K.8 alm.

Transformasi / Operator Linear

✓ Definisi

$L : V \to W$ transformasi / operator linear jika:

$$\boxed{L(\alpha v_1 + \beta v_2) = \alpha L(v_1) + \beta L(v_2)}$$

untuk semua $v_1, v_2 \in V$, $\alpha \beta$ skalar.

✓ Teorema.

1. $L(v_1 + v_2) = L(v_1) + L(v_2)$, $\forall v_1, v_2 \in V$,
2. $L(\alpha v) = \alpha L(v)$, u/ $v \in V$, α skalar.

✓ Contoh Transformasi Linear.

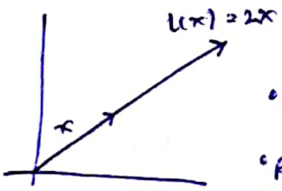
1. Perentangan / penyusutan di R^2.

$L(x) = 2x$, u/ setiap $x \in R^2$.

↳ transformasi linear karena

$L(x+y) = 2(x+y) = 2x + 2y = L(x) + L(y)$

$L(\alpha x) = 2(\alpha x) = \alpha(2x) = \alpha L(x)$



$L(x) = \beta x$

- β positif $> 1 \to$ perentangan
- β positif $< 1 \to$ penyusutan
- β negatif $\to$ arah berlawanan

2. Proyeksi pada sumbu x_1 di R^2.

$L(x) = x_1 e_1 = \begin{pmatrix} x_1 \\ 0 \end{pmatrix}$, untuk $x \in R^2$.

misalkan $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ dan $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$, maka

$\alpha x + \beta y = \begin{pmatrix} \alpha x_1 + \beta y_1 \\ \alpha x_2 + \beta y_2 \end{pmatrix}$, sehingga

$$\begin{aligned} L(\alpha x + \beta y) &= (\alpha x_1 + \beta y_1, 0)^T \\ &= (\alpha x_1, 0)^T + (\beta y_1, 0)^T \\ &= L(\alpha x) + L(\beta y) \end{aligned}$$

∴ transformasi linear.

3. Pencerminan terhadap sumbu x_1

$L(x) = (x_1, -x_2)^T = \begin{pmatrix} x_1 \\ -x_2 \end{pmatrix}$, $\forall x = (x_1, x_2)^T \in R^2$.

4. Pemutaran vektor sebesar $90°$ dalam arah berlawanan dg arah jam

$L(x) = (-x_2, x_1)^T$, u/ setiap $x = (x_1, x_2)^T \in R^2$.

✓ Contoh Bukan Transformasi linear.

misal: $M(x) = \sqrt{x_1^2 + x_2^2}$

Perhatikan:

$$\begin{aligned} M(\alpha x) &= \sqrt{(\alpha x_1)^2 + (\alpha x_2)^2} \\ &= \sqrt{\alpha^2 (x_1^2 + x_2^2)} \\ &= \sqrt{\alpha^2} \sqrt{x_1^2 + x_2^2} \\ &= |\alpha| \sqrt{x_1^2 + x_2^2} \to \text{selalu positif.} \end{aligned}$$

$\alpha M(x) = \alpha \sqrt{x_1^2 + x_2^2} \to$ bisa negatif.

∴ $M(\alpha x) \neq \alpha M(x)$, sehingga M bukan transformasi linear.

✓ Transformasi linear dari R^n ke R^m.

$L : R^2 \to R^3$.

$L(x_1, x_2)^T = \begin{pmatrix} x_2 \\ x_1 \\ x_1 + x_2 \end{pmatrix}$

misal: $x, y \in R^2$

$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

(i) $x + y = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \end{pmatrix}$

$L(x+y) = \begin{pmatrix} x_2 + y_2 \\ x_1 + y_1 \\ (x_1 + y_1) + (x_2 + y_2) \end{pmatrix}$

$L(x) + L(y) = \begin{pmatrix} x_2 \\ x_1 \\ x_1 + x_2 \end{pmatrix} + \begin{pmatrix} y_2 \\ y_1 \\ y_1 + y_2 \end{pmatrix} = \begin{pmatrix} x_2 + y_2 \\ x_1 + y_1 \\ x_1 + y_1 + x_2 + y_2 \end{pmatrix}$

$\therefore L(x+y) = L(x) + L(y)$ sehingga transformasi linear.

(ii) $\alpha x = \begin{pmatrix} \alpha x_1 \\ \alpha x_2 \end{pmatrix}$ utuk α skalar sembarang

$$\left.\begin{array}{l} L(\alpha x) = \begin{pmatrix} \alpha x_2 \\ \alpha x_1 \\ \alpha x_1 + \alpha x_2 \end{pmatrix} \\ \alpha L(x) = \alpha \begin{pmatrix} x_2 \\ x_1 \\ x_1 + x_2 \end{pmatrix} \end{array}\right\} L(\alpha x) = \alpha L(x)$$

Dan (i) & (ii), L merupakan transformasi linear.

atau

$$\alpha x + \beta y = \begin{pmatrix} \alpha x_1 + \beta y_1 \\ \alpha x_2 + \beta y_2 \end{pmatrix}$$

$$L(\alpha x + \beta y) = \begin{pmatrix} \alpha x_2 + \beta y_2 \\ \alpha x_1 + \beta y_1 \\ \alpha x_1 + \beta y_1 + \alpha x_2 + \beta y_2 \end{pmatrix}$$

$$\alpha L(x) + \beta L(y) = \alpha \begin{pmatrix} x_2 \\ x_1 \\ x_1 + x_2 \end{pmatrix} + \beta \begin{pmatrix} y_2 \\ y_1 \\ y_1 + y_2 \end{pmatrix}$$

$$= \begin{pmatrix} \alpha x_2 + \beta y_2 \\ \alpha x_1 + \beta y_1 \\ \alpha x_1 + \alpha x_2 + \beta y_1 + \beta y_2 \end{pmatrix}$$

$\therefore L(\alpha x + \beta y) = \alpha L(x) + \beta L(y)$

L transformasi linear.

atau.

$L: R^2 \to R^3$ dapat ditulis sbg:

$$L(x) = \begin{pmatrix} x_2 \\ x_1 \\ x_1 + x_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = Ax,$$

catatan:

$$\boxed{L: R^n \to R^m \quad \to \quad A_{m \times n}}$$

$\therefore$ transformasi linear karena

$$L_A(\alpha x + \beta y) = A(\alpha x + \beta y)$$
$$= \alpha Ax + \beta Ay$$
$$= \alpha L_A(x) + \beta L_A(y)$$

contoh:

$\boxed{1}$ $L: R^3 \to R^2$

$$L(x) = (x_2, x_3)^T = \begin{pmatrix} x_2 \\ x_3 \end{pmatrix} \quad u/\ x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\begin{pmatrix} x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$L_A(x) = A \cdot x.$$

$\therefore$ dapat ditunjukkan bahwa $L_A(x) = Ax$ merupakan transformasi linear.

contoh:

catatan: $\bar{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ $\quad L(x) = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

$\boxed{2}$ $L: R^3 \to R^2$

$$L(x) = L(x_1, x_2, x_3)^T = \begin{pmatrix} 1 + x_1 \\ x_2 \end{pmatrix}$$

misal: $x, y \in R^3$ dg $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, $y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$

$$x + y = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \end{pmatrix}$$

$$L(x+y) = \begin{pmatrix} 1 + (x_1 + y_1) \\ x_2 + y_2 \end{pmatrix}$$

$$L(x) + L(y) = \begin{pmatrix} 1 + x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1 + y_1 \\ y_2 \end{pmatrix}$$

$$= \begin{pmatrix} 2 + x_1 + y_1 \\ x_2 + y_2 \end{pmatrix} \neq L(x+y)$$

$\therefore$ L bukan transformasi linear.

✓ Ruang Nol (Kernel) dan Transformasi linear.

$L: V \to W$

$$\boxed{Ker(L) = \{ v \in V \mid L(v) = 0_W \}}$$

K-8 alra

contoh:

1. $L : R^2 \to R^2$ dg $L(x,y) = (y,x)^T$

$L(x,y)^T = \begin{pmatrix} y \\ x \end{pmatrix}$

$ker(L) = \{ v \in R^2 \mid L(v) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \}$

$v = \begin{pmatrix} x \\ y \end{pmatrix}$

$v \in ker(L) \to L(v) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\begin{pmatrix} y \\ x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$y = 0$ dan $x = 0$

$ker(L) = \{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \}$

2. $L : R^2 \to R^2$

$L(x) = \begin{pmatrix} x_2 \\ x_2 \end{pmatrix}$ dg $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$x \in ker(L) \to L(x) = 0$

$\begin{pmatrix} x_2 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$x_2 = 0$ dg $x \in R$ sembarang

$ker(L) = \{ \begin{pmatrix} x_1 \\ 0 \end{pmatrix} \mid x_1 \in R \}$

contoh elemen : $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1/2 \\ 0 \end{pmatrix}$

3. $L : R^3 \to R^3$

$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ dan $L(x) = \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix}$

$x \in ker(L) \to L(x) = 0$

$\begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ $x_1 = 0$, $x_2 = 0$, $x_3 \in R$ sembarang

$ker(L) = \{ \begin{pmatrix} 0 \\ 0 \\ t \end{pmatrix} \mid t \in R \}$

catatan : $\begin{pmatrix} 0 \\ 0 \\ t \end{pmatrix} = t \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

basis $ker(L) = \{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \}$

Jangkauan dan Peta

$L : V \to W$ transformasi linear dan S adalah rb V. Jangkauan dari S = L(S) adalah

$$\boxed{L(S) = \{ w \in W \mid w = L(v),\ v \in S \}}$$

Jangkauan dari V, yaitu L(V) disebut Peta (range) dari L.

contoh:

diberikan transformasi linear $L : R^3 \to R^3$.

dg $L(x) = (x_1, x_2, 0)^T$

Tentukan:

I. Jangkauan dari $S = \{ (1,2,3)^T, (1,0,-1)^T, (2,-2,2)^T \}$

$L : R^3 \to R^3$

$L(x) = \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix}$

a) $L \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$

$L \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$L \begin{pmatrix} 2 \\ -2 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ 0 \end{pmatrix}$

$L(S) = \{ \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ -2 \\ 0 \end{pmatrix} \}$

b) $L(x) = \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix}$ u/ sembarang $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in R^3$

Peta dari $L = \{ \begin{pmatrix} a \\ b \\ 0 \end{pmatrix} \mid a, b \in R^2 \}$

Jawab : $S \subseteq R^3$

$S = \{ \alpha e_1 + \beta e_2 \mid \alpha, \beta \in R \}$

$= \{ \alpha \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \}$

$= \left\{ \begin{pmatrix} \alpha \\ \beta \\ 0 \end{pmatrix}, \alpha \beta \in R \right\}$

① $L(x) = \begin{pmatrix} x_3 \\ x_2 \\ x_1 \end{pmatrix}$

$L(s) = \left\{ \begin{pmatrix} 0 \\ \beta \\ \alpha \end{pmatrix} \right\}$

② $L(x) = \begin{pmatrix} x_1 \\ x_1 \\ x_1 \end{pmatrix}$

$L(s) = \left\{ \begin{pmatrix} \alpha \\ \alpha \\ \alpha \end{pmatrix} \middle| \alpha \in R \right\}$

$= \left\{ \alpha \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \middle| \alpha \in R \right\}$

kg. alin.

$\boxed{\text{soal:}}$

$L: R^3 \to R^2$

$L(x) = (x_3, x_1 + x_2)$

$L(x) = \begin{pmatrix} x_3 \\ x_1 + x_2 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$

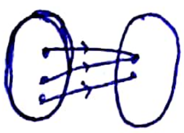
$a, b \in R$ sembarang.

$L(R^3) = R^2$

∴ L transformasi linear yg pada (onto).

translin pada ≡ surjektif

✓ $\boxed{\text{catatan:}}$



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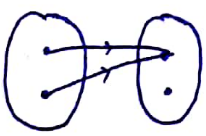

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kg.

Lambang Matriks Translin

✓ Teorema lambang matriks.

$L: V \to W$

(basis: E) (basis: F)

terdapat matriks A berorde m×n

sehingga: $\boxed{[L(v)]_F = A\,[v]_E}$

untuk setiap $\bar{v} \in V$.

$\boxed{a_j = [L(v_j)]_F \text{ untuk } j = 1, 2, \ldots, n.}$

contoh:

1. $L: R^3 \to R^2$ $u_1 = (1, 0, -1)^T$ $u_2 = (-1, -2, 1)^T$ $u_3 = (-1, -1, 1)^T$, $b_1 = (1,1)$ $b_2 = (2,1)^T$

$\underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in R^3$. $L(\underline{x}) = \begin{pmatrix} x_1 + x_2 \\ x_1 - x_3 \end{pmatrix}$

- Basis $E = \{u_1, u_2, u_3\}$

$L(u_1) = \begin{pmatrix} 1 + 0 \\ 1 - (-1) \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$L(u_2) = \begin{pmatrix} 1 + (-2) \\ 1 - 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$

$L(u_3) = \begin{pmatrix} -1 + (-1) \\ -1 - 1 \end{pmatrix} = \begin{pmatrix} -2 \\ -2 \end{pmatrix}$

a. lambang matriks untuk L:

- $\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \underset{(3)}{c_1} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \underset{(-1)}{c_2} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = L(u_1)$
- $L(u_2) = \underset{(1)}{c_1} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \underset{(-1)}{c_2} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$
- $L(u_3) = \begin{pmatrix} -2 \\ -2 \end{pmatrix} = \overset{(-2)}{c_1} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \overset{(0)}{c_2} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$A = \begin{pmatrix} 3 & 1 & -2 \\ -1 & -1 & 0 \end{pmatrix}$ //

kg. b. atm

$$\begin{pmatrix}-1\\-2\\3\end{pmatrix} = c_1\begin{pmatrix}1\\0\\-1\end{pmatrix} + c_2\begin{pmatrix}1\\-2\\1\end{pmatrix} + c_3\begin{pmatrix}-1\\-1\\1\end{pmatrix}$$

$$\left(\begin{array}{ccc|c}1&1&-1&-1\\0&-2&-1&-2\\-1&1&1&3\end{array}\right) \underset{\sim}{E_{31}(1)} \left(\begin{array}{ccc|c}1&1&-1&-1\\0&-2&-1&-2\\0&2&0&2\end{array}\right)$$

$$\underset{\sim}{E_{32}(1)} \left(\begin{array}{ccc|c}1&1&-1&-1\\0&-2&-1&-2\\0&0&-1&0\end{array}\right) \quad \boxed{c_3=0}$$

$$-2c_2=-2$$

$$\boxed{c_2=1}$$

$$c_1+c_2-c_3=-1$$

$$c_1+1-0=-1$$

$$\boxed{c_1=-2} \qquad [x]_E = \begin{pmatrix}-2\\1\\0\end{pmatrix}$$

Jadi, $[L(x)]_F = A\,[x]_E$

$$= \begin{pmatrix}3&1&-2\\-1&-1&0\end{pmatrix}\begin{pmatrix}-2\\1\\0\end{pmatrix}$$

$$= \begin{pmatrix}-5\\1\end{pmatrix}$$

catatan: $L(-1,-2,3)^T = \begin{pmatrix}-3\\-4\end{pmatrix}$

$$\begin{pmatrix}-3\\-4\end{pmatrix} = \underbrace{-5\begin{pmatrix}1\\1\end{pmatrix} + 1\begin{pmatrix}2\\1\end{pmatrix}}_{}$$

$$= [L(x)]_F \quad //$$

✓ Teorema ✓

Khusus untuk: $L: \mathbb{R}^n \to \mathbb{R}^m$.

$$\boxed{a_j = B^{-1} L(u_j), \quad \forall j = 1,2,\dots,n}$$

dengan: $B = (b_1, b_2, \dots, b_m)$.

contoh tadi:

$$B = \begin{pmatrix}1&2\\1&1\end{pmatrix} \quad B^{-1} = \frac{1}{-1}\begin{pmatrix}1&-2\\-1&1\end{pmatrix}$$

$$= \begin{pmatrix}-1&2\\1&-1\end{pmatrix}$$

- $L(u_1) = \begin{pmatrix}1\\2\end{pmatrix} \to a_1 = B^{-1}L(u_1)$

$$a_1 = \begin{pmatrix}-1&2\\1&-1\end{pmatrix}\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}3\\-1\end{pmatrix}$$

- $L(u_2) = \begin{pmatrix}-1\\0\end{pmatrix} \to a_2 = \begin{pmatrix}-1&2\\1&-1\end{pmatrix}\begin{pmatrix}-1\\0\end{pmatrix}$

$$= \begin{pmatrix}1\\-1\end{pmatrix}$$

- $L(u_3) = \begin{pmatrix}-2\\-2\end{pmatrix} \to a_3 = \begin{pmatrix}-1&2\\1&-1\end{pmatrix}\begin{pmatrix}-2\\-2\end{pmatrix}$

$$= \begin{pmatrix}-2\\0\end{pmatrix}$$

$$\therefore A = \begin{pmatrix}3&1&-2\\-1&-1&0\end{pmatrix}$$

✓ Akibat. ✓

Jika A matriks yg melambangkan translin $L: \mathbb{R}^n \to \mathbb{R}^m$. relatif thd basis² E & F.

maka bentuk eselon baris tereduksi dari

$(b_1, \dots, b_m \mid L(u_1), \dots, L(u_n)) = \underline{(I \mid A)}$

contoh akibat:

$$\left(\begin{array}{cc|ccc}1&2&1&-1&-2\\1&1&2&0&-2\end{array}\right) \underset{\sim}{E_{21}(-1)} \left(\begin{array}{cc|ccc}1&2&1&-1&-2\\0&-1&1&1&0\end{array}\right)$$

$$\underset{\sim}{E_{2}(-1)} \left(\begin{array}{cc|ccc}1&2&1&-1&-2\\0&1&-1&-1&0\end{array}\right) \underset{\sim}{E_{12}(-2)}$$

$$\left(\begin{array}{cc|ccc}1&0&3&1&-2\\0&1&-1&-1&0\end{array}\right)$$

$$\underbrace{}_{I} \quad \underbrace{}_{A}$$

eselon baris tereduksi

alin

Keserupaan

- Definisi.

misal A dan B adalah matriks $n \times$ berukuran $n \times n$. B serupa dg A jika terdapat matriks tak singular s.

$$\boxed{B = S^{-1} A S.}$$

contoh:

1. B serupa dengan A.

$B = S^{-1} A S$ *

Apakah A serupa dg B ?

Dari * : $S^{-1} A S = B$

$\underbrace{S . S^{-1}}_{I} A S = S B$

$A S = S B$

$A . \underbrace{S . S^{-1}}_{I} = S B S^{-1}$

$= S B S^{-1}$

$= \underbrace{(S^{-1})^{-1}}_{I} . B . S^{-1}$

$\therefore$ A serupa dg B.

2. misalkan $L : R^2 \to R^2$ adalah translin dg $L(x) = (-x_1, x_2)^T$

a. Tentukan matriks A yg melambangkan L relatif terhadap basis terurut $E = [e_1, e_2]$

Jawab: $E = [e_1, e_2] = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$

- $L(e_1) = \begin{pmatrix} -1 \\ 0 \end{pmatrix} = -1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 0 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

- $L(e_2) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 1 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

matriks $A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

b. tentukan matriks B yg melambangkan L relatif thd basis terurut $F = [u_1, u_2]$ dg :

$u_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ dan $u_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

- $L(u_1) = \begin{pmatrix} -1 \\ 1 \end{pmatrix} = 0 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + 1 \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

- $L(u_2) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + 0 \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$\therefore B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

c. matriks transisi S dari F ke E.

$\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 1 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\begin{pmatrix} -1 \\ 1 \end{pmatrix} = -1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 1 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\therefore S = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$

d. Periksa Apakah $B = S^{-1} A S$.

$S^{-1} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{pmatrix}$

$S^{-1} A S = \begin{pmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$

$= \begin{pmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$

$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$= B$ //

K.10 alin.

Ortogonalitas

Hasil Kali Skalar di Ruang Euclide

Definisi

misal $x = (x_1, x_2, \dots x_n)^T$

$y = (y_1, y_2, \dots y_n)^T$ di R^n.

$$\boxed{x^T y = x_1 y_1 + x_2 y_2 + \dots + x_n y_n}$$

contoh:

1. $x = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$, $y = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$

$$x^T y = 2 \times (-3) + (-1) \times 1 = -7.$$

Panjang Euclides dari Suatu Vektor

- Definisi

$$\|x\| = \sqrt{x^T x} = \begin{cases} \sqrt{x_1^2 + x_2^2}, & x \in R^2 \\ \sqrt{x_1^2 + x_2^2 + x_3^2}, & x \in R^3. \end{cases}$$

contoh:

1. $x = \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix}$, $\|x\| = \sqrt{(-2)^2 + 2^2 + 1^2}$

$= 3.$

- Teorema

Jika x dan y 2 vektor taknol di R^2 atau R^3 dan θ sudut di antaranya maka:

$$\boxed{x^T y = \|x\| \, \|y\| \cos\theta}$$

Vektor Satuan dan Sudut.

$$u = \frac{1}{\|x\|} x, \quad v = \frac{1}{\|y\|} y$$

Teorema:

$$\cos\theta = \frac{x^T y}{\|x\| \|y\|} = u^T v.$$

Pertaksamaan Cauchy - Schwarz.

$$\boxed{|x^T y| \le \|x\| \, \|y\|}$$

Berlaku jika salah satu vektor 0. atau vektor yg satu merupakan kelipatan vektor yg lain.

Ortogonal.

Vektor x dan y di R^2 atau di R^3 ortogonal jika $\boxed{x^T y = 0}$

contoh:

1. $x = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $y = \begin{pmatrix} 4 \\ -8 \end{pmatrix}$

$$x^T y = (2 \ 1) \begin{pmatrix} 4 \\ -8 \end{pmatrix} = 0$$

$\therefore$ ortogonal.

Proyeksi Skalar

Definisi: $\boxed{\alpha = \dfrac{x^T y}{\|y\|}}$

contoh:

1. $v = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$, $w = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

a. Proyeksi skalar dari v pada w.

$$\alpha = \frac{v^T w}{\|w\|} = \frac{0}{\sqrt{9+4}} = 0$$

b. Proyeksi skalar dan w pada v.

$$\beta = \frac{w^T . v}{\|v\|} = \frac{0}{\sqrt{13}} = 0$$

✓ Proyeksi Vektor.

$$\boxed{p = \alpha u = \alpha \frac{1}{\|y\|} y = \frac{x^T y}{y^T y} y}$$

✓ Vektor satuan.

$$v = \begin{pmatrix} 2 \\ -3 \end{pmatrix} \rightarrow \|v\| = \sqrt{13}$$

$$\underline{u} = \frac{1}{\|v\|} v = \frac{1}{\sqrt{13}} \begin{pmatrix} 2 \\ -3 \end{pmatrix} = \begin{pmatrix} 2\sqrt{13} \\ -3\sqrt{13} \end{pmatrix}$$

$$\|u\| = \sqrt{\frac{4}{13} + \frac{9}{13}} = 1$$

$$p = \alpha \cdot \frac{1}{\|y\|} \cdot y$$

$$= \frac{x^T y}{\|y\|} \cdot \frac{1}{\|y\|} \cdot y$$

$$= \frac{x^T . y}{(\|y\|)^2} \cdot y = \frac{x^T y}{(\sqrt{y^T y})^2} \cdot y.$$

K-11

Ruang Hasil Kali Dalam

✓ Definisi

1. $\langle x, x \rangle \geq 0$, $x = 0$
2. $\langle x, y \rangle = \langle y, x \rangle$, $\forall x, y \in V$
3. $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$, $\forall x, y, z \in V$ dan semua skalar α, β.

catatan:

Sifat 3 dapat dituliskan dalam :

1. $\langle x+y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$, $\forall x, y, z \in V$.
2. $\langle \alpha x, z \rangle = \alpha \langle x, z \rangle$, α skalar.

✓ Contoh Hasil Kali dalam

1. Di Ruang Vektor R^n.

a. Tunjukkan bahwa $x^T y$ merupakan hasil kali dalam di R^n.

Jawab:

$\langle x, y \rangle = x^T y$ di R^n.

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \in R^n.$$

$$\langle x, y \rangle = x^T y$$
$$= (x_1 \ x_2 \ \cdots \ x_n) \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$$
$$= x_1 y_1 + x_2 y_2 + \cdots + x_n y_n.$$

(i) $\langle x, x \rangle = x^T x = x_1^2 + x_2^2 + \cdots + x_n^2 \geq 0$

$\langle x, x \rangle = 0 \Leftrightarrow x_1^2 + x_2^2 + \cdots + x_n^2 = 0$

$\Leftrightarrow x_1^2 = x_2^2 = \cdots = x_n^2 = 0$

$\Leftrightarrow x_i = 0;\ i = 1, 2, \cdots, n$

$\Leftrightarrow x = 0$

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lanjutan soal.

(ii) $\langle x, y \rangle = x^T y = \sum_{i=1}^{n} x_i \cdot y_i$

$= \sum_{i=1}^{n} y_i \cdot x_i$

$= y^T x = \langle y, x \rangle$

$\therefore \langle x, y \rangle = \langle y, x \rangle$

(iii) $\alpha x + \beta y = \begin{pmatrix} \alpha x_1 + \beta y_1 \\ \alpha x_2 + \beta y_2 \\ \vdots \\ \alpha x_n + \beta y_n \end{pmatrix}, \; z = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix}$

$\langle \alpha x + \beta y, z \rangle = (\alpha x_1 + \beta y_1) z_1 + \dots$
$+ (\alpha x_n + \beta y_n) z_n$
$= \alpha x_1 z_1 + \beta y_1 z_1 + \dots + \alpha x_n z_n + \beta y_n z_n$
$= \alpha x_1 z_1 + \dots + \alpha x_n z_n + \beta y_1 z_1 + \dots + \beta y_n z_n$
$= \alpha (x_1 z_1 + \dots + x_n z_n) + \beta (y_1 z_1 + \dots + y_n z_n)$
$= \alpha \langle x, z \rangle + \beta \langle y, z \rangle$

b. $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \; y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

$\langle x, y \rangle = 3 x_1 y_1 + 2 x_2 y_2$

(i) $\langle x, x \rangle = 3 x_1 x_1 + 2 x_2 x_2$
$= 3 x_1^2 + 2 x_2^2 \geq 0$

$\vdots$

2 Di Ruang Vektor $R^{m \times n}$

Jika $A, B \in R^{m \times n}$, maka:

$$\boxed{\langle A, B \rangle = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} \cdot b_{ij}}$$

contoh:

a. misal: $A = \begin{pmatrix} 1 & 2 & 3 \\ -2 & 3 & 5 \end{pmatrix}$

$B = \begin{pmatrix} 4 & 6 & -3 \\ 2 & 0 & -1 \end{pmatrix}$

tentukan $\langle A, B \rangle$?

Jawab:

$\langle A, B \rangle = \underbrace{1 \times 4 + 2 \times 6 + 3 \times (-3)}_{i=1}$
$+ (-2) \times 2 + 3 \times 0 + 5 \times (-1)$
$= 7 + (-9) = -2$

✓ Sifat - Sifat.

Jika $v \in V$, maka panjang / norm dari v adalah:

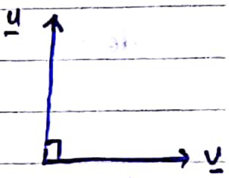
$$\boxed{\|v\| = \sqrt{\langle v, v \rangle}}$$

dua vektor u dan v dikatakan ortogonal jika $\boxed{\langle u, v \rangle = 0}$

✓ Sifat Vektor-Vektor Ortogonal

Teorema:

Hukum pythagoras.



$\|v\|^2 = \langle v, v \rangle$

$$\boxed{\|u + v\|^2 = \|u\|^2 + \|v\|^2}$$

✓ Norm Frobenius

notasi: $\| \cdot \|_F$

Jika $A \in R^{m \times n}$, maka.

$$\boxed{\|A\|_F = \sqrt{\langle A, A \rangle}}$$

$= \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}^2}$

contoh:

(1) $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}$

maka:

$\|A\|_F = \sqrt{1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2}$

$= \sqrt{91}$ //

Proyeksi Skalar dan Proyeksi Vektor.

Def:

Jika u dan v adalah vektor di ruang hasil kali dalam V dan $v \neq 0$. maka:

- proyeksi skalar dari u pada v.

$$\boxed{\alpha = \frac{\langle u, v\rangle}{\|v\|}},$$

- proyeksi vektor dari u pada v,

$$\boxed{p = \alpha \left(\frac{1}{\|v\|} v\right) = \frac{\langle u, v\rangle}{\langle v, v\rangle} v}$$

contoh:

1. misalkan $A, B \in R^{2\times 2}$ dg:

$$A = \begin{pmatrix} 1 & 2 \\ 1 & -3 \end{pmatrix}, B = \begin{pmatrix} 0 & -1 \\ 1 & -5 \end{pmatrix}$$

dengan menggunakan hasil kali dalam

$$\langle A, B\rangle = tr(A^T B)$$

tentukan:

a. vektor p, yaitu proyeksi vektor dari A pada B.

Jawab: $p = \frac{\langle A, B\rangle}{\langle B, B\rangle} B$

(i) $\langle A, B\rangle = tr(A^T B)$

$$A^T B = \begin{pmatrix} 1 & 1 \\ 2 & -3 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & -5 \end{pmatrix} = \begin{pmatrix} 1 & -6 \\ -3 & 13 \end{pmatrix}$$

$\langle A, B\rangle = 1 + 13 = 14.$

(ii) $\langle B, B\rangle = tr(B^T B)$

$$B^T B = \begin{pmatrix} 0 & 1 \\ -1 & -5 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & -5 \end{pmatrix} = \begin{pmatrix} 1 & -5 \\ -5 & 26 \end{pmatrix}$$

$\langle B, B\rangle = tr(B^T B) = 1 + 26$

$= 27$ //

$\therefore p = \frac{14}{27}\begin{pmatrix} 0 & -1 \\ 1 & -5 \end{pmatrix}$ //

Pengamatan

Jika $v \neq 0$ dan p adalah proyeksi vektor dari u pada v, maka:

1. u−p dan p ortogonal.
2. u = p jhj $u = \beta v$, β skalar.

Teorema (Cauchy-Schwarz)

$$\boxed{|\langle u, v\rangle| \leq \|u\| \|v\|}$$

berlaku jhj u & v bergantung linear.

Akibat Ketaksamaan Cauchy-Schwarz

Jika u dan v vektor taknol.

$$\boxed{\cos\theta = \frac{\langle u, v\rangle}{\|u\| \|v\|}}$$

Norma / Panjang (norm)

Definisi

dikatakan ruang linear bernorma jika:

1. $\|v\| \geq 0$, $v = 0$
2. $\|\alpha v\| = |\alpha| \|v\|$, α skalar
3. $\|v + w\| \leq \|v\| + \|w\|$, $\forall v, w \in V$ (pertaksamaan segitiga).

Teorema

Jika V suatu ruang hasil kali dalam.

$$\boxed{\|v\| = \sqrt{\langle v, v\rangle}}$$

alin. K.12.

No.

Date

- Norma lain di $\mathbb{R}^n$.

1. $\|x\|_\infty = \max_{1 \le i \le n} |x_i|$

2. $\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}$, $\forall$

setiap bil. real $p \geq 1$

a. $\|x\|_1 = \sum_{i=1}^{n} |x_i|$

b. $\|x\|_2 = \sqrt{\sum_{i=1}^{n} |x_i|^2} = \sqrt{\langle x, x \rangle}$

Jika $p \neq 2$, maka $\|\cdot\|_p$ tidak berkorespodensi dg hasil kali dalam manapun.

Definisi

Jarak antara x & y → $\|x - y\|$

Himpunan Ortonomal

Definisi

$\langle v_i, v_j \rangle = 0$ bila $i \neq j$

maka $\{v_1, v_2, \dots v_n\}$ dikatakan himpunan ortogonal dari vektor**.

contoh:

① Himp. $\{(1,1,1)^T, (2\ 1\ -3)^T, (4\ -5\ 1)^T\}$ ortogonal di $\mathbb{R}^T$ karena:

$(1,1,1)(2,1,-3)^T = 0$

$(1,1,1)(4,-5,1)^T = 0$

$(2,-1,3)(4,-5,1)^T = 0$

Teorema

Jika $\{v_1, v_2, \dots, v_n\}$ adalah himpunan ortogonal dari vektor** taknol pada ruang hasil kali dalam V, maka $v_1, v_2, \dots v_n$ adalah bebas linear.

Definisi

Sebuah himpunan ortonormal dari vektor** adalah sebuah himpunan ortogonal dari vektor** satuan.

contoh:

① $v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $v_2 = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$, $v_3 = \begin{pmatrix} 4 \\ -5 \\ 1 \end{pmatrix}$

Jawab:

$u_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{3}}(1,1,1)^T$

$u_2 = \frac{v_2}{\|v_2\|} = \frac{1}{\sqrt{14}}(2,1,-3)^T$

$u_3 = \frac{v_3}{\|v_3\|} = \frac{1}{\sqrt{42}}(4,-5,1)^T$

- Proses ortogonalisasi Gram - Schmit.

misalkan

$u_1 = \left(\frac{1}{\|x_1\|}\right) x_1$

↓

$p_1 = \langle x_2, u_1 \rangle u_1$

maka: ↓

$(x_2 - p_1) \perp u_1$

$u_2 = \frac{(x_2 - p_1)}{\|x_2 - p_1\|}$

↓

$p_2 = \langle x_3, u_1 \rangle u_1 + \langle x_3, u_2 \rangle u_2$

↓

$u_3 = \frac{(x_3 - p_2)}{\|x_3 - p_2\|}$ dst...

basis ortonormalnya: $\{u_1, u_2, u_3\}$.

Dekomposisi QR

- definisi

$r_{11} = \|a_1\|$

$r_{kk} = \|a_k - P_{k-1}\|,$

$r_{ik} = q_i^T a_k$

untuk $i = 1, 2, \dots, k-1$

$k = 2, 3, \dots, n.$

didefinisikan $Q = (q_1, q_2, \dots, q_n)$ dan

$$R = \begin{pmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ 0 & r_{22} & \cdots & r_{2n} \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & r_{nn} \end{pmatrix}$$

maka : $A = QR.$

contoh:

$$A = \begin{pmatrix} 2 & 1 \\ 2 & 0 \\ 0 & 1 \end{pmatrix}$$

Jawab:

$r_{11} = \|a_1\| = \sqrt{8} = 2\sqrt{2}$

$q_1 = \frac{1}{r_{11}} a_1 = \frac{1}{2\sqrt{2}} (2, 2, 0)^T$

$r_{12} = \langle a_2, q_1 \rangle = a_2^T q_1 = q_1^T a_2$

$= \frac{1}{\sqrt{2}}$

$P_1 = r_{12} q_1$

$= \frac{1}{\sqrt{2}} \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right)^T$

$= \left(\frac{1}{2}, \frac{1}{2}, 0 \right)^T$

$a_2 - P_1 = \left(\frac{1}{2}, -\frac{1}{2}, 1 \right)^T$

$r_{22} = \|a_2 - P_1\| = \frac{1}{2}\sqrt{6}$

$q_2 = \frac{1}{r_{22}} (a_2 - P_1) = \frac{1}{\sqrt{6}} (1, -1, 2)^T$

Jadi, basis ortonormalnya $\{q_1, q_2\}$

K.13. alm

Nilai Eigen

- Nilai Eigen & Vektor Eigen

Definisi

$$\boxed{Ax = \lambda x} \rightarrow A_{n \times n}\, x_{n \times 1}$$

λ = Nilai eigen

x = vektor eigen → taknol.

contoh:

① $A = \begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix}$ $x = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

Jawab:

$$Ax = \begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \end{pmatrix} = 3x.$$

sehingga, $\lambda = 3$. → nilai eigen.

- Jika λ adalah nilai eigen dari matriks A, maka

$$Ax - \lambda x = 0$$

$$(A - \lambda I)x = 0$$

andaikan $A - \lambda I$ matriks taksingular.

$A - \lambda I$ mempunyai inven.

$$\underbrace{(A - \lambda I)^{-1} \cdot (A - \lambda I)}\underline{x} = (A - \lambda I)^{-1} \cdot \underline{0}$$

$$I\underline{x} = \underline{0}$$

$$\underline{x} = \underline{0}$$

Kontradiksi dg vektor eigen adalah vektor taknol.

∴ Haruslah $A - \lambda I$ matriks singular

∴ $\det(A - \lambda I) = 0$ //

② tentu nilai eigen, vektor eigen, dan basis dari ruang eigen u/ matriks berikut:

(i) $A = \begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix}$ (ii) $B = \begin{pmatrix} 3 & -1 & -2 \\ 2 & 0 & -2 \\ 2 & -1 & -1 \end{pmatrix}$

Jawab:

(i) untuk matriks A.

- $A - \lambda I = \begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$= \begin{pmatrix} 3-\lambda & 0 \\ 8 & -1-\lambda \end{pmatrix}$$

- $|A - \lambda I| = 0$

$$\begin{vmatrix} 3-\lambda & 0 \\ 8 & -1-\lambda \end{vmatrix} = (3-\lambda)(-1-\lambda) = 0$$

nilai eigen: $\lambda_1 = 3$ & $\lambda_2 = -1$.

- $\lambda_1 = 3$.

$$(A - 3I)x = 0$$

$$\begin{pmatrix} 0 & 0 \\ 8 & -4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

diperoleh: $8x_1 - 4x_2 = 0$

$$8x_1 = 4x_2$$

$$\boxed{x_2 = 2x_1}$$

$$x = \begin{pmatrix} x_1 \\ 2x_1 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Jadi, semua kelipatan taknol dari $(1,2)^T$ merupakan vektor eigen yg berpadanan dg $\lambda_1 = 3$.

Basis $= \{(1,2)^T\}$

- untuk $\lambda_2 = -1$

$$(A+I)x = 0$$

$$\begin{pmatrix} 4 & 0 \\ 8 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

diperoleh $x_1 = 0$.

sehingga,

$$x = \begin{pmatrix} 0 \\ x_2 \end{pmatrix} = x_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Jadi,
vektor eigen yg berpadanan dg $\lambda_2 = -1$ adalah semua kelipatan taknol dari $(0,1)^T$. basis $= \{(0,1)^T\}$ //

(ii) untuk matriks B.

$$|B - \lambda I| = 0$$

$$\begin{vmatrix} 3-\lambda & -1 & -2 \\ 2 & -\lambda & -2 \\ 2 & -1 & -1-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - 2\lambda^2 + \lambda = 0$$

$$\lambda(\lambda^2 - 2\lambda + 1) = 0$$

$$\lambda(\lambda-1)^2 = 0$$

nilai eigennya, $\lambda_1 = 0$ $\lambda_2 = \lambda_3 = 1$

- untuk $\lambda_1 = 0$

$$Ax = \begin{pmatrix} 3 & -1 & -2 \\ 2 & 0 & -2 \\ 2 & -1 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

dengan OBD diperoleh:

$$\left(\begin{array}{ccc|c} 3 & -1 & -2 & 0 \\ 0 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

sehingga,

$$2x_2 - 2x_3 = 0$$

$$\boxed{x_2 = x_3}$$

$$3x_1 - x_2 - 2x_2 = 0$$

$$\boxed{x_1 = x_2}$$

Jadi vektor eigennya,

$$x = \begin{pmatrix} x_1 \\ x_1 \\ x_1 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

vektor eigen: kelipatan taknol dari $(1,1,1)^T$

basis: $\{(1,1,1)^T\}$

- untuk $\lambda_2 = \lambda_3 = 1$

$$(A-I)x = \begin{pmatrix} 2 & -1 & -2 \\ 2 & -1 & -2 \\ 2 & -1 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

diperoleh,

$$2x_1 - x_2 - 2x_3 = 0$$

misal: $x_1 = s$, $x_3 = t$, $s,t \in \mathbb{R}$. maka, $x_2 = 2s - 2t$

Sehingga,

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} s \\ 2s-2t \\ t \end{pmatrix} = s\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + t\begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}$$

Basis: $\{(1,2,0)^T, (0,-2,1)^T\}$ //

K.B.

alin.

Hasil Kali & Jumlah Kali Eigen.

Teorema.

Misal $A_{n\times n}$, λ_i adalah nilai** eigen dari A. u/ $i = 1, 2, \cdots n$.

$$\boxed{\begin{aligned} \lambda_1 \lambda_2 \cdots \lambda_n &= \det(A), \\ \lambda_1 + \lambda_2 + \cdots + \lambda_n &= tr(A), \end{aligned}}$$

tr = Jumlah semua elemen diagonal utama dari A.

Pendiagonalan

$$X^{-1} A X = D.$$

- matriks diagonal.

Definisi:

matriks A berorde $n \times n$ disebut matriks diagonal jika $a_{ij} = 0$, untuk $i \neq j$.

contoh: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

- matriks yang dapat didiagonalkan.

Teorema.

Suatu matriks A berorde $n \times n$ dapat didiagonalkan jika dan hanya jika A memiliki n vektor eigen yg bebas linier

contoh:

dari soal sebelumnya.

$$A = \begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix}$$

vektor eigennya:

$\lambda_1 = 3$ dg vektor eigen $s\begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $s \neq 0$

$\lambda_2 = -1$ dg vektor eigen $t\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $t \neq 0$

A berordo 2×2 dan memiliki 2 nilai eigen yg berbeda. maka matriks A dapat didiagonalkan.

Perpangkatan matriks yang dapat didiagonalkan.

$$\boxed{A^k = X D^k X^{-1}}$$

Matriks Defektif.

Definisi

matriks berukuran $n \times n$ dan memiliki vektor eigen bebas linear yang kurang dari n. dinamakan matriks defektif

Eksponensial suatu matriks.

Jika D matriks diagonal.

$$D = \begin{pmatrix} \lambda_1 & 0 & & 0 \\ 0 & \lambda_2 & \ddots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}$$

maka.

$$e^D = \lim_{m \to \infty} \left(I + D + \frac{1}{2!} D^2 + \cdots + \frac{1}{m!} D^m \right)$$

$$= \begin{pmatrix} e^{\lambda_1} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & e^{\lambda_n} \end{pmatrix} //$$

Catatan:

① Jika A matriks sembarang, maka agak sulit menentukan e^A.

② Jika A matriks yg dapat didiagonalkan, maka:

$$\boxed{e^A = X e^D X^{-1}}$$

contoh:

dari contoh sebelumnya,

$$A = \begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix}$$

dapat didiagonalkan yaitu:

$$A = XDX^{-1}$$

$$= \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$$

eksponensial matriks A:

$$e^A = X e^D X^{-1}$$

$$= \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} e^3 & 0 \\ 0 & e^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} e^3 & 0 \\ -2e^{-1} + 2e^3 & e^{-1} \end{pmatrix}$$

Matriks Definit Positif.

Definisi

suatu matriks simetrik real A berorde $n \times n$ dikatakan definit positif jika $x^T A x > 0$ u/ semua x taknol di R^n.

contoh:

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 6 \end{pmatrix}$$

karena

$$(x_1 \ x_2) \begin{pmatrix} 1 & 0 \\ 0 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$= x_1^2 + 6x_2^2 > 0$$

$\therefore$ definit positif u/ semua $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Teorema

misal A matriks ~~dia~~ simetrik real berorde $n \times n$, maka A definit positif jhj semua nilai eigennya positif.

- Jika nilai eigen matriks A semuanya negatif, maka $-A$ matriks def. positif. sehingga A matriks definit negatif.
- Jika A memiliki nilai eigen berbeda tanda, maka A tak definit.

✓ Submatriks utama.

misal: $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$.

$A_1 = (1)$, $A_2 = \begin{pmatrix} 1 & 2 \\ 4 & 5 \end{pmatrix}$

$A_3 = A$.

✓ Sifat-sifat matriks definit positif.

Teorema

1. Jika A matriks def. positif simetrik, maka A tak singular.
2. Jika A matriks def. positif simetrik, maka det (A) > 0
3. Jika A matriks def. positif simetrik, maka submatriks utama dari A semuanya definit positif.

Teorema.

Jika A matriks def. positif simetrik, maka A dapat direduksi menjadi matriks segitiga atas dg OBD III ($E_{ij}(k)$) & semua elemen diagonalnya positif.

✓ Dekomposisi / Faktorisasi LU.

Teorema,

$$\boxed{A = LU}$$

L = matriks segitiga bawah dg elemen** 1 pada diagonalnya.

$\boxed{A = LDL^T}$ — D = matriks diagonal dg semua elemen diagonal utamanya positif. → $D = \begin{pmatrix} u_{11} & 0 & 0 \\ 0 & u_{22} & 0 \\ 0 & 0 & u_{3} \end{pmatrix}$ ⋮ dst.

$\boxed{L = LD^{1/2}}$ $\boxed{A = LL^T}$

↓

dekomposisi cholesky.

alm.

Transformasi Linear

- Teorema :

1. $L(v_1+v_2) = L(v_1) + L(v_2), \forall v_1, v_2 \in V.$
2. $L(\alpha v) = \alpha L(v)$, α skalar, $v \in V$, $V \to W$.

✓ Ruang Nol (Kernel).

$\ker(L) = \{ v \in V \mid L(v) = 0_W \}$

✓ Jangkauan (Image)

$L(S) = \{ w \in W \mid w = L(v)$, untuk suatu $v \in V \}$

✓ Peta (Range)

$L(V)$ adalah peta dari L, Jangkauan dari V.

contoh:

1. Periksa apakah L = transformasi linear dari R^3 ke R^2.

a. $L(x) = (x_2, x_3)^T$

- ambil $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, $y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$

$$L(x+y) = L\begin{pmatrix} x_1+y_1 \\ x_2+y_2 \\ x_3+y_3 \end{pmatrix} = \begin{pmatrix} x_2+y_2 \\ x_3+y_3 \end{pmatrix}$$

$$= \begin{pmatrix} x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} y_2 \\ y_3 \end{pmatrix}$$

$$L(x+y) = L(x) + L(y) \quad \blacksquare$$

- ambil α skalar, $v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \in R^3$.

$$L(\alpha v) = L\left(\alpha \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}\right)$$

$$= L\begin{pmatrix} \alpha v_1 \\ \alpha v_2 \\ \alpha v_3 \end{pmatrix}$$

$$= \begin{pmatrix} \alpha v_2 \\ \alpha v_3 \end{pmatrix}$$

$$= \alpha \begin{pmatrix} v_2 \\ v_3 \end{pmatrix} \qquad L(\alpha v) = \alpha L(v) \quad \blacksquare$$

2. Periksa apakah L = transformasi linear dari $R^2 \to R^2$.

c. $L(x) = (x_1, x_2, 1)^T$

- ambil $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in R^2$.

$$L(x+y) = L\begin{pmatrix} x_1+y_1 \\ x_2+y_2 \end{pmatrix}$$

$$= \begin{pmatrix} x_1+y_1 \\ x_2+y_2 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} x_1 \\ x_2 \\ 1 \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \\ 0 \end{pmatrix} \neq \begin{pmatrix} x_1 \\ x_2 \\ 1 \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \\ 1 \end{pmatrix}$$

$$L(x+y) \neq L(x) + L(y)$$

$\therefore$ L bukan transformasi linear.

b. $L(x) = (1 + x_1, x_2)^T$

- ambil $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, $y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$

$$L(x+y) = L\begin{pmatrix} x_1+y_1 \\ x_2+y_2 \\ x_3+y_3 \end{pmatrix}$$

$$= \begin{pmatrix} 1+(x_1+y_1) \\ x_2+y_2 \end{pmatrix}$$

$$= \begin{pmatrix} 1+x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0+y_1 \\ y_2 \end{pmatrix}$$

$$\neq \begin{pmatrix} 1+x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1+y_1 \\ y_2 \end{pmatrix}$$

$$L(x+y) \neq L(x) + L(y)$$

$\therefore$ L bukan transformasi linear.

d. $L(x) = (x_1, x_2, x_1 + 2x_2)^T$

- ambil $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in R^2$.

$$L(x+y) = L\begin{pmatrix} x_1+y_1 \\ x_2+y_2 \end{pmatrix}$$

$$L(x+y) = \begin{pmatrix} x_1+y_1 \\ x_2+y_2 \\ (x_1+y_1)+2(x_2+y_2) \end{pmatrix}$$

$$= \begin{pmatrix} x_1 \\ x_2 \\ x_1+2x_2 \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \\ y_1+2y_2 \end{pmatrix}$$

$L(x+y) = L(x) + L(y)$ ▨

- ambil α skalar, $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in R^2$.

$$L(\alpha x) = L\left(\alpha \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right)$$

$$= L\begin{pmatrix} \alpha x_1 \\ \alpha x_2 \end{pmatrix}$$

$$= \begin{pmatrix} \alpha x_1 \\ \alpha x_2 \\ \alpha x_1 + 2\alpha x_2 \end{pmatrix}$$

$$= \alpha \begin{pmatrix} x_1 \\ x_2 \\ x_1+2x_2 \end{pmatrix}$$

$L(\alpha x) = \alpha L(x)$ ▨

$\therefore$ L merupakan transformasi linear.

3 Tentukan kernel dan peta dari transformasi linear $L: R^3 \to R^3$, $L(x) = (x_1, x_2, 0)^T$

$Ker(L) = \{ v \in V \mid L(v) = 0_W \}$ $R^3 \to R^3$.

misal $x \in V$. ambil $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$R^3 \to R^3$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \to \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \begin{matrix} x_1 = 0 \\ x_2 = 0 \\ x_3 \in R \end{matrix}$$

misal: $x_3 = A$, $A \in R$

$$L(v) = \begin{pmatrix} 0 \\ 0 \\ A \end{pmatrix}$$

$$Ker(L) = \left\{ x \in V \mid x = \begin{pmatrix} 0 \\ 0 \\ A \end{pmatrix}, A \in R \right\}$$

Peta.

$$L(x) = L\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix}$$

misal: $x_1 = A$, $x_2 = B$, A dan $B \in R$

$$L(v) = \begin{pmatrix} A \\ B \\ 0 \end{pmatrix}$$

✓ Lambang Matriks Transfrm.

$L_A : R^n \to R^m$

$L_A(x) = Ax$, $\forall x \in R^n$.

Teorema: (lambang matriks).

$$[L(v)]_F = A[v]_E$$

- A adalah matriks yang melambangkan L relatif terhadap basis terurut E dan F dan kolom-kolom matriks A adalah: $\boxed{a_j = [L(v_j)]_F, \ j = 1, 2, \cdots, n.}$

soal:

① misal $E = [u_1, u_2, u_3]$ basis R^3

$F = [b_1, b_2]$ basis R^2.

$u_1 = (1, 0, -1)^T$ $\quad b_1 = (1, 1)^T$

$u_2 = (1, -2, 1)^T$ $\quad b_2 = (2, 1)^T$

$u_3 = (-1, -1, 1)^T$

$L: R^3 \to R^2$

$L(x) = (x_1 + x_2, x_1 - x_3)^T$

tentukan:

a. lambang matriks L.

$$L\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ x_1 - x_3 \end{pmatrix}$$

$A = (a_j)$ A matriks 2×3.

- $a_1 = [L(u_1)]_F$

R8.

$L(u_1) = L\begin{pmatrix}1\\0\\-1\end{pmatrix} = \begin{pmatrix}1+0\\1-(-1)\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix}$

$\begin{pmatrix}1\\2\end{pmatrix} = \alpha\begin{pmatrix}1\\1\end{pmatrix} + \beta\begin{pmatrix}2\\1\end{pmatrix}$

$\boxed{\alpha = 3 \quad \beta = -1}$

$[L(u_1)]_F = \begin{pmatrix}3\\-1\end{pmatrix}$

- $a_2 = [L(u_2)]_F$

$L(u_2) = L\begin{pmatrix}1\\-2\\1\end{pmatrix} = \begin{pmatrix}1+(-2)\\1-1\end{pmatrix} = \begin{pmatrix}-1\\0\end{pmatrix}$

$\begin{pmatrix}-1\\0\end{pmatrix} = \alpha\begin{pmatrix}1\\1\end{pmatrix} + \beta\begin{pmatrix}2\\1\end{pmatrix}$

$\alpha + 2\beta = -1 \rightarrow -\beta + 2\beta = -1$

$\alpha + \beta = 0$

$\boxed{\alpha = -\beta}$ $\boxed{\beta = -1}$ $\boxed{\alpha = 1}$

$[L(u_2)]_F = \begin{pmatrix}1\\-1\end{pmatrix}$

- $a_3 = [L(u_3)]_F$

$L(u_3) = L\begin{pmatrix}-1\\-1\\1\end{pmatrix} = \begin{pmatrix}-1+(-1)\\-1-1\end{pmatrix} = \begin{pmatrix}-2\\-2\end{pmatrix}$

$\begin{pmatrix}-2\\-2\end{pmatrix} = \alpha\begin{pmatrix}1\\1\end{pmatrix} + \beta\begin{pmatrix}2\\1\end{pmatrix}$

$-2 = \alpha + 2\beta$

$-2 = \alpha + \beta$

$\boxed{\beta = 0 \quad \alpha = -2}$

$[L(u_3)]_F = \begin{pmatrix}-2\\0\end{pmatrix}$

$\therefore A = [[L(u_1)]_F \; [L(u_2)]_F \; [L(u_3)]_F]$

$A = \begin{bmatrix}3 & 1 & -2\\-1 & -1 & 0\end{bmatrix}$ //

b. $[L(x)]_F = A[x]_E \quad \rightarrow x = (-1,-2,3)^T.$

$\begin{pmatrix}-1\\-2\\3\end{pmatrix} = c_1\begin{pmatrix}1\\0\\-1\end{pmatrix} + c_2\begin{pmatrix}1\\-2\\1\end{pmatrix} + c_3\begin{pmatrix}-1\\-1\\1\end{pmatrix}$

$\left(\begin{array}{ccc|c}1 & 1 & -1 & -1\\0 & -2 & -1 & -2\\-1 & 1 & 1 & 3\end{array}\right) \overset{E31(1)}{\sim} \left(\begin{array}{ccc|c}1 & 1 & -1 & -1\\0 & -2 & -1 & -2\\0 & 2 & 0 & 2\end{array}\right)$

$\overset{E32(1)}{\sim} \left(\begin{array}{ccc|c}1 & 1 & -1 & -1\\0 & -2 & -1 & -2\\0 & 0 & -1 & 0\end{array}\right)$ $\boxed{c_3 = 0}$

$-2c_2 = -2$

$\boxed{c_2 = 1}$

$c_1 + c_2 - c_3 = -1$

$c_1 + 1 - 0 = -1$

$\boxed{c_1 = -2}$

$[x]_E = \begin{pmatrix}-2\\1\\0\end{pmatrix}$

Jadi, $[L(x)]_F = A[x]_E$

$= \begin{pmatrix}3 & 1 & -2\\-1 & -1 & 0\end{pmatrix}\begin{pmatrix}-2\\1\\0\end{pmatrix}$

$= \begin{pmatrix}-5\\1\end{pmatrix}$

R9. dlln.

Lambang Matriks.

- standar
- Relatif thd suatu basis.

✓ standar :

Jika $A_{m \times n}$, maka $L_A(x) : R^n \rightarrow R^m$.

dengan $L_A(x) = Ax$.

contoh :

$R^2 \rightarrow R^3 \quad L(x) = \begin{pmatrix}x_1\\x_2\\x_1+x_2\end{pmatrix}$

$L(x) = \begin{pmatrix}1 & 0\\0 & 1\\1 & 1\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}$

✓ Relatif thd suatu Basis.

$E = [v_1, v_2, \dots, v_n]$ Basis V.

$F = [w_1, w_2, \dots, w_n]$ Basis W.

maka:

$TL : L : V \to W$

terdapat matriks $A_{m \times n}$.

$$[L(u)]_F = A\,[v]_E$$

a_j = kolom-kolom mat A.

$a_j = [L(v_j)]_F$

contoh:

1. $L : R^3 \to R^3$.

$L(x) = (2x_1 - x_2 - x_3,\ 2x_2 - x_1 - x_3,\ 2x_3 - x_1 - x_2)^T$

Tentukan lambang matriks A dari L dan gunakan A untuk mencari $L(x)$ untuk x vektor berikut.

a. $x = (1,1,1)^T$

b. $x = (-5,3,2)^T$

Jawab:

a. $L(x) = Ax$.

$$L(x) = \begin{pmatrix} 2x_1 - x_2 - x_3 \\ -x_1 + 2x_2 - x_3 \\ -x_1 - x_2 + 2x_3 \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$Ax = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \therefore L\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

b.

$$Ax = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \begin{pmatrix} -5 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} -15 \\ 9 \\ 6 \end{pmatrix}$$

2. misal : $E = \{u_1, u_2, u_3\}$

$F = \{b_1, b_2\}$

$u_1 = (1,0,-1)^T \qquad b_1 = (1,-1)^T$

$u_2 = (1,2,1)^T \qquad b_2 = (2,-1)^T$

$u_3 = (-1,1,1)^T$

$L : R^3 \to R^2$ tentukan A relatif thd basis E & F untuk.

a. $L(x) = (x_3, x_1)^T$

- $L(u_1) = \begin{pmatrix} -1 \\ 1 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

$$\left(\begin{array}{cc|c} 1 & 2 & -1 \\ -1 & -1 & 1 \end{array}\right) \underset{\sim}{E_{21}(1)} \left(\begin{array}{cc|c} 1 & 2 & -1 \\ 0 & 1 & 0 \end{array}\right) \quad \boxed{\begin{array}{l} c_2 = 0 \\ c_1 = -1 \end{array}}$$

- $L(u_2) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

$$\left(\begin{array}{cc|c} 1 & 2 & 1 \\ -1 & -1 & 1 \end{array}\right) \underset{\sim}{E_{21}(1)} \left(\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 1 & 2 \end{array}\right) \quad \begin{array}{l} c_2 = 2 \\ c_1 = -3 \end{array}$$

- $L(u_3) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

$$\left(\begin{array}{cc|c} 1 & 2 & 1 \\ -1 & -1 & -1 \end{array}\right) \underset{\sim}{E_{21}(1)} \left(\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 1 & 0 \end{array}\right) \quad \begin{array}{l} c_2 = 0 \\ c_1 = 1 \end{array}$$

$$A = \begin{pmatrix} -1 & -3 & 1 \\ 0 & 2 & 0 \end{pmatrix}$$

atau :

$(b_1\ b_2 \mid L(u_1)\ L(u_2)\ L(u_3))$

$$\left(\begin{array}{cc|ccc} 1 & 2 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 & -1 \end{array}\right) \underset{\sim}{E_{21}(1)}$$

$$\left(\begin{array}{cc|ccc} 1 & 2 & -1 & 1 & 1 \\ 0 & 1 & 0 & 2 & 0 \end{array}\right) \underset{\sim}{E_{12}(-2)}$$

$$\left(\begin{array}{cc|ccc} 1 & 0 & -1 & -3 & 1 \\ 0 & 1 & 0 & 2 & 0 \end{array}\right)$$

with the right block $= A$ //

Rg. alrn

b. $\bullet\ L(u_1) = \begin{pmatrix} 0 \\ -1 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

$$\left(\begin{array}{cc|c} 1 & 2 & 0 \\ -1 & -1 & -1 \end{array}\right) \overset{E_{21}(1)}{\sim} \left(\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 1 & -1 \end{array}\right) \quad \boxed{c_2 = -1}$$

$c_1 + (-2) = 0$

$\boxed{c_1 = 2}$

$\bullet\ L(u_2) = \begin{pmatrix} 4 \\ -1 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

$$\left(\begin{array}{cc|c} 1 & 2 & 4 \\ -1 & -1 & 3 \end{array}\right) \overset{E_{21}(1)}{\sim} \left(\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & 1 & 3 \end{array}\right) \quad \boxed{c_2 = 3}$$

$c_1 + 6 = 4$

$\boxed{c_1 = -2}$

$\bullet\ L(u_3) = \begin{pmatrix} 2 \\ 1 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

$$\left(\begin{array}{cc|c} 1 & 2 & 2 \\ -1 & -1 & 1 \end{array}\right) \overset{E_{21}(1)}{\sim} \left(\begin{array}{cc|c} 1 & 2 & 2 \\ 0 & 1 & 3 \end{array}\right) \quad \boxed{c_2 = 3}$$

$c_1 + 6 = 2$

$\boxed{c_1 = -4}$

$$\therefore\ A = \begin{pmatrix} 2 & -2 & -4 \\ -1 & 3 & 3 \end{pmatrix}$$

3. $L(x) = (2x_1 - x_2 - x_3,\ 2x_2 - x_1 - x_3,\ 2x_3 - x_1 - x_2)^T$

A relatif thd basis $[e_1, e_2, e_3]$ $\mathbb{R}^3 \to \mathbb{R}^3$.

Jika $\left.\begin{array}{l} u_1 = (1,1,0)^T \\ u_2 = (1,0,1)^T \\ u_3 = (0,1,1)^T \end{array}\right\}$ basis terurut $\mathbb{R}^3$.

a. cari matriks U yg berhubungan dg perubahan basis u ke e.

b. Tentukan matriks B yg melambangkan L relatif thd $[u_1, u_2, u_3]$ dg cara menghitung $U^{-1}AU$.

Keserupaan

misalkan :

$E = [v_1, v_2, \dots, v_n]$ dan $F = [w_1, w_2, \dots, w_n]$

$L : V \to V$, $S_{F \to E}$

A : lambang matriks L relatif thd E.

B : lambang matriks L relatif thd F.

maka : $\boxed{B = S^{-1} A S.}$

$$\begin{tikzcd}[column sep=large, row sep=large]
\text{Basis } E \arrow[r, "L", "\text{lambang } A"'] & \text{Basis } E. \arrow[d, "\text{matriks transisi: } S^{-1}"] \\
\text{Basis } F \arrow[u, "\text{matriks transisi: } S"] \arrow[r, "\text{Lambang } B", "L"'] & \text{Basis } F
\end{tikzcd}$$

- serupa (similar)

misalkan :

A & B matriks berukuran $n \times n$,

matriks B dikatakan similar dg A

Jika terdapat matriks taksingular S

sehingga : $\boxed{B = S^{-1} A S}$

Ortogonalitas.

- Hasil kali skalar di R^n (euclid)

$$\boxed{x^T y = x_1 y_1 + x_2 y_2 + \dots + x_n y_n}$$

- Panjang Euclid dari suatu Vektor.

$$\|x\| = \sqrt{x^T x} = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}$$

- Teorema

x dan y vektor taknol , θ sudut diantara x dan y.

$$\boxed{x^T y = \|x\| \, \|y\| \cos \theta}$$

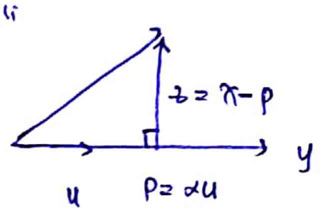
- Vektor satuan dan sudut x dan y vektor taknol.

$$\boxed{u = \frac{1}{\|x\|} x} \qquad \boxed{v = \frac{1}{\|y\|} y}$$

dan jika θ sudut antara x dan y

maka : $\boxed{\cos \theta = \dfrac{x^T y}{\|x\| \, \|y\|}}$

- ilustrasi



- Ortogonal

Vektor x dan y di R^2 atau R^3

dikatakan ortogonal ($\perp$)

Jika : $\boxed{x^T y = 0}$

- Proyeksi Skalar

$$\boxed{\alpha = \frac{x^T y}{\|y\|}}$$

- Proyeksi Vektor

$p = \alpha u$

$p = \alpha \dfrac{1}{\|y\|} y$

$$\boxed{p = \frac{x^T y}{y^T y} \cdot y}$$

R-10 dlm.

contoh:

1. misal: $L: R^3 \to R^3$ $L(x) = Ax$

$$A = \begin{pmatrix} 2 & 2 & 0 \\ 2 & 1 & 2 \\ 1 & 1 & 2 \end{pmatrix}$$

Jika A melambangkan L relatif terhadap basis baku $E = [e_1, e_2, e_3]$

Tent: B melambangkan L relatif thd $F = [y_1, y_2, y_3]$

$$y_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad y_2 = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}, \quad y_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

a. $B = S^{-1} A S$.

$S_{F \to E} = [(y_1)_E \ (y_2)_E \ (y_3)_E]$

$$S = \begin{pmatrix} 1 & -2 & 1 \\ -1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{matrix} 1 & -2 \\ -1 & 1 \\ 0 & 1 \end{matrix}$$

$\det S = (1 + 0 - 1) - (2 + 1 + 0)$

$= -3$

$a_{11} = \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0$ $a_{12} = \begin{vmatrix} -1 & 1 \\ 0 & 1 \end{vmatrix} = -1$

$a_{21} = \begin{vmatrix} -2 & 1 \\ 1 & 1 \end{vmatrix} = -3$ $a_{22} = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1$

$a_{31} = \begin{vmatrix} -2 & 1 \\ 1 & 1 \end{vmatrix} = -3$ $a_{32} = \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} = 2$

$a_{13} = \begin{vmatrix} -1 & 1 \\ 0 & 1 \end{vmatrix} = -1$

$a_{23} = \begin{vmatrix} 1 & -2 \\ 0 & 1 \end{vmatrix} = 1$

$a_{33} = \begin{vmatrix} 1 & -2 \\ -1 & 1 \end{vmatrix} = -1$

$$S^{-1} = \frac{1}{-3}\begin{bmatrix} 0 & 3 & -3 \\ 1 & 1 & -2 \\ -1 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 1 \\ -1/3 & -1/3 & 2/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}$$

$B = S^{-1} A S$

$$= \frac{1}{3}\begin{bmatrix} 0 & -3 & 3 \\ -1 & 1 & -2 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 2 & 0 \\ 2 & 1 & 2 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & -2 & 1 \\ -1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$ //

catatan: !

$E = [e_1, e_2, e_3] \to A$

$U = [u_1, u_2, u_3] \to B$.

IM

$A = [L(e_1)_E, L(e_2)_E, L(e_3)_E]$

$L(e_1) = \alpha e_1 + \beta e_2 + \gamma e_3$

$B = [L(u_1)_u \ L(u_2)_u \ L(u_3)_u]$

$L(u_1) = \alpha u_1 + \beta u_2 + \gamma u_3$

$S \to$ matriks transisi $U \to E$.

$S_{U \to E} = [(u_1)_E \ (u_2)_E \ (u_3)_E]$

2. misal: a. $x = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$, $y = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

b. $x = \begin{pmatrix} 2 \\ -5 \\ 4 \end{pmatrix}$, $y = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$

Tent: a. Sudut kedua vektor

b. Vektor p. Proyeksi vektor x pada y

c. Buktikan bahwa vektor p dan x-p ortogonal.

Jawab:

a. 1. $\cos\theta = \dfrac{x^T y}{\|x\| \|y\|} = \dfrac{(3 \ 5)\begin{pmatrix} 1 \\ 1 \end{pmatrix}}{\sqrt{9+25}\sqrt{1+1}}$

$= \dfrac{8}{\sqrt{68}}$

$\theta = \cos^{-1}\left(\dfrac{8}{\sqrt{68}}\right) = 14{,}069$ //

2. $p = \frac{x^T y}{y^T y} \cdot y$

$= \frac{(3 \quad 5)\begin{pmatrix}1\\1\end{pmatrix}}{(1 \; 1)\begin{pmatrix}1\\1\end{pmatrix}} \cdot \begin{pmatrix}1\\1\end{pmatrix}$

$= \frac{8}{2}\begin{pmatrix}1\\1\end{pmatrix}$

$= \begin{pmatrix}4\\4\end{pmatrix}$ //

3. $p^T(x-p) = 0$

$(4 \quad 4)\begin{pmatrix}3-4\\5-4\end{pmatrix} = 0$

$(4 \quad 4)\begin{pmatrix}-1\\1\end{pmatrix} = 0$

∴ ortogonal.

(b.) 1. $\cos\theta = \frac{x^T y}{\|x\| \|y\|}$

$= \frac{(2 \quad -5 \quad 4)\begin{pmatrix}1\\2\\-1\end{pmatrix}}{\sqrt{45}\sqrt{4}}$

$= \frac{2-10-4}{\sqrt{80}} = \frac{-12}{\sqrt{80}}$

$$\boxed{\theta = \arccos -\frac{12}{\sqrt{80}}}$$

2. $p = \frac{x^T y}{y^T y} \cdot y$

$= \frac{(2 \quad -5 \quad 4)\begin{pmatrix}1\\2\\-1\end{pmatrix}}{(1 \quad 2 \quad -1)\begin{pmatrix}1\\2\\-1\end{pmatrix}} \begin{pmatrix}1\\2\\-1\end{pmatrix}$

$p = \frac{2-10-4}{1+4+1}\begin{pmatrix}1\\2\\-1\end{pmatrix}$

$= -2\begin{pmatrix}1\\2\\-1\end{pmatrix}$

$= \begin{pmatrix}-2\\-4\\2\end{pmatrix}$ //

3. $p^T(x-p) = 0$

$(-2 \quad -4 \quad 2)\begin{pmatrix}2-(-2)\\-5-(-4)\\4-2\end{pmatrix} = 0$

$(-2 \quad -4 \quad 2)\begin{pmatrix}4\\-1\\2\end{pmatrix} = 0$

∴ ortogonal.

✓ **Ruang Hasil Kali Dalam.**

misalkan: V ruang vektor

operasi V yg memadankan setiap panjang vektor x dan y dr V dg sebuah bil. real $\langle x, y \rangle$, dg syarat:

1. $\langle x, x \rangle \geq 0 \Leftrightarrow x = 0$
2. $\langle x, y \rangle = \langle y, x \rangle$, $\forall x, y \in V$.
3. $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$

$\forall x, y, z \in V$, $\forall \alpha, \beta$ skalar

$$\boxed{\langle x, x \rangle = x^T x}$$

contoh:

1. R^3, $x, y \in R^3$.

misal: $\langle x, y \rangle = x_1^2 y_1^2 + x_2^2 y_2^2 + x_3^2 y_3^2$

Tunjukkan $\langle x, y \rangle = x_1^2 y_1^2 + x_2^2 y_2^2 + x_3^2 y_3^2$ adalah HKD!

Jawab:

1. ∘ $\langle x,x \rangle \geq 0$. $x = \begin{pmatrix} x_1^2 \\ x_2^2 \\ x_3^2 \end{pmatrix}$

Bukti:

$\langle x,x \rangle = x_1^2 x_1^2 + x_2^2 x_2^2 + x_3^2 x_3^2$

$= x_1^4 + x_2^4 + x_3^4$

karena x_1^4 dan x_2^4 dan x_3^4 selalu ≥ 0, maka $\langle x,x \rangle \geq 0$ //

∘ $\langle x,y \rangle = \langle y,x \rangle$?

$\langle x,y \rangle = x_1^2 y_1^2 + x_2^2 y_2^2 + x_3^2 y_2^3$

karena x,y bil. real, maka berlaku $xy = yx$ sehingga:

$= y_1^2 x_1^2 + y_2^2 x_2^2 + y_3^2 x_3^2$

$\therefore \langle x,y \rangle = \langle y,x \rangle$

∘ $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$; $y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$; $z = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}$

$\langle x+y,z \rangle = \langle x,z \rangle + \langle y+z \rangle$?

$\left\langle \begin{pmatrix} x_1+y_1 \\ x_2+y_2 \\ x_3+y_3 \end{pmatrix}, \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \right\rangle$

$= (x_1+y_1)^2 (z_1)^2 + (x_2+y_2)^2 z_2^2 + (x_3+y_3)^2 z_3^2$

$= [(x_1+y_1)z_1]^2 + [(x_2+y_2)z_2]^2 + [(x_3+y_3)z_3]^2$

$= [x_1z_1 + y_1z_1]^2 + [x_2z_2 + y_2z_2]^2 + [x_3z_3 + y_3z_3]^2$

$= \langle x,z \rangle + \langle y,z \rangle + 2pq.$

$\neq x_1^2z_1^2 + x_2^2z_2^2 + x_3^2z_3^2 + y_1^2z_1^2 + y_2^2z_2^2 + y_3^2z_3^2$

$\neq \langle x,z \rangle + \langle y,z \rangle$

$\therefore$ tidak terpenuhi.

Ruang Hasil Kali Dalam

✓ syarat:

1. $\langle x,x \rangle \geq 0$, $x=0$
2. $\langle x,y \rangle = \langle y,x \rangle$
3. $\langle \alpha x + \beta y, z \rangle$, $\alpha \langle x,z \rangle + \beta \langle y,z \rangle$

✓ contoh ruang hkd:

1. RV R^n

$\langle x,y \rangle = x^T y$

2. RV $R^{m \times n}$

$\langle x,y \rangle = \sum_{i=1}^{n} \sum_{j=1}^{n} x_{ij} y_{ij}$

3. RV $C[0,1]$

$\langle f,g \rangle = \int_a^b f(x)\, g(x)$

4. RV P_n.

$\langle p,q \rangle = \sum_{i=1}^{n} p(x_i)\, q(x_i)$

contoh:

1. misal $A,B \in R^{2 \times 3}$

$A = \begin{pmatrix} 1 & 2 & 3 \\ -2 & 3 & 5 \end{pmatrix}$ $B = \begin{pmatrix} 4 & 6 & -3 \\ 2 & 0 & 1 \end{pmatrix}$

hitung hkd?

Jawab:

$\langle A,B \rangle = a_{11}b_{11} + a_{12}b_{12} + a_{13}b_{13} + a_{21}b_{21} + a_{22}b_{22} + a_{23}b_{23}$

$= (4+12-9) + (-4+0+5)$

$= 8$

2. di $C[0,1]$ hitung $\langle e^x, e^x \rangle$

$\langle x,y \rangle = \int_0^1 e^x \cdot e^x \, dx$

$= \int_0^1 e^{2x} \, dx$

$= \int_0^2 e^u \frac{1}{2} du$

misal:
$u = 2x$
$du = 2dx$
$dx = \frac{1}{2} du$

$= \frac{1}{2} e^u \Big]_0^2 = \frac{1}{2}e^2 - \frac{1}{2}e^0 = \frac{1}{2}(e^2 - 1)$

3. misal $\langle u, v\rangle = 3$ $\quad \|u\| = 1$

$\langle v, w\rangle = -4$ $\quad \|v\| = 2$

$\langle u, w\rangle = 5$ $\quad \|w\| = 8$

tent: $\langle 2v - w, 3u + 2w\rangle$

Jawab:

$= \langle 2v, 3u + 2w\rangle + \langle -w, 3u + 2w\rangle$

$= \langle 3u + 2w, 2v\rangle + \langle 3u + 2w, -w\rangle$

$= \langle 3u, 2v\rangle + \langle 2w, 2v\rangle + \langle 3u, -w\rangle$

$+ \langle 2w, -w\rangle$

$= 6\langle u, v\rangle + 4\langle w, v\rangle - 3\langle u, w\rangle$

$-2\langle w, w\rangle$

$= 6(3) + 4(-4) - 3(5) - 2(64)$

$= -141.$

$\langle w, w\rangle = \|w\|^2$

$= 8^2$

$= 64$

• $\langle 3u, 2v\rangle$

$= 3\langle u, 2v\rangle$

$= 3\langle 2u, v\rangle$

$= 3 \cdot 2\langle v, u\rangle$

✓ Proyeksi skalar & Proyeksi vektor.

Proyeksi skalar : $\alpha = \dfrac{\langle u, v\rangle}{\|v\|}$

proyeksi vektor : $p = \dfrac{\langle u, v\rangle}{\langle u, v\rangle} \cdot v$

contoh:

① misal $A, B \in \mathbb{R}^{2\times 2}$ dengan.

$A = \begin{pmatrix} 1 & 2 \\ 1 & -3 \end{pmatrix}, \; B = \begin{pmatrix} 0 & -1 \\ 1 & -5 \end{pmatrix}$

$\langle A, B\rangle = tr\,(A^T B)$

a. vektor p, yaitu proyeksi dari A pada B.

$\langle A, B\rangle = Tr\,(A^T B) = 1 + 13 = 14$

$A^T B = \begin{pmatrix} 1 & 1 \\ 2 & -3 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 5 \end{pmatrix} = \begin{pmatrix} 1 & -6 \\ -3 & 13 \end{pmatrix}$

$\langle B, B\rangle = Tr\,(B^T B) = 1 + 26 = 27.$

$B^T B = \begin{pmatrix} 0 & 1 \\ -1 & -5 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & -5 \end{pmatrix} = \begin{pmatrix} 1 & -5 \\ -5 & 26 \end{pmatrix}$

$p = \dfrac{\langle A, B\rangle}{\langle B, B\rangle} B = \dfrac{14}{27}\begin{pmatrix} 0 & -1 \\ 1 & -5 \end{pmatrix}$

$\langle A, B\rangle = \sum\sum a_{ij} B_{ij}$

$= 0 - 1 + 2 + 15$

$= 16$

soal :

① $x_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ $\quad x_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ $\quad x_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

Jawab:

$u_1 = \dfrac{x_1}{\|x_1\|} = \dfrac{1}{\sqrt{3}}(1, 1, 1)^T$

$p_1 = \langle x_2, u_1\rangle u_1$

$= (0 \;\; 1 \;\; 1) \cdot \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix} \cdot \dfrac{1}{\sqrt{3}}(1, 1, 1)^T$

$= \dfrac{2}{\sqrt{3}}\left[\dfrac{1}{\sqrt{3}}(1, 1, 1)^T\right]$

$= \dfrac{2}{3}(1, 1, 1)^T$

$x_2 - p_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 2/3 \\ 2/3 \\ 2/3 \end{pmatrix} = \begin{pmatrix} -2/3 \\ 1/3 \\ 1/3 \end{pmatrix}$

$u_2 = \dfrac{x_2 - p_1}{\|x_2 - p_1\|} = \dfrac{3}{\sqrt{6}}\begin{pmatrix} -2/3 \\ 1/3 \\ 1/3 \end{pmatrix} = \dfrac{1}{\sqrt{6}}\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$

$p_2 = \langle x_3, u_1\rangle u_1 + \langle x_3, u_2\rangle u_2$

$= \dfrac{1}{\sqrt{3}} \cdot \dfrac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \dfrac{1}{\sqrt{6}} \cdot \dfrac{1}{\sqrt{6}}\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$

$= \dfrac{1}{3}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \dfrac{1}{6}\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$

$= \dfrac{1}{6}\begin{pmatrix} 0 \\ 3 \\ 3 \end{pmatrix} = \dfrac{1}{2}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

R.U. alin.

No.
Date

lanjutan soal:

$$u_3 = \frac{x_3 - p_2}{\|x_3 - p_2\|} = \frac{2}{\sqrt{2}}\begin{pmatrix} 0 \\ -1/2 \\ 1/2 \end{pmatrix} = \sqrt{2}\begin{pmatrix} 0 \\ -1/2 \\ 1/2 \end{pmatrix} = \begin{pmatrix} 0 \\ -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$

$\therefore$ basis ortonormal $\{u_1, u_2, u_3\}$

2. $$A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

Jawab:

- $r_{11} = \|a_1\| = \sqrt{6}$

$$q_1 = \frac{1}{\|a_1\|} \cdot a_1 = \frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

- $r_{12} = \langle a_2, q_1 \rangle = \frac{3}{\sqrt{6}}$

$$p_1 = r_{12} \cdot q_1 = \frac{1}{2}\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

$$a_2 - p_1 = (-1/2, 0, 1/2)^T$$

- $r_{22} = \|a_2 - p_1\| = \frac{1}{\sqrt{2}}$

$$q_2 = \frac{a_2 - p_1}{\|a_2 - p_1\|} = \sqrt{2}\begin{pmatrix} -1/2 \\ 0 \\ 1/2 \end{pmatrix}$$

- $r_{13} = \langle a_3, q_1 \rangle = \frac{1}{\sqrt{6}}$

- $r_{23} = \langle a_3, q_2 \rangle = \frac{1}{2}\sqrt{2}$

$$p_2 = r_{13} q_1 + r_{23} q_2$$

$$= \frac{1}{\sqrt{6}} \frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \frac{1}{\cancel{2}} \cancel{\sqrt{2}}\sqrt{2}\begin{pmatrix} -1/2 \\ 0 \\ 1/2 \end{pmatrix}$$

$$= \begin{pmatrix} 1/6 \\ 1/3 \\ 1/6 \end{pmatrix} + \begin{pmatrix} -1/2 \\ 0 \\ 1/2 \end{pmatrix}$$

$$= \begin{pmatrix} -1/3 \\ 1/3 \\ 2/3 \end{pmatrix}$$

- $r_{33} = \|a_3 - p_2\| = \frac{1}{3}\sqrt{3}$

$$q_3 = \frac{a_3 - p_2}{\|a_3 - p_2\|} = \frac{3}{\sqrt{3}}\begin{pmatrix} 1/3 \\ -1/3 \\ 1/3 \end{pmatrix} = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

$\therefore$ Basis ortonormal:

$$\underbrace{\left\{\begin{pmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{pmatrix}, \begin{pmatrix} \sqrt{2}/2 \\ 0 \\ \sqrt{2}/2 \end{pmatrix}, \begin{pmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}\right\}}_{Q}$$

$$R = \begin{pmatrix} \sqrt{6} & 3/\sqrt{6} & 1/\sqrt{6} \\ 0 & 1/\sqrt{2} & \frac{1}{2}\sqrt{2} \\ 0 & 0 & 1/\sqrt{3} \end{pmatrix}$$

Rumus:

$r_{11} = \|a_1\| \rightarrow r_{kk} = \|a_k - p_{k-1}\|$

$q_k = \frac{1}{r_{kk}} \|a_1\|$

$p_k = r_{1k+1}\, q_1 + r_{2k+1}\, q_2 + \dots + r_{kk+1}\, q_k$

$r_{ij} = \langle a_j, q_i \rangle, \; i < j$

R-12

alin

NILAI EIGEN

✓ Hasil kali & Jumlah nilai eigen

$$\prod_{i=0}^{n} \lambda_i = \lambda_1\lambda_2\cdots\lambda_n = \det(A)$$

$$\sum_{i=0}^{n} \lambda_i = \lambda_1 + \lambda_2 + \cdots + \lambda_n = tr(A).$$

contoh:

① $A = \begin{pmatrix} 10 & -9 \\ 4 & -3 \end{pmatrix}$

✓ $|A| = 6$

✓ $tr(A) = 7$

$$\begin{vmatrix} 10-\lambda & -9 \\ 4 & -3-\lambda \end{vmatrix} = 0$$

$$(10-\lambda)(-3-\lambda) + 36 = 0$$

$$-30 - 7\lambda + \lambda^2 + 36 = 0$$

$$\lambda^2 - 7\lambda + 6 = 0$$

$$(\lambda-1)(\lambda-6) = 6$$

$$\lambda_1 = 1,\ \lambda_2 = 6$$

✓ Matriks Diagonal & Matriks yg dapat didiagonalkan.

Definisi

$A_{n\times n}$ disebut matriks diagonal jika $a_{ij} = 0$ u/ $i \neq j$.

contoh: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

✓ Pendiagonalan

$$\boxed{X^{-1} A X = D}$$

contoh:

① $A = \begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix}$

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 3-\lambda & 0 \\ 8 & -1-\lambda \end{vmatrix} = 0$$

$$(3-\lambda)(-1-\lambda) = 0$$

$$\boxed{\lambda_1 = 3 \mid \lambda_2 = -1}$$

∘ $\lambda_1 = 3.$

$$\begin{pmatrix} 0 & 0 \\ 8 & -4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$8x_1 - 4x_2 = 0$$

$$8x_1 = 4x_2$$

$$\boxed{2x_1 = x_2}$$

$$x = \begin{pmatrix} x_1 \\ 2x_1 \end{pmatrix} = x_1\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

∘ $\lambda_2 = -1$

$$\begin{pmatrix} 4 & 0 \\ 8 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$x_1 = 0,\quad x = \begin{pmatrix} 0 \\ x_2 \end{pmatrix} = x_2\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

sehingga,

$$X = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \qquad X^{-1} = \frac{1}{1}\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$$

$$X^{-1} A X = D$$

$$\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} = D$$

$$\begin{pmatrix} 3 & 0 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}$$

bebas linear → Jika nilai eigen (λ) nya beda semua.

- Perpangkatan Matriks yg dapat didiagonalkan.

A dapat didiagonalkan

⇓

$$\boxed{A = XDX^{-1}}$$

akibatnya :

$$\boxed{A^k = XD^kX^{-1}}$$

- Matriks Defektif → tidak bebas linear (λ nya sama).

Definisi

- matriks definit positif
  - $\forall \lambda$ positif → definit positif.
  - $\forall \lambda$ negatif → definit negatif.
  - $\exists \lambda \oplus, \ominus$ → tak definit

- Dekomposisi / Faktorisasi LU
  - Dekomposisi LU

$$\boxed{A = LU}$$

ket : L = matriks segitiga bawah dg elemen 1 pada diagonalnya.

- Dekomposisi LDL^T

$$\boxed{A = LDL^T}$$

- Dekomposisi cholesky.

$$\boxed{A = BB^T} \quad \boxed{B = LD^{1/2}}$$

Tambahan catatan,

Definit positif

- A disebut Df jika $x^TAx > 0$, $\forall x \neq 0 \in \mathbb{R}^n$.

Teo

A simetrik dan A D⊕ ⟺ $\forall \lambda i > 0$

akibatnya

$\forall \lambda i < 0$ → -A Df

Sifat I

A def ⊕ simetrik ⇒ A tak singular.

Sifat II

A def ⊕ simetrik ⇒ $|A| > 0$

Sifat III

A def ⊕ simetrik ⇒ $|A_r| > 0$, $\forall r$

Sifat IV

A def ⊕ simetrik ⇒ dapat direduksi dg OBD dan semua elemen diagonal ⊕

contoh :

(1)

$$A = \begin{pmatrix} 2 & -2 & 1 \\ -2 & 3 & 2 \\ 1 & 2 & 10 \end{pmatrix}$$

a. tunjukkan A def ⊕

b. tentukan dekomposisi cholesky.

R.14. adm.

Jawab :

a. $|A_1| = 2 > 0$

$|A_2| = \begin{vmatrix} 2 & -2 \\ -2 & 3 \end{vmatrix} = 6-4 = 2 > 0$

$|A_3| = \begin{vmatrix} 2 & -2 & 1 \\ -2 & 3 & 2 \\ 1 & 2 & 10 \end{vmatrix} \begin{matrix} 2 & -2 \\ -2 & 3 \\ 1 & 2 \end{matrix}$

$= (60-4-4) - (3+8+40)$

$= 52 - 51 = 1 > 0$

∴ menurut teorema A def(+)

$A = A^T \rightarrow$ A simetrik

↳ sehingga bisa dilakukan

oleh dekomposisi cholesky.

OBD :

$\begin{pmatrix} 2 & -2 & 1 \\ -2 & 3 & 2 \\ 1 & 2 & 10 \end{pmatrix} \underset{E_{31}(-\frac{1}{2})}{\overset{E_{21}(1)}{\sim}} \begin{pmatrix} 2 & -2 & 1 \\ 0 & 1 & 3 \\ 0 & 3 & 19/2 \end{pmatrix} \overset{E_{32}(-3)}{\sim}$

$\begin{pmatrix} 2 & -2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1/2 \end{pmatrix} = u \rightarrow d_{ii} = u_{ii}$

$L = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1/2 & 3 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{pmatrix}$

1 cek

$LU = A$

$\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1/2 & 3 & 1 \end{pmatrix} \begin{pmatrix} 2 & -2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1/2 \end{pmatrix} =$

$\begin{pmatrix} 2 & -2 & 1 \\ -2 & 3 & 2 \\ 1 & 2 & 10 \end{pmatrix} = A$ ▨

2 cek $LDL^T = A$

$\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1/2 & 3 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{pmatrix} \begin{pmatrix} 1 & -1 & 1/2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} 2 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 3 & 1/2 \end{pmatrix} \begin{pmatrix} 1 & -1 & 1/2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} 2 & -2 & 1 \\ -2 & 3 & 2 \\ 1 & 2 & 10 \end{pmatrix} = A$ ▨

3. cek $BB^T = A$

$B = LD^{1/2} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1/2 & 3 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{pmatrix}^{1/2}$

$= \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1/2 & 3 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/\sqrt{2} \end{pmatrix}$

$= \begin{pmatrix} \sqrt{2} & 0 & 0 \\ -\sqrt{2} & 1 & 0 \\ \sqrt{2}/2 & 3 & 1/\sqrt{2} \end{pmatrix}$

∘ $BB^T = \begin{pmatrix} \sqrt{2} & 0 & 0 \\ -\sqrt{2} & 1 & 0 \\ \sqrt{2}/2 & 3 & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} \sqrt{2} & -\sqrt{2} & \sqrt{2}/2 \\ 0 & 1 & 3 \\ 0 & 0 & 1/\sqrt{2} \end{pmatrix}$

$= \begin{pmatrix} 2 & -2 & 1 \\ -2 & 3 & 2 \\ 1 & 2 & 10 \end{pmatrix} = A$ ▨