Lecture 13 Module 1

Saturday, 26 September 2020 10:51 AM

Lec13Mo...

Axiomatization of PL

We had remarked earlier that all mathematical laws must be deducible from some 'primitive' or 'unquestioned' laws. These are the *axioms*.

Any formal theory has such a distinguished set of wffs, and also rule(s) of inference to define the deduction procedure.

Axioms and rule of inference of PL that make it a formal theory? – a *Hilbert system* – underlie many mathematical theories.

Note: there are mathematical theories based on logics different from PL, such as constructive mathematics based on Intuitionistic logic.

For this part, we assume that $\neg$ and $\rightarrow$ are the primitive logical connectives in the alphabet. The rest are defined in terms of these two connectives, i.e. we introduce the following abbreviations.

Abbreviations:

(a) $\alpha \wedge \beta := \neg(\alpha \rightarrow \neg\beta)$.

(b) $\alpha \vee \beta := \neg\alpha \rightarrow \beta$.

(c) $\alpha \leftrightarrow \beta := (\alpha \rightarrow \beta) \wedge (\beta \rightarrow \alpha)$, where $\wedge$ is defined as in (a).

1 Axiom schemata

$\alpha \rightarrow (\beta \rightarrow \gamma)$
$\equiv (\alpha \wedge \beta) \rightarrow \gamma$

Let α, β, γ be wffs of PL.

A1 $\alpha \rightarrow (\beta \rightarrow \alpha)$ (*Law of affirmation of consequent*)

A2 $(\alpha \rightarrow (\beta \rightarrow \gamma)) \rightarrow ((\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \gamma))$ (*Self-distributive law of implication*)

A3 $(\neg\beta \rightarrow \neg\alpha) \rightarrow (\alpha \rightarrow \beta)$ (*Law of contraposition*)

Note that α, β, γ are *any* wffs of PL. So A1-A3 are referred to as *axiom schemata*. For example, if p is a propositional variable, the wff

$$p \rightarrow ((p \rightarrow p) \rightarrow p)$$

is an instance of axiom A1.

2 Rule of Inference

Modus Ponens (MP)

$$\frac{\begin{array}{c}\alpha \\ \alpha \to \beta\end{array}}{\beta}$$

Again, α, β are any wffs of PL.

3 The deduction procedure of PL

semantic consequence
$\models$

Syntactic consequence relation $\vdash$ of PL:

Let Γ be any set of wffs and α any wff in PL.

Definition 3.1. α is a *syntactic consequence* of Γ (denoted $\Gamma \vdash \alpha$) if and only if, there is a sequence $\alpha_1, ..., \alpha_n (:= \alpha)$ such that each $\alpha_i (i = 1, ..., n)$ is either
(i) an axiom of PL, or
(ii) a member of Γ, or
(iii) derived from some of $\alpha_1, ..., \alpha_{i-1}$ by MP.

$\Gamma \vdash \alpha_1$
$\vdots$
α_i
$\vdots$
$\alpha = \alpha_n$

Remark. If Γ is empty in the above, we simply write $\vdash \alpha$, and say that α is a *theorem*, the sequence $\alpha_1, ..., \alpha_n (:= \alpha)$ constituting a *proof* of α.

$\vdash \alpha$

Note that the α_i's here need to be either axioms, or derived from previous members of the sequence by MP. In particular, then for any axiom α, we have $\vdash \alpha$.

Proposition 3.1.

(a) If $\vdash \alpha$, then $\Gamma \vdash \alpha$,

(b) *Overlap*: if $\alpha \in \Gamma$, then $\Gamma \vdash \alpha$,

(c) *Dilution*: if $\Gamma \subseteq \Delta$ and $\Gamma \vdash \alpha$, then $\Delta \vdash \alpha$,

(d) *Cut*: if $\Delta \vdash \gamma$ for each $\gamma \in \Gamma$ and $\Gamma \vdash \alpha$, then $\Delta \vdash \alpha$.

(e) *Compactness*: If $\Gamma \vdash \alpha$, then there is a *finite* subset Γ' of Γ such that $\Gamma' \vdash \alpha$.

$\Gamma \vdash \alpha_1$
Γ'
$\vdots$
$\alpha = \alpha_n$

Proof. Exercise! □

The semantic consequence relation $\models$ we discussed earlier also satisfies these properties. However, observe that the compactness property (property (e) above) of $\vdash$ comes directly from its definition, whereas it was established for $\models$ through the Compactness theorem.

Example 3.1. Let us prove the PL-theorem $\alpha \to \alpha$, for any wff α. By the definition above, we need to provide a sequence $\alpha_1, ..., \alpha_n$ for some n, with $\alpha_n := \alpha \to \alpha$, satisfying the conditions mentioned in the Remark made at the end of ~~Lecture 15~~, viz. each α_i should be either an axiom instance, or derived by MP from previous members of the sequence. Consider the following sequence, giving such a proof.

$\alpha_1 := (\alpha \to ((\alpha \to \alpha) \to \alpha)$ (A1)
$\alpha_2 := (\alpha \to ((\alpha \to \alpha) \to \alpha) \to ((\alpha \to (\alpha \to \alpha)) \to (\alpha \to \alpha))$ (A2)
$\alpha_3 := (\alpha \to (\alpha \to \alpha)) \to (\alpha \to \alpha)$ (MP on α_1, α_2)
$\alpha_4 := \alpha \to (\alpha \to \alpha)$ (A1)
$\alpha_5 := \alpha \to \alpha$ (MP on α_3, α_4)

A1: $\alpha \to (\beta \to \alpha)$
A2: $(\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma))$
A3: $(\neg\beta \to \neg\alpha) \to (\alpha \to \beta)$.

The following theorem gives a fundamental property of $\vdash$: it, in fact, relates three levels of 'implication' – those at the 'object' level ($\rightarrow$), 'meta'-level ($\vdash$) and 'meta-meta'-level ('if..then') of discourse.

Theorem 3.2. (Deduction Theorem or D.T.) For any Γ, α, β,
if $\Gamma \cup \{\alpha\} \vdash \beta$ then $\Gamma \vdash \alpha \rightarrow \beta$.

Proof. The proof is by induction on the number n of steps of derivation of β from $\Gamma \cup \{\alpha\}$.
Basis: $n = 1$. β is an axiom, or $\beta \in \Gamma \cup \{\alpha\}$. If β is an axiom or $\beta \in \Gamma$, we have a proof of $\alpha \rightarrow \beta$ from Γ as follows:

$\Gamma \vdash$ β (Proposition 3.1 (a) or (b))
$\beta \rightarrow (\alpha \rightarrow \beta)$ (A1)
$\alpha \rightarrow \beta$ (MP).
What if β is α? (*Exercise!*)

α : ax $\rightarrow \Gamma \vdash \alpha$
$\alpha \in \Gamma \rightarrow \Gamma \vdash \alpha$

In the induction step, we consider the possibility that β is derived by MP. So there would be two earlier steps of the form $\Gamma \vdash \gamma$ and $\Gamma \vdash \gamma \rightarrow \beta$ in the proof, for some wff γ. By induction hypothesis, $\Gamma \vdash \alpha \rightarrow \gamma$ and $\Gamma \vdash \alpha \rightarrow (\gamma \rightarrow \beta)$.
Then we have the following proof of $\alpha \rightarrow \beta$ from Γ:

$\alpha_1 := \alpha \rightarrow \gamma$
$\alpha_2 := \alpha \rightarrow (\gamma \rightarrow \beta)$
$\alpha_3 := (\alpha \rightarrow (\gamma \rightarrow \beta)) \rightarrow ((\alpha \rightarrow \gamma) \rightarrow (\alpha \rightarrow \beta))$ (A2)
$\alpha_4 := (\alpha \rightarrow \gamma) \rightarrow (\alpha \rightarrow \beta)$ (MP on α_2, α_3)
$\alpha_5 := \alpha \rightarrow \beta$ (MP on α_1, α_4). □

We shall give several examples of PL-theorems, and look at some more properties of $\vdash$, observing in the process, the usefulness of the Deduction theorem.

Lecture 13 Module 2

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Lec13Mo...

Theorems and metatheorems of PL

Notation. Let $\Gamma \vdash \square$ denote that there is some wff β such that $\Gamma \vdash \beta$ and $\Gamma \vdash \neg\beta$. We can read this as saying that Γ yields a contradiction.

The next two propositions include some important theorems and metatheorems of PL. Most are given as exercises. The deduction theorem comes in handy in proving these results.

Proposition 0.1.

(a) (Hypothetical Syllogism, HS) $\{\alpha \to \beta, \beta \to \gamma\} \vdash \alpha \to \gamma$.

(b) $\neg\alpha \to (\alpha \to \beta)$.

(c) If $\Gamma \vdash \square$, then $\Gamma \vdash \alpha$, for every wff α.

(d) (Reductio ad absurdum, RAA) If $\Gamma \cup \{\neg\alpha\} \vdash \square$, then $\Gamma \vdash \alpha$.

(e) $\vdash (\neg\alpha \to \alpha) \to \alpha$.

(f) $\vdash \neg\neg\alpha \to \alpha$.

involution

(g) $\vdash (\alpha \to \neg\alpha) \to \neg\alpha$.

(h) If $\Gamma \cup \{\alpha\} \vdash \square$, then $\Gamma \vdash \neg\alpha$.

(i) $\vdash \alpha \to \neg\neg\alpha$.

(j) $\vdash (\alpha \to \beta) \to (\neg\beta \to \neg\alpha)$.

(k) $\vdash (\beta \to \alpha) \to ((\neg\beta \to \alpha) \to \alpha)$.

(l) If $\Gamma \cup \{\beta\} \vdash \alpha$ and $\Gamma \cup \{\neg\beta\} \vdash \alpha$ then $\Gamma \vdash \alpha$.

(m) (i) $\{\neg(\alpha \to \neg\beta)\} \vdash \alpha$.

D.T: If $\Gamma \cup \{\alpha\} \vdash \beta$
then $\Gamma \vdash \alpha \to \beta$.

$\alpha \wedge \beta$

(ii) $\{\neg(\alpha \to \neg\beta)\} \vdash \beta$.

(n) (i) $\{\alpha\} \vdash \neg\alpha \to \beta$.

(ii) $\{\beta\} \vdash \neg\alpha \to \beta$. $\alpha \vee \beta$

Proof. We prove a few of the results, and leave the rest as exercises.

(a) Use D.T.: It is clear that $\{\alpha \to \beta, \beta \to \gamma\} \cup \{\alpha\} \vdash \gamma$.

To show: $\{\alpha \to \beta, \beta \to \gamma\} \vdash \alpha \to \gamma$

(b) Use HS on $\neg\alpha \to (\neg\beta \to \neg\alpha)$ (A1) and $(\neg\beta \to \neg\alpha) \to (\alpha \to \beta)$ (A3).

(c) Let $\Gamma \vdash \square$. So there is some wff β such that $\Gamma \vdash \beta$ and $\Gamma \vdash \neg\beta$. We have the following proof of α from Γ:

$\neg\beta \to (\neg\alpha \to \neg\beta)$ (A1)

$\neg\beta$ (assumption)

$\neg\alpha \to \neg\beta$ (MP)

$(\neg\alpha \to \neg\beta) \to (\beta \to \alpha)$ (A3)

$\beta \to \alpha$ (MP)

β (assumption)

α (MP).

(d) Let $\Gamma \cup \{\neg\alpha\} \vdash \square$. Using (c), $\Gamma \cup \{\neg\alpha\} \vdash \alpha$, as well as $\Gamma \cup \{\neg\alpha\} \vdash \neg(\neg\alpha \to \alpha)$. Then we have the following proof of α from Γ:

$\Gamma \vdash$ $\neg\alpha \to \alpha$ (D.T.)

$\neg\alpha \to \neg(\neg\alpha \to \alpha)$ (D.T.)

$(\neg\alpha \to \alpha) \to \alpha$ (A3 on previous step and MP)

α (MP). $\square$

(c) tells us that if Γ yields a contradiction, it yields "anything" – indicating a kind of collapse. (d) may be read as proof by contradiction: if you assume "not α" and get a contradiction, you must have α. (h) gives this in another way: if you assume α and get a contradiction, you must have "not α". The last two allow us to conclude that $\{\alpha \wedge \beta\}$ yields both α and β, while both $\{\alpha\}, \{\beta\}$ yield $\alpha \vee \beta$.

Remark. We must point out that the Deduction theorem or Hypothetical syllogism just serve as efficient tools for proving theorems of PL. They are *not, per se, constructions of proofs* of theorems. There may be logical systems where these are not available, and then one may have to resort to other techniques.

Exercise 0.1.

1. Prove the following theorems of PL.
   (i) $\vdash \alpha \to (\alpha \vee \beta)$.
   (ii) $\vdash \neg(\alpha \wedge \neg\alpha)$.
   (iii) $\vdash \alpha \to \neg(\neg\alpha \wedge \beta)$. $\neg , \to$
   (iv) $\vdash (\alpha \vee \beta) \vee (\beta \to \alpha)$.

2. Prove the following:
   (i) $\{\alpha, \neg\beta\} \vdash \neg(\neg\alpha \vee \beta)$.
   (ii) $\{\neg(\alpha \vee \beta)\} \vdash \neg\alpha$.
   (iii) $\{\alpha \to \beta, \gamma \vee \alpha\} \vdash \gamma \vee \beta$.
   (iv) $\{\alpha \to (\beta \wedge \gamma)\} \vdash \alpha \to \beta$.
   (v) $\{(\alpha \vee \beta) \to \gamma\} \vdash \alpha \to \gamma$.
   (vi) $\{\alpha \to \gamma\} \vdash (\alpha \wedge \beta) \to \gamma$.

Proposition 0.2. Consider PL with axiom A3 replaced by the two axioms

A3′ $\neg\alpha \to (\alpha \to \beta)$,

A3″ $(\neg\alpha \to \alpha) \to \alpha$.

$\vdash (\neg\beta \to \neg\alpha) \to (\alpha \to \beta)$

A3 can be derived as a theorem in this new system.

Proof. Exercise! □

Notice that we have already proved A3′ and A3″ as theorems. The above thus gives an alternate and *equivalent* axiomatization of PL (with MP as the rule of inference).

Lecture 14 Module 1

Saturday, 26 September 2020 12:42 PM

Lec14Mo...

Soundness theorem and Consistency

1 Soundness of PL

$\phi \vdash \alpha$ $\phi \models \alpha$

On the one hand, we now have the collection of theorems, and on the other, tautologies. Is there a relationship?

More generally, is there a relationship between the two consequence relations we have defined, viz. $\vdash$ and $\models$? $\Gamma \vdash \alpha$

There most certainly is, and in fact, we show that *these are identical*.

In other words, we shall establish that for any Γ and α in PL, $\Gamma \vdash \alpha$, if and only if $\Gamma \models \alpha$.

In particular then, $\vdash \alpha$, if and only if $\models \alpha$, i.e. the theorems are just the tautologies.

One direction of this fundamental result of PL is easy.

Cor. If $\vdash \alpha$ Then $\models \alpha$.

$\Leftrightarrow$ If $\not\models \alpha$ Then $\not\vdash \alpha$.

Theorem 1.1. (Soundness Theorem) If $\Gamma \vdash \alpha$ then $\Gamma \models \alpha$.

Proof. The proof is by induction on the number of steps of derivation of α from Γ. In essence, one shows that (i) the axioms are valid, and (ii) MP 'preserves truth', i.e. if α and $\alpha \to \beta$ are true, then β is also true. Complete the proof as an exercise! □

α_i

$\Gamma \vdash \alpha_i \to \alpha$

α

For the converse, called the *Completeness Theorem*, we need to do some work, and we shall require the notion of *consistency* defined in the following section. Before that, here are a few exercises. Use the soundness theorem to prove the following.

Exercise 1.1.

1. Let $\Gamma := \{p_1, \neg p_2 \vee p_3\}$. Show that there is no proof of p_3 using Γ. → $\Gamma \nvdash p_3$. Show $\Gamma \nvDash p_3$.
2. Let $\Gamma := \{p_n : n \geq 2\}$. Prove the following: & Use Soundness theorem.
   (i) There is no proof of p_1 using Γ.
   (ii) There is no proof of $\neg p_1$ using Γ.
3. Let $\Gamma \cup \{\neg\alpha\}$ be satisfiable. Show that α is not a theorem of Γ. → $\Gamma \vdash \alpha$

The soundness theorem tells us that if $\{\alpha_1, \alpha_2, \ldots, \alpha_n\} \vdash \beta$, then the argument form $\alpha_1, \alpha_2, \ldots, \alpha_n \therefore \beta$ is valid, i.e. $\{\alpha_1, \alpha_2, \ldots, \alpha_n\} \models \beta$.

Exercise 1.2. Use the soundness theorem to show that the following argument forms are valid:

(i) The arguments given in Exercise 2.2 of Lecture 2.

(ii) $\alpha \vee \beta, \gamma \vee \delta, \neg\alpha \vee \neg\gamma, \neg\beta \vee \neg\delta, \beta \vee \gamma \therefore \neg\alpha \wedge (\beta \wedge (\gamma \wedge \neg\delta))$.

(iii) $\alpha \vee (\beta \vee \gamma), \alpha \vee \neg\beta, \neg\alpha \vee \delta, \gamma \vee \neg\delta \therefore \gamma$.

(iv) $\alpha \vee (\beta \vee \gamma), \neg\alpha \vee \beta, \alpha \vee \neg\gamma, \neg\beta \vee \gamma \therefore \alpha \wedge (\beta \wedge \gamma)$.

(v) $\alpha \vee \beta, \beta \to (\alpha \wedge \gamma) \therefore \alpha$.

2 Consistency

We use the term "consistency" regularly in our daily discourse, specially in relation to arguments. For instance, we say "her conclusion is consistent with the assumptions".

This would typically mean that the statements – the ones that are the assumptions, and the conclusion – can all be true together.

In other words, if the set Γ represents the set of assumptions (premisses) and α the conclusion, the set $\Gamma \cup \{\alpha\}$ should be satisfiable, or equivalently, not contradictory.

The notion of contradiction has already been introduced in the syntax, through the notation $\Box$. We now formalize the notion of consistency of any set Γ using $\vdash$ and $\Box$.

Definition 2.1. Γ is said to be *negation consistent* if and only if $\Gamma \nvdash \Box$. → There is no β : $\Gamma \vdash \beta$ & $\Gamma \vdash \neg\beta$.

Interestingly, we need not depend on negation to define consistency.

Definition 2.2. Γ is said to be *absolutely consistent* if and only if there is some wff β such that $\Gamma \nvdash \beta$.

The idea is that, if everything follows from Γ, it indicates a collapse of the theory – which may be termed as inconsistency of the theory. However, in PL, we have the following.

Proposition 2.1. A set Γ of wffs is negation consistent if and only if it is absolutely consistent. → in PL.

Proof. Easy! □

Henceforth, *consistency* in PL would mean any of the equivalent notions of negation and absolute consistency.

The following propositions are straightforward to derive.

Proposition 2.2. The set of theorems of PL is consistent.

The above result is equivalent to showing that $\emptyset$ is consistent.

Proposition 2.3. If Γ has a model, it is consistent.

Is the converse true? For that, we need the notion of *maximal* consistency, defined in the next lecture. Before that, let us go through the following exercises. Note that we shall be making use of the soundness theorem and results on consistency above.

Exercise 2.1.

1. Let Γ be the set of all wffs of the form $\neg\alpha \vee \beta$. Is Γ consistent?

2. Let Δ be a consistent set of wffs. Show that the following are equivalent:

   (i) $\alpha \in \Delta$ or $\neg\alpha \in \Delta$ for every formula α.

   (ii) If $\Delta \cup \{\alpha\}$ is consistent, then $\alpha \in \Delta$.

Lecture 14 Module 2

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Completeness Th.: If $\Gamma \models \alpha$ then $\Gamma \vdash \alpha$.

→ (*) If Γ is consistent then Γ has a model.

Maximal consistency

→ 1. If Γ is consistent then there is a max. cons. extension Δ of Γ $(\Gamma \subseteq \Delta)$

→ 2. Any max. cons. set has a model.

Definition 0.1. A set Γ of wffs is *maximally consistent*, if and only if

1 & 2 → (*)

(i) it is consistent, and (ii) $\Gamma \cup \{\alpha\}$ is *inconsistent*, whenever $\alpha \notin \Gamma$.

So such a Γ is like a "fully blown balloon" – addition of a even a single wff leads to an explosion!

Maximal consistent sets of wffs are extremely special, as we see in the following proposition.

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Proposition 0.1. Let Γ be maximally consistent, and α, β any wffs.

(a) $\Gamma \vdash \alpha$, if and only if $\alpha \in \Gamma$.

(b) $\neg\alpha \in \Gamma$ if and only if $\alpha \notin \Gamma$.

(c) $\alpha \to \beta \in \Gamma$ if and only if $\alpha \notin \Gamma$, or $\beta \in \Gamma$.

(d) $\alpha \wedge \beta \in \Gamma$ if and only if $\alpha \in \Gamma$ and $\beta \in \Gamma$.

(e) $\alpha \vee \beta \in \Gamma$ if and only if $\alpha \in \Gamma$ or $\beta \in \Gamma$.

Proof. (a) Let $\Gamma \vdash \alpha$, and $\alpha \notin \Gamma$. By definition, $\Gamma \cup \{\alpha\} \vdash \Box$, and hence $\Gamma \vdash \neg\alpha$ (using earlier proposition) – a contradiction to the consistency of Γ.

(c) Let $\alpha \to \beta \in \Gamma, \alpha \in \Gamma$. This means $\Gamma \vdash \beta$, and so by (a), $\beta \in \Gamma$.

$$\frac{\Gamma \vdash \alpha \quad \alpha \to \beta}{\beta}$$

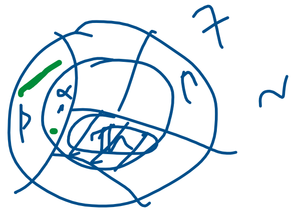
Conversely, first suppose $\alpha \notin \Gamma$. By (b), $\neg\alpha \in \Gamma$. Using earlier proposition, $\vdash \neg\alpha \to (\alpha \to \beta)$. So using (a), $\alpha \to \beta \in \Gamma$. If $\beta \in \Gamma$, use A1 and (a).

$$\frac{\Gamma \vdash \beta \quad \beta \to (\alpha \to \beta)}{\alpha \to \beta} \qquad \frac{\Gamma \vdash \neg\alpha \quad \neg\alpha \to (\alpha \to \beta)}{\alpha \to \beta}$$

We leave the other items as exercises. $\Box$

Corollary 0.2. A maximally consistent Γ contains all theorems of PL, and is closed with respect to MP. Γ may thus be viewed as a union of equivalence classes under the relation $\sim$ on $\mathcal{F}$.

We next prove a couple of fundamental results about maximal consistent sets, that will eventually lead us to the completeness theorem.



$\alpha \sim \beta$ iff $\vdash \alpha \leftrightarrow \beta$.

Proposition 0.3. If Γ is consistent, it has a maximally consistent extension.

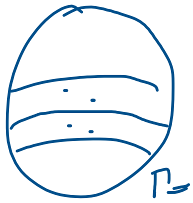
Proof. As we have mentioned earlier, the set $\mathcal{F}$ of all wffs of PL is countably infinite. Let $\alpha_o, \alpha_1, ...$ be an enumeration of all the wffs. We construct, recursively, an ascending chain of sets Γ_i, $i = 0, 1, 2, ...$ of wffs as follows.

(a) Γ_0 is Γ;

(b) For any $i \geq 0$, Γ_{i+1} is $\Gamma_i \cup \{\alpha_i\}$, if $\Gamma_i \cup \{\alpha_i\}$ is consistent.

Otherwise, Γ_{i+1} is Γ_i.

Clearly
(i) $\Gamma_0 \subseteq \Gamma_1 \subseteq \Gamma_2 \subseteq ... \subseteq \Gamma_i \subseteq ...$, and
(ii) for each i, Γ_i is consistent.



$\Gamma = \Gamma_0$

Let $\Delta := \bigcup_i \Gamma_i$. We show that Δ is the required maximally consistent extension of Γ.

(i) $\Gamma \subseteq \Delta$.

(ii) Δ is consistent: Suppose not.
Then there is a wff β with $\Delta \vdash \beta$ and $\Delta \vdash \neg\beta$. By compactness of $\vdash$, there are finite subsets Δ_1, Δ_2 of Δ such that $\Delta_1 \vdash \beta$ and $\Delta_2 \vdash \neg\beta$. One can find $i \geq 0$, such that $\Delta_1, \Delta_2 \subseteq \Gamma_i$. So $\Gamma_i \vdash \beta$ and $\Gamma_i \vdash \neg\beta$ (using dilution) – a contradiction to the consistency of Γ_i.

(iii) Δ is maximally consistent: Suppose $\alpha \notin \Delta$. To show $\Delta \cup \{\alpha\} \vdash \Box$
α is α_k, for some $k \geq 0$. Take $\Gamma_k \cup \{\alpha_k\}$. If it were consistent, $\alpha := \alpha_k \in \Gamma_k \cup \{\alpha_k\} := \Gamma_{k+1} \subseteq \Delta$, a contradiction. So $\Gamma_k \cup \{\alpha_k\} \vdash \Box$. Then by dilution, $\Delta \cup \{\alpha_k\} \vdash \Box$. $\Box$

Proposition 0.4. If Γ is maximally consistent, it has a model.

Proof. Let Γ be maximally consistent. Define v_0, the canonical valuation, as follows ical valuation, as follows: $v_0 : PV \to \{T, F\}$

For any propositional variable p, $v_0(p) := T$ if and only if $p \in \Gamma$.

To show $v_0 \models \Gamma$

We show that Claim
$v_0(\alpha) = T$ if and only if $\alpha \in \Gamma$, for *any wff* α. ...(*)

The proof is by induction on the number n of occurrences of connectives in the wff α.

Basis. n = 0. α is a propositional variable: By definition of v_0.

Induction Hypothesis. Let the property (*) of v hold for any β having less than n occurrences of connectives.

Induction Step. Let α have exactly n occurrences of connectives.

def of val.

(i) $\alpha := \neg\beta$, for some wff β: $v_0(\alpha) = v_0(\neg\beta) := \neg v_0(\beta) = T$ if and only if $v_0(\beta) = F$, i.e. if and only if $\beta \notin \Gamma$, by induction hypothesis applied on β. But $\beta \notin \Gamma$ if and only if $\alpha := \neg\beta \in \Gamma$, Γ being maximally consistent (Proposition 0.1 above).

(ii) $\alpha := \beta \to \gamma$, for some wffs β, γ: As done for (i), use the definition of extension of valuations to the set $\mathcal{F}$ of all wffs, and earlier proposition. □

Note. The canonical valuation v_0, as defined in Proposition 0.4, thus satisfies every maximally consistent set Γ. But it may not satisfy a set Γ that is not maximally consistent.

$\Gamma := \{p, \neg q\} \to$ not max. cons.

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$v_0(p) = T$, $v_0(q) = F$ $\quad v_0 \models \Gamma$?

$\Gamma := \{p, p \to q\}$ $\quad v_0(p) = T$, $v_0(q) = F$

For instance, take $\Gamma := \{\neg q \to p\}$, p, q being any propositional variables. Γ is consistent as it has a model. However, Γ is not maximally consistent, as, for example, it does not contain either of p or $\neg p$. v_0 does not satisfy Γ.

Lecture 15

Saturday, 26 September 2020 5:48 PM

Lec15

Completeness theorem

If $\Gamma \models \alpha$ Then $\Gamma \vdash \alpha$

The propositions of the previous lecture lead us to the converse of the earlier proposition viz.

Theorem 0.1. If Γ is consistent, it has a model.

Theorem 0.1 is also sometimes referred to as the completeness theorem. This is because, it is, in fact, *equivalent* to the completeness theorem as we stated it. We prove one side of the equivalence below; the other is left as an exercise.

Given Completeness Thm:
For any Γ, α, if $\Gamma \models \alpha$ Then $\Gamma \vdash \alpha$.

To show: for any consistent Γ, Γ is satisfiable.

$\rightarrow$ Γ contradictory $\Rightarrow$ $\Gamma \models \alpha$, for every wff α!

$\Downarrow$ (Completeness)

$\Gamma \vdash \alpha$, for all α

Theorem 0.2. (Completeness Theorem) If $\Gamma \models \alpha$ then $\Gamma \vdash \alpha$.

Proof. Let $\Gamma \models \alpha$, and suppose that $\Gamma \nvdash \alpha$. By earlier proposition, $\Gamma \cup \{\neg\alpha\} \nvdash \Box$, i.e. $\Gamma \cup \{\neg\alpha\}$ is consistent.

By Theorem 0.1 above, $\Gamma \cup \{\neg\alpha\}$ has a model, say v.

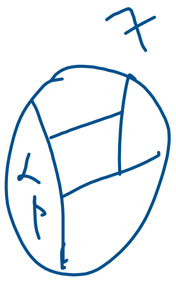
So v is a model for Γ, but $v(\neg\alpha) = T$, i.e. $v(\alpha) = F$.

Hence $\Gamma \not\models \alpha$, which is a contradiction to our assumption. $\Box$

Corollary 0.3.

(a) If $\models \alpha$ then $\vdash \alpha$, for any wff α.

(b) The quotient sets $\mathcal{F}/\sim$ and $\mathcal{F}/\equiv$ are *equal*.



Proof. (b) The soundness and completeness theorems together imply that the set of all theorems of PL is exactly the set of all tautologies. $\Box$

$\sim \Downarrow \quad \equiv$

$\alpha \sim \beta$ iff $\vdash \alpha \leftrightarrow \beta$.

$\alpha \equiv \beta$ iff $\models \alpha \leftrightarrow \beta$

Exercise 0.1.

1. For each of the following, decide whether Γ is consistent or inconsistent. To show that Γ is consistent, find a valuation that satisfies Γ. To show that Γ is inconsistent, either show that Γ is contradictory or find some wff α such that both $\Gamma \vdash \alpha$ and $\Gamma \vdash \neg\alpha$.

   (i) $\Gamma := \{p_1, p_2, \neg(p_2 \vee p_3)\}$.

   (ii) $\Gamma := \{p_1 \vee \neg p_2, \neg(p_1 \vee p_2)\}$.

   (iii) $\Gamma := \{p_1 \vee \neg p_2, \neg(p_1 \vee p_2), p_1 \to p_2\}$.

   (iv) $\Gamma := \{\neg(p_1 \vee p_2), \neg p_2 \vee p_3, (p_2 \vee p_3) \to p_4\}$.

2. Let Γ be a consistent set of wffs. Show that the following are equivalent:

   (i) For every formula α, $\Gamma \vdash \alpha$ or $\Gamma \vdash \neg\alpha$.

   (ii) There is exactly one valuation that satisfies Γ.

(i) $\Rightarrow$ (ii)

$\Leftarrow$

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