STAT 611-600: Theory of Inference **Spring 2021**

Final Exam

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Instructions:

- There are **5** questions in this exam.
- You have **165 mins** to finish the exam **AND** upload your answers to eCampus.
- By **4:45 PM CST, May 6, 2020**, you must finish writing and uploading your answers. No late submission will be allowed.
- Please make sure your exam paper has: your name, your UIN.
- This exam is open-book and open-notes but **NO googling or other online resources**. Everything must be your own work.
- Please mark your answers **clearly**. In order to enhance readability of the scanned copies, please avoid using pencils to write your solutions.
- Please check your scanned copy before submission.
- The usual punishment for students caught cheating is an F* in the class. Cheating includes, but is not limited to, communicating in any form with any other student about the questions or answers on this exam before the solutions are posted.
- Please affirm the Aggie Code of Honor with your signature on the first page of your answer sheets: "An Aggie does not lie, cheat or steal, or tolerate those who do." Lu Sun

Problem 1:

1. $E[X_i^2] = \int_0^\infty x^2 \cdot \frac{2x}{\theta} e^{-x^2/\theta} dx = \int_0^\infty -x^2 d e^{-x^2/\theta} = \int_0^\infty e^{-x^2/\theta} dx^2 \overset{t=x^2}{=} \int_0^\infty e^{-t/\theta} dt$

$= \int_0^\infty d e^{-t/\theta} (-\theta) = 0 - (-\theta) = \theta$

$\Rightarrow Bias_\theta [X_i^2] = E[X_i^2] - \theta = 0 \Rightarrow$ unbiased.

2. By 1. $\int_0^\infty 2\frac{x^3}{\theta} e^{-x^2/\theta} dx = \theta \Rightarrow \int_0^\infty x^3 e^{-x^2/\theta} dx = \frac{\theta^2}{2}$.

$E[X_i^4] = \int_0^\infty x^4 \cdot \frac{2x}{\theta} e^{-x^2/\theta} dx = \int_0^\infty -x^4 d e^{-x^2/\theta} = \int_0^\infty e^{-x^2/\theta} dx^4 = 4\int_0^\infty x^3 e^{-x^2/\theta} dx = 4 \cdot \frac{\theta^2}{2} = 2\theta^2$

$X_{(1)} \geq 0$ $E_\theta([\frac{\partial}{\partial\theta} \log f(x_i;\theta)]^2) = E_\theta([\frac{\partial}{\partial\theta}(\log 2X_i - \log\theta - \frac{X_i^2}{\theta})]^2)$

$= E_\theta([-\frac{1}{\theta} + \frac{X_i^2}{\theta^2}]^2) = E_\theta[\frac{1}{\theta^2} + \frac{X_i^4}{\theta^4} - \frac{2X_i^2}{\theta^3}]$

$= \frac{1}{\theta^2} + \frac{1}{\theta^4} \cdot 2\theta^2 - \frac{2}{\theta^3} \cdot \theta = \frac{1}{\theta^2}$

Cramér-Rao lower bound $= \frac{|\tau'(\theta)|^2}{n E_\theta([\frac{\partial}{\partial\theta} \log f(X;\theta)]^2)} = \frac{1}{n \frac{1}{\theta^2}} = \frac{\theta^2}{n}$

3. $L(\theta; X) = \frac{2\prod_{i=1}^n X_i \, 1\{X_{(1)}>0\}}{\theta^n} \cdot e^{-\frac{1}{\theta}\sum_{i=1}^n X_i^2}$

$\Rightarrow$ full rank exponential family $\Rightarrow \sum_{i=1}^n X_i^2$ is complete statistic for θ

Then $T := \frac{\sum_{i=1}^n X_i^2}{n}$ $E[T] = \frac{1}{n}\sum_{i=1}^n E[X_i^2] = \frac{n\theta}{n} = \theta$.

$\Rightarrow \frac{\sum_{i=1}^n X_i^2}{n}$ is UMVUE of θ (by completeness of UMVUE theorem)

Problem 2:

1. $\mu = E[X_i] \underset{\text{function}}{\overset{\text{beta}}{=\!=}} \frac{1}{1+\theta}$ $\quad \sigma^2 = Var(X_i) \underset{\text{function}}{\overset{\text{beta}}{=\!=}} \frac{\theta}{(1+\theta)^2(2+\theta)}$ $\quad \forall\ i = 1, 2, 3, \ldots$

$\{X_i\}$ iid, By. central limit theorem: $\sqrt{n}\left(\bar{X}_n - \frac{1}{1+\theta}\right) \xrightarrow{d} N\left(0, \frac{\theta}{(1+\theta)^2(2+\theta)}\right)$ $\forall \theta$

$\Rightarrow$ $\bar{X}_n$ is asymptotically normal.

2. suppose $m_1 = E[X_1] = \frac{1}{1+\theta}$, $\quad m_2 = E[X_1^2] = Var(X_1) + E[X_1]^2$

$= \frac{\theta}{(1+\theta)^2(2+\theta)} + \frac{1}{(1+\theta)^2} = \frac{2}{(1+\theta)(2+\theta)}$

$\frac{m_1}{m_2} = \frac{2+\theta}{2} \Rightarrow \theta = \frac{2m_1}{m_2} - 2$

Here $m_1 = \bar{X}_n$, $\quad m_2 = \frac{\sum_{i=1}^n X_i^2}{n}$

3. In part 2. $\hat{\theta} = \frac{2\bar{X}_n}{\frac{1}{n}\sum_{i=1}^n X_i^2} - 2 = \frac{2\sum_{i=1}^n X_i}{\sum_{i=1}^n X_i^2} - 2$

Suppose $\bar{X}_n^2 := \frac{1}{n}\sum_{i=1}^n X_i^2$, $\quad \bar{X}_n = \frac{1}{n}\sum_{i=1}^n X_i$

Then by central limit theorem: $\bar{X}_n \xrightarrow{d} N\left(\frac{1}{1+\theta}, \frac{\theta}{n(1+\theta)^2(2+\theta)}\right)$

$X_i \sim Beta(1, \theta)$, Then

2. suppose $\hat{m}_1 := \bar{X}_n$. $E[\hat{m}_1] = E[X_1] = \frac{1}{1+\theta}$

$\Rightarrow \hat{\theta}_{MME} := \frac{1}{\hat{m}_1} - 1$

3. By part 2. $\hat{\theta}_{MME} = \frac{1}{\bar{X}_n} - 1$ $\quad \because \bar{X}_n \xrightarrow{d} N\left(\frac{1}{1+\theta}, \frac{\theta}{n(1+\theta)^2(2+\theta)}\right)$

suppose $g(\bar{X}_n) := \frac{1}{\bar{X}_n} - 1$, then $g'(x) = -\frac{1}{x^2} \neq 0$. , suppose $\mu = \frac{1}{1+\theta}$

By delta method: $\sqrt{n}\left(g(\bar{X}_n) - g\left(\frac{1}{1+\theta}\right)\right) \xrightarrow{d} N\left(0, \frac{(1+\theta)^4 \cdot \theta}{(1+\theta)^2(2+\theta)}\right)$

$= N\left(0, \frac{(1+\theta)^2 \cdot \theta}{(2+\theta)}\right)$

$I(\theta) = E_\theta\left[\left[\frac{\partial}{\partial\theta} \log L(\theta; x)\right]^2\right] \overset{\text{interchangeable}}{=} E_\theta\left[-\frac{\partial}{\partial\theta^2} \log L(\theta; x)\right]$

$= E_\theta\left[-\frac{\partial}{\partial\theta^2}\left(n\log\theta + (\theta-1)\sum_{i=1}^n \log(1-X_i)\right)\right] = \frac{n}{\theta^2}$

$\frac{(\tau'(\theta))^2}{I(\theta)} = \frac{1}{\frac{n}{\theta^2}} = \frac{\theta^2}{n} \neq \frac{(1+\theta)^2}{(2+\theta)}\theta$ $\quad \Rightarrow \hat{\theta}_{MME}$ is not asymptotically efficient

4. $\{X_i\} \overset{iid}{\sim} Beta(1, \theta)$, $\quad \bar{X}_n \sim \chi^2_{2n}$

$95\% = 1 - \alpha = P(\text{Acceptance}) = P(0 < \theta \leq C) = P\left(\frac{1}{\bar{X}_n} - 1 \leq C\right)$

$= P\left(\frac{1}{C+1} \leq \bar{X}_n\right)$

$\Rightarrow \alpha = 0.05$ $\quad \frac{1}{C+1} = \chi^2_{2n, 0.05}$ $\quad C = \frac{1}{\chi^2_{2n, 0.05}} - 1$

$\Rightarrow 95\%$ CI is $\left(0, \frac{1}{\chi^2_{2n, 0.05}} - 1\right)$

Problem 3:

1. ① By N-P Lemma, suppose UMP test $\phi(X) := \begin{cases} 1 & \text{if } P(x;\theta_1) > k P(x;\theta_0), \text{reject} \\ 0 & \text{if } P(x;\theta_1) < k P(x;\theta_0), \text{accept} \end{cases}$

$0.04 = \alpha = E_{\theta_0}(\phi(X)) = 0.01 + 0.01 + 0.01 + 0.01$

$\Rightarrow \begin{cases} 0.03 > k \cdot 0.01 \\ 0.02 < k \cdot 0.01 \end{cases} \Rightarrow 2 < k < 3$

$\Rightarrow$ MPT is $\phi(X) := \begin{cases} 1 & \text{if } P(X;\theta_1) > k P(X;\theta_0) \quad \text{reject}, X \in R \\ 0 & \text{if } P(X;\theta_1) < k P(X;\theta_0) \quad \text{accept}, X \in R^c \end{cases}$ for some $2 < k < 3$

② Type II error $= P_{\theta_1}(X \in R^c) = P_{\theta_1}(X = 5, 6, 7) = 0.02 + 0.01 + 0.79 = 0.82.$

2. (a) ① $L(\theta \mid X) = \left(\frac{1}{\theta}\right)^n \cdot 1\{X_{(1)} \geq 0\} \cdot 1\{X_{(n)} \leq \theta\}$, Given θ, $P(X_{(n)} \leq t) = \left(\frac{t}{\theta}\right)^n$

if $X_{(n)} \leq \theta_0$, $L_R(\theta \mid X) = \left(\frac{1}{\theta_0}\right)^n$, $L_{UR}(\theta \mid X) = \left(\frac{1}{X_{(n)}}\right)^n \Rightarrow \lambda(X) = \left(\frac{X_{(n)}}{\theta_0}\right)^n$

if $X_{(n)} > \theta_0$, $L_R(\theta \mid X) = 0$ $\quad L_{UR}(\theta \mid X) = \left(\frac{1}{X_{(n)}}\right)^n \Rightarrow \lambda(X) = 0$

$R := \{\lambda(X) \leq c\} \overset{0 \leq c \leq 1}{=\!=} \left\{\left(\frac{X_{(n)}}{\theta_0}\right)^n \leq c\right\} = \{X_{(n)} \leq \theta_0 c^{\frac{1}{n}}\}$

$P_{H_0}(R) = P_{\theta_0}(X_{(n)} \leq \theta_0 c^{\frac{1}{n}}) = \left(\frac{\theta_0 c^{\frac{1}{n}}}{\theta}\right)^n \Big|_{\theta = \theta_0} = c = \alpha.$

$\Rightarrow$ LRT : $\lambda(X) = \left(\frac{X_{(n)}}{\theta_0}\right)^n$, with Restriction region $:= \{\lambda(X) \leq \alpha\} = \{X_{(n)} \leq \theta_0 \alpha^{\frac{1}{n}}\}$

② $\Pi(\theta) = P_\theta(R) = P_\theta(X_{(n)} \leq \theta_0 \alpha^{\frac{1}{n}}) = \begin{cases} \left(\frac{\theta_0 \alpha^{\frac{1}{n}}}{\theta}\right)^n & \theta \geq \theta_0 \alpha^{\frac{1}{n}} \\ 1 & \theta < \theta_0 \alpha^{\frac{1}{n}} \end{cases}$

(b) ① suppose $T = X_{(n)}$, by factorization theorem, T is sufficient statistic for θ, $P(T \leq t) = \left(\frac{t}{\theta}\right)^n$ $\quad f_T(t \mid \theta) = \frac{n t^{n-1}}{\theta^n}$

$f_T(t \mid \theta_2) / f(t \mid \theta_1) = \left(\frac{\theta_1}{\theta_2}\right)^n$ an increasing (decreasing) function of t $\forall \theta_2 > \theta_1$

Then By Karlin-Rubin theorem:

the UMP level α test is given by rejects H_0, if and only if $T = X_{(n)} > t_0$, where $\alpha = P_{\theta_0}(T > t_0) = 1 - \left(\frac{t_0}{\theta_0}\right)^n$

$\Rightarrow t_0 = \theta_0 (1-\alpha)^{\frac{1}{n}}$

② $\Pi(\theta) = P_\theta(R) = P_\theta(T > \theta_0 (1-\alpha)^{\frac{1}{n}}) = 1 - \left(\frac{\theta_0 (1-\alpha)^{\frac{1}{n}}}{\theta}\right)^n$ $\quad \theta > \theta_0$

$= 1 - \left(\frac{\theta_0}{\theta}\right)^n (1-\alpha)$

Problem 4.

1. $0 \le \theta \le 1,\ 0 \le 2\theta \le 1,\ 0 \le 1-3\theta \le 1 \quad \Rightarrow \quad 0 \le \theta \le \frac{1}{3}$

2. $l(\theta; X) = \log L(\theta; X) = \log\left[\theta^{n_0}(2\theta)^{n_1}(1-3\theta)^{n-n_0-n_1}\right]$

$$= n_0 \log\theta + n_1 \log 2\theta + (n-n_0-n_1)\cdot\log(1-3\theta)$$

Suppose test $H_0: \theta = \theta_0$, $H_1: \theta \ne \theta_0$

$$\frac{\partial l(\theta; X)}{\partial\theta} = \frac{n_0}{\theta} + \frac{n_1}{\theta} + \frac{n-n_0-n_1}{1-3\theta}\cdot(-3) = 0 \quad \Rightarrow \quad \hat\theta = \frac{n_0+n_1}{3n}$$

$$\frac{\partial^2 l(\theta; X)}{\partial\theta^2} = -\frac{n_0}{\theta^2} - \frac{n_1}{\theta^2} - \frac{3^2(n-n_0-n_1)}{(1-3\theta)^2}\Big|_{\theta = \frac{n_0+n_1}{3n}} < 0$$

$$\Rightarrow -2\log\lambda(X) = -2\,l(\theta_0; X) + 2\,l(\hat\theta; X)$$

$$= -2\log\frac{\theta_0^{n_0+n_1}(1-3\theta_0)^{n-(n_0+n_1)}\cdot(3n)^n}{(n_0+n_1)^{n_0+n_1}(3n-3(n_0+n_1))^{n-n_0+n_1}}$$

$$= -2(n_0+n_1)\log\theta_0 - 2\log(1-3\theta_0$$

Then by asymptotic properties of LRT, $-2\log\lambda(X) = -2\left(l(\theta_0; X) - l\left(\frac{n_0+n_1}{3n}; X\right)\right) \xrightarrow{d} \chi^2_1$

→ reject region

$$\alpha = P_{\theta_0}\left(-2\log\lambda(X) \ge \chi^2_{1,\alpha}\right)$$

$$\alpha = P_{\theta_0}\left(-2\left(n_0\log\frac{3\theta_0 n}{n_0+n_1} + n_1\log\frac{6\theta_0 n}{n_0+n_1} + (n-n_0-n_1)\log\frac{(1-3\theta_0)n}{n-n_0+n_1}\right) \ge \chi^2_{1,\alpha}\right)$$

Problem 5:

1. $P_{H_0}(P(X) \le \alpha) = P_{H_0}(\max_{1\le j \le k} P_j(X) \le \alpha) = P_{H_0}\left(\bigcap_{j=1}^{k} \{P_j(X) \le \alpha\}\right)$

$\le P_{H_0}(P_1(X) \le \alpha) \underset{\text{valid p-value}}{\overset{P_j \text{ is}}{\le}} \alpha$

$\Rightarrow P_{H_0}(P(X) \le \alpha) \le \alpha$

$\Rightarrow P(X)$ is a valid p-value for H_0

2. $P_{H_0}(V = t) = P_{H_0}\left(\sum_{j=1}^{k} \mathbb{1}\{\phi_j(X)=1\} = t\right)$

if $t > k \Rightarrow P_{H_0}(V=t) = 0$

if $t = 0 \Rightarrow P_{H_0}(V \le t) = P_{H_0}\left(\sum_{j=1}^{k} \mathbb{1}\{\phi_j(X)=1\} = 0\right) \overset{P_j \text{ iid}}{=} \prod_{j=1}^{k} P_{H_0}\left(\mathbb{1}\{\phi_j(X)=1\} = 0\right)$

$= \prod_{j=1}^{k} P_{H_0}(P_j(X) > \alpha)$

$\overset{\text{Unif}(0,1)}{=} (1-\alpha)^k$

if $0 < t \le k \Rightarrow P_{H_0}(V = t) = C_k^t P_{H_0}(\phi_{j,1}=1, \phi_{j,2}=1, \cdots, \phi_{j,t}=1) \cdot P_{H_0}(\phi_{j,t+1}=0, \cdots, \phi_{j,k}=0)$

$= C_k^t \alpha^t (1-\alpha)^{k-t}$

$$\Rightarrow P_{H_0}(V=t) = \begin{cases} C_k^t \alpha^t (1-\alpha)^{k-t} & t = 0, 1, 2, \cdots, k \\ 0 & \text{otherwise} \end{cases}$$