We now begin the study of the formal theory of *classical propositional logic* (PL). Let us first specify the language of PL.

1 Alphabet

The alphabet of PL consists of

1. a non-empty countable set PV of *propositional variables*, denoted as p_1,p_2,p_3,... (or simply, $p, q, r, ...$),
2. *logical connectives* $\neg$, $\wedge$, $\vee$, $\rightarrow$, and
3. parentheses (,).

In particular, the set PV could be finite. In fact, for all practical purposes, one would begin with a finite PV.
Propositional variables represent simple declarative statements (termed *propositions*), such as '2 is an even number', or 'The grass is green', which take exactly one of two *truth values* 'true' and 'false'.
Logical connectives connect simple propositions to form compound propositions. $\neg$ is a unary connective that stands for 'not' (or 'negation'), and $\wedge$, $\vee$, $\rightarrow$ are binary connectives that stand, respectively, for 'and' (or 'conjunction'), 'or' (or 'disjunction'), and 'if-then' (or 'implication'). ($\rightarrow$ is also referred to as the 'conditional'.) Formally, compound propositions are built as follows, to give *well-formed formulae* – the statements with which PL deals.

2 Well-formed formulae (wffs)

Definition 2.1. The *set $\mathcal{F}$ of wffs* is the smallest set of strings over the alphabet given above, such that

(i) any propositional variable in PV is a wff (i.e. a member of $\mathcal{F}$), called an *atomic wff*,

(ii) if α is a wff, so is $(\neg\alpha)$,

(iii) if α, β are wffs, so are $(\alpha \wedge \beta)$, $(\alpha \vee \beta)$, $(\alpha \rightarrow \beta)$.

When there is no confusion, we shall drop the outermost parentheses.

Example 2.1. $\neg(p \wedge (q \rightarrow (\neg r)))$ is a wff, $\neg \wedge p$ is not.

Observe how in the first string, we must be careful about the brackets:
$\neg(p \wedge q \rightarrow (\neg r))$ would have resulted in confusion! However, we may safely drop the brackets around $\neg r$. Normally, one follows the law of 'left association'.
The second string fails to be a wff, as we have not respected the *arity* of the connective $\wedge$: it is binary, and so needs two wffs to act upon.

Exercise 2.1. Which of the following are wffs?

(i) $p \vee (\neg q)$.

(ii) $\wedge(q \rightarrow r)$.

Proposition 2.1. The set $\mathcal{F}$ of all wffs of PL is countable.

Proof. (Hint) The alphabet of PL could be taken to be finite and given an order. For instance, one may write it as $\{\neg, \rightarrow, (,), p, '\}$, where the last symbol, along with p, can be used to generate the countable set of propositional variables. Then generate the set of all strings over this alphabet using the connectives and parentheses in a suitable order, one that would demonstrate its countability. $\square$

Abbreviation

- $\alpha \leftrightarrow \beta := (\alpha \rightarrow \beta) \wedge (\beta \rightarrow \alpha)$.

The wffs defined above may be used to express sentences in natural language. For instance, take the sentence "If Australia reached the World Cup finals, then either England slipped up or South Africa played very well." Let the propositional variable p denote "Australia reached the World Cup finals", q "England slipped up", and r "South Africa played very well". Then the given sentence is translated in PL by the wff $p \rightarrow (q \vee r)$. "m is positive, but not an integer", would simply be translated as $p \wedge \neg q$, where p stands for "m is positive", and q for "m is an integer".

Exercise 2.2. Express the following arguments given in natural language, as sequences of wffs in PL.

(i) If m is negative then n is negative. If s is positive then n is negative. If m is negative or s is positive then n is negative.

(ii) If Gödel was the Prime Minister, the parliament would pass logical laws. Gödel was not the Prime Minister. The parliament does not pass logical laws.

In order to get a sense of the 'use' of PL, let us first take a look at the *semantics* of PL, that deals with truth and falsity of propositions. We shall study the deduction procedure of PL a little later, starting from Lecture 11 of this module.

As mentioned earlier, we deal in PL with exactly two truth values 'true' (in brief, T), and 'false' (F). (Your guess is right: there are formal theories with semantics involving *more than two* truth values. These are called *many-valued logics*.) Alternatively, we may denote 'true' and 'false' respectively by 1 and 0. We shall have a close encounter with the truth value set $\mathbf{2}{:=}\{0, 1\}$ (or, $\{T, F\}$), in the part of the course on Boolean algebras.

How do we ascribe, mathematically, truth values to wffs? Given truth values of atomic wffs, is there a procedure for evaluating the truth value of a general ('compound') wff? This is done through *valuations*, defined in the next lecture. In order to define valuations, we shall have to look at operations on the set **2** that 'correspond' to the logical connectives $\neg, \wedge, \vee, \rightarrow$ of PL. The operations are defined by *truth tables*, given in the following section.

Note. We use the notations of the logical connectives for the corresponding operations on **2** as well.

3 ~~Truth tables~~

Let $x \in \mathbf{2}$.

Table for $\neg$

x	$\neg x$
T	F
F	T

Table for $\wedge$

$\wedge$	T	F
T	T	F
F	F	F

Table for $\vee$

$\vee$	T	F
T	T	T
F	T	F

Table for $\rightarrow$

$\rightarrow$	T	F
T	T	F
F	T	T

As we shall see in the next lecture, using these tables and the notion of valuations, one may compute *truth tables of wffs.*

We stress here that these truth tables are sufficient to express the mathematical connotations of the logical connectives. They fail to capture the varied usage of these connectives in natural language. For instance, very often, 'and' would be used in a temporal sense in everyday language, and hence is non-commutative: "I fell ill and went to the doctor" is certainly not treated as equivalent to "I went to the doctor and fell ill"! But, as we can see from the truth-table of $\wedge$, $\wedge$ is commutative. 'Or', given by the truth table for $\vee$, is not mutually exclusive. However, some connectives of natural language may be translated satisfactorily. "p, unless q" is expressed by the wff $\neg p \rightarrow q$. More often than not, we would say "Not p, unless q", which is then given by the wff $p \rightarrow q$ (equivalently, by $\neg q \rightarrow \neg p$, as we shall see in Lecture 3).

1 Valuations

Valuations are defined in two steps: first these assign truth-values to atomic propositions, i.e. propositional variables in PV; then using these truth values, one determines the truth values of wffs in general, by defining an *extension* of the initial assignment. Formally, the extension is done through *mathematical induction*, on the number of connectives occurring in the wff in question. The truth tables for the different connectives defined in the previous lecture, are used to determine the truth values.

Definition 1.1. A valuation is a map v: $PV \longrightarrow \{T, F\}$.
v is extended to a map on the set $\mathcal{F}$ of wffs as follows. We retain the notation v for the extension.

- $v(\neg\alpha) := \neg v(\alpha)$.
- $v(\alpha \wedge \beta) := v(\alpha) \wedge v(\beta)$.
- $v(\alpha \vee \beta) := v(\alpha) \vee v(\beta)$.
- $v(\alpha \rightarrow \beta) := v(\alpha) \rightarrow v(\beta)$.

Example 1.1. Let $\alpha := (\neg p) \wedge (\neg q \vee p)$, and $v(p) := T$, $v(q) := F$. Then $v(\alpha) = v(\neg p) \wedge v(\neg q \vee p) = \neg v(p) \wedge (\neg v(q) \vee v(p)) = \neg T \wedge (\neg F \vee T) = F \wedge (T \vee T) = F \wedge T = F$.

Exercise 1.1.

(i) Let v be the truth assignment such that $v(p_1) := T$, $v(p_2) := F$, $v(p_3) := F$. Find $v(p_1 \rightarrow (\neg p_2 \wedge p_3))$.

(ii) Let α be the wff $((p_1 \rightarrow p_2) \wedge \neg p_1) \rightarrow \neg p_2$. Find a valuation that satisfies α and one that does not.

2 ~~Truth tables of wffs~~

Now one is ready to define truth tables of wffs. Consider a wff α with n propositional variables, say $p_1, \ldots, p_n$. The truth table of α consists of $n+1$ columns. The first n entries in a row in the truth table of α consists of some truth values t_i, $i = 1, \ldots, n$ assigned to the propositional variables $p_i, i = 1, \ldots, n$ respectively. In other words, every row defines a valuation v such that

$$v(p_i) = t_i,\ i = 1, \ldots, n....(*)$$

The $(n+1)$-th entry of the row gives the truth value of α under that valuation. So, notice that for the n propositional variables in α, there would be 2^n distinct rows in its truth table. Observe also that there can be many valuations satisfying (*). However, we have the following simple but important proposition.

~~Example 2.~~1. Let us find the truth table of $(p \wedge q) \to r$.

p	q	r	$p \wedge q$	$(p \wedge q) \to r$
T	T	T	T	T
T	T	F	T	F
T	F	T	F	T
T	F	F	F	T
F	T	T	F	T
F	T	F	F	T
F	F	T	F	T
F	F	F	F	T

In the 3rd row, for instance, we consider any valuation v such that $v(p) = v(r) = T$, $v(q) = F$, and compute $v((p \wedge q) \to r)$ to get the last entry in the row.

~~Exercise 2.1~~.

(i) Find the truth table of $(\neg p \to q) \vee r$.

(i~~i) Show that $p \to q$ and $\neg p \vee q$ have the same truth~~ tables.

3 Some special wffs

We now take a look at some special classes of wffs, that will play an important role in PL.

Definition 3.1. (**Tautology**)
α is a *tautology* if and only if under every valuation v, $v(\alpha) = T$.

Example: $p \vee \neg p$, or, $p \to p$.

Exercise 3.1.

(i) Is $p \to (\neg p \to p)$ a tautology? What about $\neg p \to (\neg p \to p)$?

(ii) Show that if α and $\alpha \to \beta$ are tautologies, so is β.

(iii) Let α be a wff whose only connective is $\vee$. Can α be tautology? Justify your answer.

Exercise 3.2. (**Characterization of $\to$**)
Show that each of the following is a tautology.

(i) $(\alpha \to \beta) \leftrightarrow (\neg\alpha \vee \beta)$.

(ii) $(\alpha \to \beta) \leftrightarrow \neg(\alpha \wedge \neg\beta)$.

(iii) $(\alpha \to \beta) \leftrightarrow (\neg\alpha \vee (\alpha \wedge \beta))$.

Definition 3.2. (**Contradiction**)
α is a *contradiction* if and only if under every valuation v, $v(\alpha) = F$.

Examples: $p \wedge \neg p$, $\neg(p \to (q \to p))$.