Group theory for theoretical physics

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Chapter 1

Introduction to group theory

The main reference for this chapter and the following ones is [1].

1.1 Basic Concepts

Definition 1.1 (Group)**.** A group G is a set endowed with an internal operation $\cdot : G \times G \to G$, which satisfies the following properties:

1. associativity
$$\forall g_1, g_2, g_3 \in G, (g_1 \cdot g_2) \cdot g_3 = g_1 \cdot (g_2 \cdot g_3) \tag{1.1}$$

2. existance of neutral element
$$\exists e \in G, \forall g \in G, g \cdot e = e \cdot g = g \tag{1.2}$$

3. existance of inverse
$$\forall g \in G, \exists g^{-1} \in G, g \cdot g^{-1} = g^{-1} \cdot g = e \tag{1.3}$$

Proposition 1.2. *The following properties hold:*

1. *The identity element $e \in G$ is unique.*

2. *The inverse $g^{-1} \in G$ of each group element $g \in G$ is unique.*

3. *The inverse of the inverse is the original element: $\forall g \in G, (g^{-1})^{-1} = g$*

4. $\forall g_1, g_2 \in G, (g_1 \cdot g_2)^{-1} = g_2^{-1} g_1^{-1}$

Proof. (1) Suppose we had two identity elements $e_1, e_2 \in G$. Then, by (1.2), we have

$$e_1 = e_1 \cdot e_2 = e_2$$

(2) Consider $g \in G$ and suppose it had to inverses $h_1, h_2 \in G$. This means that $h_2 = e \cdot h_2 = (h_1 \cdot g) \cdot h_2 = h_1 \cdot (g \cdot h_2) = h_1 \cdot e = h_1$, where we used associativity (1.1).

(3) Consider an element $g \in G$. We have $g^{-1} \cdot (g^{-1})^{-1} = e = g^{-1} \cdot g$. If we multiply by g on the left, we get $(g^{-1})^{-1} = g$.

(4) Consider $g_1, g_2 \in G$. Then, by associativity, we have $(g_2^{-1} \cdot g_1^{-1}) \cdot (g_1 \cdot g_2) = g_2^{-1} \cdot (g_1^{-1} \cdot g_1) \cdot g_2 = g_2^{-1} \cdot g_2 = e$. □

Remark 1.3. Since the inverse is unique, the equation $a \cdot x = b$ for $a, b, x \in G$ implies a unique solution $x = a^{-1} \cdot b$. The same holds for $y \cdot a = b$, which yields $y = b \cdot a^{-1}$.

In particular, for $u, v \in G$, the equality $a \cdot u = a \cdot v$ implies $u = w$ since we can multiply both sides by a^{-1} on the left. Analogously, the equality $u \cdot a = v \cdot a$ implies $u = v$ after multiplying by a^{-1} on the right of both sides.

Definition 1.4 (Abelian group)**.** A group G is said to be *abelian* if its operation is commutative.

$$\forall g_1, g_2 \in G, g_1 \cdot g_2 = g_2 \cdot g_1 \tag{1.4}$$

Bibliography

[1] Wu-Ki Tung and Michael Aivazis. *Group theory in physics. 1, Hauptbd.* World Scientific, 1985.