

Inference with Many Weak Instruments and Heterogeneity

Luther Yap*

August 28, 2025

Abstract

This paper considers inference in a linear instrumental variable regression model with many potentially weak instruments, in the presence of heterogeneous treatment effects. I first show that existing test procedures, including those that are robust to either weak instruments or heterogeneous treatment effects, can be arbitrarily oversized. I propose a novel and valid test based on a score statistic and a “leave-three-out” variance estimator. In the presence of heterogeneity and within the class of tests that are functions of the leave-one-out analog of a maximal invariant, this test is asymptotically the uniformly most powerful unbiased test. In two applications to judge and quarter-of-birth instruments, the proposed inference procedure also yields a bounded confidence set while some existing methods yield unbounded or empty confidence sets.

Keywords: Many Weak Instruments, Heterogeneous Treatment Effects.

1 Introduction

Many empirical studies in economics involve instrumental variable (IV) models with many instruments. A prominent example is the examiner design: several studies argue

*Email: lyap@nus.edu.sg. Department of Economics, National University of Singapore. This paper benefited from discussions with Michal Kolesár, David Lee, Ulrich Müller, and seminar participants at Duke University, Cornell University, National University of Singapore, Princeton University, University of Notre Dame, University of Pennsylvania, and University of Southern California.

that judges or case workers are as good as randomly assigned given covariates and can affect the treatment status, so they are used as instruments to study the effects of foster care (Doyle, 2007), incarceration (Kling, 2006), disability benefits (Autor et al., 2019), and misdemeanor prosecution (Agan et al., 2023), among others. When the IV are indicators for judges, the number of instruments can be large relative to the sample size. Another example of many IV is a single instrument interacted with discrete covariates. When Angrist and Krueger (1991) used the quarter of birth as an instrument to study the returns to education, interacting the quarter of birth with the state of birth generates 150 instruments.

Despite the pervasiveness and importance of this setting, there does not yet exist an inference procedure that is robust to both heterogeneous treatment effects and weak instruments, which is a gap this paper aims to fill. Weak IV refers to a setting where the first-stage coefficients converge to zero at a rate such that no consistent estimator for the object of interest exists (following from the definition in Mikusheva and Sun (2022) rather than Chao et al. (2012)); and heterogeneous treatment effects refers to a setting where different subsets of the many IV may estimate different local average treatment effects (LATE) (Imbens and Angrist, 1994). There are several recent proposals (Crudu et al., 2021; Mikusheva and Sun, 2022; Matsushita and Otsu, 2022) that are robust to weak IV, but they assume constant treatment effects. A separate literature (Evdokimov and Kolesár, 2018) proposed variance estimators for the Jackknife IV Estimator (JIVE) (Angrist et al., 1999) that are robust to heterogeneous treatment effects, but their t -statistic test is still not robust to many weak IV. While it is clear that weak IV can lead to substantial distortions in inference (e.g., Dufour (1997); Staiger and Stock (1997)), it is less obvious if procedures developed under constant treatment effects that are robust to weak IV are still valid with heterogeneous treatment effects.

This paper makes two contributions. The first contribution is to point out that there is currently no procedure that is simultaneously robust to both many weak IV and heterogeneity. In particular, neglecting either heterogeneity or weak instruments can result in substantial distortions in inference. Section 2 presents a simple simulation that has both weak instruments and heterogeneous treatment effects. For a nominal 5% test, using the procedure from Mikusheva and Sun (2022) (MS22), which

is robust to weak instruments but not heterogeneity, can result in 100% rejection under the null, because their test statistic is not centered correctly when there is heterogeneity. This result is attributed to how their test is a joint test of both the parameter value and the null of no heterogeneity. Similarly, the procedure from [Evdokimov and Kolesár \(2018\)](#) (EK18), which is robust to heterogeneity but not weak instruments, can be severely oversized, as expected due to [Dufour \(1997\)](#). Additionally, this section documents how an empirically common practice of constructing a “leniency measure” that combines the many instruments and then using weak IV robust procedures from the just-identified IV literature is invalid. These results have not been documented in the existing literature to my knowledge, so my paper provides theoretical and simulation evidence on why they overreject.

The second contribution of this paper is to show that a modification of a Lagrange Multiplier (LM) test can yield a valid and asymptotically optimal test. [Section 3](#) proposes a procedure for valid inference. Following the many IV literature, the JIVE estimand is the object of interest — this estimand can be interpreted as a weighted average of treatment effects when there is heterogeneity (e.g., EK18). I modify the LM statistic proposed by [Matsushita and Otsu \(2022\)](#) (MO22) under constant treatment effects in two ways: first, I use a general matrix for variations of JIVE instead of the projection matrix so that we are robust to many covariates; second, I use a different variance estimator so that the statistic is robust to heterogeneity. I show that this LM (i.e., score) statistic is mean zero and asymptotically normal even with treatment effect heterogeneity. In fact, I prove a stronger normality result that a set of jackknife statistics that includes the LM is jointly normal, which is the first technical challenge of this paper. This new normality result uses an asymptotic environment that nests the asymptotic environments of EK18 and MS22 in that normality holds if either the number of instruments is large or the instruments are strong. This normality implies that, as long as the variance of LM is consistently estimable, a t -statistic can be calculated and critical values from the standard normal distribution are valid for inference. Obtaining a consistent variance estimator is the second technical challenge of the paper, since reduced-form coefficients are not consistently estimable when there are few observations per instrument. Motivated by [Anatolyev and Sølvesten \(2023\)](#) who proposed a method to jointly test the significance of many covariates in OLS, I

construct a novel leave-three-out (L3O) variance estimator for the LM variance and show that it is consistent, even when reduced-form coefficients are not consistently estimable.

Beyond validity, Section 4 shows how the proposed LM procedure is optimal. In the over-identified IV environment with normal homoskedastic errors, Moreira (2009a) showed that, if we are willing to restrict our attention to tests that are invariant to rotations of the instruments, it suffices to consider tests that are functions of three statistics. These three statistics are known as a “maximal invariant”. To be robust to non-normality and heteroskedasticity in the many IV environment, I focus on the leave-one-out (L1O) analog of this maximal invariant. The proposed LM statistic is one of the three statistics in the L1O analog, and I show that the two-sided LM test is asymptotically uniformly most powerful unbiased (UMPU) within the class of tests that are functions of this L1O analog, for the interior of the alternative space (i.e., where heterogeneity is imposed). While the UMPU test for just-identified IV has been established since Moreira (2009b), obtaining a UMPU test in the over-identified IV environment has thus far been more elusive because several statistics are informative of the object of interest. By imposing heterogeneity and many IV in the environment, the problem is (somewhat surprisingly) simplified. Since only the LM statistic in the asymptotic distribution is directly informative of the object of interest, I obtain a UMPU result that is distinct in this literature.

Simulation results in Section 5 show how the procedure is robust even with a small number of instruments, and it is reasonably powerful even with constant treatment effects. Section 6 contains two empirical applications that show how being robust to many weak IV and heterogeneity can change conclusions.¹ In the Angrist and Krueger (1991) quarter of birth application, the Matsushita and Otsu (2022) procedure that are robust to many weak IV but not heterogeneity have unbounded confidence sets while L3O has a bounded confidence set. In the Agan et al. (2023) judge application, MS22 has an empty confidence set, and the length of the L3O confidence interval is more than twice that of EK18, which is not robust to many weak IV.

This paper contributes to the following strands of literature. First, this paper contributes to a growing literature on many weak instruments. While recent procedures

¹Implementation code can be found at: <https://github.com/lutheryap/mwivhet>.

accommodate both many IV and weak IV simultaneously (e.g., [Crudu et al. \(2021\)](#); [Mikusheva and Sun \(2022\)](#); [Matsushita and Otsu \(2022\)](#); [Lim et al. \(2024\)](#)), their focus has been on the linear IV model with constant treatment effects. This paper augments their setup by allowing for heterogeneity in treatment effects, contributes new results on the limitations of their procedures under heterogeneity, and shows how heterogeneity can be understood in a framework analogous to weak instruments.

Second, this paper contributes to the literature on heterogeneity (e.g., [Kolesár \(2013\)](#); [Evdokimov and Kolesár \(2018\)](#)). The previous papers exploit consistent estimation of the object of interest to conduct inference. In contrast, this paper uses the (more general) weak IV environment where the object of interest may not be consistently estimated. Two recent papers allow weak IV and heterogeneity. [Boot and Nibbering \(2024\)](#) study a discrete instrument interacted and saturated with many covariates and target a specific weighted average of local average treatment effects (“SIVE”) instead of the JIVE. [Kleibergen and Zhan \(2025\)](#) target a continuous updating (CU) GMM estimator with a fixed number of instruments.²

Third, this paper contributes to a literature on inference when coefficients cannot be consistently estimated. Recent literature that has made substantial progress in the context of inference in OLS with many covariates: [Cattaneo et al. \(2018\)](#) and [Anatolyev and Sølvesten \(2023\)](#) proposed consistent variance estimators that are robust to heteroskedasticity, which involve inverting a large (n by n , where n is the sample size) matrix and L3O respectively. [Boot and Nibbering \(2024\)](#) adapt the [Cattaneo et al. \(2018\)](#) variance estimator for inference. In contrast, this paper adapts the approach from [Anatolyev and Sølvesten \(2023\)](#) that does not require an inversion of a large matrix, and whose L3O implementation is fast when using matrix operations.

Fourth, this paper contributes to a literature on optimal tests. There is a large literature that numerically compares various valid tests and characterizes various forms of optimality in over-identified IV (e.g., [Andrews \(2016\)](#); [Andrews et al. \(2019\)](#); [Van de Sijpe and Windmeijer \(2023\)](#); [Lim et al. \(2024\)](#)). This paper contributes new results under heterogeneity and many IV.

²They show that their CU-GMM estimator corresponds to the limited information maximum likelihood (LIML) estimator. However, it is also known that the LIML estimand may not be interpretable as a weighted average of LATE’s. ([Kolesár, 2013](#)) I am unaware of any paper that allows both weak IV and heterogeneity with a fixed number of instruments and targets a parameter that is a weighted average of LATE’s.

2 How Existing Procedures Overreject

This section shows theoretically and with simulations that existing procedures are not robust to weak IV and heterogeneity simultaneously. By considering a simple potential outcomes model without covariates that exhibits weak IV and heterogeneity, the model in this section is a special case of the general model in Section 3.

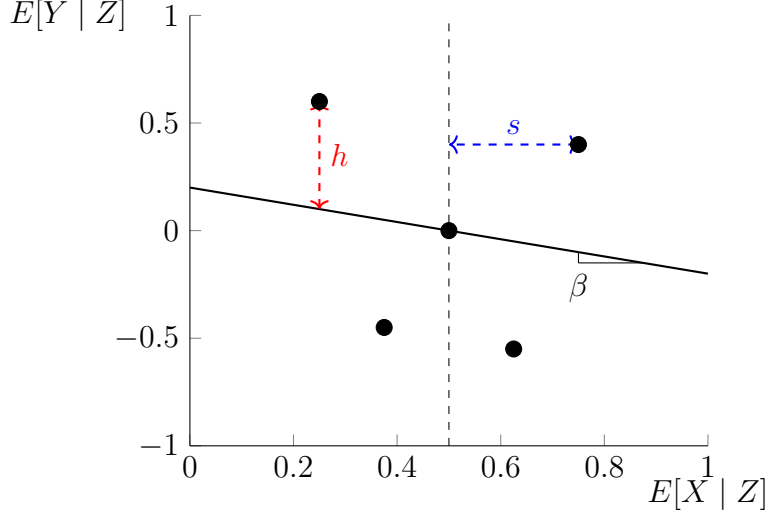
2.1 Setting for Simple Example

The simple example uses the canonical latent variable framework of Heckman and Vytlacil (2005). We are interested in the effect of $X_i \in \{0, 1\}$ (e.g., incarceration) on some outcome Y_i , for $i = 1, \dots, n$ that indexes individuals. To instrument for X_i , we use a vector of judges indicators: Z_i is a $(K + 1)$ -dimensional vector of indicators for judges, indexed $k = 1, \dots, K + 1$, each with $c = 5$ individual cases, so the vector takes value 1 for the k th component when individual i is matched to judge k , and 0 elsewhere. Then, $n = (K + 1)c$. The problem of many instruments arises when c is fixed while K increases. Let $Y_i(0)$ and $Y_i(1)$ denote the untreated and treated potential outcomes respectively, and we observe $Y_i = Y_i(X_i)$. The treatment status given some instrument value z is $X_i(z)$, and we observe $X_i(Z_i)$. The model is:

$$X_i(z) = 1\{z'\lambda > v_i\}, \text{ and } Y_i(x) = xf(v_i) + \varepsilon_i, \quad (1)$$

where $1\{\cdot\}$ is an indicator function that takes the value 1 if the argument is true and 0 otherwise. Here, $Z_i'\lambda = \lambda_{k(i)}$, where $k(i)$ is the judge that individual i is matched to. With individual unobservable $v_i \sim U[0, 1]$, the probability of treatment (i.e., $X_i = 1$) given judge k is λ_k . I set $\lambda_k = 1/2$ for the base judge, and evenly split all other K judges to take 4 different values of λ_k . Potential outcomes are $Y_i(0) = \varepsilon_i$ and $Y_i(1) = f(v_i) + \varepsilon_i$ so $Y_i(1) - Y_i(0) = f(v_i)$ is the treatment effect. The individual-specific residuals v_i and ε_i are allowed to be arbitrarily correlated. Let β_k denote the local average treatment effect (LATE) when comparing judge k to the base judge: for instance, when $\lambda_k > 1/2$, $\beta_k = \frac{1}{\lambda_k - 1/2} \int_{1/2}^{\lambda_k} f(v)dv$. The values of (λ_k, β_k) for the 4 groups of judges are $(1/2 - s, \beta - h/s)$, $(1/2(1 - s), \beta + 2h/s)$, $(1/2(1 + s), \beta - 2h/s)$, and $(1/2 + s, \beta + h/s)$. The function $f(v)$ that delivers these parameters and further details of this example are in Appendix D.1.

Figure 1: IV Strength and Heterogeneity in Reduced Form



The λ_k and β_k values are parameterized by objects s and h , which control the IV strength and heterogeneity in the model respectively. The impact of these parameters are illustrated in Figure 1 that plots the point masses for the four groups of judges in reduced-form. Parameter s controls how far $E[X | Z]$ are spread across judges, which then affects the instrument strength. Parameter h controls the distance between the mass points and a line with slope β — this slope is the object of interest. If the impact of X on Y is homogeneous, then $h = 0$, and all mass points *must* lie on a line — this implication is falsifiable by the data.

The simulation designs vary the values of s and h through the parameters:

$$E[T_{FS}] = \frac{5}{8}\sqrt{K}(c-1)s^2, \text{ and } E[T_{AR}] = \frac{5}{8}\sqrt{K}(c-1)h^2. \quad (2)$$

The statistic T_{FS} is the leave-one-out (L1O) analog of the first-stage “F” statistic in this model, and T_{AR} is similarly the L1O analog of the Anderson-Rubin statistic under the null. These objects are explained in detail in Section 3, but it suffices to mention here that, for the given c and K , there is a one-to-one mapping between $(E[T_{FS}], E[T_{AR}])$ and (s, h) . Using [Staiger and Stock \(1997\)](#) asymptotics, $E[T_{FS}]$ is the parameter that determines whether there is strong or weak identification. Where C is some positive arbitrary constant, $E[T_{FS}] \rightarrow \infty$ is an environment with strong identification where the object of interest can be estimated consistently, and $E[T_{FS}] \rightarrow$

Table 1: Rejection rates under the null for nominal size 0.05 test

| Designs | | Procedures | | | | | | | | |
|-------------|-------------|------------|-------|-------|-------|----------------|-----------------|-------|-------|-------|
| $E[T_{AR}]$ | $E[T_{FS}]$ | TSLS | EK | MS | MO | \tilde{X} -t | \tilde{X} -AR | L3O | LMorc | ARorc |
| $2\sqrt{K}$ | $2\sqrt{K}$ | 0.428 | 0.066 | NaN | 0.027 | 0.041 | 0.041 | 0.060 | 0.048 | 1.000 |
| | 2 | 0.978 | 0.040 | NaN | 0.282 | 0.114 | 0.303 | 0.052 | 0.049 | 1.000 |
| | 0 | 0.983 | 0.025 | NaN | 0.260 | 0.054 | 0.282 | 0.047 | 0.055 | 1.000 |
| 2 | $2\sqrt{K}$ | 0.984 | 0.076 | 1.000 | 0.039 | 0.046 | 0.049 | 0.044 | 0.050 | 1.000 |
| | 2 | 1.000 | 0.096 | 1.000 | 0.085 | 0.155 | 0.149 | 0.048 | 0.049 | 1.000 |
| | 0 | 1.000 | 0.128 | 1.000 | 0.103 | 0.225 | 0.177 | 0.060 | 0.051 | 1.000 |
| 0 | $2\sqrt{K}$ | 0.994 | 0.097 | 0.064 | 0.064 | 0.071 | 0.067 | 0.059 | 0.055 | 0.057 |
| | 2 | 1.000 | 0.231 | 0.059 | 0.047 | 0.179 | 0.106 | 0.049 | 0.051 | 0.055 |
| | 0 | 1.000 | 0.359 | 0.063 | 0.041 | 0.350 | 0.107 | 0.048 | 0.046 | 0.059 |

Notes: The table displays rejection rates of various procedures (in columns) for various designs (in rows). Details of the data generating process are in Appendix D.1. I use $K = 400, c = 5, \beta = 0$ with 1000 simulations. TSLS implements the standard two-stage-least-squares t -test for an over-identified IV system. EK implements the procedure in [Evdokimov and Kolesár \(2018\)](#). MS uses T_{AR} with the cross-fit procedure in [Mikusheva and Sun \(2022\)](#). MO uses the T_{LM} statistic with the variance estimator proposed in [Matsushita and Otsu \(2022\)](#). \tilde{X} -t uses a constructed instrument and runs TSLS for a just-identified IV system. \tilde{X} -AR uses the [Anderson and Rubin \(1949\)](#) (AR) procedure for a just-identified system using a constructed instrument. L3O uses the variance estimator proposed in this paper. LMorc is the infeasible theoretical benchmark that uses an LM statistic with an oracle variance. ARorc uses the AR statistic with an oracle variance.

$C < \infty$ is an environment with weak identification where no consistent estimator exists.

For every design, I generate data under the null and calculate the frequency that each inference procedure rejects the null of $\beta_0 = 0$. These procedures include the standard TSLS t -test, procedures that are robust to either weak instruments (MO, MS) or heterogeneity (EK), and procedures that use a constructed instrument (\tilde{X}). The results are presented in Table 1, which I will refer to in the remainder of this section as I explain them.

2.2 Issue with Many Weak Instruments

Many IV and weak IV are different but related issues. The many IV problem arises when the number of cases per judge c does not diverge to infinity, so that K is large relative to n . When c is small, the judge-specific λ_k and β_k cannot be consistently estimated and hence inference procedures like the TSLS t -test can be oversized. In Figure 1, a small c is attributed to the sample uncertainty surrounding each black circle. The weak IV problem arises from $E[T_{FS}]$ not diverging: since $E[T_{FS}]$ is a function of K, c , and s , the weak IV issue is related to the many IV issue.

If we simply run the TSLS t -test for an over-identified model, then the estimator can be asymptotically biased and inference is invalid, a fact already known in the literature. This fact is also evident in Table 1, where TSLS has 100% rejection in many designs. In TSLS, the first stage regresses X on Z to get a predicted $\hat{X} = Z\hat{\pi}$, where $\hat{\pi}$ is the estimated coefficient; the second stage regresses Y on \hat{X} . With constant treatment effects, the asymptotic bias of the TSLS estimator depends on $\sum_i \varepsilon_i \hat{X}_i / \sum_i \hat{X}_i^2$. When every judge only has $c = 5$ cases, the influence of v_i on $\hat{\pi}_{k(i)}$ and hence \hat{X}_i is non-negligible. Since ε_i and v_i can be arbitrarily correlated, the numerator is biased. If the instruments are weak such that the denominator $\sum_i \hat{X}_i^2$ does not diverge sufficiently quickly, then the asymptotic bias can be large. Due to the asymptotic bias, the t -statistic is not centered around β_0 when data is generated under the null, so we observe over-rejection in Table 1.

Since the bias in the TSLS estimator arises from using X_i to estimate $\hat{\pi}$, a natural solution to address that bias is to use the JIVE to estimate β . Instead of using $\hat{X}_i = Z_i' \hat{\pi}$ in the second stage, we instead use $\tilde{X}_i = Z_i' \hat{\pi}_{-i}$, where $\hat{\pi}_{-i}$ is the coefficient

from the first-stage regression that leaves out observation i . I call $\hat{\pi}_{-i}$ the leave-one-out (L1O) coefficient. With $P = Z(Z'Z)^{-1}Z'$ denoting the projection matrix, $\tilde{X}_i = Z_i'\hat{\pi}_{-i}$ can be written as $\tilde{X}_i = \sum_{j \neq i} P_{ij}X_j$. Then, the JIVE is:

$$\hat{\beta} = \frac{\sum_i Y_i \left(\sum_{j \neq i} P_{ij} X_j \right)}{\sum_i X_i \left(\sum_{j \neq i} P_{ij} X_j \right)}. \quad (3)$$

In the many IV context with constant treatment effects, the asymptotic distribution of the t -statistic of the JIVE is the same as the distribution of the t -statistic of the TSLS estimator in the just-identified environment (Mikusheva and Sun, 2022) — it is a ratio of two normally distributed random variables. It is well-known that, in the just-identified IV context with weak IV, the rejection rate of the standard t -statistic can be up to 100% for a nominal 5% test (e.g., Dufour (1997)). Hence, like the just-identified IV context, by using a structural model that has sufficiently weak instruments and high covariance, the simulation can deliver high rejection rates.

EK18 have a procedure that is robust to heterogeneity, but not weak instruments, so even if we use their variance estimator for the t -statistic, this problem is not alleviated: Corollary 3.4 from Dufour (1997) still applies as the confidence set from EK18 is bounded almost surely. This fact is evident in the EK column of Table 1, where, with a sufficiently large correlation in the individual unobservables, rejection rates can be large.³ Hence, ignoring the issue of weak instruments can lead to substantial distortions in inference. In fact, even with strong instruments, there is no guarantee that EK18 achieves the nominal rate, because their variance estimation method requires consistent estimation of the first-stage coefficients $\hat{\pi}$. A condition for consistent variance estimation is that the number of cases per judge is large, which is not $c = 5$.

Remark 1. In the literature, there have been several definitions of weak IV, which I clarify in this remark. Using Equation (2), there are three asymptotic regimes, ordered from the strongest to the weakest: (i) $\frac{1}{\sqrt{K}}E[T_{FS}] \rightarrow \infty$, (ii) $E[T_{FS}] \rightarrow \infty$, and (iii) $E[T_{FS}] \rightarrow C < \infty$. Regime (i) is a necessary condition for the TSLS estimator to be consistent, so $\frac{1}{\sqrt{K}}E[T_{FS}] \rightarrow C < \infty$ is what Stock and Yogo (2005) would refer to as weak instruments. Regime (ii) is a necessary condition for the JIVE to be consistent

³The rejection rate of EK can be 100% under the null in some simulations: one example is given in Table 7 in Online Appendix D.3.

(e.g., [Chao et al. \(2012\)](#), EK18). Regime (iii) is where no estimator is consistent (e.g., MS22). If K is fixed, then (i) and (ii) are the same asymptotically, and (iii) is the relevant weak identification asymptotic regime. If $K \rightarrow \infty$, then there is more ambiguity in what weakness means: [Chao et al. \(2012\)](#) and EK18 who assume (ii) are robust to weak instruments when defined in the [Stock and Yogo \(2005\)](#) sense, because s can converge to 0, albeit at a slower rate than \sqrt{K} . In this paper, I follow the [Staiger and Stock \(1997\)](#) standard of weak identification where no consistent estimator exists, which corresponds to (iii) that EK18 is not robust to.

2.3 Issue with Heterogeneity

Next, consider proposals for inference that are developed for contexts with many weak IV. MS22 (and [Crudu et al. \(2021\)](#)) propose using an Anderson-Rubin (AR) statistic $T_{AR} = \frac{1}{\sqrt{K}} \sum_i \sum_{j \neq i} P_{ij} e_i(\beta_0) e_j(\beta_0)$, for $e_i(\beta_0) := Y_i - X_i \beta_0$ where β_0 is the hypothesized null value. With constant treatment effects, $e_i := Y_i - X_i \beta$ is the residual. Hence, if the instrument is orthogonal to the residual, then $E[Z_i e_i] = 0$.⁴ Then, T_{AR} is the L1O analog for the quadratic form that tests the moment $E[Z_i e_i] = 0$. Since observations are independent, the critical value for the test is obtained from a mean-zero normal distribution. In this model, $E[T_{AR}] = \sqrt{K}(c-1)h^2$ under the null.⁵ Hence, when there are constant treatment effects such that $h = 0$ for all k , the statistic is unbiased. However, in the setup with heterogeneity, the T_{AR} can be biased: in fact, when h does not converge to zero, $E[T_{AR}]$ diverges, resulting in a 100% rejection rate under the null, even if the oracle variance were used. Further, there does not exist any estimand β such that $E[T_{AR}] = 0$, as stated in Lemma 1 of Appendix B.

A further problem with the feasible MS procedure is that when there is strong heterogeneity ($E[T_{AR}] = 2\sqrt{K}$) in this simulation, their cross-fit variance estimate is

⁴An equivalent way to see how heterogeneity affects inference is through the framework of [Hall and Inoue \(2003\)](#) and [Lee \(2018\)](#): $E[Z_i e_i] = 0$ is a special case of a misspecified over-identified GMM problem. The instruments are individually valid, but every component of the K moments in $E[Z_i e_i] = 0$ identifies a different treatment effect, so there is no parameter that satisfies all moments simultaneously under heterogeneity. Then, while the estimand is still interpretable as a combination of these treatment effects due to how GMM weights these moments, there are additional components in the variance that affect inference.

⁵This result can be obtained as a special case of Theorem 1 in Section 3 and using the fact that $\sum_i \sum_{j \neq i} P_{ij}^2 = \sum_i \sum_{j \neq i} (1/c^2) = \sum_i \frac{c-1}{c^2} = \sum_k \frac{c-1}{c}$.

negative for all simulation draws, as the negative heterogeneity terms are larger in magnitude than the positive variances of the residuals. The formal analysis requires more notation from Section 3, so details are deferred to Appendix B.

Another proposal in the literature that is robust to many weak instruments is [Matsushita and Otsu \(2022\)](#) (MO22) who use the statistic $T_{LM} = \frac{1}{\sqrt{K}} \sum_i \sum_{j \neq i} P_{ij} e_i(\beta_0) X_j = \frac{1}{\sqrt{K}} \sum_i e_i(\beta_0) \tilde{X}_i$. This statistic can be interpreted as the LM (or score) statistic that uses the moment $E[e\tilde{X}] = 0$. They propose the following variance estimator $\hat{\Psi}_{MO}$:

$$\hat{\Psi}_{MO} := \frac{1}{K} \sum_i \left(\sum_{j \neq i} P_{ij} X_j \right)^2 e_i(\beta_0)^2 + \frac{1}{K} \sum_i \sum_{j \neq i} P_{ij}^2 X_i e_i(\beta_0) X_j e_j(\beta_0). \quad (4)$$

While T_{LM} has zero mean under the null even with heterogeneity, a result shown later in Section 3, the MO22 variance estimator was constructed under the constant treatment effects assumption, so the variance estimand differs from the true variance. It can be shown that $E[\hat{\Psi}_{MO}] \neq \text{Var}(T_{LM})$, and $\hat{\Psi}_{MO}$ is inconsistent in general, so when it is used to construct the t -statistic of T_{LM} , the normalized statistic is not distributed $N(0, 1)$ asymptotically. Consequently, by constructing a DGP where $\hat{\Psi}_{MO}$ underestimates the variance, it is possible to get over-rejection of the MO22 procedure, as in the cases of Table 1 where $E[T_{AR}]$ diverges. As expected, when there is no heterogeneity such that $h = 0$, the rejection rate of MO22 and MS22 are close to the nominal rate.

2.4 Issue with a Constructed Instrument

In light of problems with weak identification and heterogeneity, there is a large applied literature that transforms a many IV environment into a just-identified single-IV environment. With a single IV, the [Anderson and Rubin \(1949\)](#) (AR) procedure (among others) is robust to both weak identification and heterogeneity.⁶ However, this subsection will argue that such an approach is invalid.

Due to how the JIVE is written, there are several empirical papers that treat $\tilde{X}_i = \sum_{j \neq i} P_{ij} X_j$ as the “instrument” so that $\hat{\beta} = \sum_i Y_i \tilde{X}_i / \sum_i X_i \tilde{X}_i$, and proceed with inference as if \tilde{X}_i is not constructed, but is an observed scalar instrument,

⁶Other just-identified procedures that are robust to weak IV and heterogeneity include Conditional Wald in [Moreira \(2003\)](#) and VtF in [Lee et al. \(2023\)](#).

usually referred to as a leniency measure. While the resulting estimator is numerically identical to JIVE, there are distortions in inference because the variance estimators do not account for the variability in constructing \tilde{X}_i .

If the TSLS t -statistic inference is used as if \tilde{X}_i is the instrument, then its rejection rates in designs with heterogeneity are usually higher than rejection rates of EK18 that accounts for the variance accurately, by comparing the \tilde{X} -t and EK columns in Table 1. Consequently, in the cases where EK under-rejects, \tilde{X} -t can have close to nominal rejection rates by coincidence.

Even if the weak IV robust AR procedure for just-identified IV were used, there can still be distortion in inference (see \tilde{X} -AR in Table 1). The AR t -statistic is $t_{\tilde{X}AR} := \sum_i e_i(\beta_0) \tilde{X}_i / \sqrt{\hat{V}}$, where $\hat{V} = \sum_i \tilde{X}_i^2 \hat{e}_i^2 / \left(\sum_i \tilde{X}_i^2 \right)^2$ and $\hat{e}_i = e_i(\beta_0) - \tilde{X}_i \left(\sum_i e_i(\beta_0) \tilde{X}_i \right) / \left(\sum_i \tilde{X}_i^2 \right)$. Even though $t_{\tilde{X}AR}$ is mean zero and asymptotically normal, the variance estimand is inaccurate, much like MO22. In particular, when $\beta_0 = \beta = 0$, the leading term of the variance estimand is $E \left[\sum_i \tilde{X}_i^2 e_i^2 \right]$, and it does not converge to the true variance derived in Section 3 in general. Hence, using the just-identified AR procedure with a constructed instrument results in over-rejection. There are several papers that cluster standard errors by judges, but this approach faces a similar issue.⁷

As a preview, the L3O procedure proposed in this paper has rejection rates close to the nominal rate while the other procedures can over-reject.

3 Valid Inference

This section describes a novel inference procedure and shows that it is valid. I set up a general model, then show that an LM statistic is asymptotically normal and a feasible variance estimator is consistent, which suffices for inference.

3.1 Setting: Model and Asymptotic Distribution

The general setup mimics [Evdokimov and Kolesár \(2018\)](#). With an independently drawn sample of individuals $i = 1, \dots, n$, we observe each individual's scalar outcome Y_i , scalar endogenous variable X_i , instrument Z_i , and covariates W_i , with $\dim(Z_i) =$

⁷Details of this discussion are relegated to Online Appendix D.2.

K .⁸ For every instrument value z , there is an associated potential treatment $X_i(z)$, and we observe $X_i = X_i(Z_i)$. Similarly, potential outcomes are denoted $Y_i(x)$, with $Y_i = Y_i(X_i)$. Let $R_i := E[X_i | Z_i, W_i]$ and $R_{Y_i} := E[Y_i | Z_i, W_i]$ be linear in Z_i and W_i . The model, written in the reduced-form and first-stage equations, is:

$$\begin{aligned} Y_i &= R_{Y_i} + \zeta_i, \text{ where} & R_{Y_i} &= Z_i' \pi_Y + W_i' \gamma_Y, & E[\zeta_i | Z_i, W_i] &= 0, \text{ and} \\ X_i &= R_i + \eta_i, \text{ where} & R_i &= Z_i' \pi + W_i' \gamma, & E[\eta_i | Z_i, W_i] &= 0. \end{aligned}$$

The setup implicitly conditions on Z_i, W_i , so R_i, R_{Y_i} are nonrandom.⁹ Linearity in Z and W is not necessarily restrictive when there is full saturation or when K is large.¹⁰

Define $e_i := Y_i - X_i \beta$, where β is some estimand of interest, and e_i is a linear transformation. Let $e_i(\beta_0) := Y_i - X_i \beta_0$ denote the feasible null-imposed linear transformation. Let $R_{\Delta i} := R_{Y_i} - R_i \beta$ and $\nu_i := \zeta_i - \eta_i \beta$. These definitions imply $e_i = R_{\Delta i} + \nu_i$ and $R_{\Delta i} = Z_i'(\pi_Y - \pi \beta) + W_i'(\gamma_Y - \gamma \beta)$. Since $E[\nu_i | Z_i, W_i] = 0$ from the model, $E[e_i | Z_i, W_i] = R_{\Delta i}$, which need not be zero. For data matrix A , let $H_A = A(A'A)^{-1}A'$ denote the hat (i.e., projection) matrix and $M_A = I - H_A$ its corresponding annihilator matrix. With Z, W denoting the corresponding data matrices of the instrument and covariates, let $Q = (Z, W)$, $P = H_Q$, and $M = I - P$. C denotes arbitrary constants.

Remark 2. While $E[e_i | Z_i, W_i] = R_{\Delta i}$ need not be zero under heterogeneous treatment effects, $E[e_i | Z_i, W_i] = R_{\Delta i} = 0$ under constant treatment effects. Since $R_{\Delta i} = Z_i'(\pi_Y - \pi \beta) + W_i'(\gamma_Y - \gamma \beta)$ for all i , constant treatment effects with $E[Y_i - X_i \beta | Z_i, W_i] = 0$ also implies $\pi_Y = \pi \beta$ and $\gamma_Y = \gamma \beta$ outside of edge cases (e.g., when Z_i, W_i are always 0). These R_{Δ} objects hence capture the impact of having heterogeneous treatment effects in the many instruments model.

⁸The endogenous variable X_i can be extended to a vector with some technical modifications and without conceptual complications.

⁹If we are interested in a superpopulation where Z is random, then the estimands would be defined as the probability limit of the conditional objects. Then, it suffices to have regularity conditions to ensure that the conditional object converges to the unconditional object.

¹⁰Any nonlinear function of the instruments can be arbitrarily well-approximated by a spline with a large number of pieces or a high-order polynomial. Moreover, the arguments in this paper could presumably be extended to a linear approximation of nonlinear functions as long as there are regularity conditions to ensure that higher-order terms are asymptotically negligible.

The (conditional) object of interest and its corresponding estimator are:

$$\beta_{JIVE} := \frac{\sum_i \sum_{j \neq i} G_{ij} R_{Y_i} R_j}{\sum_i \sum_{j \neq i} G_{ij} R_i R_j}, \text{ and } \hat{\beta}_{JIVE} = \frac{\sum_i \sum_{j \neq i} G_{ij} Y_i X_j}{\sum_i \sum_{j \neq i} G_{ij} X_i X_j},$$

where G is an $n \times n$ matrix that can take several forms. If there are no covariates, using the projection matrix $G = H_Z = P$ is the standard JIVE, and when there are covariates, I use the unbiased JIVE “UJIVE” (Kolesár, 2013) with $G = (I - \text{diag}(H_Q))^{-1} H_Q - (I - \text{diag}(H_W))^{-1} H_W$. In an environment with a binary instrument and many covariates interacted with the instrument, the saturated estimand “SIVE” (Chao et al., 2023; Boot and Nibbering, 2024) uses $G = P_{BN} - M_Q D_{BN} M_Q$, where $P_{BN} = M_W Z (Z' M_W Z)^{-1} Z' M_W$ and D_{BN} is defined as a diagonal matrix with elements such that $P_{BN,ii} = [M_Q D_{BN} M_Q]_{ii}$. With constant treatment effects, the estimand is the same for all the estimators: $R_{Y_i} = R_i \beta$ so $\beta_{JIVE} = \beta$. Depending on the application, the estimand is usually interpretable as some weighted average of treatment effects when using JIVE without covariates or UJIVE with covariates with a saturated regression.¹¹ (EK18) The focus of this paper is on inference, so I will not discuss the estimand in detail. The results for valid inference are established for any G that satisfies properties that will be formally stated in the theorem.

This paper restricts its attention to the following statistics:

$$(T_{AR}, T_{LM}, T_{FS})' := \frac{1}{\sqrt{K}} \sum_i \sum_{j \neq i} G_{ij} (e_i(\beta_0) e_j(\beta_0), e_i(\beta_0) X_j, X_i X_j)'. \quad (5)$$

It suffices to focus on (T_{AR}, T_{LM}, T_{FS}) for inference as they correspond to a linear transformation of the leave-one-out analog of a maximal invariant — details are in Section 4.1. T_{AR} is the (unnormalized) AR statistic used by MS22 for inference, and T_{LM} is the LM (score) statistic used by MO22. T_{FS} corresponds to a first-stage F statistic that can be used as a diagnostic for weak instruments.

¹¹In the judge example without covariates above, we have $G = P$ and $\pi_{Y_k} = \beta_k \pi_k$ where β_k is the local average treatment effect (LATE) between judge k and the base judge, so $\beta_{JIVE} = \frac{\sum_k \pi_{Y_k} \pi_k}{\sum_k \pi_k^2} = \frac{\sum_k \pi_k^2 \beta_k}{\sum_k \pi_k^2}$ is a weighted average of LATE’s.

Asymptotic theory in this paper uses $r_n/\sqrt{K} \rightarrow \infty$, where

$$r_n := \sum_i \left(\sum_{j \neq i} G_{ij} R_j \right)^2 + \sum_i \left(\sum_{j \neq i} G_{ij} R_{\Delta j} \right)^2 + \sum_i \sum_{j \neq i} G_{ij}^2, \quad (6)$$

which nests the environments of EK18, MS22, and MO22: as long as one of the three objects in Equation (6) diverges at a rate above \sqrt{K} , we obtain $r_n/\sqrt{K} \rightarrow \infty$. EK18 assume strong identification that translates to $\sum_i \left(\sum_{j \neq i} G_{ij} R_j \right)^2 / \sqrt{K} \rightarrow \infty$ in this setting, but $r_n/\sqrt{K} \rightarrow \infty$ can also be achieved if either of the latter terms in r_n diverges. MS22 and MO22 assume $K \rightarrow \infty$. Without covariates, $G = P$, so $\sum_i \sum_{j \neq i} G_{ij}^2 = O(K)$, and hence $r_n/\sqrt{K} \rightarrow \infty$. Hence, to apply the asymptotic theory in this paper, it suffices to have either strong identification, or $K \rightarrow \infty$. The only case ruled out is where K is fixed, *and* there is weak identification in that $\sum_i \left(\sum_{j \neq i} G_{ij} R_j \right)^2 / \sqrt{K}$ does not diverge.

Assumption 1. (a) There exists $C < \infty$ such that $E[\eta_i^4] + E[\nu_i^4] \leq C$ for all i .

(b) $E[\nu_i^2]$ and $E[\eta_i^2]$ are bounded away from 0 and $|\text{corr}(\nu_i, \eta_i)|$ is bounded away from 1.

(c) There exists $\underline{c} > 0$ such that for any c_1, c_2, c_3 that are not all 0,

$$\begin{aligned} & \frac{1}{r_n} \sum_i \left(c_3 \sum_{j \neq i} (G_{ij} + G_{ji}) R_j + c_2 \sum_{j \neq i} G_{ji} R_{\Delta j} \right)^2 \\ & + \frac{1}{r_n} \sum_i \left(c_1 \sum_{j \neq i} (G_{ij} + G_{ji}) R_{\Delta j} + c_2 \sum_{j \neq i} G_{ij} R_j \right)^2 \\ & + \frac{1}{r_n} \text{Var} \left(\sum_i \sum_{j \neq i} G_{ij} (c_1 \nu_i \nu_j + c_2 \nu_i \eta_j + c_3 \eta_i \eta_j) \right) \geq \underline{c}. \end{aligned}$$

(d) $\frac{1}{r_n^2} \sum_i \left(\left(\sum_{j \neq i} G_{ij} R_j \right)^4 + \left(\sum_{j \neq i} G_{ij} R_{\Delta j} \right)^4 + \left(\sum_{j \neq i} G_{ji} R_j \right)^4 + \left(\sum_{j \neq i} G_{ji} R_{\Delta j} \right)^4 \right) \rightarrow 0$.

(e) $\| \frac{1}{r_n} G_L G'_L \|_F + \| \frac{1}{r_n} G_U G'_U \|_F \rightarrow 0$, where G_L is a lower-triangular matrix with elements $G_{L,ij} = G_{ij} 1\{i > j\}$ and G_U is an upper-triangular matrix with elements $G_{U,ij} = G_{ij} 1\{i < j\}$.

Assumption 1 states high-level conditions that mimic EK18 so that a central limit theorem (CLT) can be applied. These conditions hence accommodate the G that EK18 consider with covariates. Having bounded moments in (a) is standard. Conditions (b) and (c) are sufficient to ensure that the variance is non-zero asymptotically.

In particular, (b) rules out perfect correlation: in the simulation, $\text{corr}(\eta_i, \nu_i) = -1$ is the pathological case that makes the variance zero, but $\text{corr}(\eta_i, \nu_i) = 1$ still allows non-zero variance. Conditions (d) and (e) ensure that the weights placed on the individual stochastic terms are not too large. The condition that $r_n/\sqrt{K} \rightarrow \infty$ is implied by (e) when $G = P$: due to Lemma B3 of [Chao et al. \(2012\)](#), under weak IV asymptotics where $P_{ii} \leq C < 1$, we obtain $\|G_L G'_L\|_F \leq C\sqrt{K}$.

Mechanically, if there is weak IV and fixed K , then $\|\frac{1}{r_n} G_L G'_L\|_F = \frac{1}{K} O(\sqrt{K}) \neq o(1)$, so (e) fails when r_n/\sqrt{K} does not diverge. Notably, the conditions do not require $P_{ii} \rightarrow 0$ so the π, π_Y coefficients need not be consistently estimated.

Theorem 1. *If Assumption 1 holds and $\beta = \beta_0$, then $\hat{\beta}_{JIVE} - \beta_{JIVE} = T_{LM}/T_{FS}$, and for $V = \text{Var}(T_{AR}, T_{LM}, T_{FS})$,*

$$V^{-1/2} \begin{pmatrix} T_{AR} - \frac{1}{\sqrt{K}} \sum_i \sum_{j \neq i} G_{ij} R_{\Delta i} R_{\Delta j} \\ T_{LM} \\ T_{FS} - \frac{1}{\sqrt{K}} \sum_i \sum_{j \neq i} G_{ij} R_i R_j \end{pmatrix} \xrightarrow{d} N \left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, I_3 \right). \quad (7)$$

In Theorem 1, $E[T_{FS}] = \frac{1}{\sqrt{K}} \sum_i \sum_{j \neq i} G_{ij} R_i R_j$ is the concentration parameter corresponding to the instrument strength. In the model of Section 2, the mapping to the reduced-form π can be found in Appendix D.1, so the concentration parameter is given by $E[T_{FS}] = \frac{1}{\sqrt{K}} \sum_k (c-1) \pi_k^2 = \frac{5}{8} \sqrt{K} (c-1) s^2$.¹² If the instruments are strong, then $E[T_{FS}] \rightarrow \infty$, so $\hat{\beta}_{JIVE} - \beta_{JIVE} \xrightarrow{d} 0$. With weak IV, $E[T_{FS}]$ converges to some constant $C < \infty$, so comparing the JIVE t -statistic with the standard normal distribution leads to invalid inference even in large samples.

The asymptotic distribution follows from establishing a quadratic CLT that may be of independent interest: it is proven by rewriting the leave-one-out sums as a martingale difference array, and then applying the martingale CLT. While there are existing quadratic CLT available, they do not fit the context exactly. [Chao et al.](#)

¹²This concentration parameter is comparable to the concentration parameter in just-identified IV. With slight abuse of notation, suppose the just-identified IV model has a first stage equation with $X = Z\pi + v$ where $\pi = s$. Then, omitting variance normalizations, using the notation from [Lee et al. \(2023\)](#), the concentration parameter is $f_0 = \sqrt{n}s$, which determines if the TSLS estimator is consistent. In the L1O asymptotics, $\sqrt{K}(c-1)s^2 \approx ns^2/\sqrt{K}$ by using $n = (K+1)c$ and approximations $\sqrt{K}/(K+1) \approx 1$ and $(c-1)/\sqrt{c} \approx \sqrt{c}$. By comparing the L1O concentration parameter ns^2/\sqrt{K} with the just-identified IV concentration parameter $f_0 = \sqrt{n}s$, I obtain the notions of weak identification in Remark 1.

(2012) Lemma A2 requires G to be symmetric, which works for $G = P$, but G for UJIVE is not symmetric in general. EK18 Lemma D2 is established for scalar random variables, so I extend it to random vectors.

Theorem 1 states that T_{LM} is mean zero and asymptotically normal. Hence, if we have access to the oracle variance of T_{LM} , we can simply use the statistic $T_{LM}/\sqrt{\text{Var}(T_{LM})}$ for testing because it has a standard normal distribution under the null. Obtaining a consistent estimator is an issue addressed in the next subsection.

3.2 Variance Estimation

To test the null that $H_0 : \beta = \beta_0$, we can calculate T_{LM} using the null-imposed β_0 and an estimator for the variance of $\sqrt{K}T_{LM}$, \hat{V}_{LM} , defined later in this section. Then, reject if $KT_{LM}^2/\hat{V}_{LM} \geq \Phi(1 - \alpha/2)^2$ for a size α test where $\Phi(\cdot)$ is the standard normal CDF. This procedure is valid when T_{LM} is asymptotically normal with mean zero as we have established in the previous section, and when \hat{V}_{LM} is consistent.

Before stating the variance estimator, I first decompose the variance expression in the equation below, which follows from substituting $e_i = R_{\Delta i} + \nu_i$ and $X_i = R_i + \eta_i$ into the variance. For $V_{LM} := \text{Var}\left(\sum_i \sum_{j \neq i} G_{ij} e_i X_j\right)$,

$$\begin{aligned} V_{LM} = & \sum_i \sum_{j \neq i} \sum_{k \neq i} E[\nu_i^2] G_{ij} G_{ik} R_j R_k + \sum_i \sum_{j \neq i} G_{ij}^2 E[\nu_i^2] E[\eta_j^2] + \sum_i \sum_{j \neq i} G_{ij} G_{ji} E[\eta_i \nu_i] E[\eta_j \nu_j] \\ & + 2 \sum_i \sum_{j \neq i} \sum_{k \neq i} E[\nu_i \eta_i] G_{ij} G_{ki} R_j R_{\Delta k} + \sum_i \sum_{j \neq i} \sum_{k \neq i} E[\eta_i^2] G_{ji} G_{ki} R_{\Delta j} R_{\Delta k}. \end{aligned} \quad (8)$$

With constant treatment effects, only the first line appears in the variance as $R_{\Delta} = 0$. With $G = P$, the expression for $\text{Var}\left(\sum_i \sum_{j \neq i} P_{ij} e_i X_j\right)$ matches the expression in EK18 Theorem 5.3, but their variance estimator cannot be used directly as they required consistent estimation of reduced-form coefficients. By adapting the leave-three-out (L3O) approach of Anatolyev and Solvsten (2023) (AS23), an unbiased and consistent variance estimator can be obtained. Intuitively, just as the own-observation bias in TSLS that involves a single sum can be addressed with L1O, an unbiased estimator for the variance expression that involves a triple sum can be obtained with L3O. Let $\tau := (\pi', \gamma')'$ and $\tau_{\Delta} := ((\pi_Y - \pi\beta)', (\gamma_Y - \gamma\beta'))'$ denote the coefficients on

Q when running the regression of X and e respectively. The variance estimator is:

$$\hat{V}_{LM} := A_1 + A_2 + A_3 + A_4 + A_5, \quad (9)$$

with

$$\begin{aligned} A_1 &:= \sum_i \sum_{j \neq i} \sum_{k \neq i} G_{ij} X_j G_{ik} X_k e_i(\beta_0) (e_i(\beta_0) - Q'_i \hat{\tau}_{\Delta, -ijk}), \\ A_2 &:= 2 \sum_i \sum_{j \neq i} \sum_{k \neq i} G_{ij} X_j G_{ki} e_k(\beta_0) e_i(\beta_0) (X_i - Q'_i \hat{\tau}_{\Delta, -ijk}), \\ A_3 &:= \sum_i \sum_{j \neq i} \sum_{k \neq i} G_{ji} e_j(\beta_0) G_{ki} e_k(\beta_0) X_i (X_i - Q'_i \hat{\tau}_{\Delta, -ijk}), \\ A_4 &:= - \sum_i \sum_{j \neq i} \sum_{k \neq j} G_{ji}^2 X_i \check{M}_{ik, -ij} X_k e_j(\beta_0) (e_j(\beta_0) - Q'_j \hat{\tau}_{\Delta, -ijk}), \\ A_5 &:= - \sum_i \sum_{j \neq i} \sum_{k \neq j} G_{ij} G_{ji} e_i(\beta_0) \check{M}_{ik, -ij} X_k e_j(\beta_0) (X_j - Q'_j \hat{\tau}_{\Delta, -ijk}), \end{aligned}$$

where

$$\begin{aligned} \hat{\tau}_{-ijk} &:= \left(\sum_{l \neq i, j, k} Q_l Q'_l \right)^{-1} \sum_{l \neq i, j, k} Q_l X_l, \\ \hat{\tau}_{\Delta, -ijk} &:= \left(\sum_{l \neq i, j, k} Q_l Q'_l \right)^{-1} \sum_{l \neq i, j, k} Q_l e_l(\beta_0), \\ D_{ij} &:= M_{ii} M_{jj} - M_{ij}^2, \text{ and} \\ \check{M}_{ik, -ij} &:= \frac{M_{jj} M_{ik} - M_{ij} M_{jk}}{D_{ij}} = -Q'_i \left(\sum_{l \neq i, j} Q_l Q'_l \right)^{-1} Q_k. \end{aligned}$$

Following AS23, Assumption 2 ensures that the L3O estimator is well-defined.¹³

Assumption 2. (a) $\sum_{l \neq i, j, k} Q_l Q'_l$ is invertible for every $i, j, k \in \{1, \dots, n\}$.

(b) $\max_{i \neq j \neq k \neq i} D_{ijk}^{-1} = O_P(1)$, where $D_{ijk} := M_{ii} D_{jk} - (M_{jj} M_{ik}^2 + M_{kk} M_{ij}^2 - 2M_{jk} M_{ij} M_{ik})$.

Assumption 2(a) corresponds to AS23 Assumption 1 and Assumption 2(b) corresponds to AS23 Assumption 4. For consistent variance estimation, we additionally

¹³If these conditions are not satisfied, then we can follow the modification in AS23 so that the variance estimator is conservative.

require regularity conditions that are stated in Assumption 3 of Appendix A. These conditions are satisfied when G is a projection matrix. With these conditions, Theorem 2 below claims that the variance estimator is consistent.

Theorem 2. *If $\beta = \beta_0$, Assumptions 1-2 hold, and Assumption 3 in Appendix A holds, then $E[\hat{V}_{LM}] = V_{LM}$ and $\hat{V}_{LM}/V_{LM} \xrightarrow{p} 1$.*

With many instruments and potentially many covariates, the reduced-form coefficients $\pi, \pi_Y, \gamma, \gamma_Y$ are not consistently estimable. The usual approach to constructing variance estimators calculates residuals by using the estimated coefficients, but this approach no longer works when these estimated coefficients are inconsistent. To be precise, applying Chebyshev's inequality for any $\epsilon > 0$ yields:

$$\Pr\left(\left|\frac{\hat{V}_{LM} - V_{LM}}{V_{LM}}\right| > \epsilon\right) \leq \frac{1}{\epsilon^2} \frac{\text{Var}(\hat{V}_{LM})}{V_{LM}^2} + \frac{1}{\epsilon^2} \frac{\left(E[\hat{V}_{LM}] - V_{LM}\right)^2}{V_{LM}^2}. \quad (10)$$

Without an unbiased estimator and when reduced-form coefficients cannot be consistently estimated, the second term in (10) is not necessarily asymptotically negligible. To overcome this problem, I use an unbiased variance estimator so that the second term is exactly zero. Then, it suffices to show that the variance of individual components of the variance are asymptotically small compared to V_{LM}^2 , so that the first term in (10) is $o(1)$ by applying the Cauchy-Schwarz inequality.

To obtain an unbiased estimator, I use estimators for the reduced-form coefficients $\pi, \pi_Y, \gamma, \gamma_Y$ that are unbiased and independent of objects that they are multiplied with. The leave-three-out (L3O) approach has this unbiasedness property for linear regressions: when leaving three observations out in the inner-most sum of the A expressions, the estimated coefficient $\hat{\tau}_{-ijk}$ is independent of i, j, k and is unbiased for τ . Then, when taking the expectation through a product of random variables of i, j, k and $\hat{\tau}_{-ijk}$, τ can be used in place of the $\hat{\tau}_{-ijk}$ component, and the expectations

of individual components can be isolated. For instance,

$$\begin{aligned}
& E \left[\sum_i \sum_{j \neq i} \sum_{k \neq i, j} G_{ij} X_j G_{ik} X_k e_i (e_i - Q'_i \hat{\tau}_{\Delta, -ijk}) \right] \\
&= \sum_i \sum_{j \neq i} \sum_{k \neq i, j} G_{ij} E[X_j] G_{ik} E[X_k] E[e_i (e_i - Q'_i \hat{\tau}_{\Delta, -ijk})] = \sum_i \sum_{j \neq i} \sum_{k \neq i, j} G_{ij} R_j G_{ik} R_k E[\nu_i^2],
\end{aligned} \tag{11}$$

which recovers the triple sums in the V_{LM} expression of (8). Without leaving out observations j and k , we would not be able to isolate $E[X_j]$ and $E[X_k]$ in the first equality. Without leaving out observation i , we would not be able to isolate τ_{Δ} on expectation to obtain $E[\nu_i^2]$ in the second equality. An analogous argument applies to other components of V in (7). Assuming that the residuals have zero mean conditional on Q is crucial: if we merely have $E[Q\zeta] = 0$, this argument can no longer be applied.

Remark 3. While the proposed \hat{V}_{LM} is motivated by AS23, the contexts and estimators are different. First, the statistic that we are estimating the variance for is different: AS23 demeaned their \mathcal{F} statistic using $\hat{E}_{\mathcal{F}}$, where $\hat{E}_{\mathcal{F}}$ is estimated using L1O, so they are interested in the variance of $\mathcal{F} - \hat{E}_{\mathcal{F}}$ that is mean zero; I use a mean-zero L1O statistic directly in T_{LM} . Second, the expectation of their variance estimator takes the form of their (9), which is analogous to the sum of A_1 and A_4 using the notation above, so repeated applications of their estimator is insufficient to recover all five terms. Hence, to adjust for the A_4 and A_5 terms here, I additionally require another estimator, and its form is similarly motivated by a L3O reasoning.

Inverting the test to obtain a confidence set is straightforward, as the test statistic T_{LM}^2 and variance estimator $\hat{V}_{LM} = B_0 + B_1\beta_0 + B_2\beta_0^2$ are quadratic in β_0 (for some B_0, B_1, B_2 that are functions of the data), so the confidence set is obtained by solving a quadratic inequality.¹⁴

4 How LM is Optimal

This section explains how the LM procedure is optimal. I first argue that we can restrict our attention to three statistics that are jointly normal by extending the

¹⁴Details are relegated to the online appendix.

argument from [Moreira \(2009a\)](#). Since the covariance matrix can be consistently estimated, the remainder of the section focuses on the 3-variable normal distribution with a known covariance matrix. With this asymptotic distribution, I show that the two-sided LM test is the uniformly most powerful unbiased test within the interior of the parameter space.

4.1 Sufficient Statistics and Maximal Invariant

As is standard in the literature, I consider the canonical model without covariates where the reduced-form errors are normal and homoskedastic (e.g., [Andrews et al. \(2006\)](#); [Moreira \(2009a\)](#)). Suppose (η, ζ) in the model of Section 3.1 are jointly normal with known variance:

$$\begin{pmatrix} \zeta_i \\ \eta_i \end{pmatrix} \sim N(0, \Omega) = N\left(0, \begin{bmatrix} \omega_{\zeta\zeta} & \omega_{\zeta\eta} \\ \omega_{\zeta\eta} & \omega_{\eta\eta} \end{bmatrix}\right). \quad (12)$$

Define:

$$\begin{pmatrix} s_1 \\ s_2 \end{pmatrix} := \begin{pmatrix} (Z'Z)^{-1/2} Z'Y \\ (Z'Z)^{-1/2} Z'X \end{pmatrix}.$$

I restrict attention to tests that are invariant to rotations of Z , i.e., transformations of the form $Z \rightarrow ZF'$ where F is a $K \times K$ orthogonal matrix. In particular, an invariant test $\phi(s_1, s_2)$ is one for which $\phi(Fs_1, Fs_2) = \phi(s_1, s_2)$ for all $K \times K$ orthogonal matrices F . If we focus on invariant tests, then the maximal invariant contains all relevant information from the data for inference.

Due to [Moreira \(2009a\)](#) Proposition 4.1, $(s'_1, s'_2)'$ are sufficient statistics for $(\pi'_Y, \pi')'$. Further, $(s'_1 s_1, s'_1 s_2, s'_2 s_2)$ is a maximal invariant, and

$$\begin{pmatrix} s_1 \\ s_2 \end{pmatrix} \sim N\left(\begin{pmatrix} (Z'Z)^{1/2} \pi_Y \\ (Z'Z)^{1/2} \pi \end{pmatrix}, \Omega \otimes I_K\right).$$

The maximal invariant $(s'_1 s_1, s'_1 s_2, s'_2 s_2)$ is jointly normal with a mean that depends on Ω when $K \rightarrow \infty$.¹⁵ Extending the argument to allow for heterogeneous treatment effects, the object of interest is $\beta = \frac{\pi' Z' Z \pi_Y}{\pi' Z' Z \pi}$ (following EK18), which is invariant to

¹⁵This result is stated in Online Appendix D.3.1.

rotations of the instrument.¹⁶

To be robust to many instruments, heteroskedasticity, and non-normality, I use the leave-one-out (L1O) analog of the maximal invariant (following MS22; Lim et al. (2024)).¹⁷ Without covariates such that $G = P$, the L1O analog $\frac{1}{\sqrt{K}} \sum_i \sum_{j \neq i} P_{ij}(Y_i Y_j, Y_i X_j, X_i X_j)$ is a linear transformation of (T_{AR}, T_{LM}, T_{FS}) .¹⁸ In the remainder of this section, I focus on testing the null that $\beta_0 = 0$ so $e(\beta_0) = Y$ and the L1O of the maximal invariant is exactly (T_{AR}, T_{LM}, T_{FS}) . The results are generalized in the appendix.

The asymptotic problem involving (T_{AR}, T_{LM}, T_{FS}) is:

$$\begin{pmatrix} T_{AR} \\ T_{LM} \\ T_{FS} \end{pmatrix} \sim N(\mu, \Sigma), \mu = \begin{pmatrix} \frac{1}{\sqrt{K}} \sum_i \sum_{j \neq i} P_{ij} R_{\Delta i} R_{\Delta j} \\ \frac{1}{\sqrt{K}} \sum_i \sum_{j \neq i} P_{ij} R_{\Delta i} R_j \\ \frac{1}{\sqrt{K}} \sum_i \sum_{j \neq i} P_{ij} R_i R_j \end{pmatrix}, \Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \cdot & \sigma_{22} & \sigma_{23} \\ \cdot & \cdot & \sigma_{33} \end{pmatrix}. \quad (13)$$

While $\mu_2 = 0$ under the null, μ_2 may not be zero under the alternative. There are several restrictions in the μ vector, which is assumed to be finite. Since P is a projection matrix, $\sum_i \sum_{j \neq i} P_{ij} R_i R_j = \sum_i R_i (\sum_j P_{ij} R_j - P_{ii} R_i) = \sum_i M_{ii} R_i^2$. Since the annihilator matrix M has positive entries on its diagonal, we obtain $\mu_3 \geq 0$ and a similar argument yields $\mu_1 \geq 0$. With $\mu_2 = \sum_i \sum_{j \neq i} P_{ij} R_{\Delta i} R_j = \sum_i M_{ii} R_{\Delta i} R_i$, the Cauchy-Schwarz inequality implies $\mu_2^2 \leq \mu_1 \mu_3$. Constant treatment effects implies $\mu_2^2 = \mu_1 \mu_3$, which is a special case of the environment here. Even with covariates, if the regression is fully saturated with G given by UJIVE, the same inequality restrictions hold.¹⁹ These properties do not contradict the joint normality: even though $\mu_3 \geq 0$, T_{FS} can still be negative when using the L1O statistic. The inequalities $\mu_1, \mu_3 \geq 0$ and $\mu_2^2 \leq \mu_1 \mu_3$ are also the *only* restrictions on μ , as it can be shown that there exists a structural model where there are no further restrictions.²⁰

¹⁶With rotation matrix F' such that $F'F = I$, observe that $X = ZF'F\pi + \eta$, so if we were to run the regression on ZF' instead of Z , we would obtain coefficients $F\pi$ instead of π . Then, the estimand is $\frac{\pi' F' F Z' Z F' F \pi_Y}{\pi' F' F Z' Z F' F \pi} = \frac{\pi' Z' Z \pi_Y}{\pi' Z' Z \pi}$ as before.

¹⁷With heteroskedasticity, the variances are not consistently estimable, so we cannot correct for the variances directly. These variances no longer feature in the L1O analog of the maximal invariant.

¹⁸To see that $\frac{1}{\sqrt{K}} \sum_i \sum_{j \neq i} P_{ij}(Y_i Y_j, Y_i X_j, X_i X_j)$ is a linear transformation, use the fact that $e = Y + X\beta$. Then, $\frac{1}{\sqrt{K}} \sum_i \sum_{j \neq i} P_{ij}((e_i + X_i\beta)(e_j + X_j\beta), (e_i + X_i\beta)X_j, X_i X_j) = (T_{AR} + 2T_{LM}\beta + T_{FS}\beta^2, T_{LM} - T_{FS}\beta, T_{FS})$.

¹⁹See Proposition 3 in Online Appendix D.3.1.

²⁰Online Appendix D.3 establishes that there exists a structural model where Σ is uninformative about μ , and $\mu_1, \mu_3 \geq 0$. Since the model in Section 2 is binary, it is insufficient for such a general result, and a continuous X is required. While the result establishes that there exists a structural

4.2 Optimality Result

With a size α test, the two-sided LM test against the alternative that $\mu_2 \neq 0$ rejects when $T_{LM}^2/\text{Var}(T_{LM}) > \Phi(1 - \alpha/2)^2$. I consider the benchmark of a uniformly most powerful unbiased test (e.g., [Lehmann and Romano \(2005\)](#); [Moreira \(2009b\)](#)).

Proposition 1. *Consider a restriction of the alternative μ space to the interior i.e., $\mu_1, \mu_3 > 0$ and $\mu_2^2 < \mu_1\mu_3$. Then, within the class of tests that are functions of (T_{AR}, T_{LM}, T_{FS}) , the two-sided LM test is the uniformly most powerful unbiased test for testing $H_0 : \mu_2 = 0$ against $H_1 : \mu_2 \neq 0$ in the asymptotic problem of [\(13\)](#).*

The argument for optimality applies a standard optimality result from [Lehmann and Romano \(2005\)](#) on the exponential family, which includes the normal distribution. To apply the [Lehmann and Romano \(2005\)](#) result, we require a convex parameter space and the existence of alternative values above and below the null value. It can be verified that the restricted parameter space is still convex, and the restriction to the interior ensures the latter condition is satisfied. The proposition claims optimality within the class of unbiased tests, and makes no statement about tests that are biased (i.e., where the power somewhere in the alternative space can be lower than the size).

Remark 4. Beyond the two-sided UMPU result, we may also consider other power properties. The one-sided LM test is shown to be the most powerful test against a particular subset of the alternative space. Numerically, using a covariance matrix calibrated from an empirical application, the power of the two-sided LM test is also close to that of the nearly optimal test against a weighted average over a grid of alternative values, constructed using the algorithm from [Elliott et al. \(2015\)](#). Details are in Online Appendix [D.3.3](#).

Studying optimality in the over-identified IV environment has thus far been complicated. With constant treatment effects, both s'_1s_1 and s'_1s_2 are informative of the object of interest β , because constant treatment effects implies $\mu_1 = \beta^2\mu_3$ in addition to $\mu_2 = \beta\mu_3$. However, once we impose $\mu_1 > 0$ under the null that $\beta = 0$, we rule out constant treatment effects by focusing on the interior of the alternative

model where there are no further restrictions, for any given structural model, there can still be further restrictions.

space. Then, the statistic associated with μ_1 is no longer directly informative of β . Imposing heterogeneity is hence the key to obtaining this UMPU result.

5 Simulations

This section focuses on the simple example from Section 2. Robustness checks that involve different data generating processes are relegated to the online appendix.²¹

Table 1 in Section 2 reports rejection rates under the null for a relatively large number of judges with $K = 400$, each with a small number of cases at $c = 5$. L3O performs well across various designs, while existing procedures can substantially over-reject in at least one design. The LMorc column is included as an infeasible theoretical benchmark that uses an oracle variance. The difference between LMorc and L3O is attributed to the variance estimation procedure.

Table 2 reports rejection rates under the null for a small number of judges with $K = 4$ and a large number of cases at $c = 200$. Based on the theory in Section 3, L3O should be valid when the instrument is strong, i.e., in the cases with $E[T_{FS}] = .5c$, which is what we observe. Notably, even when $E[T_{FS}] = 2$ or $E[T_{FS}] = 0$, the over-rejection for L3O is not too severe. EK performs very well in the cases with $E[T_{FS}] = .5c$ as expected in their theory. In contrast, MS and MO can over-reject severely with strong heterogeneity, even when instruments are strong.

Table 3 reports rejection rates under the alternative. When $E[T_{FS}] = 0$, the instrument should be completely uninformative about the true parameter, so we should have 0.05 rejection rate for a valid test, which is what we observe for L3O. When $E[T_{FS}] = 2\sqrt{K}$, all procedures, including L3O, are very informative. Considering the designs with $E[T_{AR}] = 0$ is most interesting, because this is an environment where MS and MO are valid, and the theoretical optimality result excludes this case. Looking at the case with $E[T_{AR}] = 0$, $E[T_{FS}] = 2$, L3O is less powerful than MS and MO in small samples, but the loss is less than 7 percentage points.

²¹There are more simulation results using several different structural models in Appendix D.5, including settings with continuous treatment X , and with covariates. The results are qualitatively similar in those simulations, suggesting that the numerical findings are not unique to the data-generating process chosen.

Table 2: Rejection rates under the null for nominal size 0.05 test

| Designs | | Procedures | | | | | | | | |
|-------------|-------------|------------|-------|-------|-------|----------------|-----------------|-------|-------|-------|
| $E[T_{AR}]$ | $E[T_{FS}]$ | TSLS | EK | MS | MO | \tilde{X} -t | \tilde{X} -AR | L3O | LMorc | ARorc |
| .5c | .5c | 0.210 | 0.049 | 1.000 | 0.212 | 0.217 | 0.217 | 0.048 | 0.052 | 1.000 |
| | 2 | 0.642 | 0.018 | 1.000 | 0.806 | 0.269 | 0.816 | 0.043 | 0.049 | 1.000 |
| | 0 | 0.498 | 0.005 | 1.000 | 0.881 | 0.330 | 0.893 | 0.062 | 0.051 | 1.000 |
| 2 | .5c | 0.075 | 0.064 | 1.000 | 0.075 | 0.073 | 0.077 | 0.063 | 0.061 | 0.913 |
| | 2 | 0.462 | 0.013 | 0.999 | 0.436 | 0.296 | 0.516 | 0.095 | 0.049 | 0.931 |
| | 0 | 0.440 | 0.008 | 1.000 | 0.448 | 0.337 | 0.576 | 0.088 | 0.052 | 0.934 |
| 0 | .5c | 0.052 | 0.048 | 0.061 | 0.045 | 0.050 | 0.046 | 0.046 | 0.048 | 0.069 |
| | 2 | 0.376 | 0.088 | 0.075 | 0.044 | 0.238 | 0.123 | 0.101 | 0.045 | 0.071 |
| | 0 | 0.590 | 0.181 | 0.076 | 0.029 | 0.431 | 0.163 | 0.075 | 0.045 | 0.080 |

Notes: $K = 4, c = 200$. Designs and procedures are otherwise identical to Table 1.

Table 3: Rejection rates under the alternative for nominal size 0.05 test

| Designs | | Procedures | | | | | | | | |
|-------------|-------------|------------|-------|-------|-------|----------------|-----------------|-------|-------|-------|
| $E[T_{AR}]$ | $E[T_{FS}]$ | TSLS | EK | MS | MO | \tilde{X} -t | \tilde{X} -AR | L3O | LMorc | ARorc |
| $2\sqrt{K}$ | $2\sqrt{K}$ | 1.000 | 1.000 | NaN | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| | 2 | 0.310 | 0.173 | NaN | 0.539 | 0.117 | 0.563 | 0.240 | 0.225 | 1.000 |
| | 0 | 0.722 | 0.029 | NaN | 0.291 | 0.055 | 0.309 | 0.048 | 0.055 | 1.000 |
| 2 | $2\sqrt{K}$ | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| | 2 | 0.244 | 0.543 | 1.000 | 0.886 | 0.163 | 0.907 | 0.789 | 0.823 | 1.000 |
| | 0 | 0.998 | 0.090 | 1.000 | 0.157 | 0.169 | 0.221 | 0.073 | 0.054 | 1.000 |
| 0 | $2\sqrt{K}$ | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| | 2 | 0.259 | 0.662 | 1.000 | 0.967 | 0.230 | 0.978 | 0.936 | 0.961 | 1.000 |
| | 0 | 1.000 | 0.373 | 0.059 | 0.048 | 0.333 | 0.117 | 0.067 | 0.055 | 0.054 |

Notes: $K = 100, \beta = 0.1, c = 5$. Designs and procedures are otherwise identical to Table 1.

6 Empirical Applications

6.1 Returns to Education

[Angrist and Krueger \(1991\)](#) were interested in the impact of years of education (X) on log weekly wages (Y). They instrument for education using the quarter of birth (QOB). I implement UJIVE using full interaction of QOB with the state of birth and year of birth (resulting in 1530 instruments) without other controls, which is similar to Table VII(2) of [Angrist and Krueger \(1991\)](#) that uses the same set of controls but without full saturation. The implementation here differs from the implementation of MS and MO in that I do not linearly partial out other covariates, but merely saturate on state and year of birth. This implementation is motivated by recent econometric research (e.g., [Blandhol et al. \(2022\)](#); [Śłoczyński \(2020\)](#)) that argue that the standard interpretation of estimands as a weighted average of LATE’s is only retained with some parametric assumptions or when the specification controls for covariates richly, which can be achieved with full saturation.²² To ensure that the different procedures are directly comparable, I adapt the MS and MO inference procedures to target UJIVE, so that the estimand is the same across all procedures and differing results can be attributed purely to inference. TSLS is consequently not a meaningful comparison as the estimand is different from the others.

The results are reported in Table 4. In addition to the aforementioned procedures, I include results from implementing the procedure in [Crudu et al. \(2021\)](#) (CMS) that uses the T_{AR} statistic like MS22, but uses a plug-in variance estimator like MO22.²³ Being robust to weak and many IV results in L3O having a longer confidence interval than EK. With full saturation, CMS and MO yield unbounded confidence sets, while L3O yields a bounded confidence set, showing how robustness to heterogeneity changes the shape of the confidence set in this context. The shape of the confidence set depends on the coefficient on β_0^2 . In particular, for $\Psi_2 := \frac{1}{K} \sum_i \left(\sum_{j \neq i} G_{ij} X_j \right)^2 X_i^2 + \frac{1}{K} \sum_{i \neq j} G_{ij}^2 X_i^2 X_j^2$, MO is unbounded when $T_{FS}^2 - q\Psi_2 < 0$ and L3O is unbounded when $T_{FS}^2 - qB_2 < 0$, where q is 3.84 for

²²A further advantage of this implementation is that the code is fast: when G is block-diagonal, it suffices to loop over blocks.

²³MS22 use a cross-fit variance estimator, while they refer to the CMS variance estimator as the “naive” variance estimator. MS22 argue that their cross-fit variance is more powerful, which corroborates how MS has a bounded confidence set while CMS does not.

a 5% test and B_2 is the coefficient on β_0^2 in the expression of \hat{V}_{LM} . Consequently, in this application, we can think of $T_{FS}^2/\Psi_2 = 0.102$ and $T_{FS}^2/B_2 = 11.8$ as first-stage statistics for MO and L3O respectively that determine whether the confidence sets are bounded.²⁴ Analogously, when solving a quartic equation in CMS, an unbounded set occurs as $T_{FS}^2/\left(\frac{2}{K} \sum_{i \neq j}^n G_{ij}^2 X_i^2 X_j^2\right) = 0.0545$, where the denominator is their coefficient on β_0^4 in their variance estimator. In contrast, the MS confidence set is bounded with $T_{FS}^2/\left(\frac{2}{K} \sum_{i \neq j}^n \frac{G_{ij}^2}{M_{ii}M_{jj}+M_{ij}^2} X_i^2 X_j^2\right) = 23.9$.

Due to the M terms in the L3O expression, it is difficult to compare the estimates directly. However, it is possible to compare the estimands of these coefficients in the judge example without covariates: $E[K\Psi_2] - E[B_2] = \sum_i M_{ii} R_i^2 (R_i^2 - 3(1 - 2P_{ii}) E[\eta_i^2])$. If $R_i^2 > 3(1 - 2P_{ii}) E[\eta_i^2]$, then MO is more likely unbounded. Intuitively, the MO estimator contains additional products of R that are not present in the true variance, and there are products of R and the error present in the true variance that MO does not account for, motivating the aforementioned difference. We can interpret this condition as MO being more likely unbounded when the signal-to-noise ratio $R_i^2/E[\eta_i^2]$ is sufficiently large. In this application, by observing that the first-stage statistic of L3O is an order of magnitude larger than that of MO (i.e., B_2 is an order of magnitude smaller), and by comparing the CMS and MO first-stage statistics, there is evidence that $R_i^2/E[\eta_i^2]$ is large.²⁵ This result does not depend on heterogeneity, because the coefficient of β_0^2 depends only on how X is combined in the variance estimator.

²⁴These statistics are “F” statistics with different variance estimators, suggesting that the instruments are meaningfully weak. The MS and MO variance estimators converge to the same object under weak identification such that $\frac{1}{K} \sum_i \sum_{j \neq i} G_{ij} R_i R_j \rightarrow 0$, which is not imposed by the asymptotic regime in this paper.

²⁵Comparing the statistics between MO and MS implies $\frac{1}{K} \sum_i \left(\sum_{j \neq i} G_{ij} X_j\right)^2 X_i^2 < \frac{1}{K} \sum_i \sum_{j \neq i} G_{ij}^2 X_i^2 X_j^2$, which can equivalently be written as $\sum_i \sum_{j \neq i} \sum_{k \neq i, j} G_{ij} G_{ik} X_j X_k X_i^2 < 0$. With full saturation, observations i, j have $G_{ij} < 0$ when they are in the same covariate group but have different instrument values. Under the MS and MO asymptotic regimes where $\frac{1}{K} \sum_i \sum_{j \neq i} G_{ij} R_i R_j \rightarrow 0$ so $R_i^2/E[\eta_i^2]$ is negligible, we obtain $\sum_i \sum_{j \neq i} G_{ij}^2 E[X_i^2] E[X_j^2] = \sum_i \sum_{j \neq i} G_{ij}^2 (R_i^2 + E[\eta_i^2]) (R_j^2 + E[\eta_j^2]) = \sum_i \sum_{j \neq i} G_{ij}^2 E[\eta_i^2] E[\eta_j^2] + o(1)$, and $\frac{1}{K} \sum_i \sum_{j \neq i} \sum_{k \neq i, j} G_{ij} G_{ik} E[X_j X_k X_i^2] = \frac{1}{K} \sum_i \sum_{j \neq i} \sum_{k \neq i, j} G_{ij} G_{ik} R_j R_k (R_i^2 + E[\eta_i^2]) = o(1)$ is asymptotically negligible. Since the difference between MS and MO is the same magnitude as the MS statistic, $\frac{1}{K} \sum_i \sum_{j \neq i} \sum_{k \neq i, j} G_{ij} G_{ik} E[X_j X_k X_i^2]$ is of similar order as $\sum_i \sum_{j \neq i} G_{ij}^2 E[X_i^2] E[X_j^2]$, so $R_i^2/E[\eta_i^2]$ is non-negligible in this application.

Table 4: 95% Confidence Sets for Returns to Education

| | EK | CMS | MS | MO | \tilde{X} -t | \tilde{X} -AR | L3O |
|----------|-------|-----------|-------|-----------|----------------|-----------------|-------|
| LB | 0.033 | $-\infty$ | 0.019 | $-\infty$ | 0.027 | 0.027 | 0.022 |
| UB | 0.173 | ∞ | 0.305 | ∞ | 0.179 | 0.189 | 0.210 |
| Estimate | 0.103 | 0.103 | 0.103 | 0.103 | 0.103 | 0.103 | 0.103 |
| Cllength | 0.140 | ∞ | 0.286 | ∞ | 0.152 | 0.161 | 0.188 |

Notes: Estimate reports the UJIVE. CMS implements the procedure from [Crudu et al. \(2021\)](#): use T_{AR} with a plug-in variance. Procedures are otherwise identical to Table 1.

Table 5: 95% Confidence Sets for Misdemeanor Prosecution

| | EK | CMS | MS | MO | \tilde{X} -t | \tilde{X} -AR | L3O |
|----------|--------|-------------|-------------|--------|----------------|-----------------|--------|
| LB | -0.151 | \emptyset | \emptyset | -0.220 | -0.187 | -0.188 | -0.201 |
| UB | -0.076 | \emptyset | \emptyset | -0.019 | -0.039 | -0.038 | -0.028 |
| Estimate | -0.113 | -0.113 | -0.113 | -0.113 | -0.113 | -0.113 | -0.113 |
| Cllength | 0.075 | \emptyset | \emptyset | 0.201 | 0.148 | 0.150 | 0.173 |

Notes: Procedures are identical to Table 4.

6.2 Misdemeanor Prosecution

[Agan et al. \(2023\)](#) were interested in the effect of misdemeanor prosecution (X) on criminal complaint in two years (Y). They instrument for misdemeanor prosecution using the assistant district attorneys (ADAs) who decide if a case should be prosecuted in the Suffolk County District Attorney’s Office in Massachusetts. As [Agan et al. \(2023\)](#) argued that as-if randomization holds conditional on court-by-time controls and that individual covariates are not required for relevance or exogeneity to hold in this context, the confidence set is constructed using full saturation of court-by-year and court-by-day-of-week fixed effects with no other controls for individual covariates.²⁶

As reported in Table 5, with full saturation, the UJIVE is -0.11 , so not prosecuting decreases the probability of criminal involvement by 11 percentage points.²⁷

²⁶Fixed effects is common in the examiner designs, so conditioning on these fixed effects is not unique to this paper.

²⁷This result is smaller than -0.36 reported in their Table III(3) that uses TSLS with a leniency measure. The result is more similar to the UJIVE robustness check in their Table A.1(5) of -0.15 with full saturation of the instrument, but their specification includes case/ defendant covariates, which results in a different estimator.

The L3O confidence interval (CI) is more than twice that of EK: unlike Section 6.1 where $n/K = 221$, we have $n/K = 11.9$ here, so a variance estimator that is robust to many IV has a larger impact on CI. MS has an empty confidence set while L3O has a bounded set, showing how being robust to heterogeneity qualitatively changes conclusions.²⁸ An empty confidence set using the AR procedure also suggests that the model with constant treatment effects is rejected, so there is meaningful heterogeneity.

The L3O CI is also shorter than MO, so being robust to heterogeneity decreases the length of the CI. Considering how the length of the MO confidence set is longer than L3O while being oversized in simulations, there is a question of when MO is conservative. While it is difficult to compare the confidence intervals or variance estimators directly, it is possible to compare the null-imposed variance estimands in the judge example without covariates. It can be shown that:

$$\begin{aligned} E \left[\hat{\Psi}_{MO} \right] - Var \left(\sum_i \sum_{j \neq i} P_{ij} e_i X_j \right) \\ = \sum_i M_{ii} R_{\Delta i}^2 (R_i^2 - (1 - 2P_{ii}) E [\eta_i^2]) - 2 \sum_i M_{ii} (1 - 2P_{ii}) E [\eta_i \nu_i] R_i R_{\Delta i}. \end{aligned}$$

Then, MO is conservative when: (i) $R_i^2 > (1 - 2P_{ii}) E [\eta_i^2]$, and (ii) $E [\eta_i \nu_i]$ is negatively correlated with $R_i R_{\Delta i}$, when $P_{ii} < 1/2$. In (i), $R_{\Delta i}^2$ only affects the magnitude of the difference, and not the sign, so this condition can be interpreted as a condition on the signal-to-noise ratio as before. Condition (ii) results from the $\sum_i M_{ii} (1 - 2P_{ii}) E [\eta_i \nu_i] R_i R_{\Delta i}$ term that MO does not account for, and covariances can be positive or negative in general.

7 Conclusion

This paper has documented how weak instruments and heterogeneity can interact to invalidate existing procedures in the environment of many instruments. Addressing both problems simultaneously, this paper contributes a feasible and robust method for

²⁸Mechanically, the confidence set for MS solves a quartic equation, so an empty set can occur, but it is difficult to characterize when this phenomenon occurs in general. Since the variances of MO and L3O converge to the same object under homogeneity, the difference between MO and L3O confidence sets also suggests that there is heterogeneity.

valid inference. The procedure is shown to be valid because the limiting distribution of commonly used statistics, including the LM statistic that generalizes [Matsushita and Otsu \(2022\)](#), is normal in an environment with many weak instruments and heterogeneity, and a leave-three-out variance estimator is consistent for obtaining the variance of the LM statistic. Beyond its validity, the LM test is also optimal, as it is the uniformly most powerful unbiased test in the asymptotic distribution for the interior of the alternative space. In light of the broader econometric literature on the value of saturated regressions and how many instruments can arise from them, this paper presents a highly applicable, robust, and powerful inference procedure for IV.

A High-level Assumptions for Inference

Following AS23, to ease notation in the L3O derivations, I define:

$$\check{M}_{il,-ijk} := \frac{M_{il} - M_{ij}\check{M}_{jl,-jk} - M_{ik}\check{M}_{kl,-jk}}{D_{ijk}/D_{jk}},$$

so that $X_i - Q'_i\hat{\tau}_{-ijk} = \sum_{l \neq k} \check{M}_{il,-ijk}X_l$, for instance.

Assumption 3 below states high-level conditions for consistency of the variance estimator. To ease notation, let R_{mi} stand for either $R_{\Delta i}$ or R_i . Denote $\tilde{R}_i := \sum_{j \neq i} G_{ij}R_j$ and $\tilde{R}_{\Delta i} := \sum_{j \neq i} G_{ij}R_{\Delta j}$. Let $h_2(i, j)$ be a product of any number of $G_{i_1 i_2}$, $i_1 \neq i_2$, and $\check{M}_{j_1 j_2}$, $j_1 \neq j_2$ with $i_1, i_2, j_1, j_2 \in \{i, j\}$. Similarly, $h_3(i, j)$ denotes a product of any number of $G_{i_1 i_2}$, $i_1 \neq i_2$, and $\check{M}_{j_1 j_2}$, $j_1 \neq j_2$ with $i_1, i_2, j_1, j_2 \in \{i, j, k\}$ such that every index in $\{i, j, k\}$ occurs at least once as an index of either $G_{i_1 i_2}$ or $\check{M}_{j_1 j_2}$. Let $h_4(i, j, k, l)$ denote a product of any number of $G_{i_1 i_2}$, $i_1 \neq i_2$ and $\check{M}_{j_1 j_2}$, $j_1 \neq j_2$ with $i_1, i_2, j_1, j_2 \in \{i, j, k, l\}$ such that every index in $\{i, j, k, l\}$ occurs at least once as an index of either $G_{i_1 i_2}$ or $\check{M}_{j_1 j_2}$, and there is no partition such that $h_4(i_1, i_2, j_1, j_2) = h_2(i_1, i_2)h_2(j_1, j_2)$, where i_1, i_2, j_1, j_2 are all different indices. For instance, $h_4(i, j, k, l)$ could be $G_{ij}\check{M}_{ik,-il}\check{M}_{lj,-ijk}$ but not $G_{ij}\check{M}_{lk,-il}$. Let $\sum_{i \neq j}^n = \sum_i \sum_{j \neq i}$ so that sums without the n superscript are still sums of individual indices, but sums with an n superscript involves the sum over multiple indices. Objects like $\sum_{i \neq j \neq k}^n$ and $\sum_{i \neq j \neq k \neq l}^n$ are defined similarly. When I refer to the p-sum, I refer to the sum over p non-overlapping indices. For instance, a 3-sum is $\sum_{i \neq j \neq k}^n$. Let F stand

for either G or G' . $1\{\cdot\}$ is an indicator function that takes the value 1 if the argument is true and 0 otherwise. $I\{\cdot\}$ is a function that takes value 1 if the argument is true and -1 if false.

Assumption 3. For some $C < \infty$,

- (a) $\sum_j F_{ij}^2 \leq C$, $\sum_{j \neq k} \left(\sum_{i \neq j, k} G_{ij} F_{ik} \right)^2 \leq C \sum_{j \neq k} G_{jk}^2$, $\sum_{j \neq k} \left(\sum_{i \neq j, k} G_{ji} G_{ki} \right)^2 \leq C \sum_{j \neq k} G_{jk}^2$, and $|R_{mi}| \leq C$.
- (b) $\sum_{i \neq j \neq k} \left(\sum_{l \neq i, j, k} h_4(i, j, k, l) R_{ml} \right)^2 \leq C \sum_i \tilde{R}_{mi}^2$, $\sum_{i \neq j} \left(\sum_{k \neq i, j} \sum_{l \neq i, j, k} h_4(i, j, k, l) R_{ml} \right)^2 \leq C \sum_i \tilde{R}_{mi}^2$, and $\sum_i \left(\sum_{j \neq i} \sum_{k \neq i, j} \sum_{l \neq i, j, k} h_4(i, j, k, l) R_{ml} \right)^2 \leq C \sum_i \tilde{R}_{mi}^2$.
- (c) $\sum_{i \neq j} \left(\sum_{k \neq i, j} h_3(i, j, k) R_{mk} \right)^2 \leq C \sum_i \tilde{R}_{mi}^2$ and $\sum_i \left(\sum_{j \neq i} \sum_{k \neq i, j} h_3(i, j, k) R_{mk} \right)^2 \leq C \sum_i \tilde{R}_{mi}^2$.
- (d) $\sum_i \left(\sum_{j \neq i} h_2(i, j) R_{mj} \right)^2 \leq C \sum_i \tilde{R}_{mi}^2$.

The first condition requires the row and column sums of the squares of the G elements to be bounded. Assumption 1(e) is insufficient because it does not rule out having $G_{ii} = K$ for some i and 0 elsewhere in the G matrix. These remaining conditions can be interpreted as (approximate) sparsity conditions on M and G as the p-sum of entries of \tilde{M} and G cannot be too large. The conditions primarily place a restriction on the types of G that can be used: for instance, a G matrix that contains all 1's is excluded. Note that other elements of the covariance matrix can be analogously shown to be consistent using the same strategy by using the lemmas from Appendix C by using $\tilde{R}_{Yi} := \sum_{j \neq i} G_{ij} R_{Yj}$ instead of $\tilde{R}_{\Delta i}$ where required.

The judges example in Section 2 satisfies this assumption when there are no covariates, $G = P$, and R values are bounded. For condition (a), $\sum_j P_{ij}^2 = P_{ii} \leq C$ and, since P is idempotent, $\sum_{j \neq k} \left(\sum_{i \neq j, k} P_{ij} P_{ik} \right)^2 = \sum_{j \neq k} \left(\sum_i P_{ij} P_{ik} - P_{jj} P_{jk} - P_{kk} P_{jk} \right)^2 = \sum_{j \neq k} \left(P_{jk} - P_{jj} P_{jk} - P_{kk} P_{jk} \right)^2 = \sum_{j \neq k} (1 - P_{jj} - P_{kk})^2 P_{jk}^2 \leq \sum_{j \neq k} P_{jk}^2$. For any \tilde{M}_{ij} and G_{ij} , these elements are nonzero only when i and j share the same judge p . Further, $R_{mi} = \pi_{mp(i)}$, where π_{mp} can denote π_p or $\pi_{\Delta p}$ in the model. Due to how the h functions are defined, when every judge has at most c cases,

$$\sum_i \left(\sum_{j \neq i} h_2(i, j) R_{mj} \right)^2 = \sum_i \left(\sum_{j \in \mathcal{N}_{p(i)} \setminus \{i\}} h_2(i, j) R_{mp(i)} \right)^2 = \sum_p \sum_{i \in \mathcal{N}_p} \left(\sum_{j \in \mathcal{N}_p \setminus \{i\}} h_2(i, j) \pi_{mp} \right)^2$$

$$= \sum_p \sum_{i \in \mathcal{N}_p} \left(\sum_{j \in \mathcal{N}_p \setminus \{i\}} h_2(i, j) \pi_{mp} \right)^2 \pi_{mp}^2 \leq C \sum_p \sum_{i \in \mathcal{N}_p} (c-1)^2 \pi_{mp}^2 = C \sum_i \tilde{R}_{mi}^2.$$

The same argument applies to the other components. For instance,

$$\begin{aligned} \sum_i \left(\sum_{j \neq i} \sum_{k \neq i, j} \sum_{l \neq i, j, k} h_4(i, j, k, l) R_{ml} \right)^2 &= \sum_p \pi_{mp}^2 \sum_{i \in \mathcal{N}_p} \left(\sum_{j \in \mathcal{N}_p \setminus \{i\}} \sum_{k \in \mathcal{N}_p \setminus \{i, j\}} \sum_{l \in \mathcal{N}_p \setminus \{i, j, k\}} h_4(i, j, k, l) \right)^2 \\ &\leq C \sum_p \sum_{i \in \mathcal{N}_p} \pi_{mp}^2 (c-1)^2 (c-2)^2 (c-3)^2 \leq C \sum_i \tilde{R}_{mi}^2. \end{aligned}$$

The upper bound is fairly loose because it merely counts the number of nonzero entries in h_4 . When every judge has a large number of cases, since h_4 contains only entries from the projection matrix, the inner sum is still bounded and the assumption is satisfied.

B Supplement for Section 2

Lemma 1. *Consider the model of Section 2. Suppose $h \neq 0$ and $Ks^2 > 0$. Then, $E[T_{AR}] \neq 0$ for all real β_0 .*

MS Variance Estimand. The proposed variance estimator is:

$$\hat{\Phi}_{MS} := \frac{2}{K} \sum_i \sum_{j \neq i} \frac{P_{ij}^2}{M_{ii}M_{jj} + M_{ij}^2} (e_i M_i \cdot e)(e_j M_j \cdot e).$$

By substituting $e_i = R_{\Delta i} + \nu_i$ and taking expectations,

$$E[e_i M_i \cdot e e_j M_j \cdot e] = R_{\Delta i} R_{\Delta j} \left(\sum_k M_{ik} M_{jk} E[\nu_k^2] \right) + (M_{ii} M_{jj} + M_{ij}^2) (E[\nu_i^2] E[\nu_j^2]).$$

This estimand can be positive or negative, but observe that $\sum_i \sum_{j \neq i} R_{\Delta i} R_{\Delta j} = (\sum_i R_{\Delta i})^2 - \sum_i R_{\Delta i}^2 = -\sum_i R_{\Delta i}^2 = -(n-c)h^2$ in the model of Section 2.1. Consequently, the negative heterogeneity component can far outweigh the positive components, resulting in a negative estimand when h does not converge to 0.

C Main Proofs

A quadratic CLT is used for Theorem 1. Let

$$T = \sum_i s'_i v_i + \sum_i \sum_{j \neq i} G_{ij} v'_i A v_j,$$

where v_i is a finite-dimensional random vector independent over $i = 1, \dots, n$ with bounded 4th moments, s_i is a nonstochastic vector, and A is a conformable matrix.

Lemma 2. *Suppose:*

1. $\text{Var}(T)^{-1/2}$ is bounded;
2. $\sum_i s_{il}^4 \rightarrow 0$; and
3. $\|G_L G'_L\|_F + \|G_U G'_U\|_F \rightarrow 0$, where G_L is a lower-triangular matrix with elements $G_{L,ij} = G_{ij} 1\{i > j\}$ and G_U is an upper-triangular matrix with elements $G_{U,ij} = G_{ij} 1\{i < j\}$.

Then, $\text{Var}(T)^{-1/2} T \xrightarrow{d} N(0, 1)$.

Proof of Theorem 1. By substituting $Y_i = R_{Yi} + \zeta_i$, $X_i = R_i + \eta_i$, $\zeta_i = \nu_i + \beta\eta_i$ and $R_{Yi} = R_{\Delta i} - R_i\beta$ into the expression for $\hat{\beta}_{JIVE}$,

$$\hat{\beta}_{JIVE} - \beta_{JIVE} = \frac{\left(\sum_i \sum_{j \neq i} G_{ij} (R_{\Delta i} \eta_j + \nu_i R_j + \nu_i \eta_j) \right)}{\sum_i \sum_{j \neq i} G_{ij} R_i R_j + \sum_i \sum_{j \neq i} G_{ij} (R_i \eta_j + R_j \eta_i + \eta_i \eta_j)}.$$

To see the equivalence with the T objects,

$$\begin{aligned} \frac{1}{\sqrt{K}} \sum_i \sum_{j \neq i} G_{ij} e_i X_j &= \frac{1}{\sqrt{K}} \sum_i \sum_{j \neq i} G_{ij} (\nu_i R_j + \nu_i \eta_j + R_{\Delta i} R_j + R_{\Delta i} \eta_j), \text{ and} \\ \sum_i \sum_{j \neq i} G_{ij} R_{\Delta i} R_j &= \sum_i \sum_{j \neq i} G_{ij} R_{Yi} R_j - \sum_i \sum_{j \neq i} G_{ij} R_i R_j \left(\frac{\sum_i \sum_{j \neq i} G_{ij} R_{Yi} R_j}{\sum_i \sum_{j \neq i} G_{ij} R_i R_j} \right) = 0, \end{aligned}$$

while T_{FS} is immediate.

Next, I show that the joint distribution of $\sqrt{\frac{K}{r_n}}(T_{AR}, T_{LM}, T_{FS})$ is asymptotically normal. Using the Cramer-Wold device, it suffices to show that $\sqrt{\frac{K}{r_n}}(c_1 T_{AR} + c_2 T_{LM} +$

$c_3 T_{FS}$) is normal for fixed c 's, where

$$\begin{aligned} \sqrt{\frac{K}{r_n}}(c_1 T_{AR} + c_2 T_{LM} + c_3 T_{FS}) &= c_1 \frac{1}{\sqrt{r_n}} \sum_i \sum_{j \neq i} G_{ij} (\nu_i R_j + \nu_i \nu_j + R_{\Delta i} R_{\Delta j} + R_{\Delta i} \nu_j) \\ &+ c_2 \frac{1}{\sqrt{r_n}} \sum_i \sum_{j \neq i} G_{ij} (\nu_i R_j + \nu_i \eta_j + R_{\Delta i} \eta_j) + c_3 \frac{1}{\sqrt{r_n}} \sum_i \sum_{j \neq i} G_{ij} (\eta_i R_j + \eta_i \eta_j + R_i R_j + R_i \eta_j). \end{aligned}$$

The object $T = \sqrt{\frac{K}{r_n}}(c_1 T_{AR} + c_2 T_{LM} + c_3 T_{FS}) - c_1 \frac{1}{\sqrt{r_n}} \sum_i \sum_{j \neq i} G_{ij} R_{\Delta i} R_{\Delta j} - c_3 \frac{1}{\sqrt{r_n}} \sum_i \sum_{j \neq i} G_{ij} R_i R_j$ can be written in the CLT form by setting:

$$\begin{aligned} v_i &= (\eta_i, \nu_i)', A = \begin{bmatrix} c_3 & 0 \\ c_2 & c_1 \end{bmatrix}, \text{ and} \\ s_i &= \begin{bmatrix} c_3 \sum_{j \neq i} (G_{ij} + G_{ji}) R_j + c_2 \sum_{j \neq i} G_{ji} R_{\Delta j} \\ c_1 \sum_{j \neq i} (G_{ij} + G_{ji}) R_{\Delta j} + c_2 \sum_{j \neq i} G_{ij} R_j \end{bmatrix}, \end{aligned}$$

so that

$$T = \frac{1}{\sqrt{r_n}} \sum_i s_i' v_i + \frac{1}{\sqrt{r_n}} \sum_i \sum_{j \neq i} G_{ij} v_i' A v_j.$$

Bounded 4th moments hold by Assumption 1(a). To apply the CLT from Lemma 2, I verify the following:

1. $\text{Var}(T)^{-1/2}$ is bounded;
2. $\frac{1}{r_n^2} \sum_i s_{il}^4 \rightarrow 0$ for all l ; and
3. $\|G_L G_L'\|_F + \|G_U G_U'\|_F \rightarrow 0$, where G_L is a lower-triangular matrix with elements $G_{L,ij} = \frac{1}{\sqrt{r_n}} G_{ij} 1\{i > j\}$ and G_U is an upper-triangular matrix with elements $G_{U,ij} = \frac{1}{\sqrt{r_n}} G_{ij} 1\{i < j\}$.

Condition (2) follows from Assumption 1(d) and applying the Cauchy-Schwarz inequality. Condition (3) follows from Assumption 1(e). For Condition (1), I show that Assumption 1(b) and (c) imply that, for any nonstochastic scalars c_1, c_2, c_3 that are finite and not all 0, $\text{Var}(T)^{-1/2}$ is bounded. Since $\text{Cov}\left(\sum_i s_i' v_i, \sum_i \sum_{j \neq i} G_{ij} v_i' A v_j\right) = 0$,

$$\text{Var}(T) = \frac{1}{r_n} \text{Var}\left(\sum_i s_i' v_i\right) + \frac{1}{r_n} \text{Var}\left(\sum_i \sum_{j \neq i} G_{ij} v_i' A v_j\right), \quad (14)$$

so it suffices to show that either term is bounded below.

The first term can be expanded as follows:

$$\begin{aligned}\text{Var} \left(\sum_i s'_i v_i \right) &= \sum_i s'_i \text{Var}(v_i) s_i = \sum_i s_{i1}^2 E[\eta_i^2] + 2s_{i1}s_{i2} E[\eta_i \nu_i] + s_{i2}^2 E[\nu_i^2] \\ &= \sum_i (1 - \rho_i)^2 E[\eta_i^2] s_{i1}^2 + \left(\rho_i s_{i1} \sqrt{E[\eta_i^2]} + s_{i2} \sqrt{E[\nu_i^2]} \right)^2 \geq \sum_i (1 - \rho_i)^2 E[\eta_i^2] s_{i1}^2.\end{aligned}$$

A similar argument yields $\text{Var}(\sum_i s'_i v_i) \geq \sum_i (1 - \rho_i)^2 E[\eta_i^2] s_{i2}^2$. Due to Assumption 1(c), at least one of the following must hold: (i) $\text{Var}(\sum_i \sum_{j \neq i} G_{ij} v'_i A v_j) \geq \underline{c}$ (ii) $\frac{1}{r_n} \sum_i s_{i1}^2 \geq \underline{c}$, or (iii) $\frac{1}{r_n} \sum_i s_{i2}^2 \geq \underline{c}$. Hence, $\text{Var}(T)^{-1/2}$ is bounded.

Finally, since ν_i, η_i are mean zero, the expectations are immediate: $E[T_{AR}] = \sum_i \sum_{j \neq i} G_{ij} R_{\Delta j} R_{\Delta i}$ and $E[T_{FS}] = \sum_i \sum_{j \neq i} G_{ij} R_j R_i$. \square

The proof of Theorem 2 involves several lemmas whose proofs are relegated to the online appendix. The proof strategy is to bound the variances above by components that are in the $h(\cdot)$ form so that Assumption 3 inequalities can be applied.

Let $V_{mi} = R_{mi} + v_{mi}$ where R_{mi} denotes the nonstochastic component while v_{mi} denotes the mean zero stochastic component. Following Equation (6), $r_n := \sum_i \tilde{R}_i^2 + \sum_i \tilde{R}_{\Delta i}^2 + \sum_i \sum_{j \neq i} G_{ij}^2$. Let C_i, C_{ij}, C_{ijk} denote nonstochastic objects that are non-negative and are bounded above by C . I use $h_4^A(\cdot)$ and $h_4^B(\cdot)$ to denote two different functions that satisfy the definition for h_4 .

Lemma 3. *Under Assumption 3, the following hold:*

- (a) $\left| \sum_{i \neq j \neq k} C_{ijk} \left(\sum_{l \neq i, j, k} h_4^A(i, j, k, l) R_{ml} \right) \left(\sum_{l \neq i, j, k} h_4^B(i, j, k, l) R_{ml} \right) \right| \leq C \sum_i \tilde{R}_{mi}^2,$
 $\left| \sum_{i \neq j} C_{ij} \left(\sum_{k \neq i, j} \sum_{l \neq i, j, k} h_4^A(i, j, k, l) R_{ml} \right) \left(\sum_{k \neq i, j} \sum_{l \neq i, j, k} h_4^B(i, j, k, l) R_{ml} \right) \right| \leq$
 $C \sum_i \tilde{R}_{mi}^2, \text{ and}$
 $\left| \sum_i C_i \left(\sum_{j \neq i} \sum_{k \neq i, j} \sum_{l \neq i, j, k} h_4^A(i, j, k, l) R_{ml} \right) \left(\sum_{j \neq i} \sum_{k \neq i, j} \sum_{l \neq i, j, k} h_4^B(i, j, k, l) R_{ml} \right) \right| \leq$
 $C \sum_i \tilde{R}_{mi}^2.$
- (b) $\left| \sum_{i \neq j} C_{ij} \left(\sum_{k \neq i, j} h_3^A(i, j, k) R_{mk} \right) \left(\sum_{k \neq i, j} h_3^B(i, j, k) R_{mk} \right) \right| \leq C \sum_i \tilde{R}_{mi}^2$
 $\text{and} \left| \sum_i C_i \left(\sum_{j \neq i} \sum_{k \neq i, j} h_3^A(i, j, k) R_{mk} \right) \left(\sum_{j \neq i} \sum_{k \neq i, j} h_3^B(i, j, k) R_{mk} \right) \right| \leq C \sum_i \tilde{R}_{mi}^2.$
- (c) $\left| \sum_i C_i \left(\sum_{j \neq i} h_2^A(i, j) R_{mj} \right) \left(\sum_{j \neq i} h_2^B(i, j) R_{mj} \right) \right| \leq C \sum_i \tilde{R}_{mi}^2.$

Lemma 4. Under Assumption 3, the following hold:

- (a) $\text{Var} \left(\sum_{i \neq j}^n G_{ij} F_{ij} V_{1i} V_{2i} V_{3j} V_{4j} \right) \leq Cr_n.$
- (b) $\text{Var} \left(\sum_{i \neq j \neq k}^n G_{ij} F_{ij} \check{M}_{ik, -ij} V_{1i} V_{2k} V_{3j} V_{4j} \right) \leq Cr_n.$
- (c) $\text{Var} \left(\sum_{i \neq j \neq l}^n G_{ij} F_{ij} \check{M}_{jl, -ij} V_{1i} V_{2i} V_{3j} V_{4l} \right) \leq Cr_n.$
- (d) $\text{Var} \left(\sum_{i \neq j \neq k \neq l}^n G_{ij} F_{ij} V_{1i} \check{M}_{ik, -ij} V_{2k} V_{3j} \check{M}_{jl, -ijk} V_{4l} \right) \leq Cr_n.$

Lemma 5. Under Assumption 3, the following hold:

- (a) $\text{Var} \left(\sum_{i \neq j \neq k}^n G_{ij} F_{ik} V_{1j} V_{2k} V_{3i} V_{4i} \right) \leq Cr_n.$
- (b) $\text{Var} \left(\sum_{i \neq j \neq k \neq l}^n G_{ij} F_{ik} \check{M}_{il, -ijk} V_{1j} V_{2k} V_{3i} V_{4l} \right) \leq Cr_n.$

Lemma 6. Under Assumption 3, the following hold:

- (a) $\text{Var} \left(\sum_{i \neq j}^n G_{ji}^2 V_{1i} V_{2i} V_{3j} V_{4j} \right) \leq Cr_n;$
- (b) $\text{Var} \left(\sum_{i \neq j \neq k}^n G_{ji}^2 \check{M}_{ik, -ij} V_{1i} V_{2k} V_{3j} V_{4j} \right) \leq Cr_n;$
- (c) $\text{Var} \left(\sum_{i \neq j \neq l}^n G_{ji}^2 \check{M}_{jl, -ij} V_{1i} V_{2i} V_{3j} V_{4l} \right) \leq Cr_n;$
- (d) $\text{Var} \left(\sum_{i \neq j \neq k \neq l}^n G_{ji}^2 V_{1i} \check{M}_{ik, -ij} V_{2k} V_{3j} \check{M}_{jl, -ijk} V_{4l} \right) \leq Cr_n;$
- (e) $\text{Var} \left(\sum_{i \neq j \neq k}^n G_{ji} F_{ki} V_{1j} V_{2k} V_{3i} V_{4i} \right) \leq Cr_n;$
- (f) $\text{Var} \left(\sum_{i \neq j \neq k \neq l}^n G_{ji} F_{ki} \check{M}_{il, -ijk} V_{1j} V_{2k} V_{3i} V_{4l} \right) \leq Cr_n.$

Proof of Theorem 2. Proof of Unbiasedness. The variance expression is:

$$\begin{aligned}
V_{LM} = & \sum_i \left(E[\nu_i^2] \left(\sum_{j \neq i} G_{ij} R_j \right)^2 + 2 \left(\sum_{j \neq i} G_{ij} R_j \right) \left(\sum_{j \neq i} G_{ji} R_{\Delta j} \right) E[\nu_i \eta_i] + E[\eta_i^2] \left(\sum_{j \neq i} G_{ji} R_{\Delta j} \right)^2 \right) \\
& + \sum_i \sum_{j \neq i} G_{ij}^2 E[\nu_i^2] E[\eta_j^2] + \sum_i \sum_{j \neq i} G_{ij} G_{ji} E[\eta_i \nu_i] E[\eta_j \nu_j].
\end{aligned} \tag{15}$$

To ease notation, let:

$$A_{1i} := \sum_{j \neq i} \sum_{k \neq i} G_{ij} X_j G_{ik} X_k e_i (e_i - Q_i' \hat{\tau}_{\Delta, -ijk}), \quad A_{2i} := \sum_{j \neq i} \sum_{k \neq i} G_{ij} X_j G_{ki} e_k e_i (X_i - Q_i' \hat{\tau}_{-ijk}),$$

$$A_{3i} := \sum_{j \neq i} \sum_{k \neq i} G_{ji} e_j G_{ki} e_k X_i (X_i - Q'_i \hat{\tau}_{-ijk}), \quad A_{4ij} := X_i \sum_{k \neq j} \check{M}_{ik, -ij} X_k e_j (e_j - Q'_j \hat{\tau}_{\Delta, -ijk}), \text{ and}$$

$$A_{5ij} := e_i \sum_{k \neq j} \check{M}_{ik, -ij} X_k e_j (X_j - Q'_j \hat{\tau}_{-ijk}).$$

Take expectation of A_1 :

$$\begin{aligned} E \left[\sum_i \sum_{j \neq i} \sum_{k \neq i} G_{ij} X_j G_{ik} X_k e_i (e_i - Q'_i \hat{\tau}_{\Delta, -ijk}) \right] \\ = \sum_i \sum_{j \neq i} \sum_{k \neq i} G_{ij} R_j G_{ik} R_k E [\nu_i^2] + \sum_i \sum_{j \neq i} G_{ij}^2 E [\nu_i^2] E [\eta_j^2]. \end{aligned}$$

Similarly, $E [A_{2i}] = \left(\sum_{j \neq i} G_{ij} R_j \right) \left(\sum_{j \neq i} G_{ji} R_{\Delta j} \right) E [\nu_i \eta_i] + \sum_{j \neq i} G_{ij} G_{ji} E [\eta_i \nu_i] E [\eta_j \nu_j]$
and $E [A_{3i}] = E [\eta_i^2] \left(\sum_{j \neq i} G_{ji} R_{\Delta j} \right)^2 + \sum_{j \neq i} G_{ji}^2 E [\eta_i^2] E [\nu_j^2]$.

For the A_4 and A_5 terms, observe that $X_i - Q'_i \hat{\tau}_{-ij} = X_i - Q'_i \sum_{k \neq i, j} \left(\sum_{l \neq i, j} Q_l Q'_l \right)^{-1} Q_k X_k = X_i + \sum_{k \neq i, j} \check{M}_{ik, -ij} X_k = \sum_{k \neq j} \check{M}_{ik, -ij} X_k$, where the final equality follows from $\check{M}_{ii, -ij} =$

1. Then,

$$\begin{aligned} E [A_{4ij}] &= E \left[X_i \sum_{k \neq j} \check{M}_{ik, -ij} X_k e_j (X_j - Q'_j \hat{\tau}_{\Delta, -ijk}) \right] = \sum_{k \neq j} E [X_i \check{M}_{ik, -ij} X_k e_j (X_j - Q'_j \hat{\tau}_{\Delta, -ijk})] \\ &= E \left[X_i \sum_{k \neq j} \check{M}_{ik, -ij} X_k \right] E [e_j (e_j - Q'_j \hat{\tau}_{\Delta, -ijk})] = E [\eta_i^2] E [\nu_j^2]. \end{aligned}$$

Similarly, $E [A_{5ij}] = E [\eta_i \nu_i] E [\eta_j \nu_j]$. Combining these expressions yields the unbiasedness result.

Proof of Consistency. By Chebyshev's inequality,

$$\begin{aligned} \Pr \left(\left| \frac{\hat{V}_{LM} - \text{Var} \left(\sum_i \sum_{j \neq i} G_{ij} e_i X_j \right)}{\text{Var} \left(\sum_i \sum_{j \neq i} G_{ij} e_i X_j \right)} \right| > \epsilon \right) \\ \leq \frac{1}{\epsilon^2} \frac{\text{Var} \left(\sum_i (A_{1i} + 2A_{2i} + A_{3i}) - \sum_i \sum_{j \neq i} G_{ji}^2 A_{4ij} - \sum_i \sum_{j \neq i} G_{ij} G_{ji} A_{5ij} \right)}{\left[\text{Var} \left(\sum_i \sum_{j \neq i} G_{ij} e_i X_j \right) \right]^2} \end{aligned}$$

Observe that the numerator can be written as the variance of the estimator only

because \hat{V}_{LM} is unbiased. I first establish the order of the denominator. Let $\tilde{R}_i := \sum_{j \neq i} G_{ij} R_j$, $\tilde{R}_{\Delta i} := \sum_{j \neq i} G_{ji} R_{\Delta j}$ and $\rho_i := \text{corr}(\eta_i \nu_i)$.

Since $E[\nu_i^2]$ and $E[\eta_i^2]$ are bounded away from zero and $|\text{corr}(\eta_i \nu_i)|$ is bounded away from one by Assumption 1(b), the first line of the V_{LM} expression in Equation (15) has order at least $\sum_i \tilde{R}_i^2 + \sum_i \tilde{R}_{\Delta i}^2$, and the second line has order at least $\sum_i \sum_{j \neq i} G_{ij}^2$. To see this, for some $\underline{c} > 0$, the first line is:

$$\begin{aligned} & \sum_i E[\nu_i^2] \tilde{R}_i^2 + 2\tilde{R}_{\Delta i} \tilde{R}_i E[\nu_i \eta_i] + \tilde{R}_{\Delta i}^2 E[\eta_i^2] = \sum_i E[\nu_i^2] \tilde{R}_i^2 + 2\tilde{R}_{\Delta i} \tilde{R}_i \rho_i \sqrt{E[\nu_i^2] E[\eta_i^2]} + \tilde{R}_{\Delta i}^2 E[\eta_i^2] \\ & \geq \sum_i \left(E[\nu_i^2] \tilde{R}_i^2 + \tilde{R}_{\Delta i}^2 E[\eta_i^2] \right) (1 - |\rho_i|) + \sum_i |\rho_i| \left(E[\nu_i^2] \tilde{R}_i^2 + \tilde{R}_{\Delta i}^2 E[\eta_i^2] - 2\tilde{R}_{\Delta i} \tilde{R}_i \sqrt{E[\nu_i^2] E[\eta_i^2]} \right) \\ & = \sum_i \left(E[\nu_i^2] \tilde{R}_i^2 + \tilde{R}_{\Delta i}^2 E[\eta_i^2] \right) (1 - |\rho_i|) + \sum_i |\rho_i| \left(\sqrt{E[\nu_i^2] \tilde{R}_i^2} - \sqrt{\tilde{R}_{\Delta i}^2 E[\eta_i^2]} \right)^2 \\ & \geq \sum_i \left(E[\nu_i^2] \tilde{R}_i^2 + \tilde{R}_{\Delta i}^2 E[\eta_i^2] \right) (1 - |\rho_i|) \geq \underline{c} \sum_i \left(\tilde{R}_i^2 + \tilde{R}_{\Delta i}^2 \right), \end{aligned}$$

and, similarly, the second line is:

$$\begin{aligned} & \sum_i \sum_{j \neq i} G_{ij}^2 E[\nu_i^2] E[\eta_j^2] + \sum_i \sum_{j \neq i} G_{ij} G_{ji} E[\eta_i \nu_i] E[\eta_j \nu_j] \\ & = \frac{1}{2} \sum_i \sum_{j \neq i} G_{ij}^2 E[\nu_i^2] E[\eta_j^2] (1 - \rho_i^2) + \frac{1}{2} \sum_i \sum_{j \neq i} G_{ji}^2 E[\nu_j^2] E[\eta_i^2] (1 - \rho_j^2) \\ & \quad + \frac{1}{2} \sum_i \sum_{j \neq i} E[\nu_i^2] E[\eta_j^2] (G_{ij}^2 \rho_i^2 + G_{ji}^2 \rho_j^2) + \sum_i \sum_{j \neq i} G_{ij} G_{ji} \rho_i \rho_j \sqrt{E[\nu_i^2] E[\eta_j^2]} \sqrt{E[\nu_j^2] E[\eta_i^2]} \\ & \geq \frac{1}{2} \sum_i \sum_{j \neq i} G_{ij}^2 E[\nu_i^2] E[\eta_j^2] (1 - \rho_i^2) + \frac{1}{2} \sum_i \sum_{j \neq i} G_{ji}^2 E[\nu_j^2] E[\eta_i^2] (1 - \rho_j^2) \geq \underline{c} \sum_i \sum_{j \neq i} G_{ij}^2. \end{aligned}$$

Consequently,

$$V_{LM} \succeq \sum_i \tilde{R}_i^2 + \sum_i \tilde{R}_{\Delta i}^2 + \sum_i \sum_{j \neq i} G_{ij}^2 =: r_n. \quad (16)$$

Hence, since $r_n \rightarrow \infty$ by Assumption 1(d), V_{LM} also diverges. By repeated application of the Cauchy-Schwarz inequality, it suffices to show that the variance of each of the 5 A terms above has order at most r_n (i.e., bounded by any of the three terms in Equation (16)). If this is true, then since the denominator has order at least r_n^2 , the

variance estimator is consistent. The A1 and A2 terms have the form:

$$\begin{aligned}
\sum_i \sum_{j \neq i} G_{ij} F_{ik} V_{1j} \sum_{k \neq i} V_{2k} V_{3i} (V_{4i} - Q'_i \hat{\tau}_{4,-ijk}) &= \sum_i \sum_{j \neq i} \sum_{k \neq i} \sum_{l \neq j,k} G_{ij} F_{ik} V_{1j} V_{2k} V_{3i} \check{M}_{il,-ijk} V_{4l} \\
&= \sum_i \sum_{j \neq i} \sum_{k \neq i,j} \sum_{l \neq i,j,k} G_{ij} F_{ik} \check{M}_{il,-ijk} V_{1j} V_{2k} V_{3i} V_{4l} + \sum_i \sum_{j \neq i} \sum_{k \neq i,j} G_{ij} F_{ik} V_{1j} V_{2k} V_{3i} V_{4i} \\
&\quad + \sum_i \sum_{j \neq i} \sum_{l \neq i,j} G_{ij} F_{ij} \check{M}_{il,-ij} V_{1j} V_{2j} V_{3i} V_{4l} + \sum_i \sum_{j \neq i} G_{ij} F_{ij} V_{1j} V_{2j} V_{3i} V_{4i}.
\end{aligned}$$

In particular, A1 uses $F = G, V_1 = X, V_2 = X, V_3 = e, V_4 = e$, while A2 uses $F = G', V_1 = X, V_2 = e, V_3 = e, V_4 = X$. By applying the Cauchy-Schwarz inequality, it suffices to show that the variance of each of the sums has order at most r_n . The terms $\sum_i \sum_{j \neq i} G_{ij} F_{ij} V_{1j} V_{2j} V_{3i} V_{4i}$ and $\sum_i \sum_{j \neq i} \sum_{l \neq i,j} G_{ij} F_{ij} \check{M}_{il,-ij} V_{1j} V_{2j} V_{3i} V_{4l}$ are identical to the result in Lemma 4, with the latter result being obtained by switching the i and j indices. The remaining terms have a variance that has a bounded order by Lemma 5. For A3, we can use G_{ji} in place of G_{ij} above, and use $F = G', V_1 = e, V_2 = e, V_3 = X, V_4 = X$ so that the order is bounded above due to Lemma 6. A4 and A5 can be written as:

$$\begin{aligned}
\sum_i \sum_{j \neq i} G_{ji} F_{ij} V_{1i} \sum_{k \neq j} \check{M}_{ik,-ij} V_{2k} V_{3j} (V_{4j} - Q'_j \hat{\tau}_{4,-ijk}) &= \sum_i \sum_{j \neq i} \sum_{k \neq j} \sum_{l \neq i,k} G_{ji} F_{ij} V_{1i} \check{M}_{ik,-ij} V_{2k} V_{3j} \check{M}_{jl,-ijk} V_{4l} \\
&= \sum_i \sum_{j \neq i} \sum_{k \neq i,j} \sum_{l \neq i,j,k} G_{ji} F_{ij} \check{M}_{ik,-ij} \check{M}_{jl,-ijk} V_{1i} V_{2k} V_{3j} V_{4l} + \sum_i \sum_{j \neq i} \sum_{k \neq i,j} G_{ji} F_{ij} \check{M}_{ik,-ij} V_{1i} V_{2k} V_{3j} V_{4j} \\
&\quad + \sum_i \sum_{j \neq i} \sum_{l \neq i,j} G_{ji} F_{ij} \check{M}_{jl,-ij} V_{1i} V_{2i} V_{3j} V_{4l} + \sum_i \sum_{j \neq i} G_{ji} F_{ij} V_{1i} V_{2i} V_{3j} V_{4j}.
\end{aligned}$$

In particular, A4 uses $F = G', V_1 = X, V_2 = X, V_3 = e, V_4 = e$, while A5 uses $F = G, V_1 = e, V_2 = X, V_3 = e, V_4 = X$. By applying the Cauchy-Schwarz inequality, it suffices to show that the variance of each of the sums has order at most r_n . This result is immediate from Lemma 4 and Lemma 6. \square

Proof of Proposition 1. Let $\mu \in \mathcal{M} = \{\mu : \mu_1 > 0, \mu_3 > 0, \mu_2^2 < \mu_1 \mu_3\}$. I first show that \mathcal{M} is convex. For $\lambda \in (0, 1)$, it suffices to show, for μ_a and μ_b that satisfy $\mu_{2a}^2 < \mu_{1a} \mu_{3a}$ and $\mu_{2b}^2 < \mu_{1b} \mu_{3b}$, that $(\lambda \mu_{2a} + (1 - \lambda) \mu_{2b})^2 < (\lambda \mu_{1a} + (1 - \lambda) \mu_{1b}) (\lambda \mu_{3a} + (1 - \lambda) \mu_{3b})$.

This set is intersected with the set that satisfies $\mu_1 > 0$ and $\mu_3 > 0$, which is clearly convex. The following is negative:

$$\begin{aligned} & (\lambda\mu_{2a} + (1-\lambda)\mu_{2b})^2 - (\lambda\mu_{1a} + (1-\lambda)\mu_{1b})(\lambda\mu_{3a} + (1-\lambda)\mu_{3b}) \\ &= \lambda^2(\mu_{2a}^2 - \mu_{1a}\mu_{3a}) + (1-\lambda)^2(\mu_{2b}^2 - \mu_{1b}\mu_{3b}) + \lambda(1-\lambda)(2\mu_{2a}\mu_{2b} - \mu_{1b}\mu_{3a} - \mu_{1a}\mu_{3b}) \\ &< \lambda(1-\lambda)(2\sqrt{\mu_{1a}\mu_{1b}\mu_{1b}\mu_{3b}} - \mu_{1b}\mu_{3a} - \mu_{1a}\mu_{3b}) < -\lambda(1-\lambda)(\sqrt{\mu_{1b}\mu_{3a}} - \sqrt{\mu_{1a}\mu_{3b}})^2 \leq 0. \end{aligned}$$

The first inequality occurs from applying $\mu_{2a}^2 < \mu_{1a}\mu_{3a}$ and $\mu_{2b}^2 < \mu_{1b}\mu_{3b}$, so \mathcal{M} is convex. Let $m \sim N(\mu, \Sigma)$ denote a statistic drawn from the asymptotic distribution, with m_i being a component of the vector m , so that m_2 is the LM statistic. Using the linear transformation from [Lehmann and Romano \(2005\)](#) Example 3.9.2 Case 3, we can transform the statistics and parameter such that m_2 is orthogonal to all other components. In particular, consider the following transformation L :

$$L := \begin{pmatrix} \sqrt{\frac{\sigma_{22}}{\sigma_{11}\sigma_{22}-\sigma_{12}^2}} & -\frac{\sigma_{12}}{\sigma_{22}}\sqrt{\frac{\sigma_{22}}{\sigma_{11}\sigma_{22}-\sigma_{12}^2}} & 0 \\ 0 & \frac{1}{\sqrt{\sigma_{22}}} & 0 \\ 0 & -\frac{\sigma_{23}}{\sigma_{22}}\sqrt{\frac{\sigma_{22}}{\sigma_{33}\sigma_{22}-\sigma_{23}^2}} & \sqrt{\frac{\sigma_{22}}{\sigma_{33}\sigma_{22}-\sigma_{23}^2}} \end{pmatrix}.$$

Then,

$$Lm \sim N \left(L\mu, \begin{pmatrix} 1 & 0 & \frac{\sigma_{13}\sigma_{22}-\sigma_{12}\sigma_{23}}{(\sigma_{11}\sigma_{22}-\sigma_{12}^2)(\sigma_{33}\sigma_{22}-\sigma_{23}^2)} \\ 0 & 1 & 0 \\ \frac{\sigma_{13}\sigma_{22}-\sigma_{12}\sigma_{23}}{(\sigma_{11}\sigma_{22}-\sigma_{12}^2)(\sigma_{33}\sigma_{22}-\sigma_{23}^2)} & 0 & 1 \end{pmatrix} \right).$$

The parameter space of $L\mu \in \mathcal{L}$ is also convex because L is a linear transformation: take any $\mu_a, \mu_b \in \mathcal{M}$, then observe that $\lambda L\mu_a + (1-\lambda)L\mu_b = L(\lambda\mu_a + (1-\lambda)\mu_b)$. Since \mathcal{M} is convex, and every element in \mathcal{M} is linearly transformed into the space on \mathcal{L} , we have $\lambda\mu_a + (1-\lambda)\mu_b \in \mathcal{M}$ and hence $L(\lambda\mu_a + (1-\lambda)\mu_b) \in \mathcal{L}$. Since Lm is normally distributed and \mathcal{L} is convex with rank 3, the problem is in the exponential class, using the definition from [Lehmann and Romano \(2005\)](#) Section 4.4. Since the joint distribution is in the exponential class and the restriction to the interior ensures that there are points in the parameter space that are above and below the null, the uniformly most powerful unbiased test follows the form

of [Lehmann and Romano \(2005\)](#) Theorem 4.4.1(iv), by using $U = m_2$ and $T = \left(\sqrt{\frac{\sigma_{22}}{\sigma_{33}\sigma_{22}-\sigma_{23}^2}}m_3 - \frac{\sigma_{23}}{\sqrt{\sigma_{22}(\sigma_{33}\sigma_{22}-\sigma_{23}^2)}}m_2, \sqrt{\frac{\sigma_{22}}{\sigma_{11}\sigma_{22}-\sigma_{12}^2}}m_1 - \frac{\sigma_{12}}{\sqrt{\sigma_{22}(\sigma_{11}\sigma_{22}-\sigma_{12}^2)}}m_2 \right)'$ in their notation. To calculate the critical values of the [Lehmann and Romano \(2005\)](#) Theorem 4.4.1(iv) result, observe that $[Lm]_2$ is orthogonal to $[Lm]_1$ and $[Lm]_3$, so the distribution of $[Lm]_2$ conditional on $[Lm]_1$ and $[Lm]_3$ is standard normal. Since $[Lm]_2$ is standard normal, it is symmetric around 0 under the null, so the solution to the critical value is ± 1.96 for a 5% test, due to simplification in [Lehmann and Romano \(2005\)](#) Section 4.2. The resulting test is hence identical to the two-sided LM test. \square

References

- AGAN, A., J. L. DOLEAC, AND A. HARVEY (2023): “Misdemeanor prosecution,” *The Quarterly Journal of Economics*, 138, 1453–1505.
- ANATOLYEV, S. AND M. SØLVSTEN (2023): “Testing many restrictions under heteroskedasticity,” *Journal of Econometrics*, 236, 105473.
- ANDERSON, T. W. AND H. RUBIN (1949): “Estimation of the parameters of a single equation in a complete system of stochastic equations,” *The Annals of mathematical statistics*, 20, 46–63.
- ANDREWS, D. W., V. MARMER, AND Z. YU (2019): “On optimal inference in the linear IV model,” *Quantitative Economics*, 10, 457–485.
- ANDREWS, D. W., M. J. MOREIRA, AND J. H. STOCK (2006): “Optimal two-sided invariant similar tests for instrumental variables regression,” *Econometrica*, 74, 715–752.
- ANDREWS, I. (2016): “Conditional linear combination tests for weakly identified models,” *Econometrica*, 84, 2155–2182.
- ANGRIST, J. D., G. W. IMBENS, AND A. B. KRUEGER (1999): “Jackknife instrumental variables estimation,” *Journal of Applied Econometrics*, 14, 57–67.

- ANGRIST, J. D. AND A. B. KRUEGER (1991): “Does compulsory school attendance affect schooling and earnings?” *The Quarterly Journal of Economics*, 106, 979–1014.
- AUTOR, D., A. KOSTØL, M. MOGSTAD, AND B. SETZLER (2019): “Disability benefits, consumption insurance, and household labor supply,” *American Economic Review*, 109, 2613–2654.
- BLANDHOL, C., J. BONNEY, M. MOGSTAD, AND A. TORGOVITSKY (2022): “When is TSLS actually late?” Tech. rep., National Bureau of Economic Research.
- BOOT, T. AND D. NIBBERING (2024): “Inference on LATEs with covariates,” *arXiv preprint arXiv:2402.12607*.
- CATTANEO, M. D., M. JANSSON, AND W. K. NEWHEY (2018): “Inference in linear regression models with many covariates and heteroscedasticity,” *Journal of the American Statistical Association*, 113, 1350–1361.
- CHAO, J. C., N. R. SWANSON, J. A. HAUSMAN, W. K. NEWHEY, AND T. WOUTERSEN (2012): “Asymptotic distribution of JIVE in a heteroskedastic IV regression with many instruments,” *Econometric Theory*, 28, 42–86.
- CHAO, J. C., N. R. SWANSON, AND T. WOUTERSEN (2023): “Jackknife estimation of a cluster-sample IV regression model with many weak instruments,” *Journal of Econometrics*, 235, 1747–1769.
- CRUDU, F., G. MELLACE, AND Z. SÁNDOR (2021): “Inference in instrumental variable models with heteroskedasticity and many instruments,” *Econometric Theory*, 37, 281–310.
- DOYLE, J. J. (2007): “Child protection and child outcomes: Measuring the effects of foster care,” *American Economic Review*, 97, 1583–1610.
- DUFOUR, J.-M. (1997): “Some impossibility theorems in econometrics with applications to structural and dynamic models,” *Econometrica: Journal of the Econometric Society*, 1365–1387.

- ELLIOTT, G., U. K. MÜLLER, AND M. W. WATSON (2015): “Nearly optimal tests when a nuisance parameter is present under the null hypothesis,” *Econometrica*, 83, 771–811.
- EVDOKIMOV, K. S. AND M. KOLESÁR (2018): “Inference in Instrumental Variables Analysis with Heterogeneous Treatment Effects,” *Working Paper*.
- HALL, A. R. AND A. INOUE (2003): “The large sample behaviour of the generalized method of moments estimator in misspecified models,” *Journal of Econometrics*, 114, 361–394.
- HECKMAN, J. J. AND E. VYTLACIL (2005): “Structural equations, treatment effects, and econometric policy evaluation 1,” *Econometrica*, 73, 669–738.
- IMBENS, G. W. AND J. D. ANGRIST (1994): “Identification and Estimation of Local Average Treatment Effects,” *Econometrica*, 62, 467–475.
- KLEIBERGEN, F. AND Z. ZHAN (2025): “Double robust inference for continuous updating GMM,” *Quantitative Economics*, 16, 295–327.
- KLING, J. R. (2006): “Incarceration length, employment, and earnings,” *American Economic Review*, 96, 863–876.
- KOLESÁR, M. (2013): “Estimation in an Instrumental Variables Model With Treatment Effect Heterogeneity,” Working Papers 2013-2, Princeton University. Economics Department.
- LEE, D. S., J. MCCRARY, M. J. MOREIRA, J. R. PORTER, AND L. YAP (2023): “What to do when you can’t use ‘1.96’ Confidence Intervals for IV,” Working Paper 31893, National Bureau of Economic Research.
- LEE, S. (2018): “A consistent variance estimator for 2SLS when instruments identify different LATEs,” *Journal of Business & Economic Statistics*, 36, 400–410.
- LEHMANN, E. L. AND J. P. ROMANO (2005): *Testing statistical hypotheses*, vol. 3, Springer.

- LIM, D., W. WANG, AND Y. ZHANG (2024): “A conditional linear combination test with many weak instruments,” *Journal of Econometrics*, 238, 105602.
- MATSUSHITA, Y. AND T. OTSU (2022): “A jackknife Lagrange multiplier test with many weak instruments,” *Econometric Theory*, 1–24.
- MIKUSHEVA, A. AND L. SUN (2022): “Inference with many weak instruments,” *The Review of Economic Studies*, 89, 2663–2686.
- MOREIRA, M. J. (2003): “A conditional likelihood ratio test for structural models,” *Econometrica*, 71, 1027–1048.
- (2009a): “A Maximum Likelihood Method for the Incidental Parameter Problem,” *The Annals of Statistics*, 37, 3660–3696.
- (2009b): “Tests with correct size when instruments can be arbitrarily weak,” *Journal of Econometrics*, 152, 131–140.
- SŁOCZYŃSKI, T. (2020): “When should we (not) interpret linear iv estimands as late?” *arXiv preprint arXiv:2011.06695*.
- STAIGER, D. AND J. H. STOCK (1997): “Instrumental Variables Regression with Weak Instruments,” *Econometrica*, 65, 557–586.
- STOCK, J. H. AND M. YOGO (2005): “Testing for Weak Instruments in Linear IV Regression,” in *Identification and Inference in Econometric Models: Essays in Honor of Thomas J. Rothenberg*, ed. by D. W. Andrews and J. H. Stock, Cambridge University Press, chap. 5, 80–108.
- VAN DE SIJPE, N. AND F. WINDMEIJER (2023): “On the power of the conditional likelihood ratio and related tests for weak-instrument robust inference,” *Journal of Econometrics*, 235, 82–104.

D Online Appendix

D.1 Data Generating Process

Data is generated from an environment with $E[\varepsilon_i] = 0$, and $\int_0^1 f(v)dv = \beta$. To run a regression on judge indicators (without an intercept) in the reduced-form system, I make a transformation $\check{X} = 2X - 1$ so that the reduced-form equations can be written as:

$$\check{X}_i = Z'_i \pi + \eta_i, \text{ and } Y_i = Z'_i \pi_Y + \zeta_i,$$

so $\pi_k = \pi_{Yk} = 0$ for the base judge. The reduced-form errors are: $\eta_i = I\{\lambda_{k(i)} - v_i \geq 0\} - \pi_{k(i)}$ and $\zeta_i = I\{\lambda_{k(i)} - v_i \geq 0\} f(v_i) + \varepsilon_i - \pi_{Yk(i)}$ respectively. With $\pi_{\Delta k} = \pi_{Yk} - \pi_k \beta$, the reduced-form parameters for the groups of judges are derived in Table 6. Since the coefficient of the base judge is normalized to zero, the implementation without covariates in simulations excludes the intercept and uses indicators for all judges, instead of omitting the base judge and having an intercept. This implementation results in a block diagonal projection matrix, which aids computational speed, while retaining the interpretation of π 's in the reduced-form model. The $f(v)$ that delivers the parameters in Table 6 is

$$f(v) = \begin{cases} -s\beta + h & v \in [0, \frac{1}{2} - s] \\ \frac{1}{s}(1-s)(-\frac{1}{2}s\beta - h) - \frac{1}{s}(1-2s)(-s\beta + h) & v \in (\frac{1}{2} - s, \frac{1}{2} - \frac{1}{2}s] \\ \frac{1}{s}(1-s)(\frac{1}{2}s\beta + h) & v \in (\frac{1}{2} - \frac{1}{2}s, \frac{1}{2}] \\ \frac{1}{s}(1+s)(\frac{1}{2}s\beta - h) & v \in (\frac{1}{2}, \frac{1}{2} + \frac{1}{2}s] \\ \frac{1}{s}(1+2s)(s\beta + h) - \frac{1}{s}(1+s)(\frac{1}{2}s\beta - h) & v \in (\frac{1}{2} + \frac{1}{2}s, \frac{1}{2} + s] \\ \frac{\beta - (\frac{1}{2}+s)(s\beta+h)}{\frac{1}{2}-s} & v \in (\frac{1}{2} + s, 1] \end{cases}. \quad (17)$$

To generate the data in the simulation, I draw $v_i \sim U[0, 1]$ as implied by the structural model, then generate $\zeta_i \mid v_i \sim N(\sigma_{\varepsilon v} v_i, \sigma_{\varepsilon \varepsilon})$. Hence, $\sigma_{\varepsilon v}$ and $\sigma_{\varepsilon \varepsilon}$ control the correlation between η_i and ζ_i , with $\sigma_{\varepsilon \varepsilon} = 0$ corresponding to perfect correlation. In the base case, I set $\sigma_{\varepsilon \varepsilon} = 0.1$ and $\sigma_{\varepsilon v} = 0.3$. With the given π_k, π_{Yk} , the observable variables are generated from $\check{X}_i = I\{\pi_{k(i)} > v_i\}$ and $Y_i = \pi_{Yk(i)} + \zeta_i$.

Table 6: Parameters for Simple Example

| | | | | | |
|------------------|-----------------------|------------------------------|---------------|------------------------------|-----------------------|
| λ_k | $\frac{1}{2} - s$ | $\frac{1}{2} - \frac{1}{2}s$ | $\frac{1}{2}$ | $\frac{1}{2} + \frac{1}{2}s$ | $\frac{1}{2} + s$ |
| β_k | $\beta - \frac{h}{s}$ | $\beta + 2\frac{h}{s}$ | NA | $\beta - 2\frac{h}{s}$ | $\beta + \frac{h}{s}$ |
| π_k | $-s$ | $-\frac{1}{2}s$ | 0 | $\frac{1}{2}s$ | s |
| π_{Yk} | $-s\beta + h$ | $-\frac{1}{2}s\beta - h$ | 0 | $\frac{1}{2}s\beta - h$ | $s\beta + h$ |
| $\pi_{\Delta k}$ | h | $-h$ | 0 | $-h$ | h |

D.2 Comparing Variance Estimands

Derivations for Constructed Instrument Using the notation for the just-identified IV AR test in Section 2.4 when $\beta = \beta_0$,

$$\begin{aligned}
\hat{\varepsilon}_i &= e_i - \tilde{X}_i \frac{\sum_i e_i \tilde{X}_i}{\sum_i \tilde{X}_i^2} = \frac{e_i \sum_i \tilde{X}_i^2 - \tilde{X}_i \sum_i e_i \tilde{X}_i}{\sum_i \tilde{X}_i^2}, \text{ and} \\
\hat{V} &= \frac{\sum_i \tilde{X}_i^2 \hat{\varepsilon}_i^2}{\left(\sum_i \tilde{X}_i^2\right)^2} = \frac{\sum_i \tilde{X}_i^2 \left(e_i \sum_j \tilde{X}_j^2 - \tilde{X}_i \sum_j e_j \tilde{X}_j\right)^2}{\left(\sum_i \tilde{X}_i^2\right)^4} \\
&= \frac{\sum_i \tilde{X}_i^2 e_i^2 \left(\sum_j \tilde{X}_j^2\right)^2 + \sum_i \tilde{X}_i^4 \left(\sum_j e_j \tilde{X}_j\right)^2 - 2 \sum_i \tilde{X}_i^3 e_i \left(\sum_j \tilde{X}_j^2\right) \left(\sum_j e_j \tilde{X}_j\right)}{\left(\sum_i \tilde{X}_i^2\right)^4}.
\end{aligned}$$

Applying the asymptotic result that $\frac{1}{n} \sum_j e_j \tilde{X}_j \xrightarrow{P} 0$ from Theorem 1,

$$\begin{aligned}
t_{AR}^2 &= \frac{\frac{(\sum_i \tilde{X}_i e_i)^2}{(\sum_i \tilde{X}_i^2)^2}}{\frac{\sum_i \tilde{X}_i^2 e_i^2 (\sum_j \tilde{X}_j^2)^2 + \sum_i \tilde{X}_i^4 (\sum_j e_j \tilde{X}_j)^2 - 2 \sum_i \tilde{X}_i^3 e_i (\sum_j \tilde{X}_j^2) (\sum_j e_j \tilde{X}_j)}{(\sum_i \tilde{X}_i^2)^4}} \\
&= \frac{\left(\frac{1}{\sqrt{n}} \sum_i \tilde{X}_i e_i\right)^2 \left(\frac{1}{n} \sum_i \tilde{X}_i^2\right)^2}{\frac{1}{n} \sum_i \tilde{X}_i^2 e_i^2 \left(\frac{1}{n} \sum_j \tilde{X}_j^2\right)^2 + \frac{1}{n} \sum_i \tilde{X}_i^4 \left(\frac{1}{n} \sum_j e_j \tilde{X}_j\right)^2 - 2 \frac{1}{n} \sum_i \tilde{X}_i^3 e_i \left(\frac{1}{n} \sum_j \tilde{X}_j^2\right) \left(\frac{1}{n} \sum_j e_j \tilde{X}_j\right)} \\
&= \frac{\left(\frac{1}{\sqrt{n}} \sum_i \tilde{X}_i e_i\right)^2 \left(\frac{1}{n} \sum_i \tilde{X}_i^2\right)^2}{\frac{1}{n} \sum_i \tilde{X}_i^2 e_i^2 \left(\frac{1}{n} \sum_j \tilde{X}_j^2\right)^2 + o_P(1)} = \frac{\left(\frac{1}{\sqrt{n}} \sum_i \tilde{X}_i e_i\right)^2}{\frac{1}{n} \sum_i \tilde{X}_i^2 e_i^2} + o_P(1), \text{ and}
\end{aligned}$$

$$\begin{aligned}
E \left[\sum_i \tilde{X}_i^2 e_i^2 \right] &= \sum_i E \left[\left(\sum_{j \neq i} P_{ij} (R_j + \eta_j) \right)^2 (R_{\Delta i} + \nu_i)^2 \right] \\
&= \sum_i E \left[\left(M_{ii}^2 R_i^2 + \left(\sum_{j \neq i} P_{ij}^2 \eta_j^2 \right) \right) (R_{\Delta i}^2 + \nu_i^2) \right] \\
&= \sum_i \left(M_{ii}^2 R_i^2 R_{\Delta i}^2 + \sum_{j \neq i} P_{ij}^2 R_{\Delta i}^2 E [\eta_j^2] + M_{ii}^2 R_i^2 E [\nu_i^2] + \sum_{j \neq i} P_{ij}^2 E [\nu_i^2] E [\eta_j^2] \right).
\end{aligned}$$

Clustering by Judges. If we just use the JIVE t-ratio with clustering, weak identification is still a problem, and we should similarly get over-rejection. The just-identified AR with clustered standard errors uses the following estimator:

$$\hat{V}_{clus} = \frac{\sum_i \sum_{j \in \mathcal{N}_i} \tilde{X}_i \hat{\varepsilon}_i \tilde{X}_j \hat{\varepsilon}_j}{\left(\sum_i \tilde{X}_i^2 \right)^2},$$

where \mathcal{N}_i is the neighborhood of i (i.e., the set of observations that share the same cluster as i). Expanding \hat{V}_{clus} using the same steps as before,

$$\hat{V}_{clus} = \frac{\sum_i \sum_{j \in \mathcal{N}_i} \tilde{X}_i \tilde{X}_j \left(e_i e_j \left(\sum_k \tilde{X}_k^2 \right)^2 - 2 \tilde{X}_i e_j \left(\sum_k e_k \tilde{X}_k \right) \left(\sum_k \tilde{X}_k^2 \right) + \tilde{X}_i \tilde{X}_j \left(\sum_k e_k \tilde{X}_k \right)^2 \right)}{\left(\sum_i \tilde{X}_i^2 \right)^4}.$$

Using the fact that $\frac{1}{n} \sum_k e_k \tilde{X}_k = o_P(1)$, the dominant term is: $\sum_i \sum_{j \in \mathcal{N}_i} \tilde{X}_i \tilde{X}_j e_i e_j$, which is analogous to the previous derivation. The expansion steps are analogous to that required to derive V_{LM} , so they are omitted. Then,

$$\begin{aligned}
&E \left[\sum_i \sum_{j \in \mathcal{N}_i} \tilde{X}_i \tilde{X}_j e_i e_j \right] \\
&= E \left[\sum_i \sum_{j \in \mathcal{N}_i} \left(\sum_{k \neq i} P_{ik} (R_k + \eta_k) \right) \left(\sum_{k \neq j} P_{jk} (R_k + \eta_k) \right) (R_{\Delta i} + \nu_i) (R_{\Delta j} + \nu_j) \right] \\
&= E \left[\sum_i \sum_{j \in \mathcal{N}_i} \left(M_{ii} R_i + \sum_{k \neq i} P_{ik} \eta_k \right) \left(M_{jj} R_j + \sum_{k \neq j} P_{jk} \eta_k \right) (R_{\Delta i} R_{\Delta j} + \nu_i R_{\Delta j} + R_{\Delta i} \nu_j + \nu_i \nu_j) \right] \\
&= \sum_i \sum_{j \in \mathcal{N}_i} \left(M_{ii} M_{jj} R_i R_j R_{\Delta i} R_{\Delta j} + R_{\Delta i} R_{\Delta j} \sum_{k \neq i, j} P_{ik} P_{jk} E [\eta_k^2] \right)
\end{aligned}$$

$$\begin{aligned}
& + 2 \sum_i \sum_{j \in \mathcal{N}_i} M_{ii} R_i R_{\Delta j} P_{ji} E[\eta_i \nu_i] + \sum_i M_{ii}^2 R_i^2 E[\nu_i^2] \\
& + \sum_i \sum_{k \neq i} P_{ik}^2 E[\nu_i^2] E[\eta_k^2] + \sum_i \sum_{j \in \mathcal{N}_i \setminus \{i\}} P_{ij}^2 E[\nu_i \eta_i] E[\nu_j \eta_j].
\end{aligned}$$

By applying the fact that the entries of the projection matrix are nonzero only when the observations share the same judge, the expression simplifies further:

$$\begin{aligned}
E \left[\sum_i \sum_{j \in \mathcal{N}_i} \tilde{X}_i \tilde{X}_j e_i e_j \right] & = \sum_i \sum_{j \in \mathcal{N}_i} M_{ii} M_{jj} R_i R_j R_{\Delta i} R_{\Delta j} + \sum_i M_{ii}^2 R_{\Delta i}^2 E[\eta_i^2] \\
& + 2 \sum_i M_{ii} R_i R_{\Delta i} E[\eta_i \nu_i] + \sum_i M_{ii}^2 R_i^2 E[\nu_i^2] \\
& + \sum_i \sum_{j \neq i} P_{ij}^2 (E[\nu_i^2] E[\eta_j^2] + E[\nu_i \eta_i] E[\nu_j \eta_j]).
\end{aligned}$$

Compared to the true variance in Equation (19), due to the own-observation bias, we have an extra $\sum_i \sum_{j \in \mathcal{N}_i} M_{ii} M_{jj} R_i R_j R_{\Delta i} R_{\Delta j}$ term, and the estimand here has $\sum_i M_{ii} R_i R_{\Delta i} E[\eta_i \nu_i]$ instead of $\sum_i M_{ii}^2 R_i R_{\Delta i} E[\eta_i \nu_i]$. Even though $\sum_i \sum_{j \in \mathcal{N}_i} M_{ii} M_{jj} R_i R_j R_{\Delta i} R_{\Delta j} \geq 0$, $\sum_i M_{ii} (1 - M_{ii}) R_i R_{\Delta i} E[\eta_i \nu_i]$ could be positive or negative, so the clustered variance estimand could either over or underestimate the true variance.

Comparing MO Variance Estimand with L3O. The Matsushita and Otsu (2022) variance estimator presented in Equation (4) is biased in general. The model of Section 2.1 implies:

$$\begin{aligned}
E[\hat{\Psi}_{MO}] & = \sum_i M_{ii}^2 R_i^2 R_{\Delta i}^2 + \sum_i M_{ii}^2 R_i^2 E[\nu_i^2] + \sum_i \sum_{j \neq i} P_{ij}^2 E[\nu_i^2] E[\eta_j^2] + \sum_i \sum_{j \neq i} P_{ij}^2 R_{\Delta i}^2 E[\eta_j^2] \\
& + \sum_i \sum_{j \neq i} P_{ij}^2 (R_i R_{\Delta i} R_j R_{\Delta j} + E[\eta_i \nu_i] R_j R_{\Delta j} + R_i R_{\Delta i} E[\eta_j \nu_j] + E[\eta_i \nu_i] E[\eta_j \nu_j]).
\end{aligned} \tag{18}$$

As a corollary of Equation (8), when $G = P$, by observing that P is symmetric, and

that since $PR = I$, we have $\sum_{j \neq i} P_{ij} R_j = \sum_{j \neq i} P_{ji} R_j = M_{ii} R_i$, so

$$\begin{aligned} \text{Var} \left(\sum_i \sum_{j \neq i} P_{ij} e_i X_j \right) &= \sum_i E [\nu_i^2] M_{ii}^2 R_i^2 + \sum_i \sum_{j \neq i} P_{ij}^2 (E [\nu_i^2] E [\eta_j^2] + E [\eta_i \nu_i] E [\eta_j \nu_j]) \\ &\quad + 2 \sum_i E [\nu_i \eta_i] M_{ii}^2 R_i R_{\Delta i} + \sum_i E [\eta_i^2] M_{ii}^2 R_{\Delta i}^2. \end{aligned} \tag{19}$$

If the R_{Δ} 's are zero, then $\hat{\Psi}_{MO}$ is unbiased. Nonetheless, heterogeneity results in many excess terms in the expectation of the variance estimator, generating bias and inconsistency in general. However, $\hat{\Psi}_{MO}$ can be consistent when forcing weak identification and weak heterogeneity. If it is assumed that $\frac{1}{\sqrt{K}} \sum_i M_{ii} R_i^2 \rightarrow C_S < \infty$ and $\frac{1}{\sqrt{K}} \sum_i M_{ii} R_{\Delta i}^2 \rightarrow C < \infty$ with weak identification and weak heterogeneity, then the excess terms in $\frac{1}{K} E [\hat{\Psi}_{MO}]$ can be written as $\frac{1}{\sqrt{K}} \frac{1}{\sqrt{K}} \sum_i M_{ii} R_i^2 = \frac{1}{\sqrt{K}} O(1) = o(1)$ and $\frac{1}{\sqrt{K}} \frac{1}{\sqrt{K}} \sum_i M_{ii} R_{\Delta i}^2 = o(1)$. However, when identification or heterogeneity is strong, $\frac{1}{K} \sum_i M_{ii} R_i^2$ or $\frac{1}{K} \sum_i M_{ii} R_{\Delta i}^2$ is nonnegligible and the variance estimator is inconsistent. The variance estimator adapted from MS22 has similar properties. In contrast, the L3O variance estimator is robust regardless of whether the identification is weak or strong.

In general, heterogeneity does not make the MO variance estimator any more or less conservative than L3O. In the simple case with judge instruments and $G = P$, we have:

$$\begin{aligned} E [\hat{\Psi}_{MO}] - \text{Var} \left(\sum_i \sum_{j \neq i} P_{ij} e_i X_j \right) &= \sum_i M_{ii}^2 R_i^2 R_{\Delta i}^2 + \sum_i \sum_{j \neq i} P_{ij}^2 R_i R_{\Delta i} R_j R_{\Delta j} \\ &\quad + \sum_i \sum_{j \neq i} P_{ij}^2 R_{\Delta i}^2 E [\eta_j^2] - \sum_i M_{ii}^2 R_{\Delta i}^2 E [\eta_i^2] + 2 \sum_i \sum_{j \neq i} P_{ij}^2 E [\eta_i \nu_i] R_j R_{\Delta j} \\ &\quad - 2 \sum_i E [\nu_i \eta_i] M_{ii}^2 R_i R_{\Delta i} \\ &= \sum_i M_{ii} R_{\Delta i}^2 R_i^2 - \sum_i M_{ii} (1 - 2P_{ii}) R_{\Delta i}^2 E [\eta_i^2] - 2 \sum_i M_{ii} (1 - 2P_{ii}) E [\eta_i \nu_i] R_i R_{\Delta i} \\ &= \sum_i M_{ii} R_{\Delta i}^2 (R_i^2 - (1 - 2P_{ii}) E [\eta_i^2]) - 2 \sum_i M_{ii} (1 - 2P_{ii}) E [\eta_i \nu_i] R_i R_{\Delta i} \end{aligned}$$

which can be positive or negative. The second equality uses the fact that P and M

are non-zero only for observations that share the same judge, and when that occurs, they have the same $R, R_Y, E[\eta_i^2]$, and $E[\zeta_i^2]$, and that $\sum_{j \neq i} P_{ij}^2 = P_{ii} M_{ii}$.

To compare the confidence sets of MO and L3O, observe that the shape of the confidence set depends on the coefficient on β_0^2 . In particular, for $\Psi_2 := \sum_i \left(\sum_{j \neq i} P_{ij} X_j \right)^2 X_i^2 + \sum_i \sum_{j \neq i} P_{ij}^2 X_i^2 X_j^2$, MO is unbounded when $\left(\sum_{i \neq j}^n P_{ij} X_i X_j \right)^2 - q \Psi_2 < 0$ and L3O is unbounded when $\left(\sum_{i \neq j}^n P_{ij} X_i X_j \right)^2 - q B_2 < 0$. In the simple judges case without covariates, the expected coefficients can be compared. With

$$E[\Psi_2] = \sum_i \left(\left(\sum_{j \neq i} P_{ij} R_j \right)^2 + \sum_{j \neq i} P_{ij}^2 E[\eta_j^2] \right) E[X_i^2] + \sum_i \sum_{j \neq i} P_{ij}^2 E[X_i^2] E[X_j^2],$$

the difference is:

$$\begin{aligned} E[\Psi_2] - E[B_2] &= \sum_i \left(\sum_{j \neq i} P_{ij} R_j \right)^2 R_i^2 + \sum_i \sum_{j \neq i} P_{ij}^2 (R_i^2 R_j^2 + 3E[\eta_i^2] R_j^2) \\ &\quad - 3 \sum_i \left(\sum_{j \neq i} P_{ij} R_j \right)^2 E[\eta_i^2] \\ &= \sum_i M_{ii} R_i^2 (R_i^2 - 3(1 - 2P_{ii}) E[\eta_i^2]), \end{aligned}$$

where the second equality uses the same trick as before.

Derivation of LM Variance.

Proof of Equation (8). Expanding the variance,

$$\begin{aligned} \text{Var} \left(\sum_i \sum_{j \neq i} G_{ij} e_i X_j \right) &= E \left[\left(\sum_i \sum_{j \neq i} G_{ij} e_i X_j \right)^2 \right] = E \left[\sum_i \sum_{j \neq i} \sum_k \sum_{l \neq k} G_{ij} e_i X_j G_{kl} e_k X_l \right] \\ &= \sum_i \sum_{j \neq i} \sum_k \sum_{l \neq k} G_{ij} G_{kl} E[\nu_i X_j \nu_k X_l] + \sum_i \sum_{j \neq i} \sum_k \sum_{l \neq k} G_{ij} G_{kl} E[\nu_i X_j R_{\Delta k} X_l] \\ &\quad + \sum_i \sum_{j \neq i} \sum_k \sum_{l \neq k} G_{ij} G_{kl} E[R_{\Delta i} X_j \nu_k X_l] + \sum_i \sum_{j \neq i} \sum_k \sum_{l \neq k} G_{ij} G_{kl} E[R_{\Delta i} X_j R_{\Delta k} X_l] \end{aligned}$$

The first term is:

$$\begin{aligned}
& \sum_i \sum_{j \neq i} \sum_k \sum_{l \neq k} G_{ij} G_{kl} E [\nu_i X_j \nu_k X_l] \\
&= \sum_i \sum_{j \neq i} \sum_k \sum_{l \neq k} G_{ij} G_{kl} E [\nu_i R_j \nu_k R_l + \nu_i \eta_j \nu_k R_l + \nu_i R_j \nu_k \eta_l + \nu_i \eta_j \nu_k \eta_l] \\
&= \sum_i \sum_{j \neq i} \sum_k \sum_{l \neq k} G_{ij} G_{kl} E [\nu_i R_j \nu_k R_l + \nu_i \eta_j \nu_k \eta_l] \\
&= \sum_i \sum_{j \neq i} \sum_{k \neq i} E [\nu_i^2] G_{ij} G_{ik} R_j R_k + \sum_i \sum_{j \neq i} \left(\sum_{l \neq i} G_{ij} G_{il} E [\nu_i \eta_j \nu_l \eta_l] + \sum_{l \neq j} G_{ij} G_{jl} E [\nu_i \eta_j \nu_l \eta_l] \right) \\
&\quad + \sum_i \sum_{j \neq i} \left(\sum_{k \neq i, j} G_{ij} G_{ki} E [\nu_i \eta_j \nu_k \eta_i] + \sum_{k \neq i, j} G_{ij} G_{kj} E [\nu_i \eta_j \nu_k \eta_j] \right) \\
&= \sum_i \sum_{j \neq i} \sum_{k \neq i} E [\nu_i^2] G_{ij} G_{ik} R_j R_k + \sum_i \sum_{j \neq i} (G_{ij}^2 E [\nu_i^2 \eta_j^2] + G_{ij} G_{ji} E [\nu_i \eta_i \eta_j \nu_j]) \\
&= \sum_i \sum_{j \neq i} \sum_{k \neq i} E [\nu_i^2] G_{ij} G_{ik} R_j R_k + \sum_i \sum_{j \neq i} (G_{ij}^2 E [\nu_i^2] E [\eta_j^2] + G_{ij} G_{ji} E [\nu_i \eta_i] E [\eta_j \nu_j])
\end{aligned}$$

In the next few terms, the expansion steps are analogous, so intermediate steps are omitted for brevity. The second to fourth terms can be expressed as:

$$\begin{aligned}
& \sum_i \sum_{j \neq i} \sum_k \sum_{l \neq k} G_{ij} G_{kl} E [\nu_i X_j R_{\Delta k} X_l] = \sum_i E [\nu_i \eta_i] \sum_{j \neq i} G_{ij} R_j \sum_{k \neq i} G_{ki} R_{\Delta k}, \\
& \sum_i \sum_{j \neq i} \sum_k \sum_{l \neq k} G_{ij} G_{kl} E [R_{\Delta i} X_j \nu_k X_l] = \sum_i \sum_{j \neq i} \sum_{l \neq i} G_{ji} G_{il} E [\eta_i \nu_i] R_{\Delta j} R_l, \text{ and} \\
& \sum_i \sum_{j \neq i} \sum_k \sum_{l \neq k} G_{ij} G_{kl} E [R_{\Delta i} X_j R_{\Delta k} X_l] = \sum_i E [\eta_i^2] \sum_{j \neq i} \sum_{k \neq i} G_{ji} G_{ki} R_{\Delta j} R_{\Delta k}.
\end{aligned}$$

The expression stated in the equation combines these expressions. \square

D.3 Further Details for Power

D.3.1 Further Analytic Results

Lemma 7. $(s'_1, s'_2)'$ are sufficient statistics for $(\pi'_Y, \pi')'$. Further, for transformations of the form $Z \rightarrow ZF'$ where F is a $K \times K$ orthogonal matrix, $(s'_1 s_1, s'_1 s_2, s'_2 s_2)$ is a

maximal invariant, and

$$\begin{pmatrix} s_1 \\ s_2 \end{pmatrix} \sim N \left(\begin{pmatrix} (Z'Z)^{1/2} \pi_Y \\ (Z'Z)^{1/2} \pi \end{pmatrix}, \Omega \otimes I_K \right).$$

Proposition 2. With Equation (12), $K \rightarrow \infty$ and $\frac{1}{\sqrt{K}} (\pi'_Y Z' Z \pi_Y, \pi' Z' Z \pi_Y, \pi' Z' Z \pi) \rightarrow (C_{YY}, C_Y, C_S)$,

$$\frac{1}{\sqrt{K}} \begin{pmatrix} s'_1 s_1 - K \omega_{\zeta\zeta} - C_{YY} \\ s'_1 s_2 - K \omega_{\zeta\eta} - C_Y \\ s'_2 s_2 - K \omega_{\eta\eta} - C_S \end{pmatrix} \xrightarrow{d} N \left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \Sigma \right) \quad (20)$$

for some variance matrix Σ . If $C_{YY}, C_Y, C_S < \infty$,

$$\Sigma = \begin{pmatrix} 2\omega_{\zeta\zeta}^2 & 2\omega_{\zeta\eta}\omega_{\zeta\zeta} & 2\omega_{\zeta\eta}^2 \\ 2\omega_{\zeta\eta}\omega_{\zeta\zeta} & \omega_{\zeta\zeta}\omega_{\eta\eta} + \omega_{\zeta\eta}^2 & 2\omega_{\zeta\eta}\omega_{\eta\eta} \\ 2\omega_{\zeta\eta}^2 & 2\omega_{\zeta\eta}\omega_{\eta\eta} & 2\omega_{\eta\eta}^2 \end{pmatrix}.$$

The proof of Proposition 2 relies on $K \rightarrow \infty$ because objects like $s'_1 s_1$ can be written as a sum of K objects. With an appropriate representation to obtain independence, a CLT can be applied to yield normality. Compared to MS22, Proposition 2 does not require constant treatment effects and characterizes the distribution without orthogonalizing the sufficient statistics. Nonetheless, the form of the covariance matrix is similar to MS22.

Instead of using the maximal invariant, we use the L1O analog

$$(T_{YY}, T_{YX}, T_{XX}) := \frac{1}{\sqrt{K}} \sum_i \sum_{j \neq i} P_{ij}(Y_i Y_j, Y_i X_j, X_i X_j),$$

which relates to the JIVE directly as $\hat{\beta}_{JIVE} = T_{YX}/T_{XX}$, so the asymptotic problem is:

$$\begin{pmatrix} T_{YY} \\ T_{YX} \\ T_{XX} \end{pmatrix} \sim N(\mu, \Sigma), \mu = \begin{pmatrix} \frac{1}{\sqrt{K}} \sum_i \sum_{j \neq i} P_{ij} R_{Yi} R_{Yj} \\ \frac{1}{\sqrt{K}} \sum_i \sum_{j \neq i} P_{ij} R_{Yi} R_j \\ \frac{1}{\sqrt{K}} \sum_i \sum_{j \neq i} P_{ij} R_i R_j \end{pmatrix}, \Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \cdot & \sigma_{22} & \sigma_{23} \\ \cdot & \cdot & \sigma_{33} \end{pmatrix}. \quad (21)$$

With abuse of notation, μ and Σ refer to the asymptotic mean and variances of (T_{YY}, T_{YX}, T_{XX}) instead of (T_{AR}, T_{LM}, T_{FS}) so that the statistics do not depend on the hypothesized null, but these are identical when $\beta_0 = 0$. Using the same argument as Section 4, $\mu_1, \mu_3 \geq 0$ and $\mu_2^2 \leq \mu_1\mu_3$.

Even with covariates, if the regression is fully saturated with G given by UJIVE, Proposition 3 below shows that the same inequality restrictions hold. To be precise, saturation is defined in Section 2 of [Evdokimov and Kolesár \(2018\)](#). All individuals can be partitioned into L covariate groups, so with group index $G_i \in \{1, \dots, L\}$, we have covariates $W_{i,l} = 1\{G_i = l\}$. We also have an instrument S_i that takes $M + 1$ possible values in each group, and these values for every group l are labeled s_{l0}, \dots, s_{lM} . Then, the vector of instruments has dimension $K = ML$ and $Z_{i,lm} = 1\{S_i = s_{lm}\}$. Adapting Equation (21) to the case with covariates,

$$\mu = \left(\frac{1}{\sqrt{K}} \sum_i \sum_{j \neq i} G_{ij} R_{Yi} R_{Yj}, \frac{1}{\sqrt{K}} \sum_i \sum_{j \neq i} G_{ij} R_{Yi} R_{Xj}, \frac{1}{\sqrt{K}} \sum_i \sum_{j \neq i} G_{ij} R_{Xi} R_{Xj} \right)'.$$

Proposition 3. *If μ is defined such that $G = (I - \text{diag}(H_Q))^{-1} H_Q - (I - \text{diag}(H_W))^{-1} H_W$, and the regression is fully saturated, then $\mu_1 \geq 0, \mu_3 \geq 0$ and $\mu_2^2 \leq \mu_1\mu_3$.*

Using the asymptotic problem of Equation (21), testing $H_0 : \mu_2/\mu_3 = \beta^*$ is identical to testing $H_0 : \mu_2 - \beta^*\mu_3 = 0$. Since β^* is fixed, and I consider alternatives of the form: $H_A : \mu_2 - \beta^*\mu_3 = h_A$. The LM statistic corresponds to $T_{YX} - \beta^*T_{XX}$, so it can be used to test the null directly. I focus on the most common case of $\beta^* = 0$, and it is analogous to extend the argument for $\beta^* \neq 0$. Let μ^A denote the mean under the alternative and μ^H under the null. The remainder of this section presents theoretical results for power, and numerical results beyond the environment covered by theory are relegated to Appendix D.3.3.

The one-sided test is the most powerful test for testing against a particular subset of alternatives $\mathcal{S} := \left\{ (\mu_1^A, \mu_2^A, \mu_3^A) : \mu_1^A - \frac{\sigma_{12}}{\sigma_{22}} \mu_2^A \geq 0, \mu_3^A - \frac{\sigma_{23}}{\sigma_{22}} \mu_2^A \geq 0 \right\}$. While \mathcal{S} may not be empirically interpretable, this set is constructed so that standard [Lehmann and Romano \(2005\)](#) arguments can be applied to conclude that the one-sided LM test is the most powerful test. The proposition makes no statement about alternative hypotheses that are not in \mathcal{S} . A more powerful test can be constructed when μ_2^A is large and covariance σ_{23}, σ_{12} are large.

Proposition 4. *The one-sided LM test is the most powerful test for testing any alternative hypothesis $(\mu_1^A, \mu_2^A, \mu_3^A) \in \mathcal{S}$ in the asymptotic problem of Equation (21).*

For a given $(\mu_1^A, \mu_2^A, \mu_3^A)$ in the alternative space, LM (which just uses the second element) is justified as being most powerful because it is identical to the Neyman-Pearson test when testing against a point null μ^H with $\mu_1^H = \mu_1^A - \frac{\sigma_{12}}{\sigma_{22}}\mu_2^A$, $\mu_2^H = 0$ and $\mu_3^H = \mu_3^A - \frac{\sigma_{23}}{\sigma_{22}}\mu_2^A$. The inequalities in \mathcal{S} are imposed so that $\mu_1^H, \mu_3^H \geq 0$, ensuring that μ^H is in the null space, so LM is the most powerful test. In contrast, if the inequalities fail in the alternative space, then $(\mu_1^A - \frac{\sigma_{12}}{\sigma_{22}}\mu_2^A, 0, \mu_3^A - \frac{\sigma_{23}}{\sigma_{22}}\mu_2^A)$ is not in the null space, and the [Lehmann and Romano \(2005\)](#) argument cannot be applied.

D.3.2 Existence of Structural Model

This section presents a structural model, then argues that any reduced-form model in the form of Equation (21) can be justified by this structural model.

Example 1. Consider a linear potential outcomes model with an instrument Z that is a vector of indicators for judges, each with $c = 5$ cases, a continuous endogenous variable X , and outcome Y :

$$X_i(z) = z'\pi + v_i, \quad Y_i(x) = x(\beta + \xi_i) + \varepsilon_i, \text{ and} \\ \begin{pmatrix} \varepsilon_i \\ \xi_i \\ v_i \end{pmatrix} \mid k(i) = k \sim N \left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma_{\varepsilon\varepsilon} & \sigma_{\varepsilon\xi} & \sigma_{\varepsilon v} \\ \cdot & \sigma_{\xi\xi} & \sigma_{\xi v k} \\ \cdot & \cdot & \sigma_{vv} \end{pmatrix} \right). \quad (22)$$

Due to the judge design, $X_i = \pi_{k(i)} + v_i$, where $k(i)$ is the judge that observation i is assigned to. The strength of the instrument is $C_S = \frac{1}{\sqrt{K}} \sum_k (c-1)\pi_k^2$. The π_k 's are constructed as such: with $s = \sqrt{C_S/\sqrt{K}/(c-1)}$, set $\pi_k = 0$ for the base judge, $\pi_k = -s$ for half the judges and $\pi_k = s$ for the other half. The heterogeneity covariances $\sigma_{\xi v k}$ are constructed so that $\sum_k \pi_k = 0$, $\sum_k \sigma_{\xi v k} = 0$, and $\sum_k \pi_k \sigma_{\xi v k} = 0$. With C_H characterizing the heterogeneity in the model, and $h = \sqrt{C_H/\sqrt{K}/(c-1)}$, set $\sigma_{\xi v k} = 0$ of the base judge; among judges with $\pi_k = s$, half of them have $\sigma_{\xi v k} = h$ and the other half $\sigma_{\xi v k} = -h$. The same construction of $\sigma_{\xi v k}$ applies for judges with $\pi_k = -s$.

In this model, the individual treatment effect is $\beta_i = \beta + \xi_i$. We can interpret v_i as the noise associated with the first-stage regression, ε_i as the noise in the intercept of the outcome equation, and ξ_i as the individual-level treatment effect heterogeneity. Further, $\sigma_{\xi vk}$ characterizes the extent of treatment effect heterogeneity. The observed outcome in a model with constant treatment effects is $Y_i(X_i) = X_i\beta + \varepsilon_i$, with $E[\varepsilon_i]=0$. When $\sigma_{\xi vk} = 0$, regardless of the values of $\sigma_{\varepsilon\xi}, \sigma_{\xi\xi}$, the observed outcome of Equation (22) can be written as $Y_i(X_i) = X_i\beta + \varepsilon_i$ where $E[\varepsilon_i] = E[X_i\xi_i + \varepsilon_i] = E[X_i E[\xi_i | X_i]] = 0$, which resembles the constant treatment effect case.

Lemma 8. *Consider the model of Example 1. If $\sqrt{K}s^2 \rightarrow \tilde{C}_S < \infty$ and $\sqrt{K}h^2 \rightarrow \tilde{C}_H < \infty$, then*

$$\begin{aligned} \sigma_{11} &= \frac{4}{\sigma_{33}} \left(\sigma_{22} - \frac{\sigma_{23}^2}{2\sigma_{33}} \right)^2 + o(1), \quad \sigma_{12} = 2 \frac{\sigma_{23}}{\sigma_{33}} \left(\sigma_{22} - \frac{\sigma_{23}^2}{2\sigma_{33}} \right) + o(1), \quad \sigma_{13} = \frac{\sigma_{23}^2}{\sigma_{33}} + o(1), \\ \sigma_{22} &= \frac{c-1}{c} (\sigma_{vv} (\sigma_{\varepsilon\varepsilon} + \sigma_{vv}\beta^2 + \sigma_{vv}\sigma_{\xi\xi} + 2\sigma_{\varepsilon v}\beta) + (\sigma_{vv}\beta + \sigma_{\varepsilon v})^2) + o(1), \\ \sigma_{33} &= 2 \frac{c-1}{c} \sigma_{vv}^2 + o(1), \quad \sigma_{23} = 2 \frac{c-1}{c} \sigma_{vv} (\sigma_{vv}\beta + \sigma_{\varepsilon v}) + o(1), \text{ and} \\ \begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{pmatrix} &= \begin{pmatrix} \sqrt{K}(c-1)(s^2\beta^2 + h^2) \\ \sqrt{K}(c-1)s^2\beta \\ \sqrt{K}(c-1)s^2 \end{pmatrix} = (c-1) \begin{pmatrix} C_S\beta^2 + C_H \\ C_S\beta \\ C_S \end{pmatrix}. \end{aligned}$$

Proposition 5. *In the model of Example 1 with $\sqrt{K}s^2 \rightarrow \tilde{C}_S < \infty$ and $\sqrt{K}h^2 \rightarrow \tilde{C}_H < \infty$, for any $\sigma_{22}, \sigma_{23}, \sigma_{33}$ such that $\sigma_{22}, \sigma_{33} > 0$, $\sigma_{23}^2 \leq \sigma_{22}\sigma_{33}$ and μ such that $\mu_1 \geq 0, \mu_3 > 0, \mu_2^2 \leq \mu_1\mu_3$, the following values of structural parameters:*

$$\begin{aligned} \tilde{C}_S &= \mu_3 / (c-1), \quad \beta = \mu_2 / \mu_3, \quad h = \sqrt{\frac{1}{\sqrt{K}} \frac{1}{c-1} \left(\mu_1 - \frac{\mu_2^2}{\mu_3} \right)}, \\ \Sigma_{SF} &= \begin{pmatrix} \sigma_{\varepsilon\varepsilon} & \sigma_{\varepsilon\xi} & \sigma_{\varepsilon v} \\ \cdot & \sigma_{\xi\xi} & \sigma_{\xi vk} \\ \cdot & \cdot & \sigma_{vv} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sigma_{vv}} \frac{c}{c-1} \left(\sigma_{22} - \frac{\sigma_{23}^2}{\sigma_{33}} \right) + \frac{\sigma_{\varepsilon v}^2}{\sigma_{vv}} & 0 & \sigma_{\varepsilon v} \\ \cdot & \frac{h}{\sigma_{vv}} & \pm h \\ \cdot & \cdot & \sigma_{vv} \end{pmatrix}, \\ \sigma_{vv} &= \sqrt{\frac{\sigma_{33}c}{2(c-1)}}, \text{ and } \sigma_{\varepsilon v} = \frac{1}{\sigma_{vv}} \left(\frac{\sigma_{23}c}{2(c-1)} - \sigma_{vv}^2\beta \right), \end{aligned}$$

satisfy the equations in Lemma 8, and $\det(\Sigma_{SF})/h \rightarrow C_D \geq 0$.

Due to Proposition 5, since the principal submatrices of Σ_{SF} are positive semidefinite asymptotically, Σ_{SF} is a symmetric positive semidefinite matrix asymptotically. The proposition thus implies that when the σ 's and μ satisfy the conditions, there exists structural parameters that can generate the given μ and Σ asymptotically. Hence, there are no further restrictions on μ from the observed Σ in Example 1.

D.3.3 Numerical Results for Power

This section presents numerical results for power in environments not covered by the theory. I first consider one-sided tests beyond the set \mathcal{S} covered by the theory, then weighted average power for two-sided tests rather than the class of unbiased tests.

The power envelope is achieved by a test that is valid across the entire composite null space, and is most powerful for testing against a particular point in the alternative space. To obtain this test, I implement the algorithm from Elliott et al. (2015) (EMW) where all weight on the alternative are placed on a single point while being valid across a composite null. Then, testing against every point in the alternative space requires a different critical value. For the numerical exercises in this subsection, I use a Σ matrix of the form:

$$\Sigma = \begin{pmatrix} 2 & 2\rho & 2\rho^2 \\ \cdot & 1 + \rho^2 & 2\rho \\ \cdot & \cdot & 2 \end{pmatrix}, \quad (23)$$

which corresponds to the Σ matrix in Proposition 2 with $\omega_{\zeta\zeta} = \omega_{\eta\eta} = 1, \omega_{\zeta\eta} = \rho$.

In the numerical exercises, I display the rejection rate across 500 independent draws from $X^* \sim N(\mu, \Sigma)$ at each point on the μ_2 axis, across several μ_1, μ_3 values for a 5% test. The composite null uses a grid of $\mu_1 \in [0, 5], \mu_3 \in [0, 5]$ in 0.5 increments, and assumes the variance is known.

Figure 2 uses a one-sided LM test, with a large covariance at $\rho = 0.9$. When data is generated from the null, since LM and EMW are valid tests, their rejection rate is at most 0.05. EMW has exact size when testing a weighted average of values in the null space and is valid across the entire space, so when data is generated from one particular point in the null, EMW can be conservative. Consistent with Proposition 4, when μ_2 is small enough for $\mu_1 = 1, \mu_3 = 4$, LM achieves the power envelope, but as

μ_2 gets larger, the gap widens substantially. This phenomenon occurs because EMW still uses the same null grid, but now it no longer needs to have correct size for testing against the point $(\mu_1^A - \frac{\sigma_{12}}{\sigma_{22}}\mu_2^A, 0, \mu_3^A - \frac{\sigma_{23}}{\sigma_{22}}\mu_2^A)$, as that point is no longer in the null space.

In Figure 3, Σ is calibrated by using the Σ matrix calculated from the Angrist and Krueger (1991) application, so after appropriate normalizations, $\rho = 0.37$. With such a low covariance, LM is basically indistinguishable from the EMW bound. Hence, even though there are gains to be made theoretically, in the empirical application considered, the gains are small.

Instead of considering a point alternative, we may be more interested in testing against a composite alternative. Here, the alternative grid for EMW places equal weight on alternatives $(\mu_1^A, \mu_2^A, \mu_3^A) \in [0, 5] \times [-2, 2] \times [0, 5]$ in increments of 0.5 (excluding $\mu_2 = 0$) subject to inequality constraints. Figures 4 and 5 present one such possibility by allowing EMW to place equal weight on several points within the alternative space. The resulting test is the nearly optimal test for a weighted average of values the null space against the uniformly weighted average of alternative values. Hence, there is no guarantee that its power is necessarily higher than the LM test at every point in the alternative space. While there are weighted-average power curves that substantially outperform LM, this result is compatible with Proposition 1. EMW is a biased test as there are points in the alternative space that are not a part of the grid where LM outperforms EMW. Nonetheless, Figure 5 suggests that, when using the empirical covariance, LM does not perform substantially worse than EMW.

D.4 Constructing Confidence Sets

Expressions for the test are given in Section 3, which can be efficiently implemented using matrix operations. Inverting the test to obtain a confidence set is also straightforward in this procedure, as the bounds of the confidence set are derived in closed-form in this section.

To invert the LM test to obtain a confidence set, use $e_i = Y_i - X_i\beta_0$ and expand the A expressions in Equation (9) so that they are written in terms of X and Y . The two-sided test rejects: $\left(\sum_i \sum_{j \neq i} G_{ij} e_i X_j\right)^2 / \hat{V}_{LM} \geq q = \Phi(1 - \alpha/2)^2$. Let $T_{YX} := \frac{1}{\sqrt{K}} \sum_i \sum_{j \neq i} G_{ij} Y_i X_j$. Then, $\sum_i \sum_{j \neq i} G_{ij} e_i X_j = \sqrt{K} (T_{YX} - T_{XX}\beta_0)$, so squaring it

Figure 2: One-sided test with $\rho = 0.9$

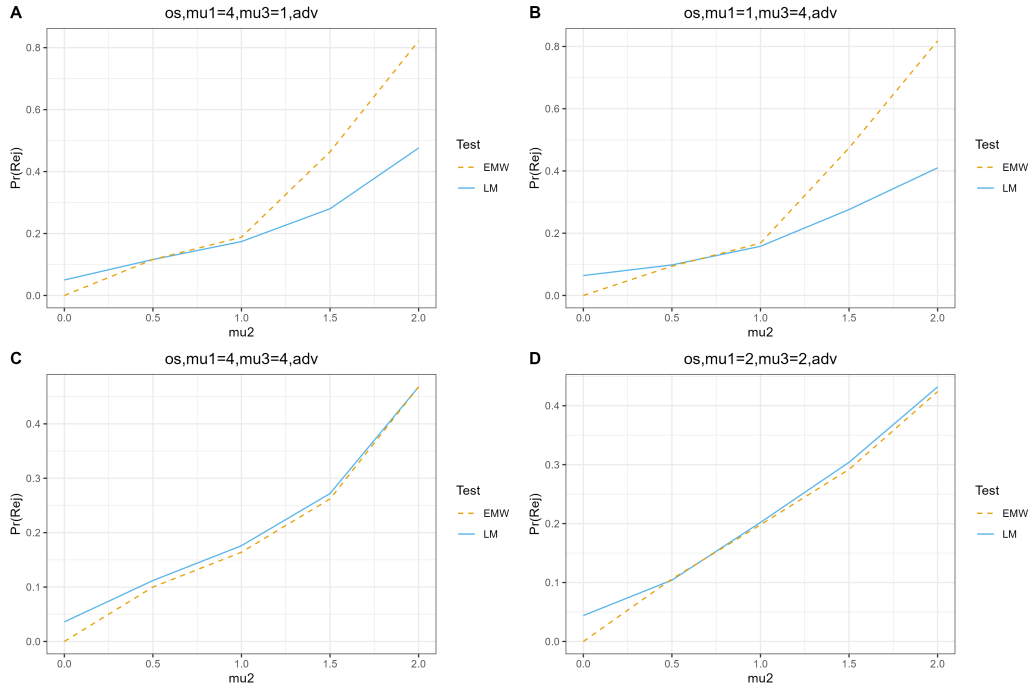


Figure 3: One-sided test with $\rho = 0.37$

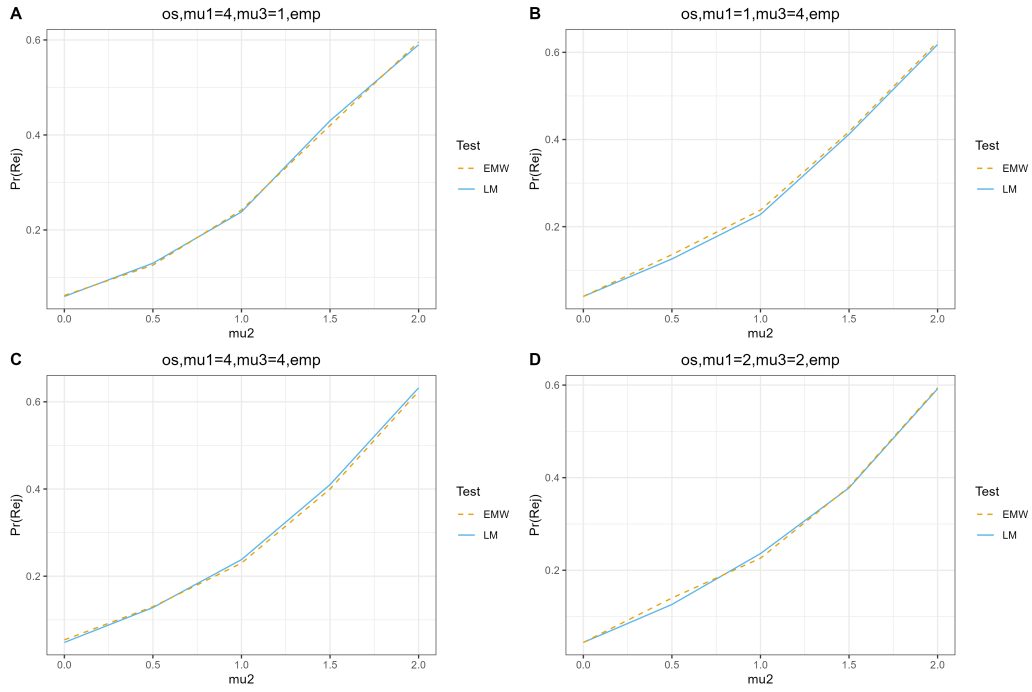


Figure 4: Uniform Weighting on grid of alternatives with $\rho = 0.9$

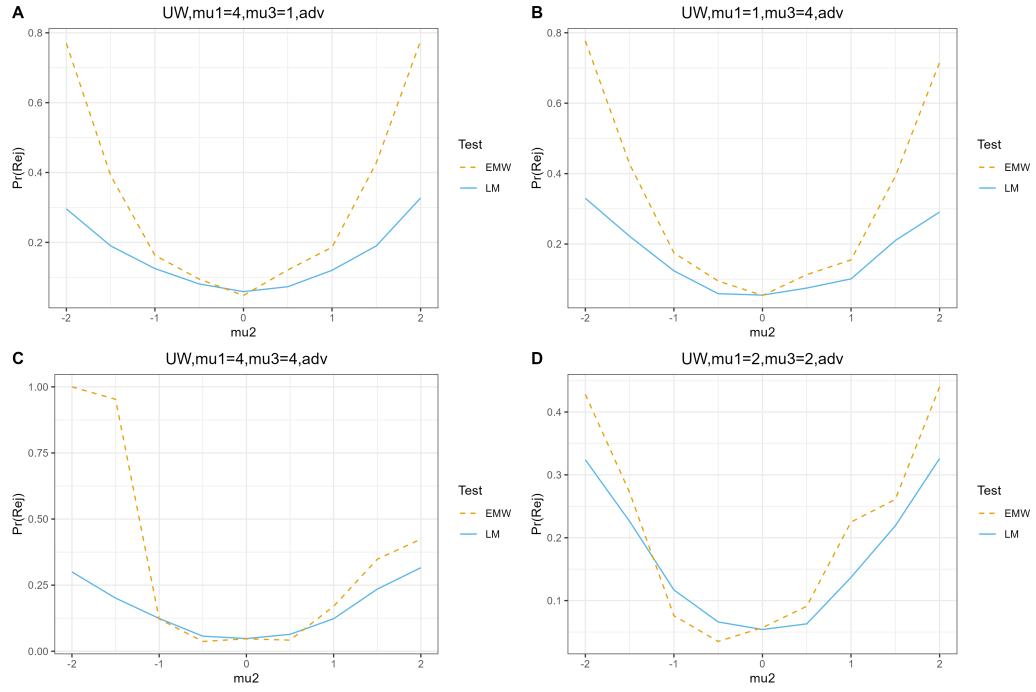
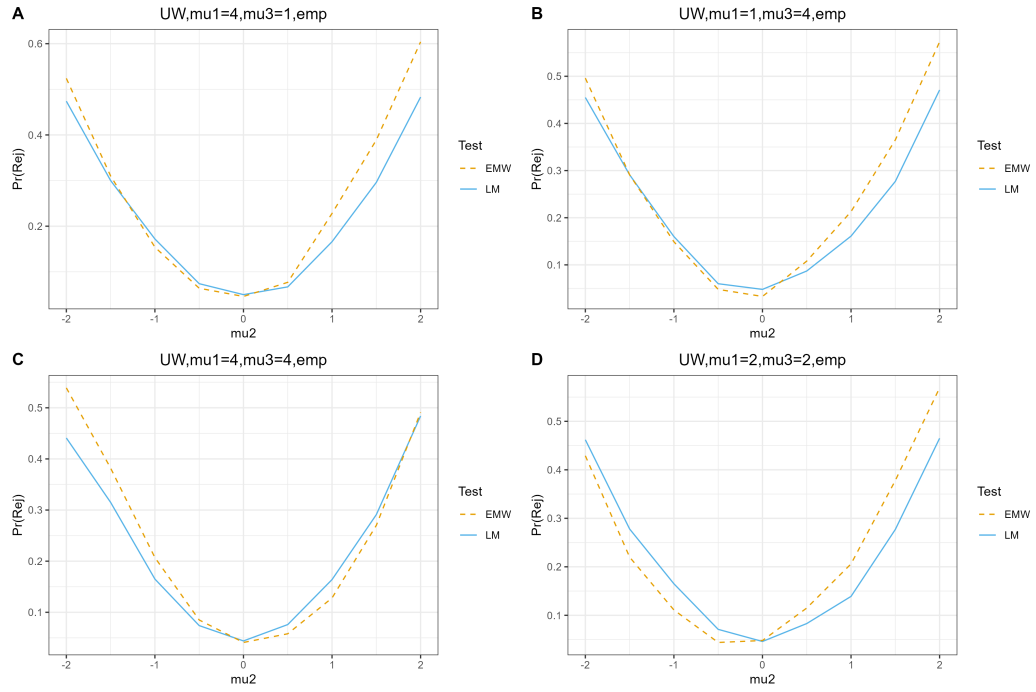


Figure 5: Uniform Weighting on grid of alternatives with $\rho = 0.37$



results in a term that is quadratic in β_0^2 . With $\hat{V}_{LM} = B_0 + C_1\beta_0 + B_2\beta_0^2$ quadratic in β_0 , the analysis for the shape of the confidence intervals is similar to the AR procedure for just-identified IV. Calculations for coefficients is similar to that of the L3O variance.

Proposition 6. *The two-sided LM test does not reject β_0 when $(KT_{XX}^2 - qB_2)\beta_0^2 - (2KT_{YX}T_{XX} + qB_1)\beta_0 + (KT_{YX}^2 - qB_0) \leq 0$. Let $D := (2KT_{YX}T_{XX} + qB_1)^2 - 4(KT_{XX}^2 - qB_2)(KT_{YX}^2 - qB_0)$. If $D \geq 0$ and $KT_{XX}^2 - qB_2 \geq 0$, then the upper and lower bounds of confidence set are:*

$$\frac{(2KT_{YX}T_{XX} + qB_1) \pm \sqrt{D}}{2(KT_{XX}^2 - qB_2)}.$$

If $D < 0$ and $KT_{XX}^2 - qB_2 < 0$, then the confidence set is empty. Otherwise, the confidence set is unbounded.

Due to $+qB_1, -qB_2$ in the expression of the upper and lower bounds, the confidence set is not necessarily centered around $\hat{\beta}_{JIVE} = T_{YX}/T_{XX}$.

D.5 Further Simulation Results

This section reports simulation results from several structural models to assess how well various procedures control for size. Since the nominal size is 0.05, and data is generated under the null, the target rejection rate is 0.05. Across the board, the L3O method performs well, and for all existing procedures, there is at least one design where they perform badly.

D.5.1 Continuous Treatment

This subsection reports results for a simulation based on Example 1 that has a continuous X . Table 7 reports results with $K = 500$ and Table 8 reports results for $K = 40$. The L3O rejection rates are closer to the nominal rate than the existing procedures in the literature, albeit worse with a smaller K . ARorc has high rejection rates with strong heterogeneity and EK has high rejection rates with weak instruments. Notably, with perfect correlation and an irrelevant instrument, EK can achieve 100% rejection in the simulation with $K = 500$. The procedures that use the LM statistic are MO,

Table 7: Rejection rates under the null for nominal size 0.05 test for continuous X

| | TSLS | EK | ARorc | MO | \tilde{X} -t | \tilde{X} -AR | L3O | LMorc |
|---|-------|-------|-------|-------|----------------|-----------------|-------|-------|
| $C_H = C_S = 3\sqrt{K}, \sigma_{\varepsilon v} = 0$ | 0.061 | 0.017 | 1.000 | 0.061 | 0.079 | 0.078 | 0.042 | 0.044 |
| $C_H = 2\sqrt{K}, C_S = 2\sqrt{K}$ | 0.952 | 0.022 | 1.000 | 0.073 | 0.087 | 0.084 | 0.058 | 0.055 |
| $C_H = 2\sqrt{K}, C_S = 2$ | 1.000 | 0.009 | 1.000 | 0.096 | 0.076 | 0.127 | 0.053 | 0.050 |
| $C_H = 2\sqrt{K}, C_S = 0$ | 1.000 | 0.006 | 1.000 | 0.103 | 0.061 | 0.127 | 0.059 | 0.052 |
| $C_H = 3, C_S = 3\sqrt{K}$ | 0.986 | 0.033 | 0.109 | 0.057 | 0.062 | 0.064 | 0.056 | 0.047 |
| $C_H = 3, C_S = 3$ | 1.000 | 0.036 | 0.168 | 0.055 | 0.078 | 0.087 | 0.055 | 0.047 |
| $C_H = 3, C_S = 0$ | 1.000 | 0.048 | 0.184 | 0.058 | 0.106 | 0.088 | 0.053 | 0.057 |
| $C_H = 0, C_S = 2\sqrt{K}$ | 1.000 | 0.089 | 0.049 | 0.063 | 0.083 | 0.080 | 0.061 | 0.058 |
| $C_H = 0, C_S = 2$ | 1.000 | 0.207 | 0.045 | 0.054 | 0.243 | 0.135 | 0.057 | 0.045 |
| $C_H = 0, C_S = 0$ | 1.000 | 0.337 | 0.051 | 0.042 | 0.413 | 0.127 | 0.045 | 0.048 |
| $C_H = C_S = 0, \sigma_{\varepsilon v} = 1$ | 1.000 | 1.000 | 0.044 | 0.042 | 1.000 | 0.157 | 0.052 | 0.044 |

Notes: Data generating process corresponds to Example 1. Unless mentioned otherwise, simulations use $K = 500, c = 5, \beta = 0, \sigma_{\varepsilon\varepsilon} = \sigma_{vv} = 1, \sigma_{\varepsilon\xi} = 0, \sigma_{\varepsilon v} = 0.8, \sigma_{\xi\xi} = 1 + h$ for $h^2 < 1$ with 1000 simulations. The table displays rejection rates of various procedures (in columns) for various designs (in rows). $C_H = 0$ uses $\xi_i = 0$ for all i , which uses $\sigma_{\xi\xi} = \sigma_{\varepsilon\xi} = \sigma_{\xi v} = 0$, corresponding to constant treatment effects. Procedures are described in Table 1.

\tilde{X} -AR, L3O and LMorc; they differ only in variance estimation. Hence, while \tilde{X} -AR and MO over-reject, the extent of over-rejection is smaller than ARorc and EK in the adversarial cases.

D.5.2 Binary Treatment

This subsection presents a structural model with a binary X . Data is generated from a judge model with $J = K + 1$ judges, each with $c = 5$ cases, and cases are indexed by i . The structural model is:

$$Y_i(x) = x(\beta + \xi_i) + \varepsilon_i, \text{ and}$$

$$X_i(z) = I\{z'\pi - v_i \geq 0\}.$$

Table 8: Rejection Rates under the null for nominal size 0.05 test for Continuous X with $K = 40$

| | TSLS | EK | ARorc | MO | \tilde{X} -t | \tilde{X} -AR | L3O | LMorc |
|---|-------|-------|-------|-------|----------------|-----------------|-------|-------|
| $C_H = C_S = 3\sqrt{K}, \sigma_{\varepsilon v} = 0$ | 0.072 | 0.022 | 0.525 | 0.051 | 0.074 | 0.068 | 0.039 | 0.055 |
| $C_H = 2\sqrt{K}, C_S = 2\sqrt{K}$ | 0.238 | 0.034 | 0.388 | 0.051 | 0.074 | 0.077 | 0.055 | 0.062 |
| $C_H = 2\sqrt{K}, C_S = 2$ | 0.547 | 0.033 | 0.475 | 0.083 | 0.096 | 0.133 | 0.077 | 0.053 |
| $C_H = 2\sqrt{K}, C_S = 0$ | 0.651 | 0.013 | 0.511 | 0.072 | 0.088 | 0.102 | 0.068 | 0.054 |
| $C_H = 3, C_S = 3\sqrt{K}$ | 0.213 | 0.025 | 0.109 | 0.048 | 0.057 | 0.063 | 0.055 | 0.046 |
| $C_H = 3, C_S = 3$ | 0.658 | 0.032 | 0.129 | 0.045 | 0.074 | 0.063 | 0.064 | 0.055 |
| $C_H = 3, C_S = 0$ | 0.849 | 0.049 | 0.127 | 0.063 | 0.109 | 0.103 | 0.087 | 0.057 |
| $C_H = 0, C_S = 2\sqrt{K}$ | 0.853 | 0.105 | 0.049 | 0.064 | 0.068 | 0.098 | 0.085 | 0.056 |
| $C_H = 0, C_S = 2$ | 0.999 | 0.152 | 0.048 | 0.045 | 0.201 | 0.132 | 0.098 | 0.037 |
| $C_H = 0, C_S = 0$ | 1.000 | 0.342 | 0.052 | 0.051 | 0.439 | 0.143 | 0.080 | 0.049 |
| $C_H = C_S = 0, \sigma_{\varepsilon v} = 1$ | 1.000 | 1.000 | 0.045 | 0.040 | 1.000 | 0.179 | 0.082 | 0.045 |

Note: Designs are identical to Table 7, but $K = 40$ here.

Our unobservables are generated as follows. Draw $v_i \sim U[-1, 1]$, then generate residuals from:

$$\varepsilon_i \mid v_i \sim \begin{cases} N(\sigma_{\varepsilon v}, \sigma_{\varepsilon \varepsilon}) & \text{if } v_i \geq 0 \\ N(-\sigma_{\varepsilon v}, \sigma_{\varepsilon \varepsilon}) & \text{if } v_i < 0 \end{cases},$$

$$\xi_i \mid v_i \geq 0 = \begin{cases} \sigma_{\xi vk} & w.p. \quad p \\ -\sigma_{\xi vk} & w.p. \quad 1 - p \end{cases}, \text{ and } \xi_i \mid v_i < 0 = \begin{cases} \sigma_{\xi vk} & w.p. \quad 1 - p \\ -\sigma_{\xi vk} & w.p. \quad p \end{cases}.$$

The process for determining s, h and $\pi_k \in \{0, -s, s\}, \sigma_{\xi vk} \in \{0, -h, h\}$ are identical to Example 1, as s controls the strength of the instrument, h the extent of heterogeneity, and β is the object of interest. Then, the problem's variances and covariances are determined by $(p, \sigma_{\varepsilon v}, \sigma_{\varepsilon \varepsilon})$. The JIVE estimand is shown to be β in Supplementary Appendix E.2. A simulation is run with $K = 100$, so the sample size is smaller than the normal experiment in Example 1.

Results are presented in Table 9, and are qualitatively similar to Section 2. The oracle test consistently obtains rejection rates close to the nominal 5% rate across all designs, in accordance with the normality result, even with heterogeneous treatment effects and non-normality of errors due to the binary setup. The L3O rejection rate

Table 9: Rejection Rates under the null for nominal size 0.05 test for binary X

| | TSLS | EK | ARorc | MO | \tilde{X} -t | \tilde{X} -AR | L3O | LMorc |
|--|-------|-------|-------|-------|----------------|-----------------|-------|-------|
| $C_H = C_S = 3\sqrt{K}, \sigma_{\varepsilon v} = 0$ | 0.046 | 0.049 | 0.059 | 0.045 | 0.045 | 0.045 | 0.049 | 0.054 |
| $C_H = 2\sqrt{K}, C_S = 2\sqrt{K}$ | 0.097 | 0.047 | 0.177 | 0.037 | 0.038 | 0.041 | 0.051 | 0.052 |
| $C_H = 2\sqrt{K}, C_S = 2$ | 0.727 | 0.059 | 1.000 | 0.127 | 0.051 | 0.143 | 0.058 | 0.051 |
| $C_H = 2\sqrt{K}, C_S = 0$ | 0.891 | 0.037 | 1.000 | 0.204 | 0.067 | 0.247 | 0.059 | 0.045 |
| $C_H = 3, C_S = 3\sqrt{K}$ | 0.092 | 0.060 | 0.051 | 0.055 | 0.057 | 0.056 | 0.055 | 0.047 |
| $C_H = 3, C_S = 3$ | 0.996 | 0.089 | 0.888 | 0.059 | 0.086 | 0.096 | 0.055 | 0.048 |
| $C_H = 3, C_S = 0$ | 1.000 | 0.124 | 0.999 | 0.101 | 0.289 | 0.181 | 0.068 | 0.052 |
| $C_H = 0, C_S = 2\sqrt{K}$ | 0.408 | 0.058 | 0.055 | 0.043 | 0.046 | 0.046 | 0.045 | 0.041 |
| $C_H = 0, C_S = 2$ | 1.000 | 0.212 | 0.052 | 0.061 | 0.188 | 0.108 | 0.078 | 0.057 |
| $C_H = 0, C_S = 0$ | 1.000 | 0.654 | 0.046 | 0.034 | 0.750 | 0.149 | 0.069 | 0.039 |
| $C_H = C_S = 0, \sigma_{\varepsilon\varepsilon} = 0$ | 1.000 | 1.000 | 0.053 | 0.057 | 1.000 | 0.173 | 0.076 | 0.053 |

Note: The data generating process corresponds to Appendix [D.5.2](#). Unless stated otherwise, designs use $K = 100, c = 5, \beta = 0, p = 7/8, \sigma_{\varepsilon\varepsilon} = 0.1, \sigma_{\varepsilon v} = 0.5$ with 1000 simulations.

is close to the nominal rate even with a smaller sample size. EK, ARorc and MO continue to have high rejection rates in the adversarial designs.

D.5.3 Incorporating Covariates

This section presents a data-generating process that involves covariates. Instead of judges, consider a model where there are K states. Let $t = 1, \dots, K$ index the state and let W denote the control vector that is an indicator for states. With a binary exogenous variable (say an indicator for birth being in the fourth quarter) $B \in \{0, 1\}$, the value of the instrument is given by $k = t \times B$. Then, the instrument vector Z is an indicator for all possible values of k . The structural model is:

$$Y_i(x) = x(\beta + \xi_i) + w'\gamma + \varepsilon_i, \text{ and}$$

$$X_i(z) = I\{z'\pi + w'\gamma - v_i \geq 0\}.$$

In the simulation, every state has 10 observations, of which 5 have $B = 1$ and the other 5 have $B = 0$. The process for generating $(v_i, \varepsilon_i, \xi_i)$, $\pi_k, \sigma_{\xi vk}$, and s, h is identical to the binary case. Hence, $\pi_0 = \sigma_{\xi v0}$ for the base group, which constitutes

Table 10: Rejection Rates under the null for nominal size 0.05 test for binary X with covariates

| | TSLS | EK | ARorc | MO | \tilde{X} -t | \tilde{X} -AR | L3O | LMorc |
|--|-------|-------|-------|-------|----------------|-----------------|-------|-------|
| $C_H = C_S = 3\sqrt{K}, \sigma_{\varepsilon v} = 0$ | 0.048 | 0.123 | 0.049 | 0.052 | 0.047 | 0.055 | 0.054 | 0.060 |
| $C_H = 2\sqrt{K}, C_S = 2\sqrt{K}$ | 0.072 | 0.111 | 0.052 | 0.044 | 0.041 | 0.046 | 0.050 | 0.053 |
| $C_H = 2\sqrt{K}, C_S = 2$ | 0.171 | 0.016 | 0.471 | 0.083 | 0.012 | 0.092 | 0.060 | 0.050 |
| $C_H = 2\sqrt{K}, C_S = 0$ | 0.259 | 0.002 | 0.960 | 0.126 | 0.008 | 0.135 | 0.047 | 0.058 |
| $C_H = 3, C_S = 3\sqrt{K}$ | 0.065 | 0.132 | 0.048 | 0.053 | 0.056 | 0.054 | 0.060 | 0.049 |
| $C_H = 3, C_S = 3$ | 0.131 | 0.015 | 0.108 | 0.040 | 0.003 | 0.042 | 0.044 | 0.050 |
| $C_H = 3, C_S = 0$ | 0.247 | 0.003 | 0.300 | 0.087 | 0.004 | 0.091 | 0.062 | 0.053 |
| $C_H = 0, C_S = 2\sqrt{K}$ | 0.084 | 0.099 | 0.054 | 0.041 | 0.036 | 0.043 | 0.048 | 0.050 |
| $C_H = 0, C_S = 2$ | 0.178 | 0.006 | 0.058 | 0.043 | 0.002 | 0.044 | 0.052 | 0.051 |
| $C_H = 0, C_S = 0$ | 0.246 | 0.006 | 0.048 | 0.063 | 0.005 | 0.069 | 0.081 | 0.050 |
| $C_H = C_S = 0, \sigma_{\varepsilon\varepsilon} = 0$ | 1.000 | 0.497 | 0.042 | 0.013 | 0.147 | 0.049 | 0.092 | 0.035 |

Note: The data generating process corresponds to Appendix D.5.3. Unless stated otherwise, designs use $K = 48, c = 5, \beta = 0, p = 7/8, \sigma_{\varepsilon\varepsilon} = 0.5, \sigma_{\varepsilon v} = 0.1$, and $g = 0.1$ with 1000 simulations.

half the observations. For $k \neq 0$, π_k is the coefficient for observations from state $t = k$ and have $B = 1$, and $\sigma_{\xi vk}$ is the corresponding heterogeneity term. Whenever $\pi_t = s$, set $\gamma_t = g$; whenever $\pi_t = -s$, set $\gamma_t = -g$. In this setup, it can be shown that the UJIVE estimand is β , and the proof is in Supplementary Appendix E.2. Table 10 reports the associated simulation results, which are qualitatively similar to the results described before.

D.6 Proofs of Lemmas 1 and 2

Proof of Lemma 1. Suppose not. Then, for some real β_0 ,

$$E[T_{AR}] = \sum_i \sum_{j \neq i} P_{ij} R_{\Delta i} R_{\Delta j} = \sum_i \sum_{j \neq i} P_{ij} (R_{Yi} R_{Yj} - R_i R_{Yj} \beta_0 - R_{Yi} R_j \beta_0 + R_i R_j \beta_0^2) = 0.$$

Solving for β_0 ,

$$\beta_0 = \frac{2 \sum_i \sum_{j \neq i} P_{ij} R_i R_{Yj} \pm \sqrt{4 \left(\sum_i \sum_{j \neq i} P_{ij} R_i R_{Yj} \right)^2 - 4 \left(\sum_i \sum_{j \neq i} P_{ij} R_i R_j \right) \left(\sum_i \sum_{j \neq i} P_{ij} R_{Yi} R_{Yj} \right)}}{2 \left(\sum_i \sum_{j \neq i} P_{ij} R_i R_j \right)}.$$

In our structural model, $R_i = \pi_{k(i)}$ and $R_{Yi} = \pi_{Yk(i)}$. The term in the square root can be written as:

$$D = 4 \left(\sum_k \pi_k \pi_{Yk} \right)^2 - 4 \left(\sum_k \pi_k^2 \right) \left(\sum_k \pi_{Yk}^2 \right)$$

Using Table 6, $\sum_k \pi_k^2 = \frac{5}{8} s^2 K$, $\sum_k \pi_{Yk}^2 = \left(\frac{5}{8} s^2 \beta^2 + h^2 \right) K$, and $\sum_k \pi_k \pi_{Yk} = \frac{5}{8} s^2 \beta K$, we obtain

$$\frac{1}{4} D = \left(\frac{5}{8} s^2 \beta K \right)^2 - \left(\frac{5}{8} s^2 K \right) \left(\frac{5}{8} s^2 \beta^2 + h^2 \right) K = -\frac{5}{8} s^2 h^2 K^2 \leq 0.$$

Since $h \neq 0$ and $K s^2 > 0$, there are no real roots of β_0 , a contradiction. \square

Proof of Lemma 2. I rewrite the quadratic term to produce a martingale difference array:

$$\begin{aligned} \sum_i \sum_{j \neq i} G_{ij} v_i' A v_j &= \sum_i \sum_{j < i} G_{ij} v_i' A v_j + \sum_i \sum_{j > i} G_{ij} v_i' A v_j \\ &= \sum_i \sum_{j < i} (G_{ij} v_i' A v_j + G_{ji} v_j' A v_i). \end{aligned}$$

Hence, $\sum_i s_i' v_i + \sum_i \sum_{j \neq i} G_{ij} v_i' A v_j = \sum_i y_i$, where

$$\begin{aligned} y_i &= s_i' v_i + \sum_{j < i} (G_{ij} v_i' A v_j + G_{ji} v_j' A v_i) = s_i' v_i + v_i' A \left(\sum_{j < i} G_{ij} v_j \right) + \left(\sum_{j < i} G_{ji} v_j' \right) A v_i \\ &= s_i' v_i + v_i' A (G_L v)_i' + (G_U' v)_i. A v_i. \end{aligned}$$

Let \mathcal{F}_i denote the filtration of y_1, \dots, y_{i-1} . To apply the martingale CLT, we require:

1. $\sum_i E [|y_i|^{2+\epsilon}] \rightarrow 0$.

2. Conditional variance converges to 1, i.e., $P(|\sum_i E[B^2 y_i^2 | \mathcal{F}_i] - 1| > \eta) \rightarrow 0$, where $B = \text{Var}(T)^{-1/2}$.

The 4th moments of v_i are bounded. With $\epsilon = 2$, we want $\sum_i E[y_i^4] \rightarrow 0$. Using Loeve's c_r inequality, it suffices that, for any element l of the v_i vector,

$$\sum_i s_{il}^4 E[v_{il}^4] \rightarrow 0, \text{ and } \sum_i E[v_{il}^4 (G_L v)_{il}^4] \rightarrow 0.$$

The first condition is immediate from condition (2). The second condition holds by condition (3) using the proof in EK18. To be precise,

$$\begin{aligned} \sum_i E[v_{il}^4 (G_L v)_{il}^4] &= \sum_i E[v_{il}^4] E[(G_L v)_{il}^4] \preceq \sum_i E[(G_L v)_{il}^4] \\ &= \sum_i \sum_j G_{L,ij}^4 E[v_{il}^4] + 3 \sum_i \sum_j \sum_{k \neq j} G_{L,ij}^2 G_{L,ik}^2 E[v_{il}^2] E[v_{jl}^2] \\ &\preceq \sum_i \sum_j \sum_k G_{L,ij}^2 G_{L,ik}^2 = \sum_i (G_L G'_L)_{ii}^2 \\ &\leq \sum_i \sum_j (G_L G'_L)_{ij}^2 = \|G_L G'_L\|_F^2. \end{aligned}$$

The argument for G_U is analogous. Now, I turn to showing convergence of the conditional variance. With abuse of notation, let $W_i = s'_i v_i$ and $X_i = v'_i A (G_L v)_{i\cdot}' + v'_i A (G'_U v)_{i\cdot}'$. Since $\text{Var}(BT) = B^2 \sum_i E[W_i^2] + B^2 \sum_i E[X_i^2] = 1$,

$$\begin{aligned} \sum_i E[B^2 y_i^2 | \mathcal{F}_i] - 1 &= B^2 \sum_i (E[X_i^2 | \mathcal{F}_i] - E[X_i^2]) + 2B^2 \sum_i E[W_i X_i | \mathcal{F}_i] \\ &\quad + B^2 \sum_i (E[W_i^2 | \mathcal{F}_i] - E[W_i^2]). \end{aligned}$$

The previous observations in the filtration do not feature, so $E[W_i^2 | \mathcal{F}_i] - E[W_i^2] = 0$. It suffices to show that the RHS converges to 0. For the $\sum_i E[W_i X_i | \mathcal{F}_i]$ term,

$$\begin{aligned} B^2 \sum_i E[W_i X_i | \mathcal{F}_i] &= B^2 \sum_i E[W_i (v'_i A (G_L v)_{i\cdot}' + v'_i A (G'_U v)_{i\cdot}') | \mathcal{F}_i] \\ &= B^2 \sum_i E[W_i v'_i A] (G_L v)_{i\cdot}' + B^2 \sum_i E[W_i v'_i A] (G'_U v)_{i\cdot}'. \end{aligned}$$

It suffices to show that the respective squares converge to 0. Due to bounded

fourth moments, and applying the Cauchy-Schwarz inequality repeatedly, for some n -vector δ_v with $\|\delta_v\|_2 \leq C$,

$$E \left[\left(\sum_i E[W_i v'_i] A (G_L v)'_{i.} \right)^2 \right] \preceq \delta'_v G_L G'_L \delta_v \leq \|\delta_v\|_2^2 \|G_L G'_L\|_2 \preceq \|G_L G'_L\|_F,$$

and the same argument can be applied to the G_U term. Finally,

$$\begin{aligned} \sum_i (E[X_i^2 | \mathcal{F}_i] - E[X_i^2]) &= \sum_i \left(E \left[(v'_i A (G_L v)'_{i.} + v'_i A (G'_U v)'_{i.})^2 | \mathcal{F}_i \right] \right. \\ &\quad \left. - E \left[(v'_i A (G_L v)'_{i.} + v'_i A (G'_U v)'_{i.})^2 \right] \right). \end{aligned}$$

It suffices to consider the G_L term, as the G_U and cross terms are analogous:

$$\begin{aligned} \sum_i \left(E \left[(v'_i A (G_L v)'_{i.})^2 | \mathcal{F}_i \right] - E \left[(v'_i A (G_L v)'_{i.})^2 \right] \right) \\ = \sum_i \left((G_L v)_{i.} A' E[v_i v'_i] A (G_L v)'_{i.} - E \left[(G_L v)_{i.} A' v_i v'_i A (G_L v)'_{i.} \right] \right). \end{aligned}$$

Since $\sum_i (G_L v)_{i.} A' E[v_i v'_i] A (G_L v)'_{i.}$ is demeaned, it suffices to show that its variance converges to 0. Due to bounded moments,

$$\text{Var} \left(\sum_i (G_L v)_{i.} A' E[v_i v'_i] A (G_L v)'_{i.} \right) \preceq \sum_i \sum_j (G_L G'_L)^2 = \|G_L G'_L\|_F^2,$$

which suffices for the result. \square

The full and latest version of the paper, including proofs of results in the appendices, can be found at https://lutheryap.github.io/files/mwiv_het_wp.pdf.

E Supplementary Appendix, Not for Publication

E.1 Proofs for Appendix C

Proof of Lemma 3. I begin with part (c). By applying the Cauchy-Schwarz inequality,

$$\begin{aligned}
& \left| \sum_i C_i \left(\sum_{j \neq i} h_2^A(i, j) R_{mj} \right) \left(\sum_{j \neq i} h_2^B(i, j) R_{mj} \right) \right| \\
& \leq \left(\sum_i C_i \left(\sum_{j \neq i} h_2^A(i, j) R_{mj} \right)^2 \right)^{1/2} \left(\sum_i C_i \left(\sum_{j \neq i} h_2^B(i, j) R_{mj} \right)^2 \right)^{1/2} \\
& \leq \max_i C_i \left(\sum_i \left(\sum_{j \neq i} h_2^A(i, j) R_{mj} \right)^2 \right)^{1/2} \left(\sum_i \left(\sum_{j \neq i} h_2^B(i, j) R_{mj} \right)^2 \right)^{1/2} \\
& \leq \max_i C_i \left(\sum_i \tilde{R}_{mi}^2 \right)^{1/2} \left(\sum_i \tilde{R}_{mi}^2 \right)^{1/2} \leq C \sum_i \tilde{R}_{mi}^2.
\end{aligned}$$

The proof of all other parts are entirely analogous. \square

Proof of Lemma 4. Proof of Lemma 4(a).

Using the decomposition from AS23,

$$\begin{aligned}
& \text{Var} \left(\sum_i \sum_{j \neq i} G_{ij} F_{ij} V_{1i} V_{2i} V_{3j} V_{4j} \right) \\
& = \sum_{i \neq j}^n G_{ij}^2 F_{ij}^2 \text{Var} (V_{1i} V_{2i} V_{3j} V_{4j}) + \sum_{i \neq j}^n G_{ij} F_{ij} G_{ji} F_{ji} \text{Cov} (V_{1i} V_{2i} V_{3j} V_{4j}, V_{1j} V_{2j} V_{3i} V_{4i}) \\
& \quad + \sum_{i \neq j \neq k}^n G_{ij} F_{ij} G_{kj} F_{kj} \text{Cov} (V_{1i} V_{2i} V_{3j} V_{4j}, V_{1k} V_{2k} V_{3j} V_{4j}) \\
& \quad + \sum_{i \neq j \neq k}^n G_{ij} F_{ij} G_{jk} F_{jk} \text{Cov} (V_{1i} V_{2i} V_{3j} V_{4j}, V_{1j} V_{2j} V_{3k} V_{4k}) \\
& \quad + \sum_{i \neq j \neq k}^n G_{ij} F_{ij} G_{ik} F_{ik} \text{Cov} (V_{1i} V_{2i} V_{3j} V_{4j}, V_{1i} V_{2i} V_{3k} V_{4k}) \\
& \quad + \sum_{i \neq j \neq k}^n G_{ij} F_{ij} G_{ki} F_{ki} \text{Cov} (V_{1i} V_{2i} V_{3j} V_{4j}, V_{1k} V_{2k} V_{3i} V_{4i}) \\
& \leq 2 \left[\max_{i,j} \text{Var} (V_{1i} V_{2i} V_{3j} V_{4j}) \right] \sum_i \left(\left(\sum_{j \neq i} G_{ij} F_{ij} \right)^2 + \left(\sum_{j \neq i} G_{ij} F_{ij} \right) \left(\sum_{j \neq i} G_{ji} F_{ji} \right) \right).
\end{aligned}$$

Notice that the terms in $\sum_{i \neq j}^n$ are absorbed into the sum over k so that the final expression can be written as $\sum_i \sum_{j \neq i} \sum_{k \neq i}$. Then, due to Assumption 3(a) and the Cauchy-Schwarz inequality,

$$\sum_i \left(\sum_{j \neq i} G_{ij} F_{ij} \right)^2 \leq \sum_i \left(\sum_{j \neq i} G_{ij}^2 \right) \left(\sum_{j \neq i} F_{ij}^2 \right) \leq C \sum_i \sum_{j \neq i} G_{ij}^2,$$

and

$$\begin{aligned} \left| \sum_i \left(\sum_{j \neq i} G_{ij} F_{ij} \right) \left(\sum_{j \neq i} G_{ji} F_{ji} \right) \right| &\leq \left(\sum_i \left(\sum_{j \neq i} G_{ij} F_{ij} \right)^2 \right)^{1/2} \left(\sum_i \left(\sum_{j \neq i} G_{ji} F_{ji} \right)^2 \right)^{1/2} \\ &\leq C \left(\sum_i \sum_{j \neq i} G_{ij}^2 \right)^{1/2} \left(\sum_i \sum_{j \neq i} G_{ji}^2 \right)^{1/2} = C \sum_i \sum_{j \neq i} G_{ij}^2. \end{aligned}$$

Proof of Lemma 4(b). Expand the term:

$$\sum_{i \neq j \neq k}^n G_{ij} F_{ij} \check{M}_{ik, -ij} V_{1i} V_{2k} V_{3j} V_{4j} = \sum_{i \neq j \neq k}^n G_{ij} F_{ij} \check{M}_{ik, -ij} (R_{1i} R_{2k} + v_{1i} R_{2k} + R_{1i} v_{2k} + v_{1i} v_{2k}) V_{3j} V_{4j}.$$

Consider the final sum with 4 stochastic terms. The 6-sums have zero covariances due to independent sampling. The 5-sums also have zero covariances, because at least one of v_1 or v_2 needs to have different indices. Within the 4-sum, the covariance is nonzero only for $j_2 \neq j$. We require i_2 to be equal to either i or k and k_2 the other index. Hence, by bounding covariances above by Cauchy-Schwarz,

$$\begin{aligned} &\text{Var} \left(\sum_{i \neq j \neq k}^n G_{ij} F_{ij} \check{M}_{ik, -ij} v_{1i} v_{2k} V_{3j} V_{4j} \right) \\ &\leq \max_{i, j, k} \text{Var} (v_{1i} v_{2k} V_{3j} V_{4j}) \sum_i \sum_{j \neq i} \sum_{k \neq i, j} \sum_{l \neq i, j, k} (G_{ij} F_{ij} G_{il} F_{il} \check{M}_{ik, -ij} \check{M}_{ik, -il} + G_{ij} F_{ij} G_{kl} F_{kl} \check{M}_{ik, -ij} \check{M}_{ki, -kl}) \\ &\quad + \max_{i, j, k} \text{Var} (v_{1i} v_{2k} V_{3j} V_{4j}) 3! \sum_{i \neq j \neq k}^n G_{ij}^2 F_{ij}^2 \check{M}_{ik, -ij}^2 \\ &\leq \max_{i, j, k} \text{Var} (v_{1i} v_{2k} V_{3j} V_{4j}) \left(\sum_{i \neq j \neq k \neq l}^n G_{ij}^2 G_{il}^2 \check{M}_{ik, -ij}^2 \right)^{1/2} \left(\sum_{i \neq j \neq k \neq l}^n F_{ij}^2 F_{il}^2 \check{M}_{ik, -ij}^2 \right)^{1/2} \\ &\quad + \max_{i, j, k} \text{Var} (v_{1i} v_{2k} V_{3j} V_{4j}) \left(\sum_{i \neq j \neq k \neq l}^n G_{ij}^2 G_{kl}^2 \check{M}_{ik, -ij}^2 \right)^{1/2} \left(\sum_{i \neq j \neq k \neq l}^n F_{ij}^2 F_{kl}^2 \check{M}_{ik, -ij}^2 \right)^{1/2} \\ &\quad + \max_{i, j, k} \text{Var} (v_{1i} v_{2k} V_{3j} V_{4j}) 3! \sum_{i \neq j \neq k}^n G_{ij}^2 F_{ij}^2 \check{M}_{ik, -ij}^2. \end{aligned}$$

To obtain the first inequality, observe that once we have fixed 3 indices, there are 3! permutations of the $v_{1i}v_{2k}V_{3j}V_{4j}$ that we can calculate covariances for. They are all bounded above by the variance. In the various combinations, we may have different combinations of G and F , but they are bounded above by the expression. To be precise, the 3-sum is:

$$\begin{aligned} & \sum_{i \neq j \neq k}^n G_{ij} F_{ij} \check{M}_{ik, -ij} (G_{ij} F_{ij} \check{M}_{ik, -ij} + G_{ik} F_{ik} \check{M}_{ij, -ik} + G_{ji} F_{ji} \check{M}_{jk, -ji}) \\ & + \sum_{i \neq j \neq k}^n G_{ij} F_{ij} \check{M}_{ik, -ij} (G_{jk} F_{jk} \check{M}_{ji, -jk} + G_{ki} F_{ki} \check{M}_{kj, -ki} + G_{kj} F_{kj} \check{M}_{ki, -kj}). \end{aligned}$$

Apply Cauchy-Schwarz to the sum and apply the commutative property of summations to obtain the upper bound. For instance,

$$\left(\sum_{i \neq j \neq k}^n G_{ij} F_{ij} \check{M}_{ik, -ij} G_{jk} F_{jk} \check{M}_{ji, -jk} \right)^2 \leq \left(\sum_{i \neq j \neq k}^n G_{ij}^2 F_{ij}^2 \check{M}_{ik, -ij}^2 \right) \left(\sum_{i \neq j \neq k}^n G_{jk}^2 F_{jk}^2 \check{M}_{ji, -jk}^2 \right).$$

Then, observe that $\sum_i \sum_{j \neq i} \sum_{k \neq i, j} G_{jk}^2 F_{jk}^2 \check{M}_{ji, -jk}^2 = \sum_j \sum_{k \neq j} \sum_{i \neq j, k} G_{jk}^2 F_{jk}^2 \check{M}_{ji, -jk}^2 = \sum_i \sum_{j \neq i} \sum_{k \neq i, j} G_{ij}^2 F_{ij}^2 \check{M}_{ik, -ij}^2$. Due to AS23 Equation (22), $\sum_l \check{M}_{il, -ijk}^2 = O(1)$, so $\sum_{i \neq j \neq k}^n G_{ij}^2 F_{ij}^2 \check{M}_{ik, -ij}^2 \leq C \sum_i \sum_{j \neq i} G_{ij}^2 F_{ij}^2 \leq C \sum_i \sum_{j \neq i} G_{ij}^2$. Similarly, $\sum_{i \neq j \neq k \neq l}^n G_{ij}^2 G_{kl}^2 \check{M}_{ik, -ij}^2 = O(1) \sum_{i \neq j \neq k}^n G_{ij}^2 \check{M}_{ik, -ij}^2 = O(1) \sum_{i \neq j}^n G_{ij}^2$, which delivers the order required.

To deal with 3 stochastic terms,

$$\begin{aligned} \text{Var} \left(\sum_{i \neq j \neq k}^n G_{ij} F_{ij} \check{M}_{ik, -ij} R_{1i} v_{2k} V_{3j} V_{4j} \right) &= \text{Var} \left(\sum_{i \neq j}^n v_{2i} V_{3j} V_{4j} \left(\sum_{k \neq i, j} G_{kj} F_{kj} \check{M}_{ki, -kj} R_{1k} \right) \right) \\ &\leq \sum_{i \neq j}^n \text{Var} (v_{2i} V_{3j} V_{4j}) \left(\sum_{k \neq i, j} G_{kj} F_{kj} \check{M}_{ki, -kj} R_{1k} \right) \left[\sum_{k \neq i, j} G_{kj} F_{kj} \check{M}_{ki, -kj} R_{1k} + \sum_{k \neq i, j} G_{ki} F_{ki} \check{M}_{kj, -ki} R_{1k} \right] \\ &+ \max_{i, j} \text{Var} (v_{2i} V_{3j} V_{4j}) \sum_{i \neq j} \sum_{l \neq i, j} \left(\sum_{k \neq i, j} G_{kj} F_{kj} \check{M}_{ki, -kj} R_{1k} \right) \left(\sum_{k \neq i, l} G_{kl} F_{kl} \check{M}_{ki, -kl} R_{1k} \right) \\ &\leq \sum_i \sum_{j \neq i} \text{Var} (v_{2i} V_{3j} V_{4j}) \left(\sum_{k \neq i, j} G_{kj} F_{kj} \check{M}_{ki, -kj} R_{1k} \right) \\ &\quad \left[\sum_{k \neq i, j} G_{kj} F_{kj} \check{M}_{ki, -kj} R_{1k} + \sum_{k \neq i, j} G_{ki} F_{ki} \check{M}_{kj, -ki} R_{1k} \right] \\ &+ \max_{i, j} \text{Var} (v_{2i} V_{3j} V_{4j}) \sum_i \left(\sum_{j \neq i} \sum_{k \neq i, j} G_{kj} F_{kj} \check{M}_{ki, -kj} R_{1k} \right)^2 \end{aligned}$$

$$\begin{aligned}
& - \max_{i,j} \text{Var} (v_{2i} V_{3j} V_{4j}) \sum_i \sum_{j \neq i} \left(\sum_{k \neq i,j} G_{kj} F_{kj} \check{M}_{ki,-kj} R_{1k} \right)^2 \\
& \leq \max_{i,j} \text{Var} (v_{2i} V_{3j} V_{4j}) \sum_i \left(\sum_{j \neq i} \sum_{k \neq i,j} G_{kj} F_{kj} \check{M}_{ki,-kj} R_{1k} \right)^2 \\
& + \sum_{i \neq j}^n \text{Var} (v_{2i} V_{3j} V_{4j}) \left(\sum_{k \neq i,j} G_{kj} F_{kj} \check{M}_{ki,-kj} R_{1k} \right) \left(\sum_{k \neq i,j} G_{ki} F_{ki} \check{M}_{kj,-ki} R_{1k} \right)
\end{aligned}$$

To get the first inequality, observe that, if for $l \neq i, j$, we have v_{2l} instead of $V_{3l} V_{4l}$, the covariance must be 0. We can then bound the order by using Assumption 3 and Lemma 3. Similarly,

$$\begin{aligned}
& \text{Var} \left(\sum_{i \neq j \neq k}^n G_{ij} F_{ij} \check{M}_{ik,-ij} v_{1i} R_{2k} V_{3j} V_{4j} \right) = \text{Var} \left(\sum_{i \neq j}^n v_{1i} V_{3j} V_{4j} \left(\sum_{k \neq i,j} G_{ij} F_{ij} \check{M}_{ik,-ij} R_{2k} \right) \right) \\
& \leq \max_{i,j} \text{Var} (v_{1i} V_{3j} V_{4j}) \sum_i \left(\sum_{j \neq i} \sum_{k \neq i,j} G_{ij} F_{ij} \check{M}_{ik,-ij} R_{2k} \right)^2 \\
& + \sum_i \sum_{j \neq i} \text{Var} (v_{1i} V_{3j} V_{4j}) \left(\sum_{k \neq i,j} G_{ij} F_{ij} \check{M}_{ik,-ij} R_{2k} \right) \left(\sum_{k \neq i,j} G_{ji} F_{ji} \check{M}_{jk,-ij} R_{2k} \right).
\end{aligned}$$

since the expansion in the intermediate steps are entirely analogous.

Turning to the sum with two stochastic objects,

$$\begin{aligned}
& \text{Var} \left(\sum_{i \neq j \neq k}^n G_{ij} F_{ij} \check{M}_{ik,-ij} R_{1i} R_{2k} V_{3j} V_{4j} \right) = \text{Var} \left(\sum_i V_{3i} V_{4i} \left(\sum_{j \neq i} \sum_{k \neq i,j} G_{ji} F_{ji} \check{M}_{jk,-ij} R_{1j} R_{2k} \right) \right) \\
& = \sum_i \text{Var} (V_{3i} V_{4i}) \left(\sum_{j \neq i} \sum_{k \neq i,j} G_{ji} F_{ji} \check{M}_{jk,-ij} R_{1j} R_{2k} \right)^2 \\
& \leq \max_i \text{Var} (V_{3i} V_{4i}) \sum_i \left(\sum_{j \neq i} \sum_{k \neq i,j} G_{ji} F_{ji} \check{M}_{jk,-ij} R_{1j} R_{2k} \right)^2.
\end{aligned}$$

With these inequalities, applying Assumption 3 suffices for the result.

Proof of Lemma 4(c). Expand the term:

$$\sum_{i \neq j \neq l}^n G_{ij} F_{ij} \check{M}_{jl,-ij} V_{1i} V_{2i} V_{3j} V_{4l} = \sum_{i \neq j \neq l}^n G_{ij} F_{ij} \check{M}_{jl,-ij} V_{1i} V_{2i} (R_{3j} R_{4l} + R_{3j} v_{4l} + v_{3j} R_{4l} + v_{3j} v_{4l}).$$

With four stochastic objects,

$$\begin{aligned}
& \text{Var} \left(\sum_{i \neq j \neq l}^n G_{ij} F_{ij} \check{M}_{jl, -ij} V_{1i} V_{2i} v_{3j} v_{4l} \right) \\
& \leq \max_{i,j,k} \text{Var} (V_{1i} V_{2i} v_{3j} v_{4l}) \sum_{i \neq j \neq l}^n \sum_{i_2 \neq i,j,l} (G_{ij} F_{ij} \check{M}_{jl, -ij} G_{i_2 j} F_{i_2 j} \check{M}_{jl, -i_2 j} + G_{ij} F_{ij} \check{M}_{jl, -ij} G_{i_2 l} F_{i_2 l} \check{M}_{lj, -i_2 l}) \\
& \quad + \max_{i,j,k} \text{Var} (V_{1i} V_{2i} v_{3j} v_{4l}) 3! \sum_{i \neq j \neq l}^n G_{ij}^2 F_{ij}^2 \check{M}_{jl, -ij}^2.
\end{aligned}$$

Simplifying the first line,

$$\begin{aligned}
& \sum_{i \neq j \neq l}^n \sum_{i_2 \neq i,j,l} (G_{ij} F_{ij} \check{M}_{jl, -ij} G_{i_2 j} F_{i_2 j} \check{M}_{jl, -i_2 j} + G_{ij} F_{ij} \check{M}_{jl, -ij} G_{i_2 l} F_{i_2 l} \check{M}_{lj, -i_2 l}) \\
& \leq \left(\sum_{i \neq j \neq l \neq i_2}^n G_{ij}^2 G_{i_2 j}^2 \check{M}_{jl, -ij}^2 \right)^{1/2} \left(\sum_{i \neq j \neq l \neq i_2}^n F_{ij}^2 F_{i_2 j}^2 \check{M}_{jl, -i_2 j}^2 \right)^{1/2} \\
& \quad + \left(\sum_{i \neq j \neq l \neq i_2}^n G_{ij}^2 G_{i_2 l}^2 \check{M}_{jl, -ij}^2 \right)^{1/2} \left(\sum_{i \neq j \neq l \neq i_2}^n F_{ij}^2 F_{i_2 l}^2 \check{M}_{lj, -i_2 j}^2 \right)^{1/2}.
\end{aligned}$$

These terms have the required order due to a proof analogous to Lemma 4(b). Next,

$$\begin{aligned}
& \text{Var} \left(\sum_{i \neq j \neq l}^n G_{ij} F_{ij} \check{M}_{jl, -ij} V_{1i} V_{2i} R_{3j} v_{4l} \right) = \text{Var} \left(\sum_i \sum_{j \neq i} V_{1i} V_{2i} v_{4j} \left(\sum_{l \neq i,j} G_{il} F_{il} \check{M}_{lj, -il} R_{3l} \right) \right) \\
& \leq \sum_i \sum_{j \neq i} \text{Var} (V_{1i} V_{2i} v_{4j}) \left(\sum_{l \neq i,j} G_{il} F_{il} \check{M}_{lj, -il} R_{3l} \right) \left[\sum_{l \neq i,j} G_{il} F_{il} \check{M}_{lj, -il} R_{3l} + \sum_{l \neq i,j} G_{jl} F_{jl} \check{M}_{li, -jl} R_{3l} \right] \\
& \quad + \max_{i,j} \text{Var} (V_{1i} V_{2i} v_{4j}) \sum_i \sum_{j \neq i} \sum_{i_2 \neq i,j} \left(\sum_{l \neq i,j} G_{il} F_{il} \check{M}_{lj, -il} R_{3l} \right) \left(\sum_{k \neq i_2, l} G_{kl} F_{kl} \check{M}_{ki_2, -kl} R_{1k} \right) \\
& \leq \max_{i,j} \text{Var} (V_{1i} V_{2i} v_{4j}) \sum_i \left(\sum_{j \neq i} \sum_{l \neq i,j} G_{il} F_{il} \check{M}_{lj, -il} R_{3l} \right)^2 \\
& \quad + \sum_i \sum_{j \neq i} \text{Var} (V_{1i} V_{2i} v_{4j}) \left(\sum_{l \neq i,j} G_{il} F_{il} \check{M}_{lj, -il} R_{3l} \right) \left(\sum_{l \neq i,j} G_{jl} F_{jl} \check{M}_{li, -jl} R_{3l} \right).
\end{aligned}$$

Further, $\text{Var} \left(\sum_{i \neq j \neq l}^n G_{ij} F_{ij} \check{M}_{jl, -ij} V_{1i} V_{2i} v_{3j} R_{4l} \right)$ can be bounded by a similar argument.

Turning to the sum with two stochastic objects,

$$\text{Var} \left(\sum_{i \neq j \neq l}^n G_{ij} F_{ij} \check{M}_{jl, -ij} V_{1i} V_{2i} R_{3j} R_{4l} \right) = \sum_i \text{Var} (V_{1i} V_{2i}) \left(\sum_{j \neq i} \sum_{l \neq i, j} G_{ij} F_{ij} \check{M}_{jl, -ij} R_{3j} R_{4l} \right)^2.$$

These inequalities suffice for the result due to Assumption 3.

Proof of Lemma 4(d). Expand the term:

$$\begin{aligned} & \sum_{i \neq j \neq k \neq l}^n G_{ij} F_{ij} \check{M}_{ik, -ij} \check{M}_{jl, -ijk} V_{1i} V_{2k} V_{3j} V_{4l} \\ &= \sum_{i \neq j \neq k \neq l}^n G_{ij} F_{ij} \check{M}_{ik, -ij} \check{M}_{jl, -ijk} V_{1i} R_{2k} (R_{3j} R_{4l} + R_{3j} v_{4l} + v_{3j} R_{4l} + v_{3j} v_{4l}) \\ &+ \sum_{i \neq j \neq k \neq l}^n G_{ij} F_{ij} \check{M}_{ik, -ij} \check{M}_{jl, -ijk} V_{1i} v_{2k} (R_{3j} R_{4l} + R_{3j} v_{4l} + v_{3j} R_{4l} + v_{3j} v_{4l}). \end{aligned}$$

Consider the v_{2k} line first. We only have the 4-sum to contend with. For 5-sum and above, at least one of the errors can be factored out as a zero expectation. Hence, by using Cauchy-Schwarz and the same argument as above,

$$\begin{aligned} & \text{Var} \left(\sum_{i \neq j \neq k \neq l}^n G_{ij} F_{ij} \check{M}_{ik, -ij} \check{M}_{jl, -ijk} V_{1i} v_{2k} v_{3j} v_{4l} \right) \\ & \leq \max_{i, j, k, l} \text{Var} (V_{1i} v_{2k} v_{3j} v_{4l}) 4! \sum_{i \neq j \neq k \neq l}^n G_{ij}^2 F_{ij}^2 \check{M}_{ik, -ij}^2 \check{M}_{jl, -ijk}^2 \\ & \leq C \sum_{i \neq j \neq k}^n G_{ij}^2 F_{ij}^2 \check{M}_{ik, -ij}^2 \leq C \sum_{i \neq j}^n G_{ij}^2 F_{ij}^2 \leq C \left(\sum_{i \neq j}^n G_{ij}^2 \right)^{1/2} \left(\sum_{i \neq j}^n F_{ij}^2 \right)^{1/2}. \end{aligned}$$

By using the same expansion step as before,

$$\begin{aligned} & \text{Var} \left(\sum_{i \neq j \neq k}^n G_{ij} F_{ij} \check{M}_{ik, -ij} V_{1i} v_{2k} v_{3j} \left(\sum_{l \neq i, j, k} \check{M}_{jl, -ijk} R_{4l} \right) \right) \\ & \leq \max_{i, j, k} \text{Var} \left(V_{1i} v_{2k} v_{3j} \left(\sum_{l \neq i, j, k} \check{M}_{jl, -ijk} R_{4l} \right) \right) \\ & \quad \sum_{i \neq j \neq k \neq i_2}^n (G_{ij} F_{ij} G_{i_2 j} F_{i_2 j} \check{M}_{ik, -ij} \check{M}_{i_2 k, -ij} + G_{ij} F_{ij} G_{i_2 k} F_{i_2 k} \check{M}_{ij, -ik} \check{M}_{i_2 j, -ik}) \\ & \quad + \max_{i, j, k} \text{Var} \left(V_{1i} v_{2k} v_{3j} \left(\sum_{l \neq i, j, k} \check{M}_{jl, -ijk} R_{4l} \right) \right) 3! \sum_i \sum_{j \neq i} \sum_{k \neq i, j} G_{ij}^2 F_{ij}^2 \check{M}_{ik, -ij}^2. \end{aligned}$$

The $\sum_{i \neq j \neq k \neq l}^n (G_{ij}F_{ij}\check{M}_{ik,-ij}\check{M}_{l2k,-ij} + G_{ij}F_{ij}G_{i2k}F_{i2k}\check{M}_{ij,-ik}\check{M}_{i2j,-ik})$ term has the required order due to the same argument as the proof of Lemma 4(b). Next,

$$\begin{aligned}
& \text{Var} \left(\sum_{i \neq j \neq k \neq l}^n G_{ij}F_{ij}\check{M}_{ik,-ij}\check{M}_{jl,-ijk}V_{1i}v_{2k}R_{3j}v_{4l} \right) = \text{Var} \left(\sum_{i \neq j \neq k}^n V_{1i}v_{2k}v_{4j} \left(\sum_{l \neq i,j,k} G_{il}F_{il}\check{M}_{ik,-il}\check{M}_{lj,-ilk}R_{3l} \right) \right) \\
& \leq \max_{i,j,k} \text{Var} (V_{1i}v_{2k}v_{4j}) \sum_{i \neq j \neq k}^n \sum_{i_2 \neq i,j,k} \left(\sum_{l \neq i,j,k} G_{il}F_{il}\check{M}_{ik,-il}\check{M}_{lj,-ilk}R_{3l} \right) \left(\sum_{l \neq i_2,j,k} G_{i_2l}F_{i_2l}\check{M}_{i_2k,-i_2l}\check{M}_{lj,-i_2lk}R_{3l} \right) \\
& \quad + \max_{i,j,k} \text{Var} (V_{1i}v_{2k}v_{4j}) \sum_{i \neq j \neq k}^n \sum_{i_2 \neq i,j,k} \left(\sum_{l \neq i,j,k} G_{il}F_{il}\check{M}_{ik,-il}\check{M}_{lj,-ilk}R_{3l} \right) \left(\sum_{l \neq i_2,j,k} G_{i_2l}F_{i_2l}\check{M}_{i_2j,-i_2l}\check{M}_{lk,-i_2lk}R_{3l} \right) \\
& \quad + \max_{i,j,k} \text{Var} (V_{1i}v_{2k}v_{4j}) 3! \sum_{i \neq j \neq k}^n \check{M}_{ik,-ij}^2 \left(\sum_{l \neq i,j,k} G_{il}F_{il}\check{M}_{ik,-il}\check{M}_{lj,-ilk}R_{3l} \right)^2 \\
& \leq \max_{i,j,k} \text{Var} (V_{1i}v_{2k}v_{4j}) \sum_k \sum_{j \neq k} \left(\sum_{i \neq k,j} \sum_{l \neq i,j,k} G_{il}F_{il}\check{M}_{ik,-il}\check{M}_{lj,-ilk}R_{3l} \right)^2 \\
& \quad - \max_{i,j,k} \text{Var} (V_{1i}v_{2k}v_{4j}) \sum_k \sum_{j \neq k} \sum_{i \neq k,j} \left(\sum_{l \neq i,j,k} G_{il}F_{il}\check{M}_{ik,-il}\check{M}_{lj,-ilk}R_{3l} \right)^2 \\
& \quad + \max_{i,j,k} \text{Var} (V_{1i}v_{2k}v_{4j}) \sum_k \sum_{j \neq k} \left(\sum_{i \neq k,j} \sum_{l \neq i,j,k} G_{il}F_{il}\check{M}_{ik,-il}\check{M}_{lj,-ilk}R_{3l} \right) \left(\sum_{i \neq k,j} \sum_{l \neq i,j,k} G_{il}F_{il}\check{M}_{ij,-il}\check{M}_{lk,-ilk}R_{3l} \right) \\
& \quad - \max_{i,j,k} \text{Var} (V_{1i}v_{2k}v_{4j}) \sum_k \sum_{j \neq k} \sum_{i \neq k,j} \left(\sum_{l \neq i,j,k} G_{il}F_{il}\check{M}_{ik,-il}\check{M}_{lj,-ilk}R_{3l} \right) \left(\sum_{l \neq i,j,k} G_{il}F_{il}\check{M}_{ij,-il}\check{M}_{lk,-ilk}R_{3l} \right) \\
& \quad + \max_{i,j,k} \text{Var} (V_{1i}v_{2k}v_{4j}) 3! \sum_{i \neq j \neq k}^n \check{M}_{ik,-ij}^2 \left(\sum_{l \neq i,j,k} G_{il}F_{il}\check{M}_{ik,-il}\check{M}_{lj,-ilk}R_{3l} \right)^2.
\end{aligned}$$

The first term in the v_{2k} line is then:

$$\begin{aligned}
& \text{Var} \left(\sum_{i \neq j \neq k \neq l}^n G_{ij}F_{ij}\check{M}_{ik,-ij}\check{M}_{jl,-ijk}V_{1i}v_{2k}R_{3j}R_{4l} \right) = \text{Var} \left(\sum_{i \neq j}^n G_{ij}F_{ij}\check{M}_{ij,-ik}V_{1i}v_{2j} \sum_{k \neq i,j} \sum_{l \neq i,j,k} \check{M}_{kl,-ijk}R_{3k}R_{4l} \right) \\
& \leq \max_{i,j} \text{Var} (V_{1i}v_{2j}) \sum_{i \neq j}^n \left(G_{ij}F_{ij} \sum_{k \neq i,j} \sum_{l \neq i,j,k} \check{M}_{ij,-ik}\check{M}_{kl,-ijk}R_{3k}R_{4l} \right)^2 \\
& \quad + \max_{i,j} \text{Var} (V_{1i}v_{2j}) \sum_{i \neq j}^n \left(G_{ij}F_{ij} \sum_{k \neq i,j} \sum_{l \neq i,j,k} \check{M}_{ij,-ik}\check{M}_{kl,-ijk}R_{3k}R_{4l} \right) \\
& \quad \left(G_{ji}F_{ji} \sum_{k \neq i,j} \sum_{l \neq i,j,k} \check{M}_{ji,-jk}\check{M}_{kl,-ijk}R_{3k}R_{4l} \right)
\end{aligned}$$

$$\begin{aligned}
& + \max_{i,j} \text{Var} (V_{1i}v_{2j}) \sum_{i \neq j \neq i_2}^n \left(G_{ij}F_{ij} \sum_{k \neq i,j} \sum_{l \neq i,j,k} \check{M}_{ij,-ik} \check{M}_{kl,-ijk} R_{3k} R_{4l} \right) \\
& \quad \left(G_{i_2j}F_{i_2j} \sum_{k \neq i_2,j} \sum_{l \neq i_2,j,k} \check{M}_{i_2j,-i_2k} \check{M}_{kl,-i_2jk} R_{3k} R_{4l} \right) \\
& \leq \max_{i,j} \text{Var} (V_{1i}v_{2j}) \sum_{i \neq j}^n \left(G_{ij}F_{ij} \sum_{k \neq i,j} \sum_{l \neq i,j,k} \check{M}_{ij,-ik} \check{M}_{kl,-ijk} R_{3k} R_{4l} \right)^2 \\
& + \max_{i,j} \text{Var} (V_{1i}v_{2j}) \sum_{i \neq j}^n \left(G_{ij}F_{ij} \sum_{k \neq i,j} \sum_{l \neq i,j,k} \check{M}_{ij,-ik} \check{M}_{kl,-ijk} R_{3k} R_{4l} \right) \\
& \quad \left(G_{ji}F_{ji} \sum_{k \neq i,j} \sum_{l \neq i,j,k} \check{M}_{ji,-jk} \check{M}_{kl,-ijk} R_{3k} R_{4l} \right) \\
& + \max_{i,j} \text{Var} (V_{1i}v_{2j}) \sum_j \left(\sum_{i \neq j} \sum_{k \neq i,j} \sum_{l \neq i,j,k} G_{ij}F_{ij} \check{M}_{ij,-ik} \check{M}_{kl,-ijk} R_{3k} R_{4l} \right)^2 \\
& - \max_{i,j} \text{Var} (V_{1i}v_{2j}) \sum_j \sum_{i \neq j} \left(\sum_{k \neq i,j} \sum_{l \neq i,j,k} G_{ij}F_{ij} \check{M}_{ij,-ik} \check{M}_{kl,-ijk} R_{3k} R_{4l} \right)^2.
\end{aligned}$$

Now, we turn back to the R_{2k} expression to complete the proof:

$$\sum_{i \neq j \neq k \neq l}^n G_{ij}F_{ij} \check{M}_{ik,-ij} \check{M}_{jl,-ijk} V_{1i} R_{2k} (R_{3j} R_{4l} + R_{3j} v_{4l} + v_{3j} R_{4l} + v_{3j} v_{4l}).$$

Consider the term with three stochastic terms first, and simplify it using the same strategy as before:

$$\begin{aligned}
& \text{Var} \left(\sum_{i \neq j \neq k \neq l}^n G_{ij}F_{ij} \check{M}_{ik,-ij} \check{M}_{jl,-ijk} V_{1i} R_{2k} v_{3j} v_{4l} \right) = \text{Var} \left(\sum_{i \neq j \neq k}^n G_{ij}F_{ij} V_{1i} v_{3j} v_{4k} \sum_{l \neq i,j,k} \check{M}_{il,-ij} \check{M}_{jk,-ijl} R_{2l} \right) \\
& \leq \max_{i,j,k} \text{Var} (V_{1i} v_{3j} v_{4k}) \left(\sum_{k \neq j}^n \left(\sum_{i \neq k,j} \sum_{l \neq i,j,k} G_{ij}F_{ij} \check{M}_{il,-ij} \check{M}_{jk,-ijl} R_{2l} \right)^2 \right. \\
& \quad \left. - \sum_{k \neq j} \sum_{i \neq k,j} \left(\sum_{l \neq i,j,k} G_{ij}F_{ij} \check{M}_{il,-ij} \check{M}_{jk,-ijl} R_{2l} \right)^2 \right) \\
& + \max_{i,j,k} \text{Var} (V_{1i} v_{3j} v_{4k}) \sum_k \sum_{j \neq k} \left(\sum_{i \neq k,j} \sum_{l \neq i,j,k} G_{ij}F_{ij} \check{M}_{il,-ij} \check{M}_{jk,-ijl} R_{2l} \right) \left(\sum_{i \neq k,j} \sum_{l \neq i,j,k} G_{ik}F_{ik} \check{M}_{il,-ik} \check{M}_{kj,-ikl} R_{2l} \right) \\
& - \max_{i,j,k} \text{Var} (V_{1i} v_{3j} v_{4k}) \sum_k \sum_{j \neq k} \sum_{i \neq k,j} \left(\sum_{l \neq i,j,k} G_{ij}F_{ij} \check{M}_{il,-ij} \check{M}_{jk,-ijl} R_{2l} \right) \left(\sum_{l \neq i,j,k} G_{ik}F_{ik} \check{M}_{il,-ik} \check{M}_{kj,-ikl} R_{2l} \right)
\end{aligned}$$

$$+ \max_{i,j,k} \text{Var} (V_{1i} v_{3j} v_{4k}) 3! \sum_{i \neq j \neq k}^n \left(G_{ij} F_{ij} \sum_{l \neq i,j,k} \check{M}_{il,-ij} \check{M}_{jk,-ijl} R_{2l} \right)^2.$$

Next,

$$\begin{aligned} & \text{Var} \left(\sum_i \sum_{j \neq i} \sum_{k \neq i,j} \sum_{l \neq i,j,k} G_{ij} F_{ij} \check{M}_{ik,-ij} \check{M}_{jl,-ijk} V_{1i} R_{2k} v_{3j} R_{4l} \right) \\ & \leq \max_{i,j} \text{Var} (V_{1i} v_{3j}) \sum_{i \neq j}^n \left(G_{ij} F_{ij} \sum_{k \neq i,j} \sum_{l \neq i,j,k} \check{M}_{ik,-ij} \check{M}_{jl,-ijk} R_{2k} R_{4l} \right)^2 \\ & + \max_{i,j} \text{Var} (V_{1i} v_{3j}) \sum_{i \neq j}^n \left(G_{ij} F_{ij} \sum_{k \neq i,j} \sum_{l \neq i,j,k} \check{M}_{ik,-ij} \check{M}_{jl,-ijk} R_{2k} R_{4l} \right) \left(G_{ji} F_{ji} \sum_{k \neq i,j} \sum_{l \neq i,j,k} \check{M}_{jk,-ij} \check{M}_{il,-ijk} R_{2k} R_{4l} \right) \\ & + \max_{i,j} \text{Var} (V_{1i} v_{3j}) \sum_j \left(\sum_{i \neq j} G_{ij} F_{ij} \sum_{k \neq i,j} \sum_{l \neq i,j,k} \check{M}_{ik,-ij} \check{M}_{jl,-ijk} R_{2k} R_{4l} \right)^2 \\ & - \max_{i,j} \text{Var} (V_{1i} v_{3j}) \sum_{j \neq i}^n \left(G_{ij} F_{ij} \sum_{k \neq i,j} \sum_{l \neq i,j,k} \check{M}_{ik,-ij} \check{M}_{jl,-ijk} R_{2k} R_{4l} \right)^2. \end{aligned}$$

Finally,

$$\begin{aligned} & \text{Var} \left(\sum_{i \neq j \neq k \neq l}^n G_{ij} F_{ij} \check{M}_{ik,-ij} \check{M}_{jl,-ijk} V_{1i} R_{2k} R_{3j} R_{4l} \right) \\ & = \sum_i \text{Var} (V_{1i}) \left(\sum_{j \neq i} \sum_{k \neq i,j} \sum_{l \neq i,j,k} G_{ij} F_{ij} \check{M}_{ik,-ij} \check{M}_{jl,-ijk} R_{2k} R_{3j} R_{4l} \right)^2. \end{aligned}$$

□

Proof of Lemma 5. Proof of Lemma 5(a). Expand the term:

$$\sum_{i \neq j \neq k}^n G_{ij} F_{ik} V_{1j} V_{2k} V_{3i} V_{4i} = \sum_{i \neq j \neq k}^n G_{ij} F_{ik} V_{3i} V_{4i} (R_{1j} R_{2k} + R_{1j} v_{2k} + v_{1j} R_{2k} + v_{1j} v_{2k}).$$

With four stochastic objects,

$$\begin{aligned} \text{Var} \left(\sum_{i \neq j \neq k}^n G_{ij} F_{ik} V_{3i} V_{4i} v_{1j} v_{2k} \right) & \leq \max_{i,j,k} \text{Var} (V_{3i} V_{4i} v_{1j} v_{2k}) \sum_{i \neq j \neq k}^n \sum_{i_2 \neq i,j,k} (G_{ij} F_{ik} G_{i_2 j} F_{i_2 k} + G_{ij} F_{ik} G_{i_2 k} F_{i_2 j}) \\ & + \max_{i,j,k} \text{Var} (V_{1i} V_{2i} v_{3j} v_{4l}) 3! \sum_{i \neq j \neq k}^n G_{ij}^2 F_{ik}^2. \end{aligned}$$

Observe that, due to Assumption 3(a),

$$\begin{aligned} \sum_{i \neq j \neq k \neq l}^n G_{ij} F_{ik} G_{lj} F_{lk} &= \sum_{j \neq k}^n \left(\sum_{i \neq j, k} G_{ij} F_{ik} \right) \left(\sum_{l \neq j, k} G_{lj} F_{lk} - G_{ij} F_{ik} \right) \\ &= \sum_{j \neq k}^n \left(\sum_{i \neq j, k} G_{ij} F_{ik} \right)^2 - \sum_{j \neq k \neq i}^n G_{ij}^2 F_{ik}^2 \end{aligned}$$

has the required order, which suffices for the bound. Next,

$$\begin{aligned} &\text{Var} \left(\sum_{i \neq j \neq k}^n G_{ij} F_{ik} V_{3i} V_{4i} R_{1j} v_{2k} \right) \\ &= \text{Var} \left(\sum_i \sum_{j \neq i} F_{ij} V_{3i} V_{4i} v_{2j} \left(\sum_{k \neq i, j} G_{ik} R_{1k} \right) \right) \\ &\leq \sum_i \sum_{j \neq i} \text{Var} (V_{3i} V_{4i} v_{2j}) \left(\sum_{k \neq i, j} F_{ij} G_{ik} R_{1k} \right) \left[\sum_{k \neq i, j} F_{ij} G_{ik} R_{1k} + \sum_{k \neq i, j} F_{ji} G_{jk} R_{1k} \right] \\ &\quad + \max_{i, j} \text{Var} (V_{3i} V_{4i} v_{2j}) \sum_i \sum_{j \neq i} \sum_{i_2 \neq i, j} \left(\sum_{k \neq i, j} F_{ij} G_{ik} R_{1k} \right) \left(\sum_{k \neq i_2, l} F_{i_2 j} G_{i_2 k} R_{1k} \right) \\ &\leq \max_{i, j} \text{Var} (V_{3i} V_{4i} v_{2j}) \sum_i \left(\sum_{j \neq i} \sum_{k \neq i, j} F_{ij} G_{ik} R_{1k} \right)^2 \\ &\quad + \sum_i \sum_{j \neq i} \text{Var} (V_{3i} V_{4i} v_{2j}) \left(\sum_{k \neq i, j} F_{ij} G_{ik} R_{1k} \right) \left(\sum_{k \neq i, j} F_{ji} G_{jk} R_{1k} \right). \end{aligned}$$

Similarly,

$$\begin{aligned} &\text{Var} \left(\sum_{i \neq j \neq k}^n G_{ij} F_{ik} V_{3i} V_{4i} v_{1j} R_{2k} \right) = \text{Var} \left(\sum_i \sum_{j \neq i} V_{3i} V_{4i} v_{1j} \left(\sum_{k \neq i, j} G_{ij} F_{ik} R_{2k} \right) \right) \\ &\leq \max_{i, j} \text{Var} (V_{3i} V_{4i} v_{1j}) \sum_i \left(\left(\sum_{j \neq i} \sum_{k \neq i, j} G_{ij} F_{ik} R_{2k} \right)^2 + \sum_{j \neq i} \left(\sum_{k \neq i, j} G_{ij} F_{ik} R_{2k} \right) \left(\sum_{k \neq i, j} G_{ji} F_{jk} R_{2k} \right) \right). \end{aligned}$$

Turning to the sum with two stochastic objects,

$$\text{Var} \left(\sum_{i \neq j \neq k}^n G_{ij} F_{ik} V_{3i} V_{4i} R_{1j} R_{2k} \right) = \text{Var} \left(\sum_i V_{3i} V_{4i} \left(\sum_{j \neq i} \sum_{k \neq i, j} G_{ij} F_{ik} R_{1j} R_{2k} \right) \right)$$

$$\leq \max_i \text{Var} (V_{3i} V_{4i}) \sum_i \left(\sum_{j \neq i} \sum_{k \neq i, j} G_{ij} F_{ik} R_{1j} R_{2k} \right)^2.$$

Proof of Lemma 5(b).

Decompose the term:

$$\begin{aligned} & \sum_{i \neq j \neq k \neq l}^n G_{ij} F_{ik} \check{M}_{il, -ijk} V_{1j} V_{2k} V_{3i} V_{4l} \\ &= \sum_{i \neq j \neq k \neq l}^n G_{ij} F_{ik} \check{M}_{il, -ijk} V_{3i} R_{1j} (R_{2k} R_{4l} + R_{2k} v_{4l} + v_{2k} R_{4l} + v_{2k} v_{4l}) \\ &+ \sum_{i \neq j \neq k \neq l}^n G_{ij} F_{ik} \check{M}_{il, -ijk} V_{3i} v_{1j} (R_{2k} R_{4l} + R_{2k} v_{4l} + v_{2k} R_{4l} + v_{2k} v_{4l}). \end{aligned}$$

Consider the v_{1j} line first.

$$\text{Var} \left(\sum_{i \neq j \neq k \neq l}^n G_{ij} F_{ik} \check{M}_{il, -ijk} V_{3i} v_{1j} v_{2k} v_{4l} \right) \leq \max_{i, j, k, l} \text{Var} (V_{3i} v_{1j} v_{2k} v_{4l}) 4! \sum_{i \neq j \neq k \neq l}^n G_{ij}^2 F_{ik}^2 \check{M}_{il, -ijk}^2.$$

Next, by using the same expansion and simplification steps as before,

$$\begin{aligned} & \text{Var} \left(\sum_{i \neq j \neq k \neq l}^n G_{ij} F_{ik} \check{M}_{il, -ijk} V_{3i} v_{1j} v_{2k} R_{4l} \right) = \text{Var} \left(\sum_{i \neq j \neq k}^n G_{ij} F_{ik} V_{3i} v_{1j} v_{2k} \sum_{l \neq i, j, k} \check{M}_{il, -ijk} R_{4l} \right) \\ & \leq \max_{i, j, k} \text{Var} (V_{3i} v_{1j} v_{2k}) \sum_k \sum_{j \neq k} \left(\left(\sum_{i \neq j, k} \sum_{l \neq i, j, k} G_{ij} F_{ik} \check{M}_{il, -ijk} R_{4l} \right)^2 - \sum_{i \neq j, k} \left(\sum_{l \neq i, j, k} G_{ij} F_{ik} \check{M}_{il, -ijk} R_{4l} \right)^2 \right) \\ & + \max_{i, j, k} \text{Var} (V_{3i} v_{1j} v_{2k}) \sum_k \sum_{j \neq k} \left(\sum_{i \neq j, k} \sum_{l \neq i, j, k} G_{ij} F_{ik} \check{M}_{il, -ijk} R_{4l} \right) \left(\sum_{i \neq j, k} \sum_{l \neq i, j, k} G_{ik} F_{ij} \check{M}_{il, -ijk} R_{4l} \right) \\ & - \max_{i, j, k} \text{Var} (V_{3i} v_{1j} v_{2k}) \sum_k \sum_{j \neq k} \sum_{i \neq j, k} \left(\sum_{l \neq i, j, k} G_{ij} F_{ik} \check{M}_{il, -ijk} R_{4l} \right) \left(\sum_{l \neq i, j, k} G_{ik} F_{ij} \check{M}_{il, -ijk} R_{4l} \right) \\ & + \max_{i, j, k} \text{Var} (V_{3i} v_{1j} v_{2k}) 3! \sum_{i \neq j \neq k}^n G_{ij}^2 F_{ik}^2 \left(\sum_{l \neq i, j, k} \check{M}_{il, -ijk} R_{4l} \right)^2 \end{aligned}$$

and

$$\text{Var} \left(\sum_{i \neq j \neq k \neq l}^n G_{ij} F_{ik} \check{M}_{il, -ijk} V_{3i} v_{1j} R_{2k} v_{4l} \right) = \text{Var} \left(\sum_{i \neq j \neq k}^n G_{ij} F_{ik} V_{3i} v_{1j} v_{4k} \sum_{l \neq i, j, k} \check{M}_{ik, -ijl} R_{2l} \right)$$

$$\begin{aligned}
&\leq \max_{i,j,k} \text{Var} (V_{3i}v_{1j}v_{4k}) \sum_k \sum_{j \neq k} \left(\left(\sum_{i \neq j,k} \sum_{l \neq i,j,k} G_{ij} F_{ik} \check{M}_{ik,-ijl} R_{2l} \right)^2 - \sum_{i \neq j,k} \left(\sum_{l \neq i,j,k} G_{ij} F_{ik} \check{M}_{ik,-ijl} R_{2l} \right)^2 \right) \\
&+ \max_{i,j,k} \text{Var} (V_{3i}v_{1j}v_{2k}) \sum_k \sum_{j \neq k} \left(\sum_{i \neq j,k} \sum_{l \neq i,j,k} G_{ij} F_{ik} \check{M}_{ik,-ijl} R_{2l} \right) \left(\sum_{i \neq j,k} \sum_{l \neq i,j,k} G_{ik} F_{ij} \check{M}_{il,-ijk} R_{2l} \right) \\
&- \max_{i,j,k} \text{Var} (V_{3i}v_{1j}v_{2k}) \sum_k \sum_{j \neq k} \sum_{i \neq j,k} \left(\sum_{l \neq i,j,k} G_{ij} F_{ik} \check{M}_{ik,-ijl} R_{2l} \right) \left(\sum_{l \neq i,j,k} G_{ik} F_{ij} \check{M}_{il,-ijk} R_{2l} \right) \\
&+ \max_{i,j,k} \text{Var} (V_{3i}v_{1j}v_{4k}) 3! \sum_{i \neq j \neq k}^n G_{ij}^2 F_{ik}^2 \left(\sum_{l \neq i,j,k} \check{M}_{ik,-ijl} R_{2l} \right)^2
\end{aligned}$$

with $\left(\sum_{l \neq i,j,k} \check{M}_{ik,-ijl} R_{2l} \right)^2 \leq C$. Finally,

$$\begin{aligned}
&\text{Var} \left(\sum_{i \neq j \neq k \neq l}^n G_{ij} F_{ik} \check{M}_{il,-ijk} V_{3i} v_{1j} R_{2k} R_{4l} \right) = \text{Var} \left(\sum_{i \neq j}^n G_{ij} V_{3i} v_{1j} \sum_{k \neq i,j} \sum_{l \neq i,j,k} F_{ik} \check{M}_{il,-ijk} R_{2k} R_{4l} \right) \\
&\leq \max_{i,j} \text{Var} (V_{3i} v_{1j}) \sum_{i \neq j}^n G_{ij} \left(\sum_{k \neq i,j} \sum_{l \neq i,j,k} F_{ik} \check{M}_{il,-ijk} R_{2k} R_{4l} \right)^2 \\
&+ \max_{i,j} \text{Var} (V_{3i} v_{1j}) \sum_{i \neq j}^n \left(\sum_{k \neq i,j} \sum_{l \neq i,j,k} G_{ij} F_{ik} \check{M}_{il,-ijk} R_{2k} R_{4l} \right) \left(\sum_{k \neq i,j} \sum_{l \neq i,j,k} G_{ji} F_{jk} \check{M}_{jl,-ijk} R_{2k} R_{4l} \right) \\
&+ \max_{i,j} \text{Var} (V_{3i} v_{1j}) \sum_j \left(\left(\sum_{i \neq j} \sum_{k \neq i,j} \sum_{l \neq i,j,k} G_{ij} F_{ik} \check{M}_{il,-ijk} R_{2k} R_{4l} \right)^2 - \sum_{i \neq j} \left(\sum_{k \neq i,j} \sum_{l \neq i,j,k} G_{ij} F_{ik} \check{M}_{il,-ijk} R_{2k} R_{4l} \right)^2 \right).
\end{aligned}$$

Now, return to the R_{1j} line: $\sum_{i \neq j \neq k \neq l}^n G_{ij} F_{ik} \check{M}_{il,-ijk} V_{3i} R_{1j} (R_{2k} R_{4l} + R_{2k} v_{4l} + v_{2k} R_{4l} + v_{2k} v_{4l})$, so

$$\begin{aligned}
&\text{Var} \left(\sum_{i \neq j \neq k \neq l}^n G_{ij} F_{ik} \check{M}_{il,-ijk} V_{3i} R_{1j} v_{2k} v_{4l} \right) = \text{Var} \left(\sum_{i \neq j \neq k}^n G_{il} F_{ik} V_{3i} v_{2k} v_{4j} \sum_{l \neq i,j,k} \check{M}_{ij,-ilk} R_{1l} \right) \\
&\leq \max_{i,j,k} \text{Var} (V_{3i} v_{2k} v_{4j}) \left(\sum_j \sum_{k \neq j} \left(\sum_{i \neq j,k} \sum_{l \neq i,j,k} G_{il} F_{ik} \check{M}_{ij,-ilk} R_{1l} \right)^2 - \sum_j \sum_{k \neq j} \sum_{i \neq j,k} \left(\sum_{l \neq i,j,k} G_{il} F_{ik} \check{M}_{ij,-ilk} R_{1l} \right)^2 \right) \\
&+ \max_{i,j,k} \text{Var} (V_{3i} v_{2k} v_{4j}) \sum_j \sum_{k \neq j} \left(\sum_{i \neq j,k} \sum_{l \neq i,j,k} G_{il} F_{ik} \check{M}_{ij,-ilk} R_{1l} \right) \left(\sum_{i \neq j,k} \sum_{l \neq i,j,k} G_{il} F_{ij} \check{M}_{ik,-ilj} R_{1l} \right) \\
&- \max_{i,j,k} \text{Var} (V_{3i} v_{2k} v_{4j}) \sum_j \sum_{k \neq j} \sum_{i \neq j,k} \left(\sum_{l \neq i,j,k} G_{il} F_{ik} \check{M}_{ij,-ilk} R_{1l} \right) \left(\sum_{l \neq i,j,k} G_{il} F_{ij} \check{M}_{ik,-ilj} R_{1l} \right)
\end{aligned}$$

$$+ \max_{i,j,k} \text{Var} (V_{3i} v_{2k} v_{4j}) 3! \sum_{i \neq j \neq k}^n \left(F_{ik} \sum_{l \neq i,j,k} G_{il} \check{M}_{ij,-ilk} R_{1l} \right)^2,$$

and

$$\begin{aligned} \text{Var} \left(\sum_{i \neq j \neq k \neq l}^n G_{ij} F_{ik} \check{M}_{il,-ijk} V_{3i} R_{1j} v_{2k} R_{4l} \right) &= \text{Var} \left(\sum_{i \neq j}^n F_{ij} V_{3i} v_{2j} \sum_{k \neq i,j} \sum_{l \neq i,j,k} G_{ik} \check{M}_{il,-ijk} R_{1k} R_{4l} \right) \\ &\leq \max_{i,j} \text{Var} (V_{3i} v_{2j}) \sum_{i \neq j}^n \left(\sum_{k \neq i,j} \sum_{l \neq i,j,k} F_{ij} G_{ik} \check{M}_{il,-ijk} R_{1k} R_{4l} \right)^2 \\ &+ \max_{i,j} \text{Var} (V_{3i} v_{2j}) \sum_{i \neq j}^n \left(\sum_{k \neq i,j} \sum_{l \neq i,j,k} F_{ij} G_{ik} \check{M}_{il,-ijk} R_{1k} R_{4l} \right) \left(\sum_{k \neq i,j} \sum_{l \neq i,j,k} F_{ij} G_{jk} \check{M}_{jl,-ijk} R_{1k} R_{4l} \right) \\ &+ \max_{i,j} \text{Var} (V_{3i} v_{2j}) \sum_j \left(\left(\sum_{i \neq j} \sum_{k \neq i,j} \sum_{l \neq i,j,k} F_{ij} G_{ik} \check{M}_{il,-ijk} R_{1k} R_{4l} \right)^2 - \sum_{i \neq j} \left(\sum_{k \neq i,j} \sum_{l \neq i,j,k} F_{ij} G_{ik} \check{M}_{il,-ijk} R_{1k} R_{4l} \right)^2 \right). \end{aligned}$$

The $\sum_{i \neq j \neq k \neq l} G_{ij} F_{ik} \check{M}_{il,-ijk} V_{3i} R_{1j} R_{2k} v_{4l}$ term is symmetric, because it does not matter which R_m we use. Finally,

$$\text{Var} \left(\sum_{i \neq j \neq k \neq l}^n G_{ij} F_{ik} \check{M}_{il,-ijk} V_{3i} R_{1j} R_{2k} R_{4l} \right) = \sum_i \text{Var} (V_{3i}) \left(\sum_{j \neq i} \sum_{k \neq i,j} \sum_{l \neq i,j,k} G_{ij} F_{ik} \check{M}_{il,-ijk} R_{1j} R_{2k} R_{4l} \right)^2.$$

□

Proof of Lemma 6. The proof of Lemma 6 is entirely analogous to Lemmas 4 and 5 just that G_{ji} is used in place of G_{ij} . □

E.2 Proofs for Appendix D

E.2.1 Proofs for Propositions in Appendix D

Proof of Proposition 2. Let

$$\begin{pmatrix} \Pi_Y \\ \Pi \end{pmatrix} := \begin{pmatrix} (Z'Z)^{1/2} \pi_Y \\ (Z'Z)^{1/2} \pi \end{pmatrix}.$$

With this definition, $(\pi_Y' Z' Z \pi_Y, \pi' Z' Z \pi_Y, \pi' Z' Z \pi) = (\Pi_Y' \Pi_Y, \Pi_Y' \Pi, \Pi' \Pi)$, and

$$\begin{pmatrix} s_1 \\ s_2 \end{pmatrix} \sim N \left(\begin{pmatrix} \Pi_Y \\ \Pi \end{pmatrix}, \Omega \otimes I_K \right).$$

Split s_1 and s_2 into the Π component and a random normal component: $s_{1k} = \Pi_{Yk} + z_{1k}$

and $s_{2k} = \Pi_k + z_{2k}$. Then, for all k ,

$$\begin{pmatrix} z_{1k} \\ z_{2k} \end{pmatrix} \sim N \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{bmatrix} \omega_{\zeta\zeta} & \omega_{\zeta\eta} \\ \omega_{\zeta\eta} & \omega_{\eta\eta} \end{bmatrix} \right), \text{ and}$$

$$\begin{aligned} \begin{pmatrix} s'_1 s_1 \\ s'_1 s_2 \\ s'_2 s_2 \end{pmatrix} &= \begin{pmatrix} \sum_k s_{1k}^2 \\ \sum_k s_{1k} s_{2k} \\ \sum_k s_{2k}^2 \end{pmatrix} = \begin{pmatrix} \sum_k (\Pi_{Yk} + z_{1k})^2 \\ \sum_k (\Pi_{Yk} + z_{1k}) (\Pi_k + z_{2k}) \\ \sum_k (\Pi_k + z_{2k})^2 \end{pmatrix} \\ &= \begin{pmatrix} \sum_k \Pi_{Yk}^2 + 2 \sum_k \Pi_{Yk} z_{1k} + \sum_k z_{1k}^2 \\ \sum_k \Pi_{Yk} \Pi_k + \sum_k \Pi_{Yk} z_{2k} + \sum_k \Pi_k z_{1k} + \sum_k z_{1k} z_{2k} \\ \sum_k \Pi_k^2 + 2 \sum_k \Pi_k z_{2k} + \sum_k z_{2k}^2 \end{pmatrix}. \end{aligned}$$

Under the assumption, $\Pi' \Pi / \sqrt{K} \rightarrow C_S$, so $\frac{1}{\sqrt{K}} \sum_k \Pi_k^2 \rightarrow C_S$. By applying the Lindeberg CLT due to bounded moments,

$$\frac{1}{\sqrt{K}} \begin{pmatrix} \sum_k \Pi_k z_{1k} \\ \sum_k \Pi_{Yk} z_{1k} \\ \sum_k \Pi_{Yk} z_{2k} \\ \sum_k \Pi_k z_{2k} \\ \sum_k z_{1k} z_{2k} \\ \sum_k z_{2k}^2 \\ \sum_k z_{1k}^2 \end{pmatrix} \stackrel{a}{\sim} N \left(\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \sqrt{K} \omega_{\zeta\eta} \\ \sqrt{K} \omega_{\eta\eta} \\ \sqrt{K} \omega_{\zeta\zeta} \end{pmatrix}, V \right),$$

where V is some variance matrix. By assumption, $\frac{1}{\sqrt{K}} \sum_k \Pi_{Yk} \Pi_k \rightarrow C_Y$ and $\frac{1}{\sqrt{K}} \sum_k \Pi_{Yk}^2 \rightarrow C_{YY}$, so

$$\begin{aligned} \frac{1}{\sqrt{K}} \begin{pmatrix} s'_1 s_1 \\ s'_1 s_2 \\ s'_2 s_2 \end{pmatrix} &= \frac{1}{\sqrt{K}} \begin{pmatrix} \sum_k \Pi_{Yk}^2 + 2 \sum_k \Pi_{Yk} z_{1k} + \sum_k z_{1k}^2 \\ \sum_k \Pi_{Yk} \Pi_k + \sum_k \Pi_{Yk} z_{2k} + \sum_k \Pi_k z_{1k} + \sum_k z_{1k} z_{2k} \\ \sum_k \Pi_k^2 + 2 \sum_k \Pi_k z_{2k} + \sum_k z_{2k}^2 \end{pmatrix} \\ &\stackrel{a}{\sim} \begin{pmatrix} C_{YY} \\ C_Y \\ C \end{pmatrix} + A \frac{1}{\sqrt{K}} \begin{pmatrix} \sum_k \Pi_k z_{1k} \\ \sum_k \Pi_{Yk} z_{1k} \\ \sum_k \Pi_{Yk} z_{2k} \\ \sum_k \Pi_k z_{2k} \\ \sum_k z_{1k} z_{2k} \\ \sum_k z_{2k}^2 \\ \sum_k z_{1k}^2 \end{pmatrix}, \text{ where} \end{aligned}$$

$$A = \begin{pmatrix} 0 & 2 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 1 & 0 \end{pmatrix}.$$

This means:

$$\frac{1}{\sqrt{K}} \begin{pmatrix} s'_1 s_1 \\ s'_1 s_2 \\ s'_2 s_2 \end{pmatrix} \stackrel{a}{\sim} N \left(\begin{pmatrix} C_{YY} + \sqrt{K} \omega_{\zeta\zeta} \\ C_Y + \sqrt{K} \omega_{\zeta\eta} \\ C + \sqrt{K} \omega_{\eta\eta} \end{pmatrix}, AVA' \right).$$

Let $\Sigma = AVA'$ to obtain the result as stated. To derive Σ explicitly, I derive V by applying the Isserlis' Theorem. As a special case of the Isserlis' Theorem for X 's that are multivariate normal and mean zero,

$$E[X_1 X_2 X_3 X_4] = E[X_1 X_2] E[X_3 X_4] + E[X_1 X_3] E[X_2 X_4] + E[X_1 X_4] E[X_2 X_3].$$

Another corollary is that if n is odd, then there is no such pairing, so the moment is always zero. Hence,

$$\begin{aligned} E[z_{1k}^2 z_{2k}^2] &= E[z_{1k}^2] E[z_{2k}^2] + 2E[z_{1k} z_{2k}] E[z_{1k} z_{2k}] = \omega_{\zeta\zeta} \omega_{\eta\eta} + 2\omega_{\zeta\eta}^2, \text{ and} \\ \text{Var}(z_{1k} z_{2k}) &= \omega_{\zeta\zeta} \omega_{\eta\eta} + \omega_{\zeta\eta}^2. \end{aligned}$$

Similarly,

$$\begin{aligned} \text{Var}(z_{2k}^2) &= E[z_{2k}^4] - \omega_{\eta\eta}^2 = 3\omega_{\eta\eta}^2 - \omega_{\eta\eta}^2 = 2\omega_{\eta\eta}^2, \\ \text{Cov}(z_{1k}, z_{1k} z_{2k}) &= E[z_{1k}^2 z_{2k}] - E[z_{1k}] E[z_{1k} z_{2k}] = 0, \\ \text{Cov}(z_{1k}^2, z_{1k} z_{2k}) &= E[z_{1k}^3 z_{2k}] - E[z_{1k}^2] E[z_{1k} z_{2k}] \\ &= 3\omega_{\zeta\eta} \omega_{\zeta\zeta} - \omega_{\zeta\zeta} \omega_{\zeta\eta} = 2\omega_{\zeta\eta} \omega_{\zeta\zeta}, \\ \text{Cov}(z_{1k}^2, z_{2k}^2) &= E[z_{1k}^2 z_{2k}^2] - \omega_{\zeta\zeta} \omega_{\eta\eta} = 2\omega_{\zeta\eta}^2, \text{ and} \end{aligned}$$

$$V = \begin{bmatrix} V_{11} & 0 \\ 0 & V_{22} \end{bmatrix}, \text{ where}$$

$$\begin{aligned} V_{11} &= \begin{bmatrix} \frac{1}{K} \sum_k \Pi_k^2 \omega_{\zeta\zeta} & \frac{1}{K} \sum_k \Pi_k \Pi_{Yk} \omega_{\zeta\zeta} & \frac{1}{K} \sum_k \Pi_k \Pi_{Yk} \omega_{\zeta\eta} & \frac{1}{K} \sum_k \Pi_k^2 \omega_{\zeta\eta} \\ \cdot & \frac{1}{K} \sum_k \Pi_{Yk}^2 \omega_{\zeta\zeta} & \frac{1}{K} \sum_k \Pi_{Yk}^2 \omega_{\zeta\eta} & \frac{1}{K} \sum_k \Pi_k \Pi_{Yk} \omega_{\zeta\eta} \\ \cdot & \cdot & \frac{1}{K} \sum_k \Pi_{Yk}^2 \omega_{\eta\eta} & \frac{1}{K} \sum_k \Pi_k \Pi_{Yk} \omega_{\eta\eta} \\ \cdot & \cdot & \cdot & \frac{1}{K} \sum_k \Pi_k^2 \omega_{\eta\eta} \end{bmatrix}, \\ V_{22} &= \begin{bmatrix} \omega_{\zeta\zeta} \omega_{\eta\eta} + \omega_{\zeta\eta}^2 & 2\omega_{\zeta\eta} \omega_{\eta\eta} & 2\omega_{\zeta\eta} \omega_{\zeta\zeta} \\ \cdot & 2\omega_{\eta\eta}^2 & 2\omega_{\zeta\eta}^2 \\ \cdot & \cdot & 2\omega_{\zeta\zeta}^2 \end{bmatrix}. \end{aligned}$$

If $\frac{1}{K} \sum_k \Pi_k^2 \rightarrow 0$, $\frac{1}{K} \sum_k \Pi_k \Pi_{Yk} \rightarrow 0$, $\frac{1}{K} \sum_k \Pi_{Yk}^2 \rightarrow 0$ under weak identification, then we obtain the Σ expression stated in the proposition. \square

Proof of Proposition 3. Use n_q^Q and n_w^W to denote the number of observations in the instru-

ment and covariate groups respectively, so

$$\begin{aligned}
\mu_3 &= \sum_i \sum_{j \neq i} G_{ij} R_i R_j = \sum_q \frac{n_q^Q}{n_q^Q - 1} \sum_{i \in \mathcal{N}_q^Q} \sum_{j \in \mathcal{N}_q^Q, j \neq i} \frac{1}{n_q^Q} R_i R_j - \sum_w \frac{n_w^W}{n_w^W - 1} \sum_{i \in \mathcal{N}_w^W} \sum_{j \in \mathcal{N}_w^W, j \neq i} \frac{1}{n_w^W} R_i R_j \\
&= \sum_q \frac{1}{n_q^Q - 1} \sum_{i \in \mathcal{N}_q^Q} \sum_{j \in \mathcal{N}_q^Q, j \neq i} R_i R_j - \sum_w \frac{1}{n_w^W - 1} \sum_{i \in \mathcal{N}_w^W} \sum_{j \in \mathcal{N}_w^W, j \neq i} R_i R_j \\
&= \sum_w \left(\sum_{q \in \mathcal{M}_w} \frac{1}{n_q^Q - 1} \sum_{i \in \mathcal{N}_q^Q} \sum_{j \in \mathcal{N}_q^Q, j \neq i} R_i R_j - \frac{1}{n_w^W - 1} \sum_{i \in \mathcal{N}_w^W} \sum_{j \in \mathcal{N}_w^W, j \neq i} R_i R_j \right) \\
&= \sum_w \left(\sum_{q \in \mathcal{M}_w} \frac{n_q^Q (n_q^Q - 1)}{n_q^Q - 1} (\pi_q + \gamma_w)^2 - \frac{1}{n_w^W - 1} \sum_{i \in \mathcal{N}_w^W} \sum_{j \in \mathcal{N}_w^W, j \neq i} R_i R_j \right).
\end{aligned}$$

Considering the second term,

$$\begin{aligned}
\sum_{i \in \mathcal{N}_w^W} \sum_{j \in \mathcal{N}_w^W, j \neq i} R_i R_j &= \sum_{i \in \mathcal{N}_w^W} \sum_{j \in \mathcal{N}_w^W, j \neq i} (\pi_{q(i)} + \gamma_w) (\pi_{q(j)} + \gamma_w) \\
&= \sum_{q \in \mathcal{M}_w} \sum_{i \in \mathcal{N}_q} \sum_{j \in \mathcal{N}_w^W, j \neq i} (\pi_{q(i)} + \gamma_w) (\pi_{q(j)} + \gamma_w) \\
&= \sum_{q \in \mathcal{M}_w} n_q^Q (n_q^Q - 1) (\pi_q + \gamma_w)^2 + \sum_{q \in \mathcal{M}_w} \sum_{i \in \mathcal{N}_q} \sum_{j \in \mathcal{N}_w^W, j \neq i} (\pi_{q(i)} + \gamma_w) (\pi_{q(j)} + \gamma_w) \\
&= \sum_{q \in \mathcal{M}_w} n_q^Q (n_q^Q - 1) (\pi_q + \gamma_w)^2 + \sum_{q \in \mathcal{M}_w} \sum_{q' \in \mathcal{M}_w, q' \neq q} n_q^Q n_{q'}^Q (\pi_q + \gamma_w) (\pi_{q'} + \gamma_w).
\end{aligned}$$

Since

$$\sum_{q \in \mathcal{M}_w} n_q^Q (\pi_q + \gamma_w)^2 - \frac{1}{n_w^W - 1} \sum_{q \in \mathcal{M}_w} n_q^Q (n_q^Q - 1) (\pi_q + \gamma_w)^2 = \sum_{q \in \mathcal{M}_w} n_q^Q \left(\frac{n_w^W - n_q^Q}{n_w^W - 1} \right) (\pi_q + \gamma_w)^2,$$

and $n_w^W = \sum_{q \in \mathcal{M}_w} n_q^Q$, we obtain

$$\begin{aligned}
\mu_3 &= \sum_w \left(\sum_{q \in \mathcal{M}_w} n_q^Q \left(\frac{n_w^W - n_q^Q}{n_w^W - 1} \right) (\pi_q + \gamma_w)^2 - \frac{1}{n_w^W - 1} \sum_{q \in \mathcal{M}_w} \sum_{q' \in \mathcal{M}_w, q' \neq q} n_q^Q n_{q'}^Q (\pi_q + \gamma_w) (\pi_{q'} + \gamma_w) \right) \\
&= \sum_w \frac{1}{n_w^W - 1} \left(\sum_{q \in \mathcal{M}_w} \sum_{q' \in \mathcal{M}_w, q' \neq q} n_q^Q n_{q'}^Q (\pi_q + \gamma_w)^2 - \sum_{q \in \mathcal{M}_w} \sum_{q' \in \mathcal{M}_w, q' \neq q} n_q^Q n_{q'}^Q (\pi_q + \gamma_w) (\pi_{q'} + \gamma_w) \right) \\
&= \sum_w \frac{1}{n_w^W - 1} \left(\sum_{q \in \mathcal{M}_w} \sum_{q' \in \mathcal{M}_w, q' \neq q} n_q^Q n_{q'}^Q (\pi_q + \gamma_w) (\pi_q - \pi_{q'}) \right).
\end{aligned}$$

Then, observe that

$$\begin{aligned}
& \sum_{q \in \mathcal{M}_w} \sum_{q' \in \mathcal{M}_w, q' \neq q} n_q^Q n_{q'}^Q (\pi_q + \gamma) (\pi_q - \pi_{q'}) \\
&= \sum_{q \in \mathcal{M}_w} \sum_{q' \in \mathcal{M}_w, q' < q} n_q^Q n_{q'}^Q (\pi_q + \gamma) (\pi_q - \pi_{q'}) + \sum_{q \in \mathcal{M}_w} \sum_{q' \in \mathcal{M}_w, q' > q} n_q^Q n_{q'}^Q (\pi_q + \gamma) (\pi_q - \pi_{q'}) \\
&= \sum_{q \in \mathcal{M}_w} \sum_{q' \in \mathcal{M}_w, q' < q} n_q^Q n_{q'}^Q (\pi_q + \gamma) (\pi_q - \pi_{q'}) - \sum_{q \in \mathcal{M}_w} \sum_{q' \in \mathcal{M}_w, q' < q} n_q^Q n_{q'}^Q (\pi_{q'} + \gamma) (\pi_q - \pi_{q'}) \\
&= \sum_{q \in \mathcal{M}_w} \sum_{q' \in \mathcal{M}_w, q' < q} n_q^Q n_{q'}^Q (\pi_q - \pi_{q'}) (\pi_q + \gamma - \pi_{q'} - \gamma) = \sum_{q \in \mathcal{M}_w} \sum_{q' \in \mathcal{M}_w, q' < q} n_q^Q n_{q'}^Q (\pi_q - \pi_{q'})^2,
\end{aligned}$$

where the second equality switches the indices of q and q' in the second element. Hence,

$$\mu_3 = \sum_w \frac{1}{n_w^W - 1} \left(\sum_{q \in \mathcal{M}_w} \sum_{q' \in \mathcal{M}_w, q' < q} n_q^Q n_{q'}^Q (\pi_q - \pi_{q'})^2 \right) \geq 0.$$

Analogously,

$$\begin{aligned}
\mu_2 &= \sum_w \left(\sum_{q \in \mathcal{M}_w} \frac{1}{n_q^Q - 1} \sum_{i \in \mathcal{N}_q^Q} \sum_{j \in \mathcal{N}_q^Q, j \neq i} R_i R_{Yj} - \frac{1}{n_w^W - 1} \sum_{i \in \mathcal{N}_w^W} \sum_{j \in \mathcal{N}_w^W, j \neq i} R_i R_{Yj} \right) \\
&= \sum_w \left(\sum_{q \in \mathcal{M}_w} n_q^Q (\pi_q + \gamma_w) (\pi_{Yq} + \gamma_w) - \frac{1}{n_w^W - 1} \sum_{q \in \mathcal{M}_w} n_q^Q (n_q^Q - 1) (\pi_q + \gamma_w) (\pi_{Yq} + \gamma_w) \right) \\
&\quad - \sum_w \frac{1}{n_w^W - 1} \sum_{q \in \mathcal{M}_w} \sum_{q' \in \mathcal{M}_w, q' \neq q} n_q^Q n_{q'}^Q (\pi_q + \gamma_w) (\pi_{Yq'} + \gamma_w) \\
&= \sum_w \left(\sum_{q \in \mathcal{M}_w} n_q^Q \left(\frac{n_w^W - n_q^Q}{n_w^W - 1} \right) (\pi_q + \gamma_w) (\pi_{Yq} + \gamma_w) \right. \\
&\quad \left. - \frac{1}{n_w^W - 1} \sum_{q \in \mathcal{M}_w} \sum_{q' \in \mathcal{M}_w, q' \neq q} n_q^Q n_{q'}^Q (\pi_q + \gamma_w) (\pi_{Yq'} + \gamma_w) \right) \\
&= \sum_w \frac{1}{n_w^W - 1} \left(\sum_{q \in \mathcal{M}_w} \sum_{q' \in \mathcal{M}_w, q' \neq q} n_q^Q n_{q'}^Q (\pi_q + \gamma_w) (\pi_{Yq} - \pi_{Yq'}) \right) \\
&= \sum_w \frac{1}{n_w^W - 1} \left(\sum_{q \in \mathcal{M}_w} \sum_{q' \in \mathcal{M}_w, q' < q} n_q^Q n_{q'}^Q (\pi_q - \pi_{q'}) (\pi_{Yq} - \pi_{Yq'}) \right),
\end{aligned}$$

and

$$\mu_1 = \sum_w \frac{1}{n_w^W - 1} \left(\sum_{q \in \mathcal{M}_w} \sum_{q' \in \mathcal{M}_w, q' < q} n_q^Q n_{q'}^Q (\pi_{Yq} - \pi_{Yq'})^2 \right) \geq 0.$$

□

Proof of Proposition 4. Fix any alternative $(\pi^A, \pi_Y^A) \in \mathcal{S}$ with a corresponding $(\mu_1^A, \mu_2^A, \mu_3^A)$. Due to the restriction in \mathcal{S} ,

$$\begin{pmatrix} \mu_1^H \\ \mu_2^H \\ \mu_3^H \end{pmatrix} = \begin{pmatrix} \mu_1^A - \frac{\sigma_{12}}{\sigma_{22}} \mu_2^A \\ 0 \\ \mu_3^A - \frac{\sigma_{23}}{\sigma_{22}} \mu_2^A \end{pmatrix}$$

is in the null space. The Neyman-Pearson test for μ^H against μ^A rejects large values of:

$$\log \frac{dN(\mu^A, \Sigma)}{dN(\mu^H, \Sigma)} = \frac{\mu_2^A}{\sigma_{22}} X_2 - \frac{1}{2} \frac{(\mu_2^A)^2}{\sigma_{22}}.$$

Hence, the most powerful test rejects large values of X_2 , which is what LM does. By [Lehmann and Romano \(2005\)](#) Theorem 3.8.1(i), since LM is valid for any distribution in the null space (by Theorem 1) and it is most powerful for some distribution in the null space, LM is most powerful for testing the composite null against the given alternative (π^A, π_Y^A) . □

Proof of Proposition 5. The first two are straightforward: $C_S = \mu_3 / (c - 1)$ and $\beta = \mu_2 / \mu_3$ imply $\mu_3 = (c - 1) C_S$ and $\mu_2 = (c - 1) C_S \beta$. For μ_1 , observe that:

$$h = \sqrt{\frac{1}{\sqrt{K}} \frac{1}{c-1} \left(\mu_1 - \frac{\mu_2^2}{\mu_3} \right)} = \sqrt{\frac{1}{\sqrt{K}} (\mu_1 - C_S \beta^2)}, \text{ and}$$

$$C_H = \sqrt{K} h^2 = \mu_1 / (c - 1) - C_S \beta^2, \text{ so}$$

$$(c - 1) (C_S \beta^2 + C_H) = (c - 1) (C_S \beta^2 + \mu_1 / (c - 1) - C_S \beta^2) = \mu_1$$

as required. Next, since $\sigma_{vv} = \sqrt{\frac{\sigma_{33}c}{2(c-1)}}$, $\sigma_{33} = 2\frac{c-1}{c}\sigma_{vv}^2$ is immediate. Similarly, with $\sigma_{\varepsilon v} = \frac{1}{\sigma_{vv}} \left(\frac{\sigma_{23}c}{2(c-1)} - \sigma_{vv}^2 \beta \right)$, $\sigma_{23} = 2\frac{c-1}{c}\sigma_{vv}(\sigma_{vv}\beta + \sigma_{\varepsilon v})$. From these two expressions, we can observe that:

$$(\sigma_{vv}\beta + \sigma_{\varepsilon v})^2 = \frac{c}{2(c-1)} \frac{\sigma_{23}^2}{\sigma_{33}}.$$

To obtain an expression for σ_{22} , rearrange $\sigma_{\varepsilon\varepsilon} = \frac{1}{\sigma_{vv}} \frac{c}{c-1} \left(\sigma_{22} - \frac{\sigma_{23}^2}{\sigma_{33}} \right) + \frac{\sigma_{\varepsilon v}^2}{\sigma_{vv}} \geq 0$:

$$\begin{aligned} \sigma_{22} &= \frac{\sigma_{23}^2}{\sigma_{33}} + \frac{c-1}{c} (\sigma_{\varepsilon\varepsilon}\sigma_{vv} - \sigma_{\varepsilon v}^2) \\ &= \frac{c-1}{c} \left(\sigma_{vv} (\sigma_{\varepsilon\varepsilon} + \sigma_{vv}\beta^2 + \sigma_{vv}\sigma_{\xi\xi} + 2\sigma_{\varepsilon v}\beta) + (\sigma_{vv}\beta + \sigma_{\varepsilon v})^2 \right) + o(1), \end{aligned}$$

where the final step uses $\sigma_{\xi\xi} = h/\sigma_{vv}$. This expression for σ_{22} is of the form required in Lemma 8. Then,

$$\det(\Sigma_{SF}) = \sigma_{\varepsilon\varepsilon}\sigma_{\xi\xi}\sigma_{vv} - \sigma_{\varepsilon\varepsilon}h^2 - \sigma_{\xi\xi}^2\sigma_{vv} + 2\sigma_{\xi\xi}\sigma_{\varepsilon v}h - \sigma_{\xi\xi}\sigma_{\varepsilon v}^2$$

$$= \sigma_{\varepsilon\varepsilon}\sigma_{\xi\xi}\sigma_{vv} - \sigma_{\varepsilon\varepsilon}h^2 - \sigma_{\xi\xi}\sigma_{\varepsilon v}^2 = \sigma_{\varepsilon\varepsilon}h - \sigma_{\varepsilon\varepsilon}h^2 - h\frac{\sigma_{\varepsilon v}^2}{\sigma_{vv}}; \text{ and}$$

$$\det(\Sigma_{SF})/h = \sigma_{\varepsilon\varepsilon} - \frac{\sigma_{\varepsilon v}^2}{\sigma_{vv}} - \sigma_{\varepsilon\varepsilon}h = \sigma_{\varepsilon\varepsilon} - \frac{\sigma_{\varepsilon v}^2}{\sigma_{vv}} + o(1).$$

An analogous argument holds for $\sigma_{\xi vk} = -h$. From the σ_{22} equation, $\sigma_{\varepsilon\varepsilon} - \frac{\sigma_{\varepsilon v}^2}{\sigma_{vv}} = \frac{c}{c-1} \left(\sigma_{22} - \frac{\sigma_{23}^2}{\sigma_{33}} \right) \geq 0$, which delivers the result that $\det(\Sigma_{SF})/h \rightarrow C_D \geq 0$. \square

Proof of Proposition 6. The A expressions can be written as:

$$\begin{aligned} A_1 &= \sum_i \sum_{j \neq i} \sum_{k \neq i} \sum_{l \neq k} \check{M}_{il, -ijk} G_{ij} X_j G_{ik} X_k (Y_i Y_l - X_i Y_l \beta_0 - Y_i X_l \beta_0 + X_i X_l \beta_0^2); \\ A_2 &= \sum_i \sum_{j \neq i} \sum_{k \neq i} \sum_{l \neq k} \check{M}_{il, -ijk} G_{ij} X_j G_{ki} X_l (Y_i Y_k - X_i Y_k \beta_0 - Y_i X_k \beta_0 + X_i X_k \beta_0^2); \\ A_3 &= \sum_i \sum_{j \neq i} \sum_{k \neq i} \sum_{l \neq k} \check{M}_{il, -ijk} X_l G_{ji} G_{ki} X_i (Y_j Y_k - X_j Y_k \beta_0 - Y_j X_k \beta_0 + X_j X_k \beta_0^2); \\ A_4 &= - \sum_i \sum_{j \neq i} \sum_{k \neq j} \sum_{l \neq i, k} \check{M}_{jl, -ijk} \check{M}_{ik, -ij} G_{ji}^2 X_i X_k (Y_j Y_l - X_j Y_l \beta_0 - Y_j X_l \beta_0 + X_j X_l \beta_0^2); \text{ and} \\ A_5 &= - \sum_i \sum_{j \neq i} \sum_{k \neq j} \sum_{l \neq i, k} \check{M}_{ik, -ij} \check{M}_{jl, -ijk} G_{ij} G_{ji} X_k X_l (Y_i Y_j - X_i Y_j \beta_0 - Y_i X_j \beta_0 + X_i X_j \beta_0^2). \end{aligned}$$

Since these terms have a quadratic form, the variance estimator is also quadratic in β_0 , i.e.,

$$\hat{V}_{LM} = B_0 + B_1 \beta_0 + B_2 \beta_0^2,$$

where the B 's can be worked out by collecting the expressions above. For instance,

$$\begin{aligned} B_0 &= \sum_i \sum_{j \neq i} \sum_{k \neq i} \sum_{l \neq k} \check{M}_{il, -ijk} G_{ij} X_j G_{ik} X_k Y_i Y_l + 2 \sum_i \sum_{j \neq i} \sum_{k \neq i} \sum_{l \neq k} \check{M}_{il, -ijk} G_{ij} X_j G_{ki} X_l Y_i Y_k \\ &\quad + \sum_i \sum_{j \neq i} \sum_{k \neq i} \sum_{l \neq k} \check{M}_{il, -ijk} X_l G_{ji} G_{ki} X_i Y_j Y_k \\ &\quad - \sum_i \sum_{j \neq i} \sum_{k \neq j} \sum_{l \neq i, k} \check{M}_{jl, -ijk} \check{M}_{ik, -ij} G_{ji}^2 X_i X_k Y_j Y_l - \sum_i \sum_{j \neq i} \sum_{k \neq j} \sum_{l \neq i, k} \check{M}_{ik, -ij} \check{M}_{jl, -ijk} G_{ij} G_{ji} X_k X_l Y_i Y_j \end{aligned}$$

B_1 and B_2 are analogous by collecting the coefficients on β_0, β_0^2 from expressions A_1 to A_5 . The test does not reject:

$$\frac{(KT_{YX} - KT_{XX}\beta_0)^2}{B_0 + B_1\beta_0 + B_2\beta_0^2} \leq q \Leftrightarrow (KT_{XX}^2 - qB_2) \beta_0^2 - (2KT_{YX}T_{XX} + qB_1) \beta_0 + (KT_{YX}^2 - qB_0) \leq 0.$$

Solutions exist when:

$$D := (2KT_{YX}T_{XX} + qB_1)^2 - 4(KT_{XX}^2 - qB_2)(KT_{YX}^2 - qB_0) \geq 0.$$

The rest of the lemma is immediate from properties of solving quadratic inequalities. \square

E.2.2 Proofs for Lemmas in Appendix D

Proof of Lemma 7. The joint distribution of $(Y', X')'$ is:

$$\begin{bmatrix} Y \\ X \end{bmatrix} \sim N \left(\begin{bmatrix} Z\pi_Y \\ Z\pi \end{bmatrix}, \begin{bmatrix} I_n\omega_{\zeta\zeta} & I_n\omega_{\zeta\eta} \\ I_n\omega_{\zeta\eta} & I_n\omega_{\eta\eta} \end{bmatrix} \right).$$

Stack them together with their predicted values $PY = Z(Z'Z)^{-1}Z'Y$ and $PX = Z(Z'Z)^{-1}Z'X$:

$$\begin{bmatrix} Y \\ X \\ P'Y \\ P'X \end{bmatrix} \sim N \left(\begin{bmatrix} Z\pi_Y \\ Z\pi \\ Z\pi_Y \\ Z\pi \end{bmatrix}, \begin{bmatrix} I_n\omega_{\zeta\zeta} & I_n\omega_{\zeta\eta} & \omega_{\zeta\zeta}P' & \omega_{\zeta\eta}P' \\ I_n\omega_{\zeta\eta} & I_n\omega_{\eta\eta} & \omega_{\zeta\eta}P' & \omega_{\eta\eta}P' \\ \omega_{\zeta\zeta}P' & \omega_{\zeta\eta}P' & \omega_{\zeta\zeta}P' & \omega_{\zeta\eta}P' \\ \omega_{\zeta\eta}P' & \omega_{\eta\eta}P' & \omega_{\zeta\eta}P' & \omega_{\eta\eta}P' \end{bmatrix} \right).$$

Then, the conditional normal distribution is:

$$\begin{aligned} \begin{bmatrix} Y \\ X \end{bmatrix} \mid \begin{bmatrix} Z(Z'Z)^{-1}Z'Y \\ Z(Z'Z)^{-1}Z'X \end{bmatrix} &\sim N \left(\begin{bmatrix} Z\pi_Y \\ Z\pi \end{bmatrix} + \begin{bmatrix} Z(Z'Z)^{-1}Z'Y - Z\pi_Y \\ Z(Z'Z)^{-1}Z'X - Z\pi \end{bmatrix}, V \right) \\ &= N \left(\begin{bmatrix} Z(Z'Z)^{-1}Z'Y \\ Z(Z'Z)^{-1}Z'X \end{bmatrix}, V \right) = N \left(\begin{bmatrix} PY \\ PX \end{bmatrix}, V \right) \end{aligned}$$

Hence, PX and PY (i.e., $Z'X$, $Z'Y$) are sufficient statistics for π_Y, π .

To show that $(s'_1s_1, s'_1s_2, s'_2s_2)$ is a maximal invariant, let F be some conformable orthogonal matrix so $F'F = I$. For invariance, let $s_1^* = Fs_1$. Then, $s_1^{*'}s_1^* = s_1'F'Fs_1 = s_1's_1$. Invariance of (s'_1s_2, s'_2s_2) is analogous. Maximality states that if $s_1^{*'}s_1^* = s_1's_1$, then $s_1^* = Fs_1$ for some F . Suppose not. This means $s_1^* = Gs_1$, and G is not an orthogonal matrix but yet $s_1^{*'}s_1^* = s_1's_1$. Since G is not an orthogonal matrix, $G'G \neq I$. Hence, $s_1^{*'}s_1^* = s_1'G'Gs_1 \neq s_1's_1$, a contradiction. To obtain the distribution,

$$\begin{bmatrix} s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} (Z'Z)^{-1/2}Z'(Z\pi_Y + \zeta) \\ (Z'Z)^{-1/2}Z'(Z\pi + \eta) \end{bmatrix} = \begin{bmatrix} (Z'Z)^{1/2}\pi_Y \\ (Z'Z)^{1/2}\pi \end{bmatrix} + \begin{bmatrix} (Z'Z)^{-1/2}Z'\zeta \\ (Z'Z)^{-1/2}Z'\eta \end{bmatrix}.$$

Since $\text{Var} \left((Z'Z)^{-1/2}Z'\eta \right) = (Z'Z)^{-1/2}Z'\omega_{\eta\eta}Z(Z'Z)^{-1/2} = I_K\omega_{\eta\eta}$,

$$\begin{bmatrix} s_1 \\ s_2 \end{bmatrix} \sim N \left(\begin{pmatrix} (Z'Z)^{1/2}\pi_Y \\ (Z'Z)^{1/2}\pi \end{pmatrix}, \Omega \otimes I_K \right).$$

□

Proof of Lemma 8. I work out the μ 's first. Using the judge structure, $\sum_i M_{ii}^2 = \sum_k \frac{(c-1)^2}{c}$, $\sum_i \sum_{j \neq i} P_{ij} = \sum_k \frac{c-1}{c}$. We have also chosen $\pi_k, \sigma_{\xi vk}$ such that $\sum_k \pi_k = 0, \sum_k \sigma_{\xi vk} = 0, \sum_k \pi_k \sigma_{\xi vk} = 0$.

Then, we get the result for means:

$$\begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{K}} \sum_k (c-1) \left(\pi_k^2 \beta^2 + 2\pi_k \beta \sigma_{\xi v k} + \sigma_{\xi v k}^2 \right) \\ \frac{1}{\sqrt{K}} \sum_k (c-1) \left(\pi_k^2 \beta + \pi_k \sigma_{\xi v k} \right) \\ \frac{1}{\sqrt{K}} \sum_k (c-1) \pi_k^2 \end{pmatrix} = \begin{pmatrix} \sqrt{K} (c-1) (s^2 \beta^2 + h^2) \\ \sqrt{K} (c-1) s^2 \beta \\ \sqrt{K} (c-1) s^2 \end{pmatrix}.$$

Using a derivation similar to that of the lemma for V_{LM} expression,

$$\begin{aligned} K\sigma_{22} &= \sum_i \sum_{j \neq i} \sum_{k \neq i} (G_{ji} G_{ki} E[\zeta_i^2] R_j R_k + 2G_{ij} G_{ki} E[\eta_i \zeta_i] R_{Yj} R_k + G_{ij} G_{ik} E[\eta_i^2] R_{Yj} R_{Yk}) \\ &\quad + \sum_i \sum_{j \neq i} (G_{ij}^2 E[\eta_i^2] E[\zeta_j^2] + G_{ij} G_{ji} E[\eta_i \zeta_i] E[\eta_j \zeta_j]); \\ K\sigma_{11} &= \sum_i \sum_{j \neq i} \sum_{k \neq i} E[\zeta_i^2] R_{Yj} R_{Yk} (G_{ji} G_{ki} + 2G_{ij} G_{ki} + G_{ij} G_{ik}) + \sum_i \sum_{j \neq i} E[\zeta_i^2] E[\zeta_j^2] (G_{ij}^2 + G_{ij} G_{ji}); \\ K\sigma_{33} &= \sum_i \sum_{j \neq i} \sum_{k \neq i} E[\eta_i^2] R_j R_k (G_{ji} G_{ki} + 2G_{ij} G_{ki} + G_{ij} G_{ik}) + \sum_i \sum_{j \neq i} E[\eta_i^2] E[\eta_j^2] (G_{ij}^2 + G_{ij} G_{ji}); \\ K\sigma_{12} &= \sum_i \sum_{j \neq i} \sum_{k \neq i} (G_{ji} G_{ki} E[\zeta_i^2] R_j R_{Yk} + 2G_{ij} G_{ki} E[\zeta_i^2] R_{Yj} R_k + G_{ij} G_{ik} E[\eta_i \zeta_i] R_{Yj} R_{Yk}) \\ &\quad + \sum_i \sum_{j \neq i} E[\eta_i \zeta_i] E[\zeta_j^2] (G_{ij}^2 + G_{ij} G_{ji}); \\ K\sigma_{23} &= \sum_i \sum_{j \neq i} \sum_{k \neq i} (G_{ji} G_{ki} E[\eta_i^2] R_{Yj} R_k + 2G_{ij} G_{ki} E[\eta_i^2] R_j R_{Yk} + G_{ij} G_{ik} E[\eta_i \zeta_i] R_j R_k) \\ &\quad + \sum_i \sum_{j \neq i} E[\eta_i \zeta_i] E[\eta_j^2] (G_{ij}^2 + G_{ij} G_{ji}); \text{ and} \\ K\sigma_{13} &= \sum_i \sum_{j \neq i} \sum_{k \neq i} E[\eta_i \zeta_i] R_{Yj} R_k (G_{ji} G_{ki} + 2G_{ij} G_{ki} + G_{ij} G_{ik}) + \sum_i \sum_{j \neq i} E[\eta_i \zeta_i] E[\eta_j \zeta_j] (G_{ij}^2 + G_{ij} G_{ji}). \end{aligned}$$

The equalities hold regardless of whether identification is strong or weak and whether heterogeneity converges or not. Without covariates, $G = P$ is symmetric and the above expressions simplify. For instance,

$$K\sigma_{22} = \sum_k \frac{(c-1)^2}{c} (\omega_{\zeta \zeta k} \pi_k^2 + 2\omega_{\zeta \eta k} \pi_k \pi_{Yk} + \omega_{\eta \eta k} \pi_{Yk}^2) + \sum_k \frac{c-1}{c} (\omega_{\eta \eta k} \omega_{\zeta \zeta k} + \omega_{\zeta \eta k}^2).$$

Evaluate the terms in the expression. For higher moments of π_k , $\sum_k \pi_k^2 = Ks^2$, $\sum_k \pi_k^3 = 0$, and $\sum_k \pi_k^4 = Ks^4$. Similarly, $\sum_k \pi_k^3 \sigma_{\xi v} = 0$. Treating the heterogeneity in the same way, $\sum_k \sigma_{\xi v}^2 = Kh^2$. Then,

$$\begin{aligned} \sum_k \omega_{\zeta \zeta k} \pi_k^2 &= \sum_k (\pi_k^2 \sigma_{\xi \xi} + 2\pi_k \beta \sigma_{\xi v k} + 2\pi_k \sigma_{\varepsilon \xi} + \sigma_{\varepsilon \varepsilon} - \sigma_{\xi v k}^2 + \sigma_{vv} \beta^2 + \sigma_{vv} \sigma_{\xi \xi} + 2\sigma_{\xi v k}^2 + 2\sigma_{\varepsilon v} \beta) \pi_k^2 \\ &= s^2 K (s^2 \sigma_{\xi \xi} + \sigma_{\varepsilon \varepsilon} + \sigma_{vv} \beta^2 + \sigma_{vv} \sigma_{\xi \xi} + h^2 + 2\sigma_{\varepsilon v} \beta); \text{ and} \end{aligned}$$

$$\begin{aligned}
\sum_k \omega_{\zeta\eta k} \pi_k \pi_{Yk} &= \sum_k (\pi_k \sigma_{\xi vk} + \sigma_{vv} \beta + \sigma_{\varepsilon v}) \pi_k (\pi_k \beta + \sigma_{\xi vk}) \\
&= \sum_k (\sigma_{vv} \beta^2 \pi_k^2 + \sigma_{\varepsilon v} \pi_k^2 \beta + \pi_k^2 \sigma_{\xi vk}^2) = s^2 K (\sigma_{vv} \beta^2 + \sigma_{\varepsilon v} \beta + h^2).
\end{aligned}$$

Now, for the P_{ij}^2 part,

$$\begin{aligned}
\sum_k \omega_{\eta\eta k} \omega_{\zeta\zeta k} &= \sum_k \sigma_{vv} (\pi_k^2 \sigma_{\xi\xi} + 2\pi_k \beta \sigma_{\xi vk} + 2\pi_k \sigma_{\varepsilon\xi} + \sigma_{\varepsilon\varepsilon} - \sigma_{\xi vk}^2 + \sigma_{vv} \beta^2 + \sigma_{vv} \sigma_{\xi\xi} + 2\sigma_{\xi vk}^2 + 2\sigma_{\varepsilon v} \beta) \\
&= \sum_k \sigma_{vv} (\pi_k^2 \sigma_{\xi\xi} + \sigma_{\varepsilon\varepsilon} - \sigma_{\xi vk}^2 + \sigma_{vv} \beta^2 + \sigma_{vv} \sigma_{\xi\xi} + 2\sigma_{\xi vk}^2 + 2\sigma_{\varepsilon v} \beta) \\
&= K \sigma_{vv} (s^2 \sigma_{\xi\xi} + \sigma_{\varepsilon\varepsilon} + \sigma_{vv} \beta^2 + \sigma_{vv} \sigma_{\xi\xi} + h^2 + 2\sigma_{\varepsilon v} \beta); \text{ and} \\
\sum_k \omega_{\zeta\eta k}^2 &= \sum_k (\pi_k \sigma_{\xi vk} \pi_k \sigma_{\xi vk} + \sigma_{vv} \beta \pi_k \sigma_{\xi vk} + \sigma_{\varepsilon v} \pi_k \sigma_{\xi vk} + \pi_k \sigma_{\xi vk} \sigma_{vv} \beta + \sigma_{vv} \beta \sigma_{vv} \beta + \sigma_{\varepsilon v} \sigma_{vv} \beta) \\
&\quad + \sum_k (\pi_k \sigma_{\xi vk} \sigma_{\varepsilon v} + \sigma_{vv} \beta \sigma_{\varepsilon v} + \sigma_{\varepsilon v}^2) \\
&= \sum_k (\pi_k^2 \sigma_{\xi vk}^2 + \sigma_{vv}^2 \beta^2 + \sigma_{\varepsilon v} \sigma_{vv} \beta + \sigma_{vv} \beta \sigma_{\varepsilon v} + \sigma_{\varepsilon v}^2) = K (s^2 h^2 + (\sigma_{vv} \beta + \sigma_{\varepsilon v})^2).
\end{aligned}$$

Combine the expressions for σ_{22} and impose asymptotics where $s \rightarrow 0$ and $h \rightarrow 0$:

$$\begin{aligned}
\sigma_{22} &= \frac{1}{K} \sum_k \frac{(c-1)^2}{c} h^2 + \frac{1}{K} \sum_k \frac{c-1}{c} (\sigma_{vv} (\sigma_{\varepsilon\varepsilon} + \sigma_{vv} \beta^2 + \sigma_{vv} \sigma_{\xi\xi} + h^2 + 2\sigma_{\varepsilon v} \beta) + (\sigma_{vv} \beta + \sigma_{\varepsilon v})^2) + o(1) \\
&= \frac{c-1}{c} (\sigma_{vv} (\sigma_{\varepsilon\varepsilon} + \sigma_{vv} \beta^2 + \sigma_{vv} \sigma_{\xi\xi} + 2\sigma_{\varepsilon v} \beta) + (\sigma_{vv} \beta + \sigma_{\varepsilon v})^2) + o(1).
\end{aligned}$$

Next, evaluate a few more sums that feature in the other σ expressions:

$$\begin{aligned}
\sum_k \omega_{\zeta\zeta} \pi_{Yk}^2 &= \sum_k (\pi_k^2 \sigma_{\xi\xi} + 2\pi_k \beta \sigma_{\xi vk} + 2\pi_k \sigma_{\varepsilon\xi} + \sigma_{\varepsilon\varepsilon} + \sigma_{vv} \beta^2 + \sigma_{vv} \sigma_{\xi\xi} + \sigma_{\xi vk}^2 + 2\sigma_{\varepsilon v} \beta) \\
&\quad (\pi_k^2 \beta^2 + 2\pi_k \sigma_{\xi vk} + \sigma_{\xi v}^2) \\
\frac{1}{K} \sum_k \omega_{\zeta\zeta} \pi_{Yk}^2 &= \frac{1}{K} \sum_k \sigma_{\xi v}^2 (\pi_k^2 \sigma_{\xi\xi} + 2\pi_k \beta \sigma_{\xi vk} + 2\pi_k \sigma_{\varepsilon\xi} + \sigma_{\varepsilon\varepsilon} + \sigma_{vv} \beta^2 + \sigma_{vv} \sigma_{\xi\xi} + \sigma_{\xi vk}^2 + 2\sigma_{\varepsilon v} \beta) \\
&= h^2 (\sigma_{\varepsilon\varepsilon} + \sigma_{vv} \beta^2 + \sigma_{vv} \sigma_{\xi\xi} + h^2 + 2\sigma_{\varepsilon v} \beta) = o(1); \\
\frac{1}{K} \sum_k \omega_{\zeta\zeta}^2 &= \frac{1}{K} \sum_k (\pi_k^2 \sigma_{\xi\xi} + 2\pi_k \beta \sigma_{\xi vk} + 2\pi_k \sigma_{\varepsilon\xi} + \sigma_{\varepsilon\varepsilon} + \sigma_{vv} \beta^2 + \sigma_{vv} \sigma_{\xi\xi} + \sigma_{\xi vk}^2 + 2\sigma_{\varepsilon v} \beta)^2 \\
&= \frac{1}{K} \sum_k (\sigma_{\varepsilon\varepsilon} + \sigma_{vv} \beta^2 + \sigma_{vv} \sigma_{\xi\xi} + \sigma_{\xi vk}^2 + 2\sigma_{\varepsilon v} \beta)^2 = (\sigma_{\varepsilon\varepsilon} + \sigma_{vv} \beta^2 + \sigma_{vv} \sigma_{\xi\xi} + 2\sigma_{\varepsilon v} \beta)^2; \\
\frac{1}{K} \sum_k \omega_{\zeta\eta} \pi_{Yk}^2 &= \frac{1}{K} \sum_k (\pi_k \sigma_{\xi vk} + \sigma_{vv} \beta + \sigma_{\varepsilon v}) (\pi_k^2 \beta^2 + 2\pi_k \sigma_{\xi vk} + \sigma_{\xi v}^2) \\
&= h^2 (\sigma_{vv} \beta + \sigma_{\varepsilon v}) = o(1); \text{ and}
\end{aligned}$$

$$\begin{aligned}
\frac{1}{K} \sum_k \omega_{\zeta\eta} \omega_{\zeta\zeta} &= \frac{1}{K} \sum_k (\pi_k \sigma_{\xi vk} + \sigma_{vv} \beta + \sigma_{\varepsilon v}) \\
&\quad (\pi_k^2 \sigma_{\xi\xi} + 2\pi_k \beta \sigma_{\xi vk} + 2\pi_k \sigma_{\varepsilon\xi} + \sigma_{\varepsilon\varepsilon} + \sigma_{vv} \beta^2 + \sigma_{vv} \sigma_{\xi\xi} + \sigma_{\xi vk}^2 + 2\sigma_{\varepsilon v} \beta) \\
&= \frac{1}{K} \sum_k (\sigma_{vv} \beta + \sigma_{\varepsilon v}) (\sigma_{\varepsilon\varepsilon} + \sigma_{vv} \beta^2 + \sigma_{vv} \sigma_{\xi\xi} + \sigma_{\xi vk}^2 + 2\sigma_{\varepsilon v} \beta) \\
&= (\sigma_{vv} \beta + \sigma_{\varepsilon v}) (\sigma_{\varepsilon\varepsilon} + \sigma_{vv} \beta^2 + \sigma_{vv} \sigma_{\xi\xi} + 2\sigma_{\varepsilon v} \beta) + o(1).
\end{aligned}$$

Using these results,

$$\begin{aligned}
\sigma_{22} &= \frac{c-1}{c} \left(\sigma_{vv} (\sigma_{\varepsilon\varepsilon} + \sigma_{vv} \beta^2 + \sigma_{vv} \sigma_{\xi\xi} + 2\sigma_{\varepsilon v} \beta) + (\sigma_{vv} \beta + \sigma_{\varepsilon v})^2 \right) + o(1); \\
\sigma_{11} &= 2 \frac{c-1}{c} (\sigma_{\varepsilon\varepsilon} + \sigma_{vv} \beta^2 + \sigma_{vv} \sigma_{\xi\xi} + 2\sigma_{\varepsilon v} \beta)^2 + o(1); \\
\sigma_{33} &= 2 \frac{c-1}{c} \sigma_{vv}^2 + o(1); \\
\sigma_{12} &= 2 \frac{c-1}{c} (\sigma_{vv} \beta + \sigma_{\varepsilon v}) (\sigma_{\varepsilon\varepsilon} + \sigma_{vv} \beta^2 + \sigma_{vv} \sigma_{\xi\xi} + 2\sigma_{\varepsilon v} \beta) + o(1); \\
\sigma_{23} &= 2 \frac{c-1}{c} \sigma_{vv} (\sigma_{vv} \beta + \sigma_{\varepsilon v}) + o(1); \text{ and} \\
\sigma_{13} &= 2 \frac{c-1}{c} (\sigma_{vv} \beta + \sigma_{\varepsilon v})^2 + o(1).
\end{aligned}$$

Hence, $\sigma_{13} = \sigma_{23}^2 / \sigma_{33} + o(1)$ is immediate. Further, for σ_{12} ,

$$\begin{aligned}
2 \frac{\sigma_{23}}{\sigma_{33}} \left(\sigma_{22} - \frac{\sigma_{23}^2}{2\sigma_{33}} \right) &= 2 \frac{\sigma_{vv} \beta + \sigma_{\varepsilon v}}{\sigma_{vv}} \left(\sigma_{22} - \frac{(2 \frac{c-1}{c} \sigma_{vv} (\sigma_{vv} \beta + \sigma_{\varepsilon v}))^2}{2 \times 2 \frac{c-1}{c} \sigma_{vv}^2} \right) + o(1) \\
&= 2 \frac{c-1}{c} (\sigma_{vv} \beta + \sigma_{\varepsilon v}) (\sigma_{\varepsilon\varepsilon} + \sigma_{vv} \beta^2 + \sigma_{vv} \sigma_{\xi\xi} + 2\sigma_{\varepsilon v} \beta) + o(1) = \sigma_{12} + o(1).
\end{aligned}$$

Finally, the σ_{11} can be obtained:

$$\begin{aligned}
\frac{4}{\sigma_{33}} \left(\sigma_{22} - \frac{\sigma_{23}^2}{2\sigma_{33}} \right)^2 &= \frac{2}{\frac{c-1}{c} \sigma_{vv}^2} \left(\frac{c-1}{c} (\sigma_{vv} (\sigma_{\varepsilon\varepsilon} + \sigma_{vv} \beta^2 + \sigma_{vv} \sigma_{\xi\xi} + 2\sigma_{\varepsilon v} \beta)) \right)^2 + o(1) \\
&= 2 \frac{c-1}{c} (\sigma_{\varepsilon\varepsilon} + \sigma_{vv} \beta^2 + \sigma_{vv} \sigma_{\xi\xi} + 2\sigma_{\varepsilon v} \beta)^2 + o(1) = \sigma_{11} + o(1).
\end{aligned}$$

□

E.2.3 Derivations for Simulations

Derivation for continuous setup without covariates.

This subsection derives expressions for objects in the reduced-form model. Comparing the first-stage equations, $\eta_i = v_i$. As a corollary, for all i , $E[\eta_i^2] = \sigma_{vv}$. Then, $\zeta_i = Z_i'(\pi\beta_i - \pi_Y) + v_i\beta_i + \varepsilon_i$. Define π_Y using $E[\zeta_i] = 0$ and $E[v_i\beta_i] = E[v_i(\beta + \xi_i)] = \sigma_{\xi vk(i)}$,

which implies $\pi_{Yk} = \pi_k \beta + \sigma_{\xi vk}$. Hence, we can rewrite ζ_i as:

$$\zeta_i = \pi_{k(i)} \xi_i - \sigma_{\xi vk(i)} + v_i \beta + v_i \xi_i + \varepsilon_i.$$

By substituting the expression for ζ_i , the covariance is $E[\eta_i \zeta_i | k] = \pi_k \sigma_{\xi vk} + \sigma_{vv} \beta + E[v_i^2 \xi_i] + \sigma_{\varepsilon v}$. By Isserlis' theorem, $E[v_i^2 \xi_i] = 0$, so $E[\eta_i \zeta_i | k] = \pi_k \sigma_{\xi vk} + \sigma_{vv} \beta + \sigma_{\varepsilon v}$. The variance of ζ_i can be derived analogously. Since $E[v_i^2 \beta_i^2] = \sigma_{vv} \beta^2 + \sigma_{vv} \sigma_{\xi \xi} + 2\sigma_{\xi vk}^2$ by applying Isserlis' theorem, with $\omega_{\eta \eta k} := E[\eta_i^2 | k(i) = k]$, $\omega_{\zeta \eta k} := E[\zeta_i \eta_i | k(i) = k]$, and $\omega_{\zeta \zeta k} := E[\zeta_i^2 | k(i) = k]$, we obtain:

$$\begin{aligned} \omega_{\eta \eta k} &= \sigma_{vv}^2, \\ \omega_{\zeta \eta k} &= \pi_k \sigma_{\xi vk} + \sigma_{vv} \beta + \sigma_{\varepsilon v}, \text{ and} \\ \omega_{\zeta \zeta k} &= \pi_k^2 \sigma_{\xi \xi} + 2\pi_k \beta \sigma_{\xi vk} + 2\pi_k \sigma_{\varepsilon \xi} + \sigma_{\varepsilon \varepsilon} + \sigma_{\xi vk}^2 + \sigma_{vv} \beta^2 + \sigma_{vv} \sigma_{\xi \xi} + 2\sigma_{\varepsilon v} \beta. \end{aligned} \quad (24)$$

In this model, the local average treatment effect (LATE) of judge k relative to the base judge 0 is:

$$LATE_k = \frac{\pi_{Yk}}{\pi_k} = \beta + \frac{\sigma_{\xi vk}}{\pi_k}. \quad (25)$$

Derivation for binary setup without covariates.

The reduced-form residuals are given by:

$$\eta_i | v_i = \begin{cases} 1 - \pi_k & \text{if } v_i \leq \pi_k \\ -\pi_k & \text{if } v_i > \pi_k \end{cases}, \text{ and } \zeta_i = \pi_{k(i)} \beta_i - \pi_{Yk(i)} + \eta_i \beta_i + \varepsilon_i.$$

Imposing $E[\zeta_i] = 0$, $\pi_{Yk(i)} = \pi_{k(i)} \beta + E[\eta_i \beta_i]$, where $E[\eta_i \beta_i] = -(1-s)(2p-1)\sigma_{\xi vk}$. Hence,

$$\pi_{Yk} = \pi_k \beta - (1-s)(2p-1)\sigma_{\xi vk}.$$

Due to the judge setup, the estimand is:

$$\frac{\sum_k \pi_{Yk} \pi_k}{\sum_k \pi_k^2} = \frac{\sum_k (\pi_k \beta - (1-s)(2p-1)\sigma_{\xi vk}) \pi_k}{\sum_k \pi_k^2} = \beta$$

because $\sum_k \sigma_{\xi vk} \pi_k = 0$ by construction.

Derivation for binary setup with covariates.

Consider the structural model:

$$\begin{aligned} Y_i(x) &= x(\beta + \xi_i) + w' \gamma + \varepsilon_i, \text{ and} \\ X_i(z) &= I\{z' \pi + w' \gamma - v_i \geq 0\}. \end{aligned}$$

Let \mathcal{N}_t denote the set of observations in state t . Then, using the G that corresponds to UJIVE,

$$\sum_{i \in \mathcal{N}_t} \sum_{j \in \mathcal{N}_t \setminus i} G_{ij} R_{Yi} R_j = \sum_{i \in \mathcal{N}_t} \sum_{j \in \mathcal{N}_t \setminus i} G_{ij} (\pi_{Yk(i)} + \gamma_{t(i)}) (\pi_{k(j)} + \gamma_{t(j)})$$

$$\begin{aligned}
&= \sum_{i \in \mathcal{N}_t} \sum_{j \in \mathcal{N}_t \setminus i} G_{ij} (\pi_{Yk(i)} \pi_{k(j)} + \gamma_{t(i)} \pi_{k(j)} + \pi_{Yk(i)} \gamma_{t(j)} + \gamma_{t(i)} \gamma_{t(j)}) \\
&= \frac{1}{1-1/5} \sum_{k \in \{0,t\}} 5 \times 4 \times \frac{1}{5} (\pi_{Yk} \pi_k + \gamma_t \pi_k + \pi_{Yk} \gamma_t + \gamma_t^2) \\
&\quad - \frac{1}{1-1/10} \sum_{i \in \mathcal{N}_t} \sum_{j \in \mathcal{N}_t \setminus i} \frac{1}{10} (\pi_{Yk(i)} \pi_{k(j)} + \gamma_t \pi_{k(j)} + \pi_{Yk(i)} \gamma_t + \gamma_t^2) \\
&= \sum_{k \in \{0,t\}} 5 (\pi_{Yk} \pi_k + \gamma_t \pi_k + \pi_{Yk} \gamma_t + \gamma_t^2) - \frac{1}{9} \sum_{k \in \{0,t\}} 5 \times 4 (\pi_{Yk} \pi_k + \gamma_t \pi_k + \pi_{Yk} \gamma_t + \gamma_t^2) \\
&\quad - \frac{1}{9} 5 \times 5 (\pi_{Yt} \pi_0 + \gamma_t \pi_0 + \pi_{Yt} \gamma_t + \gamma_t^2) - \frac{1}{9} 5 \times 5 (\pi_{Y0} \pi_t + \gamma_t \pi_t + \pi_{Y0} \gamma_t + \gamma_t^2) \\
&= 5 \left(\frac{5}{9} \right) (\pi_{Y0} \pi_0 + \pi_{Yt} \pi_t - \pi_{Yt} \pi_0 - \pi_{Y0} \pi_t).
\end{aligned}$$

Using the result that $\pi_{Yk} = \pi_k \beta - (1-s)(2p-1)\sigma_{\xi vk}$,

$$\sum_{i \in \mathcal{N}_t} \sum_{j \in \mathcal{N}_t \setminus i} G_{ij} R_{Yi} R_j = 5 \left(\frac{5}{9} \right) (\pi_{Y0} \pi_0 + \pi_{Yt} \pi_t - \pi_{Yt} \pi_0 - \pi_{Y0} \pi_t) = \frac{25}{9} \pi_{Yt} \pi_t.$$

Analogously, $\sum_{i \in \mathcal{N}_t} \sum_{j \in \mathcal{N}_t \setminus i} G_{ij} R_i R_j = \frac{25}{9} \pi_t^2$. Hence, as long as $\sum_t \sigma_{\xi vt} \pi_t = 0$, which is the case for the construction in the main text, we still recover β as our estimand:

$$\begin{aligned}
\frac{\sum_i \sum_{j \neq i} G_{ij} R_{Yi} R_j}{\sum_i \sum_{j \neq i} G_{ij} R_i R_j} &= \frac{\sum_t \pi_{Yt} \pi_t}{\sum_t \pi_t^2} = \frac{\sum_t (\pi_t \beta - (1-s)(2p-1)\sigma_{\xi vt}) \pi_t}{\sum_t \pi_t^2} \\
&= \beta - \frac{\sum_t (1-s)(2p-1)\sigma_{\xi vt} \pi_t}{\sum_t \pi_t^2} = \beta,
\end{aligned}$$

regardless of γ_t .

E.2.4 Derivations for Variance Estimands

Proof of Equation (18).

$$\begin{aligned}
E[\hat{\Psi}_{MO}] &= E \left[\sum_i \left(\sum_{j \neq i} P_{ij} (R_j + \eta_j) \right)^2 (R_{\Delta i} + \nu_i)^2 + \sum_i \sum_{j \neq i} P_{ij}^2 (R_i + \eta_i) (R_{\Delta i} + \nu_i) (R_j + \eta_j) (R_{\Delta j} + \nu_j) \right] \\
&= E \left[\sum_i \left(\left(\sum_{j \neq i} P_{ij} R_j \right)^2 + \left(\sum_{j \neq i} P_{ij} \eta_j \right)^2 \right) (R_{\Delta i} + \nu_i)^2 \right] \\
&\quad + E \left[\sum_i \sum_{j \neq i} P_{ij}^2 (R_i R_{\Delta i} + \eta_i R_{\Delta i} + R_i \nu_i + \eta_i \nu_i) (R_j R_{\Delta j} + \eta_j R_{\Delta j} + R_j \nu_j + \eta_j \nu_j) \right]
\end{aligned}$$

$$\begin{aligned}
&= \sum_i M_{ii}^2 R_i^2 (R_{\Delta i}^2 + E[\nu_i^2]) + \sum_i R_{\Delta i}^2 E \left[\left(\sum_{j \neq i} P_{ij} \eta_j \right)^2 \right] + \sum_i E[\nu_i^2] E \left[\left(\sum_{j \neq i} P_{ij} \eta_j \right)^2 \right] \\
&\quad + \sum_i \sum_{j \neq i} P_{ij}^2 (R_i R_{\Delta i} + E[\eta_i \nu_i]) (R_j R_{\Delta j} + E[\eta_j \nu_j]) \\
&= \sum_i M_{ii}^2 R_i^2 (R_{\Delta i}^2 + E[\nu_i^2]) + \sum_i \sum_{j \neq i} P_{ij}^2 E[\eta_j^2 (R_{\Delta i}^2 + \nu_i^2)] \\
&\quad + \sum_i \sum_{j \neq i} P_{ij}^2 (R_i R_{\Delta i} + E[\eta_i \nu_i]) (R_j R_{\Delta j} + E[\eta_j \nu_j]) \\
&= \sum_i M_{ii}^2 R_i^2 R_{\Delta i}^2 + \sum_i M_{ii}^2 R_i^2 E[\nu_i^2] + \sum_i \sum_{j \neq i} P_{ij}^2 E[\nu_i^2] E[\eta_j^2] + \sum_i \sum_{j \neq i} P_{ij}^2 R_{\Delta i}^2 E[\eta_j^2] \\
&\quad + \sum_i \sum_{j \neq i} P_{ij}^2 (R_i R_{\Delta i} R_j R_{\Delta j} + E[\eta_i \nu_i] R_j R_{\Delta j} + R_i R_{\Delta i} E[\eta_j \nu_j] + E[\eta_i \nu_i] E[\eta_j \nu_j])
\end{aligned}$$

□