First-Order Logic

The formal system of first-order logic is constructed around the idea of well-formed formulas (WFFs). The simplest WFFs are,

- **Variables** are symbols used to represent unspecified objects or individuals. Variables are typically denoted by single lowercase letters (e.g., x, y, z).[1]
- **Constants** represent specific, fixed objects or individuals in the domain of discourse. The most well known constants are the Boolean constants *true* and *false*.
- **Predicates** are functions that represent properties or relations that can be true or false for individuals. Predicates are typically denoted by uppercase letters (e.g., P, Q, R) followed by a number of arguments. For example, P(x), Q(x, y), R(y, z, w) are predicates.[2]

[1] This only applies to pencil-and-paper logic. Prolog has its own rules in what constitutes a variable.

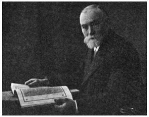
[2] This notational convention again only applies to pencil-and-paper logic.

First-Order Logic

Connectives are used to form more complex formulas from simpler ones. The most common connectives in first-order logic are,

- **Negation ($\neg$):** Given a formula A, $\neg A$ represents the negation or the opposite of A.
- **Conjunction ($\wedge$):** Given two formulas A and B, $A \wedge B$ represents the logical AND of A and B.
- **Disjunction ($\vee$):** Given two formulas A and B, $A \vee B$ represents the logical OR of A and B.
- **Implication ($\rightarrow$):** Given two formulas A and B, $A \rightarrow B$ represents the implication: if A then B. In the English grammar this can also be expressed as: B if A.
- **Equivalence ($\leftrightarrow$):** Given two formulas A and B, $A \leftrightarrow B$ represents the equivalence or "if and only if" statement between A and B, often written as A iff B.

Quantifiers are used to express statements about all or some individuals in the domain of discourse. Predicates and quantifiers as additions to modern logic are the contributions of Gottlob Frege, a logician and philosopher from the 19th century.



The two main quantifiers in first-order logic are:

- **Universal Quantifier ($\forall$):** Given a variable x and a formula $A(x)$, $\forall x[A(x)]$ represents "for all x, $A(x)$ is *true*."
- **Existential Quantifier ($\exists$):** Given a variable x and a formula $A(x)$, $\exists x[A(x)]$ represents "there exists an x such that $A(x)$ is *true*."

WFFs

Here are some example WFFs in first-order logic:

- $human(socrates)$ – here *human* is a predicate and *socrates* is a constant.
- $\forall x[human(x) \rightarrow mortal(x)]$ – x is a variable and *human* and *mortal* are predicates.
- $\forall x \exists y[human(x) \wedge parent(y, x)]$ – here $parent(y, x)$ means y is the parent of x.
- $\forall x[man(x) \wedge parent(x)]$

Deduction

We have seen that we can construct first-order logic sentences and we have an intuitive idea on how to interpret them.[3]

The question now is, how do we reason using logic?

- Deduction!

Deduction or inference is process of deriving new sentences from existing ones. The kind of deduction one is allowed to make is defined by the notation and a set of inference rules.

[3] Logician are concerned about formal ways of assigning meaning to logic formulas – model theory.

Deduction

Classical first-order logic has a quite a few deduction rules,[4]

Table 1. Inference Rules

Modus Ponens (MP) $p \Rightarrow q$ p ______ $\vdash q$	Modus Tollens (MT) $p \Rightarrow q$ $\sim q$ ______ $\vdash \sim p$
Hypothetical Syllogism (HS) $p \Rightarrow q$ $q \Rightarrow r$ ______ $\vdash p \Rightarrow r$	Disjunctive Syllogism (DS) $p \vee q$ $\sim p$ ______ $\vdash q$
Constructive Dilemma (CD) $(p \Rightarrow q) \wedge (r \Rightarrow s)$ $p \vee r$ ______ $\vdash q \vee s$	Destructive Dilemma (DD) $(p \Rightarrow q) \wedge (r \Rightarrow s)$ $\sim q \vee \sim s$ ______ $\vdash \sim p \vee \sim r$
Simplification (Simp.) $p \wedge q$ ______ $\vdash p$	Conjunction (Conj.) p q ______ $\vdash p \wedge q$
Addition (Add.) p ______ $\vdash p \vee q$	

Note: The table uses $\Rightarrow$ as implication and $\sim$ as the logical not.

[4]Source: "Symbolic Logic", I. Copi, Macmillian, 1979.

Deduction

From our perspective the **modus ponens** is the most important rule because

1. It is the main inference rule on which our natural deduction system for our semantics is based on.
2. It is the only inference rule implemented in Prolog.

Example:

$$\begin{array}{l} \forall x[human(x) \rightarrow mortal(x)] \\ human(socrates) \\ \hline \therefore\ mortal(socrates) \end{array}$$

A Word or Two about Implication

The truth table for the implication operator '$\rightarrow$' can be given as (assuming $1 \equiv$ *true* and $0 \equiv$ *false*)

	A	B	$A \rightarrow B$
(1)	1	0	0
(2)	1	1	1
(3)	0	0	1
(4)	0	1	1

Entries (1) and (2) are intuitive: When the antecedent A is true but the consequent B is false then the implication itself is false. If both the antecedent and the consequent are true then the implication is true.

However, entries (3) and (4) are somewhat counter intuitive. They state that if the antecedent A is false then the implication is true regardless of the value of the consequent. In other words, we can conclude "anything" from an antecedent that is false. In mathematical jargon we say that (3) and (4) **hold trivially**.

A Word or Two about Implication – An Example

$$\begin{array}{l} \text{If Bob is a bachelor, then he is single.} \\ \text{Bob is a bachelor.} \\ \hline \therefore \text{Bob is single.} \end{array}$$

Now consider an antecedent that is not true,

$$\begin{array}{l} \text{If Bob is a bachelor, then he is single.} \\ \text{Bob is not a bachelor.} \\ \hline \therefore \text{Bob is not single (by rule (3)).} \end{array}$$

Since the antecedent is not true rule (3) allows us to conclude the opposite of what the implication dictates. However, the following is also valid reasoning,

$$\begin{array}{l} \text{If Bob is a bachelor, then he is single.} \\ \text{Bob is not a bachelor.} \\ \hline \therefore \text{Bob is single (by rule (4)).} \end{array}$$

Not being a bachelor does not necessarily imply that Bob is not single. For example, Bob could be a widower or a divorcee.

A Word or Two about Implication

Given the truth table for implication,

	A	B	$A \rightarrow B$
(1)	1	0	0
(2)	1	1	1
(3)	0	0	1
(4)	0	1	1

this means that in order to show that an implication holds we only have to show that rule (2) holds. Rule (1) states that the implication is false and rules (3) and (4) are trivially true and therefore not interesting.

Note: We will see that in our approach to semantics (natural semantics) that is precisely what we aim to do!

Closely Related: Equivalence

We write $A \leftrightarrow B$ if A and B are equivalent.
Given the truth table for the equivalence operator is given as,

	A	B	$A \leftrightarrow B$
(1)	1	0	0
(2)	1	1	1
(3)	0	0	1
(4)	0	1	0

That is, the operator only produces a true value if A and B have the same truth assignment.
Another, and very useful, way to look at the equivalence operator is as follows:

$$A \leftrightarrow B \equiv A \rightarrow B \wedge B \rightarrow A$$

Note: The above rule gives us a powerful proof technique for equivalence!

Closely Related: Equivalence

Exercise: Prove that the following statement holds,

$$A \leftrightarrow B \equiv A \rightarrow B \wedge B \rightarrow A$$

Proof: Showing that the statement holds is easily accomplished by showing that the truth table for,

$$A \rightarrow B \wedge B \rightarrow A$$

is the same as for the equivalence operator on the previous slide,

	A	B	$A \rightarrow B$	$B \rightarrow A$	$A \rightarrow B \wedge B \rightarrow A$
(1)	1	0	0	1	0
(2)	1	1	1	1	1
(3)	0	0	1	1	1
(4)	0	1	1	0	0

Clearly, the last column shows that the truth values for our expression are the same as for the equivalence operator.$\square$

If and only If

Proposition: Show that A iff B[5] is logically equivalent to $A \leftrightarrow B$ or, more explicitly, locally equivalent to

$$B \rightarrow A \wedge A \rightarrow B$$

Proof: The proof proceeds in two parts.

1. The first part is A if B, which is just a different way of saying $B \rightarrow A$. This gives us the first half of our equivalence.
2. In the second part we need to show that A only if B is equivalent to $A \rightarrow B$, the second half of our equivalence. Now, there exists no such thing as a 'only if' logical operator but we can reformulate A only if B as the logical statement $\neg B \rightarrow \neg A$. However, this is the logically equivalent contrapositive statement to $A \rightarrow B$. This gives us the second half of our equivalence.$\square$

[5] The term 'iff' a shortening of the phrase 'if and only if'.

Sets

Sets[6] are unordered collections of objects and are usually denoted by capital letters. For example, let a, b, c denote some objects then the set A of these objects is written as,

$$A = \{a, b, c\}.$$

There are a number of standard sets which come in handy,

- $\emptyset$ denotes the empty set, i. e. $\emptyset = \{\}$,
- $\mathbb{N}$ denotes the set of all natural numbers including 0, e. g. $\mathbb{N} = \{0, 1, 2, 3, \cdots\}$,
- $\mathbb{I}$ denotes the set of all integers, $\mathbb{I} = \{\cdots, -2, -1, 0, 1, 2, \cdots\}$,
- $\mathbb{R}$ denotes the set of all reals,
- $\mathbb{B}$ denotes the set of boolean values, $\mathbb{B} = \{\mathit{true}, \mathit{false}\}$.

[6] Read Sections 2.1 and 2.2 in the book by David Schmidt.

The most fundamental property in set theory is the notion of **belonging**,

$$a \in A \text{ iff } a \text{ is an element of the set } A.$$

The notion of belonging allows us to define **subsets**,

$$Z \subseteq A \text{ iff } \forall e \in Z.\, e \in A.$$

We define set **equivalence** as,

$$A = B \text{ iff } A \subseteq B \wedge B \subseteq A$$

We can construct new sets from given sets using **union**,

$$A \cup B = \{e \mid e \in A \vee e \in B\},$$

and **intersection**,

$$A \cap B = \{e \mid e \in A \wedge e \in B\}.$$

There is another important set construction called the **cross product**,

$$A \times B = \{(a, b) \mid a \in A \wedge b \in B\},$$

$A \times B$ is the set of all ordered pairs where the first component of the pair is drawn from the set A and the second component of the pair is drawn from B. (

Exercise: Let $A = \{a, b\}$ and $B = \{c, d\}$, construct the set $A \times B$.

A construction using subsets is the **powerset** of some set X,

$$\mathcal{P}(X),$$

The **powerset** of set X is set of all subsets of X. For example, let $X = \{a, b\}$, then

$$\mathcal{P}(X) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}.$$

Note: $\emptyset \subset X$

Exercise: What would $\mathcal{P}(X \times X)$ look like?

The fact that $\emptyset \subset X$ for any set X is interesting in its own right. Let's see if we can prove it.

Proposition: The subset relation $\emptyset \subset X$ holds for all sets X.

Proof: Proof by contradiction. Assume X is any set. Assume that $\emptyset$ is not a subset of X. Then the definition of subsets,

$$A \subseteq B \leftrightarrow \forall e \in A . e \in B,$$

implies that there exist at least one element in $\emptyset$ that is not also in X. But that is not possible because $\emptyset$ has no elements – a contradiction. Therefore, our assumption the that $\emptyset$ is not a subset of X must be wrong and we can conclude that $\emptyset \subset X$.$\square$

Relations

A (binary) relation is a set of ordered pairs. If R is a relation that relates the elements of set A to the elements B, then

$$R \subseteq A \times B.$$

This means if $a \in A$ is related to $b \in B$ via the relation R, then $(a, b) \in R$. We often write

$$a \, R \, b.$$

Consider the relational operator $\leq$ applied to the set $\mathbb{N} \times \mathbb{N}$. This induces a relation, call it $\leq \subseteq \mathbb{N} \times \mathbb{N}$, with $(a, b) \in \leq$ (or $a \leq b$ in our relational notation) if $a \in \mathbb{N}$ is less or equal to $b \in \mathbb{N}$.

Relations

The first and second components of each pair in some relation R are drawn from different sets called the **projections** of R onto the first and second **coordinate**, respectively. We introduce the operators **domain** and **range** to accomplish these projections. Let $R \subseteq A \times B$, then,

$$\mathrm{dom}(R) = A,$$

and

$$\mathrm{ran}(R) = B.$$

In this case we talk about a relation **from** A **to** B. The range is often called the co-domain. If $R \subseteq X \times X$, then

$$\mathrm{dom}(R) = \mathrm{ran}(R) = X.$$

Here we talk about a relation **in** X.

Equality Relation

Let $R \subseteq X \times X$ such that $(a, b) \in R$ iff $a = b$. That is, R is the **equality relation** in X. (What do the elements of the equality relation look like for $\mathbb{N} \times \mathbb{N}$?)

A relation $R \subseteq X \times X$ is an **equivalence relation** if the following conditions hold,

- R is **reflexive**[7] – $x \, R \, x$,
- R is **symmetric** – $x \, R \, y \Rightarrow y \, R \, x$,
- R is **transitive** – $x \, R \, y \wedge y \, R \, z \Rightarrow x \, R \, z$,

where $x, y, z \in X$.

The **smallest** equivalence relation in some set X is the equality relation defined above. The **largest** equivalence relation is some set is the cross product $X \times X$. (Consider the smallest/largest equiv. relation in $\mathbb{I}$)

[7] Recall that $x \, R \, x \equiv (x, x) \in R$

Functions

A **function** f from X to Y is a relation $f \subseteq X \times Y$ such that

$$\forall x \in X, \exists y, z \in Y. (x, y) \in f \wedge (x, z) \in f \Rightarrow y = z.$$

In other words, each $x \in X$ has a unique value $y \in Y$ with $(x, y) \in f$ or functions are constrained relations.

We let $X \to Y$ denote the **set of all functions** from X to Y (i. e. $X \to Y \subset \mathcal{P}(X \times Y)$, why is the subset strict? Hint: it is not a relation), then the customary notation for specifying functions can be defined as follows,

$$f : X \to Y \text{ iff } f \in X \to Y.$$

Functions

For **function application** it is customary to write

$$f(x) = y$$

for $(x, y) \in f$. In this case we say that the function is **defined** at point x. Otherwise we say that the function is **undefined** at point x and we write $f(x) = \perp$.

Note that $f(\perp) = \perp$ and we say the f is **strict**.

We say that $f : X \to Y$ is a **total** function if f is defined for all $x \in X$. Otherwise we say that f is a **partial** function.

Functions

We can now make the notion of a predicate formal – a predicate is a function whose range (co-domain) is restricted to the boolean values:

$$P : X \to \mathbb{B}$$

where P is a predicate that returns true or false for the objects in set X.

Example: Let U be the set of all possible objects – a universe if you like, and let,

$$human : U \to \mathbb{B}$$

be the predicate that returns true if the object is a human and will return false otherwise, then

$$human(\text{socrates}) = true$$
$$human(\text{car}) = false$$

Exercises

1. In your own words explain what the function $m : X \times Y \to Z$ does.
2. How would you describe the function $c : X \to (Y \to Z)$?
3. In your own words explain what the relation $R \subseteq (X \times Y) \times (Z \times W)$ does.