ALIN

transformasi linear

CH. 8

* Misalkan $L : V \to W$ suatu transformasi linear, maka :

1) $L(v_1 + v_2) = L(v_1) + L(v_2)$, $\forall v_1, v_2 \in V$
2) $L(\alpha v) = \alpha L(v)$, untuk $v \in V$ dan skalar α.

* ruang nol (Kernel)

1) $Ker(L) = \{ v \in V \mid L(v) = \bar{0} \}$

* jangkauan / peta dari S

$L(S) = \{ w \in W \mid w = L(v)$ untuk suatu $v \in V \}$

contoh soal :

1) Tunjukan bahwa

$$L(f) = \int_0^1 f(x)\, d(x)$$

adalah TL dari $C[0,1]$ ke R^1.

1) $L(f+g) = L(f) + L(g)$
2) $L(\alpha f) = \alpha L(f)$

1) $L(f+g) = \int_0^1 (f+g)x \; dx$

$= \int_0^1 (f(x) + g(x))\, dx$

$= \int_0^1 f(x)\, dx + \int_0^1 g(x)\, dx$

$= L(f) + L(g)$ terbukti

2) $L(\alpha f) = \int_0^1 \alpha f(x)\, dx$

$= \alpha \int_0^1 f(x)\, dx$

$= \alpha L(f)$ terbukti

2) periksa apakah L merupakan TL dari R^3 ke R^2

$L(x) = (1 + x_1, x_2)^T$

$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$

$x + y = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \end{pmatrix}$

1) $L(x+y) = L(x) + L(y)$

$L(x+y) = L\begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \end{pmatrix}$

$= \begin{pmatrix} 1 + x_1 + y_1 \\ x_2 + y_2 \end{pmatrix}$

$L(x) + L(y) = L\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + L\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$

$= \begin{pmatrix} 1 + x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1 + y_1 \\ y_2 \end{pmatrix}$

$= \begin{pmatrix} 2 + x_1 + y_1 \\ x_2 + y_2 \end{pmatrix}$

$L(x+y) \neq L(x) + L(y)$ shg

$L(x) = (1 + x_1, x_2)^T$ bukan TL.

3) $L : P_3 \to P_3$ dengan $L(P(x)) = P(0)x + P(1)$

Ambil sembarang $p(x)$ dan $q(x) \in P_3$

(1) $L(p(x) + q(x)) = L(p(x)) + L(q(x))$

$L(p(x) + q(x)) = L((p+q)(x))$

$= (p+q)(0)(x) + (p+q)(1)$

$= p(0)x + q(0)x + p(1) + q(1)$

$= p(0)x + p(1) + q(0)x + q(1)$

$= L(p(x)) + L(q(x))$

(2) $L(\alpha p(x)) = \alpha L(p(x))$

$L(\alpha p(x)) = L((\alpha p)(x))$

$= (\alpha p)(0)(x) + (\alpha p)(1)$

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$= \alpha (P(0)x + P(1))$

$= \alpha L(P(x))$

$\therefore L(P(x))$ meruparan TL

b) Kernel

$= \{P(x) \mid L(P(x)) = 0\}$

$= \{P(x) \mid P(0)x + P(1) = 0\}$

$P(x) = ax^2 + bx + c$

$P(0) = c$

$P(1) = a + b + c$

$P(0)x + P(1) = 0$

$cx + a + b + c = 0x + 0$

$c = 0 \qquad a + b + c = 0$

$a + b = 0$

$b = -a$

misal $a = t$

$b = -t$

$c = 0$

kernel $= \{P(x) \mid c = 0, a = t, b = -t\}$

$= \{P(x) = tx^2 - tx\}$

c) $L\, P_3(x) = \{q(x) \mid q(x) = L(P(x))\}$

$= \{q(x) \mid q(x) = P(0)x + P(1)\}$

$= \{q(x) \mid q(x) = cx + a + b + c\}$

mis : $\ell = a + b + c$

$= \{q(x) \mid q(x) = cx + \ell\}$

Latihan soal

1. periksa apakah L merupakan TL dari R^2 ke R^3

a) $L(x) = (x_1, x_2, 1)^T$

$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \qquad L(x) = \begin{pmatrix} x_1 \\ x_2 \\ 1 \end{pmatrix}$

$y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \qquad x + y = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \end{pmatrix}$

$L(x+y) = L(x) + L(y)$

$L\begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ 1 \end{pmatrix}$

$L\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + L\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ 1 \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \\ 1 \end{pmatrix}$

$= \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ 2 \end{pmatrix}$

karna $L(x+y) \neq L(x) + L(y)$

maka $L(x) = (x_1, x_2, 1)^T$ bukan TL

b) $L(x) = (x_1, x_2, x_1 + 2x_2)^T$

$x = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \qquad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

$(x + y) = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \end{pmatrix}$

$\to L(x+y) = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_1 + y_1 + 2x_2 + 2y_2 \end{pmatrix}$

$L(x) + L(y) = \begin{pmatrix} x_1 \\ x_2 \\ x_1 + 2x_2 \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \\ y_1 + 2y_2 \end{pmatrix}$

$= \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_1 + y_1 + 2x_2 + 2y_2 \end{pmatrix}$

Terbukti

$\to L(\alpha \bar{x}) = \alpha L(\bar{x})$

$L(\alpha \bar{x}) = L\left(\alpha \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right)$

$= L\begin{pmatrix} \alpha x_1 \\ \alpha x_2 \end{pmatrix}$

$= \begin{pmatrix} \alpha x_1 \\ \alpha x_2 \\ \alpha x_1 + 2\alpha x_2 \end{pmatrix}$

$= \alpha \begin{pmatrix} x_1 \\ x_2 \\ x_1 + 2x_2 \end{pmatrix}$

$= \alpha L(x)$ Terbukti

shg $L(x) = (x_1, x_2, x_1 + 2x_2)^T$ adalah TL

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2. Periksa apakah L merupakan TL dari P_2 ke P_3

$L(P(x)) = P(x) + x\,P(x) + x^2\,P'(x)$

$P(x) + q(x) = (P+q)(x)$

$L(P(x) + q(x)) = L((P+q)x)$

$= (P+q)(x) + x(P+q)(x) + x^2(P+q)'(x)$

$= P(x) + q(x) + x\,(P(x) + q(x)) + x^2(P'(x) + q'(x))$

$= P(x) + xP(x) + x^2P'(x) + q(x) + x\,q(x) + x^2q'(x)$

$= L(P(x)) + L(q(x))$ terbukti

$*\ L(\alpha P(x)) = L((\alpha P)(x))$

$= (\alpha P)(x) + x\,(\alpha P)(x) + x^2(\alpha P)'(x)$

$= \alpha P(x) + \alpha\,x\,P(x) + \alpha\,x^2P'(x)$

$= \alpha\,(P(x) + x.P(x) + x^2\,P'(x))$

$= \alpha\,L(P(x))$ terbukti

$\therefore L(P(x)) = P(x) + x\,P(x) + x^2\,P'(x)$ merupakan TL.

3. Tentukan ruang nol dari TL

$L : P_3 \to P_3$ dengan $L(P(x)) = P(x) - P'(x)$

$L(P(x)) = 0$

$P(x) - P'(x) = 0$

ambil $P(x)$ dari P_3

$P(x) = ax^2 + bx + c$

$P'(x) = 2ax + b$

$P(x) - P'(x) = 0$

$ax^2 + bx + c - 2ax - b = 0$

$ax^2 + (b-2a)x + c - b = 0$

$a = 0 \qquad b - 2a = 0 \qquad c - b = 0$

$\qquad\qquad b = 0 \qquad\qquad c = 0$

kernel $= \{P(x) \mid a=0, b=0, c=0\}$

$= \{P(x) \mid P(x) = 0\}$

4. Tentukan peta dari TL

a) $L : R^3 \to R^3$ dengan $L(x) = (x_3, x_2, x_1)^T$

$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad L(x) = \begin{pmatrix} x_3 \\ x_2 \\ x_1 \end{pmatrix}$

mis: $x_1 = a \quad x_3 = c$

$x_2 = b$

Peta dari L: $\left\{ \begin{pmatrix} c \\ b \\ a \end{pmatrix} \middle|\ a, b, c \in R \right\}$

b) $L : P_3 \to P_3$ dengan $L(P(x)) + x.(P'(x))$

ambil $P(x)$ dan $q(x) \in P^3$

$P(x) = ax^2 + bx + c$

$P'(x) = 2ax + b$

$= \{q(x) \mid q(x) = L.P(x)\}$

$= \{q(x) \mid q(x) = x.P'(x)\}$

$= \{q(x) \mid q(x) = x\,(2ax + b)\}$

$= \{q(x) \mid q(x) = 2ax^2 + bx\}$

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Responsi - 9

LAMBANG MATRIKS

Jika $E = \{v_1, v_2, \ldots, v_n\}$ dan $F = \{w_1, w_2, \ldots w_m\}$ berturut - turut adalah basis berurut untuk ruang vektor V & W, maka untuk setiap TL : $L : V \to W$ terdapat matriks A berorde $m \times n$ shg :

$$[L(v)]_F = A[v]_E$$

untuk setiap $v \in V$

$$a_j = [L(v_j)]_F \; ; \; j = 1, 2, \ldots, n$$

Khusus untuk $L : R^n \to R^m$

$E = [u_1, u_2, u_3, \ldots, u_n]$

$F = [b_1, b_2, \ldots, b_m]$

maka : $a_j = B^{-1} L(u_j)$; $j = 1, 2, \ldots, n$

dengan $B = (b_1, b_2, \ldots b_m)$

$E = [v_1, v_2, \ldots, v_n]$
$F = [w_1, w_2, \ldots, w_m]$) basis terurut.

$L \to$ operator linear

$S \to$ matriks transisi (perubahan basis dari F ke E)

$A \to$

$B \to$ lambang matriks dari L relatif thd F

maka : $B = S^{-1} A . S$

misalkan A & B matriks" $n \times n$ matriks B dikatakan serupa (similar) dgn A jika terdapat matriks tak singular S shg

$$B = S^{-1} A S$$

CONTOH SOAL :

1) misalkan $E = [u_1, u_2, u_3]$ dan $F = [b_1, b_2]$ dengan :

$u_1 = (1, 0, -1)^T$, $u_2 = (1, 2, 1)^T$, $u_3 = (-1, 1, 1)^T$

$b_1 = (1, -1)^T$, $b_2 = (2, -1)^T$

dengan transformasi $R^3 \to R^2$ dan

$L(x) = (2x_2, -x_1)$

carilah lambang matriks L relatif thd basis - basis terurut E dan F.

$$L(u_1) = \begin{pmatrix} 0 \\ -1 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

$$\begin{array}{rl} \alpha + 2\beta &= 0 \\ -\alpha - \beta &= -1 \quad + \\ \hline \beta &= -1 \\ \alpha &= 2 \end{array} \qquad \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

$$L(u_2) = \begin{pmatrix} 4 \\ -1 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

$$\begin{array}{rl} \alpha + 2\beta &= 4 \\ -\alpha - \beta &= -1 \quad + \\ \hline \beta &= 3 \\ \alpha &= 4 - 6 = -2 \end{array} \qquad \begin{pmatrix} -2 \\ 3 \end{pmatrix}$$

$$L(u_3) = \begin{pmatrix} 2 \\ 1 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

$$\begin{array}{rl} \alpha + 2\beta &= 2 \\ -\alpha - \beta &= 1 \quad + \\ \hline \beta &= 3 \\ \alpha &= 2 - 6 = -4 \end{array} \qquad \begin{pmatrix} -4 \\ 3 \end{pmatrix}$$

$$A = \begin{pmatrix} 2 & -2 & -4 \\ -1 & 3 & 3 \end{pmatrix}$$

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cara 2 :

$B = \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}$

$B^{-1} = \frac{1}{1} \begin{pmatrix} -1 & -2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & -2 \\ 1 & 1 \end{pmatrix}$

$a_j = B^{-1} \; L(u)$

$L(u_1) = \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 & -2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

$L(u_2) = \begin{pmatrix} 4 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 & -2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 4 \\ -1 \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$

$L(u_3) = \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 & -2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -4 \\ 3 \end{pmatrix}$

$A = \begin{pmatrix} 2 & -2 & -4 \\ -1 & 3 & 3 \end{pmatrix}$

cara 3 :

akibat $(\underbrace{b_1, b_2}_{\text{dijadiin identitas}} \mid L(u_1), L(u_2), L(u_3))$

$\left(\begin{array}{cc|ccc} 1 & 2 & 0 & 4 & 2 \\ -1 & -1 & -1 & -1 & 1 \end{array}\right)$

$E_{12(2)} \left(\begin{array}{cc|ccc} -1 & 0 & -2 & 2 & 4 \\ -1 & -1 & -1 & -1 & 1 \end{array}\right)$

$E_{21(-1)} \left(\begin{array}{cc|ccc} -1 & 0 & -2 & 2 & 4 \\ 0 & -1 & 1 & -3 & 3 \end{array}\right)$

$\begin{matrix} E_{1(-1)} \\ E_{2(-1)} \end{matrix} \left(\begin{array}{cc|ccc} 1 & 0 & 2 & -2 & -4 \\ 0 & 1 & -1 & 3 & -3 \end{array}\right)$

2) misalkan $D[p(x)] = p'(x)$ pada P_3

a) tent matriks A yang melambangkan D terhadap $E = [1, x, x^2]$

$E = [1, x, x^2]$

$F = [1, 2x, 4x^2 - 2]$

$\rightarrow D(1) = 0 = \alpha(1) + \beta(x) + \gamma(x^2)$

$\downarrow$

turunan 1

$\alpha = 0 \quad \gamma = 0$

$\beta = 0$

$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\rightarrow D(x) = 1 = \alpha(1) + \beta(x) + \gamma(x^2)$

$\alpha = 1 \quad \beta = 0 \quad \gamma = 0$

$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$\rightarrow D(x^2) = 2x = \alpha(1) + \beta(x) + \gamma(x^2)$

$\alpha = 0 \quad \beta = 2 \quad \gamma = 0$

$\begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$

$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

b) tent matriks B yang melambangkan D thd $F = [1, 2x, ax^2 - 2]$

$D(1) = 0 = \alpha(1) + \beta(2x) + \gamma(4x^2 - 2)$

$\alpha = 0 \quad \beta = 0 \quad \gamma = 0$

$D(2x) = 2 = \alpha(1) + \beta(2x) + \gamma(4x^2 - 2)$

$\alpha = 2 \quad \beta = 0 \quad \gamma = 0$

$D(4x^2 - 2) = 8x = \alpha(1) + \beta(2x) + \gamma(4x^2 - 2)$

$\alpha = 0 \quad \beta = 4 \quad \gamma = 0$

$B = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix}$

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Responsi — ALIN

SEPULUH

ORTOGONALITAS

* hasil kali skalar dari vektor $x = (x_1, \dots, x_n)^T$ dan $y = (y_1, \dots, y_n)^T$ di R^n.

$$x^T y = x_1 y_1 + \dots + x_n y_n$$

* jika x dan y dua vektor tak nol di R^2 atau R^3 dan θ sudut di antaranya, maka :

$$x^T y = \|x\| . \|y\| . \cos\theta$$

$$\|x\| = \sqrt{(x^T x)} = \sqrt{x_1^2 + \dots + x_n^2}$$

PERTIDAKSAMAAN CAUCHY - SCHWARZ

Jika x dan y vertek di R^n maka

$$\|x^T y\| \leq \|x\| . \|y\|$$

$$\hookrightarrow -1 \leq \frac{x^T y}{\|x\| . \|y\|} \leq 1$$

dengan persamaan berlaku jika salah satu vektor tersebut vektor nol atau vektor yang satu merupakan kelipatan dari vektor yg lain.

* vektor 2x x dan y di R^n disebut ortogonal jika :

$$\boxed{x^T y = 0}$$

* sudut diantara dua vektor x dan y

$$\boxed{\cos\theta = \frac{x^T y}{\|x\| . \|y\|}}$$

$$0 \leq \theta \leq \pi$$

* proyeksi skalar dari x pada y

$$\boxed{\alpha = \frac{x^T y}{\|y\|}}$$

* Proyeksi vektor dari x pada y :

$$\boxed{p = \alpha . u = \alpha . \frac{1}{\|y\|} . y = \frac{x^T y}{y^T y} . y}$$

RUANG HASIL KALI DALAM

Aksioma :

1) $\langle x, x \rangle \geq 0$ dengan pers berlaku jika $x = 0$.
2) $\langle x, y \rangle = \langle y, x \rangle$, $\forall x, y \in V$
3) $\langle x+y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$
4) $\langle \alpha x, z \rangle = \alpha \langle x, z \rangle$

contoh soal :

1. $x = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$ $y = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

a) tentukan sudut antara kedua vektor

$$\cos\theta = \frac{x^T y}{\|x\| . \|y\|} = \frac{3(1) + 5(1)}{\sqrt{3^2+5^2} . \sqrt{1^2+1^2}}$$

$$(3 \quad 5)\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \frac{3+5}{\sqrt{34} . \sqrt{2}}$$

$$= \frac{8}{\sqrt{2.17} . \sqrt{2}}$$

$$= \frac{8}{2\sqrt{17}} = \frac{4}{\sqrt{17}}$$

$$\theta = \arccos\left(\frac{4}{\sqrt{17}}\right)$$

$$\theta = \cos^{-1}\left(\frac{4}{\sqrt{17}}\right)$$

b) tentukan proyeksi skalar dan proyeksi vektor (p)

$$\alpha = \frac{x^T y}{\|y\|} = \frac{8}{\sqrt{2}} . \frac{\sqrt{2}}{\sqrt{2}} = 4\sqrt{2}$$

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$$p = \frac{x^T . y}{y^T . y} . y = \frac{8}{2}\begin{pmatrix}1\\1\end{pmatrix}$$

$$= 4\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}4\\4\end{pmatrix}$$

sng

$$p = \begin{pmatrix}4\\4\end{pmatrix}$$

c) Tunjukkan bahwa p dan $x - p$ ortogonal

$$x - p = \begin{pmatrix}3\\5\end{pmatrix} - \begin{pmatrix}4\\4\end{pmatrix} = \begin{pmatrix}-1\\1\end{pmatrix}$$

$$p^T (x-p) = (4 \quad 4)\begin{pmatrix}-1\\1\end{pmatrix}$$

$$= -4 + 4$$

$$= 0$$

∴ sng terbukti bahwa p dan $x - p$ ortogonal.

② Tunjukkan $x^T y$ merupakan ruang hasil kali dalam!

a) adb : $\langle x, x \rangle \geq 0$, mis: $x = \begin{pmatrix}x_1\\x_2\end{pmatrix}$

$y = \begin{pmatrix}y_1\\y_2\end{pmatrix}$; $z = \begin{pmatrix}z_1\\z_2\end{pmatrix}$

$\langle x, x \rangle = x^T . x \longrightarrow \begin{pmatrix}x_1\\x_2\end{pmatrix}$ $(x_1 \; x_2)$

$$= \underbrace{x_1^2 + x_2^2}_{\text{positif}} \geq 0$$

$$\langle x, x \rangle = 0$$

$$x_1^2 + x_2^2 = 0$$

$$x_1 = x_2 = 0$$

Terbukti

b) adb : $\langle x, y \rangle = \langle y, x \rangle$

$$\langle x, y \rangle = x^T . y$$

$$= x_1 y_1 + x_2 . y_2$$

$$= y_1 . x_1 + y_2 . x_2$$

Terbukti $= y^T x = \langle y, x \rangle$

c) Adb : $\langle x+y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$

$$\langle x+y, z \rangle = (x+y)^T . z$$

$$= \begin{pmatrix}x_1+y_1\\x_2+y_2\end{pmatrix}^T . \begin{pmatrix}z_1\\z_2\end{pmatrix}$$

$$= (x_1 + y_1) z_1 + (x_2 + y_2) z_2$$

$$= x_1 . z_1 + y_1 z_1 + x_2 z_2 + y_2 z_2$$

$$= x_1 . z_1 + x_2 . z_2 + y_1 z_1 + y_2 z_2$$

$$= x^T z + y^T . z$$

terbukti $= \langle x, z \rangle + \langle y, z \rangle$

d) $\langle \alpha x, z \rangle = (\alpha x)^T . z = (\alpha x_1 \quad \alpha x_2)\begin{pmatrix}z_1\\z_2\end{pmatrix}$

$$= \alpha x_1 . z_1 + \alpha x_2 . z_2$$

$$= \alpha (x_1 . z_1 + x_2 z_2)$$

$$= \alpha (x^T z)$$

Terbukti $= \alpha \langle x, z \rangle$

LATIHAN SOAL

1] Untuk setiap pasang vektor x & y, tent

a) sudut antara kedua vektor

b) vektor p, yaitu proyeksi vektor dari x pada y

c) buktikan bahwa p dan $x - p$ adalah ortogonal.

i) $x = \begin{pmatrix}4\\1\end{pmatrix}$; $y = \begin{pmatrix}3\\2\end{pmatrix}$

ii) $x = \begin{pmatrix}2\\-5\\4\end{pmatrix}$; $y = \begin{pmatrix}1\\2\\-1\end{pmatrix}$

i) a) $\cos \theta = \dfrac{x^T . y}{\|x\| . \|y\|} = \dfrac{(4 \quad 1)\begin{pmatrix}3\\2\end{pmatrix}}{(\sqrt{4^2+1^2})(\sqrt{3^2+2^2})}$

$$= \frac{14}{\sqrt{17} . \sqrt{13}} = \frac{14}{\sqrt{221}}$$

$$\theta = \arc . \cos\left(\frac{14}{\sqrt{221}}\right) = \cos^{-1}\left(\frac{14}{\sqrt{221}}\right)$$

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b) $p = \frac{x^T . y}{y^T . y} . y$

$= \frac{(4 \quad 1)\begin{pmatrix}3\\2\end{pmatrix}}{(3 \quad 2)\begin{pmatrix}3\\2\end{pmatrix}} \begin{pmatrix}3\\2\end{pmatrix}$

$= \frac{14}{13}\begin{pmatrix}3\\2\end{pmatrix} = \begin{pmatrix}42/13\\28/13\end{pmatrix}$

c) $p^T(x-p) = 0$

$= (42/13 \quad 28/13)\left(\begin{pmatrix}\frac{52}{13}\\\frac{17}{13}\end{pmatrix} - \begin{pmatrix}\frac{42}{13}\\\frac{28}{13}\end{pmatrix}\right)$

$= (42/13 \quad 28/13)\begin{pmatrix}10/17\\-15/17\end{pmatrix}$

$= \frac{420}{169} - \frac{420}{169}$

$= 0$

Terbukti

ii) a) θ ?

$\cos\theta = \frac{x^T . y}{\|x\| . \|y\|} = \frac{-12}{\sqrt{45}.\sqrt{6}} = \frac{-4}{\sqrt{270}}$

$= \frac{-4}{3\sqrt{30}}$

$\theta = \text{arc} . \cos\left(\frac{-4}{3\sqrt{30}}\right)$

$= \cos^{-1}\left(-\frac{4}{3\sqrt{30}}\right)$

b) $p = \alpha . u = \frac{x^T . y}{y^T . y} . y = \frac{-12}{6}\begin{pmatrix}1\\2\\-1\end{pmatrix}$

$= -2\begin{pmatrix}1\\2\\-1\end{pmatrix} = \begin{pmatrix}-2\\-4\\2\end{pmatrix}$

$p = \begin{pmatrix}-2\\-4\\2\end{pmatrix}$

c) $p^T(x-p) = 0$

$(x-p) : \begin{pmatrix}4\\-1\\2\end{pmatrix}$

$p^T(x-p)$

$= -8+4+4$

$= 0$

terbukti ortogonal

2] Tent $\langle x, x^2\rangle$ dengan hasil kali dalam $\langle p, q\rangle = \sum_{i=1}^{n} p(x_i) . q(x_i)$ di P_5 dan $x_i = \frac{i-3}{2}$, untuk $i = 1, 2, 3, 4, 5$.

Jwb:

$\boxed{i = 1}$

$x_i = \frac{1-3}{2} = -1$ $\quad \langle x, x^2\rangle$

$\langle -1, (-1)^2\rangle = \langle -1, 1\rangle$

$\boxed{i = 2}$

$x_i = \frac{2-3}{2} = -\frac{1}{2}$

$\langle -\frac{1}{2}, (-\frac{1}{2})^2\rangle = \langle -\frac{1}{2}, \frac{1}{4}\rangle$

$\boxed{i = 3}$

$x_i = \frac{3-3}{2} = 0$

$\langle 0, 0\rangle$

$\boxed{i = 4}$

$x_i = \frac{4-3}{2} = \frac{1}{2}$

$\langle \frac{1}{2}, \frac{1}{4}\rangle$

$\boxed{i = 5}$

$x_i = \frac{5-3}{2} = 1$

$\langle 1, 1\rangle$

$\langle x, x^2\rangle = \sum_{i=1}^{5} p(x_i) . q(x_i)$

$= (-1 . 1) + (-\frac{1}{2} . \frac{1}{4}) + (0 . 0) + (\frac{1}{2} . \frac{1}{4}) + (1 . 1)$

$= -1 - \frac{1}{8} + 0 + \frac{1}{8} + 1$

$= 0$

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ALIN

Ruang hasil kali dalam

CH. 11

AKSIOMA

① $\langle x, x\rangle \geq 0$

② $\langle x, y\rangle = \langle y, x\rangle$

③ $\langle \alpha x + \beta y, z\rangle = \alpha\langle x, z\rangle + \beta\langle y, z\rangle$

- $\langle x+y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$
- $\langle \alpha x, z\rangle = \alpha\langle x, z\rangle$

1] Di ruang vektor R^n

$\langle x, y\rangle = x^T y$

Jika diberikan vektor w dgn entri" positif w_i,

$$\langle x, y\rangle = \sum_{i=1}^{n} x_i y_i w_i$$

Juga hasil kali dalam. Entri" $w_i \to$ bobot.

contoh:

① Tunjukan $x^T y$ merupakan hasil kali dalam

$\langle x, y\rangle = x^T y$ di R^n

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

$$\langle x, y\rangle = x^T y = (x_1\ x_2\ x_3)\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

$$= (x_1 . y_1 + x_2 y_2 + x_3 y_3)$$

i) $\langle x, x\rangle = x^T x = x_1^2 + x_2^2 + \dots + x_n^2 \geq 0$

$\langle x, x\rangle = 0 \leftrightarrow x_1^2 + x_2^2 + \dots + x_n^2 = 0$

$x_1^2 = x_2^2 = x_3^2 = 0$

$x = 0$

ii) $\langle x, y\rangle = x^T y = \sum_{i=1}^{n} x_i . y_i$

$= \sum_{i=1}^{n} y_i . x_i = y^T x = \langle y, x\rangle$

$\therefore \langle x, y\rangle = \langle y, x\rangle$

iii) $\alpha x + \beta y = \begin{pmatrix} \alpha x_1 + \beta y_1 \\ \alpha x_2 + \beta y_2 \\ \vdots \\ \alpha x_n + \beta y_n \end{pmatrix}, z = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix}$

$\langle \alpha x + \beta y, z\rangle = (\alpha x_1 + \beta y_1) z_1 + \dots$

$= \alpha x_1 z_1 + \beta y_1 z_1 + \alpha x_2 z_2 + \beta y_2 z_2$

$= \alpha(x_1 z_1 + x_2 z_2 + \dots) + \beta(y_1 z_1 + y_2 z_2 + \dots)$

$= \alpha\langle x, z\rangle + \beta\langle y, z\rangle$

② misalkan $x = (x_1, x_2)^T$ dan $y = (y_1, y_2)^T$ vektor" di R^2

tunjukan bahwa

$$\langle x, y\rangle = 3x_1 y_1 + 2x_2 y_2$$

adalah suatu hasil kali dalam terboboti.

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$\langle x, y\rangle = 3x_1 y_1 + 2x_2 y_2$

(i) $\langle x, x\rangle = 3x_1 x_1 + 2x_2 x_2$

$= 3x_1^2 + 2x_2^2 \geq 0$

2] Di ruang vektor $R^{m\times n}$

Jika $A, B \in R^{m\times n}$, maka

$$\langle A, B\rangle = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij} b_{ij}$$

adalah hasil kali dalam di $R^{m\times n}$

misalkan:

ex: $A = \begin{pmatrix} 1 & 2 & 3 \\ -2 & 3 & 5 \end{pmatrix}, B = \begin{pmatrix} 4 & 6 & -3 \\ 2 & 0 & -1 \end{pmatrix}$

tentukan $\langle A, B\rangle$

$\langle A, B\rangle = 1.4 + 2.6 + 3(-3) + (-2).2 + 3.0 + 5.(-1)$

$= -2$ //

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SIFAT - SIFAT

1) Jika $v \in V$, maka panjang / norm dari v adalah

$$\| v \| = \sqrt{\langle v, v \rangle}$$

2) u dan v dikata ortogonal jika

$$\langle u, v \rangle = 0$$

SIFAT VEKTOR" ORTOGONAL

1) Teorema Pythagoras

$$\| u + v \|^2 = \| u \|^2 + \| v \|^2$$

PROYEKSI SKALAR DAN PROYEKSI VEKTOR

- proyeksi skalar dari u dan v

$$\alpha = \frac{\langle u, v \rangle}{\| v \|}$$

- Proyeksi vektor

$$p = \frac{\langle u, v \rangle}{\langle v, v \rangle} \cdot v.$$

ex :

① misalkan $A, B \in R^{2 \times 2}$ dengan

$$A = \begin{pmatrix} 1 & 2 \\ 1 & -3 \end{pmatrix} \text{ dan } B = \begin{pmatrix} 0 & -1 \\ 1 & -5 \end{pmatrix}$$

dengan menggunakan

i) $A^T B = \begin{pmatrix} 1 & 1 \\ 2 & -3 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & -5 \end{pmatrix}$

$$= \begin{pmatrix} 1 & -6 \\ -3 & 13 \end{pmatrix}$$

$\langle A, B \rangle = 1 + 13 = 14$

$\langle B, B \rangle = tr(B^T B)$

ii) $B^T B = \begin{pmatrix} 0 & 1 \\ -1 & -5 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & -5 \end{pmatrix}$

$$= \begin{pmatrix} 1 & -5 \\ -5 & 26 \end{pmatrix}$$

$\langle B, B \rangle = tr(B^T B) = 1 + 26 = 27$

$$P = \frac{\langle A, B \rangle}{\langle B, B \rangle} \cdot B$$

$$= \frac{14}{27} \begin{pmatrix} 0 & -1 \\ 1 & -5 \end{pmatrix}$$

catatan : untuk slide 29 - 30 pelajari sendiri + ketaksamaan Cauchy - Schwarz

HIMPUNAN ORTONORMAL → ortogonal yg panjangnya 1

$\langle v_i, v_j \rangle = 0$ bila $i \neq j$

maka $\{ v_1, v_2, \dots v_n \}$ dikatakan himp. ortonormal.

Contoh :

$v_1 = (1, 1, 1)^T$ $v_2 = (2, 1, -3)^T$

$v_3 = (4, -5, 1)^T$ maka $\{ v_1, v_2, v_3 \}$

himp ~~ortonormal~~ ortogonal tapi bukan ortonormal di R^3?

jika → panjangnya bukan 1

$$u_1 = \frac{v_1}{\| v_1 \|} = \frac{1}{\sqrt{3}} (1, 1, 1)^T$$

$$u_2 = \frac{v_2}{\| v_2 \|} = \frac{1}{\sqrt{14}} (2, 1, -3)^T.$$

$$u_3 = \frac{v_3}{\| v_3 \|} = \frac{1}{\sqrt{42}} (4, -5, 1)^T$$

PROSES ORTOGONALISASI GRAM - SCHMIDT

- proses pembentukan basis ortonormal untuk suatu ruang hkd

$$u_1 = \left(\frac{1}{\| x_1 \|} \right) x_1$$

rentang (u_1) = Rentang (x_1)

misalkan :

$p_1 = \langle x_2, u_1 \rangle u_1$

maka

$$(x_2 - p_1) \perp u_1.$$

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contoh :

diberikan vektor" basis R^3

$x_1 = (1,1,1)^T$, $x_2 = (0,1,1)^T$

$x_3 : (0,0,1)^T$.

Jwb :

$u_1 : \frac{x_1}{\|x_1\|} = \frac{1}{\sqrt{3}}(1,1,1)^T$

$p_1 = \langle x_2, u_1 \rangle u_1$

$= (0 \; 1 \; 1) \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix} \cdot \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}$

$= \frac{2}{\sqrt{3}}\left(\frac{1}{\sqrt{3}}(1,1,1)^T\right)$

$= \frac{2}{3}(1,1,1)^T$

$x_2 - p_1 = \left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right)^T$

$u_2 = \frac{x_2 - p_1}{\|x_2 - p_1\|}$

$= \frac{1}{\sqrt{6/9}}\left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right)^T = \left(-\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)^T$

$p_2 = \langle x_3, u_1 \rangle u_1 + \langle x_3, u_2 \rangle u_2$

$= \frac{1}{\sqrt{3}}\left[\frac{1}{\sqrt{3}}(1,1,1)^T\right] + \frac{1}{\sqrt{6}}\left(-\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)^T$

$= \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)^T + \left(-\frac{1}{3}, \frac{1}{6}, \frac{1}{6}\right)^T$

$= \left(0, \frac{1}{2}, \frac{1}{2}\right)^T$

$x_3 - p_2 = (0,0,1)^T - \left(0, \frac{1}{2}, \frac{1}{2}\right)^T$

$= \left(0, -\frac{1}{2}, \frac{1}{2}\right)^T$

$u_3 = \frac{x_3 - p_2}{\|x_3 - p_2\|}$

$= \frac{(0, -\frac{1}{2}, \frac{1}{2})^T}{\sqrt{\frac{1}{2}}} = \sqrt{2}\left(0, -\frac{1}{2}, \frac{1}{2}\right)^T$

$u_3 = \left(0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)^T$

shg

basis ortonormal $\{u_1, u_2, u_3\}$

ujung kiri bawah selalu 0

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Responsi 12 — ALIN

ORTONORMAL & ORTOGONALISASI GRAM SCHMIDT

Definisi: Misal $v_1, v_2, \dots, v_n$ vektor" di ruang hasil kali dalam V. Jika $\langle v_1, v_j \rangle = 0$ bila $i \neq j$ maka $\{v_1, v_2, \dots, v_n\}$ dikatakan himp ortogonal di vektor".

Tent: Jika $\{v_1, v_2, \dots, v_n\}$ adl himp ortogonal dari vektor" tak nol pada ruang hasil kali dalam V_1

→ $v_1, v_2, \dots, v_n$ adalah bebas linear.

- Sebuah himp. ortonormal dari vektor" adalah sebuah himp. ortogonal dari vektor" satuan.
- Misalkan $\{x_1, x_2, \dots, x_n\}$ adalah basis untuk ruang hasil kali dalam V. Misalkan

$$\rightarrow u_1 = \left(\frac{1}{\|x_1\|}\right) x_1$$

didefinisikan $u_2, u_3, \dots u_n$ secara rekursif

$$u_k = \frac{1}{\|x_k - p_{k-1}\|}(x_k - p_{k-1})$$

$\forall k = 1, 2, \dots, n-1$.

$$\rightarrow p_k = \langle x_{k+1}, u_1 \rangle u_1 + \dots + \langle x_{k+1}, u_k \rangle u_k$$

p_k adalah proyeksi dari x_{k+1} pada rentang $(u_1, u_2, \dots, u_k)$ Himp $\{u_1, u_2, \dots, u_n\}$ merupakan basis ortonormal u/ V.

- Teorema Dekomposisi QR

Jika A adalah sebuah matriks $m \times n$ dengan rank n, maka A dapat difaktorkan ke dalam sebuah hasil kali QR, dengan Q adl sebuah matriks $m \times n$ dengan kolom" ortonormal dan R adl matriks $m \times n$ yg merupakan matriks segitiga atas dan tak singular.

Contoh:

1. $x_1 = (1,1,1)^T$ $\quad x_3 = (0,0,1)^T$
$x_2 = (0,1,1)^T$
Dg proses GS tent basis ortonormal u/ $\mathbb{R}^3$?

Jwb:

- $u_1 = \dfrac{x_1}{\|x_1\|} = \dfrac{1}{\sqrt{3}}(1,1,1)^T$
- $p_1 = \langle x_2, u_1 \rangle u_1$

$$= (0,1,1)\begin{pmatrix} \frac{1}{\sqrt3} \\ \frac{1}{\sqrt3} \\ 1/\sqrt3 \end{pmatrix} u_1$$

$$= \frac{2}{\sqrt3}\left(\frac{1}{\sqrt3}(1,1,1)^T\right)$$

$$= \frac{2}{3}(1,1,1)^T$$

- $x_2 - p_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 2/3 \\ 2/3 \\ 2/3 \end{pmatrix} = \begin{pmatrix} -2/3 \\ 1/3 \\ 1/3 \end{pmatrix}$
- $u_2 = \dfrac{x_2 - p_1}{\|x_2 - p_1\|} = \dfrac{1}{\sqrt{\frac{6}{9}}}\left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right)^T$

$$= \left(-\frac{2}{\sqrt6}, \frac{1}{\sqrt6}, \frac{1}{\sqrt6}\right)^T$$

- $p_2 = \langle x_3, u_1 \rangle u_1 + \langle x_3, u_2 \rangle u_2$

$$= \frac{1}{\sqrt3}\left[\frac{1}{\sqrt3}(1,1,1)^T\right] + \frac{1}{\sqrt6}\left(-\frac{2}{\sqrt6}, \frac{1}{\sqrt6}, \frac{1}{\sqrt6}\right)^T$$

$$= \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)^T + \left(-\frac{1}{3}, \frac{1}{6}, \frac{1}{6}\right)^T$$

$$= \left(0, \frac{1}{2}, \frac{1}{2}\right)^T$$

- $x_3 - p_2 = (0,0,1)^T - (0, 1/2, 1/2)^T = (0, -1/2, 1/2)$
- $u_3 = \dfrac{x_3 - p_2}{\|x_3 - p_2\|} = \dfrac{(0, -1/2, 1/2)^T}{\sqrt{1/2}} = \sqrt2\left(0, -\frac{1}{2}, \frac{1}{2}\right)^T$

$$= \left(0, -\frac{1}{\sqrt2}, \frac{1}{\sqrt2}\right)^T$$

basis ortonormalnya adl $\{u_1, u_2, u_3\}$.

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$$Q = \begin{array}{c} \begin{array}{ccc} u_1 & u_2 & u_3 \end{array} \\ \begin{bmatrix} | & | & | \\ | & | & | \end{bmatrix} \end{array}$$

catatan

$$R = \begin{bmatrix} \langle u_1, x_1 \rangle & \langle u_1, x_2 \rangle & \langle u_1, x_3 \rangle \\ 0 & \langle u_2, x_2 \rangle & \langle u_2, x_3 \rangle \\ 0 & 0 & \langle u_3, x_3 \rangle \end{bmatrix}$$

↳ segitiga atas

$A = Q.R$

$$\Rightarrow R = \begin{bmatrix} \|x_1\| & \langle x_2, u_1 \rangle & \langle x_3, u_1 \rangle \\ 0 & \|x_2 - p_1\| & \langle x_3, u_2 \rangle \\ 0 & 0 & \|x_3 - p_2\| \end{bmatrix}$$

2) Gunakan GS untuk mengubah basis dari P_3 yaitu $\{1, x, x^2\}$ menjadi basis ortonormal dgn HKD (hasil kali dalam) di P_3 adl

$$\langle p, q \rangle = \int_0^1 (p(x) \cdot q(x)\, dx)$$

$x_1 = 1 \qquad x_2 = x \qquad x_3 = x^2$

$$\langle p, q \rangle = \int_0^1 p(x) \cdot q(x)\, dx$$

$$\|x_1\| = \sqrt{\langle x_1, x_1 \rangle}$$

$$= \sqrt{\int_0^1 1 \cdot 1\, dx} = \sqrt{x|_0^1} = \sqrt{1} = 1$$

$$u_1 = \frac{x_1}{\|x_1\|} = \frac{1}{1} = 1$$

$$p_1 = \langle x_2, u_1 \rangle u_1$$

$$= \left(\int_0^1 x \cdot 1\, dx\right) = \left(\tfrac{1}{2} x^2 |_0^1\right) = \tfrac{1}{2}$$

$$x_2 - p_1 = x - 1/2$$

$$\|x_2 - p_1\| = \sqrt{\langle x_2 - p_1, x_2 - p_1 \rangle}$$

$$= \sqrt{\int_0^1 (x - \tfrac{1}{2})^2\, dx}$$

$$= \sqrt{\int_0^1 (x^2 - x + \tfrac{1}{4})\, dx}$$

$$= \sqrt{\tfrac{1}{3}x^3 - \tfrac{1}{2}x^2 + \tfrac{1}{4}x |_0^1}$$

$$= \sqrt{1/3 - 1/2 + 1/4} = \frac{1}{2\sqrt{3}}$$

$$u_2 = \frac{x_2 - p_1}{\|x_2 - p_1\|} = \frac{x - 1/2}{1/2\sqrt{3}} = 2\sqrt{3}x - \sqrt{3}$$

$$p_2 = \langle x_3, u_1 \rangle u_1 + \langle x_3, u_2 \rangle u_2$$

$$= \left(\int_0^1 x^2 \cdot 1\, dx\right) + \left(\int_0^1 x^2 (2\sqrt{3}x - \sqrt{3})\right).$$

$$2\sqrt{3}x - \sqrt{3}$$

$$= \tfrac{1}{3} x^3 |_0^1 + \left(\left(\tfrac{\sqrt{3}}{2} - \tfrac{\sqrt{3}}{3}\right) 2\sqrt{3}x - \sqrt{3}\right)$$

$$= x - 1/6$$

dari soal nomor 1

$$Q = (u_1, u_2, u_3)$$

$$= \begin{pmatrix} 1/\sqrt{3} & -2/\sqrt{6} & 0 \\ 1/\sqrt{3} & 1/\sqrt{6} & -\sqrt{2}/2 \\ 1/\sqrt{3} & 1/\sqrt{6} & \sqrt{2}/2 \end{pmatrix}$$

$$R = \begin{pmatrix} \|x_1\| & \langle x_2, u_1 \rangle & \langle x_3, u_1 \rangle \\ 0 & \|x_2 - p_1\| & \langle x_3, u_2 \rangle \\ 0 & 0 & \|x_3 - p_2\| \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{3} & 2/\sqrt{3} & 1/\sqrt{3} \\ 0 & \tfrac{1}{3}\sqrt{6} & 1/\sqrt{6} \\ 0 & 0 & 1/\sqrt{2} \end{pmatrix}$$

$$A = \begin{pmatrix} \frac{1}{\sqrt{3}} & -2/\sqrt{6} & 0 \\ 1/\sqrt{3} & 1/\sqrt{6} & -\sqrt{2}/2 \\ 1/\sqrt{3} & 1/\sqrt{6} & \sqrt{2}/2 \end{pmatrix} \begin{pmatrix} \sqrt{3} & 2/\sqrt{3} & 1/\sqrt{3} \\ 0 & 1/3\sqrt{6} & 1/\sqrt{6} \\ 0 & 0 & 1/\sqrt{2} \end{pmatrix}$$

$= \ldots$

$$x_3 - p_2 = x^2 - x + 1/6$$

$$\|x_3 - p_2\| = \sqrt{\langle x^2 - x + \tfrac{1}{6}, x^2 - x + \tfrac{1}{6} \rangle}$$

$$= \sqrt{\int_0^1 (x^2 - x + \tfrac{1}{6})^2\, dx}$$

$$= \sqrt{\int_0^1 x^4 - 2x^3 + \tfrac{8}{6}x^2 - \tfrac{2}{6}x + \tfrac{1}{36}\, dx}$$

$$= \sqrt{\frac{1}{180}} = \frac{1}{6\sqrt{5}}$$

$$u_3 = \frac{x_3 - p_2}{\|x_3 - p_2\|} = \frac{x^2 - x + \frac{1}{6}}{1/6\sqrt{5}} = 6\sqrt{5}(x^2 - x + 1/6)$$

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3) $A = \begin{pmatrix} 3 & 2 & 2 \\ 4 & 6 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

a) Tent basis ortonormal dan ruang kolom A
(ruang yg direntang oleh vektor" kolom A)

b) Tent dekomposisi QR dan matriks A

Jwb:

a) $x_1 = (3, 4, 0)^T$

$x_2 = (2, 6, 0)^T$

$x_3 = (2, 1, 1)^T$

$\|x_1\| = \sqrt{\langle x_1, x_1 \rangle}$

$= \sqrt{3^2 + 4^2 + 0^2} = 5$

$u_1 = \frac{x_1}{\|x_1\|} = \begin{pmatrix} 3/5 \\ 4/5 \\ 0 \end{pmatrix}$

$P_1 = \langle x_2, u_1 \rangle . u_1$

$= (2\ 6\ 0) \begin{pmatrix} 3/5 \\ 4/5 \\ 0 \end{pmatrix} \begin{pmatrix} 3/5 \\ 4/5 \\ 0 \end{pmatrix}$

$= 6 \begin{pmatrix} 3/5 \\ 4/5 \\ 0 \end{pmatrix} = \begin{pmatrix} 18/5 \\ 24/5 \\ 0 \end{pmatrix}$

$x_2 - P_1 = \begin{pmatrix} 2 \\ 6 \\ 0 \end{pmatrix} - \begin{pmatrix} 18/5 \\ 24/5 \\ 0 \end{pmatrix} = \begin{pmatrix} -8/5 \\ 6/5 \\ 0 \end{pmatrix}$

$u_2 = \frac{x_2 - P_1}{\|x_2 - P_1\|} = \frac{1}{2} \begin{pmatrix} -8/5 \\ 6/5 \\ 0 \end{pmatrix} = \begin{pmatrix} -4/5 \\ 3/5 \\ 0 \end{pmatrix}$

$P_2 = \langle x_3, u_2 \rangle u_2 + \langle x_3, u_1 \rangle u_1$

$= (2\ 1\ 1) \begin{pmatrix} -4/5 \\ 3/5 \\ 0 \end{pmatrix} \begin{pmatrix} -4/5 \\ 3/5 \\ 0 \end{pmatrix} + (2\ 1\ 1) \begin{pmatrix} 3/5 \\ 4/5 \\ 0 \end{pmatrix} \begin{pmatrix} 3/5 \\ 4/5 \\ 0 \end{pmatrix}$

$= (2\ 1\ 0)^T$

$x_3 - P_2 = (2, 1, 1)^T - (2\ 1\ 0)^T$

$= (0\ 0\ 1)^T$

$u_3 = \frac{x_3 - P_2}{\|x_3 - P_2\|} = \frac{1}{1} (0\ 0\ 1)^T = (0\ 0\ 1)^T$

$\therefore$ basis ortonormal : $\{u_1, u_2, u_3\}$

b) $A = Q . R$

$Q = \begin{pmatrix} 3/5 & -4/5 & 0 \\ 4/5 & 3/5 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$R = \begin{pmatrix} \|x_1\| & \langle x_2, u_1 \rangle & \langle x_3, u_1 \rangle \\ 0 & \|x_2 - P_1\| & \langle x_3, u_2 \rangle \\ 0 & 0 & \|x_3 - P_2\| \end{pmatrix}$

$= \begin{pmatrix} 5 & 6 & 2 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{pmatrix}$

$A = Q . R$

$= \begin{pmatrix} 3/5 & -4/5 & 0 \\ 4/5 & 3/5 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 5 & 6 & 2 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{pmatrix}$

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nilai eigen, vektor

eigen dan diagonalisasi

→ $Ax = (\lambda)x$ (nilai eigen: λ; vektor: x)

Teorema : 1) λ adalah nilai eigen dr A

2) $(A - \lambda I)x = 0$ memp solusi tak trivial

3) $N(A - \lambda I) \neq \{0\}$

4) $A - \lambda I$ adl matriks singular

5) $\det(A - \lambda I) = 0$

→ $\det(A - \lambda I) = 0$ → pers. karakteristik

$p(\lambda) = \det(A - \lambda I)$ → polinom karakteristik

→ misal λ_1 adl nilai" eigen dari A maka

$\lambda_1 \cdot \lambda_2 \dots \lambda_n = \det(A)$

$\lambda_1 + \lambda_2 + \dots + \lambda_n = tr(A)$

→ jika A dan B matriks $n \times n$ & B serupa dgn A maka A dan B memp polinom karakteristik dan nilai eigen yg sama.

→ jika $\lambda_1, \lambda_2, \dots, \lambda_n$ adalah nilai² eigen dari A dan $x_1, x_2, x_3, \dots, x_n$ adalah vektor eigen dari A maka $x_1, x_2, \dots x_n$ bebas linear

→ matriks $A_{n \times n}$ Dapat didiagonalkan jika ada matriks X tak singular dan D matriks diagonal, maka

$$\boxed{X^{-1}AX = D}$$

→ matriks X dikatakan mendiagonalkan A

→ $A^k = X D^k X^{-1}$

→ suatu matriks $A_{n \times n}$ dpt didiagonalkan jika dan hanya jika A memp n vektor eigen yg bebas linear.

$X^{-1}AX = D$

$A = XDX^{-1}$

$A^2 = X \cdot D \cdot X^{-1} \cdot X \cdot D \cdot X^{-1}$

$= X \cdot D^2 X^{-1}$

Contoh:

1) $A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$

Tent nilai eigen & vektor eigen.

Jwb:

$|A - \lambda I| = 0$ → determinan = 0

$= \left| \begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right|$

$= \left| \begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right| = \left| \begin{pmatrix} 3-\lambda & 0 \\ 8 & -1-\lambda \end{pmatrix} \right|$

$(3-\lambda)(-1-\lambda) - 0 = 0$

$(3-\lambda)(-1-\lambda) = 0$

$\lambda_1 = 3 \qquad \lambda_2 = -1$

$\underline{\lambda_1 = 3}$

$(A - \lambda I)\vec{v} = \vec{0}$

$\begin{pmatrix} 3-3 & 0 \\ 8 & -1-3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\begin{pmatrix} 0 & 0 \\ 8 & -4 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$8v_1 - 4v_2 = 0$

$4v_2 = 8v_1$

$v_2 = 2v_1$

mis: $v_1 = s$

VE = $\begin{pmatrix} s \\ 2s \end{pmatrix} = s \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

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$\lambda_2 = -1$

$$\begin{pmatrix} 3+1 & 0 \\ 8 & -1+1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 4 & 0 \\ 8 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$4v_1 = 0$

$v_1 = 0$

$v_2 = k$

$$V_E = k\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$RE : \left\{ x \mid x = \left\{ s\begin{pmatrix} 1 \\ 2 \end{pmatrix} + k\begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} \right\}$$

2) $B = \begin{bmatrix} 3 & -1 & -2 \\ 2 & 0 & -2 \\ 2 & -1 & -1 \end{bmatrix}$

Tent nilai eigen, vektor eigen dan matriks diagonal.

$|A - \lambda I| = 0$

$$\begin{vmatrix} 3-\lambda & -1 & -2 \\ 2 & -\lambda & -2 \\ 2 & -1 & -1-\lambda \end{vmatrix} = 0$$

$\Rightarrow [(3-\lambda)(-\lambda)(-1-\lambda) + 4 + 4] - [4\lambda + 2(3-\lambda) + (-1-\lambda)(-2)]$

$\Rightarrow [(\lambda+\lambda^2)(3-\lambda) + 8] - [4\lambda + 6 \cancel{-2\lambda} + 2 \cancel{+2\lambda}]$

$= [3\lambda - \lambda^2 + 3\lambda^2 - \lambda^3 \cancel{+8}] - [4\lambda \cancel{+8}]$

$= -\lambda + 2\lambda^2 - \lambda^3 = 0$

$-\lambda(\lambda^2 - 2\lambda + 1) = 0$

$-\lambda(\lambda^2 - 2\lambda + 1) = 0$

$-\lambda(\lambda - 1)(\lambda - 1) = 0$

$\lambda_1 = 0 \quad \lambda_2 = 1 \quad \lambda_3 = 1$

$\lambda_1 = 0$

$(A - \lambda_1)\bar{u} = 0$

$$\begin{pmatrix} 3 & -1 & -2 \\ 2 & 0 & 2 \\ 2 & -1 & -1 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

OBD

$$\left(\begin{array}{ccc|c} 3 & -1 & -2 & 0 \\ 2 & 0 & -2 & 0 \\ 2 & -1 & -1 & 0 \end{array}\right) \begin{matrix} E_{21}(-\frac{2}{3}) \\ E_{31}(-\frac{2}{3}) \end{matrix} \left(\begin{array}{ccc|c} 3 & -1 & -2 & 0 \\ 0 & 2/3 & -2/3 & 0 \\ 0 & -1/3 & 1/3 & 0 \end{array}\right)$$

$$E_{32}(1/2)\left(\begin{array}{ccc|c} 3 & -1 & -2 & 0 \\ 0 & 2/3 & -2/3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$\frac{2}{3}u_2 - \frac{2}{3}u_3 = 0$

$u_2 - u_3 = 0$

$u_3 = u_2$

$3u_1 - u_2 - 2u_3 = 0$

$3u_1 - u_2 - 2u_2 = 0$

$3u_1 - 3u_2 = 0$

$u_1 = u_2$

mis : $u_1 = s$

$$VE = s\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$\lambda_2 = \lambda_3 = 1$

$[A - \lambda I]\bar{v} = 0$

$$\begin{pmatrix} 2 & -1 & -2 \\ 2 & -1 & -2 \\ 2 & -1 & -2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$2v_1 - v_2 - 2v_3 = 0$

$2v_1 = v_2 + 2v_3$

$v_1 = \frac{1}{2}v_2 + v_3$

mis : $v_2 = t$

$v_3 = u$

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$$VE : \begin{pmatrix} \frac{1}{2}t + u \\ t \\ u \end{pmatrix}$$

$$VE : t \begin{pmatrix} 1/2 \\ 1 \\ 0 \end{pmatrix} + u \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$D : \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \rightarrow \text{matriks diagonal}$$

$$X : \begin{pmatrix} 1 & 1/2 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \rightarrow \text{kumpulan vektor eigen}$$

$$B : X D X^{-1}$$

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CH. 14 ALIN

perpangkatan

MATRIKS YG DPT DIDIAGONALKAN

suatu matriks A dikatakan dapat didiagonalkan jika :

$A = XDX^{-1}$

shg

$A^k = XD^kX^{-1}$

ex :

$$A = \begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}}_{\text{VE}} \underbrace{\begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}}_{\text{nilai eigen}} \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}^{-1}$$

MATRIKS DEFEKTIF

matriks berukuran $n \times n$ memiliki vektor eigen yg kurang dari n

$$A = \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}$$

$$= \left| \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right|$$

$$= \begin{pmatrix} 2-\lambda & 0 \\ 1 & 2-\lambda \end{pmatrix}$$

$\Leftrightarrow (2-\lambda)(2-\lambda) - 0 = 0$

$4 - 4\lambda + \lambda^2 = 0$

$\lambda^2 - 4\lambda + 4 = 0$

$(\lambda - 2)(\lambda - 2) = 0$

$\underbrace{\lambda_1 = 2 \qquad \lambda_2 = 2}_{\text{nilainya sama}}$

$A_{n \times n}$ memiliki nilai eigen yg beda $\rightarrow$ maka dia bebas linear $\rightarrow$ dapat didiagonalkan.

CONTOH

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 1 & 0 & 2 \end{pmatrix} \quad B = \begin{pmatrix} 2 & 6 & 0 \\ -1 & 4 & 0 \\ -3 & 6 & 2 \end{pmatrix}$$

$A \downarrow 2 \qquad B \downarrow 3$

a) tunjukan bahwa A dan B memiliki eigen yg sama

b) tunjukan A defektif, B tidak.

EKSPONENSIAL MATRIKS DIAGONAL

$$D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}$$

maka

$$e^D = \lim_{m \to \infty} \left(I + D + \frac{1}{2!} D^2 + \dots + \frac{1}{m!} D^m \right)$$

$=$

$$e^A = X e^D X^{-1}$$

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$$: \begin{pmatrix} e^{\lambda_1} & 0 & 0 \\ 0 & e^{\lambda_2} & 0 \\ 0 & 0 & \ddots \; e^{\lambda_n} \end{pmatrix}$$

$A = XPX^{-1}$

$A^k = XP^kX^{-1}$; untuk $k = 1, 2, \ldots$

ex:

$$A = \begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix}$$

$$A = XDX^{-1} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$$

λ_1 λ_2

$$e^A = Xe^DX^{-1}$$

$$= \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} e^3 & 0 \\ 0 & e^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} e^3 & 0 \\ -2e^{-1} + 2e^3 & e^{-1} \end{pmatrix}$$

MATRIKS-MATRIKS DEFINIT POSITIF

↳ jika $x^TAx > 0$ untuk semua x tak nol di R^n.

- A dikatakan definit positif jika dan hanya jika semua nilai eigennya positif.

ex:

$A = \begin{pmatrix} 1 & 0 \\ 0 & 6 \end{pmatrix}$ maka

$|A - \lambda I| = 0$

$\begin{vmatrix} 1-\lambda & 0 \\ 0 & 6-\lambda \end{vmatrix} = 0$

$(1-\lambda)(6-\lambda) = 0$

$\lambda = 1 \quad \lambda = 6$

- A dapat direduksi menjadi segitiga atas semua elemen diagonalnya positif
- A dpt difaktorkan menjadi: $A = LDL^T$
- A dpt difaktorkan menjadi $A = LL^T$

SUBMATRIKS UTAMA

A berordo $n \times n$. A_r adalah matriks yg menghilangkan $n-r$ baris dan kolom terakhir.

ex:

$$A = \begin{array}{c} \begin{array}{ccc} A_1 & A_2 & A_3 \end{array} \\ \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \end{array}$$

$A_1 = 1$

$A_2 = \begin{pmatrix} 1 & 2 \\ 4 & 5 \end{pmatrix}$

$A_3 = A$

1) Jika A def positif simetrik maka:
- A tak singular $|A| \neq 0$
- det $(A) > 0$
- submatriks utama $|A_1|, |A_2| \ldots A_n$ semuanya definit positif

SIFAT" MATRIKS DEFINIT POSITIF

Jika A matriks definit positif → dpt dijadiin matriks segitiga atas → hanya dgn OBD ke-3 ($E_{ij}(k)$) dan semua elemen diagonal (poros) adl positif.

$$A = \begin{pmatrix} 2 & -2 & 1 \\ -2 & 3 & 2 \\ 1 & 2 & 10 \end{pmatrix} \begin{matrix} E_{21}(1) \\ E_{31}(-\frac{1}{2}) \end{matrix} \begin{pmatrix} 2 & -2 & 1 \\ 0 & 1 & 3 \\ 0 & 3 & 9\frac{1}{2} \end{pmatrix}$$

① ②

$$E_{32}(-3) \begin{pmatrix} 2 & -2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1/2 \end{pmatrix}$$

DEKOMPOSISI / FAKTORISASI LU

$I_{21} = -1$

↳ $E_{21}(1)$ ditukar tandanya

$I_{31} = \frac{1}{2}$

↳ $e_{31}(-\frac{1}{2})$ tandanya dituker

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matriks L dinyatakan

$$L = \begin{pmatrix} 1 & 0 & 0 \\ \boxed{-1} & 1 & 0 \\ \boxed{\frac{1}{2}} & \boxed{3} & 1 \end{pmatrix}$$

dikali berapa dia supaya 0

$$LU = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 1 \\ 1/2 & 3 & 1 \end{pmatrix}\begin{pmatrix} 2 & -2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1/2 \end{pmatrix}$$

$$A = \begin{pmatrix} 2 & -2 & 1 \\ -2 & 3 & 2 \\ 1 & 2 & 10 \end{pmatrix}$$

ex:

$$A = \begin{pmatrix} 4 & 8 & 4 \\ 8 & 6 & 0 \\ 16 & 0 & 7 \end{pmatrix}$$

1) dekomposisi LU
2) dekomposisi LDL^T
3) dekomposisi Cholesky

hubungan antara faktorisasi LU dgn faktorisasi LDL^T

$$U = \begin{pmatrix} 2 & 3 \\ 0 & 4 \end{pmatrix}$$

$$U = \begin{pmatrix} 1 & \frac{u_{12}}{u_{11}} & \frac{u_{13}}{u_{11}} \\ 0 & 1 & \\ 0 & 0 & 1 \end{pmatrix}$$

kalo baris ke-n dibagi sama u_{nn}

$$= \underbrace{\begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}}_{D}\underbrace{\begin{pmatrix} 1 & 3/2 \\ 0 & 1 \end{pmatrix}}_{U_1}$$

$$\frac{3}{u_{11}} = \frac{3}{2}$$

$$\lambda_1 = 2$$

$$\begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

DEKOMPOSISI CHOLESKY

$$A = LL^T$$

$$L_1 = LD^{1/2}$$

$$D^{1/2} = \begin{pmatrix} \sqrt{u_{11}} & 0 & 0 \\ 0 & \sqrt{u_{22}} & 0 \\ 0 & 0 & \ddots \ \sqrt{u_{nn}} \end{pmatrix}$$

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