Inter-renewal times $\{X_n\}$ $P\{X_1 \le t\} = F(t)$

$X_n \ge 0$

$$E[X_1] = \mu = \int_0^\infty [1 - F(t)]\,dt$$

Renewal times (epochs) $\{S_n\}$ $P\{S_n \le t\} = F_n(t)$

Renewal process $\{N(t)\}$

Renewal function: $m(t) = E[N(t)] = \sum_{n=1}^{\infty} F_n(t)$

$$F * m(t) = m * F(t) = m(t) - F(t)$$

Renewal Eq.

$$h(t) \stackrel{!}{=} g(t) + F * h(t) \quad \forall\, t \ge 0 \text{ with } g(t) \ge 0$$

Solution: $h(t) = g(t) + m * g(t)$

Feb 10-8:08 AM

For transient $\{N(t)\}$ with $L = \sup\{n : S_n < \infty\}$

$$P\{L \le t\} = (1 - F(\infty))(1 + m(t))$$

Solve $P\{L > t\}$

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Laplace transforms

$$\tilde{F}(s) = \int_{[0,\infty)} e^{-st} F(dt) = E\left[e^{-sX_1}\right]$$

$$E\left[e^{-s(X_1+\cdots+X_n)}\right] = E\left[e^{-sX_1} e^{-sX_2} \cdots e^{-sX_n}\right] = E\left[e^{-sX_1}\right]^n$$

Laplace transform of convolution is product of the transforms.

$F(t) = 1 - e^{-\lambda t}$ for $t \geq 0$

$$\tilde{F}(s) = \frac{\lambda}{\lambda + s}$$

$$\tilde{g}(s) = \int_{[0,\infty)} e^{-st} g(dt)$$

$g(t) = at$ for $t \geq 0 \Longrightarrow \tilde{g}(s) = \frac{a}{s}$

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$$m(t) = \sum_{n=1}^{\infty} F_n(t)$$

$$\tilde{m}(s) = \sum_{n=1}^{\infty} \left(\tilde{F}(s)\right)^n = \tilde{F}(s) \sum_{n=0}^{\infty} \left(\tilde{F}(s)\right)^n$$

$$= \frac{\hat{F}(s)}{1 - F(s)}$$

For $F(t) = 1 - e^{-\lambda t}$ for $t \geq 0$

$\tilde{m}(s) = \lambda / s \Longrightarrow m(t) = \lambda t$

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If $\{N(t)\}$ is transient

$$\lim_{t\to\infty} m * g(t) = m(\infty)\, g(\infty)$$

If $\{N(t)\}$ is aperiodic, recurrent

$$\lim_{t\to\infty} m * g(t) = \frac{1}{\mu}\int_0^\infty g(x)\,dx$$

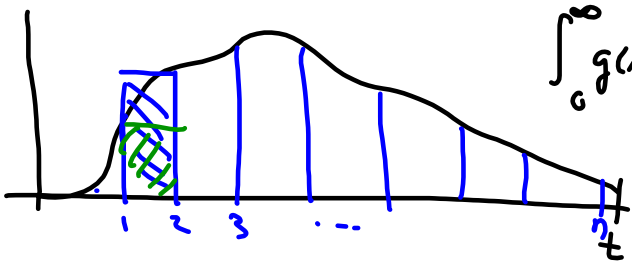
for g a directly Riemann integrable function.

Key Renewal Thm.

sufficient

1) $g(x) \geq 0 \quad \forall x \geq 0$
2) $g(x)$ is non-increasing (beyond some finite point)
3) $\int_0^\infty g(x)\,dx < \infty$

Feb 10-8:41 AM



$$\int_0^\infty g(x)\,dx \to \lim_{t\to\infty}\int_0^t g(x)\,dx$$

Upper sum $= U(g,t,n)$

Lower sum $= L(g,t,n)$

$$\lim_{t\to\infty}\lim_{n\to\infty} U(g,t,n) = \lim_{t\to\infty}\lim_{n\to\infty} L(g,t,n)$$

Directly Riemann Integrable

$$\lim_{n\to\infty}\lim_{t\to\infty} U(g,t,n) = \lim_{n\to\infty}\lim_{t\to\infty} L(g,t,n)$$

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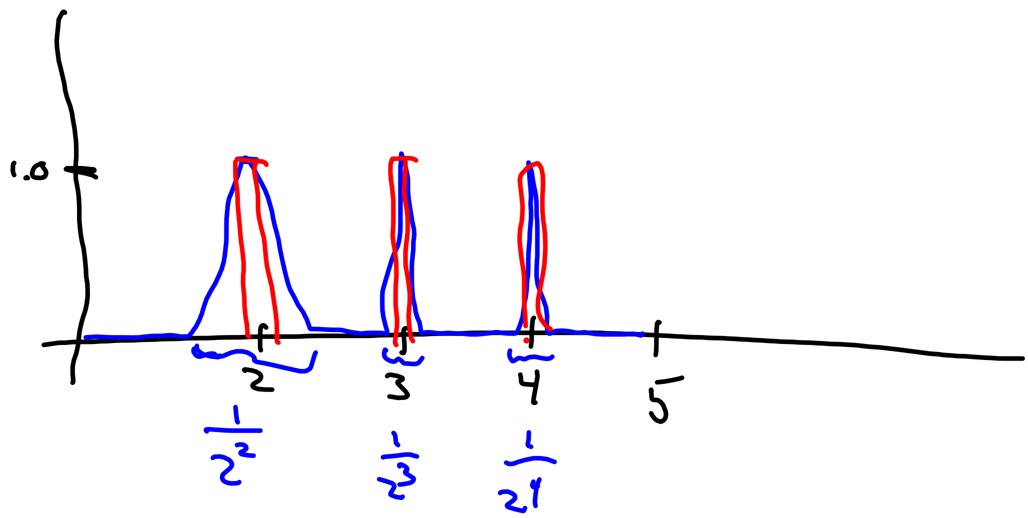

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Regenerative Process

The process $\{Z(t)\}$ is said to be a regenerative process provided there exists a sequence $\{S_1, S_2, \dots\}$ of stopping times such that

a) $\{N(t)\}$ is a renewal process where
$$N(t) = \sum_{n=1}^{\infty} I_{[0,t]}(S_n)$$

and

b) the future of the process $\{Z(t)\}$ at a given renewal point is a probablistic replicate of $\{Z(t)\}$.

Feb 10-9:02 AM

$\{Z(t)\}$ is a regenerative process provided

a) there is an imbedded renewal process where the renewal times are stopping times

b) the future of the process is independent of the past at the renewal points

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$\{N(t)\}$ be a renewal process and let

$V(t) = S_{N(t)+1} - t$. $\Rightarrow$ $V(t)$ is future recurrence time from point t.

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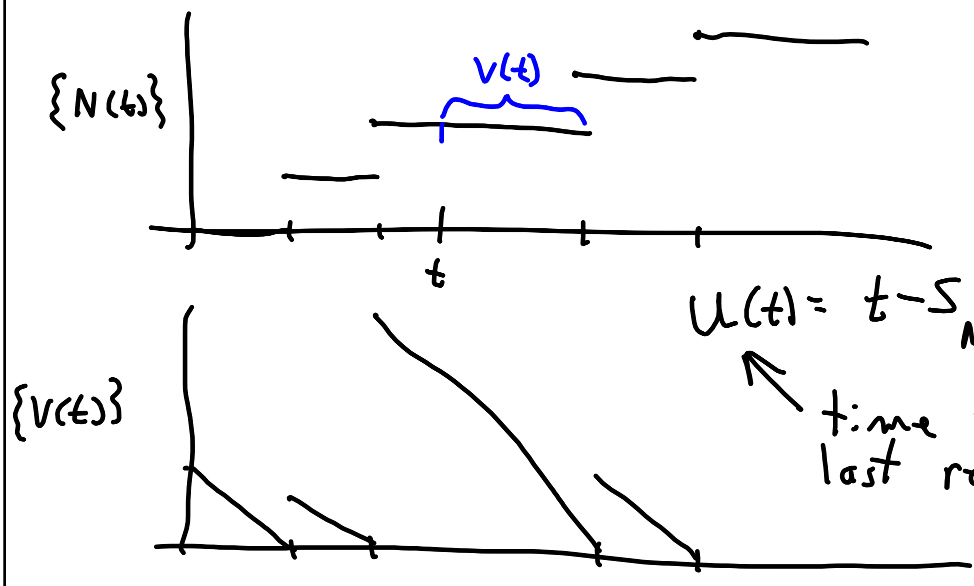

$U(t) = t - S_{N(t)}$

time since last renewal

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