An argument in any daily discourse typically takes the form "Γ, hence α", where Γ is a set (usually finite!) of statements, viz. the premisses of the argument, and α is the conclusion. We would say that such an argument is "valid", if the conclusion is true whenever the premisses are all true. We represent this in PL through the notion of *semantic consequence* as follows.

Let $\Gamma \cup \{\alpha\}$ be a set of wffs of PL.

Definition 0.1. Γ is said to be *satisfied* by a valuation v, if and only if for every $\gamma \in \Gamma$, $v(\gamma) = T$ (in brief, $v \models \Gamma$). In this case, v is also called a *model* of Γ. (When Γ is $\{\alpha\}$ for some wff α, we simply say v is a model of α.)
Γ is said to be *satisfiable*, if there is a valuation v that satisfies it.
On the other hand, Γ is *contradictory*, if and only if Γ is not satisfiable, i.e. there is no valuation v that satisfies *all* the wffs in Γ.

Example 0.1. Any set $\{p\}$, $p \in PV$, is clearly satisfiable.
Any set $\{\alpha\}$, where α is a contradiction, is of course contradictory, and so is the set $\mathcal{F}$ of all wffs of PL. What about $\{p, \neg q, p \to q\}$? $\{p, \neg q, q \to p\}$?

Definition 0.2. α is said to be a *semantic consequence of the set* Γ, denoted $\Gamma \models \alpha$, when, for all valuations v, if v is a model of Γ, it is also a model of α, i.e. $v(\gamma) = T$ for each $\gamma \in \Gamma$ implies that $v(\alpha) = T$.

Remark. In the special case that Γ is $\emptyset$, we simply write $\models \alpha$. It is not difficult to see that $\models \alpha$, provided every valuation v satisfies α, i.e. α is a tautology. We say, in this case, that α is *valid*.

Exercise 0.1. Check if the arguments represented by the sequences of statements you translated in Exercise 2.2 of Lecture 2, are valid: in other words, can you represent the arguments legitimately in the form of $\Gamma \models \alpha$, for some Γ and α?

Exercise 0.2. Prove the following for any set $\Gamma \cup \Delta \cup \{\alpha\}$ of wffs in PL.

(i) If $\models \alpha$ then $\Gamma \models \alpha$.

(ii) $\Gamma \models \alpha$, if and only if $\Gamma \cup \{\neg\alpha\}$ is contradictory.

(iii) If Γ is satisfiable and $\Delta \subseteq \Gamma$ then Δ is satisfiable.

(iv) If Γ is contradictory and $\Gamma \subseteq \Delta$ then Δ is contradictory.

(v) (Overlap) If $\alpha \in \Gamma$ then $\Gamma \models \alpha$.

(vi) (Dilution) If $\Gamma \models \alpha$, $\Gamma \subseteq \Delta$ then $\Delta \models \alpha$.

(vii) (Cut) If $\Gamma \models \alpha$ and $\Delta \models \beta$, for each $\beta \in \Gamma$ then $\Delta \models \alpha$.

(viii) $\Gamma \cup \{\alpha\} \models \beta$, if and only if $\Gamma \models \alpha \to \beta$.

(ix) If $\Gamma \models \alpha \wedge \neg\alpha$ for some wff α, then $\Gamma \models \beta$, for any wff β.

A set Γ of premisses need not always be finite. But do we need to use all the members of an infinite Γ to obtain a conclusion α? This is the question of *compactness* of the theory. It turns out in PL that if $\Gamma \models \alpha$, there is always a finite subset Γ' of Γ such that $\Gamma' \models \alpha$. But there are other, equivalent, ways of making this assertion. To see this, let us define *finite satisfiability.*

Let Γ be any set of wffs.

Definition 0.3. Γ is *finitely satisfiable*, if and only if every finite subset of Γ is satisfiable.

What is the relation of finite satisfiability with satisfiability? Is there any relation of this question with that of compactness raised above? The first is answered by the Compactness theorem to be presented in the next lecture. The second is answered by the following proposition, which gives a relation between all the notions introduced in this lecture.

Proposition 0.1. The following are equivalent.

(i) Γ is satisfiable, if and only if Γ is finitely satisfiable.

(ii) Γ is contradictory, if and only if there is a finite subset of Γ that is contradictory.

(iii) For any wff α, $\Gamma \models \alpha$, if and only if there is a finite subset Δ of Γ such that $\Delta \models \alpha$.

Proof. Hint: take the help of results proved in Exercise 0.2 above. $\square$

1 The theorem

Theorem 1.1. (Compactness Theorem) Γ is satisfiable, if and only if Γ is finitely satisfiable.

Proof. One direction is immediate: if Γ is satisfied by v and Γ' is a finite a subset of Γ, v works for Γ' as well.
For the converse, we assume that the set PV of propositional variables is countably infinite, say $PV := \{p_1, p_2 ....\}$.
(How would you argue for the case when PV is finite? *Exercise!*)
Let Γ be finitely satisfiable. We need to find a valuation v such that $v(\gamma) = T$, for each $\gamma \in \Gamma$. This is achieved in two steps.
(i) We define a sequence $\{\epsilon_n\}_{n \in N}$ (N the set of natural numbers), such that $\epsilon_n \in \{T, F\}$, and for any $n \in N$, the following property is satisfied.
Property(R_n): For every finite subset Γ' of Γ, there is a valuation v such that $v \models \Gamma'$ and $v(p_i) = \epsilon_i, i = 1, ..., n$.
(ii) Next we define a valuation $v_0 : PV \to \{T, F\}$ such that $v_0(p_n) = \epsilon_n$, $i \in N$. v_0 is shown to satisfy Γ.
The definition of the ϵ_ns is by induction on n.
Base Case: $n = 1$. Two cases are possible.

- **Case 1:** For every finite subset Γ' of Γ, there exists v such that $v \models \Gamma'$ and $v(p_1) = F$.
  In this case, we set $\epsilon_1 := F$.

- **Case 2:** There exists a finite subset Γ_1 of Γ such that for all v, $v \models \Gamma_1$ implies $v(p_1) = T$.
  Here, we set $\epsilon_1 := T$.

In Case 2, we have the following important observation, which gives us an expression similar to Case 1.
Observation: For every finite subset Γ' of Γ, there exists v such that $v \models \Gamma'$ and $v(p_1) = T$.
Proof of Observation: Consider Γ_1 as in Case 2, and take a finite subset Γ' of Γ. $\Gamma' \cup \Gamma_1$ is then also a finite subset of Γ. As Γ is finitely satisfiable, there exists v such that $v \models \Gamma' \cup \Gamma_1$. So $v \models \Gamma'$, and $v \models \Gamma_1$ implies that $v(p_1) = T$, from Case 2.
Thus we may summarize the above facts obtained from Cases 1 and 2 as:
Property(R_1): For every finite subset Γ' of Γ, there exists v such that $v \models \Gamma'$ and $v(p_1) = \epsilon_1$.
Induction hypothesis: Suppose $\epsilon_1, ..., \epsilon_n$ have been defined such that the property (R_n) holds.
Induction step: We need to define ϵ_{n+1} such that property (R_{n+1}) is satisfied.

- **Case 1**: For every finite subset Γ' of Γ, there exists v such that $v \models \Gamma'$ and $v(p_i) = \epsilon_i, i = 1, ..., n$, and $v(p_{n+1}) = F$.
  In this case, set $\epsilon_{n+1} := F$.

- **Case 2**: There exists a finite subset Γ_{n+1} of Γ such that for all v, if $v \models \Gamma_{n+1}$ and $v(p_i) = \epsilon_i, i = 1, ..., n$ then $v(p_{n+1}) = T$.
  For this case, set $\epsilon_{n+1} := T$.

As before, we have the following observation, that will lead us to (R_{n+1}).
Observation: For every finite subset Γ' of Γ, there is v such that $v \models \Gamma'$, $v(p_i) = \epsilon_i, i = 1, ..., n$ and $v(p_{n+1}) = T$.
Proof of Observation: Exercise.
Thus the sequence $\{\epsilon_n\}_{n \in N}$ is defined, and moreover, it is such that for each n, the property (R_n) is satisfied. Now define the valuation $v_0 : PV \to \{T, F\}$ such that $v_0(p_n) := \epsilon_n$, $n \in N$. We show that v_0 satisfies Γ.
Let $\gamma \in \Gamma$, and suppose γ has all its propositional variables occurring in the list $p_1, \ldots, p_k$, for some k. Now $\{\gamma\}$ is a finite subset of Γ, and the property (R_k) is satisfied. So there is a valuation v such that $v \models \gamma$, i.e. $v(\gamma) = T$, and $v(p_i) = \epsilon_i, i = 1, ..., k$. But $v_0(p_i) = \epsilon_i, i = 1, ..., k$, also. By Proposition 2.1 of Lecture 3, we have $v_0(\gamma) = v(\gamma) = T$. □

2 An application

Let us illustrate how the compactness theorem may be used in different mathematical settings; we pick the domain of graph theory.

Consider a graph G on a set V: i.e. G is a symmetric and antireflexive binary relation on V. (Antireflexivity of G: $(x, x) \notin G$, for any $x \in V$.)
A *finite restriction* G' of G is a graph such that $G' \subseteq G$ and G' is finite.

Definition 2.1. Let $n (\neq 0)$ be any natural number. The graph G is said to be *n-colourable*, if and only if there is a function $f : V \longrightarrow \{1, \ldots, n\}$ such that $f(x) \neq f(y)$, for all $(x, y) \in G$. f is called a *colour assignment* of G.

We could interpret this as saying that no two adjacent vertices of G have the same colour under the colour assignment f.

Proposition 2.1. G is n-colourable, if and only if each finite restriction of G is n-colourable.

Hint

Proof. The compactness theorem will be used to prove the 'if' part of the result. (The other direction is obvious.)
Given n and any graph G, let us define the set PV of propositional variables as the set $\{p_{xi} : (x, i) \in V \times \{1, \ldots, n\}\}$. Now consider the following set of wffs on this PV.

$$\mathcal{S}(V, G, n) := \{\alpha_v : v \in V\} \cup \{\beta_{vu} : (v, u) \in G\},$$

where