§8.1 Markov Inequality

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1 Block Functions

Definition 1.1. A analytic function $g(z)$ on $\mathbb{D}$ is called a **Block Function** if

$$||g||_B = \sup_{z\in\mathbb{D}} |g'(z)|(1-|z|^2) < \infty$$

Remark

- $||g||_B$ is called Block norm.
- If $T(z) = \lambda\frac{z+a}{1+\bar{a}z}, a \in \mathbb{D}, \lambda \in \partial\mathbb{D}$, then $||g \circ T||_B = ||g||_B$.
- If $Re(g)$ is bounded, then $g \in B$, here B is the set of Block Functions.

Theorem 1.1. *Let $g(z)$ be analytic on $\mathbb{D}$. Then $g \in B$ if and only if g is Lipschitz continuous as a map from hyperbolic metric on $\mathbb{D}$ to the Euclidean metric on $\mathbb{D}$. Namely,*

$$||g||_B = \sup_{z,w\in\mathbb{D}} \frac{|g(z)-g(w)|}{\rho(z,w)}$$

$$\rho_{\mathbb{D}}(z_1,z_2) = \inf \int_{z_1}^{z_2} \frac{|dz|}{1-|z|^2}$$

2 Law of Herated Logarithm

In this section, law of Herated logarithm for Block functions is shown:

Theorem 2.1. ***(Markov)*** *$\exists C > 0$, s.t. whenever $g \in B$ on $\mathbb{D}$ a.e. on $\partial\mathbb{D}$,*

$$\limsup_{r\to 1} \frac{|g(re^{i\theta})|}{\sqrt{\log(\frac{1}{1-r})\log\log\log(\frac{1}{1-r})}} \le C||g||_B$$

Theorem 2.2. *If $||g||_B \le 1$ and if $\exists \beta > 0$ and $M < \infty$, s.t. for all $z_0 \in \mathbb{D}$,*

$$\sup_{\{z:\rho(z,z_0)<M\}} (1-|z|^2)|g'(z)| \ge \beta, \tag{0a}$$

then a.e. on $\partial\mathbb{D}$

$$\limsup_{r\to 1} \frac{Re\big(g(re^{i\theta})\big)}{\sqrt{\log(\frac{1}{1-r})\log\log\log(\frac{1}{1-r})}} \ge C(\beta, M) > 0$$

Theorem 2.3. ***Hardy-Littlewood maximal theorem****(H.L.theorem):*
If $f \in L^p(R^n)$ for $n > 1, 1 < p \le \infty$, then $\exists$ a constant $C_{p,n} > 0$, s.t.

$$\left|\left|\sup_{r>0}\left|\frac{1}{|B(x,r)|}\int_{B(x,r)} f(y)dy\right|\right|\right|_{L^p(R^n)} \le C_{p,n}||f||_{L^p(R^n)}.$$

Remark

- If conformal map ψ mapping $\mathbb{D}$ to the domain Ω inside the snowflake curve, then $(0a)$ holds for the global function $g = \log(\psi')$ and vice versa.
- If $g(z) = \sum_{n=1}^{\infty} z^{2^n}$, and $g_N(z) = \sum_{n=1}^{N} z^{2^n}$ is the partial sum of $g(z)$ with $\limsup_{r\to 1} \frac{g_N(re^{i\theta})}{\sqrt{N\log\log N}} = 1$. Then,

$$\limsup_{r\to 1} \frac{|g(re^{i\theta})|}{\sqrt{\log(\frac{1}{1-r})\log\log\log(\frac{1}{1-r})}} = 1$$

proof. *Proof for Theorem 2.1.*
WLOG: $g(0) = 0$ and $||g||_B = 1$.
Let $p \in \mathbb{N}$ and consider

$$I_p(r) = \frac{1}{2\pi}\int_0^{2\pi} |g(re^{i\theta})|^{2p} d\theta,$$

then,

$$\frac{d}{dr}\big(rI_p'(r)\big) = \frac{4p^2 r}{2\pi}\int_0^{2\pi} |g(re^{i\theta})|^{2p-2} * |g'(re^{i\theta})|^2 d\theta.$$

By Hardy's inequality:

$$I_p(r) \le p!\big(\log(\frac{1}{1-r^2})\big)^p \le p!\big(\log(\frac{1}{1-r^2})\big)^p, \tag{1}$$

we have:

$$\begin{aligned}
\tfrac{d}{dr}\big(rI_p'(r)\big) &= \tfrac{4p^2 r}{2\pi}\textstyle\int_0^{2\pi} |g(re^{i\theta})|^{2p-2} * |g'(re^{i\theta})|^2 d\theta \\
&\overset{By\ ||g||_B=1}{\le} \tfrac{4p^2 r}{(1-r)^2} I_{p-1}(r) \\
&\overset{By\ (1)}{\le} \tfrac{4pp! r}{(1-r^2)^2}\big(\log\tfrac{1}{1-r^2})^{p-1}\big) \\
&\le p!\tfrac{d}{dr}\big(r\tfrac{d}{dr}(\log\tfrac{1}{1-r^2})^p\big)
\end{aligned}$$

by ind assumption integrating both sides two times yields (1).

Applying H.L.theorem to $|g(re^{i\theta})|^p \in L^2$ *with* $g_r^*(e^{i\theta}) = \sup_{\rho<r} |g(\rho e^{i\theta})|$, *we get:*

$$\frac{1}{2\pi}\int_0^{2\pi} |g_r^*(e^{i\theta})|^{2p} d\theta \le Cp!\big(\log\frac{1}{1-r}\big)^p. \tag{2}$$

Let $\alpha > 1$ *and* $A_p(r) = \frac{1}{1-r}\frac{1}{\left(\log\frac{1}{1-r}\right)^{p+1}}\frac{1}{\left(\log\log\frac{1}{1-r}\right)^{\alpha}}$, *then we get:*

$$\int_r^1 A_p(s)ds \ge \frac{C}{p}\frac{1}{\left(\log\frac{1}{1-r}\right)^{p+1}}\frac{1}{\left(\log\log\frac{1}{1-r}\right)^{\alpha}}. \tag{3}$$

Consider:

$$\begin{aligned}\int_r^1 A_p(s)\int_0^{2\pi}|g_r^*(e^{i\theta})|^{2p}d\theta ds &\overset{By(2)\ \&\ def\ of\ A_p(z)}{\le} Cp!\int_r^1 \frac{1}{\left(\log\log\frac{1}{1-s}\right)^{\alpha}}\frac{1}{\log\frac{1}{1-s}}\frac{ds}{1-s}\\ &\le C_\alpha p!\end{aligned} \tag{4}$$

Let $E_p := \{\theta : \int_r^1 A_p(s)|g_s^*(e^{i\theta})|^{2p}ds > C_\alpha p^2 p!\}$. *By Chebyshev and Fubini, we have* $|E_p| \le \frac{1}{p^2}$. *If* $\theta \notin \cup_{n>p}E_n$, *then*

$$\begin{aligned}|g(re^{i\theta})|^{2p} &\le \frac{\int_r^1 A_p(s)|g_s^*(e^{i\theta})|^{2p}ds}{\int_r^1 A_p(s)ds}\\ &\overset{By\ (3)\ and\ E_p}{\le} C_\alpha p^2 p!\frac{p(\log\frac{1}{1-r})^p(\log\log\frac{1}{1-r})^{\alpha}}{C}\end{aligned} \tag{5}$$

By (5) we have:

$$\frac{|g(re^{i\theta})|}{\sqrt{\log(\frac{1}{1-r})\log\log\log(\frac{1}{1-r})}} \le \frac{C^{-\frac{1}{2p}}C_\alpha^{-\frac{1}{2p}}p^{\frac{3}{2p}}(p!)^{\frac{1}{2p}}(\log\log\frac{1}{1-r})^{\frac{\alpha}{2p}}}{\sqrt{\log\log\log\frac{1}{1-r}}} = (*) \tag{6}$$

By stirling's formula:

$$p! \sim \sqrt{2\pi p}(\frac{p}{e})^p,$$

and setting $p = \log\log\log\frac{1}{1-r}$, *we have*

$$\begin{aligned}(*) &= \frac{C^{-\frac{1}{2p}}C_\alpha^{-\frac{1}{2p}}p^{\frac{3}{2p}}\left(\sqrt{2\pi p}(\frac{p}{e})^p\right)^{\frac{1}{2p}}(e^p)^{\frac{\alpha}{2p}}}{\sqrt{p}}\\ &= \big(\frac{\sqrt{2\pi}}{CC_\alpha}\big)^{\frac{1}{2\alpha}}p^{\frac{3}{2p}}(\sqrt{e})^{\alpha-1} \sim (\sqrt{e})^{\alpha-1}\end{aligned} \tag{7}$$

With $\alpha > 1$, $||g||_B = 1$, *we get the constant with* $C = 1, |E_p| < \frac{1}{2p} \to 0$. *Finally, by equation* (6) (7), *we get the Markov inequality in Theorem 2.1.*