# FVM Practical Assignment 1

Solve 1D Poison problem with boundary conditions

**Luu Giang Nam**

**Email**: **luugiangnam96@gmail.com**

**University of Science, Ho Chi Minh City**

**31st March 2017**

# Contents

# List of Figures

# Problem 1
# Dirichlet boundary condition

## Introduction

We use FVM to solve this equation

$$u''_{xx}(x) = f(x), \quad x \in \Omega(x) \tag{Eq1.1}$$

with initial condition:

$$u(0) = a \tag{Condition 1}$$

$$u(1) = b \tag{Condition 2}$$

We have 2 situations for solving:

## 1.1. Regular grid and the control point is the mid point of each control volume

$$x_i = \frac{x_{i-\frac{1}{2}} + x_{i+\frac{1}{2}}}{2}$$

Let us choose $N+1$ points $\{x_{i+\frac{1}{2}}\}_{i=\overline{0,N}}$ in $[0,1]$ such that:

$$0 = x_{\frac{1}{2}} < x_{\frac{3}{2}} < ... < x_{N+\frac{1}{2}} = 1$$

We set $T_i = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$, $|T_i| = x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}} \forall i \in \overline{1,N}$, $h = \max_{i \in \overline{1,N}}\{|T_i|\}$ and

$$\begin{cases} x_0 = 0, x_{N+1} = 1, \\ x_i = \dfrac{x_{i-\frac{1}{2}} + x_{i+\frac{1}{2}}}{2} \end{cases}$$

From equation Eq1.1 we have:

$$\frac{1}{|T_i|}\int_{T_i} -u_{xx}dx = \frac{1}{|T_i|}\int_{T_i} f(x)dx \tag{Eq1.2}$$

Because of Green's formula we obtain:

$$\frac{1}{|T_i|}\int_{T_i} -u_{xx}dx = \frac{-u_{x_{i+\frac{1}{2}}} + u(x_{i-\frac{1}{2}})}{|T_i|} \tag{Eq1.3}$$

And let

$$f_i = \frac{1}{|T_i|}\int_{T_i} f(x)dx \tag{Eq1.4}$$

From Eq1.2, (Eq1.3) and (Eq1.4) we obtain:

$$\frac{-u(x_{i+\frac{1}{2}}) + u(x_{i-\frac{1}{2}})}{|T_i|} = f_i \tag{Eq1.5}$$

Use Taylor expression we have:

$$u(x_{i+1}) = u(x_{i+\frac{1}{2}}) + u_x(x_{i+\frac{1}{2}})(x_{i+1} - x_{i+\frac{1}{2}}) + \frac{u_{xx}(x_{i+\frac{1}{2}})}{2!}(x_{i+1} - x_{i+\frac{1}{2}})^2 + O(h^3) \tag{Eq1.6}$$

$$u(x_i) = u(x_{i+\frac{1}{2}}) + u_x(x_{i+\frac{1}{2}})(x_i - x_{i+\frac{1}{2}}) + \frac{u_{xx}(x_{i+\frac{1}{2}})}{2!}(x_i - x_{i+\frac{1}{2}})^2 + O(h^3) \tag{Eq1.7}$$

Then let (Eq1.6) to subtract (Eq1.7) we have:

$$u(x_{i+1}) - u(x_i) = u_x(x_{i+\frac{1}{2}})(x_{i+1} - x_i) + \left((x_{i+1} - x_{i+\frac{1}{2}})^2 - (x_i - x_{i+\frac{1}{2}})\frac{u_{xx}(x_{i+\frac{1}{2}})}{2} + O(h^3)\right) \tag{Eq1.8}$$

Because of regular grid and $x_i = \frac{x_{i-\frac{1}{2}} + x_{i+\frac{1}{2}}}{2}$ we imply that $x_{i+\frac{1}{2}}$ is midpoint of $[x_i, x_{i+1}]$.
Then, we have:

$$u(x_{i+1}) - u(x_i) = u_x(x_{i+\frac{1}{2}})(x_{i+1} - x_i) + O(h^3)$$

It means

$$u_x(x_{i+\frac{1}{2}}) = \frac{u(x_{i+1}) - u(x_i)}{x_{i+1} - x_i} + O(h^2)$$

We have the approximate of the term $u_x(x_{i+\frac{1}{2}})$ is:

$$u_x(x_{i+\frac{1}{2}}) = \frac{u(x_{i+1}) - u(x_i)}{|D_{i+\frac{1}{2}}|} \tag{Eq1.9}$$

where $D_{i+\frac{1}{2}} = x_{i+1} - x_i$.
From (Eq1.5) and (Eq1.9) we have:

$$\frac{-u_{i-1}}{|D_{i-\frac{1}{2}}||T_i|} + \left[\frac{1}{|D_{i+\frac{1}{2}}||T_i|} + \frac{1}{|D_{i-\frac{1}{2}}||T_i|}\right]u_i - \frac{u_{i+1}}{|D_{i+\frac{1}{2}}||T_i|} = f_i \tag{Eq1.10}$$

Now we set

$$\alpha_i = \frac{-1}{|D_{i-\frac{1}{2}}||T_i|}$$

$$\beta_i = \frac{1}{|D_{i-\frac{1}{2}}||T_i|} + \frac{1}{|D_{i+\frac{1}{2}}||T_i|}$$

$$\gamma_i = \frac{-1}{|D_{i+\frac{1}{2}}||T_i|}$$

Then (Eq1.10) is changed into:

$$\alpha_i u_{i-1} + \beta_i u_i + \gamma_i u_{i+1} = f_i, \forall i \in \overline{1, N} \tag{Eq1.11}$$

Combining with the boundary conditions, we get the scheme for the cell-center finite volume method:

$$\begin{cases} \alpha_i u_{i-1} + \beta_i u_i + \gamma_i u_{i+1} = f_i, \forall i = \overline{1, N}, \\ u_0 = 0, \\ u_{N+1} = 1 \end{cases} \tag{Eq1.12}$$

From (Eq1.12) have have a linear system for the scheme:

$$\begin{cases} i = 1 : \alpha_1 u_0 + \beta_1 u_1 + \gamma_1 u_2 & = f_1 \\ i = 2 : \quad \alpha_2 u_1 + \beta_2 u_2 + \gamma_2 u_3 & = f_1 \\ . \\ . \\ . \\ i = N-1 : \quad \alpha_{N-1} u_{N-2} + \beta_{N-1} u_{N-1} + \gamma_{N-1} u_N & = f_{N-1} \\ i = N : \quad \alpha_N u_{N-1} + \beta_N u_N + \gamma_N u_{N+1} & = f_N \end{cases}$$

Replace initial condition into linear system we have matrix form

$$AU = F$$

where $A \in \mathbb{R}^N \times \mathbb{R}^N$, $U, F \in \mathbb{R}^N$ satisfy:

$$\begin{bmatrix} \beta_1 & \gamma_1 & 0 & \dots & 0 & 0 & 0 \\ \alpha_2 & \beta_2 & \gamma_2 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \alpha_{N-1} & \beta_{N-1} & \gamma_{N-1} \\ 0 & 0 & 0 & \dots & 0 & \alpha_N & \beta_N \end{bmatrix} \tag{Matrix A.1}$$

$$U = \begin{bmatrix} u_1 \\ u_2 \\ \ddots u_{N-1} \\ u_N \end{bmatrix} \tag{cellU.1}$$

$$F = \begin{bmatrix} f_1 - a\alpha_1 \\ f_2 \\ \ddots f_{N-1} \\ f_N - \beta\gamma_N \end{bmatrix} \tag{cellF.1}$$

We can prove that (Matrix A.1) is invertible matrix so there is only one roof $(u_i)_{i=\overline{1,N}}$.

**Now we see graphics of exact and approximate solution with two problems**

**Problem 1:**

We set up with following exact solution $u$ and function $f$

$$\begin{cases} u(x) &= x^4 + 2x^3 - 10x^2 + 2 \\ f(x) &= -(12x^2 + 12x - 20) \end{cases}$$

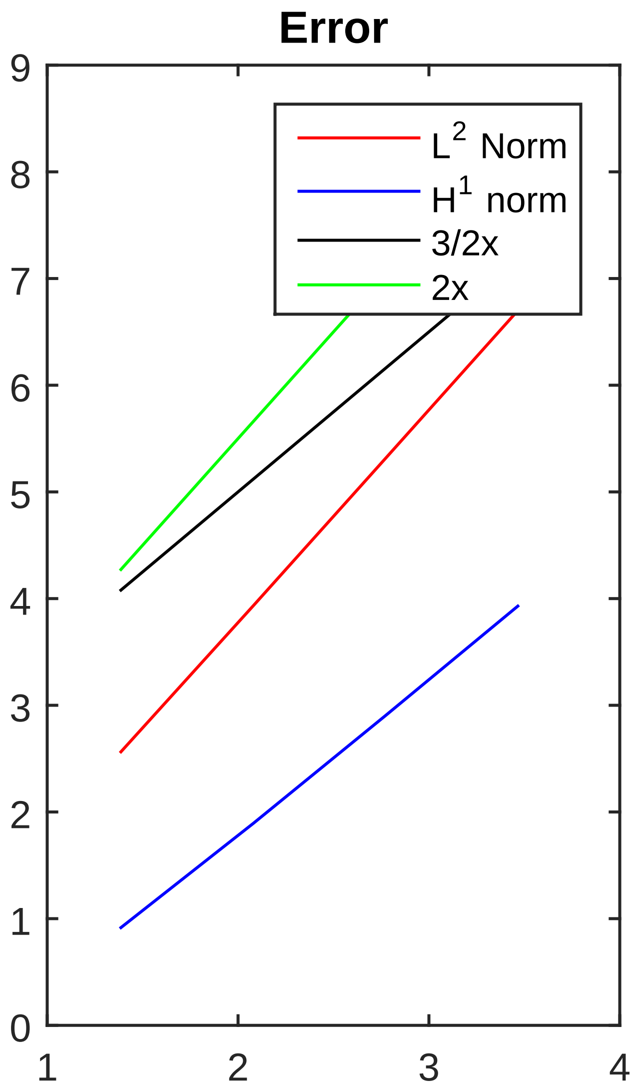



Figure 1: Error of Problem 1 - Question 1a

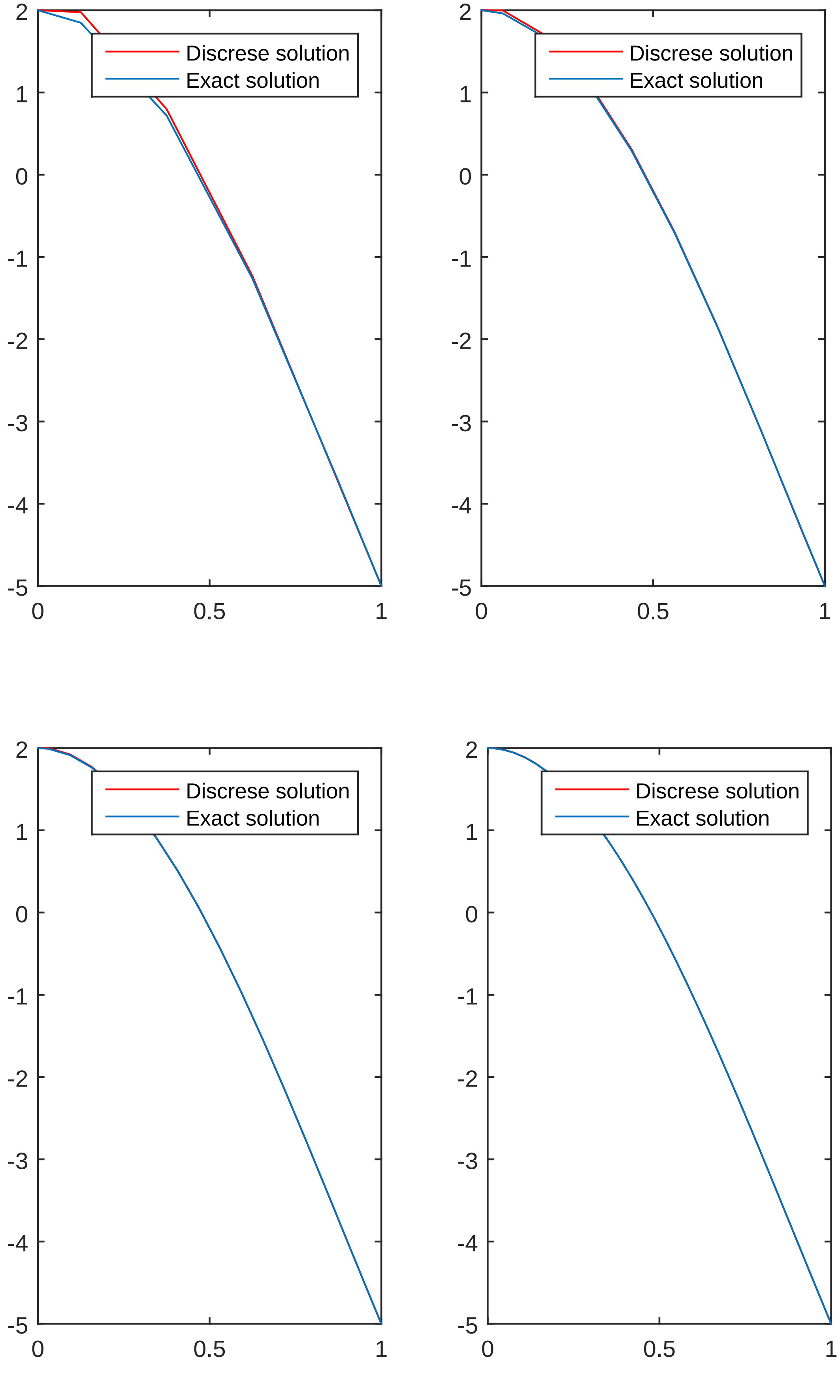





Figure 2: Approximate solution of Problem 1 - Question 1a.

**Problem 2:**

We set up with following exact solution $u$ and function $f$

$$\begin{cases} u(x) &= \sin(x^3) \\ f(x) &= 9x^4\sin(x^3) - 6x\cos(x^3) \end{cases}$$

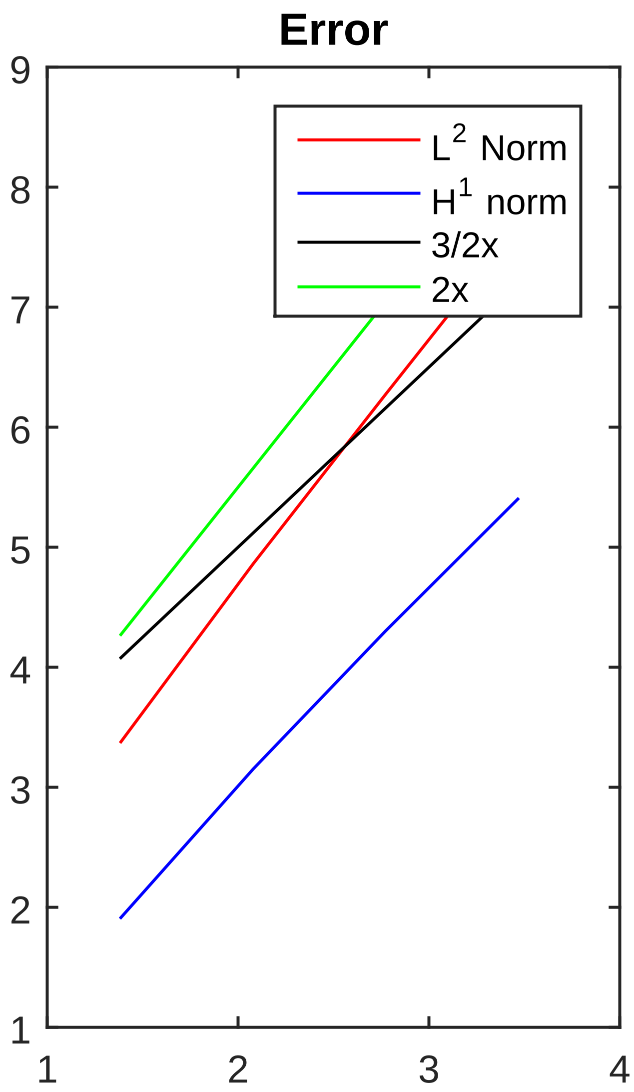
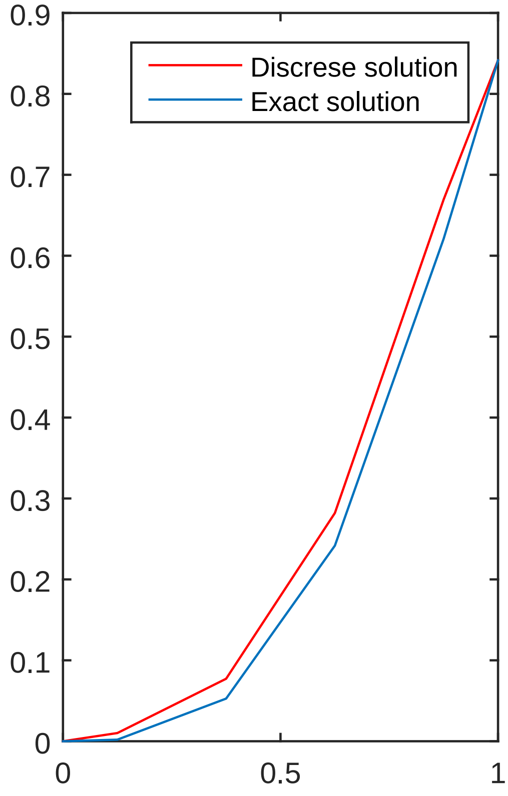
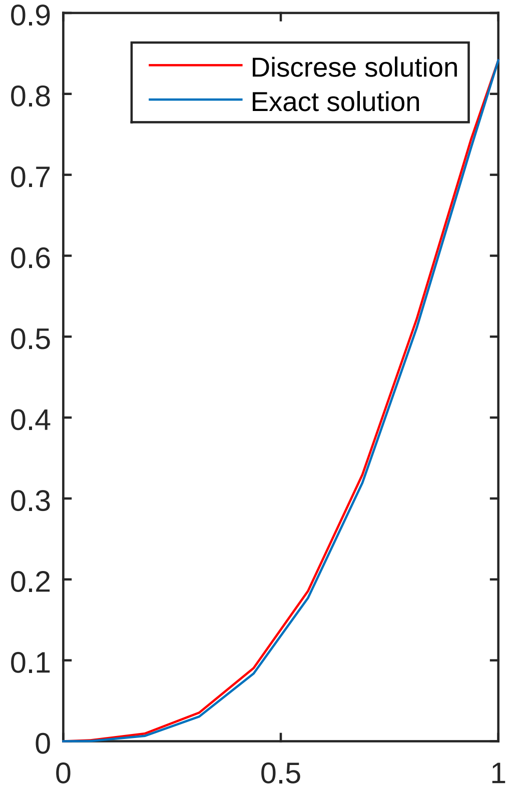


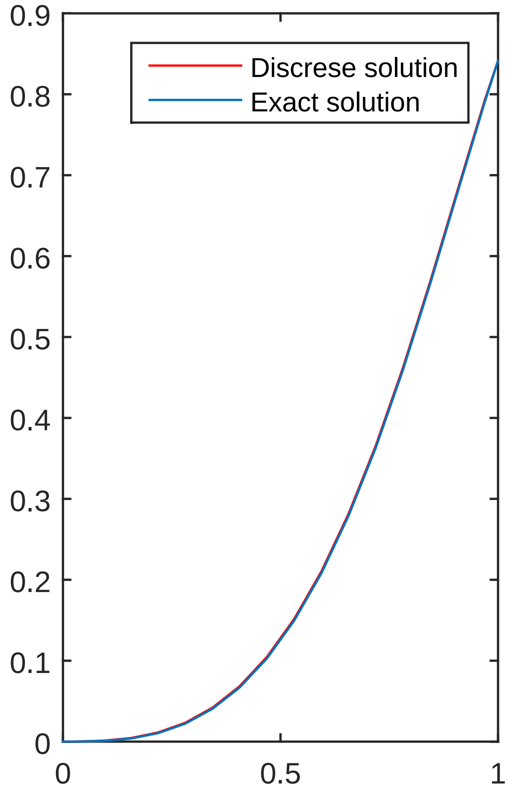

Figure 3: Error of Problem 2 - Question 1a



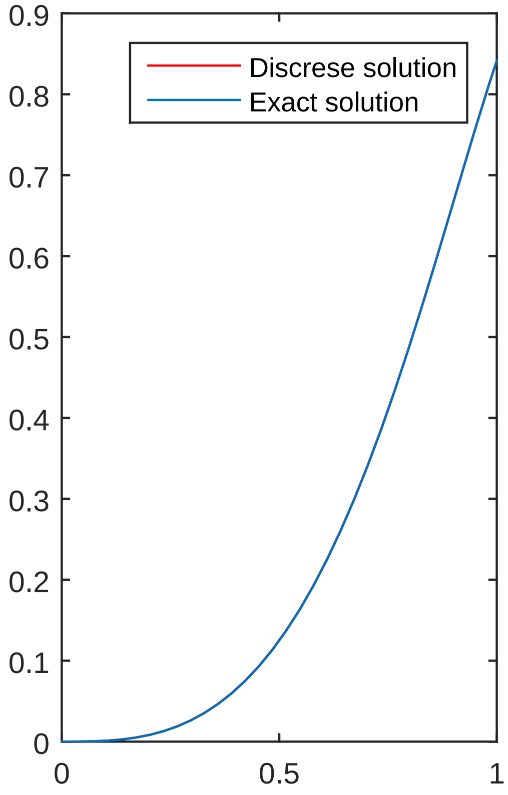

(a) 11




(b) 12




(c) 13




(d) 14

Figure 4: Approximate solution of Problem 2 - Question 1a.

## 1.2. Regular grid and the control point is $1/3$ from the left of each control volume

$$x_i = \frac{2}{3}x_{i-\frac{1}{2}} + \frac{1}{3}x_{i+\frac{1}{3}}$$

We also create a grid similarly to Problem 1.1,
Let us choose $N+1$ points $\{x_{i+\frac{1}{2}}\}_{i=\overline{0,N}}$ in $[0,1]$ such that:

$$0 = x_{\frac{1}{2}} < x_{\frac{3}{2}} < ... < x_{N+\frac{1}{2}} = 1$$

We set $T_i = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$, $|T_i| = x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}} \forall i \in \overline{1,N}$, $h = \max_{i\in\overline{1,N}}\{|T_i|\}$ and

$$\begin{cases} x_0 = 0, x_{N+1} = 1, \\ x_i = \dfrac{2}{3}x_{i-\frac{1}{2}} + \dfrac{1}{3}x_{i+\frac{1}{3}} \end{cases}$$

Continue similar process, we have result:

$$u(x_{i+1}) - u(x_i) = u_x(x_{i+\frac{1}{2}})(x_{i+1} - x_i) + \left( (x_{i+1} - x_{i+\frac{1}{2}})^2 - (x_i - x_{i+\frac{1}{2}})\frac{u_{xx}(x_{i+\frac{1}{2}})}{2} + O(h^3) \right) \quad \text{(Eq1.13)}$$

Because $x_{i+\frac{1}{2}}$ is not midpoint of $[x_i, x_{i+1}]$ then

$$u_x(x_{i+\frac{1}{2}}) = \frac{u(x_{i+1}) - u(x_i)}{x_{i+1} - x_i} + O(h)$$

We also have the approximate of the term $u_x(x_{i+\frac{1}{2}})$ is:

$$u_x(x_{i+\frac{1}{2}}) = \frac{u(x_{i+1}) - u(x_i)}{|D_{i+\frac{1}{2}}|} \quad \text{(Eq1.14)}$$

where $D_{i+\frac{1}{2}} = x_{i+1} - x_i$.

It is the only difference of two problem 1.1 and 1.2, we imply to the same the result:

$$AU = F$$

where $A \in \mathbb{R}^N \times \mathbb{R}^N$, $U, F \in \mathbb{R}^N$ satisfy:

$$\begin{bmatrix} \beta_1 & \gamma_1 & 0 & \dots & 0 & 0 & 0 \\ \alpha_2 & \beta_2 & \gamma_2 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \alpha_{N-1} & \beta_{N-1} & \gamma_{N-1} \\ 0 & 0 & 0 & \dots & 0 & \alpha_N & \beta_N \end{bmatrix} \quad \text{(Matrix A.2)}$$

$$U = \begin{bmatrix} u_1 \\ u_2 \\ \ddots u_{N-1} \\ u_N \end{bmatrix} \tag{cellU.2}$$

$$F = \begin{bmatrix} f_1 - a\alpha_1 \\ f_2 \\ \ddots f_{N-1} \\ f_N - \beta\gamma_N \end{bmatrix} \tag{cellF.2}$$

We can prove that (Matrix A.2) is invertible matrix so there is only one roof $(u_i)_{i=\overline{1,N}}$.

**Now we see graphics of exact and approximate solution with two problems**

**Problem 1:**

We set up with following exact solution $u$ and function $f$

$$\begin{cases} u(x) &= x^4 + 2x^3 - 10x^2 + 2 \\ f(x) &= -(12x^2 + 12x - 20) \end{cases}$$

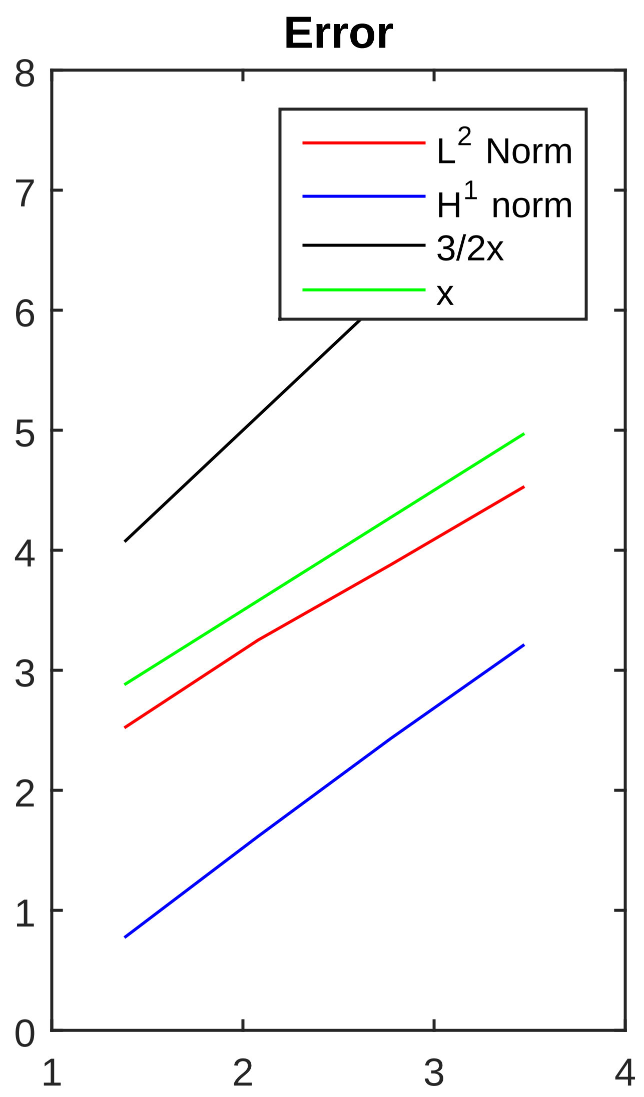
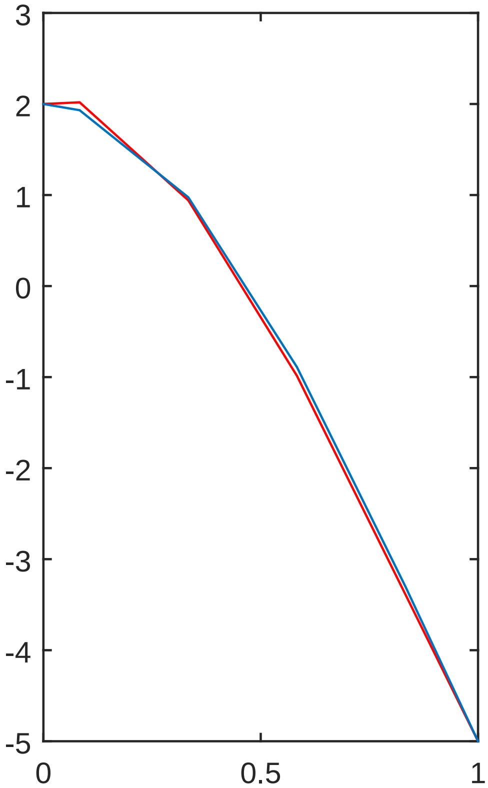
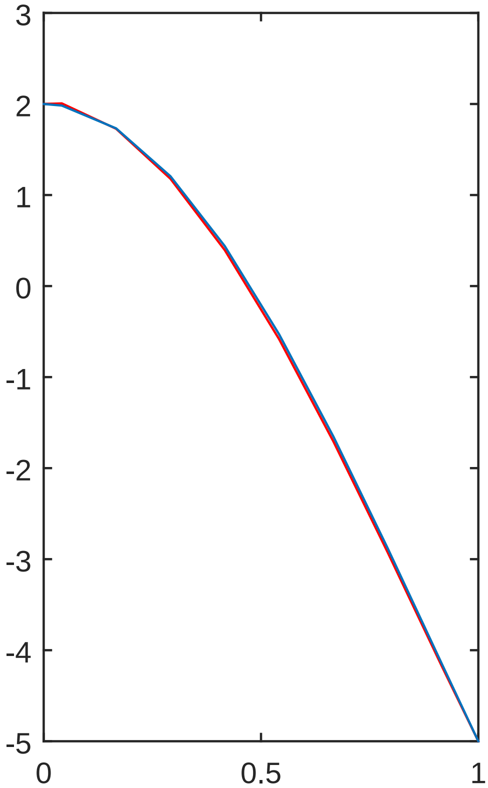


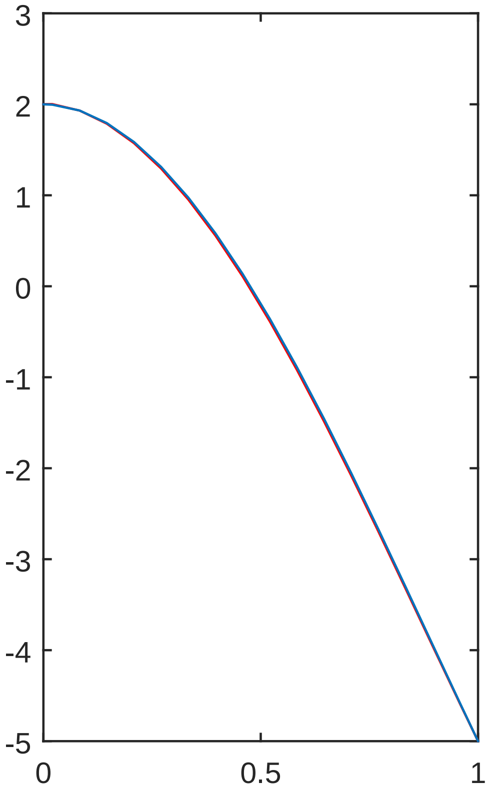

Figure 5: Error of Problem 1 - Question 1b.



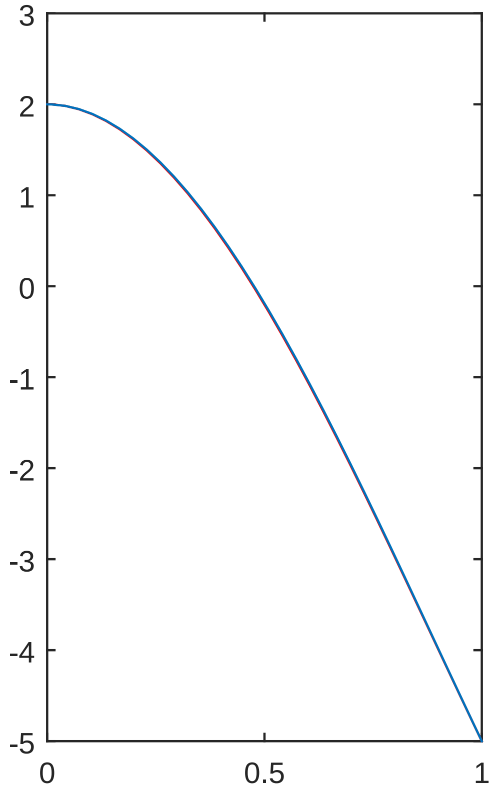

(a) 21




(b) 22




(c) 23




(d) 24

Figure 6: Approximate solution of Problem 1 - Question 1b.

**Problem 2:**

We set up with following exact solution $u$ and function $f$

$$\begin{cases} u(x) &= \sin(x^3) \\ f(x) &= 9x^4\sin(x^3) - 6x\cos(x^3) \end{cases}$$

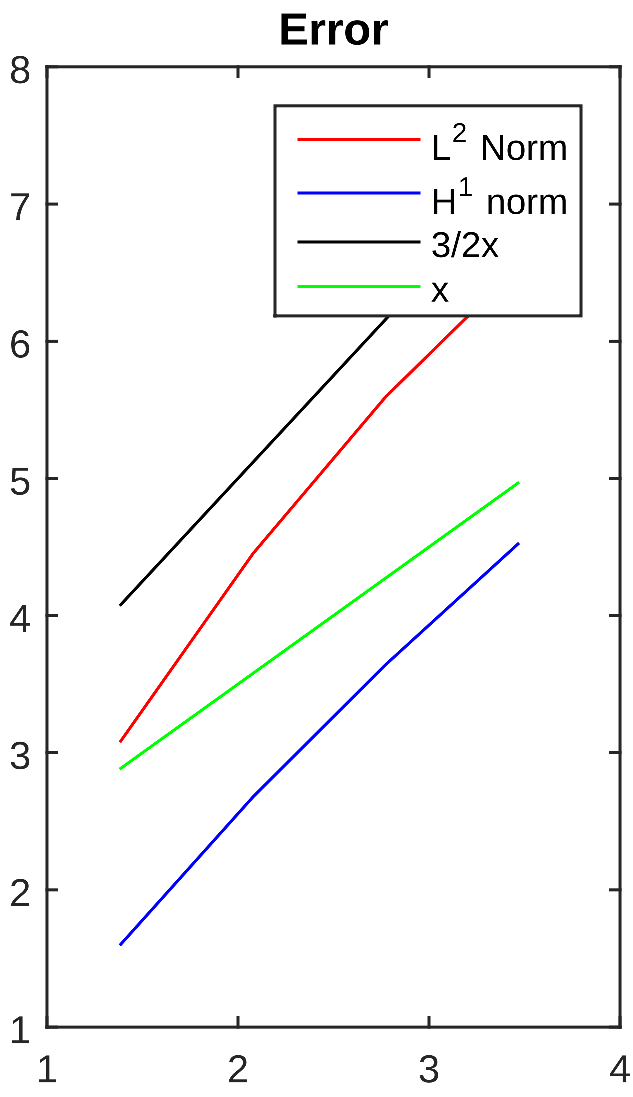



Figure 7: Error of Problem 2 - Question 1b.

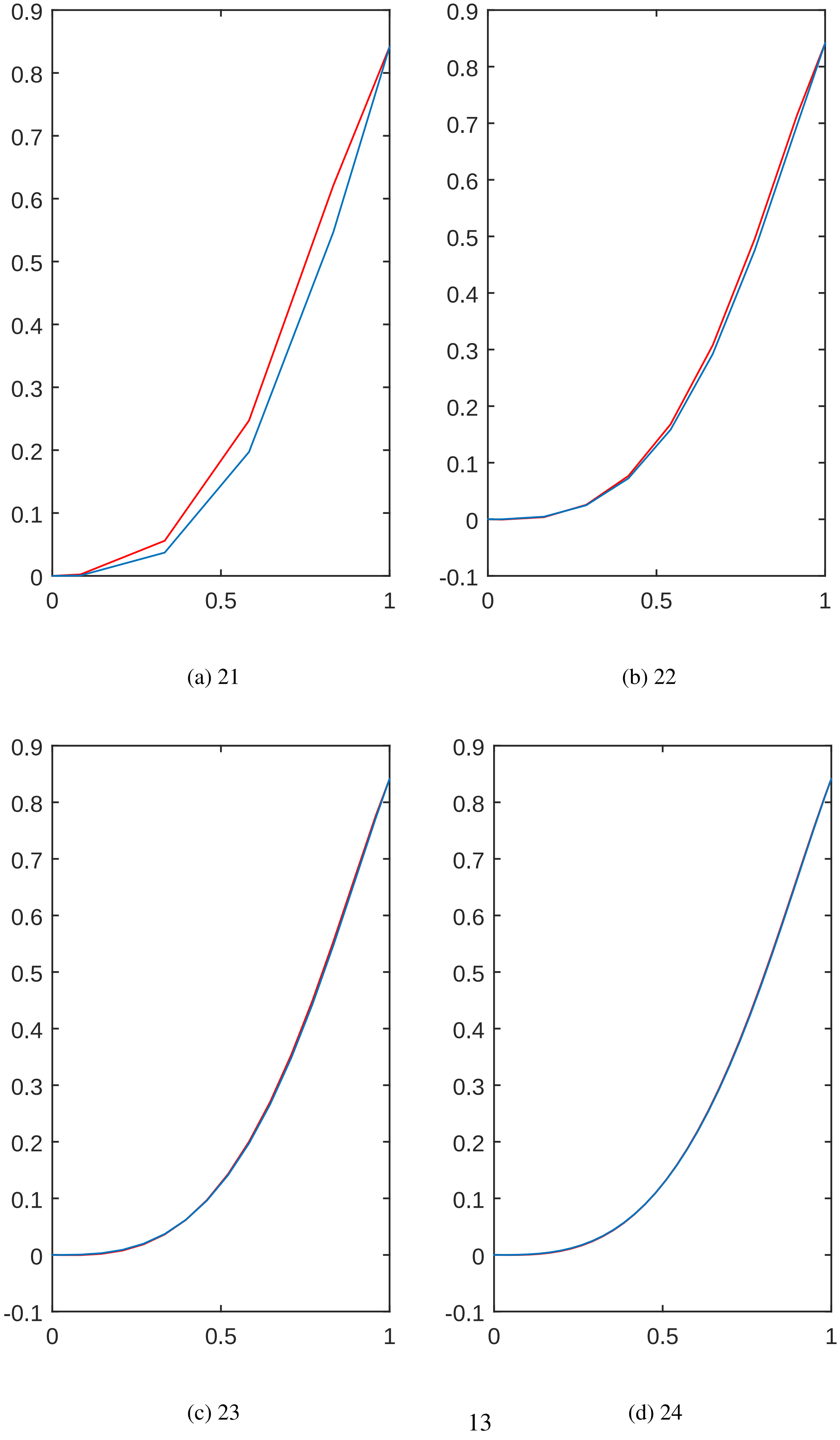





Figure 8: Approximate solution of Problem 2 - Question 1b.

## 1.3. Approximate the mean-value of $f$ over $T_i$

### 1.3.1. Trapezoidal rule

The 2-point formula

$$\int_{T_i} f(x)dx \approx \frac{1}{2}|T_i|(f(x_{i-\frac{1}{2}}) + f(x_{i+\frac{1}{2}}))$$

Then $f_i$ can approximate by:

$$f_i = \frac{1}{|T_i|}\int_{T_i} f(x)dx \approx \frac{1}{2}(f(x_{i-\frac{1}{2}}) + f(x_{i+\frac{1}{2}}))$$

Now we use problem 1 of 1.a:
We set up with following exact solution $u$ and function $f$

$$\begin{cases} u(x) &= x^4 + 2x^3 - 10x^2 + 2 \\ f(x) &= -(12x^2 + 12x - 20) \end{cases}$$

we have graphics of Error and Discrete Solution:

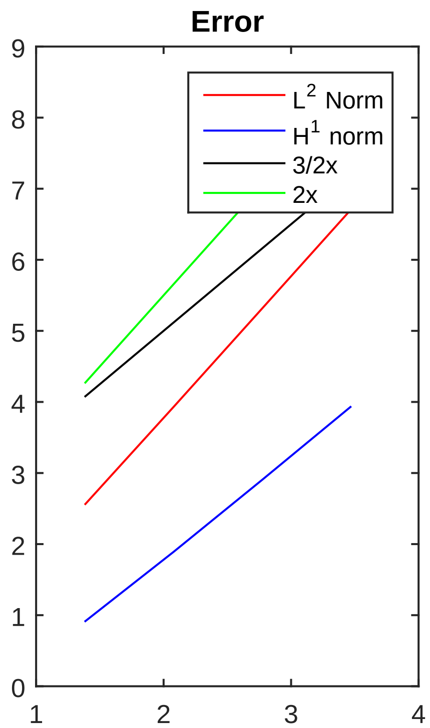
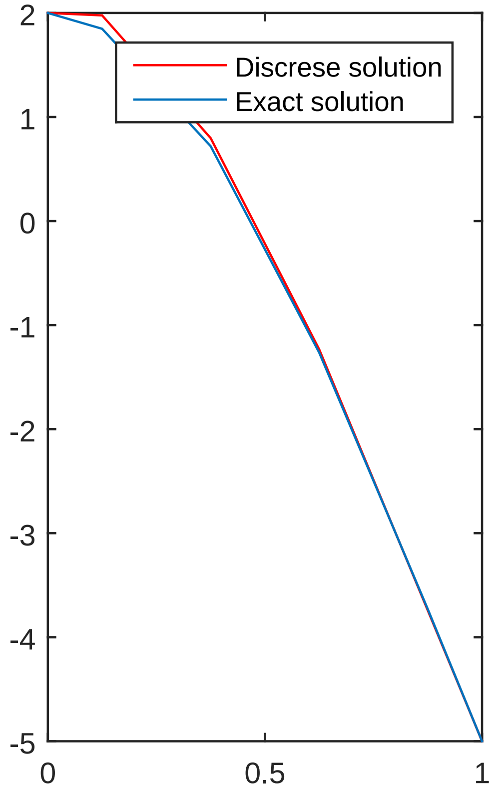
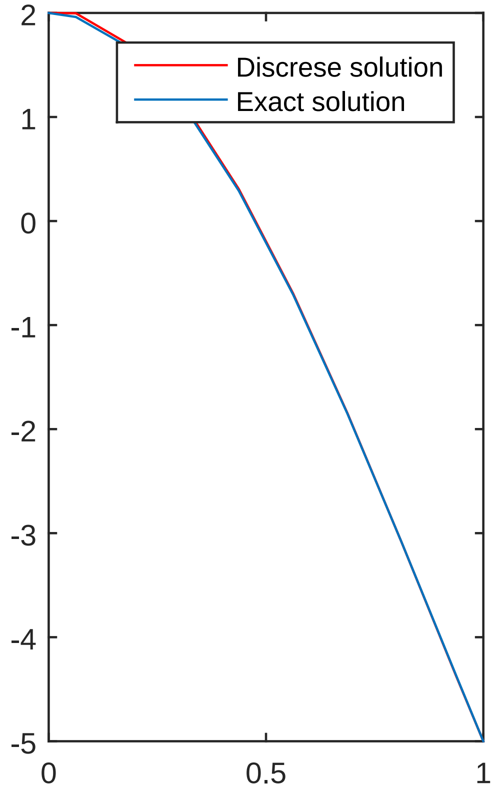


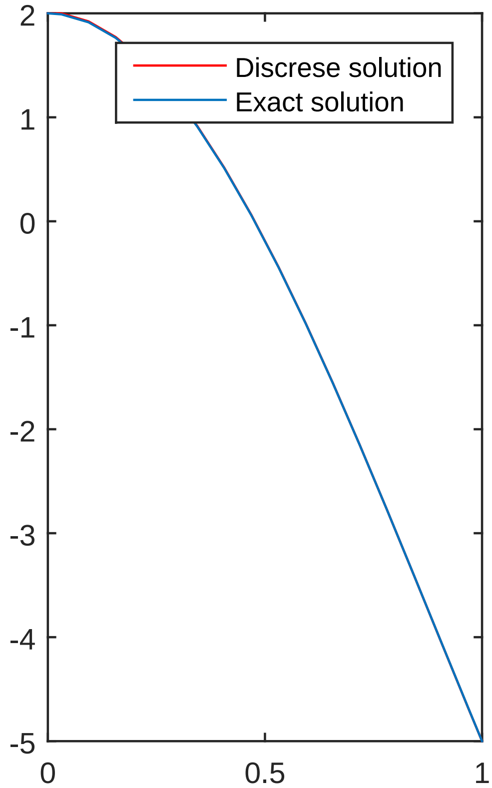

Figure 9: Error for Trapezoidal rule.



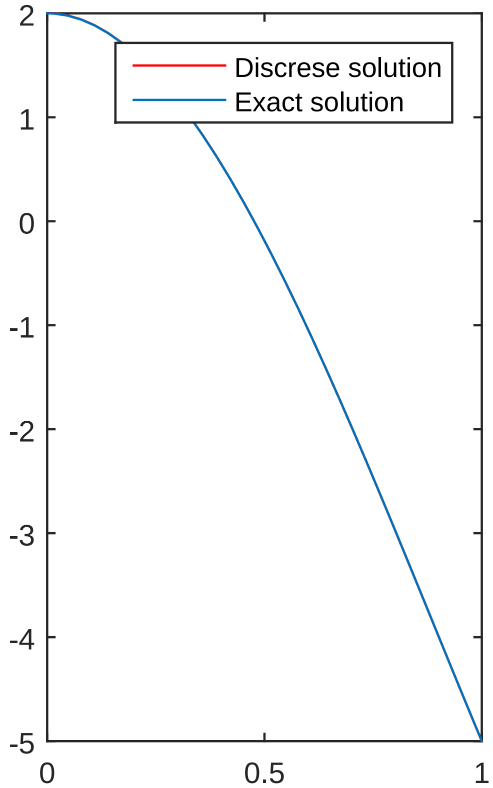

(a) 21




(b) 22




(c) 23




(d) 24

Figure 10: Approximate solution for Trapezoidal rule.

### 1.3.2. Simpson's rule

The 3-point formula

$$\int_{T_i} f(x)dx \approx \frac{1}{6}|T_i|(f(x_{i-\frac{1}{2}}) + 4f(\frac{x_{i-\frac{1}{2}} + x_{i+\frac{1}{2}}}{2}) + f(x_{i+\frac{1}{2}}))$$

Then $f_i$ can approximate by:

$$f_i = \frac{1}{|T_i|}\int_{T_i} f(x)dx \approx \frac{1}{6}(f(x_{i-\frac{1}{2}}) + 4f\left(\frac{x_{i-\frac{1}{2}} + x_{i+\frac{1}{2}}}{2}\right) + f(x_{i+\frac{1}{2}}))$$

Now we use problem 1 of 1.a:
We set up with following exact solution $u$ and function $f$

$$\begin{cases} u(x) &= x^4 + 2x^3 - 10x^2 + 2 \\ f(x) &= -(12x^2 + 12x - 20) \end{cases}$$

we have graphics of Error and Discrete Solution:

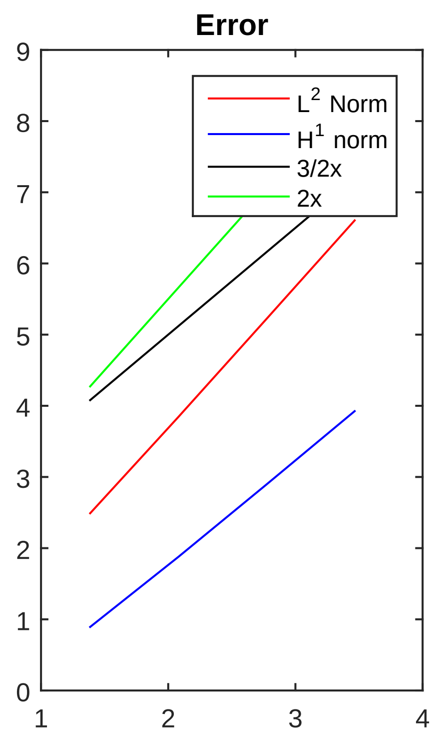
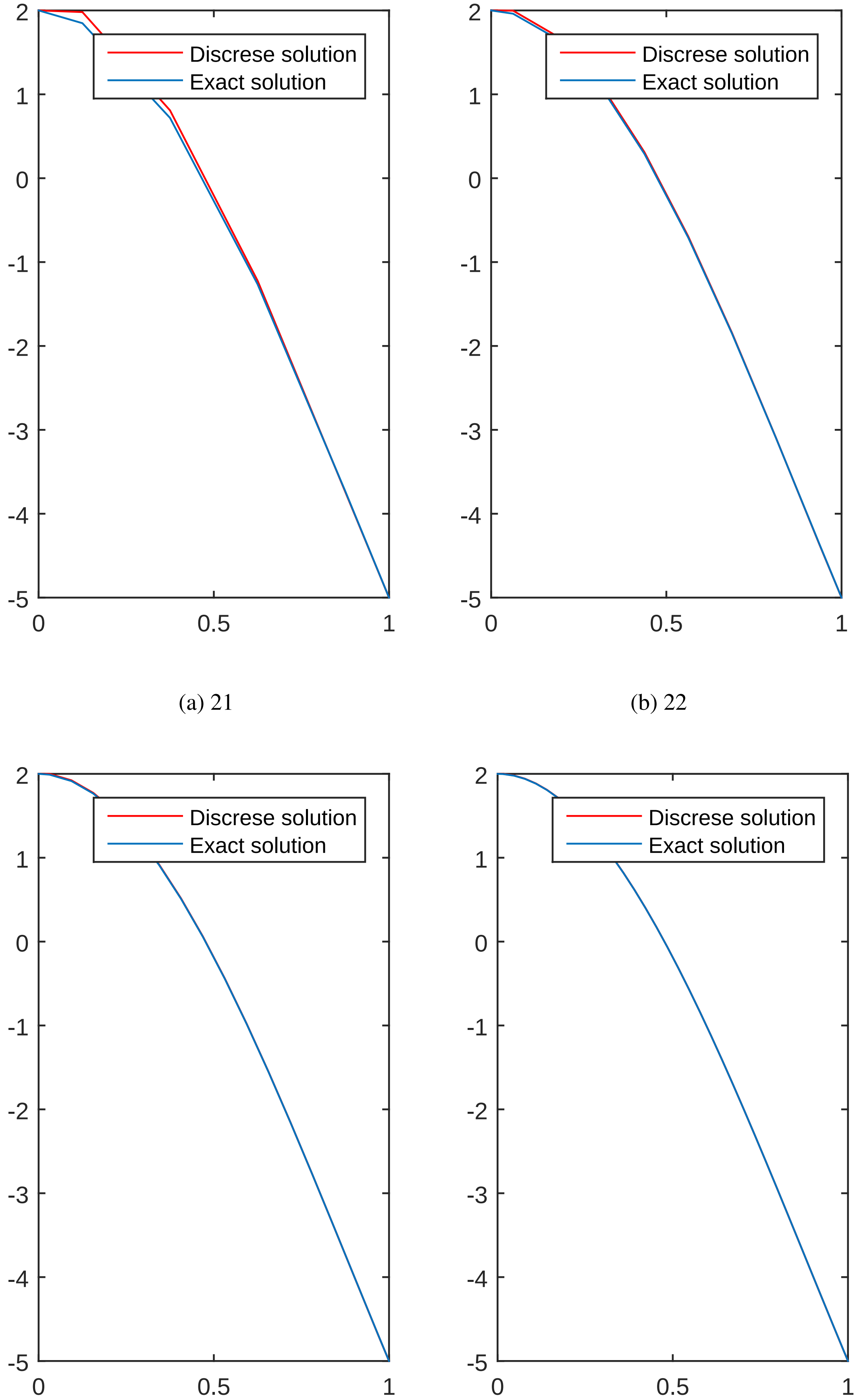



Figure 11: Error for Simpson's rule.




(a) 21

(b) 22

(c) 23

(d) 24



Figure 12: Approximate solution for Simpson's rule.

### 1.3.3. Simpson's rule $\frac{3}{8}$

The 4-point formula

$$\int_{T_i} f(x)dx \approx \frac{1}{8}|T_i|(f(x_{i-\frac{1}{2}}) + 3\left(\frac{2x_{i-\frac{1}{2}} + x_{i+\frac{1}{2}}}{3}\right) + 3f\left(\frac{x_{i-\frac{1}{2}} + 2x_{i+\frac{1}{2}}}{3}\right) + f(x_{i+\frac{1}{2}}))$$

Then $f_i$ can approximate by:

$$f_i = \frac{1}{|T_i|}\int_{T_i} f(x)dx \approx \frac{1}{8}(f(x_{i-\frac{1}{2}}) + 3\left(\frac{2x_{i-\frac{1}{2}} + x_{i+\frac{1}{2}}}{3}\right) + 3f\left(\frac{x_{i-\frac{1}{2}} + 2x_{i+\frac{1}{2}}}{3}\right) + f(x_{i+\frac{1}{2}}))$$

Now we use problem 1 of 1.a:
We set up with following exact solution $u$ and function $f$

$$\begin{cases} u(x) &= x^4 + 2x^3 - 10x^2 + 2 \\ f(x) &= -(12x^2 + 12x - 20) \end{cases}$$

we have graphics of Error and Discrete Solution:

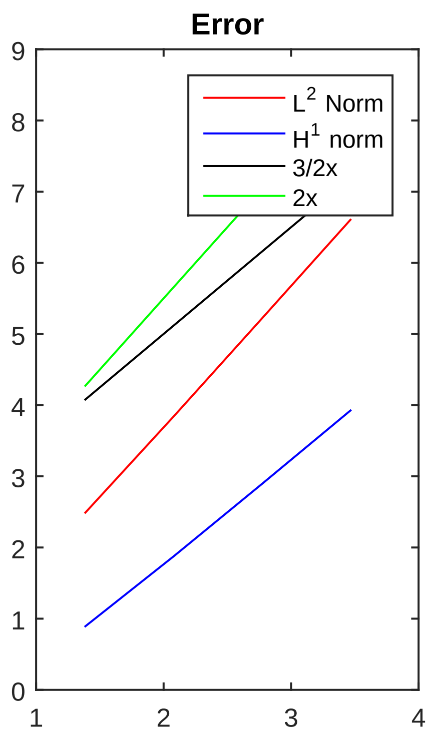
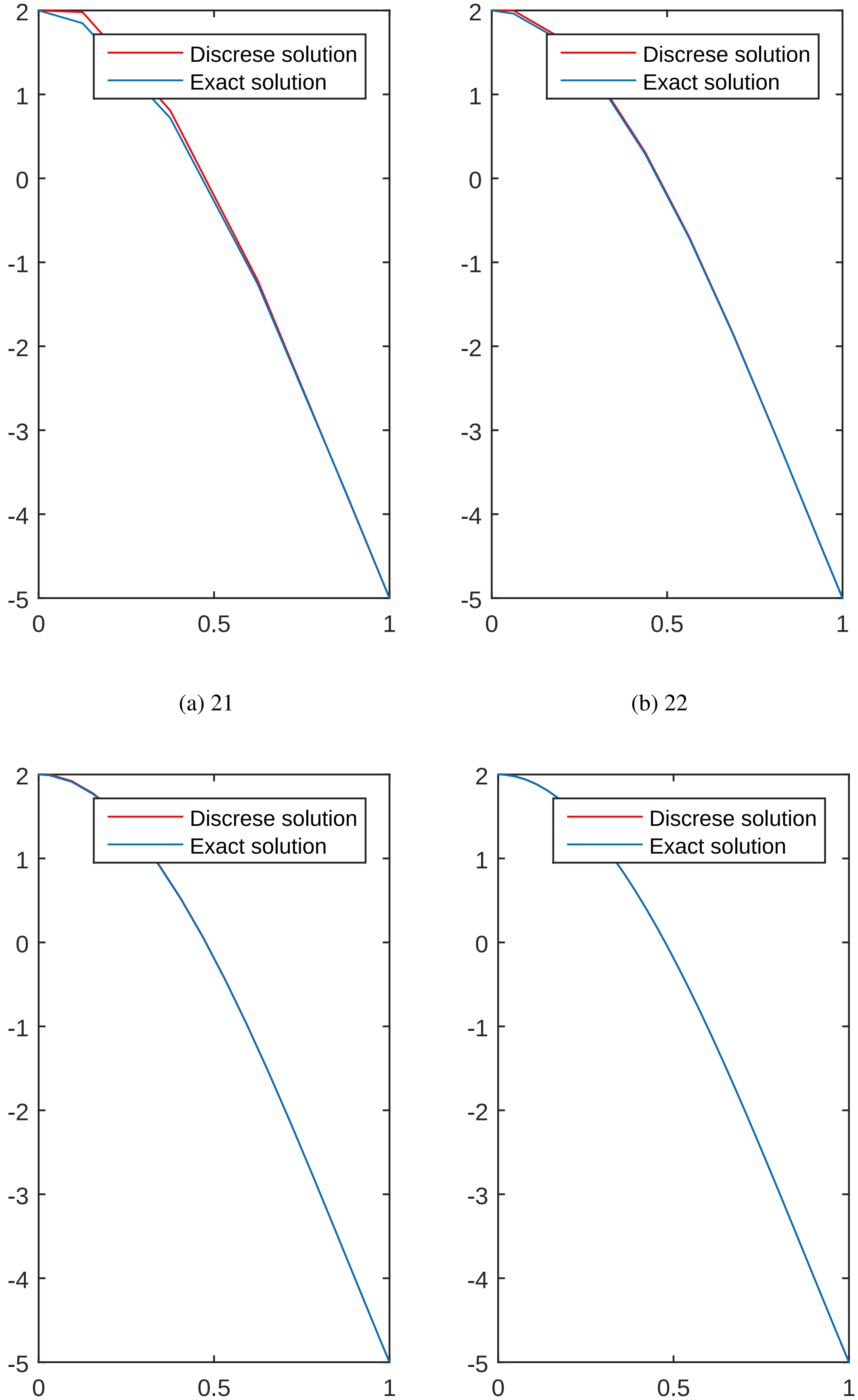



Figure 13: Error for Simpson's rule $\frac{3}{8}$.




(a) 21

(b) 22

(c) 23

(d) 24



Figure 14: Approximate solution for Simpson's rule $\frac{3}{8}$ .

## Compare some ways approximation

By looking the graphics of Errors we can remark that the fastest convergence is Simpson's rule $\frac{3}{8}$ and the lowest convergence is Trapezoidal rule.
We can explain this result to be if we use more point to approximate a function or in this case is approximate integral of function in a interval, the error will be smaller.
However, the differences are not large and Trapezoidal rule is the easiest way to approximate. So it is the reason why we use Trapezoidal first.

## 1.4. Case of only singular grid, not uniform grid

We also create a grid similarly to Problem 1.1,
Let us choose $N+1$ points $\{x_{i+\frac{1}{2}}\}_{i=\overline{0,N}}$ in $[0,1]$ such that:

$$0 = x_{\frac{1}{2}} < x_{\frac{3}{2}} < ... < x_{N+\frac{1}{2}} = 1$$

We set $T_i = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$, $|T_i| = x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}} \forall i \in \overline{1,N}$, $h = \max_{i\in\overline{1,N}}\{|T_i|\}$ and

$$\begin{cases} x_0 = 0, x_{N+1} = 1, \\ x_i = \dfrac{2}{3}x_{i-\frac{1}{2}} + \dfrac{1}{3}x_{i+\frac{1}{3}} \end{cases}$$

Continue similar process, we have result:

$$u(x_{i+1}) - u(x_i) = u_x(x_{i+\frac{1}{2}})(x_{i+1} - x_i) + \left((x_{i+1} - x_{i+\frac{1}{2}})^2 - (x_i - x_{i+\frac{1}{2}})\frac{u_{xx}(x_{i+\frac{1}{2}})}{2} + O(h^3)\right) \tag{Eq1.13}$$

Because $x_{i+\frac{1}{2}}$ is not midpoint of $[x_i, x_{i+1}]$ then

$$u_x(x_{i+\frac{1}{2}}) = \frac{u(x_{i+1}) - u(x_i)}{x_{i+1} - x_i} + O(h)$$

We also have the approximate of the term $u_x(x_{i+\frac{1}{2}})$ is:

$$u_x(x_{i+\frac{1}{2}}) = \frac{u(x_{i+1}) - u(x_i)}{|D_{i+\frac{1}{2}}|} \tag{Eq1.14}$$

where $D_{i+\frac{1}{2}} = x_{i+1} - x_i$.

It is the only difference of two problem 1.1 and 1.2, we imply to the same the result:

$$AU = F$$

where $A \in \mathbb{R}^N \times \mathbb{R}^N$, $U, F \in \mathbb{R}^N$ satisfy:

$$\begin{bmatrix} \beta_1 & \gamma_1 & 0 & \dots & 0 & 0 & 0 \\ \alpha_2 & \beta_2 & \gamma_2 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \alpha_{N-1} & \beta_{N-1} & \gamma_{N-1} \\ 0 & 0 & 0 & \dots & 0 & \alpha_N & \beta_N \end{bmatrix} \tag{Matrix A.2}$$

$$U = \begin{bmatrix} u_1 \\ u_2 \\ \ddots u_{N-1} \\ u_N \end{bmatrix} \tag{cellU.2}$$

$$F = \begin{bmatrix} f_1 - a\alpha_1 \\ f_2 \\ \ddots f_{N-1} \\ f_N - \beta\gamma_N \end{bmatrix} \tag{cellF.2}$$

We can prove that (Matrix A.2) is invertible matrix so there is only one roof $(u_i)_{i=\overline{1,N}}$.

Now we use problem 1 of 1.a:
We set up with following exact solution $u$ and function $f$

$$\begin{cases} u(x) &= x^4 + 2x^3 - 10x^2 + 2 \\ f(x) &= -(12x^2 + 12x - 20) \end{cases}$$

And use $u_{i+\frac{1}{2}} = x_i + \left(1 - \cos\left(\frac{\pi i}{2N}\right)\right)(x_{i+1} - x_i)$ we have graphics of Error and Discrete Solution:

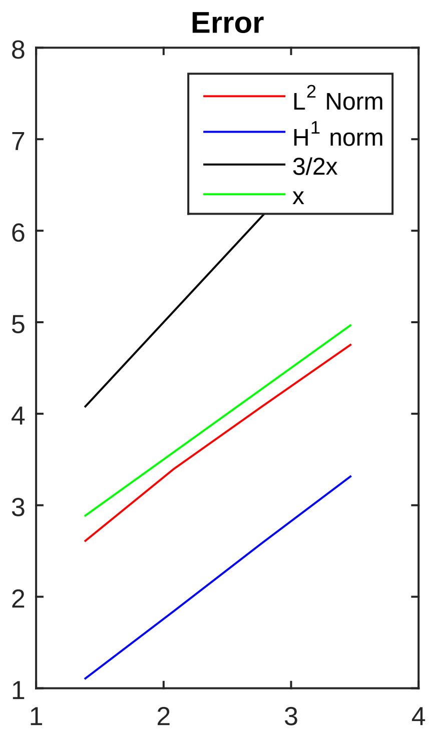
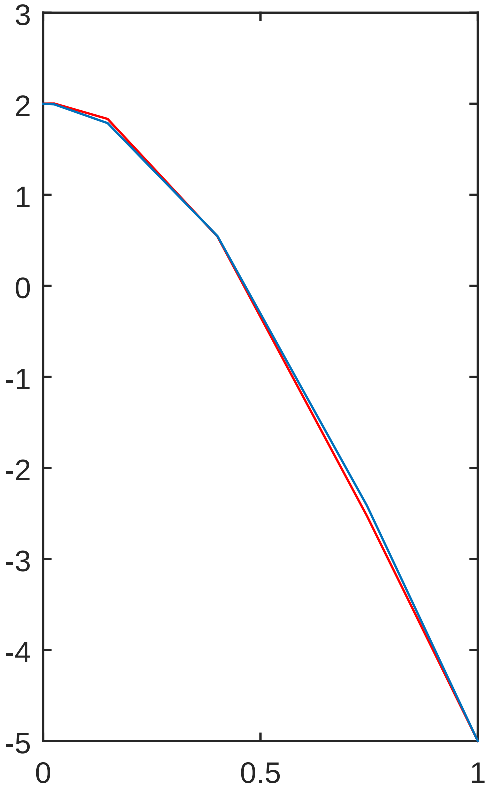
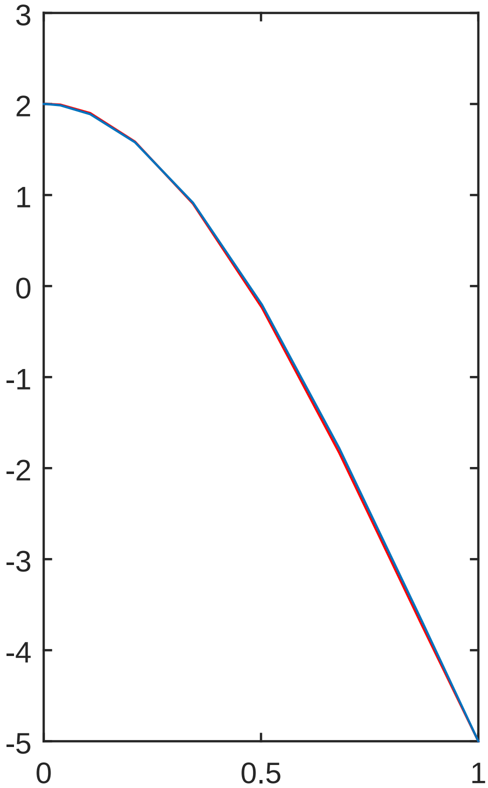


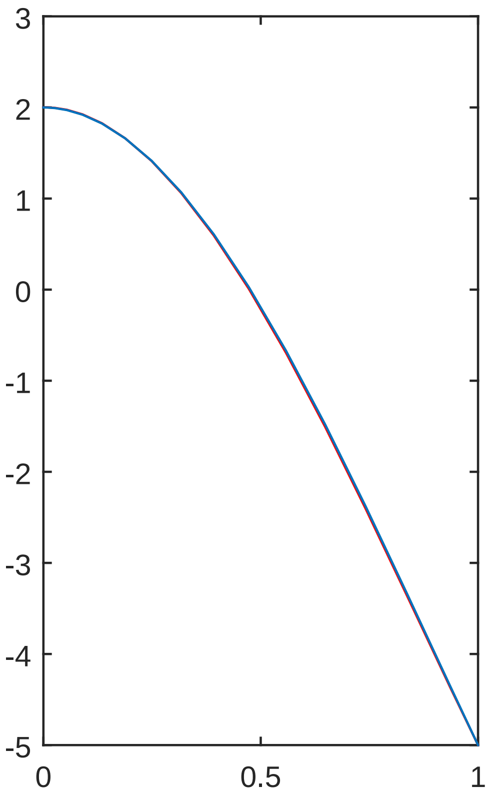

Figure 15: Error of Problem 1 - Question 1d.



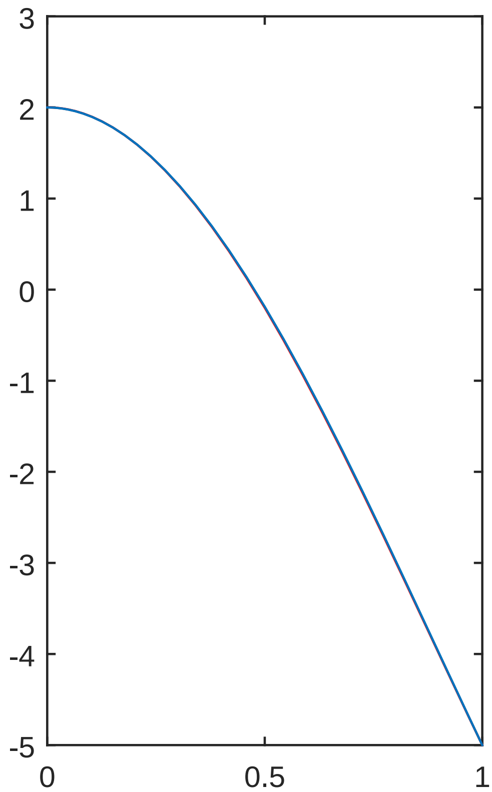

(a) 21




(b) 22




(c) 23




(d) 24

Figure 16: Approximate solution of Problem 1 - Question 1d.

# Problem 2
# Neumann boundary condition

## 2.1. Introduction

We use FVM to solve this equation

$$u''_{xx}(x) = f(x), x \in \Omega(x) \tag{Eq2.1}$$

with boundary condition:

$$u'(0) = 0 \tag{Condition 1}$$

$$u'(1) = 0 \tag{Condition 2}$$

necessary condition

$$\int_\Omega f(x)dx = 0 \tag{necessary}$$

and condition to determine unique solution

$$\int_\Omega f(x)dx = 0 \tag{unique solution}$$

We only consider case of the control point is the mid point of each control volume $x_i = \dfrac{x_{i-\frac{1}{2}} + x_{i+\frac{1}{2}}}{2}$.

## 2.2. Mesh

Let us choose $N+1$ points $\{x_{i+\frac{1}{2}}\}_{i=\overline{0,N}}$ in $[0,1]$ such that:

$$0 = x_{\frac{1}{2}} < x_{\frac{3}{2}} < ... < x_{N+\frac{1}{2}} = 1$$

We set $T_i = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$, $|T_i| = x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}} \forall i \in \overline{1,N}$, $h = \max_{i\in\overline{1,N}}\{|T_i|\}$ and

$$\begin{cases} x_0 = 0, x_{N+1} = 1, \\ x_i = \dfrac{x_{i-\frac{1}{2}} + x_{i+\frac{1}{2}}}{2} \end{cases}$$

## 2.3. Scheme

Now, use similar process of Dirichlet condition, we also have the scheme for cell-center finite volume method:

$$\begin{cases} \alpha_i u_{i-1} + \beta_i u_i + \gamma_i u_{i+1} = f_i, \forall i = \overline{1, N}, \\ u'(0) = 0, \\ u'(1) = 1 \end{cases} \tag{Eq2.2}$$

Two below equation can be discretized into:

$$\frac{x_1 - x_0}{\left|D_{\frac{1}{2}}\right|} = 0 \text{ and } \frac{x_{N+1} - x_N}{\left|D_{N+\frac{1}{2}}\right|} = 0$$

$$\Leftrightarrow x_0 = x_1 \text{ and } x_N = x_{N+1}$$

And condition to determine (unique solution) is discretized into:

$$\sum_{i=1}^{N} |T_i| u_i = 0$$

Now we have:

$$\alpha_i u_{i-1} + \beta_i u_i + \gamma_i u_{i+1} = f_i, \forall i = \overline{1, N} \tag{Eq2.3}$$

$$\begin{cases} u_0 = u_1 \\ u_N = u_{N+1} \end{cases} \tag{Eq2.4}$$

$$\sum_{i=1}^{N} |T_i| u_i = 0 \tag{Eq2.5}$$

Thus, there are $N + 3$ equations and but only $N + 2$ unknowns. However, the set of equations (Eq2.3) and (Eq2.4) are not independent.
We have:

$$\sum_{i=1}^{N} \left[ -\frac{x_{i+1} - u_i}{\left|D - i + \frac{1}{2}\right|} + \frac{x_i - x_{i-1}}{\left|D_{i-\frac{1}{2}}\right|} \right] = \sum_{i=1}^{N} |T_i| f_i$$

$$\frac{x_{N+1} - x_N}{\left|D_{N+\frac{1}{2}}\right|} + \frac{x_1 - x_0}{\left|D_{\frac{1}{2}}\right|} = \sum_{i=1}^{N} |T_i| \frac{1}{|T_i|} \int_{T_i} f(x) dx$$

It is very to see that two side of above equation is vanished by $a_1 = a_0$, $a_N = a_{N+1}$ and $\int_\Omega f(x)dx$.
Now we have 2 case:

## 2.4. Numerical experiments

### 2.4.1. Case of regular grid

In case of regular grid with control point be the mid point of each control volume $x_i = \dfrac{x_{i-\frac{1}{2}} + x_{i+\frac{1}{2}}}{2}$. It implies that $T_1 = T_2 = ... = T_N$, so we have:

$$\sum_{i=1}^{N} u_i = 0$$

$$\Rightarrow u_N = -(u_1 + u_2 + ... + u_{N-1})$$

In conclusion, we have a system of $N$ equations and $N$ unknowns:

$$\begin{cases} i=1: -\gamma_1 u_1 + \gamma_1 u_2 & = f_1 \\ i=2: \alpha_2 u_1 + \beta_2 u_2 + \gamma_2 u_3 & = f_2 \\ i=2: \qquad \alpha_3 u_2 + \beta_3 u_3 + \gamma_3 u_4 & = f_2 \\ \qquad \vdots \\ i=N-1: \qquad \alpha_{N-1} u_{N-2} + \beta_{N-1} u_{N-1} + \gamma_{N-1} u_N & = f_{N-1} \\ i=N: \alpha_N u_1 + \alpha_N u_2 + \alpha_N u_3 + \cdots + \alpha_N u_{N-2} + 2\alpha_N u_{N-1} & = f_N \end{cases}$$

And we can write under form of matrix:

$$AU = F$$

where $A \in \mathbb{R}^N \times \mathbb{R}^N$, $U, F \in \mathbb{R}^N$ satisfy:

$$\begin{bmatrix} -\gamma_1 & \gamma_1 & 0 & 0 & \dots & 0 & 0 & 0 \\ \alpha_2 & \beta_2 & \gamma_2 & 0 & \dots & 0 & 0 & 0 \\ 0 & \alpha_3 & \beta_3 & \gamma_3 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & 0 & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \alpha_{N-1} & \beta_{N-1} & \gamma_{N-1} \\ \alpha_N & \alpha_N & \alpha_N & \alpha_N & \dots & \alpha_N & 2\alpha_N & 0 \end{bmatrix} \tag{Matrix A.3}$$

$$U = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{N-1} \\ u_N \\ u_{N+1} \end{bmatrix} \tag{cellU.3}$$

$$F = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_{N-1} \\ f_N \\ f_{N+1} \end{bmatrix} \tag{cellF.3}$$

We can prove that (Matrix A.3) is invertible matrix so there is only one roof $(u_i)_{i=\overline{1,N}}$.

**Now we see graphics of exact and approximate solution with a problem**

We set up with following exact solution $u$ and function $f$

$$\begin{cases} u(x) &= \dfrac{-x^3}{3} + \dfrac{x^2}{2} - \dfrac{1}{12} \\ f(x) &= 2x - 1 \end{cases}$$

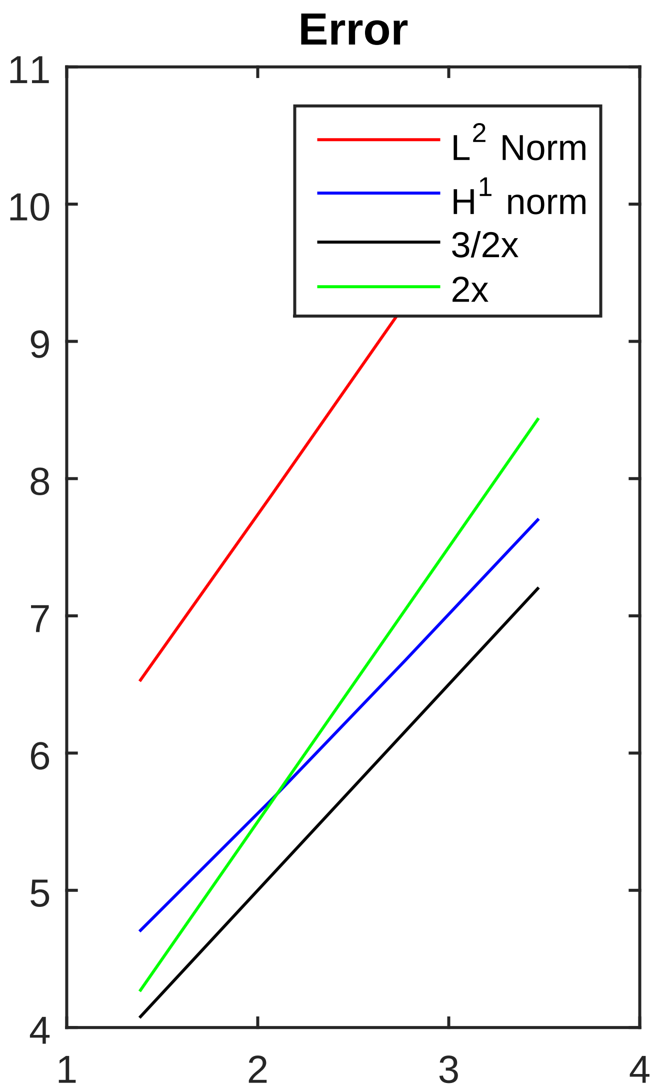
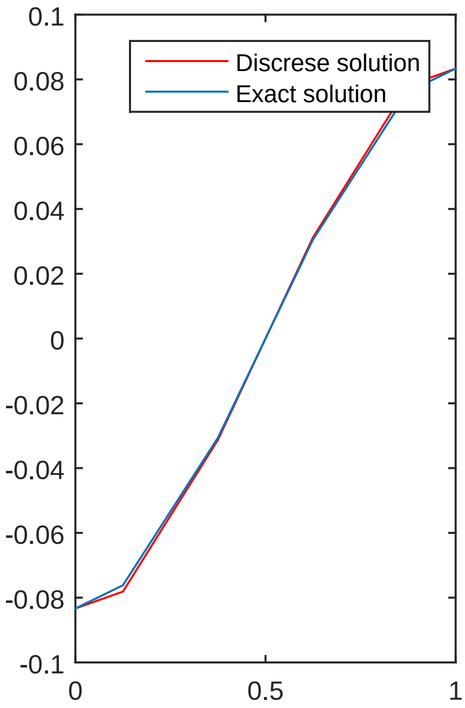
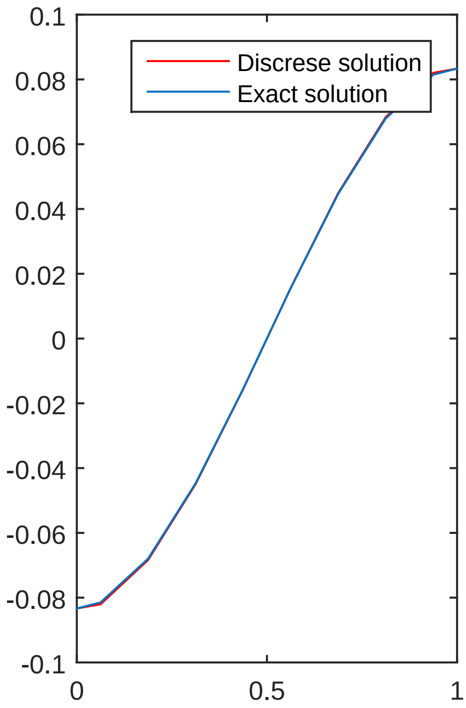


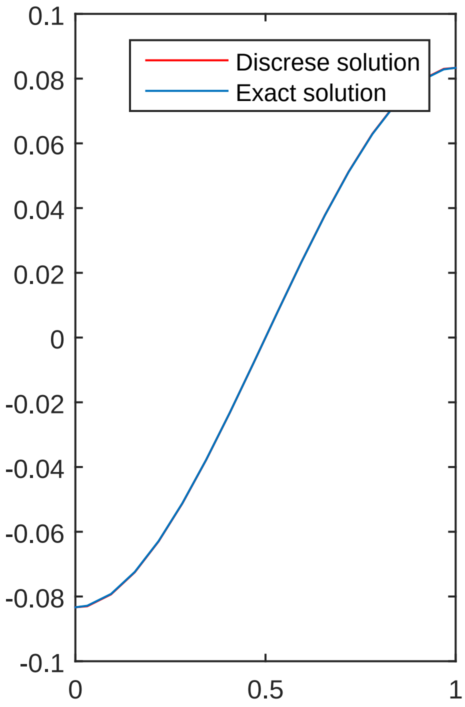

Figure 17: Error for Neumann boundary condition - Regular grid



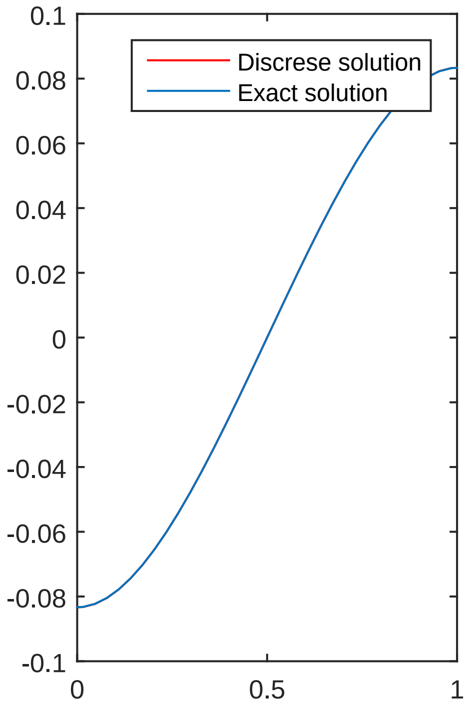

(a) 11




(b) 12




(c) 13




(d) 14

Figure 18: Approximate solution for Neumann boundary condition - Regular grid

### 2.4.2. Case of singular grid

In case of singular grid with control point be the mid point of each control volume $x_i = \dfrac{x_{i-\frac{1}{2}} + x_{i+\frac{1}{2}}}{2}$.
We have a system of N+2 equations:

$$AU = F$$

where $A \in \mathbb{R}^N \times \mathbb{R}^N$, $U, F \in \mathbb{R}^N$ satisfy:

$$\begin{bmatrix} -\gamma_1 & \gamma_1 & 0 & \dots & 0 & 0 & 0 & 0 \\ \alpha_2 & \beta_2 & \gamma_2 & \dots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \\ 0 & 0 & 0 & \dots & \alpha_{N-1} & \beta_{N-1} & \gamma_{N-1} & 0 \\ 0 & 0 & 0 & \dots & 0 & \alpha_N & \beta_N & \gamma_N \\ T_1 & T_2 & T_3 & \dots & T_{N-2} & T_{N-1} & T_N & 0 \end{bmatrix} \tag{Matrix A.3}$$

$$U = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{N-1} \\ u_N \\ u_{N+1} \end{bmatrix} \tag{cellU.2}$$

$$F = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_{N-1} \\ f_N \\ f_{N+1} \end{bmatrix} \tag{cellF.2}$$

We can prove that (Matrix A.2) is invertible matrix so there is only one roof $(u_i)_{i=\overline{1,N}}$.

**Now we see graphics of exact and approximate solution with some problems**

We set up with following exact solution $u$ and function $f$

$$\begin{cases} u(x) &= \dfrac{-x^3}{3} + \dfrac{x^2}{2} - \dfrac{1}{12} \\ f(x) &= 2x - 1 \end{cases}$$

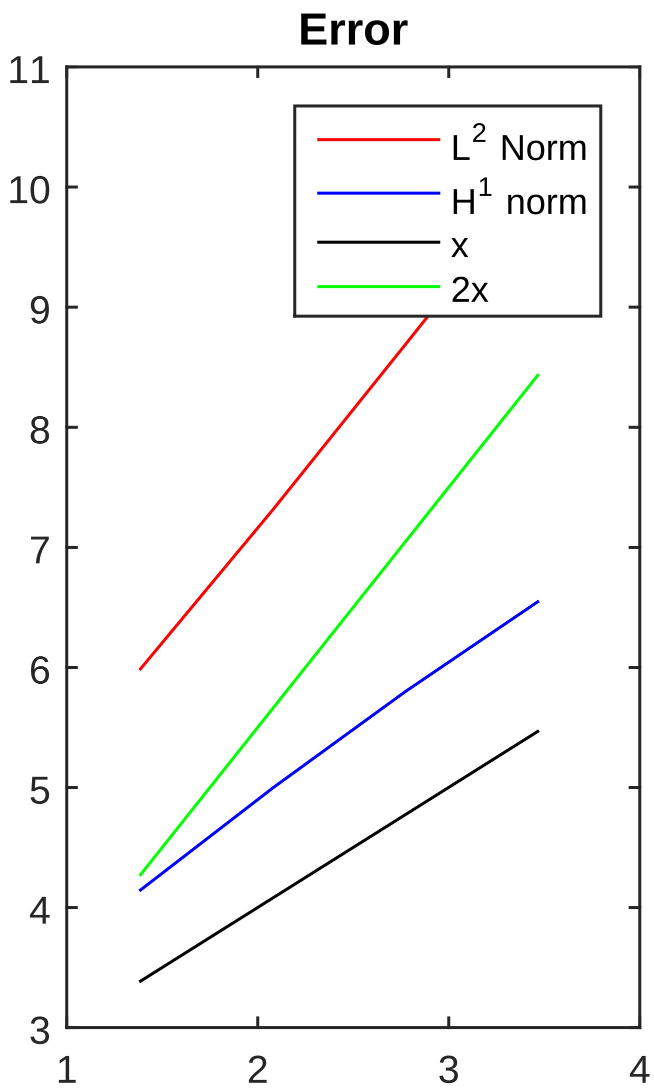
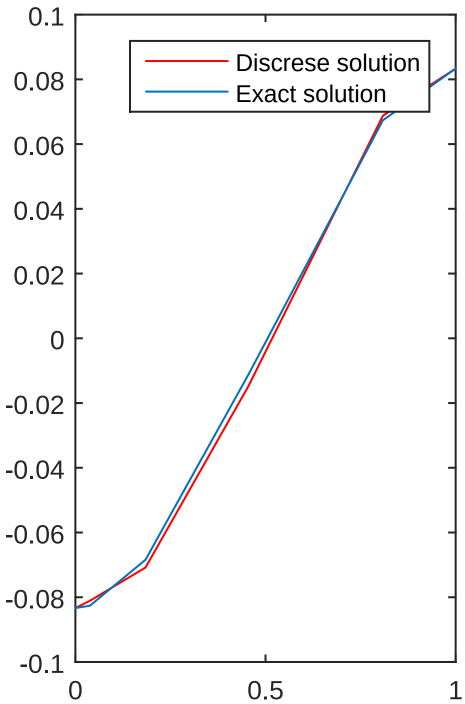
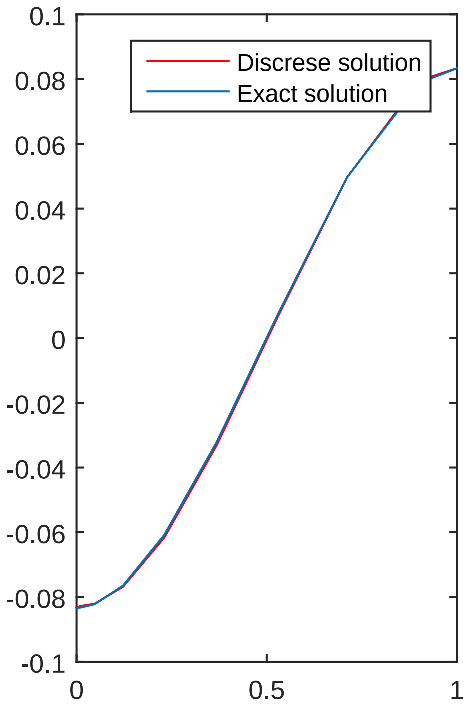


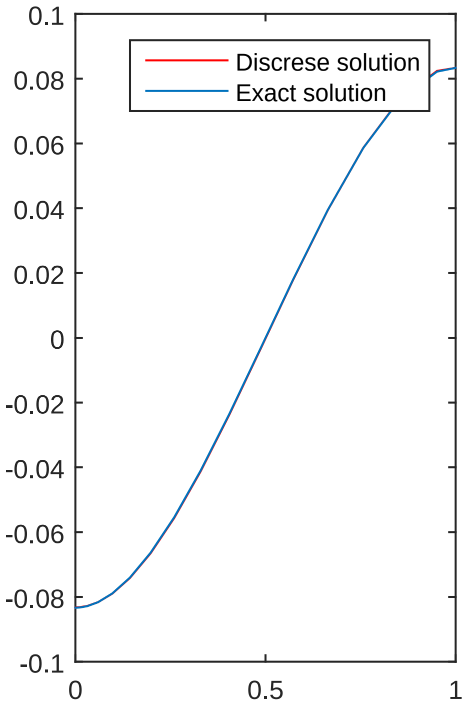

Figure 19: Error for Neumann boundary condition - Singular grid



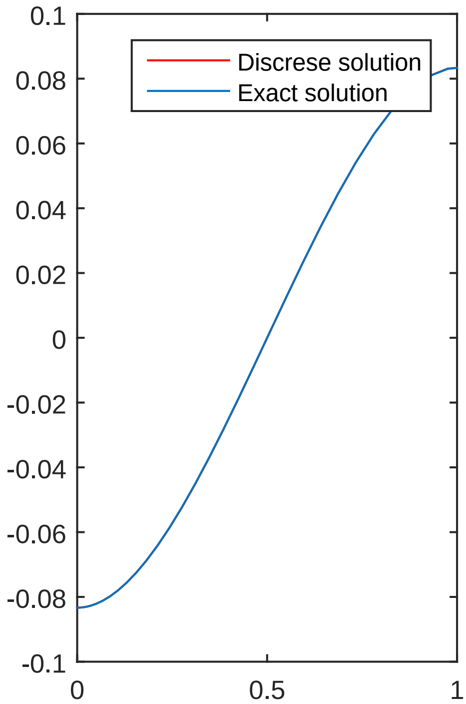

(a) 11




(b) 12




(c) 13




(d) 14

Figure 20: Approximate solution - Singular grid