Fungsi 2 peubah / lebih & Turunan Parsial

- Fungsi 2 Peubah

$(x,y) \xrightarrow{f} z = f(x,y) \in R$

R^2

$D_f = \{ z = f(x,y) \mid (x,y) \in R^2 \}$

- Bila tidak dinyatakan secara eksplisit, D_f himpunan terbesar dari f, yang membuat f terdefinisi.

contoh:

1. $f(x,y) = \dfrac{2\sqrt{x}}{y-1}$

   $D_f = \{ (x,y) \in R^2 \mid x \geq 0,\ y \in R - \{1\} \}$

2. $f(x,y) = \dfrac{x}{y}$

   $D_f = \{ (x,y) \in R^2 \mid y \neq 0 \}$

3. $f(x,y) = \sqrt{4 - x^2 - y^2}$

   $D_f = \{ (x,y) \in R^2 \mid x^2 + y^2 \leq 4 \}$

   $W_f = \{ z \in R \mid z \geq 0 \}$

4. $f(x,y) = \dfrac{\sqrt{x+y+1}}{x-1}$ $\quad f(2,1) = \dfrac{2}{1} = 2$

   $D_f = \{ (x,y) \in R^2 \mid x+y \geq -1,\ x \neq 1 \}$

5. $f(x,y) = x \ln (y^2 - x)$

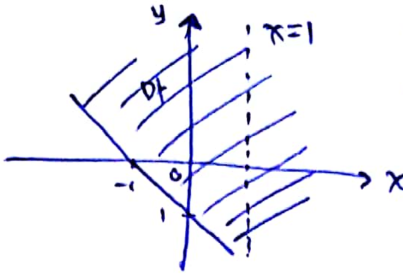
   $f(2,1) = 2 \ln (1^2 - 2)$ → tdk ada nilai

   $D_f = \{ (x,y) \in R^2 \mid y^2 - x > 0 \}$

   $x < y^2$

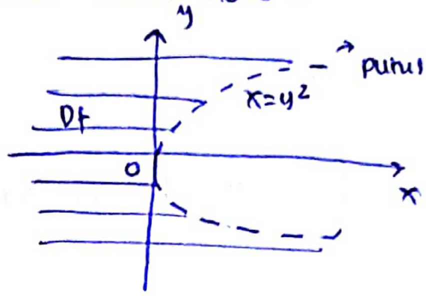
∴ titik (2,1) tidak memenuhi D_f, maka nilainya tdk terdefinisi.

kurva domain no 4 :



yg garis putus² tidak boleh.

kurva domain no 5 :



putus-putus karena > bukan ≥.

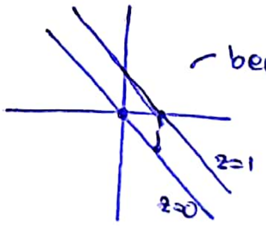
soal :

① $z = f(x,y) = x + y$

$k = -7, -2, 0, 1, 2, 7$

untuk k tentukan pers. kurva ketinggiaan.

$z = 0 \quad x + y = 0 \quad x = -y$ $\qquad z = 1 \quad x + y = 1 \quad x = 1 - y$



bentuknya terus naik ke atas.

② $f(x,y) = \frac{1}{3}\sqrt{36 - 9x^2 - 4y^2}$

$k = 0, 1, 1{,}5, 1{,}75, 2$

$0 = \frac{1}{9}(36 - 9x^2 - 4y^2)$

$0 = 36 - 9x^2 - 4y^2$

$36 = 9x^2 + 4y^2$

$9 = 36 - 9x^2 - 4y^2$

$2 = 9x^2 - 4y2$

Turunan

- Turunan Parsial

misal $z = f(x,y)$ fungsi 2 variabel.

$$\frac{\partial}{\partial x} f(x,y) = f_x(x,y) = \lim_{h \to 0} \left[\frac{f(x+h,y) - f(x,y)}{h} \right]$$

↳ turunan parsial $z = f(x,y)$ thdp x.

$$\boxed{\frac{\partial}{\partial y} f(x,y) = \lim_{h \to 0} \left[\frac{f(x,y+k) - f(x,y)}{k} \right]}$$

↳ turunan parsial $z = f(x,y)$ thd y.

contoh:

① $f(x,y) = x^2 + y^2$ thd x.

$$\frac{\partial F}{\partial x} = F_x(x,y) = \lim_{h \to 0} \left[\frac{(x+h)^2 + y^2 - (x^2 + y^2)}{h} \right]$$

$$= \lim_{h \to 0} \frac{\cancel{x^2} + 2hx + h^2 + \cancel{y^2} - \cancel{x^2} - \cancel{y^2}}{\cancel{h}}$$

$$= \lim_{h \to 0} 2x + h$$

$$= 2x$$

② $f(x,y) = x^2 y + x$ thd y

$$\frac{\partial F}{\partial y} = F_y(x,y) = \lim_{h \to 0} \frac{x^2(y+h) + x - (x^2 y + x)}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{x^2 y} + x^2 \cancel{h} + \cancel{x} - \cancel{x^2 y} - \cancel{x}}{\cancel{h}}$$

$$= \lim_{h \to 0} x^2$$

$$= x^2$$

③ $f(x,y) = x^4 + 6\sqrt{y} - 10$

$$\frac{\partial F}{\partial x} = F_x(x,y) = 4x^3$$

$$\frac{\partial F}{\partial y} = F_y(x,y) = 6 \cdot \frac{1}{2} y^{-1/2}$$

$$= \frac{3}{\sqrt{y}}.$$

④ $f(x,y) = x^2 y + \sin xy^2$

$F_x(x,y) = 2xy + y^2 \cos(xy^2)$

$F_y(x,y) = x^2 + 2xy \cos(xy^2)$

⑤ $f(x,y) = (x+y)^{1/2}$

$$F_x(x,y) = \frac{1}{2}(x+y)^{-1/2} \cdot 1 = \frac{1}{2\sqrt{x+y}}$$

$$F_y(x,y) = \frac{1}{2}(x+y)^{-1/2} \cdot 1 = \frac{1}{2\sqrt{x+y}}$$

⑥ $f(x,y,z) = xz + e^{y^2 z} + \sqrt{xy^2 z^3}$

$F_x(x,y,z) = z + \frac{1}{2}(xy^2z^3)^{-1/2} (y^2 z^3)$

$F_y(x,y,z) = 2yze^{y^2 z} + \frac{1}{2}(xy^2z^3)^{-1/2} (2xyz^3)$

$F_z(x,y,z) = x + y^2 e^{y^2 z} + \frac{1}{2}(xy^2z^3)^{-1/2} (3xy^2z^2)$

⑦ $f(x,y) = \underbrace{\cos\left(\frac{4}{x}\right)}_{u} \underbrace{e^{x^2 y - 5y^3}}_{v}$

$$F_x(x,y) = \left(\frac{-4}{x^2}\right)\left(-\sin\left(\frac{4}{x}\right)\right) \cdot e^{x^2 y - 5y^3} +$$
$$\cos\left(\frac{4}{x}\right)(2xy)\, e^{x^2 y - 5y^3}$$

$$F_y(x,y) = \cos\left(\frac{4}{x}\right) e^{x^2 y - 5y^3} (x^2 - 15y^2)$$

K.8 PMK

Turunan Parsial Orde Tinggi

$$f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$$

$$f_{xy} = (f_x)_y = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$$

$$f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}$$

$$f_{yx} = (f_y)_x = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}$$

contoh:

① $F(x,y) = \sqrt{1-x^2-y^2}$

$$F_x = \frac{1}{2}(1-x^2-y^2)^{-1/2}(-2x) = -x(1-x^2-y^2)^{-1/2}$$

$$F_y = \frac{1}{2}(1-x^2-y^2)^{-1/2}(-2y) = -y(1-x^2-y^2)^{-1/2}$$

$$F_{xy} = \frac{\partial^2 F}{\partial y \partial x} = -x\left(-\frac{1}{2}\right)(1-x^2-y^2)^{-3/2}(-2y)$$

$$= -xy(1-x^2-y^2)^{-3/2}$$

$$F_{yx} = \frac{\partial^2 F}{\partial x \partial y} = -y\left(-\frac{1}{2}\right)(1-x^2-y^2)^{-3/2}(-2x)$$

$$= -xy(1-x^2-y^2)^{-3/2}$$

catatan!
$F_{xy} = F_{yx}$ selalu

→ Teorema Clairaut.

K9. PMK.

Bidang Singgung & Hampiran linear.

- Persamaan bidang singgung yg melalui titik $P(x_0, y_0, z_0)$ =

$$\boxed{A(x-x_0) + B(y-y_0) + C(z-z_0) = 0}$$

- Persamaan bidang singgung di P:

$$z - z_0 = a(x-x_0) + b(y-y_0)$$

$$\boxed{z - z_0 = f_x(x_0,y_0).(x-x_0) + f_y(x_0,y_0)(y-y_0)}$$

↓ turunan parsial

↳ Persamaan Umum

contoh:

1. $z = 2x^2 + y^2$ di titik $(1,1,3)$

$f_x(x,y) = 4x = 4$

$f_y(x,y) = 2y = 2$

Pers: $z-3 = 4(x-x_0) + 2(y-y_0)$

$z-3 = 4(x-1) + 2(y-1)$

$z-3 = 4x-4+2y-2$

$z = 4x+2y-3$

- Hampiran linear.

Bila f mempunyai turunan di titik $p=(a,b)$ maka f mempunyai hamparan linear.

- Hampiran linear di sekitar (a,b):

$$\boxed{F(x,y) \approx f(a,b) + dz}$$

↓ pers. bidang singgung

contoh:

1. $f(x,y) = z = x^2 + y^2$ di titik $P(1,2)$ menghampiri nilai (1,1 dan 1,9)

$dx = 1{,}1 - 1 = 0{,}1$

$dy = 1{,}9 - 2 = -0{,}1$

$f_x = 2x$ | $dz = f_x(1,2)dx + f_y(1,2)dy$

$f_x(1,2) = 2$ | $dz = 2dx + 4dy$

$f_y = 2y$ | $dz = 2(0{,}1) + 4(-0{,}1)$

$f_y(1,2) = 4$ | $dz = 0{,}2 - 0{,}4$

$\boxed{dz = -0{,}2}$ → diferensial

$F(1,2) \approx 5 - 0{,}2 = 4{,}8$ // → hampiran lin.

2. $L(x,y) = x \cdot y$ → x = panjang, L → lebar.

Saat : $x=30$, $y=29$ → $f_x = 29$

$dx = 0{,}1$ $dy = 0{,}1$ $f_y = 30$

$dL = f_x(30,29)\,dx + f_y(30,29)\,dy$

$= 29\,(0{,}1) + 30\,(0{,}1)$

$= 5{,}9\ cm^2$ → kesalahan maks.

$F(x,y) = f(a,b) + dz$

$= (30 \cdot 29) + 5{,}9$

$= 725{,}9$ //

✓ Aturan Rantai

misal $z = f(x,y)$ dg $x = x(t)$, $y = y(t)$

$$\boxed{\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}}$$

→ Jenis 1.

contoh:

1. $z = x^3 y$, $x = 2t$, $y = t^2$.

$F_x = 3x^2 y$ $\quad \frac{dx}{dt} = 2$ $\quad \frac{dy}{dt} = 2t$

$f_y = x^3$

$\frac{dz}{dt} = (3x^2 y \cdot 2) + (x^3 \cdot 2t)$

$\frac{dz}{dt} = 6x^2 y + 2x^3 t$

$= 6(2t)^2 (t^2) + 2(2t)^3 \cdot t$

$= 24t^4 + 16t^4$

$$\boxed{\frac{dz}{dt} = 40t^4}$$

✓ Aturan Rantai (Jenis 2)

misal : $z = f(x,y)$, $x = x(s,t)$

$y = (s,t)$

$$\boxed{\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\,\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial s}}$$

$$\boxed{\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\,\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial t}}$$

contoh:

1. $z = 3x^2 - y^2$, $x = 2s + 7t$, $y = 5st$

$\frac{\partial z}{\partial x} = 6x$ $\quad \frac{\partial x}{\partial s} = 2$

//

K.10

Aturan Rantai

✓ Untuk Fungsi implisit 2 peubah

$$\boxed{\frac{dy}{dx} = -\frac{\partial F/\partial x}{\partial F/\partial y}}$$

contoh:

1. $f(x,y) = x^3 - x^2 y - 10y^4 = 0$

◦ menggunakan turunan implisit.

$3x^2 \frac{dx}{dx} - 2xy\frac{dx}{dx} - x^2\frac{dy}{dx} - 40y^3\frac{dy}{dx} = 0$

$\frac{dy}{dx}(-x^2 - 40y^3) = 2xy - 3x^2$

$\frac{dy}{dx} = \frac{2xy - 3x^2}{-x^2 - 40y^3}$

$\frac{dy}{dx} = \frac{3x^2 - 2xy}{x^2 + 40y^3}$ //

◦ menggunakan aturan rantai.

$\frac{dy}{dx}$?

K-10 PMK.

- lanjutan jawaban soal

$$\frac{\partial F}{\partial x} = 3x^2 - 2xy$$

$$\frac{\partial F}{\partial y} = -x^2 - 40y^3$$

$$\frac{dy}{dx} = -\frac{(3x^2 - 2xy)}{-x^2 - 40y^3}$$

$$\frac{dy}{dx} = \frac{-3x^2 + 2xy}{-x^2 - 40y^3} \quad //$$

✓ Untuk fungsi implisit 3 peubah.

misal $f(z, x, y) = 0$

$$\boxed{\frac{\partial z}{\partial x} = -\frac{\partial F/\partial x}{\partial F/\partial z} \quad \Bigg| \quad \frac{\partial z}{\partial y} = -\frac{\partial F/\partial y}{\partial F/\partial z}}$$

Turunan Berarah

✓ $D_u f(x,y) = \langle F_x(x,y), F_y(x,y) \rangle . u$

$$\boxed{u = \frac{a}{|a|} \quad \Bigg| \quad |a| = \sqrt{a_i^2 + a_j^2}}$$

contoh:

1. $f(x,y) = 4x^2 - xy + 3y^2$.

$a = 4i + 3j$

turunan berarah? f di $(2,-1)$

$|a| = \sqrt{4^2 + 3^2}$

$= \sqrt{25}$

$= 5$

$u = \frac{a}{|a|} = \frac{4i + 3j}{5}$

$= \frac{4}{5}i + \frac{3}{5}j$

$\rightarrow u = \langle 4/5, 3/5 \rangle$

- $f_x(x,y) = 8x - y$

$f_x(2,-1) = 8 \cdot 2 - (-1) = 17$

- $f_y(x,y) = -x + 6y$

$f_y(2,-1) = -2 + 6(-1) = -8$

$\langle 17, -8 \rangle$

Sehingga,

$D_u f(x,y) = \langle F_x(x,y), F_y(x,y) \rangle^T \langle u \rangle$

$$= \begin{pmatrix} 17 & -8 \end{pmatrix} \begin{pmatrix} 4/5 \\ 3/5 \end{pmatrix}$$

$$= 17 \cdot \frac{4}{5} + -8 \cdot \frac{3}{5}$$

$$= 44/5 \quad //$$

2. $f(x,y,z) = xy \sin z$.

$a = 1i + 2j + 2k$

turunan berarah? f di $(1, 1, \frac{\pi}{2})$

$|a| = \sqrt{1^2 + 2^2 + 2^2}$

$= \sqrt{9}$

$= 3$

$u = \frac{a}{|a|} = \frac{1i + 2j + 2k}{3}$

$= \frac{1}{3}i + \frac{2}{3}j + \frac{2}{3}k$

$\rightarrow u = \langle \frac{1}{3}, \frac{2}{3}, \frac{2}{3} \rangle$

- $f_x(x,y,z) = y \sin z$

$f_x(1,1,\frac{\pi}{2}) = 1 \cdot \sin\frac{\pi}{2}$

$= 1$

- $f_y(x,y,z) = x \sin z$

$f_y(1,1,\frac{\pi}{2}) = 1 \cdot \sin\frac{\pi}{2}$

$= 1$

- $f_z(x,y,z) = xy \cos z$

$f_z(1,1,\frac{\pi}{2}) = 1 \cdot 1 \cdot \cos\frac{\pi}{2}$

$= 0$

$\langle 1, 1, 0 \rangle$

sehingga,

$$D_u f(x,y,z) = \begin{pmatrix} 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1/3 \\ 2/3 \\ 2/3 \end{pmatrix}$$

$$= \frac{1}{3} + \frac{2}{3} + 0$$

$$= 3/3$$

$$= 1$$

//

✓ Laju Perubahan Maksimum

$D_u f(P)$ maksimum ketika $\theta = 0$

minimum ketika $\theta = \frac{\pi}{2}$ //

$D_u f(P) = u \cdot \nabla f(P) =$

$\|u\| \cdot \|\nabla f(P)\| \cos\theta =$

$\|\nabla f(P)\| \cos\theta$

$\boxed{\nabla f = \text{gradien.}}$

contoh:

1. Seekor kutu berada pada parabola hiperbola $z = x^2 - y^2$ di titik $P(1,1,0)$. Pada arah mana seharusnya kutu bergerak untuk panjatan paling curam? kemiringan?

Jawab:

$D_u f' = \|\nabla f(x,y)\| \cos\theta$

$\theta = 0$

$D_u f' = \|\nabla f(x,y)\|$

$z = x^2 - y^2$

$f(x,y) = x^2 - y^2$

$f_x = 2x \rightarrow f_x(1,1,0) = 2$

$f_y = -2y \rightarrow f_y(1,1,0) = -2$ 1.

$D_u f = f_x \cdot a + f_y \cdot b$

$\nabla f = 2\hat{i} - 2\hat{j}$ //

$\|\nabla f\| = \sqrt{2^2 + (-2)^2}$

$= \sqrt{8}$

$= 2\sqrt{2}$ //

K.11. PMK

Maksimum & Minimum Fungsi

misal: $f_x(a,b) = 0$

$f_y(a,b) = 0$

$$\boxed{D = D(a,b) = f_{xx}(a,b)\, f_{yy}(a,b) - [f_{xy}(a,b)]^2}$$

1. Jika $D > 0$ dan $f_{xx}(a,b) > 0$ maka $f(a,b)$ minimum lokal.
2. Jika $D > 0$ dan $f_{xx}(a,b) < 0$ maka $f(a,b)$ maksimum lokal.
3. Jika $D < 0$, maka $f(a,b)$ bukan maksimum / minimum lokal.

Maks & Min Lokal.

catatan :

1. Jika $D < 0 \to$ titik (a,b) disebut titik pelana f.
2. Jika $D = 0$. pengujian tidak memberikan informasi.
3. $$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$$
$$= f_{xx} f_{yy} - (f_{xy})^2 \;//.$$

contoh :

① $f(x,y) = xy.$

$f_x(x,y) = y = 0$

$f_y(x,y) = x = 0$

titik kritis : $(0,0)$

$f_{xx}(x,y) = 0 \to f_{xx}(0,0) = 0$

$f_{yy}(x,y) = 0 \to f_{yy}(0,0) = 0$

$f_{xy}(x,y) = 1 \to f_{xy}(x,y) = 1.$

untuk titik $(0,0)$ $D = -1 \to$ titik pelana

② $f(x,y) = 3x^3 + y^2 - 9x + 4y.$

$f_x(x,y) = 9x^2 - 9 = 0$

$x^2 = 1$

$\boxed{x = \pm 1}$

$f_y(x,y) = 2y + 4 = 0$

$2y = -4$

$\boxed{y = -2}$

titik kritis : $(1,-2)$ & $(-1,-2)$

$f_{xx}(x,y) = 18x \to f_{xx}(1,-2) = 18$

$f_{xx}(-1,-2) = -18$

$f_{yy}(x,y) = 2$

$f_{xy}(x,y) = 0$

Sehingga :

untuk titik $(1,-2)$

$D = 18 \cdot 2 - 0^2 = 36$

$D > 0$, $f_{xx}(1,-2) > 0$ } minimum lokal

untuk titik $(-1,-2)$

$D = -18 \cdot 2 - 0^2 = -36$

$D < 0 \to$ titik pelana.

③ $f(x,y) = x^2 + y^2 + x^2 y + 4.$

$f_x(x,y) = 2x + 2xy$

$= 2x(1+y) = 0$

$2x = 0 \to 2x + 2xy = 0$

$\boxed{x = 0}$ $\quad 2 \cdot 0 + 2 \cdot 0 y = 0$

$\boxed{y = 0}$

$1 + y = 0$

$\boxed{y = -1}$

$f_y(x,y) = 2y + x^2 = 0$

$2(-1) + x^2 = 0$

$x^2 = 2$

$\boxed{x = \pm\sqrt{2}}$

titik kritis : $(0,0)$ & $(\pm\sqrt{2}, -1)$

Sehingga :

$f_{xx}(x,y) = 2+2y$

- $f_{xx}(0,0) = 2$
- $f_{xx}(-\sqrt{2},-1) = 0$
- $f_{xx}(\sqrt{2},-1) = 0$

$f_{yy}(x,y) = 2$

$f_{xy}(x,y) = 2x$

- $f_{xy}(0,0) = 0$
- $f_{xy}(\sqrt{2},-1) = 2\sqrt{2}$
- $f_{xy}(-\sqrt{2},-1) = -2\sqrt{2}$

Sehingga,

- $(0,0)$

  $D = 4$, $f_{xx}(x,y) > 0$

  $\hookrightarrow$ minimum lokal.

- $(\sqrt{2},-1)$

  $D = -(2\sqrt{2})^2 = -8$

  $D < 0 \rightarrow$ pelana

- $(-\sqrt{2},-1)$

  $D = -8$, $D < 0 \rightarrow$ pelana

Nilai Maks & Min Mutlak

misal. dari contoh di atas:

$f(x,y) = x^2 + y^2 + x^2y + 4.$

$D = \{(x,y) \mid |x| \leq 1, |y| \leq 1\}$

titik $(0,0)$ adalah titik kritis,

sedangkan $(\pm\sqrt{2},-1)$ diluar batas

ada 4 batas yg harus di periksa:

$x = \pm 1$, $y = \pm 1$

✓ untuk $x = -1$

$f(-1,y) = 1 + y^2 + y + 4$

$= y^2 + y + 5$, $|y| \leq 1$

Tk: $y = -0{,}5$, $f(-1,-0{,}5) = 4{,}75.$

Tus: $y = -1$ dan $y = 1$, $f(-1,-1) = 5$,

$f(-1,1) = 7.$

✓ untuk $x = 1$

$f(1,y) = 1 + y^2 + y + 4 = y^2 + y + 5$, $|y| \leq 1$

Tk: $y = -0{,}5$, $f(1,-0{,}5) = 4{,}75$

Tus: $y = -1$ dan $y = 1$, $f(1,-1) = 5$,

$f(1,1) = 7.$

✓ untuk $y = -1$

$f(x,-1) = x^2 + 1 - x^2 + 4 = 5$

Tus: $x = -1$ dan $x = 1$, $f(1,-1) = 5$

$f(1,-1) = 5$

✓ untuk $y = 1$

$f(x,1) = x^2 + 1 + x^2 + 4 = 2x^2 + 5$

Tk: $x = 0$. $f(0,1) = 5$

Tus: $x = -1$ dan $x = 1$, $f(-1,1) = 7$,

$f(1,1) = 7.$

Pada titik $(0,0)$ nilai $f(x,y) = f(0,0) = 4$

Pada: $x = -1$, nilai terbesar $= 7$, terkecil $= 4{,}75$

Pada: $x = 1$, nilai terbesar $= 7$, terkecil $= 4{,}75$

Pada $y = -1$, terbesar $= 5$, terkecil $= 5$

Pada $y = 1$, terbesar $= 7$, terkecil $= 5.$

$\therefore$ maks global $= 7$, min global $= 4.$

K.12 PMK

Metode Pengali Lagrange

- Kasus 2 kendali

a. tentukan titik $(x,y,z) \in D$.

Sehingga, $\nabla f(x,y,z) = \lambda \nabla g(x,y,z) + \mu \nabla h(x,y,z)$

dengan: $g(x,y,z) = k$ dan $h(x,y,z) = c$.

b. hitung f di semua titik (x,y,z) dari (a).

Terbesar → nilai maks f.

Terkecil → nilai min f.

catatan.

$\nabla f(x,y,z) = \lambda \cdot \nabla g(x,y,z) + \mu \nabla h(x,y,z)$

dengan $g(x,y,z) = k$ dan $h(x,y,z) = c$

berakibat,

$f_x(x,y,z) = \lambda g_x(x,y,z) + \mu h_x(x,y,z)$

$f_y(x,y,z) = \lambda g_y(x,y,z) + \mu h_y(x,y,z)$

$f_z(x,y,z) = \lambda g_z(x,y,z) + \mu h_z(x,y,z)$

$g(x,y,z) = k \qquad h(x,y,z) = c$

contoh.

① memaksimumkan $f(x,y,z) = 4y - 2z$ dengan kendala $2x - y - z = 2$ dan $x^2 + y^2 = 1$

Jawab:

- $f_x = \lambda g_x + \mu h_x$

$0 = 2\lambda + 2x\mu \quad \to f_x$

$4 = -\lambda + 2y\mu \quad \to f_y$

$-2 = -\lambda \quad \to f_z$

$\boxed{\lambda = 2}$

- $0 = 2\lambda + 2x\mu$

$0 = 2 \cdot 2 + 2x\mu$

$-\frac{4}{2\mu} = x$

$\boxed{x = -\frac{2}{\mu}}$

- $4 = -\lambda + 2y\mu$

$4 + 2 = 2y\mu$

$\boxed{\frac{3}{\mu} = y}$

- sehingga, $x^2 + y^2 = 1$

$\left(\frac{-2}{\mu}\right)^2 + \left(\frac{3}{\mu}\right)^2 = 1$

$\frac{4}{\mu^2} + \frac{9}{\mu^2} = 1$

$\frac{13}{\mu^2} = 1$

$\mu^2 = 13$

$\boxed{\mu = \pm\sqrt{13}}$

- untuk $\mu = \sqrt{13} \to x = -\frac{2}{\sqrt{13}} \quad y = \frac{3}{\sqrt{13}}$

- untuk $\mu = -\sqrt{13} \to x = \frac{2}{\sqrt{13}} \quad y = -\frac{3}{\sqrt{13}}$

Bila disubstitusikan ke pers:

$2x - y - z = 2$

① $-\frac{4}{\sqrt{13}} - \frac{3}{\sqrt{13}} - z = 2$

$\boxed{z = -2 - \frac{7}{\sqrt{13}}}$

$\mu = -\sqrt{13}$

$x = \frac{2}{\sqrt{13}}$

$y = -\frac{3}{\sqrt{13}}.$

② $\frac{4}{\sqrt{13}} + \frac{3}{\sqrt{13}} - z = 2$

$$\boxed{z = -2 + \frac{7}{\sqrt{13}}}$$

maka $f(x,y,z) = 4y - 2z$.

- $f\left(-\frac{2}{\sqrt{13}}, \frac{3}{\sqrt{13}}, -2-\frac{7}{\sqrt{13}}\right) = 4 + \frac{26}{\sqrt{13}}$

$= 11{,}211 \rightarrow \max$

- $f\left(\frac{2}{\sqrt{13}}, -\frac{3}{\sqrt{13}}, -2+\frac{7}{\sqrt{13}}\right) = 4 - \frac{26}{\sqrt{13}}$

$= -3.2111 \rightarrow \min$

latihan.

1. $f(x,y,z) = x+2y+3z$ terhadap.

$g(x,y,z) = x^2+y^2-2=0$ dan

$h(x,y,z) = y+z-1=0$

max & min?

Jawab :

Pers. lagrange :

1. $1 = 2x\lambda + 0\mu$
2. $2 = 2y\lambda + \mu$
3. $3 = 0\lambda + \mu \rightarrow \boxed{\mu = 3.}$
4. $x^2+y^2 = 2$
5. $y+z=1$

dan pers (3) subtitusi ke (2)

1. $2x\lambda = 1 \rightarrow x = \frac{1}{2\lambda}$

2. $2y\lambda + 3 = 2 \rightarrow y = \frac{-1}{2\lambda}$

Substitusi ke. 4 :

$$\left(\frac{1}{2\lambda}\right)^2 + \left(\frac{-1}{2\lambda}\right)^2 = 2$$

$$\frac{1}{4\lambda^2} + \frac{1}{4\lambda^2} = 2$$

$$\frac{2}{4\lambda^2} = 2$$

$$\boxed{\lambda = \pm\frac{1}{2}}$$

✓ Jika $\lambda = \frac{1}{2}$

maka $x = \frac{1}{2\lambda} \Rightarrow x = 1$

$y = \frac{-1}{2\lambda} \Rightarrow y = -1$

✓ Jika $\lambda = \frac{-1}{2}$

maka $x = \frac{1}{2\lambda} \rightarrow x = -1$

$y = \frac{-1}{2\lambda} \Rightarrow y = 1$

bila disubstitusikan ke. pers:

$f(x,y,z) = x+2y+3z$

- $y+z-1=0$
  $-1+z-1=0$
  $\boxed{z = 2}$

- $y+z-1=0$
  $1+z-1=0$
  $\boxed{z = 0}$

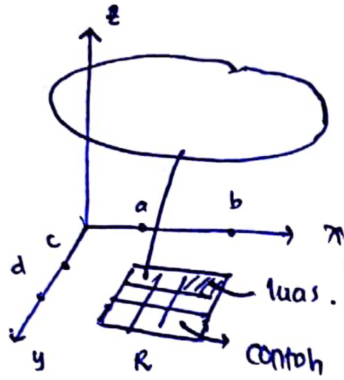
$f(1,-1,2) = 1+2(-1)+3(2)$

$= 5 \rightarrow \max$

$f(-1,1,0) = -1+2(1)+3(0)$

$= 1 \rightarrow \min.$

Integral Lipat



luas. $\boxed{\Delta A = \Delta x \cdot \Delta y}$

contoh titik sampel (peletakkan bebas)

$$\boxed{R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]}$$

$[a,b]$ dibagi sebanyak m.

$$\boxed{\Delta x = \frac{(b-a)}{m}}$$

$[c,d]$ dibagi sebanyak n

$$\boxed{\Delta y = \frac{(d-c)}{n}}$$

$$\boxed{V = \iint_R f(x,y)\, dA}$$

$$V = \lim_{m,n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A$$

contoh:

① $R = [0,2] \times [0,2]$ → bujur sangkar.

$z = 16 - x^2 - 2y^2$

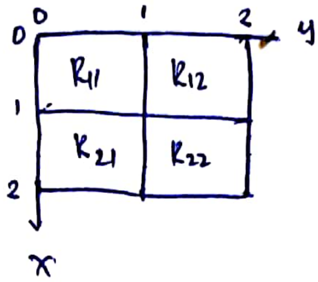
sampel : pojok kanan atas.

Volume ?

Jawab :

$R = [0,2] \times [0,2] \rightarrow m = 2$, $n = 2$

$f(x,y) = 16 - x^2 - 2y^2$



$\Delta x = \frac{2-0}{2} = 1$

$\Delta y = \frac{2-0}{2} = 1$

$\Delta A = \Delta x \cdot \Delta y$

$= 1$

$R_{11} \rightarrow$ sampel $(0,1)$ $\quad R_{21} \rightarrow (1,1)$

$R_{12} \rightarrow$ sampel $(0,2)$ $\quad R_{22} \rightarrow (1,2)$

$$V = \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A$$

$= f(0,1) \cdot 1 + f(0,2) \cdot 1 +$

$\quad f(1,1) \cdot 1 + f(1,2) \cdot 1$

$= 14 + 8 + 13 + 7$

$= 42$ //

K. 13.

Pmk.

Integral Berulang.

Definisi

$\Delta V = A(y) \cdot \Delta y$

$V = \int_c^d A(y)\, dy.$

$\downarrow$

$\int_a^b f(x,y)\, dx$

$\downarrow$

$V = \int_c^d \int_a^b f(x,y)\, dx\, dy$

atau,

$V = \int_a^b \int_c^d f(x,y)\, dy\, dx$

contoh :

① $\iint_R 6xy^2\, dA = \int_2^4 \left[\int_1^2 6xy^2\, dy \right] dx$

$= \int_2^4 \left[\frac{6}{3} xy^3 \right]_1^2 dx$

$= \int_2^4 \left[2 \cdot 2^3 x - 2 \cdot 1^3 x \right] dx$

$= \int_2^4 14x\, dx$

$= 7x^2 \Big]_2^4$

$= 7 \cdot 16 - 7 \cdot 4$

$= 84$ //

② $\iint_R 2x - 4y^3\, dA = \int_{-5}^4 \left[\int_0^3 2x - 4y^3\, dy \right] dx$

$= \int_{-5}^4 \left[2xy - y^4 \right]_0^3 dx$

$= \int_{-5}^4 \left[6x - 81 \right] dx$

$= 3x^2 - 81x \Big]_{-5}^4$

$= (3 \cdot 4^2 - 81 \cdot 4) - (3(-5)^2 - 81(-5))$

$= -756$ //

③ $\iint_R x^2y^2 + \cos(\pi x) + \sin(\pi y)\, dA$

$= \int_{-2}^{-1} \int_0^1 x^2y^2 + \cos(\pi x) + \sin(\pi y)\, dy\, dx$

$= \int_{-2}^{-1} \left[\frac{x^2y^3}{3} + y\cos(\pi x) + \left(-\frac{1}{\pi}\cos(\pi y)\right) \right]_0^1 dx$

$= \int_{-2}^{-1} \left(\frac{x^2}{3} + \cos(\pi x) - \frac{1}{\pi} \underset{-1}{\underbrace{\cos \pi}} \right) - \left(0 + 0 - \frac{1}{\pi}\cos 0 \right) dx$

$= \int_{-2}^{-1} \frac{x^2}{3} + \cos(\pi x) + \frac{1}{\pi} - \left(-\frac{1}{\pi}\right) dx$

$= \int_{-2}^{-1} \frac{x^2}{3} + \cos(\pi x) + \frac{2}{\pi}\, dx$

$= \frac{x^3}{9} + \frac{1}{\pi}\sin(\pi x) + \frac{2}{\pi} x \Big]_{-2}^{-1}$

$= \left(\frac{(-1)^3}{9} + \frac{1}{\pi}\sin(-\pi) + \frac{2}{\pi}(-1) \right) - \left(\frac{(-2)^3}{9} + \frac{1}{\pi}\sin(-2\pi) + \left(-\frac{4}{\pi}\right) \right)$

$= \left(-\frac{1}{9} + \frac{1}{\pi} \cdot 0 - \frac{2}{\pi} \right) - \left(-\frac{8}{9} + \frac{1}{\pi} \cdot 0 - \frac{4}{\pi} \right)$

$= \frac{7}{9} + \frac{2}{\pi}$ //

4. $\iint_R \frac{1}{(2x+3y)^2}\, dA$

$= \int_0^1 \int_1^2 (2x+3y)^{-2}\, dy\, dx$

$= \int_0^1 -\frac{1}{3(2x+3y)} \Big]_1^2 \, dx$

$= \int_0^1 \frac{-1}{6x+18} - \left(-\frac{1}{6x+9} \right) dx$

$= \int_0^1 -\frac{1}{6x+18} + \frac{1}{6x+9}\, dx$

$= -\frac{1}{6}\left[\ln(6x+18)\right] + \frac{1}{6}\left[\ln(6x+9)\right]\Big]_0^1$

$= \left[-\frac{1}{6}(\ln 24 - \ln 18) + \frac{1}{6}(\ln 15 - \ln 9)\right]$ //

✓ Integral Lipat 2 pada daerah

Umum

- Jenis I

$D = \{(x,y) \mid a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$

$= \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\, dy\, dx.$

- Jenis 2

$D = \{(x,y) \mid h_1(y) \le x \le h_2(y),\ c \le y \le d\}$

$= \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\, dx\, dy$

contoh:

① $\iint (4x + 10y)\, dA$

$D = \{(x,y) \mid 3 \le x \le 5,\ -x \le y \le x^2\}$

Jawab:

$\int_3^5 \int_{-x}^{x^2} (4x + 10y)\, dy\, dx$

$= \int_3^5 4xy + 5y^2 \Big|_{-x}^{x^2} dx$

$= \int_3^5 4x^3 + 5x^4 - (-4x^2 + 5x^2)\, dx$

$= \int_3^5 5x^4 + 4x^3 - x^2\, dx$

$= x^5 + x^4 - \frac{1}{3}x^3 \Big|_3^5$

$= \left(3125 + 625 - \frac{125}{3}\right) - \left(243 + 81 - 9\right)$

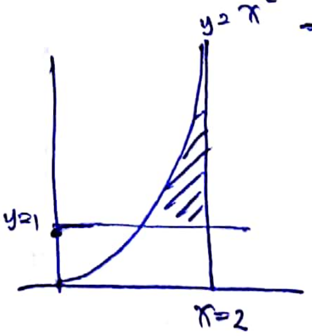
$= 3393,33$

② $\iint (x^2 + y^2)\, dA$

dibatasi oleh : $y = x^2$ & $x = 2$ & $y = 1$.

Jawab:

$y = x^2 \rightarrow x = \sqrt{y}$



Jenis 1 : $D = 1 \le x \le 2$

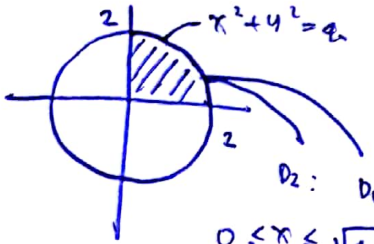
$1 \le y \le x^2$

Jenis 2 : $D = 1 \le y \le 4$

$\sqrt{y} \le x \le 2$

contoh :

① carilah volume benda pejal di oktan I ($x \geq 0$, $y \geq 0$, $z \geq 0$) yang dibatasi oleh parabola $z = \underbrace{x^2 + y^2}_{f(x,y)}$, tabung $x^2 + y^2 = 4$.



D_1 : $0 \leq x \leq 2$; $0 \leq y \leq \sqrt{4 - x^2}$

D_2 : $0 \leq x \leq \sqrt{4 - y^2}$; $0 \leq y \leq 2$

R8.

PMK

Aturan Rantai.

- Jenis 1.

misal : $z(x,y)$, $x(t)$, $y(t)$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial t}.$$

soal:

1. Tentukan $\frac{dz}{dt}$. Jika $z = x^4 y^2$, $x=t$, $y = et^2$.

Jawab:

$$\frac{dz}{dt} = \frac{\partial(x^4y^2)}{\partial x} \cdot \frac{\partial(t)}{\partial t} + \frac{\partial(x^4y^2)}{\partial y} \cdot \frac{\partial(et^2)}{\partial t}$$

$$= 4x^3y^2 \cdot 1 + 2x^4y \cdot 2te$$

$$= 4t^3 e^2 t^4 + 4t^4 et^2 \cdot te$$

$$= 4t^7 e^2 + 4t^7 \cdot e^2$$

$$= 8t^7 e^2 \quad //$$

2. tentukan $\frac{dz}{dt}$, Jika $z = 3xy$, $x = 4t$, $y=2$

Jawab:

$$\frac{dz}{dt} = \frac{\partial(3xy)}{\partial x} \cdot \frac{\partial(4t)}{\partial t} + \frac{\partial(3xy)}{\partial y} \cdot \frac{\partial(2)}{\partial t}$$

$$= 3y \cdot 4 + 3x \cdot 0$$

$$= 12y$$

$$= 24 \quad //$$

3. $z = x^2 + 2y^2$, $x = 3s + 4t$. $y = 6s^2t^3$

a. $\frac{\partial z}{\partial x} = 2x$, $\frac{\partial z}{\partial y} = 4y$

$$\frac{\partial z}{\partial t} = 2x \cdot 4 + 4y \cdot 18s^2t^2$$

$$= 8x + 72ys^2t^2$$

$$= 8(3s+4t) + 72(6s^2t^3)s^2t^2$$

$$= 24s + 32t + 432s^4t^5 \quad //$$

b. $$\frac{\partial z}{\partial s} = 2x \cdot 3 + 4y \cdot 12st^3$$

$$= 6x + 48yst^3$$

$$= 6(3s+4t) + 48(6s^2t^3)st^3$$

$$= 18s + 24t + 288s^3t^6 \quad //$$

- Aturan Rantai Untuk Fungsi Eksplisit.

misal terdapat $f(z,x,y)$

$\frac{\partial z}{\partial x} = -\frac{\partial F/\partial x}{\partial F/\partial z}$	$\frac{\partial z}{\partial y} = -\frac{\partial F/\partial y}{\partial F/\partial z}$

contoh:

1. tentukan $\frac{dy}{dx}$ jika $F(x,y) = x^3 - 4xy^2 = 0$

Jawab:

$\frac{\partial F}{\partial x} = 3x^2 - 4y^2 = 0$

$\frac{\partial F}{\partial y} = -8xy = 0$

$$\frac{dy}{dx} = \frac{-\partial F/\partial x}{\partial F/\partial y} = \frac{(3x^2 - 4y^2)}{-8xy}$$

2. $z = x^2 + 2y$.

$x^2 + 2y - z = 0$

$\frac{\partial F}{\partial x} = 2x$

$\frac{\partial F}{\partial y} = 2$

$\frac{\partial F}{\partial z} = -1$

$$\frac{dy}{dx} = \frac{-\partial F/\partial x}{\partial F/\partial y} = \frac{-2x}{2} = -x$$

- Turunan Berarah.

- $D_u F(x,y) = \langle F_x(x,y), F_y(x,y) \rangle \cdot u$
- $u = \frac{a}{|a|}$ • $|a| = \sqrt{a_1^2 + a_2^2}$

contoh:

1. Turunan berarah F di (3,1) pada arah vektor $a = 4i + 3j$, Jika $F(x,y) = 4x^2 - 3yx^2$.

Jawab:

$|a| = \sqrt{4^2+3^2}$
$= \sqrt{25}$
$= 5$

$u = \frac{a}{|a|}$
$= \frac{4i+3j}{5}$
$= \frac{4}{5}i + \frac{3}{5}j$

$\boxed{u = \langle 4/5, 3/5 \rangle}$

$Fx(x,y) = 8x - 6yx$.
$Fy(x,y) = -3x^2$.

F di (3,1) → $Fx(3,1) = 8.3 - 6.1.3$
$= 6$
$Fy(3,1) = -3.3^2$
$= -27$.

$D_u F(x,y) = \langle Fx(x,y), Fy(x,y) \rangle . u$
$= \langle 6, -27 \rangle \langle 4/5, 3/5 \rangle$
$= 6.\frac{4}{5} - 27.\frac{3}{5}$
$= -\frac{57}{5}$ //

K.10.

Persamaan Bidang Singgung

$$z - z_0 = Fx(x_0, y_0)(x - x_0) + Fy(x_0, y_0)(y - y_0)$$

contoh:

1. Tentukan bidang singgung kurva $z = 2x^2 + y^2$ di titik (1,1,3)

Jawab:

$fx(x,y) = 4x$ $\quad fy(x,y) = 2y$
$fx(1,1) = 4$ $\quad fy(1,1) = 2$

Pers. bidang singgung di titik (1,1,3)

$z - z_0 = fx(x_0, y_0)(x - x_0) + fy(x_0, y_0)(y - y_0)$
$z - 3 = 4(x-1) + 2(y-1)$
$z = 4x - 4 + 2y - 2 + 3$
$z = 4x + 2y - 3$ //

2. Tentukan bidang singgung kurva $z = 3x^3 - 2xy^2$ di titik (2,-1,4)

Jawab:

$fx(x,y) = 9x^2 - 2y^2$ $\quad fy(x,y) = -4xy$
$fx(2,-1) = 36 - 2$ $\quad fy(2,-1) = 8$.
$= 34$

Pers bidang singgungnya:

$z - 4 = 34(x-2) + (8)(y - (-1))$
$z - 4 = 34x - 68 + (8y + 8)$
$z - 4 = 34x + 8y - 60$
$z = 34x + 8y - 56$

R.10. PMK.

Hampiran Linear.

$\Delta z = z_2 - z_1$

$$\boxed{dz = f_x(x,y)\,dx + f_y(x,y)\,dy}$$

$dx = \Delta x$, $dy = \Delta y$.

contoh:

1. $z = f(x,y) = 2x^3 + xy - y^3$.

Hitung Δz dan dz bila (x,y) berubah dari $(2,1)$ ke $(2{,}03,\ 0{,}98)$.

Jawab:

$\Delta z = z_1 - z_0$

$= f(x_1, y_1) - f(x_0, y_0)$

$= [2(2{,}03)^3 + (2{,}03)(0{,}981) - (0{,}981)^3] - [2(2)^3 + (2)(1) - (1)^3]$

$= 17{,}779062 - 17$

$= 0{,}779062$

atau.

$dz = f_x(x,y)\,dx + f_y(x,y)\,dy$

$= \underbrace{(6x^2+y)}_{x_0,y_0}(2{,}03-2) + \underbrace{(x-3y^2)}_{x_0,y_0}(0{,}98-1)$

$= (6(2^2)+1)(0{,}03) + (2-3(1^2))(-0{,}02)$

$= 0{,}75 + 0{,}02$

$= 0{,}77$

2. dik: $p = 30$ cm, $dp = 0{,}1$
$l = 24$ cm, $dl = 0{,}1$

$L = p \times l = 30 \times 24 = 720\ cm^2$

$dL = f_p(p,l)\,dp + f_l(p,l)\,dl$

$= l.\ \Delta p + p\ \Delta l$

$= 24(0{,}1) + 30(0{,}1)$

$= 2{,}4 + 3$

$= 5{,}4$.

$L\ max = 720 + 5{,}4$

$L\ min = 720 - 5{,}4$.

Aturan Rantai Untuk F. implisit.

$$\frac{\partial z}{\partial x} = -\frac{\partial F/\partial x}{\partial F/\partial z} \quad \& \quad \frac{\partial z}{\partial y} = -\frac{\partial F/\partial y}{\partial F/\partial z}$$

Contoh:

1. $f(x,y) = x^3 - 4xy^2 = 0$ $\quad \dfrac{dy}{dx}$?

Jawab:

$$\frac{dy}{dx} = -\frac{\partial F/\partial x}{\partial F/\partial y} = -\frac{3x^2 - 4y^2}{-8xy}.$$

Turunan Berarah.

$$\boxed{D_u f(x,y) = f_x(x,y)\,a + f_y(x,y)\,b.}$$

contoh:

1. Turunan berarah f di $(3,1)$ pada vektor $a = 4i + 3j$ jika $f(x,y) = 4x^2 - 3yx^2$.

$|a| = \sqrt{4^2+3^2}$
$= \sqrt{25}$
$= 5$

$u = \dfrac{a}{|a|} = \dfrac{4i+3j}{5}$

$= \frac{4}{5}i + \frac{3}{5}j$

$u = \langle 4/5,\ 3/5 \rangle$

- $f_x(x,y) = 8x - 6xy$

$f_x(3,1) = 8(3) - 6(3)(1)$

$= 6$

- $f_y(x,y) = -3x^2$

$f_y(3,1) = -3(3)^2$

$= -27$

$\langle 6, -27 \rangle$

$D_u f(x,y) = \langle F_x(x,y), F_y(x,y) \rangle \langle u \rangle$

$= \langle f_x(x,y), f_y(x,y) \rangle^T (u)$

$= (6 \quad -27) \begin{pmatrix} 4/5 \\ 3/5 \end{pmatrix}$

$= 6 \cdot 4/5 + -27 \cdot 3/5$

$= -57/5$ //

RII.

pmk

Maximum & Minimum Lokal

untuk $f(x,y)$

- Uji turunan 1

$$\boxed{f_x(x,y) = 0 \ \& \ f_y(x,y) = 0}$$

- Uji turunan 2

$$\boxed{D = f_{xx}(x,y) \cdot f_{yy}(x,y) - [f_{xy}(x,y)]^2}$$

1. Jika $D > 0$ dan $f_{xx}(x,y) > 0$, maka $f(x,y)$ adalah min. lokal.
2. Jika $D > 0$ dan $f_{xx}(x,y) < 0$, maka $f(x,y)$ adalah maks. lokal.
3. Jika $D < 0$, maka $f(x,y)$ bukan maks / min. lokal.

catatan:

- dalam kasus (3) titik (x,y) disebut titik pelana f.
- Jika $D = 0$, tidak memberi informasi.

contoh:

1. $f(x,y) = 3x^3 + y^2 - 9x + 4y$

Jawab:

- uji turunan 1

$f_x = 9x^2 - 9 = 0$

$\boxed{x = \pm 1}$

$f_y = 2y + 4 = 0$

$2y = -4$

$\boxed{y = -2}$

titik: $(1,-2)$ dan $(-1,-2)$

- $f_{xx} = 18x \rightarrow f_{xx}(1,-2) = 18$

$f_{xx}(-1,-2) = -18$

$f_{yy} = 2 \rightarrow f_{yy}(1,-2) = 2$

$f_{yy}(-1,-2) = 2$

$f_{xy} = 0 \rightarrow f_{xy}(1,-2) = 0$

$f_{xy}(-1,-2) = 0$

$D_1 = 18 \cdot 2 - 0 = 36$

$D_2 = -18 \cdot 2 - 0 = -36 \rightarrow D < 0$ titik pelana.

$f_{xx} > 0$, $D > 0$

- Nilai Maksimum & Minimum

contoh:

1. $f(x,y) = x^2 + y^2 + x^2 y + 4$

Jawab: $f_x(x,y) = 2x + 2xy$

$= 2x(1+y) = 0$

$x = 0$

$y = -1$

$f_y(x,y) = 2y + x^2 = 0$

$y = 0$

$x = \pm\sqrt{2}$

titik kritis: $(0,0)$ & $(\pm\sqrt{2}, -1)$

R.11. pmk

lanjutan soal:

$f_{xx}(x,y) = 2+2y \rightarrow f_{xx}(0,0)=2$

$f_{xy}(x,y) = 2x \rightarrow f_{xy}(0,0)=0$

$f_{yy}(x,y) = 2 \rightarrow f_{yy}(0,0)=2$

$f_{xx}(-\sqrt{2},-1) = -2$	$f_{xx}(\sqrt{2},-1) = -2$
$f_{xy}(-\sqrt{2},-1) = -2\sqrt{2}$	$f_{xy}(\sqrt{2},-1) = 2\sqrt{2}$
$f_{yy}(-\sqrt{2},-1) = 2$	$f_{yy}(\sqrt{2},-1) = 2$

- untuk titik $(0,0)$, $D=4 \rightarrow$ minimum
  untuk titik $(-\sqrt{2},-1)$ $D=-4 \rightarrow$ pelana
  untuk titik $(\sqrt{2},-1)$ $D=-4 \rightarrow$ pelana

✓ Nilai maksimum & Minimum

Mutlak

contoh:

1. $f(x,y) = x^3 - 12x + y^3 + 3y^2 - 9y$
   pada $[-2,2] \times [-2,2]$

Jawab:

$f_x(x,y) = 3x^2 - 12 = 0$

$f_y(x,y) = 3y^2 + 6y - 9 = 0$

$3x^2 - 12 = 0$ | $3y^2 + 6y - 9 = 0$

$3x^2 = 12$ | $y^2 + 2y - 3 = 0$

$x^2 = 4$ | $(y+3)(y-1) = 0$

$\boxed{x = \pm 2}$ | $y = -3$ $\boxed{y=1}$

↳ tidak masuk selang.

titik kritis: $\{(-2,1), (2,1)\}$

$f(2,1) = -21$

$f(-2,1) = -11$

- Ada 3 batas.

$\boxed{x=2}$

$F(2,y) = -16 + y^3 + 3y^2 - 9y = g(y)$

$g'(y) = 3y^2 + 6y - 9 = 0$

$y^2 + 2y - 3 = 0$

$(y+3)(y-1) = 0$

$y = -3$ $\boxed{y=1}$

$\boxed{\text{TUS} = 1} \rightarrow \boxed{g(1) = -21}$

$\boxed{x=-2}$

$F(-2,y) = 16 + y^3 + 3y^2 - 9y = g(y)$

$g'(y) = 3y^2 + 6y - 9 = 0$

$(y+3)(y-1) = 0$

$y = -3$ $\boxed{y=1}$

$\boxed{g(1) = 11}$

$y = 1$

$F(x,1) = x^3 - 12x - 5 = g(x)$

$g'(x) = 3x^2 - 12 = 0$

$x^2 - 4 = 0$

TUS $\rightarrow$ $\boxed{x = \pm 2}$

$\boxed{\begin{array}{l} g(2) = -21 \\ g(-2) = 11 \end{array}}$

- Pada titik $(2,1) \rightarrow f(x,y) = -21$
  pada titik $(-2,1) \rightarrow f(x,y) = 11$

Pada $\boxed{x=2}$

maks $= -21$

min $= -21$

Pada $\boxed{x=-2}$

maks $= 11$

min $= 11$

pada $\boxed{y=1}$

maks $= 11$

min $= -21$ //

R.12 pmk

Metode Pengali Lagrange.

menentukan max / min $f(x, y, z)$,

dengan kendala $g(x, y, z)$.

1. $f_x(x, y, z) = g_x(x, y, z)\lambda$
2. $f_y(x, y, z) = g_y(x, y, z)\lambda$
3. $f_z(x, y, z) = g_z(x, y, z)\lambda$
4. $g(x, y, z) = k$.

contoh:

① $f(x, y) = xy$, kendala $g(x, y) = \sqrt{x^2 + y^2} = 2$

$x^2 + y^2 = 4$.

Sehingga:

a. $f_x = y = \lambda(2x)$

b. $f_y = x = \lambda(2y)$

c. $= 2(2x\lambda)\lambda$

$x = 4x\lambda^2$

$\frac{1}{4} = \lambda^2$

$\boxed{\lambda = \frac{1}{2}}$

$y = 2x\lambda$

$y = 2x\left(\frac{1}{2}\right)$

$\boxed{x = y}$

d. $x^2 + y^2 = 4$

$x^2 + x^2 = 4$

$x^2 = 2$

$y = \boxed{x = \sqrt{2}}$

$\boxed{f(\sqrt{2}, \sqrt{2}) = 2.}$

$\downarrow$ luas.

② $f(x, y) = 5x - 3y$, kendala $= x^2 + y^2 = 136$.

a. $f_x = 5 = 2x\lambda \rightarrow \lambda = \frac{5}{2x}$

b. $f_y = -3 = 2y\lambda \rightarrow \lambda = \frac{-3}{2y}$

c. $\frac{5}{2x} = \frac{-3}{2y} \rightarrow \boxed{x = -\frac{5}{3}y}$

d. $g(x, y) = x^2 + y^2 = 136$.

$\left(-\frac{5}{3}y\right)^2 + y^2 = 136$

$\left(\frac{25}{9} + 1\right) y^2 = 136$.

$y^2 = \frac{136 \cdot 9}{34}$

$y^2 = 36$

$\boxed{y = \pm 6}$

- $x = -\frac{5}{3}(\pm 6)$

$\boxed{x = \pm 10}$

Sehingga, $f(10, 6) = 32$

$f(10, -6) = -68 \rightarrow$ min.

$f(-10, 6) = 68 \rightarrow$ maks

$f(-10, -6) = -32$

③ $f(x, y) = y^2 - x^2$

kendala : $g(x) = \frac{x^2}{4} + y^2 = 1$

a. $-2x = \lambda\left(\frac{x}{2}\right) \rightarrow \lambda(2y) - 2y = 0$

b. $2y = \lambda(2y)$

$2y(\lambda - 1) = 0$

$\boxed{y = 0}$ $\boxed{\lambda = 1}$

c. $\frac{x^2}{4} + y^2 = 1$

$\frac{x^2}{4} + 0 = 1 \rightarrow x^2 = 4$

$\boxed{x = \pm 2}$

R.13.

Integral Lipat Dua

contoh:

① $\iint_A (4x+10y)\, dA$,

$D = \{(x,y) \mid 3 \le x \le 5, -x \le y \le x^2\}$

Jawab:

$\int_3^5 \int_{-x}^{x^2} (4x+10y)\, dy\, dx =$

$\int_3^5 [4xy + 5y^2]_{-x}^{x^2}\, dx$

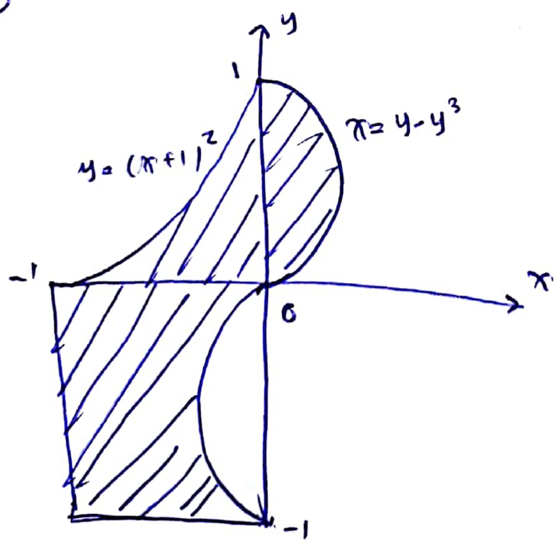
$= \int_3^5 [(4x(x^2)+5(x^2)^2) - (4x(-x)+5(-x)^2)]\, dx$

$= \int_3^5 5x^4 + 4x^3 - x^2\, dx.$

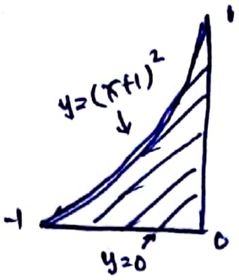
$= x^5 + x^4 - \frac{1}{3}x^3]_3^5$

$= 3393,33$ //

②



Jawab:



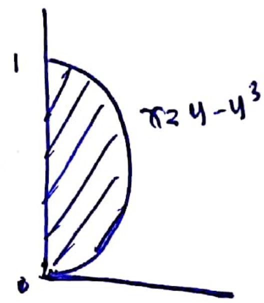
$= \int_{-1}^{0} \int_{0}^{(x+1)^2} y\, dy\, dx$

$= \int_{-1}^{0} [\frac{1}{2}y^2]_0^{(x+1)^2}\, dx$

$= \int_{-1}^{0} \frac{1}{2}(x+1)^4\, dx$

$= \frac{1}{10}(x+1)^5 \Big|_{-1}^{0}$

$= \frac{1}{10}$ //



$0 \le y \le 1$

$0 \le x \le y-y^3$

$= \int_0^1 \int_0^{y-y^3} y-y^3\, dx\, dy$

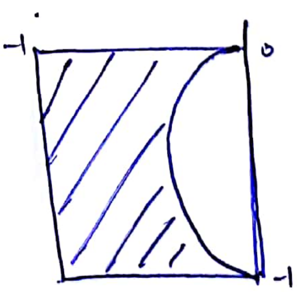
$= \int_0^1 xy - xy^3]_0^{y-y^3}\, dy$

$= \int_0^1 (y^2 - 2y^4 + y^6)\, dy$

$= \frac{1}{3}y^3 - \frac{2}{5}y^5 + 17]_0^1$

$= \frac{1}{3} - \frac{2}{5} + 17$

$= \frac{22}{105}$ //



$= \int_{-1}^{0} \int_{-1}^{y-y^3} y\, dx\, dy$

$= \int_{-1}^{0} [xy]_{-1}^{y-y^3}\, dy$

$= \int_{-1}^{0} (y^2 - y^4) - (-y)\, dy$

$= [\frac{1}{3}y^3 - \frac{1}{5}y^5 + \frac{1}{2}y^2]_{-1}^{0}$

$= -\frac{1}{30}$ //

R. 14.

pmk

Integral lipat 2 pada daerah Polar.

koordinat polar (r,θ) suatu titik

dikaitkan thd koordinat siku² (x,y)

Oleh pers:

$$\boxed{\begin{aligned} r^2 &= x^2+y^2 \\ x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}}$$

✓ perubahan ke koordinat polar dalam integral lipat 2.

$R = \{(r,\theta) \mid 0 \le a \le r \le b,\ \alpha \le \theta \le \beta\}$

dengan $0 \le \beta - \alpha \le 2\pi$, maka

$$\boxed{\iint_R f(x,y)\,dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta}$$

contoh:

① $z = e^{x^2+y^2}$, $R = \{(r,\theta),\ 1 \le r \le 3,\ 0 \le \theta \le \frac{\pi}{4}\}$

Jawab:
$r^2 = x^2+y^2$
$x = r\cos x$
$y = r\sin x$

$$V = \iint_R e^{x^2+y^2}\,dA$$

$$= \int_0^{\pi/4}\int_1^3 e^{r^2}\, r\,dr\,d\theta$$

$$= \int_0^{\pi/4}\int_1^9 e^u\, \cancel{r}\cdot\frac{du}{2\cancel{r}}\,d\theta$$

$$\boxed{\begin{aligned} &\text{mis: } u = r^2 \\ &du = 2r\,dr \\ &dr = \frac{du}{2r} \end{aligned}}$$

$$= \int_0^{\pi/4}\int_1^9 \frac{e^u}{2}\,du\,d\theta$$

$$= \frac{1}{2}\int_0^{\pi/4} \left[e^u\right]_1^9 d\theta$$

$$= \frac{1}{2}\int_0^{\pi/4} (e^9 - e^1)\,d\theta$$

$$= \frac{1}{2}\left[(e^9 - e)\,\theta\right]_0^{\pi/4}$$

$$= \frac{1}{2}(e^9 - e)\frac{\pi}{4} \quad //$$