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Print Full name

Test Three

Open Book, Open Notes

Instructions: This test represents your independent work. You may not talk to anyone else about this test until after you have turned it in. Answer the questions neatly so that I can read your work. Please return a scan or a photo of the test to me at richf@tamu.edu after insuring that the scan or photo will be readable. The test must be sent to me before 10:00 AM; thus, you have 2 hours for the test. (Every minute past 10:00 will cost five points.) If you have any questions regarding the meaning of a problem, make what you feel is an appropriate assumption and write down your assumption. You will then be graded on the correctness of the problem based on your assumption and the appropriateness of the assumption. Since I cannot see what you enter into your calculator, it is important to show all your work on the test. Guesses do not count, so if your work is not present, I'll assume a lucky guess which deserves no credit. Answers to the parts of Problem 1 should be rounded and accurate to four digits to the right of the decimal. Put a box around your answers to insure I see your results.

Sign the statement below and when you return the test, please return this page with your test. (If you do not have a printer, copy the statement below on your own paper and then sign it.)

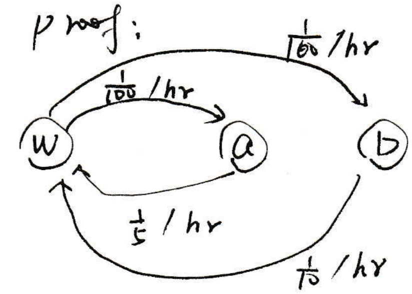
Aggie Honor Code: "An Aggie does not lie, cheat, or steal or tolerate those who do." On my honor as an Aggie, I have neither given nor received unauthorized aid on this test.

Lu Sun

signature

Prob 1: proof:

a.



Assume t in hours

Then generator matrix:

$$-0.0163 \quad ss \qquad\qquad 0.0063 \quad ss$$

$$G := \begin{array}{c c} & \begin{array}{ccc} w & a & b \end{array} \\ \begin{array}{c} w \\ a \\ b \end{array} & \begin{bmatrix} -\frac{13}{800} & \frac{1}{100} & \frac{1}{160} \\ \frac{1}{5} & -\frac{1}{5} & 0 \\ \frac{1}{10} & 0 & -\frac{1}{10} \end{bmatrix} \end{array} = \begin{bmatrix} -0.01625 & 0.01 & 0.00625 \\ 0.2 & -0.2 & 0 \\ 0.1 & 0 & -0.1 \end{bmatrix}$$

b. Assume P is long-run probability, then $P \cdot G = 0$

$$\Rightarrow \begin{cases} P_1 + P_2 + P_3 = 1 \\ \frac{P_1}{100} - \frac{1}{5} P_2 = 0 \\ \frac{P_1}{160} - \frac{P_3}{10} = 0 \end{cases} \Rightarrow P = (P_1 \ P_2, \ P_3) = \left(\frac{80}{89}, \frac{4}{89}, \frac{5}{89}\right)$$

$$\boxed{P_{\text{working}} = \frac{80}{89} \approx 0.8989}$$

c. $Q(a, w, t) = P(Z_1 \le t) = 1 - e^{-\lambda_3 t} = 1 - e^{-\frac{1}{5} t}$.

$Q(b, w, t) = P(Z_2 \le t) = 1 - e^{-\lambda_4 t} = 1 - e^{-\frac{1}{10} t}$.

$Q(w, a, t) = P(S_1 \le S_2, T \le t) = P(S_1 \le S_2) \cdot P(T \le t) = \frac{\lambda_1}{\lambda_1 + \lambda_2}\left(1 - e^{-(\lambda_1+\lambda_2) t}\right)$

$= \frac{8}{13}\left(1 - e^{-\frac{13}{800} t}\right)$

$Q(w, b, t) = P(S_2 \le S_1, T \le t) = P(S_2 \le S_1) \cdot P(T \le t) = \frac{\lambda_2}{\lambda_1+\lambda_2} \cdot \left(1 - e^{-(\lambda_1+\lambda_2) t}\right)$

$= \frac{5}{13}\left(1 - e^{-\frac{13}{800} t}\right)$

$$\Rightarrow Q(t) = \begin{array}{c c} & \begin{array}{ccc} w & a & b \end{array} \\ \begin{array}{c} w \\ a \\ b \end{array} & \begin{bmatrix} 0 & \frac{8}{13}\left(1 - e^{-\frac{13}{800} t}\right) & \frac{5}{13}\left(1 - e^{-\frac{13}{800} t}\right) \\ 1 - e^{-\frac{t}{5}} & 0 & 0 \\ 1 - e^{-\frac{t}{10}} & 0 & 0 \end{bmatrix} \end{array}$$

d. $$Q(10) = \begin{bmatrix} 0 & \frac{8}{13}\left(1 - e^{-\frac{13}{800} \cdot 10}\right) & \frac{5}{13}\left(1 - e^{-\frac{13}{800} \cdot 10}\right) \\ 1 - e^{-\frac{10}{5}} & 0 & 0 \\ 1 - e^{-\frac{10}{10}} & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0.0923 & 0.0577 \\ 0.8647 & 0 & 0 \\ 0.6321 & 0 & 0 \end{bmatrix}$$

e. $\hat{P} := \lim_{t\to\infty} Q(t) = \begin{bmatrix} 0 & \frac{8}{13} & \frac{5}{13} \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

$$v \cdot \hat{P} = v \Rightarrow \begin{cases} v_2 + v_3 = v_1 \\ \frac{8}{13} v_1 = v_2 \\ \frac{5}{13} v_1 = v_3 \end{cases} \Rightarrow \begin{cases} v_1 = \frac{13}{800} \\ v_2 = \frac{1}{100} \\ v_3 = \frac{1}{160} \end{cases}$$

$$\mu(w) = \int_0^\infty \frac{8}{13} e^{-\frac{13}{800}t} + \frac{5}{13} e^{-\frac{13}{800}t} \, dt = \frac{800}{13}$$

$$\mu(a) = \int_0^\infty e^{-\frac{t}{5}} \, dt = 5$$

$$\mu(b) = \int_0^\infty e^{-\frac{t}{10}} \, dt = 10$$

$$\lim_{t\to\infty} P\{Y(t) = w\} = \frac{v(w)\cdot\mu(w)}{v\cdot\mu} = \frac{\frac{800}{13}\cdot\frac{13}{800}}{\frac{800}{13}\cdot\frac{13}{800} + \frac{5}{100} + \frac{10}{160}} = \frac{1}{1+\frac{5}{100}+\frac{10}{160}}$$

$$= \frac{80}{89} \approx \boxed{0.8989}$$

Prob 2 : proof:

a. $P_i\{Y(t)=j, V(t)>y\} = P_i\{Y(t)=j, V(t)>y, T_1>t\}$
$+ P_i\{Y(t)=j, V(t)>y, T_1<t\}$

$$P_i\{Y(t)=j, V(t)>y, T_1>t\} = I(i,j)\cdot P_i\{T_1>t+y\}$$
$$= I(i,j)(1-P_i\{T_1\le t+y\})$$
$$= I(i,j)\cdot(1-\sum_{\hat{k}\in E} Q(i,\hat{k},t+y))$$

$$P_i\{Y(t)=j, V(t)>y, T_1<t\} = \sum_k \int_{[0,t]} Q(i,k,ds)\cdot P_k\{Y(t-s)=j, V(t-s)>y\}$$

Assume : $h(i,t) := P_i\{Y(t)=j, V(t)>y\}$
$g(i,t) := I(i,j)(1-\sum_{\hat{k}\in E} Q(i,\hat{k},t+y))$

Then By renewal theorem, we have

$$P_i\{Y(t)=j, V(t)>y\} = \sum_k \int_{[0,t]} R(i,k,ds)\cdot I(k,j)[1-\sum_{\hat{k}\in E} Q(k,\hat{k},t-s+y)]$$
$$= \boxed{\int_{[0,t]} R(i,j,ds)\cdot[1-\sum_{\hat{k}\in E} Q(j,\hat{k},t-s+y)]}$$

b. $\lim_{t\to\infty} P_i\{Y(t)=j, V(t)>y\} = \boxed{\frac{v(j)}{v\cdot\mu}\int_0^\infty 1-\sum_{\hat{k}\in E} Q(j,\hat{k},t+y)\,dt}$

Here $v\cdot\lim_{t\to\infty} Q(t) = v$, $\mu(i) = \int_0^\infty 1-\sum_{k\in E} Q(i,k,t)\,dt$

c. Assume. $y=10$

$$\lim_{t\to\infty} P_i\{Y(t)=w, V(t)>10\} = \frac{\frac{13}{800}}{\frac{800}{13}\cdot\frac{13}{800}+\frac{5}{100}+\frac{10}{160}}\cdot\int_0^\infty 1-\frac{8}{13}(1-e^{-\frac{13}{800}(t+10)})-\frac{5}{13}(1-e^{-\frac{13}{800}(t+10)})\,dt$$

$$= \frac{13}{890}\cdot\int_0^\infty e^{-\frac{13}{800}t}\cdot e^{-\frac{13}{800}\cdot 10}\,dt$$

$$= \frac{13}{890}\cdot e^{-\frac{13}{80}}\cdot\frac{800}{13} = \frac{80}{89}e^{-\frac{13}{80}} \approx 0.7640594069$$

$$\approx \boxed{0.7641}$$