Formal Proof

$\vdash (A \Rightarrow A)$

$$\frac{\dfrac{}{A \vdash A} \, {}^{hyp}}{\vdash (A \Rightarrow A)} \Rightarrow goal$$

$\vdash (B \Rightarrow C) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C))$

$$\frac{\dfrac{\dfrac{}{B \Rightarrow C, A \vdash A} \, {}^{hyp} \quad \dfrac{\dfrac{}{A,B \vdash B} \, {}^{hyp} \quad \dfrac{}{A,B,C \vdash C} \, {}^{hyp}}{B \Rightarrow C, A, B \vdash C} \Rightarrow hyp}{\dfrac{B \Rightarrow C, A \Rightarrow B, A \vdash C}{\dfrac{B \Rightarrow C, A \Rightarrow B \vdash A \Rightarrow C}{\dfrac{B \Rightarrow C \vdash (A \Rightarrow B) \Rightarrow (A \Rightarrow C)}{\vdash (B \Rightarrow C) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C))} \Rightarrow goal} \Rightarrow goal} \Rightarrow goal} \Rightarrow hyp}{} $$

Lambda Calculus

$((\lambda f. \lambda g. \lambda x. f (g\ x))\ (\lambda x.x))\ (\lambda x.x) \quad \therefore \beta$

$= (\lambda g. \lambda x. f (g\ x) [f := \lambda y.y])\ (\lambda x.x)$

$\quad (\lambda g. \lambda x. \lambda y. y (g\ x))\ (\lambda x.x) \quad \therefore \beta$

$= (\lambda g. \lambda x. g\ x)\ (\lambda x.x) \quad \therefore \beta$

$= (\lambda g. \lambda x. g\ x [g := \lambda z.z])$

$\quad \lambda x. \lambda z. z\ x$

$= \lambda x. x \quad \therefore \beta$