simulate MRP with semi-Markov kernel

$$Q(t) = \begin{bmatrix} 0.6\,\underline{(1-e^{-5t})} & 0.4\,\underline{(1-e^{-3t})} \\ 0.5\,(1-e^{-2t}) & 0.5\,(1-e^{-4t}) \end{bmatrix}$$

which yields

$$\longrightarrow \quad P = \begin{bmatrix} 0.6 & 0.4 \\ 0.5 & 0.5 \end{bmatrix}$$

As a check on the simulation, look at long-run probabilities, namely, find $\pi P = \pi$ and $\vec{\pi}\mathbf{1} = 1$

$\Rightarrow \pi_a = 5/9$ and $\pi_b = 4/9$

Summary

$\{S_n^j : n = 0, 1, \dots\}$ successive visits to state j

$\{N^j(t) : t \geq 0\}$ (possibly delayed) renewal process associated with $\{S_n^j\}$

$$F(i,j,t) = \begin{cases} P_i\{S_0^j \leq t\} & \text{if } i \neq j \\ P_i\{S_1^i \leq t\} & \text{if } i = j \end{cases}$$

$\leftarrow$ d.f. for first visit to j

$\leftarrow$ d.f. for first return

n-fold MRP convolution

✓ $$R(i,j,t) = E_i[N^j(t)] = \sum_{n=0}^{\infty} Q^n(i,j,t)$$

also $$R(i,j,t) = \begin{cases} \int_{[0,t]} F(i,j,ds)\, R(j,j,t-s) & \text{if } i \neq j \\ \sum_{n=0}^{\infty} F_n(i,i,t) & \text{if } i = j \end{cases}$$

ordinary n-fold convolution

MRP Convolution operator

✓ $$Q * h(i,t) = \sum_k \int_{[0,t]} Q(i,k,ds)\, h(k,t-s)$$

✓ $$R * g(i,t) = \sum_k \int_{[0,t]} R(i,k,ds)\, g(k,t-s)$$

✓ $$Q^{n+1}(i,j,t) = \sum_k \int_{[0,t]} Q^n(i,k,ds)\, Q(k,j,t-s)$$

The equation $h = g + Q * h$ has solution given by $h = R * g$

MRP type equation

$$R = \sum_{n=0}^{\infty} Q^n \implies R * Q = R - I$$

what is the distribution for T_1

$$P_i\{T_1 \le t\} = \sum_{j\in E} Q(i,j,t)$$

☆ $$\mu(i) = E_i[T_1] = \int_0^\infty \left[1 - \sum_k Q(i,k,t)\right] dt$$

$$\eta(i) = \frac{1}{E_i[S_1^i]}$$ ← mean rate between visits to state i

Assume the imbedded M.C. is irreducible, recurrent aperiodic and $0 < \mu(i) < \infty$

$$\lim_{t\to\infty} \left[R(i,j,t+\tau) - R(i,j,t)\right] = \tau\,\eta(j)$$

Let ν be an invariant measure of X; that is, let ν be a solution of $\nu P = \nu$.

$$\eta(i) = \frac{\nu(i)}{\nu\cdot\mu}$$

Note: $\nu\cdot\mu = \sum_k \nu(k)\mu(k)$ (ν does not need to be normed)

Mean ~~time~~ recurrance time can be calculated without knowing the distribution of return times!

Calculate the $E_b[S_1^b]$ ← mean recurrence for state b.

$\nu P = \nu$

$0.6\nu_a + 0.5\nu_b = \nu_a \Rightarrow \nu_b = 0.8\nu_a$

let $r_a = 1 \Rightarrow r_b = 0.8$

$$\mu(a) = \int_0^\infty \left[0.6 e^{-5t} + 0.4 e^{-3t}\right] \times 60 \; dt = 15.2 \text{ min.}$$

$$\mu(b) = \int_0^\infty \left[0.5 e^{-2t} + 0.5 e^{-4t}\right] \times 60 \; dt = 22.5 \text{ min}$$

$$E_b\left[S_1^b\right] = \frac{1}{\eta(b)} = \frac{1 \times 15.2 + 0.8 \times 22.5}{0.8} = 41.5 \text{ min}$$

Find collection of functions $\{h(i,\cdot) : i \in E\}$ such that

$$h(i,t) = g(i,t) + \sum_k \int_{[0,t]} Q(i,k,ds)\, h(k,t-s)$$

$$\Rightarrow h(i,t) = \sum_k \int_{[0,t]} R(i,k,ds)\, g(k,t-s)$$

OR $h = R * g$

$$\lim_{t\to\infty} \sum_k \int_{[0,t]} R(i,k,ds)\, g(k,t-s) = \frac{1}{\nu\cdot\mu} \sum_k \nu(k) \int_0^\infty g(k,t)\,dt$$

where $\nu P = \nu$, $\mu(i) = E_i[T_1]$

$$\nu\cdot\mu = \sum_k \nu(k)\mu(k)$$

Let $H(i,t) = P_i\{T_1 \le t\} = \sum_k Q(i,k,t)$

$\overline{H}(i,t) = 1 - H(i,t) = P_i\{T_1 > t\}$

$\{X_n, T_n\} \to$ M.R.P.

$\{Y(t)\} \to$ SMP $\quad Y(t) = X_n$ if $T_n \le t < T_{n+1}$

$$\lim_{n\to\infty} T_n = \infty$$

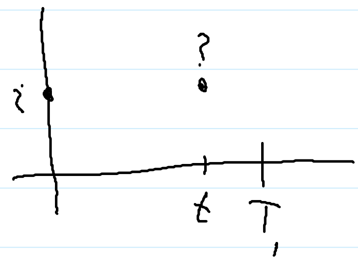
If the semi-Markov kernel has the form

$$Q(i,j,t) = P(i,j)\left[1 - e^{-\lambda(i)t}\right]$$

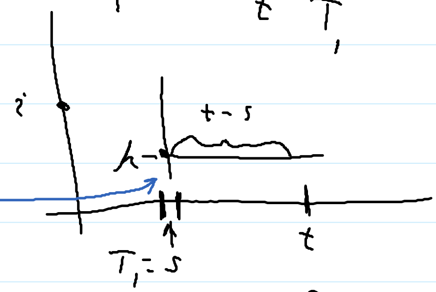
then $\{Y(t)\}$ is a Markov process; otherwise, it is not Markovian.

$$P_t(i,j) = P_i\{Y(t)=j\} = P_i\{Y(t)=j, T_1 > t\} + P_i\{Y(t)=j, T_1 \le t\}$$

$$P_i\{Y(t)=j, T_1 > t\} = I(i,j)\bar{H}(i,t)$$



$$P_i\{Y(t)=j, T_1 \le t\} = \sum_k \int_{[0,t]} Q(i,k,ds)\, P_k\{Y(t-s)=j\}$$



Fixed j

$$\underbrace{P_t(i,j)}_{h(i,t)} = \underbrace{I(i,j)\bar{H}(i,t)}_{g(i,t)} + \sum_k \int_{[0,t]} Q(i,k,ds)\, P_{t-s}(k,j) \quad \text{for } i \in E$$

$$P_t(i,j) = \sum_k \int_{[0,t]} R(i,k,ds)\, I(k,j)\, \bar{H}(k,t-s) \longleftarrow$$

$$= \int_{[0,t]} R(i,j,ds)\, \bar{H}(j,t-s)$$

$$\lim_{t\to\infty} P_t(i,j) = \frac{1}{\nu \cdot \mu} \sum_k \nu(k) \int_{[0,\infty)} I(k,j)\, \bar{H}(k,t)\, dt$$

$$= \frac{1}{\nu\mu}\, \nu(j) \boxed{\int_0^\infty \bar{H}(j,t)\, dt} \longrightarrow \mu(j)$$

$$= \frac{1}{\nu \mu} \nu(j) \boxed{\int_0^\infty \bar{H}(j, t)\, dt} \longrightarrow \mu(j)$$

$$\lim_{t \to \infty} P_i\{ Y(t) = j \} = \nu(j) \cdot \mu(j) \Big/ \sum_k \nu(k) \mu(k)$$