Group theory for theoretical physics

Luca Morelli, Damiano Scevola

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Chapter 1

Introduction to group theory

The main reference for this chapter and the following ones is [1].

1.1 Basic Concepts

Definition 1.1 (Group)**.** A group G is a set endowed with an internal operation $\cdot : G \times G \to G$, which satisfies the following properties:

1. associativity
$$\forall g_1, g_2, g_3 \in G, (g_1 \cdot g_2) \cdot g_3 = g_1 \cdot (g_2 \cdot g_3) \tag{1.1}$$

2. existance of neutral element
$$\exists e \in G, \forall g \in G, g \cdot e = e \cdot g = g \tag{1.2}$$

3. existance of inverse
$$\forall g \in G, \exists g^{-1} \in G, g \cdot g^{-1} = g^{-1} \cdot g = e \tag{1.3}$$

If it is non-ambiguous, one can omit the multiplication symbol $g \cdot h = gh$.

Proposition 1.2. *The following properties hold:*

1. *The identity element $e \in G$ is unique.*
2. *The inverse $g^{-1} \in G$ of each group element $g \in G$ is unique.*
3. *The inverse of the inverse is the original element: $\forall g \in G, (g^{-1})^{-1} = g$*
4. $\forall g_1, g_2 \in G, (g_1 g_2)^{-1} = g_2^{-1} g_1^{-1}$

Proof. 1. Suppose we had two identity elements $e_1, e_2 \in G$. Then, by (1.2), we have

$$e_1 = e_1 \cdot e_2 = e_2$$

2. Consider $g \in G$ and suppose it had to inverses $h_1, h_2 \in G$. This means that $h_2 = e \cdot h_2 = (h_1 \cdot g) \cdot h_2 = h_1 \cdot (g \cdot h_2) = h_1 \cdot e = h_1$, where we used associativity (1.1).

3. Consider an element $g \in G$. We have $g^{-1} \cdot (g^{-1})^{-1} = e = g^{-1} \cdot g$. If we multiply by g on the left, we get $(g^{-1})^{-1} = g$.

4. Consider $g_1, g_2 \in G$. Then, by associativity, we have $(g_2^{-1} \cdot g_1^{-1}) \cdot (g_1 \cdot g_2) = g_2^{-1} \cdot (g_1^{-1} \cdot g_1) \cdot g_2 = g_2^{-1} \cdot g_2 = e$.

□

Remark 1.3. Since the inverse is unique, the equation $a \cdot x = b$ for $a, b, x \in G$ implies a unique solution $x = a^{-1} \cdot b$. The same holds for $y \cdot a = b$, which yields $y = b \cdot a^{-1}$.

In particular, for $u, v \in G$, the equality $a \cdot u = a \cdot v$ implies $u = w$ since we can multiply both sides by a^{-1} on the left. Analogously, the equality $u \cdot a = v \cdot a$ implies $u = v$ after multiplying by a^{-1} on the right of both sides.

Definition 1.4 (Abelian group)**.** A group G is said to be *abelian* if its operation is commutative.

$$\forall g_1, g_2 \in G, g_1 \cdot g_2 = g_2 \cdot g_1 \tag{1.4}$$

Definition 1.5 (Subgroup)**.** Suppose we have a group $(G, \cdot)$ and $H \subset G$. Then, we say that H is a subgroup of G if it is closed under the group operation.

$$\forall h_1, h_2 \in H, h_1 \cdot h_2 \in H$$

Definition 1.6 (Homomorphism)**.** Let $(G, \cdot)$ and $(H, *)$ be two groups and $\phi : G \to H$. Then, we say that ϕ is a *homomorphism* if

$$\forall g_1, g_2 \in G, \phi(g_1 \cdot g_2) = \phi(g_1) * \phi(g_2)$$

Definition 1.7 (Conjugation)**.** Let G be a group and $a, b \in G$. We say that a and b are conjugate if there exists $g \in G$ such that

$$b = g \cdot a \cdot g^{-1}$$

Example 1.1. *In the case of $G = GL(N)$, which is the set of $N \times N$ invertible matrices, two matrices are conjugate if they are related by similarity $B = PAP^{-1}$.*

Proposition 1.8. *Conjugacy is an equivalence relation.*

Proof.

1. Reflexivity: $a = gag^{-1}$ when $g = \mathbb{I} \in G$
2. Symmetry: assume $b = gag^{-1}$ for some $g \in G$. Then, if we multiply by g^{-1} on the left and g on the right both sides, we get $g^{-1}bg = a$
3. Transitivity: assume $b = gag^{-1}$ and $c = hbh^{-1}$, then $c = (hg)ag^{-1}h^{-1} = (hg)a(hg)^{-1}$

□

Definition 1.9 (Conjugacy class)**.** Given an element $a \in G$, we define the following equivalence class, called *conjugacy class* of a

$$Cl(a) = \{h \in G | \exists g \in G, a = ghg^{-1}\}$$

Remark 1.10. If the group G is abelian, all its conjugacy classes are singlets, since $a = ghg^{-1} = hgg^{-1} = h$.

Bibliography

[1] Wu-Ki Tung and Michael Aivazis. *Group theory in physics. 1, Hauptbd.* World Scientific, 1985.