STAT 611 Homework 5 Solutions

1. (a) Denote by α_1 and α_2 the sizes of ϕ_1 and ϕ_2, respectively. Then $\alpha_1 = P(X_1 > 0.95|\theta = 0) = 0.05$ and if $1 \le C \le 2$, then

$$\alpha_2 = P(X_1 + X_2 > C|\theta = 0) = \int_{1-C}^{1} \int_{C-x_1}^{1} 1 dx_2 dx_1 = \frac{(2-C)^2}{2}$$

Setting this equal to α we obtain that $C = 2 - \sqrt{2\alpha}$. Thus, for $\alpha = 0.05$, $C = 2 - \sqrt{0.1} \approx 1.68$.

(b) For the first test, the power function is

$$\beta_1(\theta) = P_\theta(X_1 > 0.95) = \begin{cases} 0 & \theta \le -0.05 \\ \theta + 0.05 & -0.05 < \theta \le 0.95 \\ 1 & 0.95 < \theta \end{cases}$$

The distribution of $Y = X_1 + X_2$ is

$$f_Y(y|\theta) = \begin{cases} y - 2\theta & 2\theta \le y < 2\theta + 1 \\ 2\theta + 2 - y & 2\theta + 1 \le y < 2\theta + 2 \\ 0 & \text{o.w.} \end{cases}$$

Hence, the power function of the second test is

$$\beta_2(\theta) = P_\theta(Y > C) = \begin{cases} 0 & \theta \le \frac{C}{2} - 1 \\ (2\theta + 2 - C)^2/2 & \frac{C}{2} - 1 < \theta \le \frac{C-1}{2} \\ 1 - (C - 2\theta)^2/2 & \frac{C-1}{2} < \theta \le C/2 \\ 1 & \frac{C}{2} < \theta \end{cases}$$

2. By the CLT, the random variable

$$Z = \frac{\sum_i X_i - np}{\sqrt{np(1-p)}}$$

is approximately $\text{n}(0, 1)$. For a test that rejects H_0 when $\sum_i X_i > c$, we need a c and n that satisfy

$$P\left(Z > \frac{c - n(0.49)}{\sqrt{n(0.49)(0.51)}}\right) = 0.01 \quad \text{and} \quad P\left(Z > \frac{c - n(0.51)}{\sqrt{n(0.51)(0.49)}}\right) = 0.99$$

Thus, we want

$$\frac{c - n(0.49)}{\sqrt{n(0.49)(0.51)}} = 2.33 \quad \text{and} \quad \frac{c - n(0.51)}{\sqrt{n(0.51)(0.49)}} = -2.33$$

which results in $n = 13,567$ and $c = 6,783.5$.

3. (a) The power function for this test can be derived as follows.

$$
\begin{aligned}
\beta(\theta) &= P_\theta\left(\frac{|\bar{X}-\theta_0}{\sigma/\sqrt{n}} > c\right) = 1 - P_\theta\left(\frac{|\bar{X}-\theta_0}{\sigma/\sqrt{n}} < c\right) \\
&= 1 - P_\theta\left(-\frac{c\sigma}{\sqrt{n}} \leq \bar{X}-\theta_0 \leq \frac{c\sigma}{\sqrt{n}}\right) \\
&= 1 - P_\theta\left(\frac{-c\sigma/\sqrt{n}+\theta_0-\theta}{\sigma/\sqrt{n}} \leq \frac{\bar{X}-\theta}{\sigma/\sqrt{n}} \leq \frac{c\sigma/\sqrt{n}+\theta_0-\theta}{\sigma/\sqrt{n}}\right) \\
&= 1 - P\left(-c+\frac{\theta_0-\theta}{\sigma/\sqrt{n}} \leq Z \leq c+\frac{\theta_0-\theta}{\sigma/\sqrt{n}}\right) \\
&= 1 + \Phi\left(-c+\frac{\theta_0-\theta}{\sigma/\sqrt{n}}\right) - \Phi\left(c+\frac{\theta_0-\theta}{\sigma/\sqrt{n}}\right)
\end{aligned}
$$

where Z is the standard normal random variable and Φ is its cdf.

(b) We have that $\beta(\theta_0) = 0.05 = 1 + \Phi(c) - \Phi(c)$ which implies that $c = 1.96$. The power is

$$
0.75 \leq \beta(\theta_0+\sigma) = 1 + \Phi(-c-\sqrt{n}) - \Phi(c-\sqrt{n}) = 1 + \underbrace{\Phi(-1.96-\sqrt{n})}_{\approx 0} - \Phi(1.96-\sqrt{n})
$$

$\Phi(-0.675) \approx 0.25 \Rightarrow 1.96 - \sqrt{n} = -0.675 \Rightarrow n = 6.943 \approx 7.$