{\int_{[0,t]} F(ds)\, P\{L>t-s\}}_{F*h(t)}$$

3. $$P\{L>t\} = F(\infty)-F(t) + \int_{[0,t]} m(ds)\,[F(\infty)-F(t-s)]$$

4. Can this be simplified

$$P\{L>t\} = F(\infty)-F(t) + F(\infty)\,m(t) - m*F$$

Use $m*F = m - F$

$$\Rightarrow\ P\{L\le t\} = [1-F(\infty)][1+m(t)]$$

$\{N(t)\}$ is recurrent aperiodic

Blackwell's thm

$$\lim_{t\to\infty}[m(t+s)-m(t)] = \frac{s}{\mu}$$

Elementary renewal thm

$$\lim_{t\to\infty}\frac{m(t)}{t} = \frac{1}{\mu}$$

$$\lim_{t\to\infty} m*g(t) = \begin{cases} m(\infty)\,g(\infty) & \text{if } \{N(t)\} \text{ transient} \\ \frac{1}{\mu}\int_0^\infty g(t)\,dt & \text{if } \{N(t)\} \text{ recurrent and } g \text{ is directly Riemann integrable} \end{cases}$$

Key renewal thm.

Laplace-Stieljes transform

$$\tilde{F}(s) = \int_{[0,\infty)} e^{-st} F(dt)$$

$$\Rightarrow \tilde{m}(s) = \frac{\tilde{F}(s)}{1-\tilde{F}(s)}$$

if $F(t) = 1 - e^{-\lambda t} \Rightarrow \frac{\lambda}{\lambda + s}$

$F * G(t) \rightarrow F(s) \cdot G(s)$

Regenerative Process

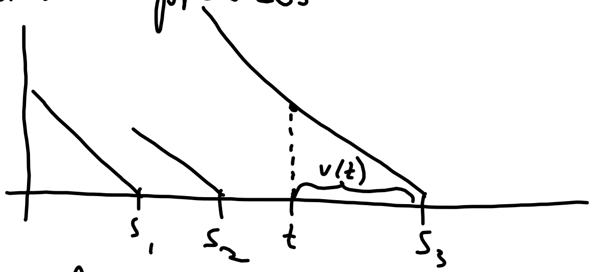
The process $\{Z(t)\}$ is said to be a regenerative process provided there exists a sequence of $\{S_1, S_2, \dots\}$ of stopping times such that

a) The counting process of $\{S_n\}$ is a renewal process

b) The future of the process $\{Z(t)\}$ at a given renewal point is a probablistic replicate of $\{Z(t)\}$.

Examples of Regenerative process

$V(t) = S_{N(t)+1} - t$



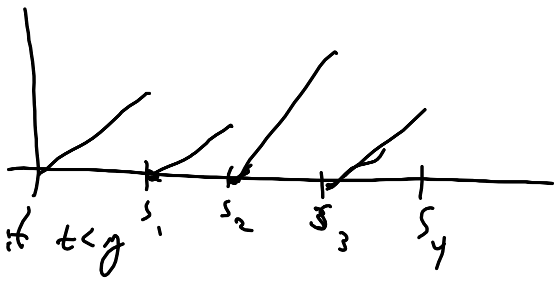
$$P\{V(t) > y\} = 1 - F(t+y) + \int_{[0,t]} m(ds)\left[1 - F(t+y-s)\right]$$

$$\lim_{t\to\infty} P\{V(t) > y\} = \frac{1}{\mu}\int_y^\infty \left[1 - F(x)\right] dx$$

$$E\left[V(t)\right] = \mu\left(m(t)+1\right) - t$$

$$\lim_{t\to\infty} E\left[V(t)\right] = \frac{1}{2}\frac{E[X_1^2]}{E[X_1]}$$

$U(t) - t - S_{N(t)}$



$$P\{U(t) > y\} = \begin{cases} 0 & \text{if } t < y \\ 1 - F(t) + \int_{[0,t-y]} m(ds)\left[1 - F(t-s)\right] & \end{cases}$$

$$\lim_{t\to\infty} P\{U(t) > y\} = \frac{1}{\mu}\int_y^\infty \left[1 - F(x)\right] dx$$

Delay renewal process

First renewal time different than other inter-renewals

Renewal Reward Process

Let R_n be reward at time of n^{th} renewal with $\{(X_n, R_n)\}$ i.i.d. random vectors

(X_n, R_n) is independent and identically distributed with (X_m, R_m) $n \neq m$.

but X_n & R_n may be dependent

$$R(t) = \sum_{n=1}^{N(t)} R_n$$

$$\lim_{t\to\infty} R(t)/t = \lim_{t\to\infty} \frac{E[R(t)]}{t} = \frac{E[R_1]}{E[X_1]}$$