a) $f(n) = \frac{n^2}{\log n}$ $\quad g(n) = n.(\log n)^2$

$$\lim_{n\to\infty} \frac{n^2/\log n}{n.(\log n)^2} \Rightarrow \lim_{n\to\infty} \frac{n^{\cancel{2}}}{\cancel{n}.(\log n)^3}$$

$\frac{\infty}{\infty}$!

Rule: $\lim_{n\to\infty} \frac{f(n)}{g(n)} = M$

if $M=0$; $f(n) \in O(g(n))$

if $M=\infty$; $f(n) \in \Omega(g(n))$

if $M=$ constant; $f(n) \in \Theta(g(n))$

Use the L'Hospital Rule:

$$\lim_{n\to\infty} = \frac{1}{3.(\log n)^2 . \frac{1}{n.\ln 2}} \Rightarrow \lim_{n\to\infty} \frac{n.\ln 2}{3(\log n)^2}$$

$\frac{\infty}{\infty}$

Use the L'Hospital Rule again:

$$\lim_{n\to\infty} \frac{1.\ln 2}{6(\log n).\frac{1}{n.\ln 2}} \Rightarrow \lim_{n\to\infty} \frac{n.(\ln 2)^2}{6(\log n)} \Rightarrow \lim_{n\to\infty} \frac{1.(\ln 2)^2}{6.\frac{1}{n.\ln 2}} \Rightarrow \lim_{n\to\infty} \frac{n.(\ln 2)^3}{6}$$

$\frac{\infty}{\infty}$!

$$\lim_{n\to\infty} \frac{n.(\ln 2)^3}{6} = \infty$$

$f \in \Omega(g)$

b) $f(n) = (\log n)^{\log n}$ $\quad g(n) = n/\log n$

$$\lim_{n\to\infty} \frac{(\log n)^{\log n}}{n/\log n} \rightarrow \text{Simplify} \quad \frac{(\log n)^{\log n+1}}{n} \Rightarrow \text{L'Hospital Rule}$$

$$\lim_{n\to\infty} \frac{((\log n)+1).(\log n)^{\log n}.\frac{1}{n\ln 2}}{1} \Rightarrow \text{Simplify} \lim_{n\to\infty} \frac{((\log n)+1).(\log n)^{\log n}}{n.\ln 2} = \infty$$

$f(n) \geq g(n)$ $\quad f \in \Omega(g)$

c) $f(n) = \sqrt{n}$ $g(n) = (\log n)^3$

$\lim_{n\to\infty} \frac{\sqrt{n}}{(\log n)^3}$ $\frac{\infty}{\infty}$ $\Rightarrow$ Use L'Hopital's Rule $\lim_{n\to\infty} \frac{\frac{1}{2\sqrt{n}}}{3.(\log n)^2.\frac{1}{n.\ln 2}}$

Simplify $\Rightarrow \lim_{n\to\infty} \frac{\sqrt{n}.\ln 2}{6.(\log n)^2}$ $\xRightarrow{\text{L'Hospital's Rule}}$ $\lim_{n\to\infty} \frac{\frac{\ln 2}{2\sqrt{n}}}{12(\log n)\frac{1}{n\ln 2}}$

$\Rightarrow \lim_{n\to\infty} \frac{(\ln 2)^2.\sqrt{n}}{24(\log n)}$ $\xRightarrow{\text{L'Hospital's Rule}}$ $\lim_{n\to\infty} \frac{\frac{(\ln 2)^2}{2\sqrt{n}}}{24.\frac{1}{n.\ln 2}}$

$\Rightarrow \lim_{n\to\infty} \frac{(\ln 2)^3.\sqrt{n}}{48} = \infty$; so $f(n) \in \Omega(g(n))$

d) $f(n) = (\log n)^{\log n}$ $g(n) = 2^{(\log_2 n)^2}$

$\lim_{n\to\infty} \frac{(\log n)^{\log n}}{2^{(\log n)^2}} = 0$ Because $(\log n)^{\log n}$ has similar growth rate to logarithmic growth rate, and $2^{(\log n)^2}$ has exponential growth rate. So $2^{(\log n)^2}$ grows faster than $(\log n)^{\log n}$.

$f(n) \in O(g(n))$

e) $f(n) = \sum_{i=1}^{n} i^k$ $g(n) = n^{k+1}$

According to summation's rule $f(n)$ equals to $\rightarrow 1^k + 2^k + 3^k + 4^k + 5^k + 6^k - - - - + (n-1)^k + n^k$

$\lim_{n\to\infty} \frac{1^k + 2^k + - - - + (n-1)^k + n^k}{n^{k+1}} = C$ This limit equals constant. Because while the degree of the numerator is k, the denominator is $k+1$.

$\underbrace{(n^k + n^k + \_\_\_ + n^k)}_{n \text{ tane}}$

$\underbrace{f(n) = \Theta(g(n))}$

f) $f(n) = 100n + \log n$ $\quad g(n) = n + (\log n)^2$

$$\lim_{n\to\infty} \frac{100n+\log n}{n+(\log n)^2} \quad \frac{\infty}{\infty}$$

Use the L'Hospital's Rule

$$\lim_{n\to\infty} \frac{100 + \frac{1}{n.\ln 2}}{1 + 2\cdot\frac{\log n}{n.\ln 2}} \xRightarrow{\text{simplify.}} \lim_{n\to\infty} \frac{100n\ln 2 + 1}{n\ln 2 + 2\log n} \quad \frac{\infty}{\infty}$$

$\Rightarrow$ Use the L'Hapitol's Rule agam.

$$\lim_{n\to\infty} \frac{100\ln 2}{\ln 2 + 2\cdot\frac{1}{n\ln 2}} \xRightarrow{\text{simplify}} \lim_{n\to\infty} \frac{100\ln 2}{\frac{n.\ln 2 + 2}{n.\ln 2}} \Rightarrow \lim_{n\to\infty} \frac{100n(\ln 2)^2}{n\ln 2 + 2}$$

$\frac{\infty}{\infty}$: Use L'Hospital's Rule

$$\Rightarrow \lim_{n\to\infty} \frac{100(\ln 2)^2}{\ln 2} \Rightarrow 100.\ln 2 \quad \text{Constant.}$$

So $f(n) \in \Theta(g(n))$.

g) $f(n) = n^{1/2}$ $\quad g(n) = 5^{\log_2 n}$

$$\lim_{n\to\infty} \frac{n^{1/2}}{5^{\log_2 n}} \xRightarrow{\text{simplify}} \lim_{n\to\infty} n^{(\underbrace{\log_2\sqrt{2} - \log_2 5}})$$

This statement equals to the $\log_2\left(\frac{\sqrt{2}}{5}\right)$, and the $\log_2\left(\frac{\sqrt{2}}{5}\right)$ equals to $-1{,}84$.

$n^{1/2}$ equals to $\log_2\sqrt{2}$,

$5^{\log_2 n}$ equals to $n^{\log_2 5}$

$$\Rightarrow \lim_{n\to\infty} n^{-1{,}84} \Rightarrow \lim_{n\to\infty} \frac{1}{n^{1{,}84}} = 0$$

$f(n) \in O(g(n))$

h) $f(n) = n^{0.1}$ $\quad g(n) = (\log n)^{10}$

$$\lim_{n\to\infty} \frac{n^{0.1}}{(\log n)^{10}} \; \infty/\infty \xRightarrow{\text{L'Hospital}} \lim_{n\to\infty} \frac{0.1 \cdot n^{-0.9}}{10\cdot(\log n)^9 \cdot \frac{1}{n.\ln 2}} \xRightarrow{\text{Simplify}} \lim_{n\to\infty} \frac{(0{,}1)\cdot n^{0.1}\cdot \ln 2}{10(\log n)^9} \xRightarrow{\text{L'Hospital}}$$

$\frac{\infty}{\infty}$!

$$\lim_{n\to\infty} \frac{(0{,}1)^2 , \ln 2 , n^{-0.9}}{10.9.(\log n)^8 \cdot \frac{1}{n.\ln 2}} \xRightarrow{\text{Simplify}} \lim_{n\to\infty} \frac{(0{,}1)^2(\ln 2)^2 \cdot n^{0.1}}{10.9.(\log n)^8}$$

– – – – – – – Same steps

$$\lim_{n\to\infty} \frac{(0{,}1)^3 , (\ln 2)^3 , n^{0.1}}{10! \, (\log n)^7} \xRightarrow{\text{L'Hospital}} \lim_{n\to\infty} \frac{(0{,}1)^{10} . (\ln 2)^{10} \cdot n^{0.1}}{10!} = \infty$$

$f(n) \; \Omega(g(n))$