Lecture 23 Module 1

Friday, 30 October 2020 5:05 PM

Lec23Mo...

Introduction to First order logic (FOL): the Syntax

Let us begin with the following argument.

All horses are animals.
Hence all heads of horses are heads of animals.

Can we represent this in classical propositional logic, PL? Try! Remember that it needs to be given the form of a proper *argument* of PL.

It will not work. PL has very limited expressibility: one cannot express natural language phrases within sentences such as 'for all' or 'there is' (called 'quantification'), or express variables, constants, relations, functions within sentences.

So we now proceed to study a logical system that has a language far richer than PL, and is able to express these kinds of sentences: *First Order Logic*, or, in brief, FOL.

Note, however, that we would not like to lose the nice features of PL. In particular, FOL would yield PL-theorems, and more.

1 The syntax of FOL

We define an FOL-language, by first laying down its alphabet. Next, just as for natural languages, one builds 'terms' (words) using the alphabet, which in turn are used to define the wffs (sentences) of FOL. Both terms and wffs are defined recursively.

1.1 The alphabet

(a) A countably infinite set $\mathcal{V}$ of *variables* $x_1, x_2, \ldots$

(b) A countable (non-empty) set of *predicate symbols* $p_1^n, p_2^n, \ldots$, for any $n (\in \mathbf{N})$, where n denotes the *arity* of the predicate symbol p_i^n

(c) A countable (possibly empty) set of *function symbols* $f_1^n, f_2^n, \ldots$, for any $n (\in \mathbf{N})$, where n denotes the arity of the function symbol f_i^n

(d) A countable (possibly empty) set of *constant symbols* $c_1, c_2, \ldots$

(e) *Logical Connectives* $\neg$, $\rightarrow$

(f) *Universal quantifier* $\forall$ ('for all') $\forall x$ —

(g) Parantheses (,)

The interpretation of the symbols in the alphabet is done in a non-empty domain of discourse, the latter considered to be a set. Predicate symbols are intended to be interpreted as relations over the domain, function symbols as functions on the domain, and constant symbols as distinguished elements of the domain. Let us give a couple of examples of FOL-languages, indicating the constituents of the alphabet apart from the mandatory logical connectives, quantifier and parantheses.

For simplicity, we may use the symbols $p, q, r, \ldots$ to denote predicate symbols, along with a mention of the arities. Similarly, we may use $f, g, h, \ldots$ for function symbols (mentioning the arities), or $x, y, z, \ldots$ for variables.

Example 1.1.

(a) The alphabet may simply contain variables and a unary predicate symbol p.

(b) Suppose the interpretation of an FOL-language is to be done in the domain of natural numbers. Then, instead of the abstract symbols mentioned in the alphabet, we may use familiar operator or relation symbols. So, for instance, the alphabet may contain
variables,
binary predicate symbols $=, <$,
binary function symbols $+, \times$, and
the constant symbol 9.

Note that countability of the sets in the alphabet is not a necessary assumption – an FOL-language may also involve uncountable sets in its alphabet.

1.2 Terms

As mentioned earlier, terms are defined recursively.

Definition 1.1. The *set $\mathcal{T}$ of terms* is the smallest set of strings (finite sequences of symbols) over the sets of variables, constants and function symbols, such that

(a) any variable is a term (i.e. a member of $\mathcal{T}$),

(b) any constant is a term,

(c) if f^n is a function symbol of arity n and $t_1, t_2, ..., t_n$ are terms, $f^n t_1 t_2 ... t_n$ is a term.

A term is said to be *closed*, if it involves constants only.

Example 1.2. Consider the FOL-language of Example 1.1 (b). Some examples of terms, apart from the variables and constant 9 would be:

$f^2_1 xz$ $x + z$, $(x + y) \times 9$, $(x+z) \times 9$ $+ : f^2_1$

$y \times ((y \times y) + y))$, or $\times : f^2_2$

$9 + 9$, $f^2_2 f^2_1 xz9$

where x, y, z are variables.

The last one is a closed term, and so is 9.

1.3 Well-formed formulae (wffs)

Terms, by themselves, are not sentences. They are used to build wffs of FOL, the simplest of which involve predicate symbols and are called *atomic*. The definition of wffs, like that of terms, is recursive.

Definition 1.2. The *set* $\mathcal{F}$ *of wffs* is the smallest set of strings over the alphabet given above, such that

(a) if p^n is a predicate symbol of arity n and $t_1, t_2, ..., t_n$ are terms, $p^n t_1 t_2 ... t_n$ is a wff, called an *atomic wff*,

$\forall\alpha$

$\forall x\alpha$

(b) if α is a wff, so is $\neg\alpha$,

(c) if α, β are wffs, $\alpha \to \beta$ is also a wff,

(d) if x is a variable and α is a wff, $\forall x\alpha$ is a wff.

As in PL, we have the usual abbreviations defining the connectives of conjunction ('and'), disjunction ('or') and bi-implication ('if and only if'). The new entrant in FOL is the *existential quantifier*, 'there exists', given in (d) below.

Abbreviations

(a) $\alpha \wedge \beta := \neg(\alpha \to \neg\beta)$.

(b) $\alpha \vee \beta := \neg\alpha \to \beta$.

(c) $\alpha \leftrightarrow \beta := (\alpha \to \beta) \wedge (\beta \to \alpha)$.

(d) $\exists x\alpha := \neg\forall x(\neg\alpha)$.

$p^n t_1 \ldots t_n$

Example 1.3. Consider the FOL-language of Example 1.1 (a). Wffs of the form px, x any variable, are the only atomic wffs. Some other wffs:

$px \to px$,
$\forall x px$,
$\exists x px \wedge \neg(\forall x px)$.

$\forall y px$

atomic

In Example 1.1 (b), some atomic wffs for variables x, y, z:
$(x + z) = (y \times z)$, $((x + y) \times 9) < 9$. Others:
$\forall y((y \times ((y \times y) + y)) < y)$, or
$(9 = 9) \leftrightarrow \exists z(z = 9)$.

$(=) p_1^2 \; f_1^2 xz \; f_2^2 yz$

To simplify notations, we may drop some parantheses. Note that one usually follows left association, e.g. $x + z = y \times z$

Proposition 1.1. The sets $\mathcal{T}, \mathcal{F}$ of all terms and wffs respectively of FOL are countable.

The proof is exactly in the lines of the one establishing that the set of all PL-wffs is countable: show that the terms (and wffs) can be generated from a finite alphabet.

Variables may occur in a wff in different ways, in the context of the positions of quantifiers in the wff. We look at this in the next lecture.

Lecture 23 Module 2

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Semantics of FOL

Let us now point out different ways in which variables may occur in a wff, in the context of the positions of quantifiers in the wff. This will prove to be critical for the FOL-semantics that we shall define in this lecture.

Definition 0.1.

$\forall x (\;)\ \exists x (\;)$

(i) The *scope* of the quantifier $\forall$ in a wff $\forall x\alpha$ is defined to be α. The scope of $\exists$ in $\exists x\alpha$ is also defined to be α.

(ii) An *occurrence* of a variable x is *bound* in a wff α if, it is immediately preceded by a quantifier, or it lies within the scope of a quantifier that is immediately followed by x.
For instance, both the occurrences of x in $\forall xpx$ are bound.

$\forall x p y$ scope

(iii) Otherwise, the *occurrence is free* in α.
Example: consider the wff $\forall xpx \to \exists yqxy$, where p is a unary predicate symbol in the language and q, binary. The first two occurrences of x in the wff are bound, while the third is free.

(iv) A *variable is free* in α, if it has at least one occurrence free in α. It is bound in α, if it has at least one occurrence bound in α.
If α has free variables from the list $x_1, \ldots, x_n$, we write $\alpha(x_1, \ldots, x_n)$ in place of α.
Clearly, a variable may be both free and bound in a

Clearly, a variable may be both free and bound in a wff.

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"α with t for x"

(v) $\alpha(t/x)$ denotes the wff obtained from α when *all free occurrences* of x in α are replaced by the term t.
Example: consider the term $t := fyfcy$, where f is a binary function symbol and c a constant symbol in the language. Take the wff $\alpha := \forall xpx \to \exists yqxy$ as in (ii).
Then $\alpha(t/x) := \forall xpx \to \exists yqfyfcyy$.

(vi) The *term t is free for the variable x in α*, if no occurrence of a variable of t becomes bound when any free occurrence of x in α is replaced by t.
Example: Take t and α as in (v) above. t is not free for x in α. However, if we take $t := fxx$ instead, $\alpha(t/x) := \forall xpx \to \exists yqfxxy$, and this t is free for x in α.

(vii) A wff is said to be *closed*, if it has no free variables. — sentences
For instance, $(9 = 9) \leftrightarrow \exists z(z = 9)$ is closed.

We observe the following, using vacuous arguments.

(a) A variable is free for itself in any wff.

(b) A closed term is always free for any variable in any wff.

1 The semantics: interpretations and valuations

In the previous lecture, we mentioned, informally, the intended interpretations for the symbols in an FOL-language. Let us now present the FOL-semantics formally.

Definition 1.1. An *interpretation* $\mathcal{I}$ of FOL consists of a non-empty domain D of discourse, and assignments mapping
(i) each constant symbol c in the alphabet to a distinguished element $\mathcal{I}(c)$ in D,
(ii) each function symbol f_i^n of arity n to an n-ary function $\mathcal{I}(f_i^n) : D^n \to D$, and
(iii) each predicate symbol p_i^n to an n-ary relation $\mathcal{I}(p_i^n) \subseteq D^n$.

$D = \mathbb{N}$

p_2^2 $\mathcal{I}(p_2^2) \subseteq \mathbb{N} \times \mathbb{N}$

Let us look at a couple of examples of interpretations.

Example 1.1.

$\mathcal{I}(p) \subseteq \mathbb{N}$

(a) Consider the FOL-language in previous lecture. An interpretation $\mathcal{I}$ with domain $D := \mathcal{N}$, the set of natural numbers, could assign to the unary predicate symbol p, the unary relation E on $\mathcal{N}$ (a subset of $\mathcal{N}$) such that $E := \{n \in \mathcal{N} : n \text{ is even}\}$.

(b) For the language of the next example, we could take an interpretation $\mathcal{I}$ with the same domain, $\mathcal{N}$, and make assignments to the symbols of the alphabet as indicated by the notations chosen for them: the natural number 9 to the lone constant symbol, the binary relations $<$ and $=$ on $\mathcal{N}$ to the two binary predicate symbols, and the binary operations $+, \times$ on $\mathcal{N}$ to the two binary function symbols.

We next specify assignments for terms of the language, in a given interpretation $\mathcal{I}$. Recall that $\mathcal{T}$ denotes the set of all terms.

p_1^2, p_2^2 $\mathcal{I}_1$ $\mathcal{I}_2$

$+ \quad \times$

$\mathcal{I}_1(p_1^2) = <$ $\mathcal{I}_2(p_1^2) = \geq$

Definition 1.2. A map $v : \mathcal{T} \to D$ is said to be a *valuation*, when
(i) $v(c) := \mathcal{I}(c)$ for any constant symbol c in $\mathcal{T}$, and
(ii) for a term $f_i^n t_1 \ldots t_n$ in $\mathcal{T}$,
$v(f_i^n t_1 \ldots t_n) := \mathcal{I}(f_i^n)(v(t_1), \ldots, v(t_n))$,
where $v(t_i)$, $i = 1, \ldots, n$, are known.

Example 1.2. Consider the interpretation $\mathcal{I}$ in Example 1.1(b). An example of a valuation v in $\mathcal{I}$: define $v(x) :=$ 5, $v(y) := 1000$, $v(z) := 2$.
Then using Definition 1.2(ii) for the term $(x \overset{f_1^2}{+} y) \overset{f_2^2}{\times} 9$, we have
$v((x + y) \times 9) := (v(x) + v(y)) \times 9 = (5 + 1000) \times 9.$ $v(f_2^2 f_1^2 x y 9)$

Note that we may have different valuations in the same interpretation, but Definition 1.2 forces all of them to assign the same value to all constant symbols, and as a result, to all the closed terms (see Exercise 1.1(b) below). So they can only differ on variables of the language.

Exercise 1.1. Let v, v' be valuations in an interpretation $\mathcal{I}$.

(a) If $v(x) = v'(x)$, for all variables x occurring in a term t then $v(t) = v'(t)$.

(b) For any closed term t, $v(t) = v'(t)$.

Observe that (b) in the exercise is a direct consequence of (a). (a) is proved by induction on the number of function symbols occurring in the term t.

We proceed to define the notions of truth and validity in the next lecture.

t: n: no of fn. symbols in t.

Base $n = 0$ — $t = y$

— $t = c$

Ind hyp- t has n fn. symbols.
v, v': $v(x) = v'(x)$ for all var. x in t Then $v(t) = v'(t)$.

Ind. Step- t has $n+1$ fn. symbols.
v, v': $v(x) = v'(x)$ for all var. x occuring in t. To show $v(t) = v'(t)$.

$t = f^m t_1 \ldots t_m$ $\quad I(f^m) = F$

By assumption $v(x) = v'(x)$ for all var. x in all the t_i's, $i = 1 \ldots m$.
By ind hyp, $v(t_i) = v'(t_i)$,
$\therefore v(t) = F(v(t_1), \ldots, v(t_m))$
$= F(v'(t_1), \ldots, v'(t_m)) = v'(t)$

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Satisfaction, Truth and Validity

given $\underline{I}$, val. $v : V \to D$

We now proceed to the definitions of (i) *satisfaction* of a wff by a valuation in an FOL-interpretation, (ii) *truth* and *falsity* of a wff in an interpretation, and finally, (iii) *validity* of a wff. Before we get to these, we need to look at some special kinds of valuations.

Definition 0.1. Valuations v, v' in an interpretation $\mathcal{I}$ are said to be x-*equivalent* to each other for a variable x, provided $v(y) = v'(y)$, for all variables $y \neq x$.

It is immediate that every valuation is x-equivalent to itself, for every variable x.

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1 Satisfaction in an interpretation $\mathcal{I}$

Let $\mathcal{I}$ be an FOL-interpretation as given previous lecture, v a valuation in $\mathcal{I}$, and α a wff. "v satisfies α in $\mathcal{I}$" is written in notation as $\mathcal{I}, v \models \alpha$. The definition of satisfaction is recursive, considering the number of connectives and quantifiers in the wff α.

Definition 1.1.

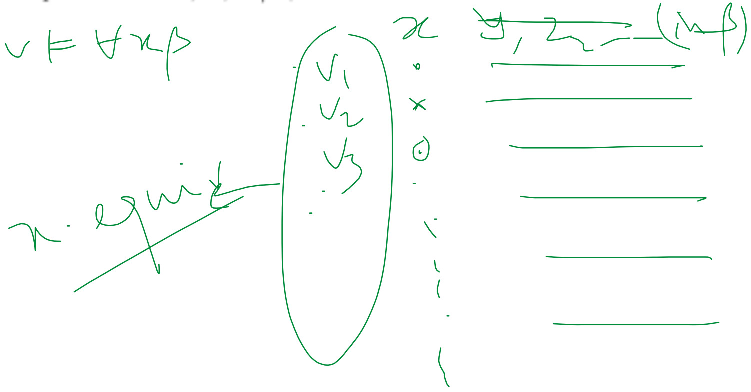
(i) α is atomic, i.e. $\alpha := p^n t_1 \ldots t_n$:
$\mathcal{I}, v \models \alpha$, if and only if $(v(t_1), \ldots, v(t_n)) \in \mathcal{I}(p^n)$. $\subseteq D^n$

(ii) $\alpha := \neg\beta$:
$\mathcal{I}, v \models \alpha$, if and only if $\mathcal{I}, v \not\models \beta$.

(iii) $\alpha := \beta \to \gamma$:
$\mathcal{I}, v \models \alpha$, if and only if whenever $\mathcal{I}, v \models \beta$, we have $\mathcal{I}, v \models \gamma$.

(iv) $\alpha := \forall x\beta$:
$\mathcal{I}, v \models \alpha$, if and only if for each valuation v' that is x-equivalent to v, $\mathcal{I}, v' \models \beta$.

$\mathcal{I}, v \models \forall x\beta$



$\exists x\beta := \neg\forall x(\neg\beta)$

Remark.

(a) If $\alpha := \exists x\beta$, $\mathcal{I}, v \models \alpha$, if and only if *there exists* a valuation v' that is x-equivalent to v such that $\mathcal{I}, v' \models \beta$. This is easily proved using the definition.

(b) (iv) in Definition 1.1 essentially means that the wff

(b) (iv) in Definition 1.1 essentially means that the wff β is satisfied by every possible value that the variable x may take in the domain (values to all other variables in the wff being those assigned by v) – in other words, by *every* element of the domain. By the same token, for satisfaction of $\exists x\beta$, we need *some* element of the domain to satisfy β.

Example 1.1. Consider the earlier example of interpretation $\mathcal{I}$, the wff $\forall y(z < y)$, and the valuation v as before: $v(x) := 5$, $v(y) := 1000$, $v(z) := 2$. Then $\mathcal{I}, v \models \forall y(z < y)$ means that for each valuation v' that is y-equivalent to v, $\mathcal{I}, v' \models z < y$, i.e. $v'(z) < v'(y)$. As v' and v are y-equivalent, the requirement boils down to $2 = v(z) < v'(y)$, for all possible v' y-equivalent to v. Thus, to get $\mathcal{I}, v \models \forall y(z < y)$, we effectively need $2 < n$, *for all* $n \in \mathcal{N}$ – which is not the case, so $\mathcal{I}, v \not\models \forall y(z < y)$.

However, suppose we consider the wff $\exists y(z < y)$ instead.
Then we get $\mathcal{I}, v \models \exists y(z < y)$:
take, for instance, v' y-equivalent to v such that $v'(y) = 3$ – then $\mathcal{I}, v' \models z < y$, as $v'(z) = v(z) = 2 < 3 = v'(y)$.

Consider now the wff $\exists y(y < z)$. Then $\mathcal{I}, v \models \exists y(y < z)$, but if we take $v(z) := 0$ instead, $\mathcal{I}, v \not\models \exists y(y < z)$.

Observe that if z were replaced by any closed term in the above example, the valuation v would not play any role in determining whether the wffs are satisfied or not (by v). (Test this by taking the wff $\forall y(9 < y)$).
We shall see that if a wff is *closed*, it is either satisfied by all valuations, or by none.
The last part of the example suggests that, the values assigned to the *free* variables by a valuation, determine whether the wff in question is satisfied by it or not. This shall be established formally as well.

2 Truth and validity; semantic consequence '$\models$'

Definition 2.1. Let $\mathcal{I}$ be any interpretation, and $\Gamma \cup \{\alpha\}$ any set of wffs.

$\mathcal{I}, v \models \alpha$

(i) α is *true* in $\mathcal{I}$, written $\mathcal{I} \models \alpha$, if and only if $\mathcal{I}, v \models \alpha$, for all valuations v in $\mathcal{I}$.

(ii) α is *false* in $\mathcal{I}$, if and only if there is no valuation v such that $\mathcal{I}, v \models \alpha$. In other words, α is false in $\mathcal{I}$, if and only if $\mathcal{I} \models \neg\alpha$.

(iii) If every wff of Γ is true in $\mathcal{I}$, $\mathcal{I}$ is called a *model* for Γ.

$\mathcal{I}$

(iv) α is a *semantic consequence* of Γ, written $\Gamma \models \alpha$, if and only if every model for Γ is a model for α. If Γ is empty, α is said to be *valid*, written $\models \alpha$.

Let us give some simple examples to illustrate these notions.

Example 2.1.

(a) Continuing Example above, the wff $\exists y(y < z)$, as we saw, is neither true nor false in $\mathcal{I}$.
However, the wff $\exists y(z = y)$ is true in $\mathcal{I}$, while $\forall y(y < z)$, or $\forall y(y < 9)$, is false in $\mathcal{I}$.

(b) Consider $\Gamma := \{\alpha\}$. Then $\Gamma \models \forall x\alpha$: let $\mathcal{I} \models \alpha$. As every v satisfies α in $\mathcal{I}$, we get $\mathcal{I} \models \forall x\alpha$.
However, $\not\models \alpha \rightarrow \forall x\alpha$. Consider the simple language and interpretation $\mathcal{I}$ in earlier Lecture, and take $\alpha := px$. Let v on $\mathcal{I}$ be defined such that $v(x) := 2$. Then $\mathcal{I}, v \models px$, but $\mathcal{I}, v \not\models \forall xpx$ (which is a closed wff), as

$\mathcal{I}, v \models px$, but $\mathcal{I}, v \not\models \forall x px$ (which is a closed wff), as, clearly, there exists a valuation v' (x-equivalent to v) on $\mathcal{I}$ such that $\mathcal{I}, v' \not\models px$.

$\Gamma \models \beta$ $\qquad$ $\{\alpha\} \models \forall x \alpha$

Here are a few easy exercises, based directly on the definitions.

$\forall x_1 \forall x_2 \, p x_1, x_2$: closure

Exercise 2.1.

(i) For any interpretation $\mathcal{I}$ and wff α, $\mathcal{I} \models \alpha$, if and only if $\mathcal{I} \models \forall x \alpha$.
In fact, if α^* denotes the *closure* of $\alpha(x_1, \ldots, x_n)$, i.e. $\alpha^* := \forall x_1 \ldots \forall x_n \alpha$ (fix any order for the quantification), we have

$p x_1, x_2$

$$\mathcal{I} \models \alpha, \text{ if and only if } \mathcal{I} \models \alpha^*.$$

$px \to (qx \to px)$

(ii) All PL axioms are valid in FOL, i.e. e.g. $\models \alpha \to (\beta \to \alpha)$, where α, β are FOL-wffs. Similarly for axioms A2, A3 of PL.

(iii) (a) $\models \forall x(\alpha \to \beta) \to (\forall x \alpha \to \forall x \beta)$.

(b) $\not\models (\forall x \alpha \to \forall x \beta) \to \forall x(\alpha \to \beta)$.
To show this, find a suitable counterexample. In other words, choose wffs α, β, interpretation $\mathcal{I}$ and valuation v in $\mathcal{I}$ such that $\mathcal{I}, v \models \forall x \alpha \to \forall x \beta$, but $\mathcal{I}, v \not\models \forall x(\alpha \to \beta)$.

(iv) $\models \forall x(\alpha \wedge \beta) \leftrightarrow (\forall x \alpha \wedge \forall x \beta)$.

$\models \exists x(\alpha \vee \beta) \leftrightarrow (\exists x \alpha \vee \exists x \beta)$

(v) (a) $\models \exists x(\alpha \wedge \beta) \to (\exists x \alpha \wedge \exists x \beta)$.

(b) $\not\models (\exists x \alpha \wedge \exists x \beta) \to \exists x(\alpha \wedge \beta)$.
(Again, find a suitable counterexample.)

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Some fundamental results on the semantics

We state below a few fundamental propositions related to interpretations and valuations.

Consider any interpretation $\mathcal{I}$, and a FOL-wff α.

Proposition 0.1. Let t be a term with at least one occurrence of the variable x, and s another term. Further, let t' be the term obtained from t by replacing every occurrence of x by the term s.
Let u, v be x-equivalent valuations on $\mathcal{I}$ such that $v(x) = u(s)$.
Then $v(t) = u(t')$.

$v(t)$ $u(t')$
$x \ldots x \ldots$ $s \ldots s$

Example 0.1. Let us return to earlier example.
Let $t := (x + y) + (9 \times x)$, $s := z \times z$. Then $t' := ((z \times z) + y) + (9 \times (z \times z))$. If valuations u, v are x-equivalent and such that $v(x) = u(s)$, we see that
$v(t) = (v(x)+v(y))+(9\times v(x)) = (u(s)+u(y))+(9\times u(s)) = u(t')$.

Proof. The proof is by induction on the structure of the term t, i.e. on the number n of function symbols in t. For the basis case ($n = 0$), it is clear that $t := x,\ t' := s$, and we get the result trivially.
For the induction step, assume that $t := f^m t_1 t_2 ... t_m$, and let u, v be x-equivalent valuations on $\mathcal{I}$ such that $v(x) = u(s)$. For $i = 1, ..., m$, let t_i' denote the term obtained from t_i by replacing every occurrence of x by the term s. Note that x occurs in at least one of the t_i's. Then $t' := f^m t_1' t_2' ... t_m'$.
By induction hypothesis applied on the t_i's that contain x, we have $v(t_i) = u(t_i')$. For the t_i's that do not contain x, we use earlier Exercise, to get $v(t_i) = u(t_i')$. The rest follows by definition of valuations on terms with function symbols. □

Let us 'extend' Proposition 0.1 to wffs.

Proposition 0.2. Suppose α has at least one free occurrence of the variable x, and let t be a term free for x in α. Let u, v be x-equivalent valuations on $\mathcal{I}$ such that $v(x) = u(t)$.
Then $\mathcal{I}, v \models \alpha(x)$, if and only if $\mathcal{I}, u \models \alpha(t/x)$.

Example 0.2. Consider again earlier example.
Take the atomic wff $\alpha := ((x + y) \times x) < (x \times y)$, and any term t, say, $t := 2 \times z$. Then $\alpha(t/x) := ((t + y) \times t) < (t \times y)$. Now assume that u, v are x-equivalent such that $v(x) = u(t)$. We can see that $\mathcal{I}, v \models \alpha$, if and only if $((v(x)+v(y)) \times v(x)) < (v(x) \times v(y))$, which is if and only if $((u(t)+u(y)) \times u(t)) < (u(t) \times u(y))$, and that is just saying $\mathcal{I}, u \models \alpha(t/x)$. One can also test the proposition with the quantified wff $\forall y((x \times y) < (x + y))$, with $t := 2 \times y$, say.
On the other hand, observe that if $\alpha(x) := \forall y(y > x)$ and $t := y$, the proposition would not apply to the wffs $\alpha(x)$ and $\alpha(t/x) := \forall y(y > y)$ as t is not free for x in $\alpha(x)$.

Proof. The proof is by induction on the structure of the wff α, i.e. on the number n of connectives and quantifiers in α. Basis case: $n = 0$, and α is atomic. Let $\alpha := p^m t_1 t_2 ... t_m$, and u, v be x-equivalent valuations on $\mathcal{I}$ such that $v(x) = u(t)$.
As in Proposition 0.1, let t_i' denote the term obtained from t_i by replacing every occurrence of x by the term t. Again, note that x occurs in at least one of the t_i's, and that every occurrence of x is free. Then $\alpha(t/x) := p^m t_1' t_2' ... t_m'$.
By Proposition 0.1 applied on the t_i's, we have $v(t_i) = u(t_i')$. The result in this case then follows from the definition of satisfaction of atomic wffs.

In the induction step, we look at the quantifier case – when $\alpha := \forall y \beta$. The Boolean cases – when α is $\neg\beta$, or $\beta \to \gamma$ for some wffs β, γ, are straightforward (*exercise!*). Observe that (i) $y \neq x$, and (ii) y does not occur in t, because of the assumptions.
Let u, v be x-equivalent valuations on $\mathcal{I}$ such that $v(x) = u(t)$. We prove one direction of the result (the other direction is left as an exercise):
if $\mathcal{I}, v \models \forall y \beta$, then $\mathcal{I}, u \models \forall y \beta(t/x)$.
So let $\mathcal{I}, v \models \forall y \beta$, and consider a valuation u' y-equivalent to u. We define a valuation v' y-equivalent to v, and x-equivalent to u': $v'(x) := v(x) = u(t)$, $v'(y) := u'(y)$. For all other variables z, $v'(z) := v(z)$. Using observations (i) and (ii) above and earlier Exercise, we find that the assumptions of the proposition are satisfied for β, v', u'. So we can use induction hypothesis on β. The definition of satisfaction for universally quantified wffs gives the result. $\square$

The following proposition tells us that if valuations match on all free variables in a wff, then they match on satisfaction of the wff too.

Proposition 0.3. Let u, v be valuations on $\mathcal{I}$ such that

$u(x) = v(x)$, for each free variable x in α. Then $\mathcal{I}, u \models \alpha$, if and only if $\mathcal{I}, v \models \alpha$.

Proof. Exercise. In the quantifier case, when $\alpha := \forall y \beta$, observe that the variable y is not free in α, but may well be free in β. $\square$

Example 0.3. If α is atomic, the claim is obvious: consider $\alpha := ((x + y) \times x) < (x \times y)$ in Example 0.2. All variables are free in an atomic wff. So if valuations u, v are such that $u(x) = v(x)$ and $u(y) = v(y)$, we get $\mathcal{I}, u \models \alpha$, if and only if $\mathcal{I}, v \models \alpha$.
Now consider the quantified wff in the example, viz.
$\alpha := \forall y((x \times y) < (x + y))$ – x is free here. If valuations u, v are such that $u(x) = v(x)$, then it is clear that $\mathcal{I}, u' \models (x \times y) < (x+y)$, if and only if $\mathcal{I}, v' \models (x \times y) < (x+y)$, for any u', v' y-equivalent to u, v respectively such that $u'(y) = v'(y)$. This is enough to show that $\mathcal{I}, u \models \alpha$, if and only if $\mathcal{I}, v \models \alpha$.

v, u [:| $v(x) = u(x)$ for all || free x in α

Proposition 0.3 leads to the result that closed wffs are either true or false in any interpretation, as we had pointed out earlier.

Corollary 0.4. *If α is closed, either $\mathcal{I} \models \alpha$, or $\mathcal{I} \models \neg\alpha$.*

So a wff like $\forall x p x$ will be satisfied by either all valuations or by none, in any interpretation.

Using Propositions 0.2 and 0.3, we can prove the following (*exercise!*). α

Proposition 0.5 $\models \forall x \alpha \Rightarrow \alpha$ $\forall y\, p x$

Proposition 6.3.

(i) $\models \forall x\alpha \to \alpha(t/x)$, if t is free for x in α.

(ii) $\models \alpha \to \forall x\alpha$, if x is not free in α.

$I, v \models \alpha \iff I, u \models \alpha$

$\models \alpha \leftrightarrow \forall x \alpha$

x: not free in α

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