A method for computing winding number: the Alexander numbering rule

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1 Introduction

In complex analysis, the winding number is particularly useful in a number of applications, for example, in computing integrals via the General Cauchy Integral Formulas and the Residue Theorem, or testing if a holomorphic function has zeros or poles in a subset of the plane using the Argument Principle, as in [5, 6].

For curves that are relatively simple, the winding number can be computed fairly easily from the definition by hand. However, for more complicated curves, direct computation becomes tedious. Even visual inspection can prove challenging! Figure 1 shows an example of this.

Our main goal in this paper is to prove the *Alexander numbering rule* for computing winding numbers of piecewise-smooth curves from the "crossings" of an auxiliary curve through the given curve. [5, Exercise 57] The procedure is roughly as follows: we first pick an auxiliary piecewise-smooth curve starting from the point we wish to compute the winding number around, to a point whose winding number we already know, which intersects the given curve in a "nice" way. Then, to each intersection point (the number of which we assume to be finite) we assign *intersection numbers* ± 1 based on the direction (left to right or vice versa) that the auxiliary curve "cuts" through the given curve. The sum of these intersection numbers and the given winding number is then the unknown winding number.

We obtain from this a way to number the regions of the plane cut out by the given curve with the winding numbers of a point lying in those regions, as in Figure 1. Such a numbering (applied to knot diagrams rather than plane curves) is an example of the *Alexander numbering* used by Alexander to formulate the first knot theory invariant, the Alexander polynomial. [2]

In §2 we start with some standard definitions and lemmas that we will need, and we prove the Alexander numbering rule in §3. We discuss some further applications and remaining questions in §4

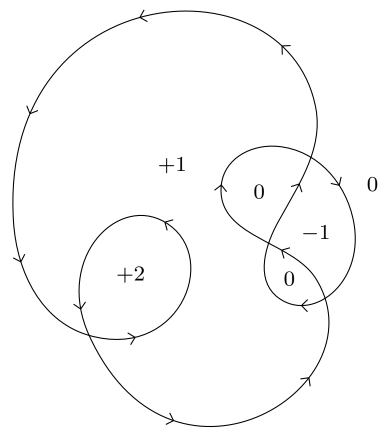

Figure 1: Winding numbers of points in regions cut out by a curve.

2 Preliminary results

We let $\gamma : [a, b] \to \mathbb{C}$ be an arbitrary closed curve (oriented counterclockwise) throughout this section unless otherwise specified, and denote its image in $\mathbb{C}$ with γ^*. In particular, the complement of its image in the plane is $\mathbb{C} \setminus \gamma^*$.

Definition 2.1. The *winding number* of an arbitrary closed curve γ around a point $z_0 \in \mathbb{C} \setminus \gamma^*$ is defined to be

$$W_\gamma(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{1}{z - z_0}\, dz$$

Remark. Taking $1/(z - z_0)$ as a function of z, it is holomorphic on $\mathbb{C} \setminus \{z_0\}$, so $W_\gamma(z_0) = W_{\gamma'}(z_0)$ for any closed curve γ' homotopic to γ (up to reparameterization) in $\mathbb{C} \setminus \{z_0\}$ by the homotopy version of Cauchy's Theorem.

In particular, the integral in the above definition, and therefore the winding number, can be made to always exist (e.g. in the case γ is not rectifiable) by declaring that

$$\int_\gamma \frac{1}{z - z_0}\, dz = \int_{\gamma'} \frac{1}{z - z_0}\, dz$$

for γ' a polygonal curve homotopic to γ in $\mathbb{C} \setminus \{z_0\}$, as in [5].

We have shown in the remark:

Lemma 2.2. *Let $z_0 \in \mathbb{C}$, and let γ, γ' be homotopic closed curves (up to reparameterization) in $\mathbb{C} \setminus \{z_0\}$. Then $W_\gamma(z_0) = W_{\gamma'}(z_0)$, i.e. the winding number of a curve around z_0 is invariant under homotopy in $\mathbb{C} \setminus \{z_0\}$.*

We notice that properties of the integral give us immediately:

Lemma 2.3. *Let $\gamma_1 : [a, b] \to \mathbb{C}$, and $\gamma_2 : [c, d] \to \mathbb{C}$ be two closed curves such that $\gamma_1(b) = \gamma_2(c)$. Let $z_0 \notin \gamma_1^* \cup \gamma_2^*$ be a point outside of both their images. Then*

$$W_{\gamma_1+\gamma_2}(z_0) = W_{\gamma_1}(z_0) + W_{\gamma_2}(z_0)$$

In words, the winding number of the concatenation $\gamma_1 + \gamma_2$ around z_0 is the sum of the winding numbers of γ_1 and γ_2 around z_0.

We also assume the following standard result on winding number (see [5]):

Lemma 2.4. *Let z_0 be a point outside the image of γ. Then $W_\gamma(z_0)$ is an integer.*

Intuitively, we know that any closed curve γ should subdivide the plane into distinct regions separated by γ (see Figure 1), similar to the pictures you might find in a "paint by numbers" book. We define these regions via (path-)connected components (as in [4]):

Definition 2.5. Let $U \subseteq \mathbb{C}$ be an arbitrary open set, and $z \in U$. Then the *connected component* of U containing z is the set of all points $w \in U$ that can be joined to z via a curve lying entirely in U, i.e. there exists a curve $\alpha : [0, 1] \to U$ such that $\alpha(0) = z$ and $\alpha(1) = w$.[1]

The connected components are precisely the equivalence classes of the relation given by $z \sim w$ if z can be joined by a curve lying entirely in U to w, so the connected components of an open set U form a *partition* of U. That is, U is the union of all of its connected components, and every two components either coincide or are disjoint. So by taking connected components of $\mathbb{C} \setminus \gamma^*$, we can partition the plane into connected "cells" according to γ.

Furthermore, exactly one of these components is unbounded. Indeed, since $[a, b]$ is compact, and γ continuous, the image of γ, γ^* is compact in $\mathbb{C}$, so γ^* is closed and bounded. Then picking $R > 0$ sufficiently large such that $\gamma^* \subseteq B(0, R)$, we have that $\mathbb{C} \setminus B(0, R) \subseteq \mathbb{C} \setminus \gamma^*$, where $\mathbb{C} \setminus B(0, R)$ is unbounded and connected. Thus there exists exactly one unbounded connected component—namely, the one containing the complement of $B(0, R)$, and it makes sense to speak of *the* unbounded component. [3, Chapter 3]

In particular, for any such component, we expect that the winding number of γ around any two points lying in that region should be the same. And, indeed, we have the following lemma.

Lemma 2.6. *Let γ be a closed curve. Then for each connected component C of $\mathbb{C} \setminus \gamma^*$ and any $z_0, z_1 \in C$, $W_\gamma(z_0) = W_\gamma(z_1)$, i.e. the winding number is constant on connected components.*

[1] Strictly speaking, the above defines the *path-connected* component of U containing z, but in $\mathbb{C}$, the equivalence of path-connectedness and connectedness means that path-connected and connected components coincide.

Proof. We follow the proof given in [3, Chapter 3].

We fix C some connected component of $\mathbb{C} \setminus \gamma^*$, points $z_0, z_1 \in C$, and a curve $\alpha : [0,1] \to C$ starting at $\alpha(0) = z_0$, and ending at $\alpha(1) = z_1$.

We define for each $s \in [0,1]$ a map

$$\begin{aligned} \sigma_s : [0,1] &\to \mathbb{C} \\ t &\mapsto \gamma(t) - \alpha(s) \end{aligned}$$

which is never zero as $\alpha([0,1]) \subseteq C \subseteq \mathbb{C} \setminus \gamma^*$. The σ_s are translations of γ towards the origin by $\alpha(s)$ in such a way that the image of the component C contains the origin.

It is clear that the map

$$\begin{aligned} \sigma : [0,1] \times [0,1] &\to \mathbb{C} \\ (s,t) &\mapsto \sigma_s(t) = \gamma(t) - \alpha(s) \end{aligned}$$

is continuous in both s and t (as γ and α are), so we have a homotopy $\sigma_0 = \gamma - z_0 \simeq \sigma_1 = \gamma - z_1$ in $\mathbb{C} \setminus \{0\}$. In particular, we have

$$W_\gamma(z_0) = W_{\gamma - z_0}(0) = W_{\gamma - z_1}(0) = W_\gamma(z_1)$$

by Lemma 2.2 and translation. □

We have immediately:

Corollary 2.7. *If w lies in the unbounded component of $\mathbb{C} \setminus \gamma^*$, then $W_\gamma(w) = 0$.*

Proof. By the earlier discussion, choose $R > 0$ sufficiently large such that $\gamma^* \subseteq B(0,R)$, and let $z_0 \in \mathbb{C} \setminus B(0,R)$ be some point in its complement. Since $B(0,R)$ is simply connected, we know that γ is homotopic (as closed curves) to a constant curve β in $B(0,R) \subseteq \mathbb{C} \setminus \{z_0\}$. Then $W_\gamma(z_0) = W_\beta(z_0) = 0$ from the definition and Lemma 2.2.

Applying the above lemma, for any w in the unbounded component (but not necessarily outside of $B(0,R)$), $W_\gamma(w) = W_\gamma(z_0) = 0$. □

Given these results, it makes sense to try to determine how winding numbers between adjacent connected components (meaning, those who share a boundary) are related. We do this by considering the case where a (potentially small) segment of the boundary between adjacent components is "linear". For a curve over a line segment z_0 to z_1, we write $\gamma_{z_0 \to z_1} : [0,1] \to \mathbb{C}$ where $\gamma_{z_0 \to z_1}(t) = (1-t)z_0 + tz_1$.

This is the key lemma we will need:

Lemma 2.8. *Let $\gamma : [a,b] \to \mathbb{C}$ be a closed curve oriented counterclockwise. Suppose that (see Figure 2 (i)):*

(i) There is an interval $(t_0, t_1) \subseteq [a,b]$ such that the image of γ restricted to (t_0, t_1) is contained within a disc $D := B(z_0, \delta)$, where $z_0 \in \gamma^*$ and $\delta > 0$, and such that for $t \notin (t_0, t_1)$, $\gamma(t) \notin D$.

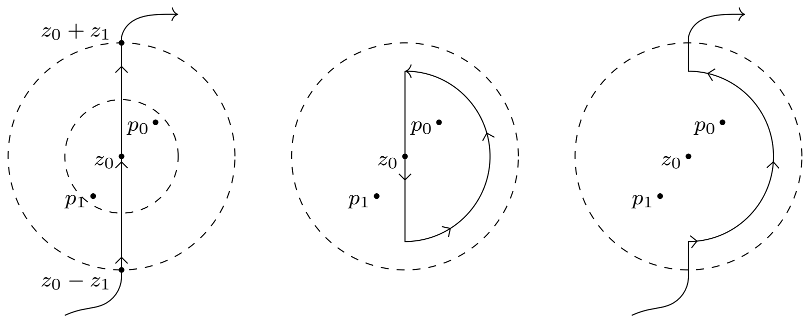

Figure 2: The curves used for Lemma 2.8

(ii) There is a complex number $z_1 \in \mathbb{C}$ such that $|z_0 - z_1| = \delta$, $\gamma(t_0) = z_0 - z_1$, $\gamma(t_1) = z_0 + z_1$ and that γ travels the straight line path crossing through $z_0 - z_1$, z_0 and $z_0 + z_1$, i.e up to reparameterization, $\gamma|_{[t_0,t_1]} \equiv \gamma_{z_0 - z_1 \to z_0 + z_1}$.

Then there exists points $p_0, p_1 \in D$ such that p_1 "lies to the left" of γ (meaning, $z_1 = \lambda e^{i\theta}(p_1 - z_0)$ for $\lambda > 1$ in $\mathbb{R}$ and $\theta \in (-\pi, 0)$) and p_0 "lies to the right" of γ (meaning similarly that $z_1 = \lambda e^{i\theta}(p_0 - z_0)$ for $\lambda > 1$ and $\theta \in (0, \pi)$), and $W_\gamma(p_1) = W_\gamma(p_0) + 1$. [3, Chapter 3]

Remark. The definition of "lying to the left (resp. right)" says that by a dilation and a rotation clockwise (resp. counterclockwise), one can transform the vector $p_1 - z_0$ (resp. $p_0 - z_0$) onto z_1 which is the "direction" that γ is traveling. We will see this again in the next section.

Proof. We follow the proof given in [3, Chapter 3].

It is clear that such points $p_0, p_1 \in D$ exist, and in fact, we can force $|p_i - z_0| < \delta/2$ for $i = 1, 2$. Indeed, we can obtain complex numbers say $\rho_0 = (1/4)e^{-i\pi/2} z_1$ and $\rho_1 = (1/4)e^{i\pi/2} z_1$ such that $\rho_0 + z_0, \rho_1 + z_0 \in B(z_0, \delta/2)$ and which lie to the right and left of γ respectively. By assumption, these points do not lie on γ^*.

Fix such $p_0, p_1 \in B(z_0, \delta/2)$. We construct an auxiliary closed curve θ with initial point $z_0 - \frac{3}{4} z_1$ that traces through the semicircle of radius $\frac{3}{4}\delta$ to $z_0 + \frac{3}{4} z_1$ counterclockwise, and then follows the straight line path back down to $z_0 - \frac{3}{4} z_1$. (see Figure 2 (ii)).

Up to homotopy in $\mathbb{C} \setminus \gamma^*$, (and reparameterization), we may assume γ has base point $z_0 - \frac{3}{4} z_1$ (i.e. that point is the initial and final point of γ). In particular, the concatenation $\theta + \gamma$ tracing first over θ then γ makes sense.

We define also a closed curve $\tilde{\gamma} : [0, 1]$ such that $\tilde{\gamma}$ agrees with $\theta + \gamma$ except on (c, d) (i.e. $\tilde{\gamma}(t) = \gamma(t)$ on $[0, 1] \setminus (c, d)$). On $[c, d]$, we force $\tilde{\gamma}(t) = z_0 + \frac{3}{4} z_1$. In effect, we replace the line segment $z_0 - \frac{3}{4} z_1$ to $z_0 + \frac{3}{4} z_1$ in γ with a semicircular arc of radius $\frac{3}{4}\delta$ heading counterclockwise like the one defined above (see Figure 2 (iii)).

We notice that $\theta + \gamma$ is homotopic to $\tilde{\gamma}$ as closed curves in $\mathbb{C} \setminus \{p_0, p_1\}$. Indeed, it is enough to continuously deform the portion of $\theta + \gamma$ that traces down the line segment $z_0 + \frac{3}{4}z_1$ to $z_0 - \frac{3}{4}z_1$ and then back up to a constant curve at $z_0 + \frac{3}{4}z_1$, and this can be done by scaling back the length of the line segments via linear homotopy in $\mathbb{C} \setminus \{p_0, p_1\}$ as p_0, p_1 are not on the diameter $z_0 - \frac{3}{4}z_1$ to $z_0 + \frac{3}{4}z_1$ (since the rest of the curve can remain constant).

Meanwhile, we have that $W_\theta(p_0) = 1$ and $W_\theta(p_1) = 0$ by construction as points in the interior of θ must make an angle between $(-\pi, 0)$ relative to z_1. Also, since $B(z_0, \delta/2) \subseteq \mathbb{C} \setminus \tilde{\gamma}^*$ by construction, p_0 and p_1 lie in the same connected component of $\tilde{\gamma}$.

Then, by Lemmas 2.2, 2.3 and 2.6:

$$\begin{aligned} W_\gamma(p_1) &= W_{\theta+\gamma}(p_1) - W_\theta(p_1) = W_{\theta+\gamma}(p_1) = W_{\tilde{\gamma}}(p_1) \\ &= W_{\tilde{\gamma}}(p_0) = W_{\theta+\gamma}(p_0) = W_\gamma(p_0) + W_\theta(p_0) = W_\gamma(p_0) + 1 \end{aligned}$$

as required. □

Remark. This makes intuitive sense: given a pair of points, one on the left side of the curve, and the other on the right (where the curve was oriented counterclockwise), we would expect that the curve on the left is being "wrapped" around by one more counterclockwise loop than the one on the right.

If every segment of a curve has a section that satisfies the above, along with Corollary 2.7 that fixes the winding number of the unbounded component to 0, we can use this rule to number every region of the plane with the corresponding winding number: from a given connected component, find a "path" of adjacent components that ends in the unbounded one, and then count the number of times one passes from right to left, or left to right. In fact, even if a curve does not satisfy this property, we can do this to a polygonal curve homotopic to the given one and still obtain a correct numbering. [3]

This is what we will do to show the Alexander numbering rule.

3 Alexander numbering rule

Throughout this section, we let $\gamma : [a, b] \to \mathbb{C}$ be a closed contour (i.e. a closed piecewise-smooth curve), say $\gamma = \gamma_1 + \cdots + \gamma_n$ for $\gamma_i : [a_i, b_i] \to \mathbb{C}$ some smooth curve for $i = 1, \ldots, n$. We let also $\sigma : [c, d] \to \mathbb{C}$ be a piecewise-smooth contour, say $\sigma = \sigma_1 + \cdots + \sigma_m$ for $\sigma_i : [c_j, b_j]$ some smooth curve, $j = 1, \ldots, m$.

Following the above discussion, we want to calculate the winding number of a given connected component of $\mathbb{C} \setminus \gamma^*$ by finding a "path" from that component to the unbounded component. We do this by looking at the intersection the auxiliary curve σ makes with γ. We begin by restricting to curves with sufficiently "nice" intersection points.

Definition 3.1. Let γ and σ be curves as above. We say σ has *smooth, simple, transverse intersection* at a point $w \in \mathbb{C}$ if

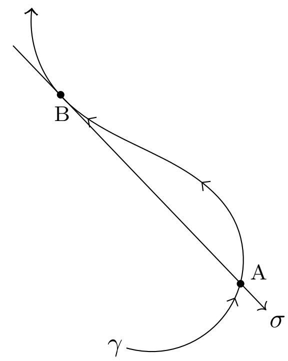

Figure 3: Transverse "cutting" intersection (A) versus nontransverse intersection (B). [3, Chapter 3]

(i) w is in the interior of some γ_i, i.e. $w = \gamma_i(t_i)$ for $a_i < t_i < b_i$.

(ii) w is in the interior of some σ_i, i.e $w = \sigma_j(s_j)$ for $c_j < s_j < d_j$.

(iii) w is traversed only once by σ and γ. That is, there do not exist $t \neq t'$ in $[c, d]$ such that $\sigma(t) = \sigma(t') = w$, and similarly, no $t \neq t'$ in $[a, b]$ satisfy $\gamma(t) = \gamma(t') = w$ either.

(iv) The derivatives $\gamma_i'(t_i)$ and $\sigma_j'(s_j)$ are linearly independent, where $\gamma_i(t_i) = \sigma_j(s_j) = w$.

Moreover, if this is true for all points w_i where the images of σ and γ intersect, we say σ is *transverse* to γ. [5]

Remark. The above definition ensures that at each intersection point, we can determine the two connected components that we are crossing in and out of. The crux is part (iv). Linear independence just means that neither derivative is 0, and that viewing as vectors in $\mathbb{R}^2$, the derivatives are not parallel. This forces σ to "cut" through γ at each intersection point (as compared to "bouncing off") as in Figure 3. Comparing the derivatives gives us a way to determine if σ cuts through from "left to right" or "right to left".

Definition 3.2. Suppose σ has a smooth simple transverse intersection with γ at a point w, where $s_j \in [c_j, d_j]$ and $t_i \in [a_i, b_i]$ satisfy $\gamma_i(t_i) = \sigma_j(s_j) = w$. We say that σ *crosses* γ *from the right* (to the left) if $\sigma_j'(s_j) = \lambda e^{i\theta}\gamma_i'(t_i)$ for some $\lambda > 0$ and $0 < \theta < \pi$. Similarly, we say that σ *crosses* γ *from the left* (to the right) if $\sigma_j'(s_j) = \lambda e^{i\theta}\gamma_i'(t_i)$ for $\lambda > 0$ and $-\pi < \theta < 0$. [5]

Remark. This is corresponds to how we defined points "lying to the left (right)" in Lemma 2.8. Informally, if we view the derivatives as vectors in $\mathbb{R}^2$, we can treat the derivatives as indicating the "direction" σ and γ are moving at an intersection point; the definition of crossing from the right tells us that the "direction" σ is moving in can be obtained by rotating the "direction" γ is moving in clockwise by an angle less than π, in particular the derivative points to the "left" side of that portion of γ. We can say something similar about crossing from the left (to the right).

Theorem 3.3 (Alexander numbering rule). *Let $\gamma : [a, b] \to \mathbb{C}$ be a closed contour, say $\gamma = \gamma_1 + \cdots + \gamma_n$ for $\gamma_i : [a_i, b_i] \to \mathbb{C}$ some smooth curve for $i = 1, \ldots, n$ and let $\sigma : [c, d] \to \mathbb{C}$ be a piecewise-smooth contour, say $\sigma = \sigma_1 + \cdots + \sigma_m$ for $\sigma_i : [c_j, b_j]$ some smooth curve, $j = 1, \ldots, m$. Let $z_0 := \sigma(c)$ be its initial point, and $z_1 := \sigma(d)$ be its final point.*

Suppose there are only finitely many points $w_1, \ldots, w_k$ where the images of γ and σ intersect, and that σ is transverse to γ. Define for each w_ℓ, $\ell = 1, \ldots, k$ the intersection number

$$i(w_\ell) = \begin{cases} +1 & \text{if } \sigma \text{ crosses } \gamma \text{ from the left (to the right)} \\ -1 & \text{if } \sigma \text{ crosses } \gamma \text{ from the right (to the left)} \end{cases}$$

Then

$$W_\gamma(z_0) - W_\gamma(z_1) = \sum_{\ell=1}^{k} i(w_\ell)$$

That is, $W_\gamma(z_0) - W_\gamma(z_1)$ is equal to the number of crossings from the left minus the number of crossings from the right. [3, 5]

Proof. By assumption, we know that σ and γ cannot intersect at z_0 and z_1 which are endpoints of σ_1 and σ_m respectively, so $z_0, z_1 \notin \gamma^*$ and the minimum distance $\operatorname{dist}(z_i, \gamma^*) > 0$ for $i = 0, 1$. Because of this and since there are finitely many intersection points, we can pick $\epsilon > 0$ sufficiently small so that the ball of radius ϵ around each w_ℓ does not contain z_0, z_1 or any other $w_{\ell'}$ for $\ell' \neq \ell$.

Fix ℓ, i, j such that $w_\ell = \gamma_i(t_i) = \sigma_j(s_j)$, some $t_i \in [a_i, b_i]$, $s_j \in [c_j, d_j]$. By assumption that γ doesn't traverse w_ℓ more than once, we can assume that the maximal interval $[t_0, t_1] \subseteq [a, b]$ containing t_ℓ such that $\gamma((t_0, t_1)) \subseteq B(w_\ell, \epsilon)$ satisfies that for $t \notin (t_0, t_1)$, $\gamma(t) \notin B(w_\ell, \epsilon)$. Indeed, if not, say, over some time interval (t_2, t_3) outside of (t_0, t_1), the image of γ lies in the ball, we must have $\gamma((t_2, t_3)) \subseteq B(w_\ell, \epsilon) \setminus \{w_\ell\}$. We can shrink ϵ (and the interval $[t_0, t_1]$) to force $\gamma((t_2, t_3)) \cap B(w_\ell, \epsilon) = \varnothing$. If no ϵ sufficiently small works, then w_ℓ would have to be a limit point, and by continuity lies in $\gamma([t_2, t_3])$, which forces $[t_2, t_3] \subseteq (t_0, t_1)$.

We scale $\gamma_i'(t_i)$ by appropriate $\lambda > 0$ such that $w_\ell + \lambda\gamma_i'(t_i)$ and $w_\ell - \lambda\gamma_i'(t_i)$ lie within the ball of radius ϵ. Let $W := \lambda\gamma_i'(t_i)$ Then we define a new curve $\tilde{\gamma} : [a, b] \to \mathbb{C}$ such that over $[a, b] \setminus (t_0, t_1)$, $\tilde{\gamma}(t) = \gamma(t)$, and over $[t_0, t_1]$, $\tilde{\gamma}|_{[t_0, t_1]} = \theta_1 + \theta_2 + \theta_3$, where θ_1, θ_2, and θ_3 are reparameterizations of the curves $\gamma_{\gamma(t_0) \to w_\ell - W}$, $\gamma_{w_\ell - W \to w_\ell + W}$, and $\gamma_{w_\ell + W \to \gamma(t_1)}$ respectively so that $\tilde{\gamma}(t_i) = \gamma(t_i) = w_\ell$.

By choice of ϵ, a linear homotopy does not pass through z_0, z_1, so $\tilde{\gamma}$ is homotopic to γ in $\mathbb{C} \setminus \{z_0, z_1\}$ as closed curves, and in a way such that σ intersects γ at w_ℓ still. We can repeat the same process for each w_ℓ to force γ to be locally "straight" at each intersection point, so that we can apply Lemma 2.8 on each $B(w_\ell, |\lambda\gamma_i'(t_i)|)$.

Suppose first that we have a crossing from the left to the right at w_ℓ In particular, we take $\lambda > 0$ sufficiently small such that $p_0 \coloneqq w_1 + \lambda\sigma_j'(s_j)$ and $p_1 \coloneqq w_1 - \lambda\sigma_j'(s_j)$ lie in the ball above. In particular, by linear approximation, for small enough λ, there are times $L < s_j < R$ such that p_1 is contained in a ball around $\sigma_j(L)$ not intersecting the image of γ, and similarly, such that p_0 is contained in a ball around $\sigma_j(R)$. In particular, the straight line paths $\gamma_{\sigma_j(L)\to p_1}$ and $\gamma_{p_0\to\sigma_j(R)}$ lie entirely in $\mathbb{C}\setminus\gamma^*$, that is, these pairs lie in the same connected components.

By definition of crossing from the left, $\lambda\sigma_j'(s_j) = ce^{i\theta}\gamma_i'(t_i)$ for some constant $c > 0$ and $\theta \in (-\pi, 0)$. Equivalently, $\gamma_i'(t_i) = Ce^{-i\theta}(p_0 - w_\ell) = Ce^{i\theta}(p_1 - w_\ell)$ for $\theta \in (-\pi, 0)$, i.e. p_0 lies to the right of γ, and p_1 lies to the left of γ.

Similarly, defining p_0, p_1 as above in the case we have a crossing from the right to the left at w_ℓ, p_0 lies to the left of γ, while p_1 lies to right of γ.

We may assume up to relabeling that the w_ℓ are numbered such that $s_{\ell_1} < s_{\ell_2}$ if $\ell_1 < \ell_2$. By induction on the number of intersection points k, for $k = 1$, if $w_1 = \sigma_j(s_j) = \gamma_i(t_i)$ some i, j, is a crossing from the left to right (i.e. with $i(w_1) = +1$) then by Lemma 2.8, $W_\gamma(p_1) = W_\gamma(p_0) + 1$. But by choice of p_0, p_1, there is a path connecting z_0 to p_1 through $\sigma_j(L)$ and similarly, a path connecting p_0 to z_1 through $\sigma_j(R)$. Thus, $W_\gamma(z_0) - W_\gamma(z_1) = i(w_1) = +1$ by Lemma 2.6. If instead we had a crossing from right to left, $W_\gamma(p_0) = W_\gamma(p_1)+1$, and $W_\gamma(z_0) - W_\gamma(z_1) = i(w_1) = -1$.

Then for arbitrary k, we can split σ into two halves $\sigma|_{[c,R]}$ and $sigma|_{[R,d]}$ where R is defined as above for w_1. The result is satisfied over both halves by induction, so:

$$W_\gamma(z_0) - W_\gamma(\sigma(R)) = i(w_1)$$

$$W_\gamma(z_0) - (W(z_1) - \sum_{\ell=2}^{k} i(w_\ell)) = i(w_1)$$

$$W_\gamma(z_0) - W(z_1) = \sum_{\ell=1}^{k} i(w_\ell)$$

as required. □

4 Some further remarks, questions and closing

While the Alexander numbering rule gives us an efficient way to compute winding numbers given that we have the nice auxiliary curve σ, it would be useful to have a way of finding such a σ for any given γ. This might be especially useful for computer algorithms that rely on winding number. For example, some "point in polygon" algorithms use a similar "axis-crossing" rule to compute winding numbers to determine whether or not a point is "inside" of a polygon. These proceed by updating the winding number by how an edge of a polygon cuts through the positive x-axis which corresponds to the method here, besides allowing for non-transverse intersection. [1] Could something similar be done for a more arbitrary closed curve?

Restricting to the case where σ is just a line segment (a "ray"), it can be shown that in the case of smooth γ, there is a way to homotopically deform γ so that σ intersects γ at finitely many points. [3] It can also be shown that almost all rays are "nice" and transverse to γ. [3] So, picking a ray at random in the smooth case is a viable option. Is there an analogous result for the non-smooth case?

Finally, the statement and proof of the Alexander numbering rule above relied heavily on the assumption that γ and σ were piecewise smooth so that we could take derivatives and compare their "directions" to check whether we had a crossing from the left, or a crossing from the right. Can we generalize the rule for arbitrary curves?

References

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