$G = (V, E)$

$E \subseteq V \times V$

$V \rightarrow$ set of vertices

→ ~~P → set of wff's.~~

~~→ ∄ ∃ v ⊨ P~~

~~P → set of wffs.~~

Thm
G is n-colorable iff each finite restriction of G is n-colorable.

proof

($\Leftarrow$)

denote our propositional vbles as
$PV = \{p_{xi} : x \in V, i \in [n]\}$

Part I

→ $\Gamma = \{\alpha_v : v \in V\} \cup \{\beta_{vu} : (v,u) \in E\}$

$\alpha_v = \left(\bigvee_{i=1}^{n} p_{vi}\right) \wedge \left(\bigwedge_{i=1}^{n}\left(\bigwedge_{j=i+1}^{n}\left(\sim(p_{vi} \wedge p_{vj})\right)\right)\right)$

$\beta_{vu} = \bigwedge_{i=1}^{n} \sim(p_{vi} \wedge p_{ui})$

Suppose, every finite restriction of $G = (V,E)$ is n-colorable.

So, for (V', E') s.t. $V' \subseteq V$ finite & $E' \subseteq E$ finite,

(V', E') is n-colorable.

$\therefore \exists f : V' \to [n]$ s.t. $f(u) \neq f(v)$ if $(u,v) \in E'$

Thus, define $v : PV \to \{T, F\}$ as

$v(p_{xi}) = T$ iff $f(x) = i$

& for rest of the propositional variable set $v(p_{x'i}) = F$ if $x \notin V'$ $\forall\ 1 \leq i \leq n$

Scanned with CamScanner

Now, take any finite subset Δ of Γ, $\Delta \subseteq \Gamma$ finite.

Thus, notice, ~~...~~ the set

$$V' = \{v : \alpha_v \in \Delta, v \in V\}$$

$\therefore$ as Δ finite,

V' finite.

Thus, consider the edge set

$$E' = \{(v,u) : B_{vu} \in \Delta \ \& \ v,u \in V'\}$$

E' is again finite.

Hence, ~~E', V'~~ (V', E') is a finite restriction of (V, E).

Therefore, it is n-colorable.

Hence, take v as shown before,

Thus, notice $\rightarrow \Delta'$

$$v \models \{\{\alpha_v : v \in V'\} \cup \{B_{vu} : (v,u) \in E'\}$$

but does E satisfy Δ.

Yes, b/c

$$\frac{\Delta}{\Delta'} = \{B_{vu} : v \notin V' \text{ or } u \notin V' \ \& \ B_{vu} \in \Delta\}$$

Thus, notice how we made v for vertices not in V'.

As false, $\therefore$ B_{vu} is true.

Hence,

$v \models \Delta$

$\therefore$ Every finite subset of Γ is satisfiable.

$\therefore$ Γ is satisfiable

($\Rightarrow$) Trivial