Group theory for theoretical physics

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Chapter 1

Formal group theory and representations

The goal of this chapter is to develop the abstract tools to deal with groups, and therefore it aims to be general. The main references for this chapter and the following ones are [1] and [2].

1.1 Basic Concepts

The choice of the basic concepts follows [1].

Definition 1.1 (Group)**.** A group G is a set endowed with an internal operation $\cdot : G \times G \to G$, which satisfies the following properties:

1. associativity
$$\forall g_1, g_2, g_3 \in G, (g_1 \cdot g_2) \cdot g_3 = g_1 \cdot (g_2 \cdot g_3) \tag{1.1}$$
2. existance of neutral element
$$\exists e \in G, \forall g \in G, g \cdot e = e \cdot g = g \tag{1.2}$$
3. existance of inverse
$$\forall g \in G, \exists g^{-1} \in G, g \cdot g^{-1} = g^{-1} \cdot g = e \tag{1.3}$$

If it is non-ambiguous, one can omit the multiplication symbol $g \cdot h = gh$.

Proposition 1.2. *The following properties hold:*

1. *The identity element $e \in G$ is unique.*

2. *The inverse $g^{-1} \in G$ of each group element $g \in G$ is unique.*

3. *The inverse of the inverse is the original element: $\forall g \in G, (g^{-1})^{-1} = g$*

4. $\forall g_1, g_2 \in G, (g_1 g_2)^{-1} = g_2^{-1} g_1^{-1}$

Proof.

1. Suppose we had two identity elements $e_1, e_2 \in G$. Then, by (1.2), we have

$$e_1 = e_1 \cdot e_2 = e_2$$

2. Consider $g \in G$ and suppose it had to inverses $h_1, h_2 \in G$. This means that $h_2 = e \cdot h_2 = (h_1 \cdot g) \cdot h_2 = h_1 \cdot (g \cdot h_2) = h_1 \cdot e = h_1$, where we used associativity (1.1).

3. Consider an element $g \in G$. We have $g^{-1} \cdot (g^{-1})^{-1} = e = g^{-1} \cdot g$. If we multiply by g on the left, we get $(g^{-1})^{-1} = g$.

4. Consider $g_1, g_2 \in G$. Then, by associativity, we have $(g_2^{-1} \cdot g_1^{-1}) \cdot (g_1 \cdot g_2) = g_2^{-1} \cdot (g_1^{-1} \cdot g_1) \cdot g_2 = g_2^{-1} \cdot g_2 = e$.

□

Remark 1.3. Since the inverse is unique, the equation $a \cdot x = b$ for $a, b, x \in G$ implies a unique solution $x = a^{-1} \cdot b$. The same holds for $y \cdot a = b$, which yields $y = b \cdot a^{-1}$.

In particular, for $u, v \in G$, the equality $a \cdot u = a \cdot v$ implies $u = w$ since we can multiply both sides by a^{-1} on the left. Analogously, the equality $u \cdot a = v \cdot a$ implies $u = v$ after multiplying by a^{-1} on the right of both sides.

From now on, we will denote the identity element as $\mathbb{1}$, and if the group to which it belongs is ambiguous, we add a subscript $\mathbb{1}_g \in G$.

Definition 1.4 (Abelian group)**.** A group G is said to be *abelian* if its operation is commutative.

$$\forall g_1, g_2 \in G, g_1 \cdot g_2 = g_2 \cdot g_1 \tag{1.4}$$

Definition 1.5 (Subgroup)**.** Suppose we have a group $(G, \cdot)$ and $H \subset G$. Then, we say that H is a subgroup of G if H itself is a group (with identity element $\mathbb{1}_H = \mathbb{1}_G$),which is closed under the group operation

$$\forall h_1, h_2 \in H, h_1 \cdot h_2 \in H$$

Definition 1.6 (Conjugation)**.** Let G be a group and $a, b \in G$. We say that a and b are conjugate if there exists $g \in G$ such that

$$b = g \cdot a \cdot g^{-1}$$

Example 1.1. *In the case of $G = GL(N)$, which is the set of $N \times N$ invertible matrices, two matrices are conjugate if they are related by similarity $B = PAP^{-1}$.*

Proposition 1.7. *Conjugacy is an equivalence relation.*

Proof.

1. Reflexivity: $a = gag^{-1}$ when $g = \mathbb{I} \in G$
2. Symmetry: assume $b = gag^{-1}$ for some $g \in G$. Then, if we multiply by g^{-1} on the left and g on the right both sides, we get $g^{-1}bg = a$
3. Transitivity: assume $b = gag^{-1}$ and $c = hbh^{-1}$, then $c = (hg)ag^{-1}h^{-1} = (hg)a(hg)^{-1}$

□

Definition 1.8 (Conjugacy class)**.** Given an element $a \in G$, we define the following equivalence class, called *conjugacy class* of a

$$Cl(a) = \{h \in G | \exists g \in G, a = ghg^{-1}\}$$

Remark 1.9. If the group G is abelian, all its conjugacy classes are singlets, since $a = ghg^{-1} = hgg^{-1} = h$.

Definition 1.10 (Invariant subgroup)**.** Let G be a group and $H \subset G$ a subgroup. We say that H is an *invariant subgroup* if it is closed under G-conjugacy, i.e.

$$\forall g \in G, \forall h \in H, ghg^{-1} \in H$$

Remark 1.11. The subgroups $\{\mathbb{1}\}$ and G itself are the trivial invariant subgroups.

Remark 1.12. All the subgroups of an abelian group are necessarily invariant subgroups, since conjugacy does not affect their elements.

Definition 1.13 (Simple and semi-simple group)**.** A group G is said to be *simple* if it has no non-trivial invariant subgroups. It is said *semi-simple* if it has no non-trivial *abelian* invariant subgroups.

Definition 1.14 (Direct product)**.** A group G is the *direct product* of two of its invariant subgroups $G = G_1 \otimes G_2$ if

1. the only common element between G_1 and G_2 is the identity: $G_1 \cap G_2 = \{\mathbb{1}_G\}$
2. each element of G_1 commutes with each element of G_2: $g_1 g_2 = g_2 g_1 \forall g_1 \in G_1, g_2 \in G_2$
3. every element of $g \in G$ admits a unique decomposition in terms of an element of G_1 and an element of G_2: $\forall g \in G, \exists! g_1 \in G_1, g_2 \in G_2, g = g_1 g_2$

Definition 1.15 (Coset)**.** Let H be a subgroup of G and $g \in G \setminus H$. We define the *left coset* and *right coset* of H respectively

$$gH = \{gh|h \in H\}$$
$$Hg = \{hg|h \in H\}$$

Proposition 1.16. *Let H be an invariant subgroup of G. Then, the left and right cosets of H coincide $gH = Hg$.*

Proof. We need to show that $\forall l \in gH, l \in Hg$ and viceversa.

Indeed, if $l \in gH$ then there exists $h \in H$ such that $l = gh$. Since H is an invariant subgroup, it means that $ghg^{-1} \in H$, and therefore $h' = lg^{-1} \in H$. Now, since $h'g \in Hg$, we have $lg^{-1}g = l \in Hg$.

Conversely, if $r \in Hg$, then there exists $h \in H$ such that $r = hg$. Since H is an invariant subgroup, it means that $g^{-1}hg \in H$ and therefore $h' = g^{-1}r \in H$. Since $gh' \in gH$, we have $g^{-1}gr = r \in gH$. □

Definition 1.17 (Quotient group)**.** Given an invariant subgroup H of a group G, we define the quotient group G/H as the coset of H alongside with the identity $G/H = gH \cup \{\mathbb{1}H\} = Hg \cup \{\mathbb{1}H\}$ with the multiplication law $\forall g_1, g_2 \in G \setminus H \cup \{\mathbb{1}\}$

$$g_1H \cdot g_2H = (g_1 \cdot g_2)H$$

1.2 Homomorphisms

Definition 1.18 (Homomorphism)**.** Let $(G, \cdot)$ and $(H, *)$ be two groups and $\phi : G \to H$. Then, we say that ϕ is a *homomorphism* if

$$\forall g_1, g_2 \in G, \phi(g_1 \cdot g_2) = \phi(g_1) * \phi(g_2)$$

Definition 1.19 (Endomorphism)**.** An *endomorphism* is a homomorphism from a group G to itself.

Definition 1.20 (Isomorphism)**.** An *isomorphism* is a bijective homomorphism.

Remark 1.21. Isomorphism between two groups is an equivalence relation.

Definition 1.22 (Automorphism)**.** An *automorphism* is a bijective endomorphism, or equivalently an isomorphism from a group G to itself.

Definition 1.23 (Kernel)**.** The kernel of a homomorphism $\phi : G \to H$ is the set $\ker G$ of all elements of G that are mapped to the identity of H.

Proposition 1.24. *The kernel* $\ker G$ *of a homomorphism* $\phi : G \to H$ *is an invariant subgroup of* G.

Proof. We need to show that $\forall k \in \ker G, \forall g \in G, gkg^{-1} \in \ker G$. Let us evaluate $\phi(gkg^{-1})$. By definition of homomorphism and by associativity we indeed have

$$\phi(gkg^{-1}) = \phi(g)\phi(k)\phi(g)^{-1} = \phi(g)\mathbb{1}_H\phi(g)^{-1} = \mathbb{1}_H$$

□

1.3 Classical groups

This section references [3], chapters 2 and 3.

Definition 1.25 (Function group)**.** Let A be a set. We define the function group $S(A)$ as the set of all invertible functions

$$S(A) = \{f : A \to A | \exists f^{-1}\}$$

Proposition 1.26. *The set* $S(A)$ *endowed with the composition operation* $(f_1 \circ f_2)(a) \equiv f_1(f_2(a))$ *forms a group, with the identity being the function* $Id(a) = a$.

Definition 1.27 ($GL(V)$)**.** Let V be a finite-dimensional linear space over the field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. The *general linear group* $GL(V)$ is defined as the set of all invertible endomorphisms in V.

$$GL(V) = \mathrm{End}(V) \cap S(V)$$

Proposition 1.28. $GL(V)$ *is a subgroup of* $S(V)$*, hence itself a group.*

Remark 1.29. Since all the elements in $GL(V)$ are invertible, then they have non-vanishing determinant.

Definition 1.30 ($SL(V)$)**.** Let V be a finite-dimensional linear space over the field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. We define the *special linear group* as

$$SL(V) = \{X \in GL(V) | \det X = 1\}$$

Proposition 1.31. $SL(V)$ *is a subgroup of* $GL(V)$.

1.4 Representations

1.4.1 Basic concepts

Definition 1.32 (Representation)**.** Let G be a group and V a linear space over real or complex scalars. A *representation* of G on V is any group homomorphism $D : G \to \mathrm{Aut}(V)$.

Remark 1.33. By definition of homomorphism, we have that $\forall g, h \in G$,

1. $D(gh) = D(g)D(h)$
2. $D(g^{-1}) = D(g)^{-1}$
3. $D(\mathbb{1}_G) = \mathbb{1}_V$

Proposition 1.34 (Direct sum of representations)**.** *Let G be a group and V_i with $i \in \{1, \dots, N\}$ be linear spaces on the same field $\mathbb{K}$. Let also $D_i : G \to V_i$ be representations of G. Then, the mapping $R : G \to Aut(\oplus_i V_i)$ defined by*

$$R(g) = \oplus_i D_i(g)$$

for all $g \in G$, is a representation of G in $\oplus_i V_i$, called direct sum *of the representations D_i and indicated with $R = \oplus_i D_i$*

Proposition 1.35 (Direct product of representations)**.** *Let G be a group and V_i with $i \in \{1, \dots, N\}$ be linear spaces on the same field $\mathbb{K}$. Let also $D_i : G \to V_i$ be representations of G. Then, the mapping $R : G \to Aut(\otimes_i V_i)$ defined by*

$$R(g) = \otimes_i D_i(g)$$

for all $g \in G$, is a representation of G in $\otimes_i V_i$, called direct product *of the representations D_i and indicated with $R = \otimes_i D_i$*

Remark 1.36. Recall that the direct sum S of two matrices A and B, respectively $(m \times n)$ and $(p \times q)$ is given by a $(m + p) \times (n + q)$ block-diagonal matrix whose blocks are the two starting matrices in the order they enter the sum (therefore, the direct sum is not commutative, strictly speaking), that is

$$S = A \oplus B = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

Also recall that the direct product P of A and B is given by a $(m \cdot p) \times (n \cdot q)$ matrix such that

$$P = A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{pmatrix}$$

Definition 1.37 (Equivalent representations)**.** Let G be a group, and let D_1 and D_2 be two representations of G on the linear spaces V_1 and V_2 over the field $\mathbb{K}$. We say that the two representations are equivalent and write $D_1 \simeq D_2$ if there exists an isomorphism $A \in \text{Iso}(V_1, V_2)$ such that

$$\forall g \in G, D_2(g) = AD_1(g)A^{-1}$$

Proposition 1.38. *The representation equivalence is an equivalence relation.*

Proof. We need to show that $\simeq$ is reflexive, symmetric and transitive.

1. Reflexivity: of course $D_1 \simeq D_1$ since we can choose $A = \mathbb{1}$
2. Symmetry: suppose $\exists A \in \text{Iso}(V_1, V_2) \forall g \in G, D_2(g) = AD_1(g)A^{-1}$, then we have that $\forall g \in G, D_1(g) = A^{-1}D_2(g)A$, and we know that $A^{-1} \in \text{Iso}(V_2, V_1)$, which implies $D_2 \simeq D_1$
3. Transitivity: suppose $\exists A \in \text{Iso}(V_1, V_2) \forall g \in G, D_2(g) = AD_1(g)A^{-1}$ and $\exists B \in \text{Iso}(V_2, V_3) \forall g \in G, D_3(g) = BD_2(g)B^{-1}$. Then we can write $\forall g \in G, D_3(g) = BAD_1(g)A^{-1}B^{-1} = (BA)D_1(g)(BA)^{-1}$, where $BA \in \text{Iso}(V_1, V_3)$, which implies $D_1 \simeq D_3$.

□

Remark 1.39. We can thus build a quotient set of inequivalent representations by considering the set of equivalence classes which group equivalent representations.

Proposition 1.40. *Let D be a representation of a group G in the linear space V, and let $L \neq 0$ be a subspace of V which is invariant under $D(g) \forall g \in G$. The mapping $D|_L : G \to Aut(L)$ defined as*

$$(D|_L)(g) = D(g)|_L \forall g \in G$$

is a representation of G on L.

Definition 1.41 (Reducible representation)**.** A representation D of a group G on a linear space V is said to be *reducible* if there is a non-trivial invariant subspace L of V which is invariant under $D(g)$ $\forall g \in G$.

Definition 1.42 (Completely reducible representation)**.** A representation D of a group G on a linear space V is said to be *completely reducible* if there are non-trivial non-intersecting subspaces $L_1, L_2 \subset V$ invariant under $D(g)$ $\forall g \in G$ such that $V = L_1 \oplus L_2$.

Proposition 1.43. *A representation D of a group G is completely reducible if and only if there exist two representations D_1, D_2 such that*

$$D \simeq D_1 \oplus D_2$$

Definition 1.44 (Character)**.** Let D be a representation of G on the linear space V over the field $\mathbb{K}$. The character of D is the function $\chi_D : G \to \mathbb{K}$ defined by

$$\chi_D(g) = \mathrm{tr} D(g), \ \forall g \in G$$

1.4.2 Unitary representations

Definition 1.45 (Unitary representation)**.** A representation D of a group G on a Hilbert space V over the field $\mathbb{C}$ is said to be *unitary* if

$$\forall g \in G, \ D(g)^{\dagger} = D(g)^{-1}$$

Theorem 1.46 (Weyl)**.** *Let D be a unitary representation of G on the linear space V over $\mathbb{C}$. Then, D is reducible if and only if it is completely reducible.*

Proof. If D is completely reducible, the it is trivially also reducible. Let D be reducible, then there exists a non-trivial subspace L of V invariant under D. Since V is a Hilbert space, the inner product is defined, and we can consider the orthogonal non-trivial complement $L^{\perp}$ of L such that $V = L^{\perp} \oplus L$. Let us show that $L^{\perp}$ is invariant under D. Let $x \in L^{\perp}$ and $y \in L$. Then

$$\langle D(g)x, y \rangle = \langle x, D(g)^{\dagger} y \rangle = \langle x, D(g^{-1}) y \rangle = 0$$

where, in the last equality, we used the fact that $D(g^{-1})y \in L$, being L invariant under D. Therefore we showed that $D(g)x$ is orthogonal to y for all $y \in L$ and for all $g \in G$. This means that $D(g)x \in L^{\perp}$ for all $x \in L^{\perp}$ and for all $g \in G$. Therefore, $L^{\perp}$ is invariant under D. This means that D is completely reducible. □

Proposition 1.47. *Let D be a unitary representation of G over a finite linear space V. Then, there exsist a finite number of irreducible inequivalent unitary representations $D_i, \ i \in \{1, \ldots, p\}$ such that $D \simeq \oplus_{i=1}^{p} D_i$.*

Chapter 2

Lie groups

2.1 Lie algebras

Definition 2.1 (Lie algebra)**.** A *Lie algebra* over the field $\mathbb{K}$ is a vector space $\mathfrak{a}$ over $\mathbb{K}$ equipped with a bilinear map $[\cdot,\cdot] : \mathfrak{a} \times \mathfrak{a} \to \mathfrak{a}$, called *Lie brackets* with the following properties:

1. Antisymmetry: $\forall X, Y \in \mathfrak{a},\ [X, Y] + [Y, X] = 0$
2. Jacobi identity: $\forall X, Y, Z \in \mathfrak{a},\ [X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$

Definition 2.2 (Abelian Lie algebra)**.** A Lie algebra $\mathfrak{a}$ is said to be *abelian* if

$$\forall X, Y, \in \mathfrak{a},\ [X, Y] = 0$$

Definition 2.3 (Structure constants)**.** Let $\{t_a \mid a \in \{1, \dots, n\}\}$ a basis of a Lie algebra $\mathfrak{a}$. Since Lie brackets always output an element of $\mathfrak{a}$, we can expand it onto the basis, so we can write

$$[t_a, t_b] = f^c_{ab} t_c$$

with summed index $c \in \{1, \dots, n\}$, and where $f^c_{ab} \in \mathbb{K}$.

Proposition 2.4. *The* structure constants $f^c_{ab} \in \mathbb{K}$ *of a Lie algebra* $\mathfrak{a}$ *with respect to the basis* $\{t_a\}_{a=1}^n$ *satisfy the following properties:*

1. $f^c_{ab} + f^c_{ba} = 0$
2. $f^d_{ae} f^e_{bc} + f^d_{be} f^e_{ca} + f^d_{ce} f^e_{ab} = 0$
3. *if*

where the first is given by antisymmetry of Lie brackets, and the second follows from Jacobi identity.

Definition 2.5 (Lie subalgebra)**.** Let $\mathfrak{a}$ be a Lie algebra and let $\mathfrak{b} \subseteq \mathfrak{a}$ be a non-null linear subspace. Then, $\mathfrak{b}$ is a *Lie subalgebra* of $\mathfrak{a}$ if it is closed under Lie brackets of $\mathfrak{a}$, that is

$$\forall X, Y \in \mathfrak{b},\ [X, Y] \in \mathfrak{b}$$

In this case, $\mathfrak{b}$ itself is a Lie algebra, with Lie brackets inherited by $\mathfrak{a}$.

Definition 2.6 (Lie ideal)**.** Let $\mathfrak{a}$ be a Lie algebra and let $\mathfrak{b} \subseteq \mathfrak{a}$ be a non-null linear subspace. Then, $\mathfrak{b}$ is a *Lie ideal* of $\mathfrak{a}$ if it is invariant under $\mathfrak{a}$, that is

$$\forall X \in \mathfrak{b},\ \forall Y \in \mathfrak{a},\ [X, Y] \in \mathfrak{b}$$

Proposition 2.7. *A Lie ideal $\mathfrak{b}$ of a Lie algebra $\mathfrak{a}$ is also a subalgebra of $\mathfrak{a}$, as implied by its definition.*

Proposition 2.8. *The union of arbitrary subalgebras (ideals) of a Lie algebra $\mathfrak{a}$ is itself a subalgebra (ideal) of $\mathfrak{a}$.*

Definition 2.9 (Lie algebra homomorphism)**.** A *Lie algebra homomorphism* between Lie algebras $\mathfrak{a}, \mathfrak{b}$ is a linear map $\phi : \mathfrak{a} \to \mathfrak{b}$ which preserves Lie brackets, that is

$$\forall X, Y \in \mathfrak{a},\ \phi([X, Y]) = [\phi(X), \phi(Y)]$$

2.2 Lie algebra representations

To define representations of Lie algebras, we need the definition of the classical Lie algebra $\mathfrak{gl}(V)$.

Definition 2.10 (General linear Lie algebra)**.** Let V be a linear space over a field $\mathbb{K}$ and $X, Y \in \mathrm{End}(V)$. Then we define $\mathfrak{gl}(V) = \mathrm{End}(V)$ as the Lie algebra (over $\mathbb{K}$) having as Lie brackets the commutator

$$\forall X, Y \in \mathrm{End}(V),\ [X, Y] = XY - YX$$

Now we can move to representations.

Definition 2.11 (Lie algebra representation)**.** Let $\mathfrak{a}$ be a Lie algebra over the field $\mathbb{K}$ and let V be a linear space over $\mathbb{L}$ with $\mathbb{K} \subseteq \mathbb{L}$. Then, a *Lie algebra representation* of $\mathfrak{a}$ on V is any Lie algebra homomorphism $\phi \in \mathrm{Hom}(\mathfrak{a}, \mathfrak{gl}(V))$.

Definition 2.12 (Adjoint representation)**.** Let $\mathfrak{a}$ be a Lie algebra and $X \in \mathfrak{a}$. We define $\mathrm{ad} : \mathfrak{a} \to \mathrm{End}(\mathfrak{a}) = \mathfrak{gl}(\mathfrak{a})$ such that, for all $Y \in \mathfrak{a}$,

$$\mathrm{ad}(X)Y = [X, Y]$$

Proposition 2.13. *ad is a representation of $\mathfrak{a}$ on the linear space $\mathfrak{a}$ (indeed we know by definition that a Lie algebra is a linear space).*

Proof. We need to show that $\mathrm{ad} \in \mathrm{Hom}(\mathfrak{a}, \mathfrak{gl}(\mathfrak{a}))$, that is satisfies the definition 2.9. Indeed, since Lie brackets are linear in the first argument, then also ad is linear by construction. Now we verify that it preserves Lie brackets. Let $X, Y, Z \in \mathfrak{a}$, then

$$\begin{aligned}
\mathrm{ad}([X,Y])Z &= [[X,Y],Z] = -[Z,[X,Y]] = [X,[Y,Z]] + [Y,[Z,X]] \\
&= [X,[Y,Z]] - [Y,[X,Z]] = \\
&= \mathrm{ad}(X)\mathrm{ad}(Y)Z - \mathrm{ad}(Y)\mathrm{ad}(X)Z = [\mathrm{ad}(X), \mathrm{ad}(Y)]Z
\end{aligned}$$

Since Z is arbitrary, we showed $\mathrm{ad}([X,Y]) = [\mathrm{ad}(X), \mathrm{ad}(Y)]$, and thus ad is indeed a homomorphism, and therefore a Lie algebra representation. $\square$

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