

# Logic as a Programming Language

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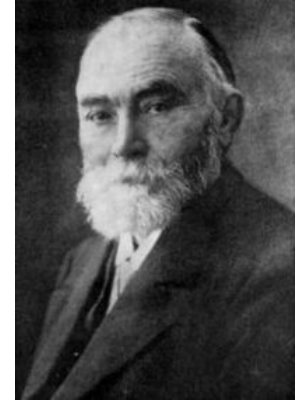
- Fundamental Question:
  - Can First Order Logic be considered a computational model?
- Another way of asking the same question:
  - Is First Order Logic *Turing Complete*?

**Turing Completeness:** A system of data-manipulation rules is said to be *Turing complete* if it can be used to simulate any Turing machine (*i.e.* compute any algorithm).

-- Wikipedia

# Quantification

- In 1879 Gottlob Frege introduced the predicate calculus ('Begriffsschrift')
- Today predicate calculus is more commonly known as First Order Logic (FOL).
- This logic is characterized by three structures:
  - predicates,
  - universal quantification, and
  - existential quantification.



Friedrich Ludwig Gottlob Frege  
Philosopher and Logician

# First-Order Logic

- Quantified Variables

- Universally quantified variables

$\forall X$  – for All objects  $X$

- Existentially quantified variables

$\exists Y$  – there Exists an object  $Y$

# First-Order Logic

- Predicates

- Predicates are functions that map their arguments into true/false
- The signature of a predicate  $p(X)$  is

$p: \text{Objects} \rightarrow \{ \text{true}, \text{false} \}$

- Example:  $\text{human}(X)$ 
  - $\text{human}: \text{Objects} \rightarrow \{ \text{true}, \text{false} \}$
  - $\text{human}(\text{tree}) = \text{false}$
  - $\text{human}(\text{paul}) = \text{true}$
- Example:  $\text{mother}(X,Y)$ 
  - $\text{mother}: \text{Objects} \times \text{Objects} \rightarrow \{ \text{true}, \text{false} \}$
  - $\text{mother}(\text{betty}, \text{paul}) = \text{true}$
  - $\text{Mother}(\text{giraffe}, \text{peter}) = \text{false}$

# First-Order Logic

- We can combine predicates and quantified variables to make statements on sets of objects
  - $\exists X[\text{mother}(X, \text{paul})]$ 
    - there exists an object  $X$  such that  $X$  is the mother of Paul
  - $\forall Y[\text{human}(Y)]$ 
    - for all objects  $Y$  such that  $Y$  is human

# First-Order Logic

- Logical Connectives: and, or, not
  - $\exists F \forall C [\text{parent}(F, C) \text{ and } \text{male}(F)]$ 
    - There exists an object F for all object C such that F is a parent of C and F is male.
  - $\forall X [\text{day}(X) \text{ and } (\text{wet}(X) \text{ or } \text{dry}(X))]$ 
    - For all objects X such that X is a day and X is either wet or dry.

# First-Order Logic

- If-then rules:  $A \rightarrow B$ 
  - $\forall X \forall Y [\text{parent}(X, Y) \text{ and } \text{female}(X) \rightarrow \text{mother}(X, Y)]$ 
    - For all objects  $X$  and for all objects  $Y$  such that if  $X$  is a parent of  $Y$  and  $X$  is female then  $X$  is a mother.
  - $\forall Q [\text{human}(Q) \rightarrow \text{mortal}(Q)]$ 
    - For all objects  $Q$  such that if  $Q$  is human then  $Q$  is mortal.



# Horn Clause Logic

In Horn clause logic the form of the WFF's is restricted:

$$P_1 \wedge P_2 \wedge \dots \wedge P_{n-1} \wedge P_n \rightarrow P_0$$

Single predicate in consequent

Conjunctions only!

The diagram shows the formula  $P_1 \wedge P_2 \wedge \dots \wedge P_{n-1} \wedge P_n \rightarrow P_0$ . A red oval encircles the antecedent part  $P_1 \wedge P_2 \wedge \dots \wedge P_{n-1} \wedge P_n$ , with an arrow pointing to it from the text "Conjunctions only!". Another red circle encircles the consequent  $P_0$ , with an arrow pointing to it from the text "Single predicate in consequent".

Where  $P_0, P_1, P_2, \dots, P_{n-1}, P_n$  are predicates.

Horn Clause Logic is a *Turing complete* subset of First Order Logic.

# Horn Clause Logic

- Proof outline that Horn Clause Logic (HCL) is Turing complete:
  - $\text{HCL} \Leftrightarrow \mu\text{-recursive functions} \Leftrightarrow \text{Turing Machine}$

# Computational Logic

- Horn Clauses
  - Restricted FOL
- Unification
  - *Pattern matching* where both pattern and subject term can have variables.
  - *E.g.*  $f(X, g(b)) \sim f(a, g(Y)) : \{ X = a, Y = b \}$
- (SLD) Resolution
  - Modus Ponens kind of reasoning.
  - Think: search top to bottom, left to right, *backtrack* if you can't find what you are looking for.
  - **Closed World Assumption:** assumes false if the entity Prolog is searching for cannot be found/deduced.

→ Prolog

# Proving things is computation!

Advantage: All this can be done mechanically (Alan Robinson, 1965)

“Deduction is Computation”

# Basic Prolog Programs

Facts - a fact constitutes a declaration of a truth; in Prolog it has to be a positive assertion.

Prolog Programs - a Prolog program is a collection of facts (...and rules, as we will see later).

Example: a simple program

```
man(phil).  
man(john).  
woman(betty).
```

} Facts, Prolog will treat these as true and enters them into its knowledgebase.

We execute Prolog programs by posing queries on its knowledgebase:

Prompt → ?- man(phil).  
          true - because Prolog can use its knowledgebase to prove true.  
          ?- woman(phil).  
          false - this fact is not in the knowledgebase.

# Prolog - Queries & Goals

A query is a way to extract information from a logic program.

Given a query, Prolog attempts to show that the query is a logical consequence of the program; of the collection of facts.

In other words, a query is a goal that Prolog is attempting to satisfy (prove true).

When queries contain variables they are existentially quantified, consider

`?- parent(X,liz).`

The interpretation of this query is: prove that there is at least one object X that can be considered a parent of liz, or formally, prove that

$\exists x[\text{parent}(x,\text{liz})]$

holds.

NOTE: Prolog will return all objects for which a query evaluates to true.

# A Prolog Program

```
% a simple prolog program
```

```
woman(pam).
```

```
woman(liz).
```

```
woman(ann).
```

```
woman(pat).
```

```
man(tom).
```

```
man(bob).
```

```
man(jim).
```

```
parent(pam,bob).
```

```
parent(tom,bob).
```

```
parent(tom,liz).
```

```
parent(bob,ann).
```

```
parent(bob,pat).
```

```
parent(pat,jim).
```

## Example Queries:

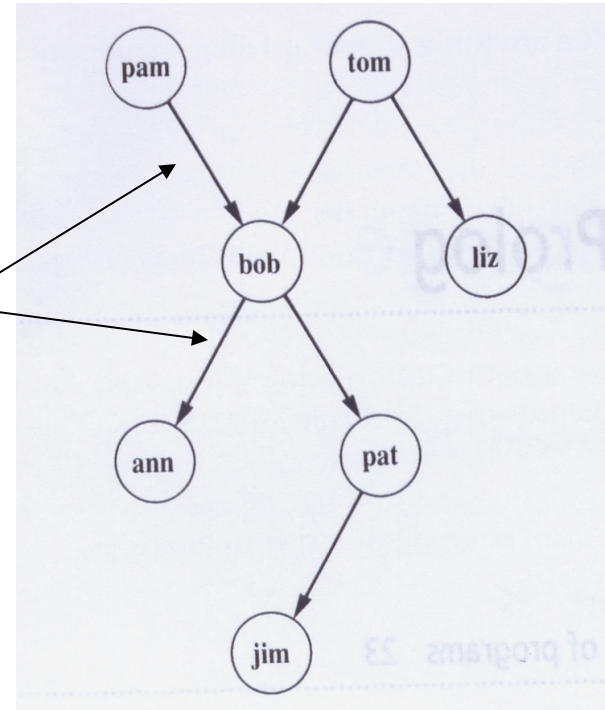
```
?- woman(pam).
```

```
?- woman(X).            $\exists X[\text{woman}(X)]?$ 
```

```
?- parent(tom,Z).
```

```
?- father(Y).
```

Parent  
Relation



A Family Tree

# Compound Queries

A compound query is the conjunction of individual simple queries.

Stated in terms of goals: a compound goal is the conjunction of individual subgoals each of which needs to be satisfied in order for the compound goal to be satisfied. Consider:

?- parent(X,Y) , parent(Y,ann).

or formally,

$\exists X,Y[\text{parent}(X,Y) \wedge \text{parent}(Y,\text{ann})]$

When Prolog tries to satisfy this compound goal, it will make sure that the two Y variables always have the same values.

Prolog uses unification and backtracking in order to find all the solutions which satisfy the compound goal.



# Prolog Rules

Prolog rules are Horn clauses, but they are written “backwards”, consider:

$$\forall X,Y[\text{woman}(X) \wedge \text{parent}(X,Y) \rightarrow \text{mother}(X,Y)]$$

is written in Prolog as

mother(X,Y) :- woman(X), parent(X,Y) .

Diagram illustrating the structure of the Prolog rule:

- The head of the rule is `mother(X,Y)`.
- The body of the rule is `woman(X), parent(X,Y)`.
- The colon dash `:-` is labeled “Implies (‘think of  $\leftarrow$ ’)”.
- The comma `,` is labeled “and”.
- The entire rule is labeled “head” and “body”.

Prolog rules are implicitly universally quantified!

You can think of a rule as introducing a new “fact” (the head), but the fact is defined in terms of a compound goal (the body). That is, predicates defined as rules are only true if the associated compound goal can be shown to be true.

# Prolog Rules

```
% a simple prolog program
woman(pam).
woman(liz).
woman(ann).
woman(pat).

man(tom).
man(bob).
man(jim).

parent(pam,bob).
parent(tom,bob).
parent(tom,liz).
parent(bob,ann).
parent(bob,pat).
parent(pat,jim).

mother(X,Y) :- woman(X),parent(X,Y).
```

Queries:

?- mother(pam,bob).

?- mother(Z,jim).

?- mother(P,Q).

# Prolog Rules

The same predicate name can be defined by multiple rules:

```
sibling(X,Y) :- sister(X,Y) .  
sibling(X,Y) :- brother(X,Y).
```

Use backtracking to find alternative definitions.

# Another Simple Prolog Program

Consider the program relating humans to mortality:

```
mortal(X) :- human(X).  
human(socrates).
```

We can now pose the query:

```
?- mortal(socrates).
```

True or false?

# Exercise

- Exercise #1

# Declarative vs. Procedural Meaning

When interpreting rules purely as Horn clause logic statement → declarative


When interpreting rules as “search” → procedural

Observation: We design programs with declarative meaning in our minds, but the execution is performed in a procedural fashion.

Consider:

```
mother(X,Y) :- woman(X),parent(X,Y).
```

# Lists & Pattern Matching

- The unification operator:  $=/2$   arity
  - The expression  $A=B$  is true if A and B are terms and unify (look identical)

?- a = a.

true

?- a = b.

false

?- a = X.

X = a

?- X = Y.

true

# Lists & Pattern Matching

- Lists – a convenient way to represent abstract concepts
  - Prolog has a special notation for lists.

[a]  
[a,b,c]  
[]

↖  
Empty  
List

[ bmw, vw, mercedes ]  
[ chicken, turkey, goose ]



# Lists & Pattern Matching

- Pattern Matching in Lists

$?- [a, b] = [a, X].$   
 $X = b$

$?- [a, b] = X.$   
 $X = [a, b]$

But:

$?- [a, b] = [X].$   
**no**

- The Head-Tail Operator:  $[H|T]$

$?- [a,b,c] = [X|Y];$   
 $X = a$   
 $Y = [b,c]$

$?- [a] = [Q|P];$   
 $Q = a$   
 $P = []$

# Lists - the First Predicate

The predicate `first/2`: accept a list in the first argument and return the first element of the list in second argument.

```
first(List,E) :- List = [H|T], E = H.
```

or

```
first([H|_],H).
```

# Lists - the Last Predicate

The predicate last/2: accept a list in the first argument and return the last element of the list in second argument.

Recursion: there are always two parts to a recursive definition; the base and the recursive step.

```
last([A],A).  
last([ _ |L],E) :- last(L,E).
```

# Member Predicate

Write a predicate `member/2` that takes a list as its first argument and an element as its second element. This predicate is to return true if the element appears in the list otherwise it returns false.

```
member([E|_],E).  
member(_|T,E) :- member(T,E).
```

# Exercise

- Write a predicate that compares two lists and returns true if the lists are the same and false otherwise.

# Prolog – Arithmetic

- Prolog is a programming language, therefore, arithmetic is implemented as expected.
- The only difference to other programming languages is that assignment is done via the predicate is rather than the equal sign, since the equal sign has been used for the unification operator.

## Examples:

?- X is 10 + 5;  
X = 15

?- X is 10 + 5 \* 6 / 3;  
X = 20

← Precedence and associativity  
of operators are respected.

# Prolog – Arithmetic

Example: write a predicate definition for length/2 that takes a list in its first argument and returns the length of the list in its second argument.

```
length([ ], 0).
```

```
length(L, N) :- L = [H|T], length(T,NT), N is NT + 1.
```

# Prolog – I/O

- `write(term)`
  - is true if term is a Prolog term, writes term to the terminal.
- `read(X)`
  - is true if the user types a term followed by a period, X becomes unified to the term.
- `nl`
  - is always true and writes a newline character on the terminal.

☞ Extra-logical predicates due to the side-effect of writing/reading to/from the terminal.



# Prolog – I/O

```
?- write(tom).  
tom
```

```
?- write([1,2]).  
[1, 2]
```

Prolog I/O Prompt

```
?- read(X).  
|: boo.  
X = boo
```

```
?- read(Q).  
|: [1,2,3].  
Q = [1, 2, 3]
```

# Exercises

- (1) Define a predicate `max/3` that takes two numbers as its first two arguments and unifies the last argument with the maximum of the two.
- (2) Define a predicate `maxlist/2` takes a list of numbers as its first argument and unifies the second argument with the maximum number in the list. The predicate should fail if the list is empty.
- (3) Define a predicate `ordered/1` that takes a list of numbers as its argument and succeeds if and only if the list is in non-decreasing order.

# A Translation Program

Write a program that takes simple English statements and translates them into German. The sentences are given as lists of words.

```
% the dictionary
lookup(logic,logik).
lookup(is,macht).
lookup(fun,spass).

% the translation procedure
translate([],[]).
translate([W|S],G):-
    lookup(W,GW),
    translate(S,GS),
    G=[GW|GS].
```

# Interaction Loops

Write a program that prompts a user for a list, then reads the list, reverses the elements of the list and then prints out the reversed list to the terminal. It then returns to prompting the user for a new list, etc.

```
interact:-  
    nl,  
    write('gimme a list> '),  
    read(X),  
    reverse(X,Y),  
    write('this is the reverse: '),  
    write(Y),  
    nl,  
    !,interact.
```



A Cut

# Exercise

- Exercise 2

- Prolog excels at natural language processing. This exercise lets you explore this a bit.