Lecture 25 Module 1

Friday, 30 October 2020 5:08 PM

Lec25Mo...

Deduction in FOL and Soundness theorem

We now present an axiomatization for FOL. As we had said, FOL is an enhancement of PL – all PL-theorems are provable in FOL. To ensure that, all PL axioms are made part of the axiom schemata for FOL; the rule of inference is again Modus Ponens (MP), and the syntactic consequence $\vdash$ is defined identically too.

1 Axiom schemata of FOL

A1 $\alpha \to (\beta \to \alpha)$.

A2 $(\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma))$.

A3 $(\neg\beta \to \neg\alpha) \to (\alpha \to \beta)$.

A4 $\forall x(\alpha \to \beta) \to (\forall x\alpha \to \forall x\beta)$.

A5 $\forall x\alpha \to \alpha(t/x)$, if t is free for x in α.

A6 $\alpha \to \forall x\alpha$, if x is not free in α.

$\forall y(px \to (qy \to px))$

$\forall x \forall y(px \to (qy \to px))$

Ax Gen: If α is an axiom, and x is free in α then $\forall x\alpha$ is also an axiom.

Observe the role of Ax Gen: as we 'close' each free variable in an axiom, we get a new axiom from it; finally, the closure of the axiom also gets to be an axiom.

Rule of inference: *Modus Ponens (MP)*

$$\frac{\begin{array}{c}\alpha\\ \alpha \to \beta\end{array}}{\beta}$$

β

2 The syntactic consequence relation '$\vdash$'

Let Γ be any set of wffs and α any wff in FOL.

Definition 2.1.

(i) $\Gamma \vdash \alpha$, if and only if there is a sequence $\alpha_1, ..., \alpha_n (:= \alpha)$ such that each $\alpha_i (i = 1, ..., n)$ is either (a) an axiom, or (b) a member of Γ, or (c) derived from some of $\alpha_1, ..., \alpha_{i-1}$ by MP.

(ii) If Γ is empty in the above, we write $\vdash \alpha$, and say that α is a *theorem*, the sequence $\alpha_1, ..., \alpha_n (:= \alpha)$ constituting a *proof* of α.

Proposition 2.1. $\vdash$ satisfies the following properties:

(i) *Overlap*: if $\alpha \in \Gamma$, then $\Gamma \vdash \alpha$,

(ii) *Dilution*: if $\Gamma \subseteq \Delta$ and $\Gamma \vdash \alpha$, then $\Delta \vdash \alpha$,

(iii) *Cut*: if $\Delta \vdash \gamma$ for each $\gamma \in \Gamma$ and $\Gamma \vdash \alpha$, then $\Delta \vdash \alpha$.

(iv) *Compactness*: If $\Gamma \vdash \alpha$, there is a *finite* subset Γ' of Γ such that $\Gamma' \vdash \alpha$.

The deduction theorem (D.T.) continues to hold, with identical proof as in PL.

Theorem 2.2. (Deduction) For any Γ, α, β, if $\Gamma \cup \{\alpha\} \vdash \beta$ then $\Gamma \vdash \alpha \to \beta$.

It should be mentioned here that there are other equivalent axiomatizations for FOL (e.g. refer Kelly). For instance, axioms A4, A6 and Ax Gen are replaced by the single axiom:

$\forall x(\alpha \to \beta) \to (\alpha \to \forall x \beta)$, whenever x is not free in α,

and along with the rule MP, the *generalization* rule is considered:

$$\frac{\alpha}{\forall x \alpha}$$

Even though this alternate axiomatization looks simpler, it comes with a price: the Deduction Theorem cannot be applied as stated above. We get $\Gamma \vdash \alpha \to \beta$, when a restriction is imposed on the deduction of β from $\Gamma \cup \{\alpha\}$, namely that there should be no application of the above generalization rule, to any variable occurring free in α.
Note that in the system adopted here, the axiom above becomes a *theorem*.

As all of PL is kept intact inside FOL, it is no surprise that we have the following.

Proposition 2.3.

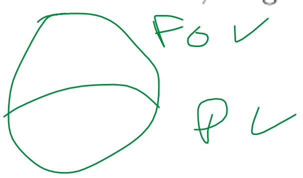
(i) All the theorems of PL are theorems in FOL, for FOL-wffs.

(ii) All PL metatheorems hold for FOL-wffs.

Moreover, it is important to note that in FOL also, one can replace any subwff of a wff by a logically equivalent wff such that the result is a wff logically equivalent to the original.

Proposition 2.4. (Equivalence theorem) If α is a wff, β a subformula of α, γ a wff such that $\vdash \gamma \leftrightarrow \beta$, and α' the result of replacing some occurrences of β in α by γ, then $\vdash \alpha \leftrightarrow \alpha'$.

So, for example, using the Equivalence theorem and PL-theorems, we get $\vdash \forall x(\alpha \leftrightarrow \beta) \leftrightarrow \forall x(\neg\beta \leftrightarrow \neg\alpha)$.



3 Soundness theorem

Theorem 3.1. (Soundness) If $\Gamma \vdash \alpha$ then $\Gamma \models \alpha$.

Proof. The proof is by induction on the number of steps of derivation of α from Γ. In essence, one shows that (i) the axioms are valid, and (ii) MP 'preserves truth'. To show the former, we use the results marked as exercises in previous Lectures. □

Lecture 25 Module 2

Friday, 30 October 2020 5:08 PM

Lec25Mo...

Some further results in FOL

1 Some derived rules in FOL

The following proposition lists what may be termed as the 'introduction' and 'elimination' rules for the quantifiers. The Specialization and Choice rules fall under the former, while the Existential and Generalization rules fall under the latter.

Proposition 1.1. The following rules may be derived in *FOL*.

(i) (**Specialization (Spec)**)

$\forall x\alpha \rightarrow \alpha(t(x))$

$$\frac{\Gamma \vdash \forall x\alpha}{\Gamma \vdash \alpha(t/x)}$$

if the term t is free for x in α.

(ii) (**Existential rule (Rule $\exists$)**)

$$\frac{\Gamma \vdash \alpha(t/x)}{\Gamma \vdash \exists x\alpha}$$

if the term t is free for x in α.

(iii) (**Generalization rule (Gen)**)

$$\frac{\Gamma \vdash \alpha}{\Gamma \vdash \forall x \alpha}$$

if x is not free in Γ (i.e. in any wff of Γ).

(iv) (**Choice rule (Rule C)**)

$$\frac{\Gamma \cup \{\alpha\} \vdash \beta \quad \Gamma \vdash \exists x \alpha}{\Gamma \vdash \beta}$$

if x is not free in Γ, nor in β.

Proof. Spec and Rule $\exists$ are straightforward. Gen is proved by induction on the length of derivation of α from Γ.
For Rule C, we first prove $\vdash \forall x(\alpha \to \beta) \to (\exists x\alpha \to \beta)$, whenever x is not free in β:
$\Gamma := \{\forall x(\alpha \to \beta), \exists x\alpha, \neg\beta\}$
$\vdash \alpha \to \beta$ (Spec)
$\vdash \neg\beta \to \neg\alpha$ (PL)
$\vdash \neg\beta$ (Overlap)
$\vdash \neg\alpha$ (MP)
$\vdash \forall x(\neg\alpha)$ (Gen, as x is not free in β)
$\vdash \neg\exists x\alpha$ (PL, and definition of $\exists$).
Hence $\Gamma \cup \{\neg\beta\} \vdash \Box$, whence by PL and two uses of D.T., we get the required theorem.
Now, using D.T. on $\Gamma \cup \{\alpha\} \vdash \beta$, the first premise of Rule C, we get $\Gamma \vdash \alpha \to \beta$. Next, using the assumption of Rule C, we get $\Gamma \vdash \forall x(\alpha \to \beta)$, by Gen. Finally, using the assumption of the rule, the theorem we just proved, and the second premise of Rule C, we get $\Gamma \vdash \beta$. $\square$

Observe some special cases of the rules.

Corollary 1.2.

(i) From Spec:

$$\frac{\Gamma \vdash \forall x\alpha}{\Gamma \vdash \alpha}$$

$$\frac{\Gamma \vdash \forall x\alpha}{\Gamma \vdash \alpha(t|x)}$$

Another special case arises, when $\Gamma = \emptyset$.

(ii) From Rule $\exists$:

$$\frac{\Gamma \vdash \alpha}{\Gamma \vdash \exists x\alpha}$$

$$\frac{\Gamma \vdash \alpha(t|x)}{\Gamma \vdash \exists x\alpha}$$

Again, another special case arises, when $\Gamma = \emptyset$.

(iii) From Gen:

$$\frac{\vdash \alpha}{\vdash \forall x\alpha}$$

$$\frac{\Gamma \vdash \alpha}{\Gamma \vdash \forall x\alpha} \quad x \text{: not free in } \Gamma$$

(iv) From Rule C:

$$\frac{\{\exists x\alpha\} \cup \{\alpha\} \vdash \beta}{\{\exists x\alpha\} \vdash \beta} \quad \text{if } x \text{ is not free in } \beta.$$

$$\frac{\Gamma \cup \{\alpha\} \vdash \beta \qquad \Gamma \vdash \exists x\alpha}{\Gamma \vdash \exists x\beta}$$

x: not free in Γ, β

The next result is an important one, as it tells us that we can replace a variable in a wff uniformly by another one to get a logically equivalent wff, under certain conditions.

Proposition 1.3. (**Change of bound variables**)

$$\vdash \forall x\alpha(x) \leftrightarrow \forall y\alpha(y/x),$$

provided (i) y is free for x in α, and (ii) y is not free in $\alpha(x)$.

Proof. The proof simply uses Spec followed by Gen in both the directions. Conditions (i) and (ii) ensure that on replacing y by x in $\alpha(y/x)$, one gets back $\alpha(x)$. This fact is used in the converse direction. $\square$

Observe that Proposition 1.3 holds, in particular, for any variable y that *does not occur* in α, as conditions (i) and (ii) are both vacuously satisfied for such a y.

$\alpha \mapsto \alpha(y/x)$

$x \dots x \quad y \dots y$

$\vdash \forall x\, \overset{pyy}{pxx} \leftrightarrow \forall y\, pyy$

$\alpha(x) \qquad \alpha(y/x)$

$\forall x\, pxy \not\leftrightarrow \forall y\, pyy$

$\vdash \forall x \alpha(x) \leftrightarrow \forall y \alpha(y/x)$

$\{\forall x \alpha(x)\} \vdash \alpha(y/x)$ Spec

$\forall y \alpha(y/x)$ Gen.

$\{\forall y \alpha(y/x)\} \vdash \beta(x/y)$ Spec

$\forall y \beta(y)$

$\forall x \beta(x/y)$ Gen

$\forall x \alpha(x)$

Lecture 26

Sunday, 8 November 2020 11:01 AM

Lec28Mo...

Completeness of FOL

Much in the same lines as for PL, the completeness of FOL is proved with respect to the semantics defined earlier. As we have enriched the language substantially, there is more work involved. Completeness is proved for a set $\Gamma \cup \{\alpha\}$ of *closed* wffs.

We begin with the notion of consistency, extend it to maximal consistency;

In PL (i) cons's $\xrightarrow{\text{ext}}$ max. cons.
(ii) max cons $\rightarrow$ model

extend the language by adding constants and introduce the notion of 'Henkin-closed' sets; prove that

(a) consistency is preserved in the extended language,

(b) any consistent set in the extended language can be extended to a maximally consistent and Henkin-closed set, and

Γ. max cons.: $\Gamma \vdash \alpha$ iff $\alpha \in \Gamma$

(c) any maximally consistent and Henkin-closed set of closed wffs must have a model.

$\neg\alpha$ $\alpha \wedge \beta$

$\alpha(x)$ $\alpha(t/x)$

$$\frac{\Gamma \vdash \alpha(t/x),\ \text{all } t \text{ (closed terms)}}{\Gamma \vdash \forall x\, \alpha}$$

As any model of an extension of a set also serves as a model of the set itself, (a)-(c) lead to the result that if Γ is a consistent set of closed wffs, it has a model.

This, in turn, leads to the completeness theorem, in the same way as in PL.

Theorem 0.1. If Γ is a consistent set of closed wffs, it has a model.

Theorem 0.1 leads to the completeness theorem of FOL for closed wffs (same proof as in PL).

Theorem 0.2. (Completeness) For a set Γ of closed wffs and a closed wff α, if $\Gamma \models \alpha$ then $\Gamma \vdash \alpha$.

Suppose $\Gamma \nvdash \alpha$. $\Gamma \cup \{\neg\alpha\}$: consistent
[If incons, then $\Gamma \vdash \alpha$]

By Th. 0.1, there is $\mathfrak{I}, v$
s.t. $\mathfrak{I}, v \models \Gamma$, $\mathfrak{I}, v \models \neg\alpha \Rightarrow \mathfrak{I}, v \not\models \alpha$.

Lecture 28

Sunday, 8 November 2020 11:01 AM

Lec28Mo...

First order Theories

Any *first order theory* T is defined as FOL having some distinguished predicate, function or constant symbols in the alphabet, along with special axioms involving these symbols (apart from all the FOL-axioms listed earlier). The distinguished symbols are termed the *proper symbols* of T, and the special axioms, which are closed wffs, are called the *proper axioms* of T.

Let us note the following.

- In an abuse of notation, the set of proper axioms of T will also be denoted by T.
- When we say *α is a wff of T*, we mean that all the proper symbols occurring in α (if any) come from the set of proper symbols of T.
- *$\mathcal{I}$ is an interpretation* or *a model of* T means that $\mathcal{I}$ is an interpretation or a model for all the proper axioms of T.

$\Gamma \vdash \alpha$

- *α is a theorem of* T, sometimes written as $\vdash_T \alpha$, means that $T \vdash \alpha$, i.e. α is deduced (through the standard FOL deduction procedure) from the proper axioms of T and the FOL-axioms. This would imply that, in particular, all the FOL-theorems are also T-theorems.
- All the results proved about any set Γ of closed wffs in earlier Lectures, apply to all first order theories T with countable language, considering T to also denote the set of proper axioms (which are closed wffs) of the theory. So we would have, e.g. if T is consistent, it has a model, or if $T \models \alpha$ then $T \vdash \alpha$, for any closed wff α in T (theorems done earlier).

1 First order theories with equality

A *first-order theory with equality* has in its language a special binary predicate symbol $=$, and as its axioms (i) the *FOL* axioms, (ii) a set of *proper axioms*, and (iii) the *equality axioms* given below.

$x < y$ $x + y$ $x' + y'$
$x' < y'$ x' y'

The equality axioms:

E1 $\forall x(x = x)$.

E2 $\forall x_1 ... \forall x_n \forall y_1 ... \forall y_n ((x_1 = y_1 \wedge ... \wedge x_n = y_n) \rightarrow (p^n x_1 ... x_n \leftrightarrow p^n y_1 ... y_n))$,
for each n-ary predicate symbol p^n in the language.

E3 $\forall x_1 ... \forall x_n \forall y_1 ... \forall y_n ((x_1 = y_1 \wedge ... \wedge x_n = y_n) \rightarrow f^n x_1 ... x_n = f^n y_1 ... y_n)$,
for each n-ary function symbol f^n in the language.

p^2

Exercise 1.1. Using the axioms, prove symmetry and transitivity of $=$, i.e.

(i) $\forall x \forall y (x = y \rightarrow y = x)$, and

(ii) $\forall x \forall y \forall z ((x = y \wedge y = z) \rightarrow x = z)$.

Hint: In axiom E2 above, consider the special case when $n = 2$, and p^2 is $=$ itself, viz.
$\forall x_1 \forall x_2 \forall y_1 \forall y_2 ((x_1 = y_1 \wedge x_2 = y_2) \rightarrow (x_1 = x_2 \leftrightarrow y_1 = y_2))$.

E1 and (i)-(ii) of the exercise above guarantee that, in any model $\mathcal{I}$ of a first order theory T with equality, the predicate symbol $=$ would be interpreted at least as an equivalence relation $=_{\mathcal{I}}$. E2 and E3 reflect the property of 'substitutivity' that is a salient feature of the equality relation. This makes $=_{\mathcal{I}}$ what is called a *congruence* relation. Are there models $\mathcal{I}$ where $=_{\mathcal{I}}$ is the 'actual' equality relation?

Can the substitutivity given by axioms E2 and E3 be extended to *any* wffs and terms? We state the following results without proof.

Proposition 1.1.

(i) $\vdash r = s \to t_r = t_s$, where r, s are any terms, t_r contains r and t_s is obtained by replacing one or more occurrences of r in t_r by s.

(ii) $\vdash r = s \to (\alpha_r \leftrightarrow \alpha_s)$, where r, s are any terms, α_r contains r, no bound variable of α_r occurs in r or s, and α_s is obtained by replacing one or more occurrences of r in α_r by s.

It should be remarked here that sometimes only the reflexivity axiom E1 and a version of (ii) in the above proposition are taken as the equality axioms (e.g. cf. Mendelson).

2 Examples of first order theories with equality

We just specify the set $\mathcal{P}$ of proper symbols, and the proper axioms in the following.

(i) **Groups**
$\mathcal{P} := \{=, +, 0\}$, $+$: binary function symbol and 0: constant;
Proper axioms:
$\forall x \forall y \forall z((x+y)+z = x+(y+z))$
$\forall x(x+0=x)$
$\forall x \exists y(x+y=0)$
Observe that the equality axiom E3 applied on $+$ takes the form:
$\forall x_1 \forall x_2 \forall y_1 \forall y_2((x_1 = y_1 \wedge x_2 = y_2) \rightarrow (x_1 + x_2 = y_1 + y_2))$.

(ii) **Abelian Groups**: Groups with the additional proper axiom
$\forall x \forall y(x+y=y+x)$

(iii) **Partially Ordered Sets**

$\mathcal{P} := \{=, \leq\}$, $\leq$: binary predicate symbol;

Proper axioms:

$\forall x(x \leq x)$

$\forall x \forall y((x \leq y \wedge y \leq x) \rightarrow x = y)$

$\forall x \forall y \forall z((x \leq y \wedge y \leq z) \rightarrow x \leq z)$

(iv) **Lattices**

$\mathcal{P} := \{=, \leq, \cup, \cap\}$, $\leq$: binary predicate symbol, $\cup, \cap$: binary function symbols;

Proper axioms:

Proper axioms for Partially Ordered Sets;

$\forall x \forall y(x \leq x \cup y \wedge y \leq x \cup y)$

$\forall x \forall y((x \cap y \leq x \wedge x \cap y \leq y)$

$\forall x \forall y \forall z((x \leq z \wedge y \leq z) \rightarrow x \cup y \leq z)$

$\forall x \forall y \forall z((z \leq x \wedge z \leq y) \rightarrow z \leq x \cap y)$

Total order

(v) **Dense Linearly Ordered Sets without End-Points**

$\mathcal{P} := \{=, <\}$, $<$: binary predicate symbol;

Proper axioms:

$\forall x \neg(x < x)$

$\forall x \forall y (x < y \vee y < x \vee x = y)$

$\forall x \forall y \forall z ((x < y \wedge y < z) \to x < z)$

$\forall x \forall y (x < y \to \exists z (x < z \wedge z < y))$

$\forall x \exists y (y < x)$

$\forall x \exists y (x < y)$

(vi) **Number Theory**

$\mathcal{P} := \{=, ', +, \times, 0\}$, $+, \times$: binary function symbols, $'$: unary function symbol (interpreted as the 'successor' function);

successor

Proper axioms:

Peano Arithmetic

$\forall x \forall y (x' = y' \to x = y)$

$\forall x (x' \neq 0)$

$\forall x (x + 0 = x)$

$\forall x \forall y (x + y' = (x + y)')$

$\forall x (x \times 0 = 0)$

$\forall x \forall y (x \times y' = (x \times y) + x)$

For all wffs $\alpha(x)$ in the language, the closure of $(\alpha(0/x) \wedge \forall x(\alpha(x) \to \alpha(x'/x))) \to \forall x \alpha(x)$

Exercise 2.1. Formulate the first order theories of (i) Rings, (ii) Fields, and (iii) Boolean Algebras.

Lecture 29 Module 2

Saturday, 14 November 2020 10:40 AM

Lec29Mo...

Equality

1 Uniqueness

Without equality, we cannot express the notion of *uniqueness*.

We use the notation $\exists! x\alpha(x)$ to denote "there exists a unique x such that $\alpha(x)$ holds". Formally, $\exists! x\alpha(x)$ is an abbreviation of the wff

$$\overbrace{\exists x\alpha(x)}^{\text{existence}} \wedge \overbrace{\forall x\forall y((\alpha(x) \wedge \alpha(y)) \rightarrow x = y)}^{\text{uniqueness}},$$

where y is the first variable in the enumeration of variables that is not in $\alpha(x)$.

In fact, $\exists! x\alpha(x)$ defined as above, is logically equivalent to

$$(i)\ \exists x(\alpha(x) \wedge \forall y(\alpha(y) \rightarrow y = x)),$$

and also to

$$(ii)\ \exists x\forall y(x = y \leftrightarrow \alpha(y)).$$

It is a non-trivial exercise to prove the equivalence, but you can try (syntactically, or semantically – as you wish).

Example 1.1. An easy consequence: $\vdash \forall x \exists! y(\overbrace{y = x}^{\alpha(y)})$.

Proof. Note that

$$\exists! y(\overbrace{y = x}^{\beta(x)}) := \exists y(y = x) \wedge \forall y \forall z((\overbrace{y = x}^{\alpha(y)} \wedge \overbrace{z = x}^{\alpha(z)}) \to y = z).$$

To prove the first part of the conjunct, use reflexivity ($x = x$), followed by Rule $\exists$. For the other part, the properties of symmetry and transitivity of equality proved earlier, give $(y = x \wedge z = x) \to y = z$. Two uses of Gen give $\forall y \forall z((y = x \wedge z = x) \to y = z)$. $\square$

$$\frac{\vdash \phi(x)}{\vdash \forall x\, \phi(x)} \text{ (Gen)}$$

E1 $\vdash \forall x(x = x)$

Spec $\vdash x = x$ $\quad t = x$

$\exists$-rule $\vdash \exists y(y = x) \to \alpha(t|y)$

$$\frac{\vdash \alpha(t/y)}{\vdash \exists y\, \alpha(y)}$$

$\exists x \exists y (\underline{\alpha(x) \wedge \alpha(y)} \wedge x \neq y)$

2 *At least, at most, exactly*

We can express notions such as "at least" or "at most", with the help of equality.

Example 2.1.

(i) The wff $\exists x \exists y(\alpha(x) \wedge \alpha(y) \wedge x \neq y)$ expresses that "there exist at least two x's such that $\alpha(x)$. More simply, the wff $\exists x \exists y(x \neq y)$ expresses that there exist at least two elements. $\underline{\mathcal{I} \models \exists x \exists y (x \neq y)} =$

In other words, for any interpretation $\mathcal{I}$ of a first order theory T with equality, iff $\underline{\exists d_1, d_2 \in D, d_1 \neq d_2}$

$\mathcal{I} \models \underline{\exists x \exists y(x \neq y)}$, if and only if $\mathcal{I}$ has at least two elements.

In general, for $n \geq 2$, we denote by '$\underline{\exists n}$' the wff
$\exists x_1 \ldots \exists x_n(x_1 \neq x_2 \wedge x_1 \neq x_3 \wedge \ldots \wedge x_1 \neq x_n \wedge x_2 \neq x_3 \wedge \ldots \wedge x_{n-1} \neq x_n)$,
giving the following.

(a) For any interpretation $\mathcal{I}$ of T,
$$\mathcal{I} \models \exists n, \text{ if and only if } |D| \geq n,$$
where $|D|$ denotes the cardinality of the domain D of $\mathcal{I}$.

(b) If $n \geq k$, $\vdash \exists n \to \exists k$.

(ii) Let us enhance the wff $\exists n$, to express the existence of *exactly* n elements. Define the wff '$\exists! n$' as follows.

$$\exists x_1 \dots \exists x_n ((x_1 \neq x_2 \wedge x_1 \neq x_3 \wedge \dots \wedge x_1 \neq x_n \wedge x_2 \neq x_3 \wedge \dots \wedge x_{n-1} \neq x_n) \wedge \forall y (y = x_1 \vee y = x_2 \vee \dots \vee y = x_n)).$$

Here we would have, for any interpretation $\mathcal{I}$ of T,

$$\mathcal{I} \models \exists! n, \text{ if and only if } |D| = n.$$

Exercise 2.1. Express through wffs in FOL with equality, that

(i) there exist at most two x's such that $\alpha(x)$,

(ii) there exist exactly two x's such that $\alpha(x)$.

Lecture 32 Module 1

Saturday, 21 November 2020 1:04 PM

Lec32Mo...

Completeness of first order theories

This is a *property of* a first order theory that we shall define – as opposed to the meta-theorem we proved *about* first order theories, viz. the 'completeness theorem'.

Definition 0.1. A first order theory T is said to be *complete*, provided, for all closed wffs α of T, either $T \vdash \alpha$ or $T \vdash \neg\alpha$.

for all I, $I \models \alpha$, or for all I, $I \models \neg\alpha$

We immediately observe the following consequences of the definition.

$T \vdash \alpha$, for all α

(i) If T is inconsistent, it is complete.

(ii) If T is maximally consistent, it is complete.

Moreover, as a consequence of (ii) and earlier Proposition,

(iii) if T is consistent, it has a consistent and complete extension.

A useful characterization of the notion of completeness is given by the following.

Proposition 0.1. T is complete, if and only if every closed wff of T that is true in one model of T is true in all models of T.

$T \models \neg\alpha$

Proof. ($\Rightarrow$) Suppose $\mathcal{I} \models \alpha$, where $\mathcal{I}$ is a model of T. As T is complete, either $T \vdash \alpha$ or $T \vdash \neg\alpha$. But the latter cannot be the case, as $\mathcal{I} \models \alpha$. Then $T \vdash \alpha$ implies that $\mathcal{I}' \models \alpha$, for all models $\mathcal{I}'$ of T.

($\Leftarrow$) Divide into two cases: (i) T is inconsistent, or (ii) T is consistent.
(i) trivially gives completeness, as noted above.
(ii) T has a model, say $\mathcal{I}$, by earlier Theorem. For any closed α in T, either $\mathcal{I} \models \alpha$ or $\mathcal{I} \models \neg\alpha$. Using the assumption and the completeness theorem, the first leads to $T \vdash \alpha$, and the second to $T \vdash \neg\alpha$. $\square$

$T \models \alpha$ $T \models \neg\alpha$

Using Proposition 0.1, we can easily produce examples of *incomplete* first order theories.

For instance, the theory AG of Abelian groups is incomplete, as the closed wff $\forall x \forall y (x = y)$ is true in singleton models of AG, but false in all others.

Clearly, this argument would apply to all theories having models of different finite cardinalities, making them incomplete. We shall see examples of complete theories in the last lecture.

Recall Number Theory, denoted $\aleph$, formulated as a first order theory earlier. We end this module by stating two fundamental results about $\aleph$, due to Kurt Gödel. One may refer to Margaris, for a sketch of the proofs.

Theorem 0.2. (Gödel's incompleteness theorem) If $\aleph$ is consistent, it is an incomplete theory.

Any closed α, $T \vdash \alpha$ or $T \vdash \neg\alpha$

The second theorem (stated informally): The consistency of $\aleph$ cannot be proved in $\aleph$.

The second theorem is a consequence of the first.

These two results are particularly significant in that they can be applied to theories containing $\aleph$, or those containing theories even weaker than $\aleph$.

Let us end this lecture with the following exercise.

Exercise 0.1.

(i) Let $\mathcal{I}$ be a model of T. Let the theorems of T' be precisely the wffs of T that are true in $\mathcal{I}$. Show that T' is a consistent and complete extension of T.

(ii) Prove that a closed wff α of T is a theorem of T if and only if α is a theorem of every consistent and complete extension of T.