STAT 611 Homework 3 Solutions

1. (a) The joint pdf is

$$
\begin{aligned}
f(\mathbf{x}|\theta) &= \prod_{i=1}^{n} \frac{\alpha}{\beta^\alpha} X_i^{\alpha-1} \mathbf{1}_{[0,\beta]}(X_i) \\
&= \left(\frac{\alpha}{\beta^\alpha}\right)^n \left(\prod_{i=1}^{n} X_i\right)^{\alpha-1} \mathbf{1}_{(-\infty,\beta]}(X_{(n)}) \mathbf{1}_{[0,\infty)}(X_{(1)}) \\
&= L(\alpha, \beta|\mathbf{x})
\end{aligned}
$$

By the Factorization Theorem,

$$
T(\mathbf{x}) = \left(\prod_{i=1}^{n} X_i, X_{(n)}\right)
$$

(b) Fix α. Then $L(\alpha, \beta|\mathbf{x}) = 0$ if $\beta < X_{(n)}$ and L is a decreasing function in β when $\beta \geq X_{(n)}$. Hence, $X_{(n)}$ is the MLE of β. Now,

$$
\begin{aligned}
\frac{\partial}{\partial \alpha} \log L &= \frac{\partial}{\partial \alpha}\left[n \log \alpha - n\alpha \log \beta + (\alpha - 1) \log \prod_{i=1}^{n} X_i\right] \\
&= \frac{n}{\alpha} - n \log \beta + \log \prod_{i=1}^{n} X_i
\end{aligned}
$$

Set this equal to zero and take $\hat{\beta} = X_{(n)}$ to obtain

$$
\hat{\alpha} = \frac{n}{n \log X_{(n)} - \log \prod_{i=1}^{n} X_i} = \left[\frac{1}{n} \sum_{i=1}^{n} (\log X_{(n)} - \log X_i)\right]^{-1}
$$

Because the second derivative, $-n\alpha^2$ is negative, the expression above is the MLE for α.

(c) From the data we see

$$
X_{(n)} = 25.0 \quad \text{and} \quad \prod_{i=1}^{n} X_i = \sum_{i=1}^{n} \log X_i = 43.95
$$

Therefore,

$$
\hat{\beta} = 25.0 \quad \text{and} \quad \hat{\alpha} = 12.59
$$

2. (a) First, the joint pdf is

$$f(\mathbf{x}|\theta) = \prod_{i=1}^{n} \theta X_i^{\theta-1} = \theta^n \left(\prod_{i=1}^{n} X_i\right)^{\theta-1} = L(\theta|\mathbf{x})$$

and so

$$\frac{d}{d\theta}\log L = \frac{d}{d\theta}\left[n\log\theta + (\theta-1)\log\prod_{i=1}^{n} X_i\right] = \frac{n}{\theta} + \sum_{i=1}^{n}\log X_i$$

Setting equal to zero and solving for θ, we have

$$\hat{\theta} = \left[-\frac{1}{n}\sum_{i=1}^{n}\log X_i\right]^{-1}$$

and because the second derivative, $-n/\theta^2$ is negative, $\hat{\theta}$ is the MLE for θ. Note that since $-\log X_i \sim \text{Exponential}(\theta^{-1})$, then $-\sum_i \log X_i \sim \text{Gamma}(n, \theta^{-1})$. Thus,

$$\hat{\theta} = \frac{n}{T}$$

where $T \sim \text{Gamma}(n, \theta^{-1})$ meaning that $\hat{\theta}$ is an inverted Gamma random variable. Hence,

$$\mathbb{E}\hat{\theta} = \frac{n\theta}{n-1}$$

and

$$\text{Var}\hat{\theta} = \frac{n^2\theta^2}{(n-1)^2(n-2)}$$

which tends to zero as $n \to \infty$.

(b) Since $X \sim \text{Beta}(\theta, 1)$,

$$\mathbb{E}X = \frac{\theta}{\theta+1}$$

So the method of moments estimator is the solution to

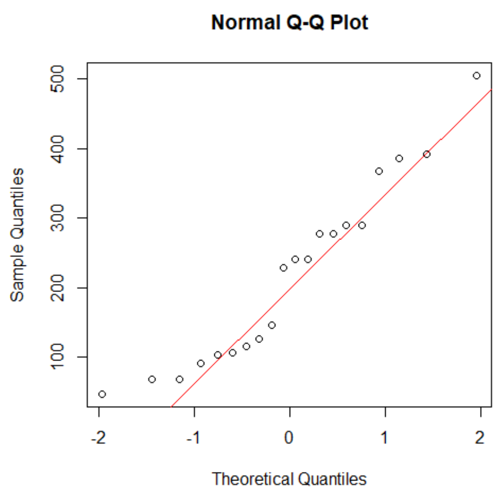
$$\frac{1}{n}\sum_{i=1}^{n} X_i = \frac{\theta}{\theta+1}$$

Therefore,

$$\tilde{\theta} = \frac{\sum_{i=1}^{n} X_i}{n - \sum_{i=1}^{n} X_i}$$

3. (a) The code below constructs a normal probability plot.

```
> loadlife <- read.csv("LoadLife.csv", head = FALSE)
> names(loadlife) <- "bearing"
> qqnorm(loadlife$bearing)
> qqline(loadlife$bearing, col = 'red')
```



The appearence of clustering is due to the limited amount of data we have. Since most of the points fall near the qqline, the plot does not indicate any strong deviation from normality.

(b) The cumulative distribution function of the Weibull distribution is

$$F(x) = 1 - e^{-(x/a)^b}$$

and so the quantile function is

$$F^{\leftarrow}(p|a, b) = b(\log(1-p))^{1/a} \tag{1}$$

To find the sample median and third quartile, use `summary()`.

```
> summary(loadlife)
 bearing
 Min.   : 47.1
 1st Qu.:105.4
 Median :234.5
 Mean   :218.2
 3rd Qu.:289.0
 Max.   :505.0
```

Denote the sample median and third quartile by x_M and x_{TQ}, respectively. Equating theoretical median and third quantiles to sample median and third quartile results in the parameter estimates

$$\hat{b} = \frac{x_M}{\ln(2)^{1/\hat{a}}}$$

and

$$\hat{a} = \frac{\ln(2)}{\ln(x_{TQ}/x_M)}$$

This results in the estimates

$$\hat{a} = 3.317$$

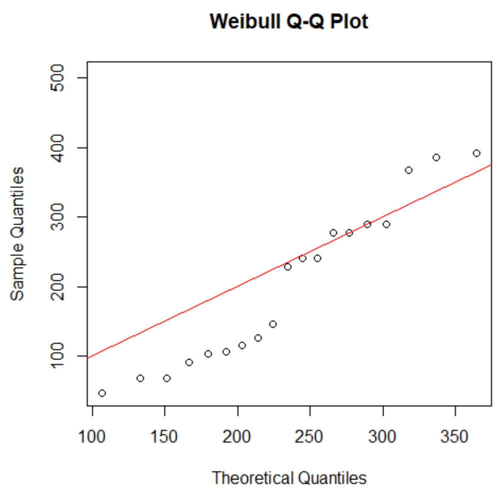
and

$$\hat{b} = 261.9$$

Now, create the Weibull probability plot.

```
> N <- length(loadlife$bearing)
> w_quantile <- qweibull((1:N / N), a_hat, b_hat)
> plot(w_quantile, sort(loadlife$bearing),
>       xlab = 'Theoretical Quantiles',
>       ylab = 'Sample Quantiles',
>       main = 'Weibull Q-Q Plot')
> abline(0, 1, col = 'red')
```

This appears somewhat worse than the normal QQ plot.


Weibull Q-Q Plot
Sample Quantiles
100
200
300
400
500
100
150
200
250
300
350
Theoretical Quantiles