Hypothesis Testing

Fakultas Ilmu Komputer

Universitas Indonesia

2014

References

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Sub topics

- There is one normal population
  - Test Concerning the **Mean** of a Normal Population
    - Case of Known Variance
    - Case of Unknown Variance
- There are two normal populations
  - Testing the **Equality of Means** of Two Normal Population
    - Independent Samples
      - Case of Known Variances
      - Case of Unknown Variances, but The relation is known
        - $\boldsymbol{\sigma^1 = \sigma^2}$
        - $\boldsymbol{\sigma^1 \neq \sigma^2}$
      - Case of Unknown Variances, but The relation is unknown (**σ¹ ? σ²**)
    - Dependent Samples (**paired t-test**)
  - Testing for the **Equality of Variances** of Two Normal Population (for Independent Samples)

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Sebuah jurnal mengklaim bahwa rata-rata tinggi badan dari semua mahasiswa UI adalah 165 CM.

Kemudian, Anda diminta untuk memverifikasi klaim tersebut !

Bagaimana caranya ?

Rektorat UI menyatakan bahwa rata-rata nilai SBMPTN seluruh mahasiswa UI adalah 930. Seseorang kemudian mengambil beberapa mahasiswa untuk menjadi sampel dan menemukan bahwa rata-rata nilai SBMPTN pada sampel mahasiswa tersebut adalah 960.

Bisakah kita simpulkan bahwa pernyataan yang dikeluarkan oleh pihak Rektorat UI adalah salah ?

Seorang pakar mengatakan bahwa rata-rata IPK mahasiswa FASILKOM lebih tinggi dari mahasiswa FH.

Bagaimana caranya menguji pernyataan tersebut ?

Diketahui ada 2 buah metode pengajaran, yaitu **metode Tradisional** dan **metode Baru**.

Seorang guru ingin mengetahui apakah metode pengajaran baru lebih baik dari metode pengajaran tradisional.

Bagaimana caranya ?

Perusahaan ingin mengetahui apakah **pelatihan motivasi** dapat meningkatkan kinerja karyawan atau tidak berpengaruh sama sekali.

Bagaimana caranya ?

Pertanyaan pada 5 kotak sebelumnya dapat dijawab menggunakan **hypothesis testing**.

Kita ingin melakukan verifikasi sebuah klaim atau hipotesis !

Hypothesis Testing = Pengadilan

Test-Statistic & prosedur uji hipotesis

Hakim & Decision-making procedure

Bukti & Saksi

Klaim tentang sesuatu

Sample

Hipotesis

Contoh untuk sebuah proses uji hipotesis

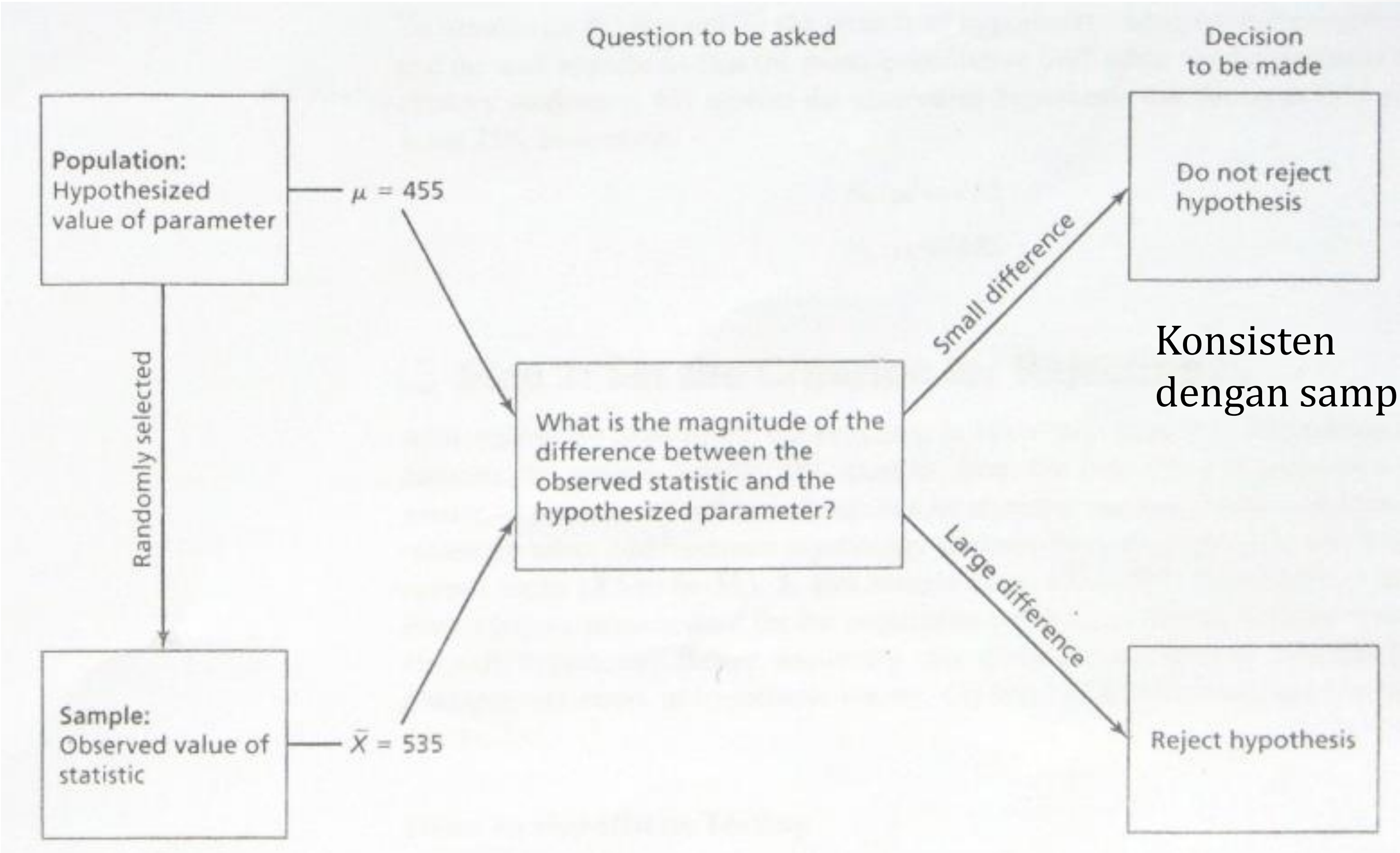


FIGURE 8.1
Log[illegible] hypothesis testing

Konsisten dengan sampel

Tidak konsisten dengan sampel

Ingat !
Ini bukan masalah BENAR atau SALAH !

Hypothesis testing in **inferential statistics** involves **making inferences about the nature of the population** (parameter) on the basis of **observations of a sample** drawn from the population.

The steps in testing a Hypothesis

- **Step 1:** State the hypothesis
- **Step 2:** Set the criterion for rejecting Null Hypothesis ($\boldsymbol{H_0}$)
- **Step 3:** Compute the Test-Statistic
- **Step 4:** Decision about Null Hypothesis ($\boldsymbol{H_0}$)
    - **Accept** or **Reject** $\boldsymbol{H_0}$
    - **Write the conclusion !**

STEP 1 : State the Hypothesis

A **hypothesis** is a conjecture about one or more **population parameter**. This conjecture **may** or **may not be true**.

There are two types of hypothesis for each situation:

Null Hypothesis (H_0) : is a statistical hypothesis that states that **there is no difference** between a parameter and a specific value, or that **there is no difference** between two parameters.

Alternative Hypothesis (H_1) : is a statistical hypothesis that states **the existence of a difference** between a parameter and a specific value, or states that **there is a difference** between two parameters.

- Can be supported only by rejecting the null hypothesis

We need them both !

STEP 1 : State the Hypothesis

Example for **One-Sample Case**, from one population

Seorang pakar psikologi merasa bahwa memutar musik saat ujian dapat mengubah nilai ujian siswa. Dari pengalaman yang telah lalu, rata-rata nilai ujian siswa adalah 68.

Nyatakan hipotesisnya (Null & Alternatif) !

$$H_0 : \mu = 68 \qquad \textbf{Versus} \qquad H_1 : \mu \neq 68$$

We call this as **Two-Tailed Test** !

To state hypotheses correctly, You must translate the *conjecture* or *claim* from words into mathematical symbols !

STEP 1 : State the Hypothesis

Example for **One-Sample Case**, from one population

Sebuah jurnal mengklaim bahwa rata-rata tinggi badan dari semua mahasiswa UI kurang dari 190 CM.

Nyatakan hipotesisnya (Null & Alternatif) !

$$H_0 : \mu = 190 \quad \textbf{Versus} \quad H_1 : \mu < 190$$

We call this as **One-Tailed Test (left-tailed)** !

STEP 1 : State the Hypothesis

Example for **One-Sample Case**, from one population

Seorang dokter menduga bahwa jika seorang ibu hamil mengkonsumsi sebuah **pil vitamin**, maka berat bayi yang lahir akan meningkat.

Sejauh ini diketahui bahwa rata-rata bayi yang baru lahir adalah **8.6 pounds**.

Nyatakan hipotesisnya (Null & Alternatif) !

$$H_0 : \mu = 8.6 \quad \textbf{Versus} \quad H_1 : \mu > 8.6$$

We call this as **One-Tailed Test (right-tailed)** !

STEP 1 : State the Hypothesis

Example for **Two-Sample Case**, from two population

Diketahui ada 2 buah metode pengajaran, yaitu **metode Tradisional** dan **metode Baru**.

Seorang pakar pendidikan mengklaim bahwa siswa yang diajar menggunakan metode baru memberikan nilai ujian yang **lebih baik** dibandingkan siswa yang diajar menggunakan metode tradisional.

Nyatakan hipotesisnya (Null & Alternatif) !

$$H_0 : \mu_1 = \mu_2 \quad \textbf{Versus} \quad H_1 : \mu_1 > \mu_2$$

We call this as **One-Tailed Test (right-tailed)** !

STEP 1 : State the Hypothesis

Table 8–1 Hypothesis-Testing Common Phrases

>	<
Is greater than	Is less than
Is above	Is below
Is higher than	Is lower than
Is longer than	Is shorter than
Is bigger than	Is smaller than
Is increased	Is decreased or reduced from
=	≠
Is equal to	Is not equal to
Is the same as	Is different from
Has not changed from	Has changed from
Is the same as	Is not the same as

	One-Sample Case			Two-Sample Case		
Two-tailed test:	$H_0 : \mu = k$	**VS**	$H_1 : \mu \neq k$	$H_0 : \mu_1 = \mu_2$	**VS**	$H_1 : \mu_1 \neq \mu_2$
One-tailed test:	$H_0 : \mu = k$	**VS**	$H_1 : \mu > k$	$H_0 : \mu_1 = \mu_2$	**VS**	$H_1 : \mu_1 > \mu_2$
	$H_0 : \mu = k$	**VS**	$H_1 : \mu < k$	$H_0 : \mu_1 = \mu_2$	**VS**	$H_1 : \mu_1 < \mu_2$

k adalah sebuah nilai konstan

STEP 1 : State the Hypothesis

In each of the following situations, state whether it is a **correctly stated hypothesis** testing problem and why.

$H_0: \mu = 190$ **Versus** $H_1: \mu < 190$

$H_0: \sigma = 3.4$ **Versus** $H_1: \sigma \neq 3.4$

$H_0: \bar{x} = 4$ **Versus** $H_1: \bar{x} > 4$

$H_0: \sigma_1 = \sigma_2$ **Versus** $H_1: \sigma_1 > \sigma_2$

$H_0: \mu > 190$ **Versus** $H_1: \mu = 190$

$H_0: S = 4$ **Versus** $H_1: S > 4$

STEP 2 : Set the criterion for rejecting H_0

Suppose, we want to test the hypothesis that the mean **SAT score** for students is 455.

$$H_0 : \mu = 455 \quad \textbf{Versus} \quad H_1 : \mu \neq 455$$

We then randomly selected **144 students** as our sample, and found that the sample mean ($\bar{X}$) is **535**.

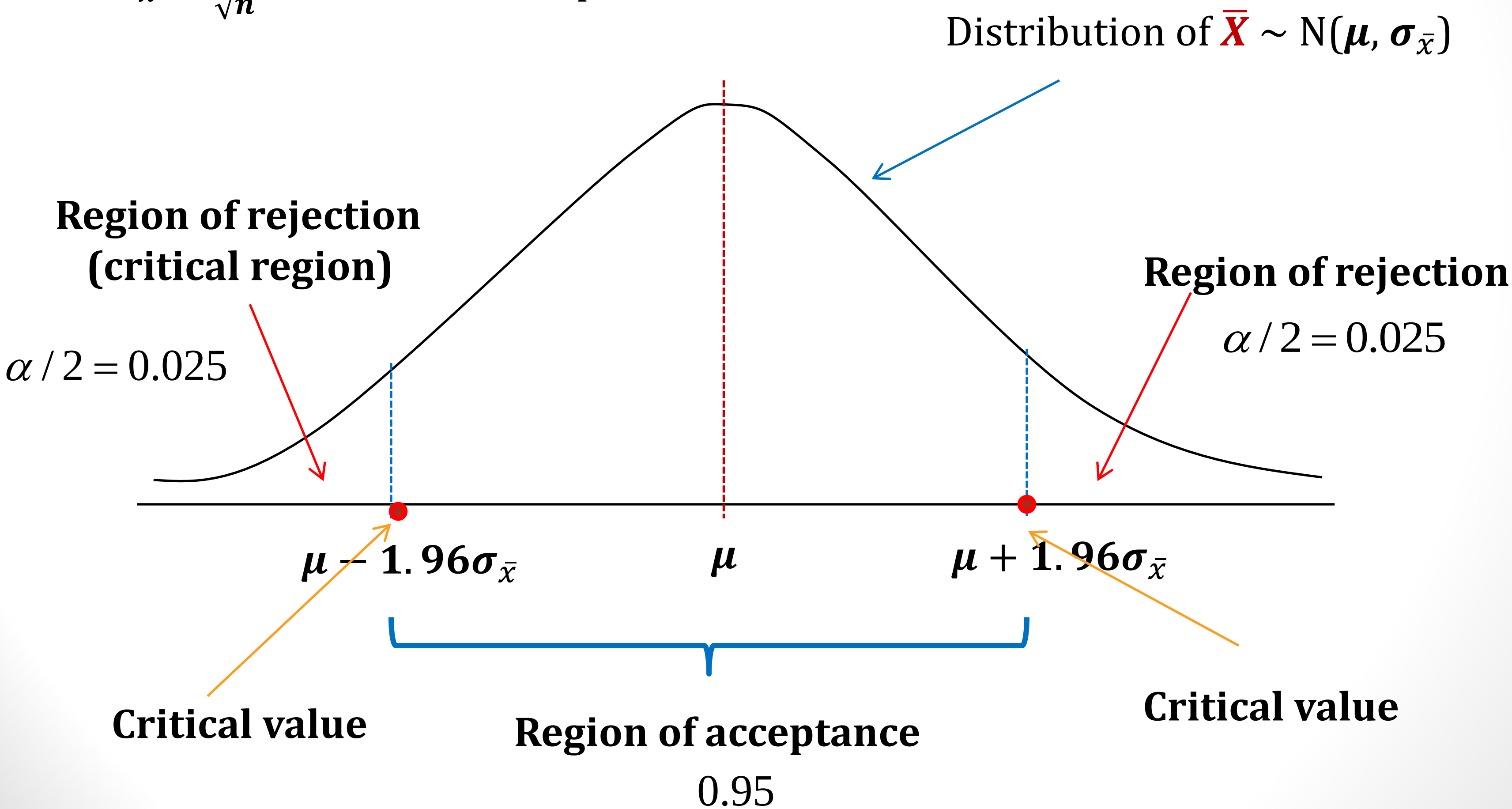
Idea:

A value of sample mean ($\bar{X}$) that falls **close** to the hypothesized value of $\mu = 455$ is **evidence** that the true mean is really 455 (supports the H_0).

STEP 2 : Set the criterion for rejecting H_0

Misal, jika $\boldsymbol{\mu - 1.96\sigma_{\bar{x}} \leq \bar{X} \leq \mu + 1.96\sigma_{\bar{x}}}$, kita **tidak akan tolak** $\boldsymbol{H_0}$.

$\boldsymbol{\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}}}$ is std. dev. of sample mean



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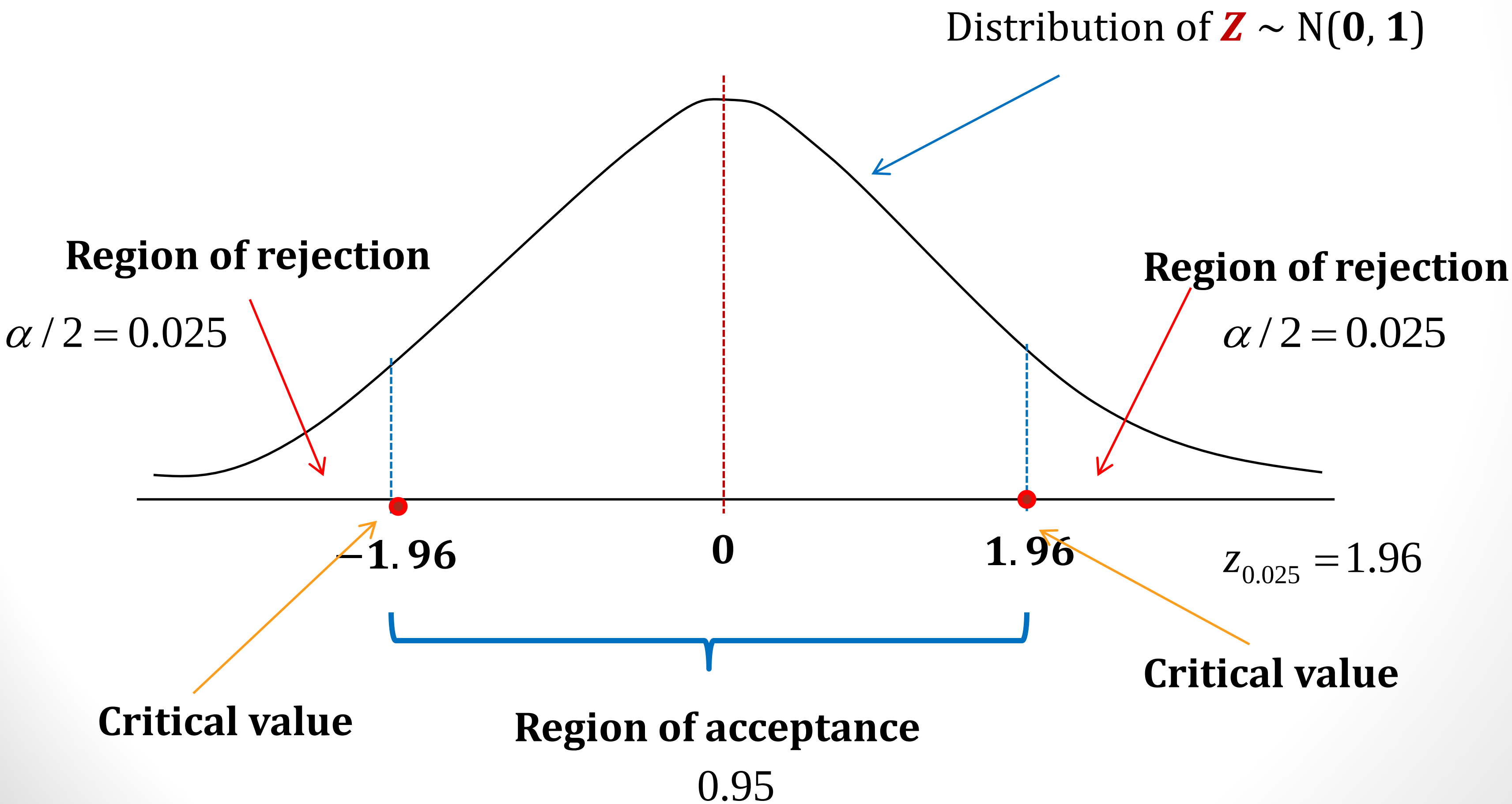
STEP 2 : Set the criterion for rejecting H_0

Now, if we use **standard-score z** to determine how different $\bar{X}$ is from $\boldsymbol{\mu}$.

$$z = \frac{\bar{X} - \mu}{\sigma / \sqrt{n}}$$

artinya, jika $\mathbf{1.96} \le Z \le \mathbf{1.96}$, kita **tidak akan tolak** H_0 !



STEP 2 : Set the criterion for rejecting H_0

The decision procedure can lead to either of **two wrong conclusions**.

Type I error : Rejecting the null hypothesis H_0 when it is true.

Type II error : Failing to reject the null hypothesis H_0 when it is false.

	H_0 true (innocent)	H_0 false (not innocent)
Reject H_0 (convict)	Type I error 1.	Correct decision 2.
Do not reject H_0 (acquit)	Correct decision 3.	Type II error 4.

STEP 2 : Set the criterion for rejecting H_0

Level of Significance (α)

Level of Significance is:

- **α**
- ***Luas area region of rejection***
- **Probability of making Type I Error**
- **Probability of rejecting H_0 when we know that H_0 is true**

Generally, the analyst **controls the type I error probability (i.e level of significance α)** when he or she **selects the critical values**.

Researchers usually **established α before** collecting any data.

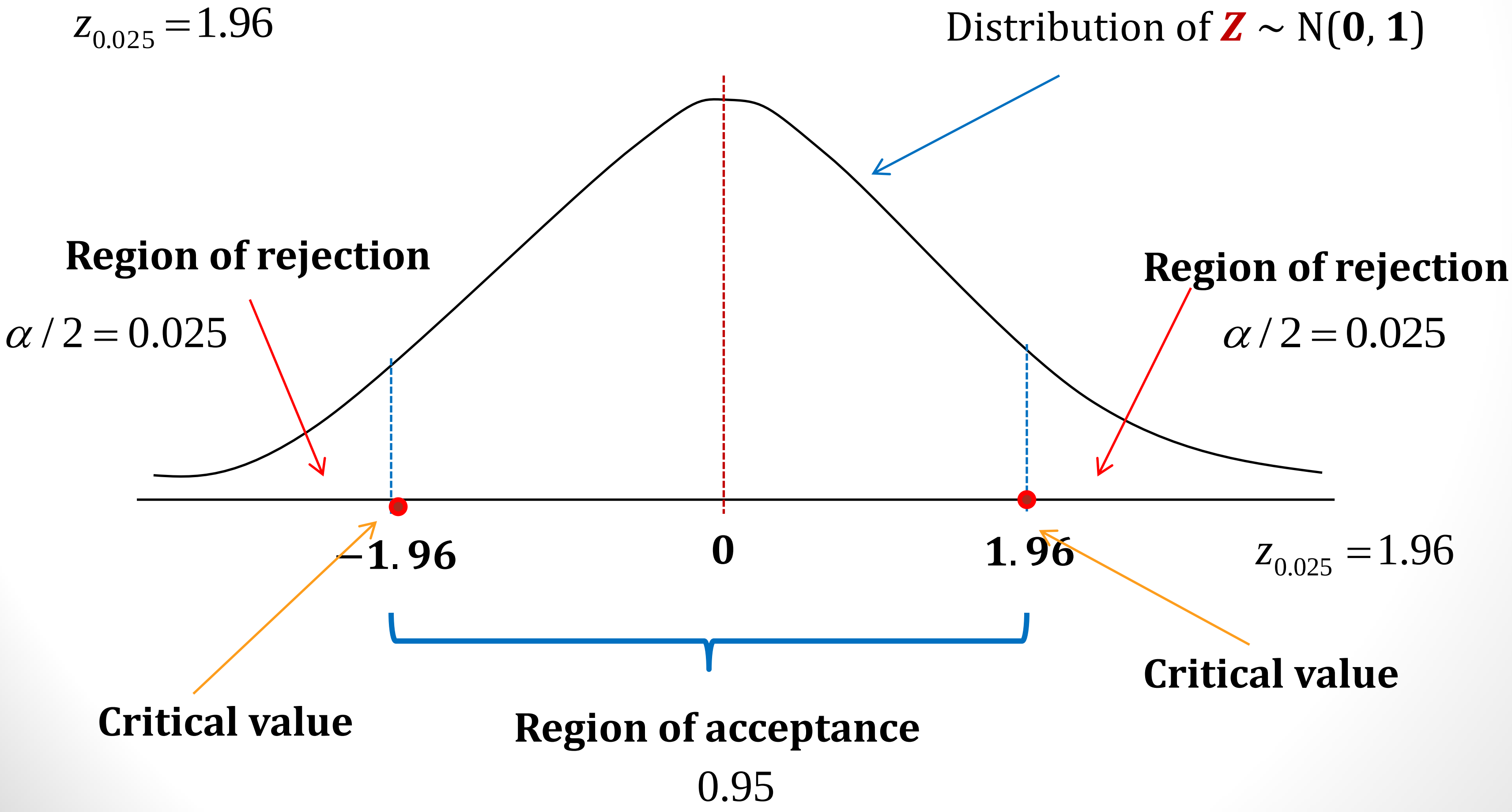
The most frequently used: 0.05 and 0.01 → the researcher knows that the decision to reject H_0 may be incorrect 5% or 1% of the time, respectively.

STEP 2 : Set the criterion for rejecting H_0

In the previous example (SAT Score), can you mention the **significance level α** being used ?

Answer: **α = 0.05 or 5%**

$z_{0.025} = 1.96$



STEP 3 : Compute the Test-Statistic (TS)

Test-Statistic : sample statistic used to decide whether to reject the **null hypothesis**.

A **statistical test** uses the data obtained from a sample to make a decision about whether the null hypothesis should be rejected.

In the previous example (SAT score), we use

$$z = \frac{\overline{X} - \mu}{\sigma / \sqrt{n}}$$

Sebuah contoh test-statistic

as our **Test-Statistic**.

- if $\mathbf{1.96} \leq \boldsymbol{Z} \leq \mathbf{1.96}$, we **accept** $\boldsymbol{H_0}$!
- otherwise

STEP 3 : Compute the Test-Statistic (TS)

Suppose, the previous example (SAT score) has the following additional information regarding the population & sample.

$$\mu = 455$$
$$n = 144$$
$$\overline{X} = 535$$
$$\sigma = 100$$

Now, we compute the Test-Statistic:

$$z = \frac{\overline{X} - \mu}{\sigma / \sqrt{n}} = \frac{535 - 455}{100 / \sqrt{144}} = 9.60$$

9.60 is a **test value**, i.e. The numerical value obtained from a statistical test.

- if $\mathbf{1.96 \le \textcolor{red}{Z} \le 1.96}$, we **accept** $\textcolor{red}{\boldsymbol{H_0}}$!
- Otherwise, we **reject**

STEP 4 : Decision about the Null Hypothesis

There are only two decisions:

- **Accept** the Null Hypothesis ($\boldsymbol{H_0}$)
- **Reject** the Null Hypothesis (Take the alternative one)

In the previous example (SAT score), the observed value of the test statistic (**+9.60**) exceeds the critical value (**± 1.96**).

We also used **level of significance α = 0.05**.

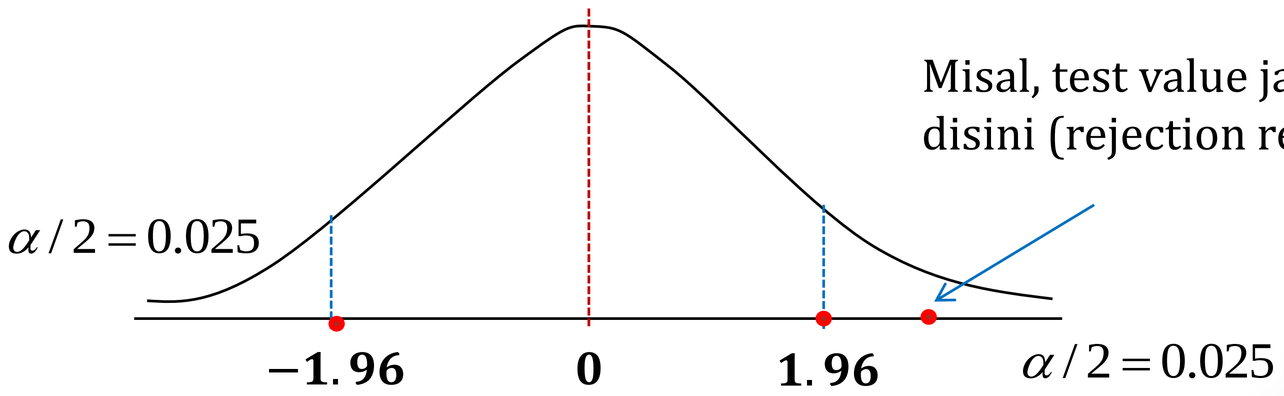
- if $\mathbf{1.96} \le \boldsymbol{Z} \le \mathbf{1.96}$, we **accept** $\boldsymbol{H_0}$!
- Otherwise, we **reject**

So, we reject $\boldsymbol{H_0}$!   But, what does it mean ??

STEP 4 : Decision about the Null Hypothesis

If the observed value of test statistic falls in the **rejection region** with, **for instance, α=0.05**, we can say for observed sample mean:

- The **sample mean** is considered as **significantly different from the H_0 at the 0.05 level of significance**.
- The probability is less than 0.05 that the observed sample mean will have occurred by chance if the null hypothesis is true → p < 0.05.

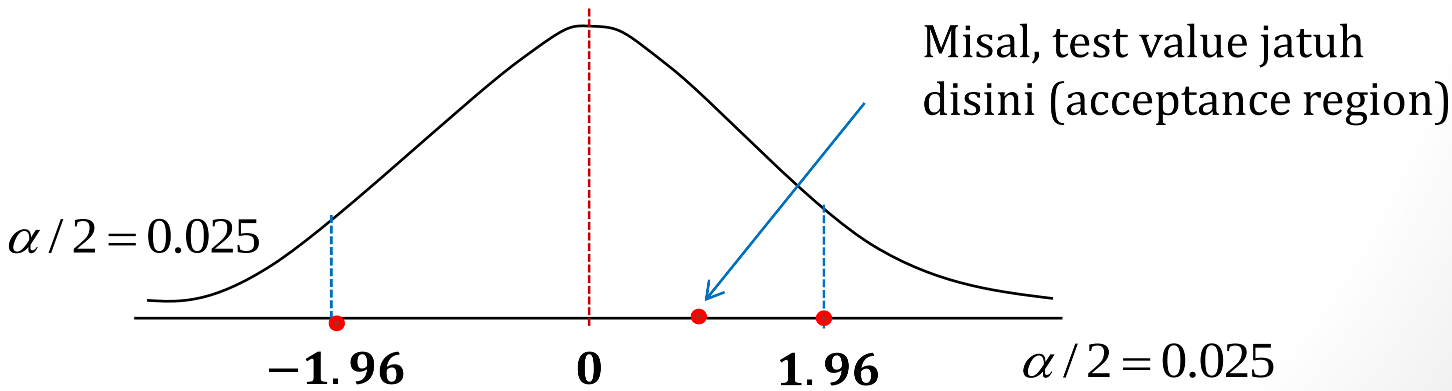


STEP 4 : Decision about the Null Hypothesis

If the observed value of test statistic does **not** fall in the **rejection region** with, **for instance, α=0.05**, we can say for observed sample mean:

- The **sample mean** is not **sufficiently different from the H_0 at the 0.05 level of significance**. (**non-significant difference**)

- The probability is greater than 0.05 that the observed sample mean will have occurred by chance if the null hypothesis is true → $p > 0.05$.



STEP 4 : Decision about the Null Hypothesis

- Choosing level of significance
  - Tendency to guard against making type I error → set too conservative α (0.05)
  - It results in retaining the null hypothesis even when p is relatively small (e.g. $p < 0.07$ but $p > 0.05$ → retain H_0)

- Statistical precision
  - Larger sample → smaller standard error → more precision → tend to reject any H_0
  - Smaller sample → bigger standard error → tend to retain any H_0 even though the difference between the hypothesized value and the observed value is seemingly large.

There is one normal population (one sample case)

Test Concerning the **Mean** of a Normal Population

$X_1, X_2, \ldots, X_n$ is a sample of size n from a normal distribution having an unknown mean μ and a known variance σ^2.

We want to test the null hypothesis:

$$H_0 : \mu = \mu_0$$

against the alternative hypothesis:

$$H_1 : \mu \neq \mu_0$$

Where μ_0 is some **specified constant**.

Ini yang disebut **Two-Tailed Test**. Nanti, kita akan belajar tentang **One-Tailed Test**

Case of Known Variance

It seems reasonable to **reject** $\boldsymbol{H_0}$ if $\bar{\boldsymbol{X}}$ is **too far** from $\boldsymbol{\mu_0}$. So the critical region **C** (region of rejection) would be

$$C = \left\{ X_1, \ldots, X_n : | \bar{X} - \mu_0 | > c \right\}$$ for some constant **c**

If we desire that the test has significance level **α** (probability of the type I error equals to **α**), then

$$P_{\mu_0}\left(| \bar{X} - \mu_0 | > c \right) = \alpha$$

The preceding probability is computed under the assumption that $\boldsymbol{\mu = \mu_0}$.

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Case of Known Variance

Hence, $$Z = \frac{\bar{X} - \mu_0}{\sigma / \sqrt{n}} \sim N(0,1)$$

Under the assumption $\boldsymbol{\mu = \mu_0}$

Now, we have the following equation from the previous one

$$P_{\mu_0}\left(| \bar{X} - \mu_0 | > c\right) = \alpha$$

$$P\left(\left|\frac{\bar{X} - \mu_0}{\sigma / \sqrt{n}}\right| > \frac{c}{\sigma / \sqrt{n}}\right) = \alpha$$

$$P\left(| Z | > \frac{c\sqrt{n}}{\sigma}\right) = \alpha$$

Case of Known Variance

Hence,

$$2P\left(Z > \frac{c\sqrt{n}}{\sigma}\right) = \alpha$$

$$P\left(Z > \frac{c\sqrt{n}}{\sigma}\right) = \alpha/2$$

We know that $P(Z > z_{\alpha/2}) = \alpha/2$

This yields $c = z_{\alpha/2}\dfrac{\sigma}{\sqrt{n}}$

Using significance level **α** test, we **reject *H*₀** if $|\bar{X} - \mu_0| > z_{\alpha/2}\dfrac{\sigma}{\sqrt{n}}$

Case of Known Variance

Or, using significance level **α** test, we **reject** ***H_0*** if $\left|\frac{\bar{X}-\mu_0}{\sigma/\sqrt{n}}\right| > z_{\alpha/2}$

Here, our **Test-Statistics (TS)** is $TS = \frac{\bar{X}-\mu_0}{\sigma/\sqrt{n}}$

Critical Value

In summary, given the following hypothesis statements and significance level **α**

$$H_0 : \mu = \mu_0 \quad \textbf{vs} \quad H_1 : \mu \neq \mu_0$$

- we **reject** ***H_0*** if $|TS| > z_{\alpha/2}$
- we **accept** ***H_0*** if $|TS| \le z_{\alpha/2}$

This is two-tailed test !

Case of Known Variance

One-tailed Test

What happens when the only alternative to $\boldsymbol{\mu}$ being equal to $\boldsymbol{\mu_0}$ is for $\boldsymbol{\mu}$ to be greater than (or lower than) $\boldsymbol{\mu_0}$?

That is, what happens when our hypothesis statement is

$$H_0 : \mu = \mu_0 \ (or\ \mu \le \mu_0) \quad \textbf{vs} \quad H_1 : \mu > \mu_0$$

OR

$$H_0 : \mu = \mu_0 \ (or\ \mu \ge \mu_0) \quad \textbf{vs} \quad H_1 : \mu < \mu_0$$

Case of Known Variance

One-tailed Test

For the following statement:

$$H_0 : \mu = \mu_0 \quad \textbf{vs} \quad H_1 : \mu > \mu_0$$

Remember that $\bar{X}$ is the point estimate of μ

It seems reasonable to **reject** H_0 if $\bar{X}$ is **much greater than** μ_0. So the critical region **C** (region of rejection) would be

$$C = \left\{X_1, \ldots, X_n : \bar{X} - \mu_0 > c\right\}$$

Tidak ada tanda mutlak !

Thus, If we desire that the test has significance level **α** (probability of the type I error equals to **α**), then

$$P_{\mu_0}\left(\bar{X} - \mu_0 > c\right) = \alpha$$

The next step is similar as before !
You can also do the same thing with the other one-sided test ($H_1 : \mu < \mu_0$)

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Case of Known Variance

summary

Our **Test-Statistics (TS)** is $$TS = \frac{\bar{X} - \mu_0}{\sigma / \sqrt{n}}$$

[Two-tailed test]

Given the following hypothesis statements and significance level **α** :

$$H_0 : \mu = \mu_0 \quad \textbf{VS} \quad H_1 : \mu \neq \mu_0$$

- we **reject** $\boldsymbol{H_0}$ if $|TS| > z_{\alpha/2}$
- we **accept** $\boldsymbol{H_0}$ if $|TS| \leq z_{\alpha/2}$

[One-tailed #1]

Given the following hypothesis statements and significance level **α** :

$$H_0 : \mu = \mu_0 \quad \textbf{VS} \quad H_1 : \mu > \mu_0$$

- we **reject** $\boldsymbol{H_0}$ if $TS > z_{\alpha}$
- we **accept** $\boldsymbol{H_0}$ if $TS \leq z_{\alpha}$

[One-tailed #2]

Given the following hypothesis statements and significance level **α** :

$$H_0 : \mu = \mu_0 \quad \textbf{VS} \quad H_1 : \mu < \mu_0$$

- we **reject** $\boldsymbol{H_0}$ if $TS < -z_{\alpha}$
- we **accept** $\boldsymbol{H_0}$ if $TS \geq -z_{\alpha}$

Case of Unknown Variance (t-test)

Previously, we assume that the population variance **σ** is known.
However, the most common situation is when **σ** is **unknown** !

Now when **σ** is **no longer known**, we use the following proposition:

$$T_{n-1} = \frac{\bar{X} - \mu_0}{S/\sqrt{n}} \sim t_{n-1}$$

Where,

$$S^2 = \frac{\sum_{i=1}^{n} \left(X_i - \bar{X}\right)^2}{n-1}$$

n : sample size

For **two-tailed test**, ***H_0*** is rejected when $\left| \frac{\bar{X} - \mu_0}{S/\sqrt{n}} \right|$ is **large** !

Case of Unknown Variance (t-test)

summary

Our **Test-Statistics (TS)** is $TS = \dfrac{\overline{X} - \mu_0}{S / \sqrt{n}}$

[Two-tailed test]

Given the following hypothesis statements and significance level **α** :

$$H_0 : \mu = \mu_0 \quad \textbf{VS} \quad H_1 : \mu \neq \mu_0$$

- we **reject** $\boldsymbol{H_0}$ if $|TS| > t_{\alpha/2, n-1}$
- we **accept** $\boldsymbol{H_0}$ if $|TS| \leq t_{\alpha/2, n-1}$

[One-tailed #1]

Given the following hypothesis statements and significance level **α** :

$$H_0 : \mu = \mu_0 \quad \textbf{VS} \quad H_1 : \mu > \mu_0$$

- we **reject** $\boldsymbol{H_0}$ if $TS > t_{\alpha, n-1}$
- we **accept** $\boldsymbol{H_0}$ if $TS \leq t_{\alpha, n-1}$

[One-tailed #2]

Given the following hypothesis statements and significance level **α** :

$$H_0 : \mu = \mu_0 \quad \textbf{VS} \quad H_1 : \mu < \mu_0$$

- we **reject** $\boldsymbol{H_0}$ if $TS < -t_{\alpha, n-1}$
- we **accept** $\boldsymbol{H_0}$ if $TS \geq -t_{\alpha, n-1}$

Seorang peneliti mengklaim bahwa rataan harga dari sepatu olahraga pria adalah kurang dari \$80. Dia kemudian memilih sampel secara random yang berisi 36 pasang sepatu dari katalog dan menemukan bahwa rataan biaya dari sepatu-sepatu pada sampel adalah \$75.

Asumsikan populasi harga sepatu mengikuti distribusi normal dengan variansi 368.84

Apakah ada cukup bukti untuk mendukung klaim dari peneliti tersebut dengan level of significance 0.1 ?

Solusi:

Step 1: State the hypothesis and identify the claim

$$H_0 : \mu = 80 \quad \textbf{VS} \quad H_1 : \mu < 80 \; (claim)$$

Step 2: Set the rejection criteria

Since **α = 0.1** and the test is one-tailed test (left) (**σ is known**),

we **reject** ***H*₀** if $TS < -z_\alpha$

The critical value is $-z_\alpha = -z_{0.1} = -1.28$

Step 3: Compute the Test-Statistics

$$TS = \frac{\overline{X} - \mu_0}{\sigma / \sqrt{n}} = \frac{75 - 80}{19.2 / \sqrt{36}} = -1.56$$

Step 4: Make the decision and conclusion

since $TS = -1.56 < -z_\alpha = -1.28$, we **reject** $\boldsymbol{H_0}$!

<u>**Conclusion:**</u> **There is enough evidence to support the claim** that the average cost of men's athletic shoes is less than $80.

Sebuah pusat rehabilitasi medis melaporkan bahwa rata-rata biaya untuk rehabilitasi penderita stroke adalah $24,672.

Untuk melihat apakah laporan ini berbeda di sebuah rumah sakit tertentu, seorang peneliti mengambil sampel secara acak yang berisi 35 penderita stroke di sebuah rumah sakit dan menemukan bahwa rataan biaya rehabilitasi mereka adalah $26.343.

Standar deviasi dari populasi adalah $3251. Pada level of significance 0.01, bisakah kita simpulkan bahwa rataan biaya rehabilitasi stroke pada rumah sakit tersebut berbeda dari $24.672 ?

Solusi:

Step 1: State the hypothesis and identify the claim

$$H_0 : \mu = 24.672 \quad \textbf{VS} \quad H_1 : \mu \neq 24.672\ (claim)$$

Step 2: Set the rejection criteria

Since **α = 0.01** and the test is two-tailed test (**σ is known**),

we **reject** ***H*₀** if $|TS| > z_{\alpha/2}$ $\quad z_{\alpha/2} = z_{0.005} = 2.58$

The critical value is $z_{0.005} = 2.58 \quad and \quad -z_{0.005} = -2.58$

Step 3: Compute the Test-Statistics

$$TS = \frac{\overline{X} - \mu_0}{\sigma / \sqrt{n}} = \frac{26.343 - 24.672}{3251 / \sqrt{35}} = 3.04$$

Step 4: Make the decision and conclusion

since $|TS| = 3.04 > z_{\alpha/2} = 2.58$, we **reject** $\boldsymbol{H_0}$!

Conclusion: **There is enough evidence to support the claim** that the average cost of rehabilitation at the particular hospital is different from $24,672.

Sebuah investigasi medis mengeluarkan klaim bahwa rata-rata banyaknya infeksi per minggu pada sebuah rumah sakit adalah 16.3

Sebuah sampel acak dari 10 minggu mempunyai rataan banyaknya infeksi 17.2. Standar deviasi dari sampel adalah 1.8

Apakah cukup bukti untuk menyangkal klaim dari si investigator pada level of significance 0.05 ?

Solusi:

Step 1: State the hypothesis and identify the claim

$$H_0 : \mu = 16.3\,(claim) \qquad \textbf{VS} \qquad H_1 : \mu \neq 16.3$$

Step 2: Set the rejection criteria

Since **α = 0.05** and the test is two-tailed test (**σ is unknown**),

we **reject** ***H*₀** if $|TS| > t_{\alpha/2,n-1}$ $\qquad t_{\alpha/2,\,n-1} = t_{0.025,\,9} = 2.262$

The critical value is $t_{0.025,9} = 2.262 \quad and \quad -t_{0.025,9} = -2.262$

Step 3: Compute the Test-Statistics (t-test)

$$TS = \frac{\overline{X} - \mu_0}{S/\sqrt{n}} = \frac{17.2 - 16.3}{1.8/\sqrt{10}} = 1.58$$

Step 4: Make the decision and conclusion

since $|TS| = 1.58 < t_{\alpha/2,n-1} = 2.262$, we **accept** ***H*₀** !

<u>**Conclusion:**</u> There is **not** enough evidence to **reject** the claim that the average number of infections is 16.3.

Berdasarkan eksperimen di lapangan, sebuah varietas baru dari gandum diharapkan dapat menghasilkan 12 Kwintal per hektar.

10 ladang yang menanam gandum baru tersebut dipilih secara acak sebagai sampel. Masing-masing ladang pada sampel dihitung dan hasilnya (Kwintal/Hektar) adalah sebagai berikut :

14.3	12.6	13.7	10.9	13.7
12.0	11.4	12.0	12.6	13.1

Asumsikan hasil "kwintal/hektar" dari gandum tersebut mengikuti distribusi **Normal**.

Apakah hasil yang diperoleh dari sampel sesuai dengan harapan ? (ujilah dengan level of significance **5%**)

There are two normal populations (Two Sample Case)

Testing for the **Equality of Variances** of Two Normal Population (for independent samples)

Let $X_1, \ldots, X_n$ be a sample of size n from a normal population having mean μ_1 and variance ${\sigma_1}^2$

Let $Y_1, \ldots, Y_m$ be a sample of size m from a different normal population having mean μ_1 and variance ${\sigma_2}^2$

Suppose that the two samples are **independent of each other**.

All parameters μ_1, ${\sigma_1}^2$, μ_2, ${\sigma_2}^2$ are **unknown** !

Now, we want to test the hypothesis:

$$H_0 : \sigma_1^2 = \sigma_2^2 \quad \textbf{vs} \quad H_1 : \sigma_1^2 \neq \sigma_2^2$$

If we let the samples' variances

$$S_1^2 = \sum_{i=1}^{n} \frac{\left(X_i - \overline{X}\right)^2}{n-1} \qquad S_2^2 = \sum_{i=1}^{m} \frac{\left(Y_i - \overline{Y}\right)^2}{m-1}$$

Then, we know that

$$(n-1)\frac{S_1^2}{\sigma_1^2} \sim \chi^2_{n-1} \qquad (m-1)\frac{S_2^2}{\sigma_2^2} \sim \chi^2_{m-1}$$

$$\frac{\chi^2_{n-1}/(n-1)}{\chi^2_{m-1}/(m-1)} = \frac{S_1^2/\sigma_1^2}{S_2^2/\sigma_2^2} \sim F_{n-1,\, m-1}$$

When we use significance level **α**, it means that **we assume that H_0 is true**:

$$\frac{S_1^2}{S_2^2} \sim F_{n-1,\, m-1}$$

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Hence, $$P_{H_0}\left(F_{1-\alpha/2,\, n-1,\, m-1} \le \frac{S_1^2}{S_2^2} \le F_{\alpha/2,\, n-1,\, m-1} \right) = 1-\alpha$$

This is the **Test-Statistic** (**F-test**) : $$TS = F = \frac{S_1^2}{S_2^2}$$

Thus, a significance level **α** test of $\boldsymbol{H_0}$ against $\boldsymbol{H_1}$ is to

- we **accept** $\boldsymbol{H_0}$ if $F_{1-\alpha/2,\, n-1,\, m-1} \le F \le F_{\alpha/2,\, n-1,\, m-1}$
- we **reject** $\boldsymbol{H_0}$ **otherwise**

To compute $F_{1-\alpha/2,\, n-1,\, m-1}$, we sometimes need the following property:

$$F_{\beta,\, n,\, m} = \frac{1}{F_{1-\beta,\, m,\, n}}$$

Versi lain yang lebih mudah secara komputasi :

Ini adalah cara yang sering digunakan secara praktis

- If $\boldsymbol{S_1^2} > \boldsymbol{S_2^2}$ then
  - we **reject** $\boldsymbol{H_0}$ if $F > F_{\alpha/2,\, n-1,\, m-1}$
  - we **accept** $\boldsymbol{H_0}$ **otherwise**

Kasus 1

- If $\boldsymbol{S_1^2} < \boldsymbol{S_2^2}$ then
  - we **reject** $\boldsymbol{H_0}$ if $F < F_{1-\alpha/2,\, n-1,\, m-1}$
  - we **accept** $\boldsymbol{H_0}$ **otherwise**

Kasus 2

TIPS: pilihlah variansi sample yang terbesar untuk menjadi indeks 1 ($\boldsymbol{S_1^2}$) sehingga kita selalu menggunakan **Kasus 1**

- **Step 1**: state the hypothesis

$$H_0 : \sigma_1^2 = \sigma_2^2 \quad \textbf{vs} \quad H_1 : \sigma_1^2 \neq \sigma_2^2$$

- **Step 2**: define rejection region
  - Using level of significance **α**
  - If $S_1^2 > S_2^2$ then
    - we **reject** H_0 if $F > F_{\alpha/2,\, n-1,\, m-1}$
  - If $S_1^2 < S_2^2$ then
    - we **reject** H_0 if $F < F_{1-\alpha/2,\, n-1,\, m-1}$

- **Step 3**: Compute Test-Statistics (F-Test)

$$TS = F = \frac{S_1^2}{S_2^2}$$

- **Step 4**: Decision about hypothesis
  - Compare the value obtained in the 3rd step with the rejection criteria described in 2nd step.

Ada 2 buah mesin penghasil ban. Sebuah ban diukur menggunakan diameternya (dalam CM). Kita ingin mengetahui apakah variansi populasi ban yang dihasilkan oleh kedua mesin berbeda atau tidak.

Untuk menguji, kita ambil 2 buah sampel dari 2 mesin tersebut. 6 buah ban diambil dari mesin pertama (dari populasi pertama) sebagai sampel, dan 31 ban diambil dari mesin kedua sebagai sampel kedua. Tabel berikut berisi informasi rinci terkait kedua sampel.

	Sample 1	**Sample 2**
Sample mean	27.20	19.60
n	6	31
S^2	145.9	62.6
S	12.07	7.09

Ujilah perbedaan variansi dari kedua populasi dengan level of significance $\alpha = 0.1$!

- **Step 1**: state the hypothesis

$$H_0 : \sigma_1^2 = \sigma_2^2 \quad \textbf{vs} \quad H_1 : \sigma_1^2 \neq \sigma_2^2$$

- **Step 2**: define rejection region
  - Using level of significance **α = 0.1**

$$F_{\alpha/2,\,n-1,\,m-1} = F_{0.05,\,6-1,\,31-1} = F_{0.05,\,5,\,30} = 2.53$$

  - $\boldsymbol{S_1^2 > S_2^2}$ then
  - we **reject** $\boldsymbol{H_0}$ if $F > F_{\alpha/2,\,n-1,\,m-1} = 2.53$

- **Step 3**: Compute Test-Statistics (F-Test)

$$TS = F = \frac{S_1^2}{S_2^2} = \frac{145.9}{62.6} = 2.33$$

- **Step 4**: Decision about hypothesis

$$F = 2.33 < F_{\alpha/2,\,n-1,\,m-1} = 2.53$$

$\boldsymbol{H_0}$ is not rejected.
Berdasarkan data observasi,
variansi kedua populasi dapat dikatakan sama
$\boldsymbol{\sigma_1^2 = \sigma_2^2}$

There are two normal populations (Two Sample Case)

Testing the **Equality of Means** of Two Normal Population (for independent samples)

Let $X_1, \dots, X_n$ be a sample of size n from a normal population having mean μ_1 and variance ${\sigma_1}^2$

Let $Y_1, \dots, Y_m$ be a sample of size m from a different normal population having mean μ_1 and variance ${\sigma_2}^2$

Suppose that the two samples are **independent of each other**.

the parameters μ_1, μ_2 are **unknown** !

Now, we want to test the hypothesis:

$$H_0 : \mu_1 = \mu_2 \quad \textbf{vs} \quad H_1 : \mu_1 \neq \mu_2$$

Case of Known Variances

The null hypothesis can be written as $H_0 : \mu_1 - \mu_2 = 0$

It seems reasonable to reject $\boldsymbol{H_0}$ when $\bar{X} - \bar{Y}$ is **far from zero**. That is,

Reject $\boldsymbol{H_0}$ if $\left|\bar{X} - \bar{Y}\right| > c$ for some value **c**

Accept $\boldsymbol{H_0}$ otherwise

We need to determine the distribution of $\bar{X} - \bar{Y}$:

$$\bar{X} - \bar{Y} \sim N\left(\mu_1 - \mu_2, \frac{\sigma_1^2}{n} + \frac{\sigma_2^2}{m}\right)$$

when $\boldsymbol{H_0}$ is true: $\mu_1 - \mu_2 = 0$

$$\frac{\bar{X} - \bar{Y}}{\sqrt{\frac{\sigma_1^2}{n} + \frac{\sigma_2^2}{m}}} \sim N(0,1) \quad \Rightarrow \quad P_{H_0}\left(-z_{\alpha/2} \le \frac{\bar{X} - \bar{Y}}{\sqrt{\frac{\sigma_1^2}{n} + \frac{\sigma_2^2}{m}}} \le z_{\alpha/2}\right) = 1 - \alpha$$

Case of Known Variances

So, we can conclude that for two-sided test

Reject H_0 if $$\left|\frac{\bar{X}-\bar{Y}}{\sqrt{\sigma_1^2/n+\sigma_2^2/m}}\right| > z_{\alpha/2}$$

One – Tailed Test

For example, for testing : $H_0 : \mu_1 = \mu_2 \ (or\ \mu_1 \le \mu_2)$ **vs** $H_1 : \mu_1 > \mu_2$

Reasonable to reject H_0 if $\bar{X}-\bar{Y} > c$ for some value **c**

Using the distribution of $\bar{X}-\bar{Y}$ and the same way as before, we conclude:

Reject H_0 if $$\frac{\bar{X}-\bar{Y}}{\sqrt{\sigma_1^2/n+\sigma_2^2/m}} > z_{\alpha}$$

Case of Known Variances

summary

Our **Test-Statistics (TS)** is $$TS = \frac{\bar{X} - \bar{Y}}{\sqrt{\sigma_1^2/n + \sigma_2^2/m}}$$

[Two-tailed test]

Given the following hypothesis statements and significance level **α** :

$$H_0 : \mu_1 = \mu_2 \qquad \textbf{VS} \qquad H_1 : \mu_1 \neq \mu_2$$

- we **reject** $\boldsymbol{H_0}$ if $|TS| > z_{\alpha/2}$
- we **accept** $\boldsymbol{H_0}$ if $|TS| \leq z_{\alpha/2}$

[One-tailed #1]

Given the following hypothesis statements and significance level **α** :

$$H_0 : \mu_1 = \mu_2 \qquad \textbf{VS} \qquad H_1 : \mu_1 > \mu_2$$

- we **reject** $\boldsymbol{H_0}$ if $TS > z_{\alpha}$
- we **accept** $\boldsymbol{H_0}$ if $TS \leq z_{\alpha}$

[One-tailed #2]

Given the following hypothesis statements and significance level **α** :

$$H_0 : \mu_1 = \mu_2 \qquad \textbf{VS} \qquad H_1 : \mu_1 < \mu_2$$

- we **reject** $\boldsymbol{H_0}$ if $TS < -z_{\alpha}$
- we **accept** $\boldsymbol{H_0}$ if $TS \geq -z_{\alpha}$

Case of Unknown Variances, but $\sigma_1^2 = \sigma_2^2 = \sigma^2$

Once again, we consider a test of $H_0 : \mu_1 = \mu_2$ **VS** $H_1 : \mu_1 \neq \mu_2$

Now, all four parameters $\mu_1, \sigma_1^2, \mu_2, \sigma_2^2$ are **unknown**, but we know that

$$\sigma_1^2 = \sigma_2^2 = \sigma^2$$

In this case, we use the following proposition:

$$\frac{\bar{X} - \bar{Y} - (\mu_1 - \mu_2)}{\sqrt{S_p^2\left(\frac{1}{n} + \frac{1}{m}\right)}} \sim t_{n+m-2} \quad \underset{\substack{\text{when } H_0 \text{ is true:} \\ \mu_1 - \mu_2 = 0}}{\Rightarrow} \quad \frac{\bar{X} - \bar{Y}}{\sqrt{S_p^2\left(\frac{1}{n} + \frac{1}{m}\right)}} \sim t_{n+m-2}$$

Where, $\mathbf{S_p^2}$ is **pooled estimator** of $\boldsymbol{\sigma^2}$ **:** $$S_p^2 = \frac{(n-1)S_1^2 + (m-1)S_2^2}{n+m-2}$$

$\mathbf{S_1^2}$ and $\mathbf{S_2^2}$ are **sample variances** for the first and second sample, respectively.

$$S_1^2 = \sum_{i=1}^{n} \frac{\left(X_i - \bar{X}\right)^2}{n-1} \qquad S_2^2 = \sum_{i=1}^{m} \frac{\left(Y_i - \bar{Y}\right)^2}{m-1}$$

Case of Unknown Variances, but $\sigma_1^2 = \sigma_2^2 = \sigma^2$

summary

Our **Test-Statistics (TS)** is $$TS = \frac{\bar{X} - \bar{Y}}{\sqrt{S_p^2\left(\frac{1}{n} + \frac{1}{m}\right)}}$$

[Two-tailed test]

Given the following hypothesis statements and significance level **α** :

$$H_0 : \mu_1 = \mu_2 \qquad \textbf{VS} \qquad H_1 : \mu_1 \neq \mu_2$$

- we **reject** $\boldsymbol{H_0}$ if $|TS| > t_{\alpha/2,\, n+m-2}$
- we **accept** $\boldsymbol{H_0}$ if $|TS| \leq t_{\alpha/2,\, n+m-2}$

[One-tailed #1]

Given the following hypothesis statements and significance level **α** :

$$H_0 : \mu_1 = \mu_2 \qquad \textbf{VS} \qquad H_1 : \mu_1 > \mu_2$$

- we **reject** $\boldsymbol{H_0}$ if $TS > t_{\alpha,\, n+m-2}$
- we **accept** $\boldsymbol{H_0}$ if $TS \leq t_{\alpha,\, n+m-2}$

[One-tailed #2]

Given the following hypothesis statements and significance level **α** :

$$H_0 : \mu_1 = \mu_2 \qquad \textbf{VS} \qquad H_1 : \mu_1 < \mu_2$$

- we **reject** $\boldsymbol{H_0}$ if $TS < -t_{\alpha,\, n+m-2}$
- we **accept** $\boldsymbol{H_0}$ if $TS \geq -t_{\alpha,\, n+m-2}$

Data 2 buah sampel menyatakan bahwa rataan sewa hotel per malam di Depok adalah $88.42 dan rataan sewa hotel di Bandung adalah $80.61. Data tersebut terdiri dari 50 hotel untuk masing-masing kota.

Standar deviasi dari kedua populasi (dalam hal harga sewa) adalah $5.62 dan $4.83 untuk Depok dan Bandung, secara berurutan.

Pada level of significance 0.05, bisakah kita simpulkan bahwa ada perbedaan signifikan dalam hal harga sewa hotel antara di kota Depok dan Bandung ?

Asumsikan populasi harga sewa hotel mengikuti distribusi normal.

Solusi:

Step 1: State the hypothesis and identify the claim

$$H_0 : \mu_1 = \mu_2 \qquad \textbf{VS} \qquad H_1 : \mu_1 \neq \mu_2 \ (claim)$$

Step 2: Set the rejection criteria

Since **α = 0.05** and the test is two-tailed test (**σ is known**),

we **reject** ***H*₀** if $|TS| > z_{\alpha/2}$ $\qquad z_{\alpha/2} = z_{0.025} = 1.96$

The critical value is $z_{0.025} = 1.96 \quad and \quad -z_{0.025} = -1.96$

Step 3: Compute the Test-Statistics (t-test)

$$TS = \frac{\bar{X} - \bar{Y}}{\sqrt{\sigma_1^2/n + \sigma_2^2/m}} = \frac{88.42 - 80.61}{\sqrt{5.62^2/50 + 4.83^2/50}} = 7.45$$

Step 4: Make the decision and conclusion

since $|TS| = 7.45 > z_{\alpha/2} = 1.96$, we **reject** ***H*₀** !

Conclusion: There is enough evidence to support the claim that the means are not equal. Hence, there is a significant difference in the rates.

Data 2 buah sampel menyatakan bahwa rata-rata luas ladang sampel dari Karawang adalah 191 m2. Rata-rata luas ladang sampel dari Palembang 199 m2. Sampel dari Karawang terdiri dari 8 ladang, dan dari Palembang ada 10 ladang.

Data diambil dari masing-masing kota dengan standar deviasi sampel 38 dan 12 meter persegi, untuk Karawang dan Palembang, secara berurutan.

Bisakah disimpulkan bahwa pada level of significance 0.05, rata-rata luas ladang di kedua kota tersebut berbeda ?

Asumsikan bahwa kedua populasi dari 2 kota berdistribusi normal dan mempunyai variansi yang sama.

Solusi:

Step 1: State the hypothesis and identify the claim

$$H_0 : \mu_1 = \mu_2 \qquad \textbf{VS} \qquad H_1 : \mu_1 \neq \mu_2 \ (claim)$$

Step 2: Set the rejection criteria

Since **α = 0.05** and the test is two-tailed test
(**σ_1^2** and **σ_2^2** **are unknown,** **$\sigma_1^2 = \sigma_2^2$**),
we **reject** ***H_0*** if $|TS| > t_{\alpha/2,n+m-2}$ $\quad t_{\alpha/2,n+m-2} = t_{0.025,16} = 2.12$

The critical value is $\quad t_{0.025,16} = 2.12 \quad and \quad -t_{0.025,16} = -2.12$

Step 3: Compute the Test-Statistics (t-test using **Pooled estimator**)

$$S_p^2 = \frac{(n-1)S_1^2 + (m-1)S_2^2}{n+m-2} = \frac{(8-1)38^2 + (10-1)12^2}{8+10-2} = 712.75$$

$$TS = \frac{\bar{X} - \bar{Y}}{\sqrt{S_p^2\left(\frac{1}{n} + \frac{1}{m}\right)}} = \frac{191 - 199}{\sqrt{712.75\left(\frac{1}{8} + \frac{1}{10}\right)}} = -0.63$$

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Step 4: Make the decision and conclusion

since $|TS| = 0.63 < t_{\alpha/2, n+m-2} = 2.12$, we **accept** ***H_0*** !

Conclusion: There is not enough evidence to support the claim that the average size of the farms is different.

Case of Unknown Variances, but $\sigma_1^2 \neq \sigma_2^2$

Once again, we consider a test of $H_0 : \mu_1 = \mu_2$ **VS** $H_1 : \mu_1 \neq \mu_2$

Now, all four parameters $\mu_1, \sigma_1^2, \mu_2, \sigma_2^2$ are **unknown**, but we know that

$$\sigma_1^2 \neq \sigma_2^2$$

This is known as **Behrens-Fisher** problem.

Since S_1^2 is the natural estimator of σ_1^2 and S_2^2 of σ_2^2, it would be reasonable to use the following quantity as our **Test-Statistics**

$$\frac{\overline{X} - \overline{Y}}{\sqrt{\frac{S_1^2}{n} + \frac{S_2^2}{m}}}$$

But, that quantity has a **complicated distribution** ! even when H_0 is true

Case of Unknown Variances, but $\sigma_1^2 \neq \sigma_2^2$

The common method is to use **Satterthwaite's approximation** for the degree of freedom (Satterthwaite, 1946)

Sattertwaite used t-distribution with approximated degree of freedom:

$$\frac{\bar{X} - \bar{Y}}{\sqrt{\frac{S_1^2}{n} + \frac{S_2^2}{m}}} \sim t_v$$

Degree of freedom ***v*** perlu diestimasi

v is approximated as follows

$$v = \frac{\left(\frac{S_1^2}{n} + \frac{S_2^2}{m}\right)^2}{\frac{\left(\frac{S_1^2}{n}\right)^2}{n-1} + \frac{\left(\frac{S_2^2}{m}\right)^2}{m-1}}$$

Jika hasilnya tidak bulat, maka ***v*** adalah bilangan bulat yang terdekat.

Satterthwaite, F.E. (1946). "An Approximate Distribution of Estimates of Variance Components".*Biometrics Bulletin*, 2, 6, pp. 110–114.

Case of Unknown Variances, but $\sigma_1^2 \neq \sigma_2^2$

summary

Our **Test-Statistics (TS)** is $$TS = \frac{\bar{X} - \bar{Y}}{\sqrt{\frac{S_1^2}{n} + \frac{S_2^2}{m}}}$$

[Two-tailed test]

Given the following hypothesis statements and significance level **α** :

$$H_0 : \mu_1 = \mu_2 \qquad \textbf{VS} \qquad H_1 : \mu_1 \neq \mu_2$$

- we **reject** H_0 if $|TS| > t_{\alpha/2, v}$
- we **accept** H_0 if $|TS| \leq t_{\alpha/2, v}$

[One-tailed #1]

Given the following hypothesis statements and significance level **α** :

$$H_0 : \mu_1 = \mu_2 \qquad \textbf{VS} \qquad H_1 : \mu_1 > \mu_2$$

- we **reject** H_0 if $TS > t_{\alpha, v}$
- we **accept** H_0 if $TS \leq t_{\alpha, v}$

[One-tailed #2]

Given the following hypothesis statements and significance level **α** :

$$H_0 : \mu_1 = \mu_2 \qquad \textbf{VS} \qquad H_1 : \mu_1 < \mu_2$$

- we **reject** H_0 if $TS < -t_{\alpha, v}$
- we **accept** H_0 if $TS \geq -t_{\alpha, v}$

Case of Unknown Variances, but σ_1^2 ? σ_2^2

Once again, we consider a test of $H_0 : \mu_1 = \mu_2$ **VS** $H_1 : \mu_1 \neq \mu_2$

Now, all four parameters $\mu_1, \sigma_1^2, \mu_2, \sigma_2^2$ are **unknown** and we don't know the relation between σ_1^2 and σ_2^2.

We don't know whether $\sigma_1^2 = \sigma_2^2$ **or** $\sigma_1^2 \neq \sigma_2^2$

In this case, we use the following step:

1. First, we test the **equality of two variances** using **F-Test**

$$H_0 : \sigma_1^2 = \sigma_2^2 \quad \textbf{vs} \quad H_1 : \sigma_1^2 \neq \sigma_2^2$$

2. After the first step, we then know the relation between σ_1^2 and σ_2^2

    1. If $\sigma_1^2 = \sigma_2^2$ then

       Test $H_0 : \mu_1 = \mu_2$ **vs** $H_1 : \mu_1 \neq \mu_2$ using **Pooled Estimator S_p^2**

    2. If $\sigma_1^2 \neq \sigma_2^2$ then

       Test $H_0 : \mu_1 = \mu_2$ **vs** $H_1 : \mu_1 \neq \mu_2$ using **Satterthwaite's Method**

Case of Unknown Variances, but $\sigma_1^2 ? \sigma_2^2$

One-Sided test

We consider a test of $H_0 : \mu_1 = \mu_2 \ (\mu_1 \le \mu_2)$ **VS** $H_1 : \mu_1 > \mu_2$

In this case, we use the following step:

1. First, we test the **equality of two variances** using **F-Test**

$$H_0 : \sigma_1^2 = \sigma_2^2 \quad \text{vs} \quad H_1 : \sigma_1^2 \ne \sigma_2^2$$

2. After the first step, we then know the relation between $\boldsymbol{\sigma_1^2}$ and $\boldsymbol{\sigma_2^2}$
    1. If $\boldsymbol{\sigma_1^2 = \sigma_2^2}$ then

       Test $H_0 : \mu_1 = \mu_2$ vs $H_1 : \mu_1 > \mu_2$ using **Pooled Estimator S_p^2**
    2. If $\boldsymbol{\sigma_1^2 \ne \sigma_2^2}$ then

       Test $H_0 : \mu_1 = \mu_2$ vs $H_1 : \mu_1 > \mu_2$ using **Satterthwaite's Method**

The similar thing happens for the test of $H_0 : \mu_1 = \mu_2$ **VS** $H_1 : \mu_1 < \mu_2$

Suatu pabrik mobil yang terkenal ingin menentukan apakah sebaiknya membeli ban merek A atau merek B untuk mobil yang mereka produksi.

Untuk itu, percobaan dilakukan dengan cara mengambil beberapa sampel dari kedua merek ban tersebut. Ban-ban yang menjadi sampel tersebut dicoba sampai aus. Hasil percobaan dari sampel tersebut adalah sebagai berikut:

	Merek A	**Merek B**
Ukuran Sampel	12 ban	12 ban
Nilai Rataan Sampel	39,8	37,9
Nilai Variansi Sampel	5,9	5,1

*angka-angka pada baris rataan dan variansi sampel merepresentasikan **"tingkat kekuatan"** ban. **Anda tidak perlu tahu satuannya**

Apakah dapat disimpulkan bahwa pada **level of significance 10%, tidak ada bedanya antara kedua merek ban tersebut ?**

Populasi dari kedua merek berdistribusi normal dengan variansi tidak diketahui.

Solusi:

Variansi kedua populasi tidak diketahui dan relasinya-pun tidak diketahui. Jadi, kita perlu lakukan uji kesamaan variansi **sebelum** uji kesamaan rataan.

First Step - UJI KESAMAAN VARIANSI

Step 1: State the hypothesis and identify the claim

$$H_0 : \sigma_1 = \sigma_2 \qquad \textbf{VS} \qquad H_1 : \sigma_1 \neq \sigma_2$$

Step 2: Set the rejection criteria

We use **$\alpha = 0.1$** and the test is two-tailed test

Merk A -> index 1; Merk B -> index 2

Since $S_1^2 > S_2^2$ we **reject** H_0 if $TS = F > F_{\alpha/2,\, n-1,\, m-1} = F_{0.05,\, 11,\, 11} = 2.82$

Step 3: Compute the Test-Statistics (F test)

$$TS = F = \frac{S_1^2}{S_2^2} = \frac{5.9}{5.1} = 1.156$$

Step 4: Conclusion since $F = 1.156 < F_{\alpha/2, n-1, m-1} = 2.82$, we **accept** H_0 !

The evidence says that the variances are the same, $\sigma_1{}^2 = \sigma_2{}^2$

Second Step - UJI KESAMAAN RATAAN

Step 1: State the hypothesis and identify the claim

$$H_0 : \mu_1 = \mu_2 \ (claim) \qquad \textbf{VS} \qquad H_1 : \mu_1 \neq \mu_2$$

Step 2: Set the rejection criteria

Since **α = 0.1** and the test is two-tailed test
(**σ_1^2** and **σ_2^2 are unknown, $\sigma_1^2 = \sigma_2^2$** – based on the previous F test),

we **reject H_0** if $|TS| > t_{\alpha/2, n+m-2}$ $\quad t_{\alpha/2,n+m-2} = t_{0.05,22} = 1.717$

The critical value is $\quad t_{0.05,22} = 1.717 \quad and \quad -t_{0.05,22} = -1.717$

Step 3: Compute the Test-Statistics (t-test using **Pooled estimator**)

$$S_p^2 = \frac{(n-1)S_1^2 + (m-1)S_2^2}{n+m-2} = \frac{(12-1)5.9 + (12-1)5.1}{12+12-2} = 5.5$$

$$TS = \frac{\bar{X} - \bar{Y}}{\sqrt{S_p^2\left(\frac{1}{n} + \frac{1}{m}\right)}} = \frac{39.8 - 37.9}{\sqrt{5.5\left(\frac{1}{12} + \frac{1}{12}\right)}} = 1.987$$

Step 4: Make the decision and conclusion

since $|TS| = 1.987 > t_{\alpha/2,n+m-2} = 1.717$, we **reject** ***H_0*** !

Conclusion: There is **not enough evidence** to **support the claim** that the average size of the **farms is equal**.

There are two normal populations (Two Sample Case)

Testing the **Equality of Means** of Two Normal Population (DEPENDENT samples)

Dependent Samples ?

In the previous tests, we had independent samples.

- The subjects were randomly selected from two populations
- And, measurements were taken.

Dependent samples: two samples of data are dependent when each score in one sample is paired with a specific score in the other sample.

There are two cases:

- A group is measured twice (**BEFORE & AFTER** situation)
- Matched samples: each subject in one sample is matched on some relevant variable with a subject in the other sample

Dependent Samples ?

Misal, seorang peneliti ingin mengetahui apakah sebuah **pelatihan khusus** mempunyai dampak terhadap nilai ujian siswa.

30 siswa dipilih untuk percobaan. **Sebelum** diberikan pelatihan, mereka disuruh untuk mengikuti ujian dan kemudian disimpan hasil ujiannya sebagai data.

Setelah itu, mereka kemudian diberikan pelatihan khusus. **Setelah** pelatihan tersebut, mereka kembali disuruh untuk mengikuti ujian dan kemudian disimpan kembali hasil ujiannya.

Siapa subjek-nya ? Ada berapa sampel ?

Apakah kedua sampel tersebut **dependent** ?

Dependent Samples ?

Misal, Anda ingin meneliti apakah **tingkat kerajinan** mahasiswa FASILKOM dan mahasiswa FISIP berbeda.

Anda kemudian memilih 50 orang mahasiswa FASILKOM secara acak dan kemudian mengukur “tingkat kerajinannya”. Ini adalah sampel yang pertama.

Setelah itu, Anda memilih 60 orang mahasiswa FISIP secara acak dan mengukur “tingkat kerajinannya”. Ini adalah sampel Anda yang kedua.

Apakah kedua sampel tersebut **dependent** ?

Dependent Samples ?

Seorang peneliti ingin mengetahui apakah **intensitas merokok** pasangan suami-istri tidak berbeda.

8 pasangan suami-istri dijadikan subjek penelitian. Hasil pengamatan untuk setiap pasangan disajikan pada tabel berikut.

Suami	Istri
16	15
20	18
10	13
15	10
8	12
19	16
14	11
15	12

Apakah dua sampel ini dependent ?

Satuan intensitas merokok : **Rokok/Hari**

In this case, we use the **difference scores across the two measurements**.

A test of $H_0 : \mu_1 = \mu_2$ **VS** $H_1 : \mu_1 \neq \mu_2$ can be considered as

$$H_0 : \delta = \mu_1 - \mu_2 = 0 \quad \textbf{VS} \quad H_1 : \delta = \mu_1 - \mu_2 \neq 0$$

Where **δ** is defined as **the mean of the difference** scores across the two measurements $\delta = \mu_1 - \mu_2$

The other symbols:

$d_i = X_i - Y_i$ is the difference for **each two-score** in the **sample**

$\bar{d} = \bar{X} - \bar{Y} = \frac{\sum_i d_i}{n}$ is **the mean of the difference** scores across the two measurements **for the sample**

$$S_d = \sqrt{\frac{\sum_{i=1}^{n}\left(d_i - \bar{d}\right)^2}{n-1}}$$

is **the std. deviation of the difference** scores across the two measurements **for the sample**

This is just the **one-sample case when the variance of population is unknown** ! **(t-test)**

$$\frac{\bar{d}-\delta}{S_d/\sqrt{n}} \sim t_{n-1} \quad \underset{\substack{\text{when } H_0 \text{ is true:} \\ \delta=\mu_1-\mu_2=0}}{\Rightarrow} \quad \frac{\bar{d}}{S_d/\sqrt{n}} \sim t_{n-1}$$

Test-statistic: $TS = \dfrac{\bar{d}}{S_d/\sqrt{n}}$

We call this as **Paired t-test** !

- Given level of significance **α**, we **reject** H_0 if $|TS| > t_{\alpha/2,\,n-1}$
- we **accept** H_0 otherwise

Paired t-test

Our **Test-Statistics (TS)** is $TS = \dfrac{\bar{d}}{S_d / \sqrt{n}}$ $\qquad \bar{d} = \bar{X} - \bar{Y}$

[Two-tailed test]

Given the following hypothesis statements and significance level **α** :

$$H_0 : \delta = \mu_1 - \mu_2 = 0 \quad \textbf{VS} \quad H_1 : \delta = \mu_1 - \mu_2 \neq 0$$

- we **reject** $\boldsymbol{H_0}$ if $|TS| > t_{\alpha/2,\, n-1}$
- we **accept** $\boldsymbol{H_0}$ if $|TS| \leq t_{\alpha/2,\, n-1}$

[One-tailed #1]

Given the following hypothesis statements and significance level **α** :

$$H_0 : \delta = \mu_1 - \mu_2 = 0 \quad \textbf{VS} \quad H_1 : \delta = \mu_1 - \mu_2 > 0$$

- we **reject** $\boldsymbol{H_0}$ if $TS > t_{\alpha,\, n-1}$
- we **accept** $\boldsymbol{H_0}$ if $TS \leq t_{\alpha,\, n-1}$

[One-tailed #2]

Given the following hypothesis statements and significance level **α** :

$$H_0 : \delta = \mu_1 - \mu_2 = 0 \quad \textbf{VS} \quad H_1 : \delta = \mu_1 - \mu_2 < 0$$

- we **reject** $\boldsymbol{H_0}$ if $TS < -t_{\alpha,\, n-1}$
- we **accept** $\boldsymbol{H_0}$ if $TS \geq -t_{\alpha,\, n-1}$

Sebuah perusahaan chip sedang menerapkan **program pelatihan keamanan** kepada seluruh karyawannya.

Rata-rata **jumlah kehilangan jam kerja** untuk satu bulan akibat kecelakaan pada 10 pabrik milik perusahaan **sebelum** dan **sesudah** diberikannya pelatihan adalah sebagai berikut.

#Pabrik	Sebelum	Sesudah
1	30.5	23
2	18.5	21
3	24.5	22
4	32	28.5
5	16	14.5
6	15	15.5
7	23.5	24.5
8	25.5	21
9	28	23.5
10	18	16.5

Pada **level of significance 5%**, apakah program pelatihan tersebut memang efektif untuk mengurangi jumlah kehilangan jam kerja ?

Solusi:

#Pabrik	Sebelum (X)	Sesudah (Y)	$d_i = X_i - Y_i$	$d_i - \bar{d}$	$(d_i - \bar{d})^2$
1	30.5	23	7.5	5.35	28.6225
2	18.5	21	-2.5	-4.65	21.6225
3	24.5	22	2.5	0.35	0.1225
4	32	28.5	3.5	1.35	1.8225
5	16	14.5	1.5	-0.65	0.4225
6	15	15.5	-0.5	-2.65	7.0225
7	23.5	24.5	-1	-3.15	9.9225
8	25.5	21	4.5	2.35	5.5225
9	28	23.5	4.5	2.35	5.5225
10	18	16.5	1.5	-0.65	0.4225

$$\bar{d} = \sum_i d_i \Big/ n = \frac{7.5 + (-2.5) + \ldots + 1.5}{10} = 2.15$$

$$\sum_{i=1}^{n} \left(d_i - \bar{d}\right)^2 = 81.025$$

$$S_d = \sqrt{\frac{\sum_{i=1}^{n} \left(d_i - \bar{d}\right)^2}{n-1}} = \sqrt{\frac{81.025}{9}} \approx 3$$

1 December 2014

Now, we test the hypothesis !

Step 1: State the hypothesis and identify the claim

$$H_0 : \delta = \mu_1 - \mu_2 = 0 \qquad \textbf{VS} \qquad H_1 : \delta = \mu_1 - \mu_2 > 0$$

Step 2: Set the rejection criteria

Since **α = 0.05** and the test is one-tailed **paired-t** test (**σ is unknown**),

we **reject** $\boldsymbol{H_0}$ if $TS > t_{\alpha,n-1}$ $\quad t_{\alpha,n-1} = t_{0.05,9} = 1.833$

The critical value is $t_{0.05,9} = 1.833$

Step 3: Compute the Test-Statistics (t-test)

$$TS = \frac{\bar{d}}{S_d / \sqrt{n}} = \frac{2.15}{3/\sqrt{10}} = 2.267$$

Step 4: Make the decision and conclusion

since $TS = 2.267 > t_{\alpha,n-1} = 1.833$, we **reject** $\boldsymbol{H_0}$!

Conclusion: There is enough evidence to support the claim that the training program is effective !

1 December 2014