Grammars

Definition: [Grammar] A *grammar* is a triple $(\Gamma, \rightarrow, \gamma)$ such that,

- $\Gamma = T \cup N$ with $T \cap N = \emptyset$, where T is a set of symbols called the *terminals* and N is a set of symbols called the *non-terminals*,[1]
- $\rightarrow$ is a set of rules of the form $u \rightarrow v$ with $u, v \in \Gamma^*$,
- γ is called the *start symbol* and $\gamma \in N$.

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[1] The fact that T and N are non-overlapping means that there will never be confusion between terminals and non-terminals.

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Example: Grammar for arithmetic expressions. We define the grammar $(\Gamma, \rightarrow, \gamma)$ as follows:

- $\Gamma = T \cup N$ with $T = \{a, b, c, +, *, (,)\}$ and $N = \{E\}$,
- $\rightarrow$ is is defined as,

$$\begin{array}{lcl} E & \rightarrow & E + E \\ E & \rightarrow & E * E \\ E & \rightarrow & (E) \\ E & \rightarrow & a \\ E & \rightarrow & b \\ E & \rightarrow & c \end{array}$$

- $\gamma = E$ (clearly this satisfies $\gamma \in N$).

With grammars, derivations always start with the start symbol. Consider,

$$E \Rightarrow E{*}E \Rightarrow (E){*}E \Rightarrow (E{+}E){*}E \Rightarrow (a{+}E){*}E \Rightarrow (a{+}b){*}E \Rightarrow (a{+}b){*}c.$$

Here, $(a + b) * c$ is a normal form often also called a *terminal* or *derived string*.

Exercise: Identify the rule that was applied at each rewrite step in the above derivation.
Exercise: Derive the string $((a))$.
Exercise: Derive the string $a + b * c$. Is the derivation unique? Why? Why not?

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We are now in the position to define exactly what we mean by a *language*.

Definition:[Language] Let $G = (\Gamma, \rightarrow, \gamma)$ be a grammar, then we define the *language of grammar* G as the set of all terminal strings that can be derived from the start symbol s by rewriting using the rules in $\rightarrow$.
Formally,

$$L(G) = \{q \mid \gamma \Rightarrow^* q \wedge q \in T^*\}.$$

Example: Let $J = (\Gamma, \rightarrow, \gamma)$ be the grammar of Java, then $L(J)$ is the set of all possible Java programs. The Java language is defined as the set of all possible Java programs.

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Observations:

- With the concept of a language we can now ask interesting questions. For example, given a grammar G and some sentence $p \in T^*$, does p belong to $L(G)$?
- If we let J be the grammar of Java, then asking whether some string $p \in T^*$ is in $L(J)$ is equivalent to asking whether p is a *syntactically correct program*.
- We can prove language membership by showing that the start symbol is equivalent to the sentence in question,

$$
\begin{tikzcd}
s \arrow[dr, Rightarrow, "*"] & \equiv & p \arrow[dl, Rightarrow, "*"'] \\
& p &
\end{tikzcd}
$$

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Observations:

- By restricting the shape of the rewrite rules in a grammar we obtain different language *classes*.
- The most famous set of language classes is the *Chomsky Hierarchy*.

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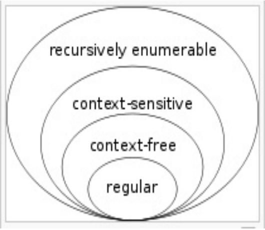
Table: The Chomsky Hierarchy

Rules	Grammar	Language	Machine
$\alpha \rightarrow \beta$	Type-0	Recursively Enumerable	Turing machine
$\alpha A \beta \rightarrow \alpha \gamma \beta$	Type-1	Context-sensitive	Linear-bounded Turing machine
$A \rightarrow \gamma$	Type-2	Context-free	Pushdown automaton
$A \rightarrow a$ and $A \rightarrow aB$	Type-3	Regular	Finite state automaton

where $\alpha, \beta, \gamma \in \Gamma^*$, $A, B \in N$, $a \in T$. In Type-1 γ is not allowed to be the empty string.



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Observation: The most convenient language class for programming language specification are the context-free languages – they are decidable – pushdown automata can be efficiently implemented in order to prove language membership.

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Example: A simple imperative language. We define grammar $G = (\Gamma, \rightarrow, \gamma)$ as follows:

- $\Gamma = T \cup N$ where

$$T = \{\mathbf{0}, \ldots, \mathbf{9}, \mathbf{a}, \ldots, \mathbf{z}, \mathbf{true}, \mathbf{false}, \mathbf{skip}, \mathbf{if}, \mathbf{then}, \mathbf{else}, \mathbf{while}, \mathbf{do}, \mathbf{end} +, -, *, =, \leq, !, \&\&, ||, :=, ;, (,)\}$$

  and

$$N = \{A, B, C, D, L, V\}.$$

- The rule set $\rightarrow$ is defined by,

$$\begin{array}{lcl}
\mathsf{A} & \rightarrow & D \mid V \mid \mathsf{A} + \mathsf{A} \mid \mathsf{A} - \mathsf{A} \mid \mathsf{A} * \mathsf{A} \mid (\mathsf{A}) \\
\mathsf{B} & \rightarrow & \mathbf{true} \mid \mathbf{false} \mid \mathsf{A} = \mathsf{A} \mid \mathsf{A} \leq \mathsf{A} \mid !\mathsf{B} \mid \mathsf{B}\&\&\mathsf{B} \mid \mathsf{B}||\mathsf{B} \mid (\mathsf{B}) \\
\mathsf{C} & \rightarrow & \mathbf{skip} \mid \mathsf{V} := \mathsf{A} \mid \mathsf{C} \,;\, \mathsf{C} \mid \mathbf{if}\ \mathsf{B}\ \mathbf{then}\ \mathsf{C}\ \mathbf{else}\ \mathsf{C}\ \mathbf{end} \mid \mathbf{while}\ \mathsf{B}\ \mathbf{do}\ \mathsf{C}\ \mathbf{end} \\
\mathsf{D} & \rightarrow & \mathsf{L} \mid -\mathsf{L} \\
\mathsf{L} & \rightarrow & \mathbf{0}\,\mathsf{L} \mid \ldots \mid \mathbf{9}\,\mathsf{L} \mid \mathbf{0} \mid \ldots \mid \mathbf{9} \\
\mathsf{V} & \rightarrow & \mathbf{a}\,\mathsf{V} \mid \ldots \mid \mathbf{z}\,\mathsf{V} \mid \mathbf{a} \mid \ldots\, \mathbf{z}
\end{array}$$

- $\gamma = \mathsf{C}$.

Observe that this is a context-free grammar!

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Here are some elements in $L(G)$,

$$x := 1; y := x$$
$$v := 1; \textbf{if } v \leq 0 \textbf{ then } v := (-1) * v \textbf{ else skip end}$$
$$n := 5; f := 1; \textbf{while } 2 \leq n \textbf{ do } f := n * f; n := n - 1 \textbf{ end}$$

Exercise: Prove that they belong to $L(G)$.

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HW#1 – see website