Name: ____

Full name

Test Three

Open Book, Open Notes

1. A certain piece of electronic equipment has two components. The time until failure for component A is described by an exponential distribution function with a mean time of 100 hours. Component B has a mean life until failure of 160 hours and is also described by an exponential distribution. When one component fails, the equipment is turned off and maintenance is performed. (With the equipment off, the other component cannot fail.) The time to fix the component is exponentially distributed with a mean time of 5 hours if it were A that failed and 10 hours if it were B that failed. Let Y be a Markov process with state space E = {w, a, b}, where State w denotes that the equipment is working, State a denotes that component A is under repair, and State b denotes that component B is under repair. For a fixed time t, the random variable Y(t) gives the state of the equipment at time t. In writing your matrices, let the first row represent the state w, the second row represent the state a, and the third row represent the state b.

   a. Give the generator matrix for this process.

$$G = \begin{matrix} -0.01625 & 0.01 & 0.00625 \\ 0.2 & -0.2 & 0.0 \\ 0.1 & 0.0 & -0.1 \end{matrix}$$

   b. Using the generator matrix, determine the long-run probability that the equipment is working?

   Equations for the second and third columns: (using $\mathbf{pG} = \mathbf{0}$ and $\mathbf{p1} = 1$)

   $0.1p(w) = 0.2p(a) \Rightarrow p(a) = 0.05p(w)$

   $0.00625p(w) = 0.1p(b) \Rightarrow p(b) = 0.0625p(w)$

   therefore, $1 = p(w) + p(a) + p(b) = 1.1125p(w) \Rightarrow p(w) = 0.8989$

   c. Every Markov process is also a semi-Markov process. Give the semi-Markov kernel, Q(t), for this process in preparation for analyzing the process using Markov renewal theory.

   First note that 0.01/(0.01+0.00625) = 0.618538 and 0.00625/(0.01+0.00625) = 0.38462

$$Q(t) = \begin{matrix} 0.0 & 0.61538(1-e^{-0.01625t}) & 0.\ 38462(1-e^{-0.01625t}) \\ 1-e^{-0.2t} & 0.0 & 0.0 \\ 1-e^{-0.1t} & 0.0 & 0.0 \end{matrix}$$

   d. To insure that I am interpreting your kernel properly, give the matrix Q(t) evaluated at t=10 hr.

   First observe $e^{-0.01625 \times 10} = 0.8500$, $e^{-0.2 \times 10} = 0.1353$, $e^{-0.1 \times 10} = 0.3679$

$$Q(t) = \begin{matrix} 0.0 & 0.0923 & 0.0577 \\ 0.8647 & 0.0 & 0.0 \\ 0.6321 & 0.0 & 0.0 \end{matrix}$$

   e. Using the methodology developed for semi-Markov processes, determine the long-run probability that the equipment is working?

   $\mu(w) = 1/0.01625 = 61.53846$; $\mu(a) = 5$; $\mu(w) = 10$

$$P = \begin{matrix} 0.0 & 0.61538 & 0.38462 \\ 1 & 0.0 & 0.0 \\ 1 & 0.0 & 0.0 \end{matrix}$$

   Solve $\boldsymbol{\nu}\mathbf{P} = \boldsymbol{\nu}$ by letting $\nu(w) = 1$ and looking at the second and third equation yields

   $\boldsymbol{\nu} = (1, 0.61538, 0.38462)$. (Notice, if you have a different vector, they should differ by a multiplicative constant.) $\boldsymbol{\nu}\,\boldsymbol{\mu} = 68.46156$ and the answer is 61.53846/68.46156 = 0.8989.

2. Let $\{X_n, T_n\}$ be a Markov renewal process with state space E, semi-Markov kernel Q and Markov renewal functions $t \to R(i, j, t)$ for $i,j \in E$. Let $\{Y(t)\}$ be the associated semi-Markov process. Let $V(t) = T_{n+1} - t$ for $T_n \le t < T_{n+1}$. (In other words, V(t) is the time interval from t until the next change of state of the Markov renewal process, or equivalently, the amount of time from t until the next change of state of the semi-Markov process.)

a. For a fixed $y > 0$ and fixed $j \in E$, derive an expression for $P\{Y(t)=j, V(t) > y \mid X_0=i\}$, or equivalently, $P_i\{Y(t)=j, V(t) > y\}$.

First observe that if $T_1>t$, then for $V(t)>y$, we must have $T_1>t+y$; therefore,

$$P_i\{Y(t)=j, V(t) > y, T_1 > t\} = I(i, j)\times P_i\{T_1 > t+y\} = I(i, j)\times[1 - H(i, t+y)],$$

where $H(i, t) = \sum_{j\in E} Q(i, j, t)$

$$P_i\{Y(t)=j, V(t) > y\} = P_i\{Y(t)=j, V(t) > y, T_1 > t\} + P_i\{Y(t)=j, V(t) > y, T_1 \le t\}$$

$$= I(i, j)\times[1 - H(i, t+y)] + \sum_{j\in E}\int_{[0,t]} Q(i,k,ds)P_k\{V(t-s) > y\}$$

$$= \sum_{j\in E}\int_{[0,t]} R(i,k,ds)I(k,j)[1 - H(k,t-s+y)] \quad = \int_{[0,t]} R(i,j,ds)[1 - H(j,r-s+y)]$$

b. For a fixed y, derive an expression for $\lim_{t\to\infty} P\{Y(t)=j, V(t) > y \mid X_0=i\}$.

$$\lim_{t\to\infty} P_i\{Y(t)=j, V(t) > y\} = (1/\boldsymbol{\nu}\,\boldsymbol{\mu}) \sum_{j\in E} v(k) \int_0^\infty I(k,j)[1 - H(k,t+y)]dt$$

$$= (1/\boldsymbol{\nu}\,\boldsymbol{\mu})\, v(j) \int_y^\infty [1 - H(j,u)]du$$

Note that we made a change of variables inside the integral with u = t+y.

c. Assume we have returned to the situation described in Problem 1 and the current time is 1000 hours after start-up (i.e., t=1000). Use your expression in (b) to obtain the (approximate) probability that the equipment is now working and the next failure will occur more than 10 hours from now. (The answer here is numerical, not an expression.)

First note that $\int_y^\infty [1 - H(j,t)]dt$ has the form $\int_y^\infty e^{-\lambda(j)t}dt = e^{-\lambda(j)y} / \lambda(j)$. Also note that $1/\lambda(j) = \mu(j)$; thus, $\int_y^\infty e^{-\lambda(j)t}dt = \mu(j)\, e^{-\lambda(j)y}$.

Therefore, $\lim_{t\to\infty} P_i\{Y(t)=w, V(t) \le y\} = 1 - \left(\nu(w)\,\mu(w)\,e^{-\lambda(w)y}\right) / \left(\sum_{k\in E} \nu(k)\mu(k) \right)$

From Problem 1d, $\boldsymbol{\nu}\,\boldsymbol{\mu} = 68.46156$ and $e^{-0.01625 \times 10} = 0.8500$; therefore,

$$\lim_{t\to\infty} P_i\{Y(t)=w, V(t) > 10\} = 1\times 61.53846\times 0.8500 / 68.46156 = 52.30769/68.46156 = 0.7640.$$